Lecture 6:

Downlink Processing, SDMA, TX Diversity

and Spatial Multiplexing/Multi-Stream TX

Overview

- UMTS up- & downlink: further considerations
- downlink processing & transmit diversity
- spatial division multiple access (SDMA)
- MIMO: Spatial Multiplexing/Multi-Stream TX

MISO case: BS Downlink Processing

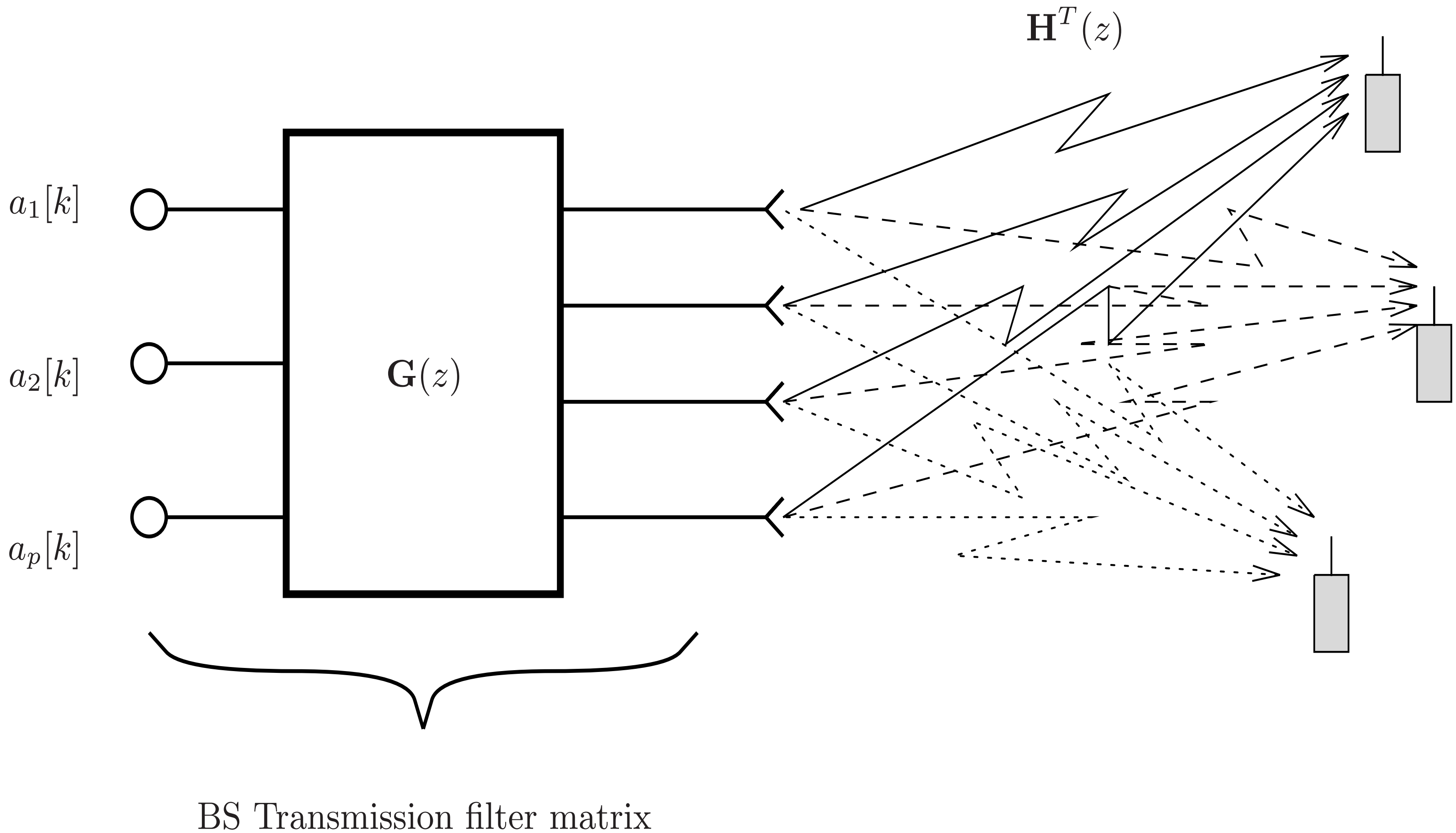
- CSIT: Channel State Information at the Transmitter (Tx)
- Full CSIT: channel known to the Tx: (TDD, channel feedback in FDD)
  - diversity: pre-RAKE
  - interference: prefilter for ISI & IUI cancellation
- no CSIT: channel unknown to the Tx (FDD): TX diversity
  - delay diversity
  - Alamouti scheme: allows to recover full diversity of 2 TX antennas

Full CSIT: Zero ISI and IUI Transmission Filter (Downlink)

- $H_{ij}(z)$ is the sampled impulse response from MS j to antenna i of the BS. Assuming channel reciprocity, the channel impulse response from antenna i to mobile j is also $H_{ij}(z)$. Hence the transfer matrix on the downlink from the m antennas to the p mobiles is $\mathbf{H}^T(z)$.
- Let $G_{ij}(z)$ be the transmitter filter linking the signal to be tansmitted to mobile j to antenna i, then $\mathbf{G}(z)$ is the overall $(m \times p)$ transmitter filter to be used on the downlink. Using this transmitter filter, the signals intended for each mobile user will arrive at the respective users without ISI or IUI if

$$\mathbf{H}^T(z)\,\mathbf{G}(z) = \operatorname{diag}\{z^{-d_1} \cdots z^{-d_p}\}$$

$\mathbf{G}(z)$ FIR can be found under the same conditions.



