Exam

For every answer you provide, try to give it in its simplest form, while answering correctly.
If you get stuck in a certain question, do not hesitate to try the other parts of the question or continue with the next question.
Results that are available in the course notes can be used and referenced and do not need to be rederived. You can answer in French or in English.

Multi-User Spatial Channels

1. MSE and SINR of (U)MMSE versus MMSE-ZF Linear Receivers

Consider the multi-user spatial channel case with received signal

$$\begin{aligned}\mathbf{y}[k] &= [\mathbf{h}_1 \cdots \mathbf{h}_p]\begin{bmatrix} a_1[k] \\ \vdots \\ a_p[k]\end{bmatrix} + \mathbf{v}[k] = \mathbf{h}\,\mathbf{a}[k] + \mathbf{v}[k] = \begin{bmatrix}\mathbf{h}_i \; \overline{\mathbf{h}}_i\end{bmatrix}\begin{bmatrix} a_i[k] \\ \overline{\mathbf{a}}_i[k]\end{bmatrix} + \mathbf{v}[k] \\ &= \mathbf{h}_i\, a_i[k] + \overline{\mathbf{h}}_i\, \overline{\mathbf{a}}_i[k] + \mathbf{v}[k]\end{aligned} \tag{1}$$

where we assume white Gaussian noise $\mathbf{v}[k] \sim \mathcal{CN}(0, \sigma_v^2 I_m)$ i.i.d. and white symbols $S_{\mathbf{aa}}(z) = \sigma_a^2\, I_p$. We shall focus on the reception of user i. Introducing a linear receiver $\mathbf{f}_i$, we get a symbol estimate

$$\widehat{a}_i[k] = \mathbf{f}_i\, \mathbf{y}[k] = \underbrace{\mathbf{f}_i\, \mathbf{h}_i\, a_i[k]}_{\text{signal}} + \underbrace{\mathbf{f}_i\, \overline{\mathbf{h}}_i\, \overline{\mathbf{a}}_i[k]}_{\text{interference}} + \underbrace{\mathbf{f}_i\, \mathbf{v}[k]}_{\text{noise}}\ . \tag{2}$$

With some abuse of notation, we shall let UMMSE and ZFMMSE denote the MSE of the UMMSE and MMSE-ZF receivers.

(a) We have seen the first equality in

$$\text{UMMSE} = (\mathbf{h}_i^H R_i^{-1} \mathbf{h}_i)^{-1} = \text{UMMSE}^I + \text{UMMSE}^N \tag{3}$$

where UMMSE^I and UMMSE^N denote the contributions to the MSE of Interference and Noise respectively. Give expressions for UMMSE^I and UMMSE^N.

(b) Using the Matrix Inversion Lemma on $R_i^{-1} = (\sigma_v^2 I_m + \sigma_a^2\, \overline{\mathbf{h}}_i \overline{\mathbf{h}}_i^H)^{-1}$, show first that $R_i^{-1} \geq \sigma_v^2 P^{\perp}_{\overline{\mathbf{h}}_i}$ and second that hence UMMSE $\leq$ ZFMMSE. Hence a fortiori $\text{UMMSE}^I \leq$ ZFMMSE and $\text{UMMSE}^N \leq$ ZFMMSE: even though the (U)MMSE receiver allows for residual interference, this degree of freedom allows the noise enhancement to reduce so much that the sum of interference and noise powers is still lower than the noise power in the MMSE-ZF receiver.

