EURECOM 28/03/2024
SP4COM Signal Processing for Communications

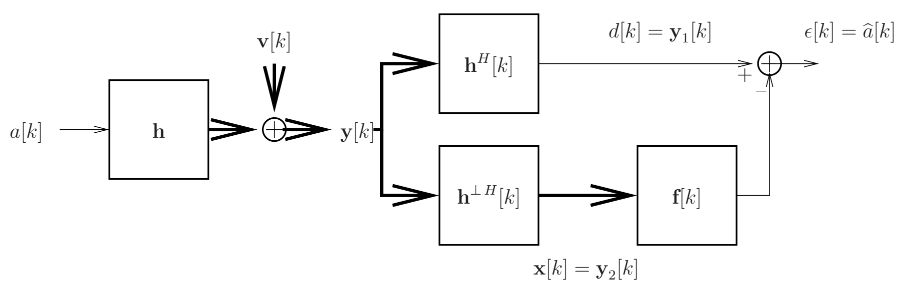
Homework 1

Due: 18/04/2024 (HW1 to be turned in by email to zilu.zhao@eurecom.fr)

Homework policy: the homework is individual. Students are encouraged to discuss with fellow students to try to find the main structure of the solution for a problem, especially if they are totally stuck at the beginning of the problem. However, they should work out the details themselves and write down in their own words only what they understand themselves. For every answer you provide, try to give it in its simplest form, while answering correctly. Results that are available in the course notes can be used and referenced and do not need to be rederived.
You can answer in French or in English. Do not forget to answer all subquestions. Word processing (Word, Latex,...) would be appreciated, or scanned readable handwritten notes.

Spatial Processing: Linear Interference Cancellation

1. Adaptation of the Spatial ICMF via LMS



Consider adaptation of the spatial Interference Canceling Matched Filter (ICMF) depicted in the figure above. The received signal $\mathbf{y}[k]$ contains m subchannels and the interference canceling filter $\mathbf{f}$ is represented as a row vector. For a generic value of $\mathbf{f}$, we get the error signal

$$\epsilon[k](\mathbf{f}) \; = \; d[k] - \mathbf{f}\,\mathbf{x}[k] \; . \tag{1}$$

For the adaptation of $\mathbf{f}$, the error signal is $\epsilon[k]$ which also provides an estimate of the transmitted symbol sequence $a[k]$ (up to a scale factor $\|\mathbf{h}\|^2$), the desired response signal is $d[k]$, which is the spatial matched filter output $\mathbf{y}_1[k]$, and the input signal is $\mathbf{x}[k]$, which is also the output $\mathbf{y}_2[k]$ of the orthogonal complement filter $\mathbf{h}^{\perp H}$. The transmitted symbol sequence $a[k]$ and the additive noise sequence $\mathbf{v}[k]$ are both considered to be temporally white and mutually independent, whereas the noise is spatially colored with covariance matrix $R_{\mathbf{VV}}$ (the noise could contain interference). The transmitted power is σ_a^2. We assume in a first instance that $\widehat{\mathbf{h}}[k] = \mathbf{h}$ (and hence $\widehat{\mathbf{h}}^{\perp}[k] = \mathbf{h}^{\perp}$).

(a) LMMSE design

Express the LMMSE filter $\mathbf{f}^o$, that minimizes σ_ϵ^2, in terms of $\mathbf{h}$, $\mathbf{h}^{\perp}$ and $R_{\mathbf{VV}}$.
Let $e[k] = \epsilon[k](\mathbf{f}^o)$ be the optimal error signal. Derive an expression for $e[k]$ in terms of the

quantities in the figure.
Introduce the matrix square root $R_{\mathbf{VV}} = R_{\mathbf{VV}}^{1/2} R_{\mathbf{VV}}^{H/2}$ and the transformed quantities $\mathbf{h}^{'} = R_{\mathbf{VV}}^{H/2}\mathbf{h}$, $\mathbf{h}^{\perp'} = R_{\mathbf{VV}}^{H/2}\mathbf{h}^{\perp}$. Note that if $R_{\mathbf{VV}}$ is not a multiple of identity, then $\mathbf{h}^{'}$ and $\mathbf{h}^{\perp'}$ are no longer orthogonal. Also introduce $\mathbf{v}^{'}[k] = R_{\mathbf{VV}}^{-1/2}\mathbf{v}[k]$ for which $R_{\mathbf{V}'\mathbf{V}'} = I_m$. Find now a simplified expression for $e[k]$ and show that the corresponding MMSE is

$$\sigma_e^2 = \sigma_a^2 \, \|\mathbf{h}\|^4 + \|P^{\perp}_{\mathbf{h}^{\perp'}}\mathbf{h}^{'}\|^2 \tag{2}$$

where $P_{\mathbf{g}} = \mathbf{g}(\mathbf{g}^H\mathbf{g})^{-1}\mathbf{g}^H$, $P^{\perp}_{\mathbf{g}} = I - P_{\mathbf{g}}$ are the projection matrices on the column space of $\mathbf{g}$ and its orthogonal complement respectively. When $R_{\mathbf{VV}} = \sigma_v^2 \, I_m$, what does $\|P^{\perp}_{\mathbf{h}^{\perp'}}\mathbf{h}^{'}\|^2$ simplify to?