Nonlinear approach: "dirty paper precoding" (multiuser version of Tomlinson-Harashima precoding)

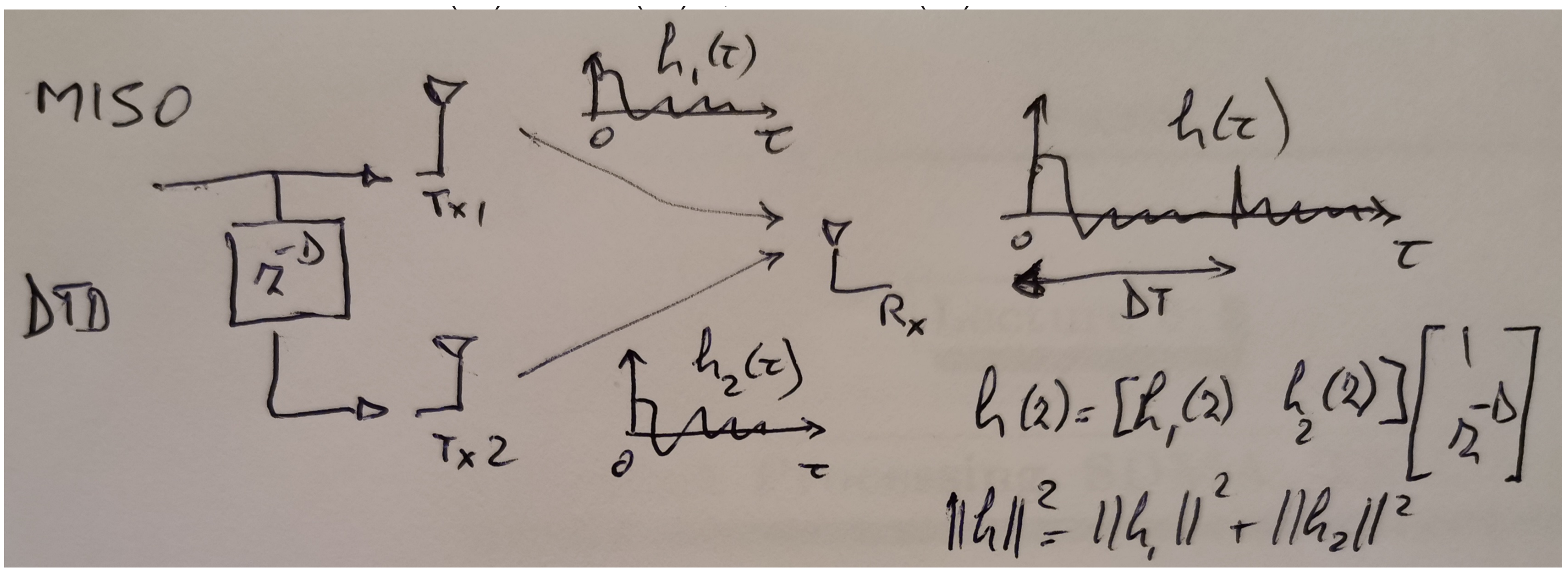
No CSIT: Transmit Diversity

Replace (multi-antenna) diversity at reception by diversity at transmission (2 Tx antennas at BS).

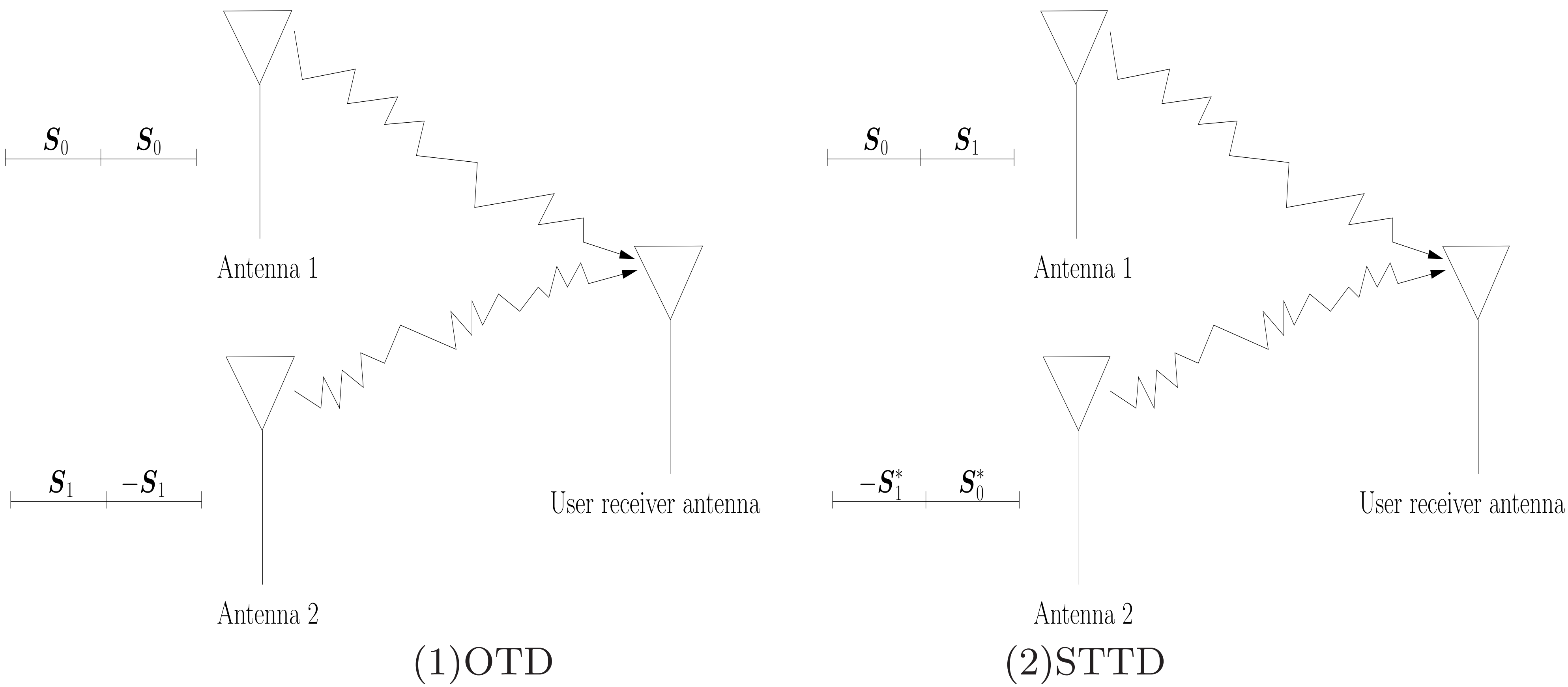
- No additional RF complexity at mobile terminal.
- Improved SNR.
- Improved SINR ? (for CDMA systems)

Transmit Diversity Modes

- DTD: Delay Transmit Diversity
  Both Tx antennas transmit the same signal, but with a differential delay D.
  h^i is the channel between Tx antenna i and the mobile and the overall channel is: $h(z) = h^1(z) + z^{-D}\, h^2(z)$
- OTD: Orthogonal Transmit Diversity
- STTD: Space-Time Transmit Diversity