(c) Now take a step backwards and focus on a single user with $\mathbf{y}[k] = \mathbf{h}\, a[k] + \mathbf{v}[k]$ with colored noise so that the matched filter bound is $\text{MFB} = \sigma_a^2\, \mathbf{h}^H R_{\mathbf{VV}}^{-1}\mathbf{h}$. Assume now that the same symbol sequence is received in two subsystems

$$\begin{array}{rcl} \mathbf{y}_1[k] & = & \mathbf{h}_1\, a[k] + \mathbf{v}_1[k] \\ \mathbf{y}_2[k] & = & \mathbf{h}_2\, a[k] + \mathbf{v}_2[k] \end{array} \tag{4}$$

where the noises $\mathbf{v}_1[k]$ and $\mathbf{v}_2[k]$ are uncorrelated and have covariance matrices $R_{\mathbf{V}_1\mathbf{V}_1}$ and $R_{\mathbf{V}_2\mathbf{V}_2}$ respectively. The two subsystems in (4) can be formulated jointly in the form $\mathbf{y}[k] = \mathbf{h}\, a[k] + \mathbf{v}[k]$ for which $\mathbf{y}[k]$, $\mathbf{h}$ and $R_{\mathbf{VV}}$?

(d) Show that for the two subsystems in (4) considered jointly, we have

$$\text{MFB} \,=\, \text{MFB}_1 + \text{MFB}_2\ . \tag{5}$$

(e) Now we return to our multi-user in white noise problem (1). We introduce the two subsystems

$$\begin{array}{rcl} \mathbf{y}_1[k] & = & \overline{\mathbf{h}}_i^{\perp H}\, \mathbf{y}[k] \\ \mathbf{y}_2[k] & = & \overline{\mathbf{h}}_i^{H}\, \mathbf{y}[k] \end{array} \tag{6}$$

in which we look at two orthogonally complementary parts of the signal. Also, we focus on user $a[k] = a_i[k]$. Considering the model in (4), give for (6) (and using (1)) $\mathbf{h}_1$, $\mathbf{h}_2$, $R_{\mathbf{V}_1\mathbf{V}_1}$ and $R_{\mathbf{V}_2\mathbf{V}_2}$, and show that $\mathbf{v}_1[k]$ and $\mathbf{v}_2[k]$ are uncorrelated so that (5) holds.

(f) Show that $\text{MFB}_1 = \text{SINR}^{MMSE-ZF}$ for user i in (1).

(g) Show that for the joint system (4) $\text{MFB} = \text{SINR}^{UMMSE}$ and hence that

$$\text{SINR}^{UMMSE} = \text{SINR}^{MMSE-ZF} + \text{MFB}_2\ (\geq\ \text{SINR}^{MMSE-ZF}) \tag{7}$$

by manipulating $(\overline{\mathbf{h}}_i^H\overline{\mathbf{h}}_i + \frac{\sigma_v^2}{\sigma_a^2} I)^{-1}$ a bit (relating it to $(\overline{\mathbf{h}}_i\overline{\mathbf{h}}_i^H)^{-1}$).

<u>CDMA</u>

2. PIC versus PE

In class, we alluded to a close relationship between PIC (Parallel Interference Cancellation) and PE (Polynomial Expansion). Indeed, we mentioned that PIC became PE when the nonlinearity of the *decision* element was removed. We shall take a closer look at this relationship.

(a) Consider the iteration expression of PIC, in which the value of the symbols $\widehat{\boldsymbol{d}}^{(m)}[n]$ gets determined in an iterative fashion. Assume the noise to be white: $R_{\boldsymbol{vv}} = \sigma_v^2\, I_L$. Show that when the decision operation becomes an identity operation, $dec\,\{\boldsymbol{d}\} = \boldsymbol{d}$, the computation at iteration m becomes

$$\widehat{\boldsymbol{d}}^{(m)}[n] = C^{-1}S^H\boldsymbol{y}[n] - C^{-1}\,Q\,C\,\widehat{\boldsymbol{d}}^{(m-1)}[n] \tag{8}$$

where $Q = S^H S - I_k$.

(b) Consider the initialization $\widehat{\boldsymbol{d}}^{(-1)}[n] = 0$. Determine the values (expressions) of $\widehat{\boldsymbol{d}}^{(0)}[n]$, $\widehat{\boldsymbol{d}}^{(1)}[n]$, $\widehat{\boldsymbol{d}}^{(2)}[n]$ and then of $\widehat{\boldsymbol{d}}^{(m)}[n]$ for a general iteration m, as a function of Q, C^{-1} and $S^H\boldsymbol{y}[n]$.