(b) LMS adaptation of $\mathbf{f}$

The LMS algorithm consists of applying one iteration, per sampling period, of the steepest-descent strategy to the instantaneous error criterion $|\epsilon[k](\mathbf{f})|^2 = \epsilon^*[k](\mathbf{f}) \; \epsilon[k](\mathbf{f})$. In the complex signals case, this becomes

$$\mathbf{f}[k] \;=\; \mathbf{f}[k{-}1] - \mu \; \left.\frac{\partial|\epsilon[k](\mathbf{f})|^2}{\partial\mathbf{f}^*}\right|_{\mathbf{f}=\mathbf{f}[k-1]} . \tag{3}$$

Work out the gradient term in this LMS update. We shall simplify the notation for the a priori error signal as $\epsilon[k] = \epsilon[k](\mathbf{f}[k{-}1])$.
To check that the gradient has indeed to be taken w.r.t. $\mathbf{f}^*$ (and not $\mathbf{f}$), express the a posteriori error signal $\epsilon[k](\mathbf{f}[k])$ as a function of the a priori error signal and observe that the update leads to a smaller error signal for a proper choice of stepsize μ.

(c) Steady-state analysis of LMS adaptation of $\mathbf{f}$

Let $\widetilde{\mathbf{f}}[k] = \mathbf{f}^o - \mathbf{f}[k]$ be the filter error. Note that due to the presumed temporal whiteness of $a[k]$ and $\mathbf{v}[k]$, also $d[k]$ and $\mathbf{x}[k]$ are temporally white. Hence $\widetilde{\mathbf{f}}[k{-}1]$ and $e[k]$ are independent (strictly speaking only uncorrelated).
Let $R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k] = \mathrm{E}\,\widetilde{\mathbf{f}}^H[k]\,\widetilde{\mathbf{f}}[k]$. We can write for the a priori error signal and MSE

$$\epsilon[k] = e[k] + \widetilde{\mathbf{f}}[k{-}1]\,\mathbf{x}[k] \;\;\Rightarrow\;\; \underbrace{\sigma^2_{\epsilon[k]}}_{\text{MSE}} \;=\; \underbrace{\sigma_e^2}_{\text{MMSE}} + \underbrace{\mathrm{tr}\{R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k{-}1]\; R_{\mathbf{XX}}\}}_{\text{EMSE}} . \tag{4}$$

The LMS update (3) for $\mathbf{f}$ leads to the following recursion for $\widetilde{\mathbf{f}}[k]$:

$$\widetilde{\mathbf{f}}[k] \;=\; \widetilde{\mathbf{f}}[k{-}1]\,(I - \mu\,\mathbf{x}[k]\,\mathbf{x}^H[k]) - \mu\,e[k]\,\mathbf{x}^H[k] \tag{5}$$

which, using the averaging analysis for small stepsize, can be approximated by

$$\widetilde{\mathbf{f}}[k] \;=\; \widetilde{\mathbf{f}}[k{-}1]\,(I - \mu\,R_{\mathbf{XX}}) - \mu\,e[k]\,\mathbf{x}^H[k] \; . \tag{6}$$

From (6), obtain the time evolution for the filter error correlation matrix $R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k]$. Assume μ small so that $I{-}\mu\,R_{\mathbf{XX}}$ is stable and neglect second-order terms in μ. In steady-state $\sigma^2_{\epsilon[\infty]} = \sigma^2_{\epsilon}$ and $R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[\infty] = R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}$. Show now from (6) that we obtain for the steady-state Excess MSE

$$\text{EMSE} \;=\; \frac{\mu}{2}\,\sigma_e^2\,\mathrm{tr}\{R_{\mathbf{XX}}\} \; . \tag{7}$$

Note that $R_{\mathbf{XX}} = \mathbf{h}^{\perp H} R_{\mathbf{VV}} \mathbf{h}^{\perp}$ is not Toeplitz and its diagonal elements are not all equal.
Using the simplified expression derived in *(a)* for $e[k]$, show the corresponding expression for the a priori error signal $\epsilon[k]$.
The signal part in $\epsilon[k]$ is the term containing $a[k]$ and all the rest is noise. Using the expression for MMSE in (2), derive an expression for the SNR in $\epsilon[k]$. Note that the noise term contains a signal part which limits the SNR attainable by the adaptive system. This is due to the fact that the signal part in the error signal $e[k]$ acts like noise for the adaptation of the filter $\mathbf{f}[k]$. This problem is generic for any adaptation algorithm and not just specific for LMS.