Transmit Diversity Modes (2)



Space-Time Transmit Diversity (STTD) (Alamouti)

- $[y_0 \;\; y_1] = [h_1 \;\; h_2] \begin{bmatrix} s_0 & s_1 \\ -s_1^* & s_0^* \end{bmatrix} + [v_0 \;\; v_1]$

- $$\begin{aligned} y_0 &= h_1\, s_0 - h_2\, s_1^* + v_0 \\ y_1 &= h_1\, s_1 + h_2\, s_0^* + v_1 \\ \Rightarrow\; y_1^* &= h_1^*\, s_1^* + h_2^*\, s_0 + v_1^* \end{aligned}$$

- $[y_0 \;\; y_1^*] = [s_0 \;\; s_1^*] \begin{bmatrix} h_1 & h_2^* \\ -h_2 & h_1^* \end{bmatrix} + [v_0 \;\; v_1^*]$ or $Y = S\, H + V$
- $H^H H = (|h_1|^2 + |h_2|^2)\, I_2$ scaled unitary
- $\hat{S} = [\widehat{s_0} \;\; \widehat{s_1^*}] = Y\, H^H = S\, H\, H^H + V\, H^H = (|h_1|^2 + |h_2|^2)\, S + V^{'}$
- $\mathrm{E}\, V^{'\,H} V^{'} = \sigma_v^2\, (|h_1|^2 + |h_2|^2)\, I_2$
- $\mathrm{SNR} = \mathrm{SINR} = \frac{\sigma_s^2}{\sigma_v^2} (|h_1|^2 + |h_2|^2)$

Full CSIT: Spatial Division Multiple Access (SDMA)

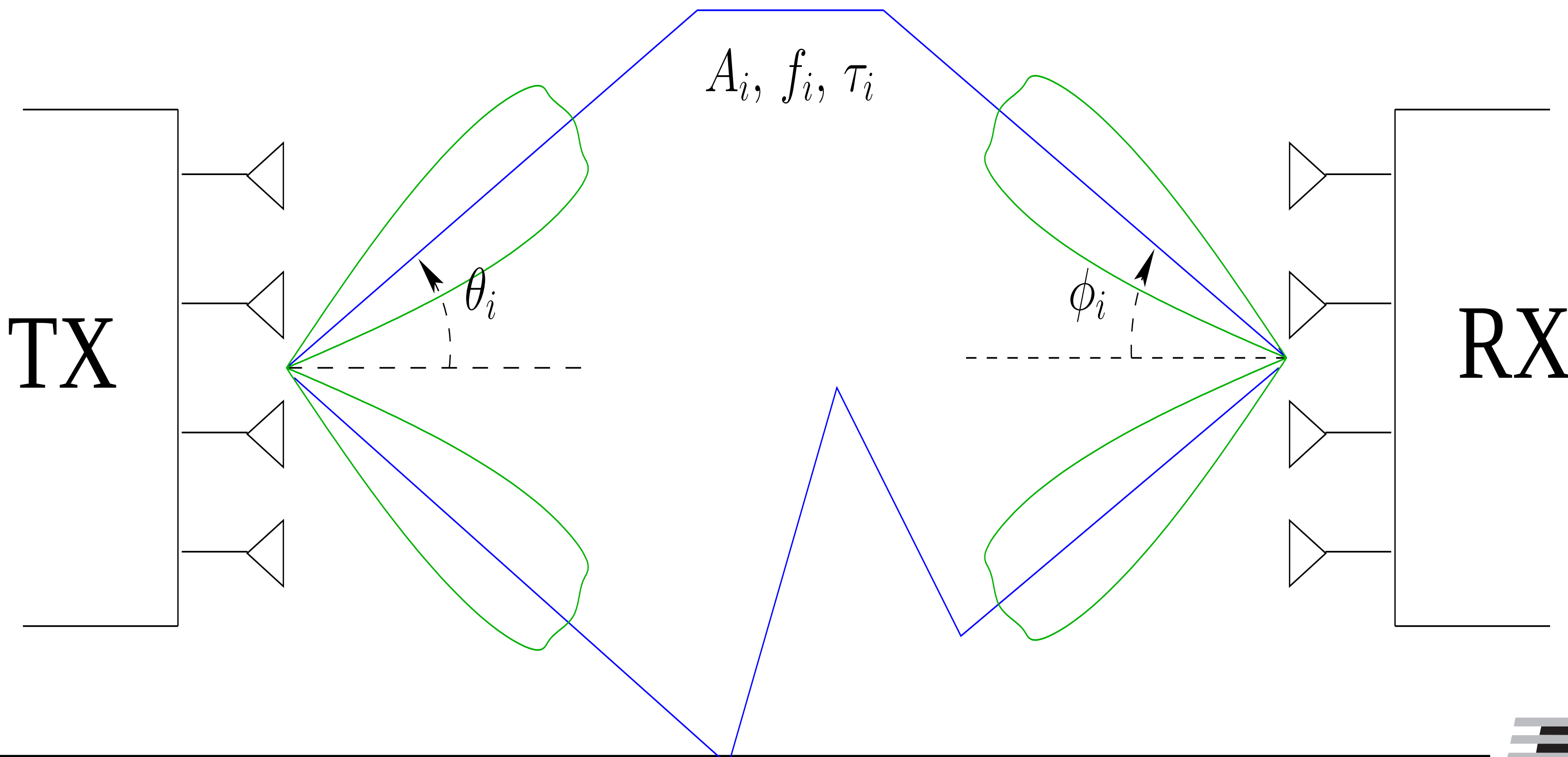
- what? distinguish multiple simultaneous users in the same frequency band through their different spatiotemporal characteristics (vector channel between mobile and BS antenna array); combines with TDMA or CDMA
- SDMA: (early '90s): based on multipath directions
  MU MIMO (Multi-User): based on channel responses
- uplink:
  - FIR ISI and IUI zero-forcing equalizers exist with m channels for up to $m-1$ users (channel and equalizer matricial)
  - non-linear MUD also an option
  - training-sequence length required for channel estimation proportional to number of users
  - in OFDM: spatial MU problem per subcarrier

Spatial Division Multiple Access (SDMA) (2)

- downlink:
  - with m FIR channels, an FIR transmission matrix filter exists so that the up to $m-1$ symbol sequences arrive at the respective mobile users without ISI nor IUI! Requires knowledge of the downlink matrix channel.
  - case of Time Division Duplex (TDD): reciprocity of the channel can be used (channel estimated from uplink)
  - otherwise (FDD): directional information has to be used. Only works well in essentially Line of Sight (LoS): requires user selection.
  - LTE: channel knowledge thru feedback $\Rightarrow$ applicable to any channel.