(c) The preceding result corresponds to PE for a linear receiver that is MMSE or MMSE-ZF?

(d) What do we get at convergence for $\widehat{\boldsymbol{d}}^{(\infty)}[n]$?

Relatively Short Questions

In case the questions below admit a simple yes/no type of answer, you should add a bit of explanation.

3. Spatial Processing / Diversity / MIMO

(a) What is a MMSE-ZF linear receiver (Minimum Mean Squared Error Zero Forcing), compared to a basic ZF receiver?
Is there a relation between the number of antennas m and the number of users p for MMSE-ZF receivers to exist?

(b) In a multi-user (MU) setting, focusing on a given user of interest, can a linear MMSE receiver be determined/estimated if only the channel of the user of interest is known?

(c) Is spatial processing relevant in the context of OFDM?

(d) (Single-User SIMO.) Does receive antenna selection achieve the full multi-antenna receive diversity order m?

(e) (Single-User SIMO.) Assume that at the output of a certain receiver, we get

$$\mathrm{SNR} = \frac{\sigma_a^2}{\sigma_v^2} \sum_{i=1}^{m} \sqrt{|h_i|} \ . \tag{9}$$

Does this receiver achieve full diversity order m?

(f) (Single-User MIMO.) Consider now a MIMO channel with N_t transmit antennas and N_r receive antennas. In an attempt to exploit the rich diversity of a MIMO channel, we are going to transmit a single stream from transmit antenna i to receive antenna j such that the channel gain $|h_{ij}|$ is maximum over all entries in the $N_r \times N_t$ MIMO channel response.

- (i) Does this approach allow to reach maximum diversity order?
- (ii) What is the maximum diversity order in this MIMO channel?
- (iii) Does this scheme enjoy spatial multiplexing gain?

(g) (Single-User MIMO.) Consider still the same $N_r \times N_t$ MIMO channel, with $N_t > N_r$.

  (i) How many streams can pass through this channel simultaneously? In other words, what is the maximum spatial multiplexing gain?

  (ii) In order to simplify transmission a bit, we are going to send N_r streams by using only the $N_r < N_t$ best transmit antennas. Does this scheme still allow full spatial multiplexing?

  (iii) How should the N_r "best" transmit antennas be chosen?

  (iv) Does this require (any) channel state information at the transmitter (CSIT)?

  (v) Does this approach suffer any diversity loss compared to an optimal approach using all N_t transmit antennas?

4. CDMA Uplink (UL)

(a) What is the MMSE-ZF receiver called in the context of CDMA? Why?

(b) Why is Polynomial Expansion (PE) well-suited for CDMA (at least in the flat channel case)?

(c) Which receiver structure(s) benefit from unequal user powers? Why?

(d) In the multipath propagation channel case, what is the matched filter receiver called in the context of CDMA? Why?

5. CDMA Downlink (DL)

(a) Why are orthogonal codes used in the downlink and not in the uplink?

(b) If channel transfer function $h(z) = \alpha\, z^{-d}$, what would the optimal receiver look like?

(c) Why is a scrambler used?

(d) Does the scrambler affect the orthogonality of the Walsh-Hadamard codes?

(e) The "chip equalizer" receiver is the cascade "channel equalizer + descrambler + correlator". Is the chip equalizer an optimal (MMSE) linear receiver structure?

(f) In lecture 4, slide 21, we find for the MMSE receiver:

$$\mathrm{SINR}_k = \left(\frac{1}{S_k^H R_{\boldsymbol{yy}}^{-1} S_k\, |c_k|^2 \sigma_d^2} - 1 \right)^{-1} = \frac{\sigma_d^2}{\mathrm{MMSE}_k} - 1\,. \tag{10}$$

Show the second equality.
Hints: for an MMSE estimator the MMSE appears on slide 35 of Lecture 1.
Also, show that $S_k^H R_{\boldsymbol{yy}}^{-1} S_k\, |c_k|^2 \sigma_d^4$ is the power at the output of the MMSE receiver.