(d) Steady-state analysis of signal compensated LMS adaptation of $\mathbf{f}$

Consider now compensating the signal part in the desired response signal $d[k]$ for the LMS adaptation. So we shall take as desired response $d[k] = \mathbf{h}^H (\mathbf{y}[k] - \mathbf{h}\, a[k]) = \mathbf{h}^H \mathbf{v}[k]$. The goal of the receiver is to detect the $a[k]$ which are hence unknown. The way this signal compensation can be implemented then is by either limiting the update for $\mathbf{f}$ to time instants at which the $a[k]$ are training symbols (used for estimating $\mathbf{h}$ also, see further) or by using the detected $a[k]$ (decision-directed (DD) strategy). In the DD strategy, the symbol $a[k]$ gets detected from $\mathbf{h}^H \mathbf{y}[k] - \mathbf{f}[k-1]\, \mathbf{x}[k]$ (or delay needs to be introduced for the updating of $\mathbf{f}$ if also channel decoding gets exploited to get more reliable $a[k]$). Making abstraction of these details, consider hence $d[k] = \mathbf{h}^H \mathbf{v}[k]$.

Does the signal compensation influence the optimal filter setting $\mathbf{f}^o$? What do the optimal error signal $e[k]$ and associated MMSE σ_e^2 become?
The signal compensation only gets done for the adaptation of $\mathbf{f}$. The thus adapted $\mathbf{f}$ then gets used in the original ICMF circuit. So, at the output of the ICMF, with the adapted $\mathbf{f}$, what does the SNR become? With the signal compensation, the SNR degradation due to the adaptation of $\mathbf{f}$ can be made arbitrarily small.

(e) LMS adaptation of $\mathbf{h}$

The transmitted symbols are in fact partitioned into known training symbols and actual data symbols. The training symbols get inserted periodically. They get used to adapt the channel estimate. From now on we shall denote the true value of the channel as $\mathbf{h}^o$ (assumed time-invariant). For the adaptation of the channel estimate, consider the error signal $\mathbf{w}[k](\mathbf{h}) = \mathbf{y}[k] - \mathbf{h}\, a[k]$. The optimal value for the error signal $\mathbf{w}[k](\mathbf{h}^o)$ has already been specified in the problem formulation. What is it?
The LMS algorithm performs one iteration of the steepest-descent strategy per training symbol to the instantaneous error criterion $\|\mathbf{w}[k](\mathbf{h})\|^2 = \mathbf{w}^H[k](\mathbf{h})\ \mathbf{w}[k](\mathbf{h})$. Derive the LMS algorithm that updates the channel estimate $\mathbf{h}[k]$ (which could have been denoted also as $\widehat{\mathbf{h}}[k]$, but let's keep $\mathbf{h}[k]$). Denote the a priori error signal as $\mathbf{w}[k]$ and the stepsize as ν. Note that the time index now is no longer the true time index but a counter for the training symbols only, since adaptation occurs only when a symbol is a training symbol.
Develop the recursion for the channel estimation error $\widetilde{\mathbf{h}}[k] = \mathbf{h}^o - \mathbf{h}[k]$. Find the steady-state value for $R_{\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}}$.

(f) Effect of channel adaptation on LMMSE ICMF operation with long-term IC estimation

Consider now the use of the adapted $\mathbf{h}[k]$ in the ICMF: for any data symbol $a[k]$, the $\mathbf{h}$ that will be used is the one adapted with LMS at the latest training symbol before the current data symbol. The main effect is that the channel estimation error $\widetilde{\mathbf{h}}$ will lead to signal leakage in the output $\mathbf{x}[k]$ of the blocking filter $\mathbf{h}^{\perp H}$. The effect of the error $\widetilde{\mathbf{h}}$ on $\mathbf{h}^{\perp}$ will depend on the choice of $\mathbf{h}^{\perp}$. Assuming the error to be small, we can perform a first-order analysis of the form

$$0 = \mathbf{h}^{\perp H}\,\mathbf{h} = (\mathbf{h}^{o\perp} - \widetilde{\mathbf{h}}^{\perp})^H(\mathbf{h}^o - \widetilde{\mathbf{h}}) \approx -\mathbf{h}^{o\perp H}\widetilde{\mathbf{h}} - \widetilde{\mathbf{h}}^{\perp H}\mathbf{h}^o \;\;\Rightarrow\;\; \widetilde{\mathbf{h}}^{\perp H}\mathbf{h}^o \approx -\mathbf{h}^{o\perp H}\widetilde{\mathbf{h}} \tag{8}$$