(Single-User) MIMO Spatial Multiplexing

- with antenna arrays at TX & RX: can separate paths on both sides and send different data streams per path simultaneously
- like SDMA with colocated terminals, N_t Tx antennas, N_r Rx antennas; MIMO: Multi-Input Multi-Output
- invented '95 (Paulraj (Stanford), Foschini (Bell Labs))



MIMO Spatial Multiplexing (2)

- number of resolvable paths for simultaneous transmission: given by channel transfer matrix rank
  paths need to be simultaneously resolvable at both TX & RX to contribute to the rank
- in contrast, it suffices that paths are resolvable at either TX or RX to contribute to diversity degree (rank of covariance matrix of vectorized channel coefficients)
- “full” data stream system: $N_s = \#$ data streams $= \min(N_t, N_r) =$ maximum channel matrix rank.
- full diversity systems: all data streams benefit from all diversity elements in the channel
- multi-stream detection = centralized multi-user detection

MIMO channel rank $f_i \equiv 0,\ \tau_i \equiv 0$

$$H = \sum_{i=1}^{L} \alpha_i \, \underline{a}_R(\varphi_i) \, \underline{a}_T^T(\theta_i) \qquad \text{rank} \le \min(M_T, M_R, L)$$

$$\underline{y}[k] = H(q)\,\underline{a}[k] + \underline{v}[k]$$
$$= H(q)\,T(q)\,\underline{b}[k] + \underline{v}[k]$$

$$S_{\underline{y}\underline{y}}(f) = H(f)\,T(f)\,S_{\underline{b}\underline{b}}(f)\,T^H(f)\,H^H(f) + S_{\underline{v}\underline{v}}(f)$$

Gaussian Capacity (MI) $C = h(\underline{y}) - h(\underline{y}|\underline{b})$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \log_2 \det\left(S_{\underline{y}\underline{y}}(f)\right) df - \int_{-\frac{1}{2}}^{\frac{1}{2}} \log_2 \det\left(S_{\underline{v}\underline{v}}(f)\right) df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \log_2 \det\left(I_{M_R} + \frac{\sigma_b^2}{\sigma_v^2} H(f)\,T(f)\,T^H(f)\,H^H(f)\right) df$$

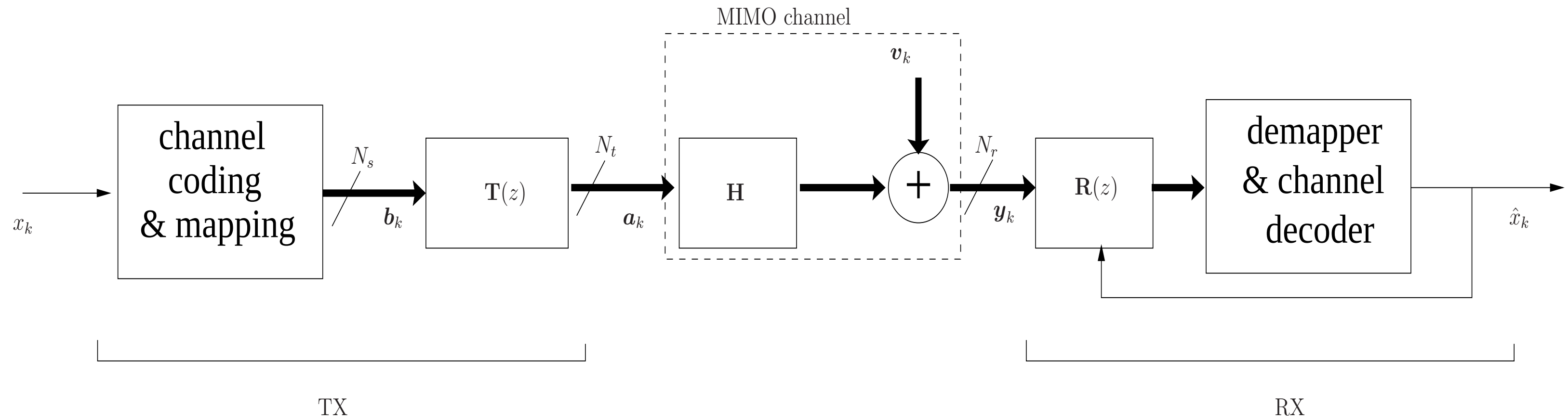
$$T(z) = D(z)\,Q = \frac{1}{\sqrt{M_t}} \begin{bmatrix} 1 & & & 0 \\ & z^{-1} & & \\ & & \ddots & \\ 0 & & & z^{-(M_t-1)} \end{bmatrix} \begin{bmatrix} 1 & \times & \cdots & \cdots \\ 1 & & & \vdots \\ \vdots & & & \vdots \\ 1 & \times & \cdots & \times \end{bmatrix} = \frac{1}{\sqrt{M_t}} \begin{bmatrix} 1 & - & \cdots \\ z^{-1} & & \\ \vdots & & \\ z^{-(M_t-1)} & \cdots & \end{bmatrix}$$

MIMO Spatial Multiplexing (3)

- channel known to TX and RX: spatial waterfilling
- channel unknown to TX, known to RX:
  channel capacity $\sim \min(N_t, N_r)$ for iid channel elements
- channel unknown to TX and RX:
  channel capacity not degraded if channel variation slow
  matrix differential encoding approaches or channel estimation

Linear Convolutive ST Precoding

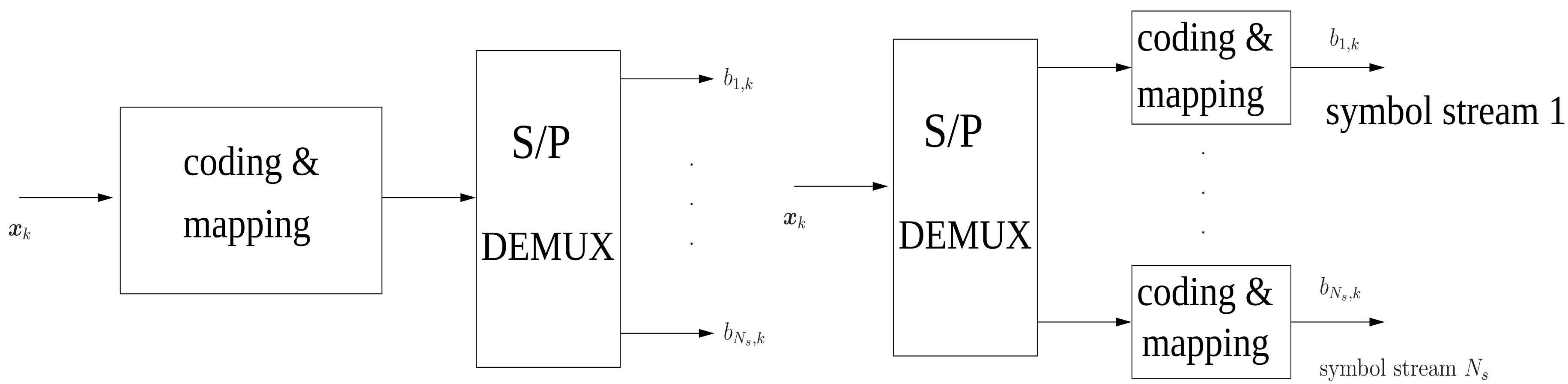
- general ST precoding setup:



inner code: linear precoding, outer code: channel coding

Linear Convolutive ST Precoding (2)

- Two channel coding, interleaving, symbol mapping and demultiplexing choices.



- linear dispersion codes: channel coding & mapping in SISO, prefiltering is done in blocks.
- no prefiltering case: channel coding & mapping is SISO, prefiltering is reduced to S/P conversion

Linear Convolutive ST Precoding (3)

- Proposed approach:
  idea: use linear prefiltering to exploit all diversity so that channel coding doesn't have to do that anymore
  channel coding & mapping starts with S/P conversion into N_s substreams, independent channel coding & mapping per substream, MIMO prefiltering of substreams before TX

  rate = # of substreams N_s
  Now basic open-loop (= no CSIT) MIMO mode of LTE: (MIMO) CDD (Cyclic Delay Diversity).

- special cases:

  VBLAST: $N_s = N_t$ ("full rate"), $\mathbf{T}(z) = \mathbf{I}_{N_t}$ (assume $N_r \geq N_t$)

  DBLAST: $N_s = 1$ ("single rate"), $\mathbf{T}(z) = [1\ \ z^{-1}, \ldots, z^{-(N_t-1)}]^T$

Capacity Considerations

- In what follows, we consider frequency flat channels and full rate systems (square $\mathbf{T}(z) : N_t \times N_t$)

- Ergodic capacity:
$$\begin{aligned}\mathbf{C} &= \mathrm{E}_H \frac{1}{2\pi j} \oint \frac{dz}{z} \log_2 \det(I + \frac{1}{\sigma_v^2} \mathbf{H}\mathbf{T}(z) S_{\boldsymbol{bb}}(z) \mathbf{T}^\dagger(z) \mathbf{H}^\dagger) \\ &= \mathrm{E}_H \frac{1}{2\pi j} \oint \frac{dz}{z} \log_2 \det(I + \rho\, \mathbf{H}\mathbf{T}(z) \mathbf{T}^\dagger(z) \mathbf{H}^\dagger)\end{aligned}$$
$$S_{\boldsymbol{bb}}(z) = \sigma_b^2\, I \,,\; S_{\boldsymbol{vv}}(z) = \sigma_v^2\, I \,,\; \rho = \frac{\sigma_b^2}{\sigma_v^2}$$

- No Channel Side Information at TX: $\displaystyle\max_{\mathbf{T}(z):\, tr\{\oint \mathbf{T}(z)\mathbf{T}^\dagger(z)\} = N_t} \mathbf{C}$

  $\Rightarrow$ $\mathbf{T}(z)$ paraunitary: $\mathbf{T}(z)\mathbf{T}(z)^\dagger = I$ to avoid capacity loss

Capacity Considerations (2)

- proposed $\mathbf{T}(z) = \mathbf{D}(z)\, Q$

  $\mathbf{D}(z) = diag\{1,\, z^{-1}, \ldots, z^{-(N_t-1)}\}$ delay diversity

  Q: unitary with $|Q_{ij}| = \frac{1}{\sqrt{N_t}}$ unitary mixing = spatial spreading

  ($z^{-1} \to z^{-N}$ for channel of length N (delay spread))

- substream k passes through equivalent SIMO channel

$$\sum_{i=1}^{N_t} z^{-(i-1)} \mathbf{H}_{:,i} Q_{ik}$$

Matched Filter Bounds (MFB)

→ multistream $MFB = MFB$ for a given substream

→ VBLAST ($\mathbf{T}(z) = \mathbf{I}$): substream i: $MFB_i = \rho||\mathbf{H}_{:,i}||_2^2$