where $\widetilde{\mathbf{h}}^{\perp}$ is not the orthogonal complement of $\widetilde{\mathbf{h}}$ but the error on $\mathbf{h}^{\perp}$.
Describe the signals $d[k]$ and $\mathbf{x}[k]$ for the ICMF operation in terms of $\mathbf{h}^o$, $\widetilde{\mathbf{h}}$ and their orthogonal complement versions, and $a[k]$ and $\mathbf{v}[k]$, neglecting products of noise terms, and using (8).
Find $R_{d\mathbf{X}}$ and $R_{\mathbf{XX}}$. Find the LMMSE filter $\mathbf{f}$ in terms of the unperturbed version $\mathbf{f}^o$.

Express the corresponding error signal $e[k]$, and MMSE in terms of $\mathbf{h}' = R_{\mathbf{VV}}^{H/2}\mathbf{h}^o$, $\mathbf{h}^{\perp'} = R_{\mathbf{VV}}^{H/2}\mathbf{h}^{o\perp}$. Give the increase in MSE due to the channel estimation error. How much is this increase when $R_{\mathbf{VV}} = \sigma_v^2 I_m$?

(g) Effect of channel adaptation on LMMSE ICMF operation with short-term IC estimation

In *(f)*, we considered the effect of channel estimation error on the operation of the ICMF when the Interference Canceling (IC) filter $\mathbf{f}$ is adapted with long-term statistics $R_{d\mathbf{X}}$, $R_{\mathbf{XX}}$. In that case, statistical averaging occurs not only over the noise and the transmitted data but also over the channel estimation error since many instances of this error will be involved and get averaged out. Another possible configuration is short-time averaging for $\mathbf{f}$, involving essentially the data between two training symbols, so that the channel estimation error remains constant in such a period. This short-term averaging only averages over noise and transmitted data. The signal leakage in the output $\mathbf{x}[k]$ of the blocking filter $\mathbf{h}^{\perp H}$ will now lead to correlation between $d[k]$ and $\mathbf{x}[k]$.
Take again the signal descriptions for $d[k]$ and $\mathbf{x}[k]$ from *(f)*, up to first order in $\widetilde{\mathbf{h}}$. Find $R_{d\mathbf{X}}$ and $R_{\mathbf{XX}}$ using averaging over noise and symbols only, up to first order in $\widetilde{\mathbf{h}}$. Find the LMMSE filter $\mathbf{f}$ up to first order in $\widetilde{\mathbf{h}}$, in terms of the unperturbed version $\mathbf{f}^o$. Note that the perturbation in $\mathbf{f}$ due to $\widetilde{\mathbf{h}}$ is proportional to signal power $\sigma_a^2\|\mathbf{h}^o\|^2$.

Express the corresponding error signal $e[k] = d[k] - \mathbf{f}\,\mathbf{x}[k]$ in terms of the unperturbed $e^o[k]$ and first-order perturbation terms in $\widetilde{\mathbf{h}}$. Note that the perturbation terms are mutually uncorrelated. Why?
Compute the corresponding MMSE, $\mathrm{E}\,\|e[k]\|^2$, by now also averaging over $\widetilde{\mathbf{h}}$, to get a simplified average expression, assuming the LMS adaptation for $\mathbf{h}[k]$ as in *(e)*.
The signal part in $e[k]$ that the receiver will assume on the basis of its knowledge of $\mathbf{h} = \mathbf{h}^o - \widetilde{\mathbf{h}}$ is $\|\mathbf{h}^o - \widetilde{\mathbf{h}}\|^2 a[k]$ while hence $e[k] - \|\mathbf{h}^o - \widetilde{\mathbf{h}}\|^2 a[k]$ is considered noise. Compute the resulting SNR with numerator and denominator averaged over $\widetilde{\mathbf{h}}$ and computed up to first order in ν. The channel estimation error leads to signal leakage in the bottom branch of the ICMF, which leads to some *signal cancellation* and ensuing loss in SNR. In the normal Generalized Sidelobe Canceler (GSC), of which the ICMF is a special instance, any error in the blocking filter ($\mathbf{h}^{\perp}$ in the ICMF case) leads to signal cancellation that becomes total when the received signal SNR increases (hence the SNR at the output of the GSC goes to zero then). In our analysis the ICMF output SNR remains bounded away from zero due to the fact that as the received SNR increases, the channel estimation error decreases also. Note the similarity with the loss in SNR in *(c)* due to IC filter adaptation by LMS without signal compensation.