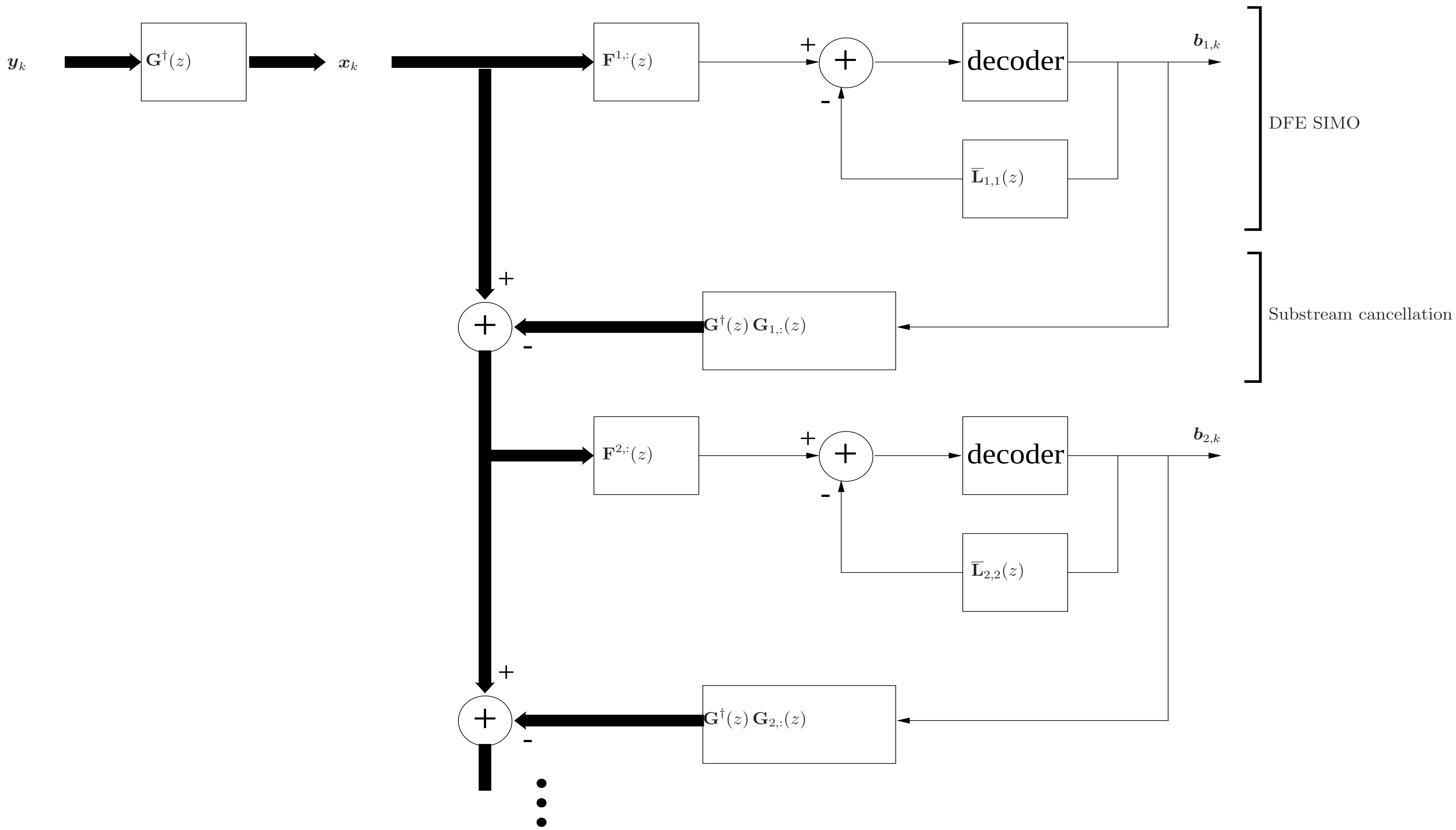
diversity limited to N_r

→ $\mathbf{T}(z) = \mathbf{D}(z)\, Q$: subtream i: $MFB_i = \rho \frac{1}{N_t} ||\mathbf{H}||_F^2$

hence this $\mathbf{T}(z)$ provides the same full diversity ($N_t N_r$) for all substreams.

MIMO MMSE ZF DFE RX

Receiver Structure ("stripping"):



MIMO DFE RX

- let $\mathbf{G}(z) = \mathbf{H}\,\mathbf{T}(z) = \mathbf{H}\,\mathbf{D}(z)\,Q$ channel + precoding
- matched filter RX:

  $$\boldsymbol{x}_k = \mathbf{G}^\dagger(q)\,\boldsymbol{y}_k = \mathbf{G}^\dagger(q)\,\mathbf{G}(q)\,\boldsymbol{b}_k + \mathbf{G}^\dagger(q)\,\boldsymbol{v}_k = \mathbf{R}(q)\,\boldsymbol{b}_k + \mathbf{G}^\dagger(q)\,\boldsymbol{v}_k$$

  where $\mathbf{R}(z) = \mathbf{G}^\dagger(z)\,\mathbf{G}(z)$, psdf of $\mathbf{G}^\dagger(q)\,\boldsymbol{v}_k$ is $\sigma_v^2\,\mathbf{R}(z)$
- DFE RX:

  $$\widehat{\mathbf{b}}_k = -\underbrace{\overline{\mathbf{L}}(q)}_{\text{feedback}}\;\mathbf{b}_k + \underbrace{\mathbf{F}(q)}_{\text{feedback}}\;\mathbf{x}_k$$

  where feedback $\overline{\mathbf{L}}(z)$ is strictly "causal"
- 2 design criteria for feedforward and feedback filters:
  MMSE ZF and MMSE

MIMO MMSE ZF DFE RX

- matrix spectral factorization:

$$\mathbf{G}^{\dagger}(z)\mathbf{G}(z) \;=\; \mathbf{R}(z) \;=\; \mathbf{L}^{\dagger}(z)\,\Sigma\,\mathbf{L}(z)$$

  $\mathbf{L}(z) = \sum_k \mathbf{L}_k\, z^{-k}$ with $diag(\mathbf{L}_0) = I$ (monic),
  $\Sigma > 0$ diagonal constant

- then $\mathbf{F}(z) \;=\; \Sigma^{-1}\,\mathbf{L}^{-\dagger}(z)$, $\;\overline{\mathbf{L}}(z) \;=\; \mathbf{L}(z) - I$

- total feedforward filter: scaled Whitened Matched Filter (WMF)

$$\mathbf{F}(z)\mathbf{G}^{\dagger}(z) = \Sigma^{-\frac{1}{2}}\;\Sigma^{-\frac{1}{2}}\mathbf{L}^{-\dagger}(z)\mathbf{G}^{\dagger}(z) = \Sigma^{-\frac{1}{2}}\;\mathbf{U}(z)$$

  where $\mathbf{U}(z)$ = paraunitary/lossless/WMF

MIMO MMSE ZF DFE RX (2)

- forward filter output

$$\mathbf{F}(q)\,\mathbf{x}_k = \mathbf{L}(q)\,\mathbf{b}_k + \mathbf{F}(q)\mathbf{G}^{\dagger}(q)\,\mathbf{v}_k = \mathbf{L}(q)\,\mathbf{b}_k + \mathbf{e}_k$$

where $\mathbf{S_{ee}}(z) \;=\; \sigma_v^2\,\Sigma^{-1}$

- at detector output i: $\quad \mathrm{SNR}_i = \rho\,\Sigma_{ii}$
- can detect the $\mathbf{b}_k$ elementwise by backsubstitution (feedback) and symbol-by-symbol detection

MIMO MMSE DFE RX

- backward channel model based on LMMSE:

$$\mathbf{b}_k = \widehat{\mathbf{b}}_k + \widetilde{\mathbf{b}}_k = \mathbf{S}_{\mathbf{bx}}(q)\,\mathbf{S}_{\mathbf{xx}}^{-1}(q)\,\mathbf{x}_k + \widetilde{\mathbf{b}}_k$$

where $\mathbf{S}_{\mathbf{bx}}(z) = \mathbf{S}_{\mathbf{bb}}(z)\,\mathbf{G}^\dagger(z)\mathbf{G}(z)$ and
$\mathbf{S}_{\mathbf{xx}}(z) = \mathbf{G}^\dagger(z)\mathbf{G}(z)\,\mathbf{S}_{\mathbf{bb}}(z)\,\mathbf{G}^\dagger(z)\mathbf{G}(z) + \sigma_v^2\,\mathbf{G}^\dagger(z)\mathbf{G}(z)$

$\Rightarrow\ \mathbf{S}_{\mathbf{bx}}(z)\,\mathbf{S}_{\mathbf{xx}}^{-1}(z) = \mathbf{R}^{-1}(z)$ with

$\mathbf{R}(z) = \mathbf{G}^\dagger(z)\mathbf{G}(z) + \sigma_v^2\,\mathbf{S}_{\mathbf{bb}}^{-1}(z) = \mathbf{G}^\dagger(z)\mathbf{G}(z) + \frac{1}{\rho}\,I$

- so $\mathbf{b}_k = \mathbf{R}^{-1}(q)\mathbf{x}_k + \widetilde{\mathbf{b}}_k$

- $\mathbf{S}_{\widetilde{\mathbf{b}}\widetilde{\mathbf{b}}}(z) = \mathbf{S}_{\mathbf{bb}}(z) - \mathbf{S}_{\mathbf{bx}}(z)\mathbf{S}_{\mathbf{xx}}^{-1}(z)\mathbf{S}_{\mathbf{xb}}(z) = \sigma_v^2\,\mathbf{R}^{-1}(z)$

MIMO MMSE DFE RX (2)

- again matrix spectral factorization:

$$\mathbf{R}(z) \;=\; \mathbf{L}^{\dagger}(z)\,\Sigma\,\mathbf{L}(z)$$

then $\quad \mathbf{b}_k \;=\; \mathbf{L}^{-1}(q)\,\Sigma^{-1}\,\mathbf{L}^{-\dagger}(q)\,\mathbf{x}_k + \widetilde{\mathbf{b}}_k$

- we get

$$\mathbf{F}(q)\,\mathbf{x}_k = \Sigma^{-1}\,\mathbf{L}^{-\dagger}(q)\,\mathbf{x}_k \;=\; \mathbf{L}(q)\,\mathbf{b}_k - \mathbf{L}(q)\,\widetilde{\mathbf{b}}_k = \mathbf{L}(q)\,\mathbf{b}_k + \mathbf{e}_k$$

where $\quad \mathbf{S}_{\mathbf{ee}}(z) \;=\; \mathbf{L}(z)\mathbf{R}^{-1}(z)\mathbf{L}^{\dagger}(z) \;=\; \sigma_v^2\,\Sigma^{-1}$

- at detector output i again: $\quad \mathrm{SNR}_i = \rho\,\Sigma_{ii}$
- $\Sigma^{MMSE} > \Sigma^{MMSEZF} \quad \Rightarrow \quad \mathrm{SNR}_i^{MMSE} > \mathrm{SNR}_i^{MMSEZF}$
- even $\quad \mathrm{SNR}_i^{UMMSE} = \mathrm{SNR}_i^{MMSE} - 1 > \mathrm{SNR}_i^{MMSEZF}$

Capacity Decomposition

- for a given channel realization

$$\begin{aligned}
\mathbf{C} &= \frac{1}{2\pi j}\oint \frac{dz}{z}\log_2\det(I_{N_r}+\rho\,\mathbf{G}(z)\mathbf{G}^\dagger(z)) \\
&= \frac{1}{2\pi j}\oint \frac{dz}{z}\log_2\det(I_{N_t}+\rho\,\mathbf{G}^\dagger(z)\mathbf{G}(z)) \\
&= \frac{1}{2\pi j}\oint \frac{dz}{z}\log_2\det(\rho\,\mathbf{R}^{MMSE}(z)) = \log_2\det(\rho\,\Sigma^{MMSE}) \\
&= \sum_{n=1}^{N_t}\log_2 \mathrm{SNR}_i^{MMSE} = \sum_{n=1}^{N_t}\log_2(1+\mathrm{SNR}_i^{UMMSE})
\end{aligned}$$

- total capacity = sum of capacities of N_t substreams output by a UMMSE DFE, taken as independent AWGN channels (Gaussian approximation of UMMSE error signal)

Triangular MIMO DFE and VBLAST

- with triangular feedback: MIMO DFE works as follows:
  1. we apply a SIMO DFE to detect a substream, the design of the SIMO DFE considers the remaining substreams as colored noise.
  2. we subtract the detected and decoded substream from the RX signal and pass on to the next substream.
- For the first substream, all remaining streams are interferers, the last substream gets detected in the single stream scenario.
- triangular MIMO DFE = extension of VBLAST to dynamic case

Triangular MIMO DFE and VBLAST (2)

- here: dynamics (temporal dispersion) have been introduced by linear convolutive precoding (introducing delay diversity).
- Advantages:

  $\rightarrow$ no ordering issue: can process streams in any order

  $\rightarrow$ higher diversity order, less dispersion of substream SNRs

Blind MIMO Channel Estimation

- channel estimation crucial component of MIMO systems
- blind SIMO channel estimation from second-order statistics works well if channel has no zeros
- similar blind MIMO channel estimation leaves a large number of inidentifiabilities
- MIMO case can be reduced to a collection of SIMO cases by coloring the different inputs (frequency domain and time domain approaches)
- to maximize (known channel) capacity though, no such coloring should be introduced

Singular Value Decomposition (SVD)

- Any rectangular $n \times m$ complex matrix $\mathbf{H}$ can be uniquely decomposed as

$$\mathbf{H} = \mathbf{U}\,\Sigma\,\mathbf{V}^H$$

where $\mathbf{U}$ is $n \times n$ unitary, $\mathbf{U}^{-1} = \mathbf{U}^H$, $\mathbf{V}$ is $m \times m$ unitary, $\mathbf{V}^{-1} = \mathbf{V}^H$, Σ is a $n \times m$ "diagonal" matrix with diagonal elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ which are called resp. the left and right singular vectors, and associated (non-zero) singular values, and r is the rank of $\mathbf{H}$.

- Relations SVD and eigen decompositions:

$$\mathbf{H}\mathbf{H}^H = \mathbf{U}\,\Sigma\Sigma^H\,\mathbf{U}^H \geq 0 \ , \quad \mathbf{H}^H\mathbf{H} = \mathbf{V}\,\Sigma^H\Sigma\,\mathbf{V}^H \geq 0$$

MIMO w Perfect CSIT: Optimal Tx Covariance

- For a given known time-invariant channel $\mathbf{H}$, in the presence of additive spatiotemporal white Gaussian noise and under a Tx power constraint, the mutual information maximizing input is a stationary temporally white Gaussian noise.

- It's spatial covariance $\mathbf{T}\mathbf{T}^H$ can be interpreted as the covariance of i.i.d. streams spatially filtered by $\mathbf{T}$.

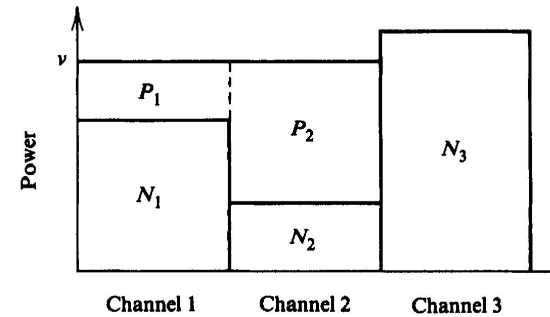
- The capacity achieving optimal Tx filter $\mathbf{T}$ can be found as

$$\begin{aligned}
\mathbf{C} &= \max_{\mathbf{T}:\mathrm{tr}\{\mathbf{T}\mathbf{T}^H\}=P} \log_2 \det(I + \tfrac{1}{\sigma_v^2}\, \mathbf{H}\mathbf{T}\mathbf{T}^H\mathbf{H}^H) \\
&= \max_{\mathbf{T}:\mathrm{tr}\{\mathbf{T}\mathbf{T}^H\}=P} \log_2 \det(I + \tfrac{1}{\sigma_v^2}\, \mathbf{U}\,\Sigma\,\mathbf{V}^H\mathbf{T}\mathbf{T}^H\mathbf{V}\,\Sigma^H\,\mathbf{U}^H) \\
&= \max_{\mathbf{T}:\mathrm{tr}\{\mathbf{T}\mathbf{T}^H\}=P} \log_2 \det(I + \tfrac{1}{\sigma_v^2}\, \Sigma\,\mathbf{V}^H\mathbf{T}\mathbf{T}^H\mathbf{V}\,\Sigma^H\,\mathbf{U}^H\mathbf{U}) \\
&= \max_{\mathbf{T}:\mathrm{tr}\{\mathbf{T}\mathbf{T}^H\}=P} \log_2 \det(I + \tfrac{1}{\sigma_v^2}\, \Sigma\,\mathbf{V}^H\mathbf{T}\mathbf{T}^H\mathbf{V}\,\Sigma^H\,) \\
&= \max_{\mathbf{P}:\mathrm{tr}\{\mathbf{P}\}=P} \log_2 \det(I + \tfrac{1}{\sigma_v^2}\, \mathbf{P}\,\Sigma^H\,\Sigma)
\end{aligned}$$

  where we used $\det(I+\mathbf{X}\mathbf{Y}) = \det(I+\mathbf{Y}\mathbf{X})$, $\mathbf{U}^H\mathbf{U} = I$, and we introduced the transformation $\mathbf{T}\mathbf{T}^H = \mathbf{V}\,\mathbf{P}\,\mathbf{V}^H$ in which $\mathbf{P} = \mathbf{P}^H \geq 0$.

- Note that for the diagonal part:
  $\mathrm{diag}\{I + \frac{1}{\sigma_v^2}\,\mathbf{P}\,\Sigma^H\,\Sigma\} = I + \frac{1}{\sigma_v^2}\,\Sigma^H\,\Sigma\,\mathrm{diag}\{\mathbf{P}\}$, whereas the power constraint only depends on $\mathrm{diag}\{\mathbf{P}\}$. On the other hand, for given (fixed) $\mathrm{diag}\{\mathbf{A}\}$ with $\mathbf{A} = \mathbf{A}^H \geq 0$, $\det(\mathbf{A}) \leq \det(\mathrm{diag}\{\mathbf{A}\})$ (off-diagonal elements lower the determinant). Hence the optimal $\mathbf{P}$ is diagonal. Optimal Tx filter: $\mathbf{T} = \mathbf{V}\,\mathbf{P}^{\frac{1}{2}}$.

MIMO w Perfect CSIT: Water Filling



- Stream power optimization (with $r = \min(N_t, N_r)$)

$$\begin{aligned} \mathbf{C} &= \max_{\mathbf{P}:\mathrm{tr}\{\mathbf{P}\}=P} \log_2 \det(I + \frac{1}{\sigma_v^2}\,\mathbf{P}\,\Sigma^H\,\Sigma) \\ &= \max_{p_i \geq 0:\,\sum_{i=1}^r p_i = P} \sum_{i=1}^r \log_2(1 + \frac{p_i\,\sigma_i^2}{\sigma_v^2}) \end{aligned}$$

- Lagrangian : $\sum_{i=1}^r \log_2(1 + \frac{p_i\,\sigma_i^2}{\sigma_v^2}) + \lambda(P - \sum_{i=1}^r p_i)$ of which the derivative w.r.t. p_i gives $\frac{\sigma_i^2}{\sigma_v^2}/(1 + \frac{p_i\,\sigma_i^2}{\sigma_v^2}) = \lambda \ln 2$ (if $p_i > 0$). Together with the requirement $p_i \geq 0$ this leads to

$$p_i = \left[\frac{1}{\lambda \ln 2} - \frac{\sigma_v^2}{\sigma_i^2}\right]_+ \quad = [\text{ v - Ni }]_+$$

where $[.]_+$ denotes the non-negative part of the argument, and the Lagrange multiplier λ can be determined by the bisection method to satisfy $\sum_{i=1}^r p_i = P$.