

Exam

For every answer you provide, try to give it in its simplest form, while answering correctly.

If you get stuck in a certain question, do not hesitate to try the other parts of the question or continue with the next question.

Results that are available in the course notes can be used and referenced and do not need to be rederived. You can answer in French or in English.

Multi-User Spatial Channels

1. MSE and SINR of (U)MMSE versus MMSE-ZF Linear Receivers

Consider the multi-user spatial channel case with received signal

$$\begin{aligned} \mathbf{y}[k] &= [\mathbf{h}_1 \cdots \mathbf{h}_p] \begin{bmatrix} a_1[k] \\ \vdots \\ a_p[k] \end{bmatrix} + \mathbf{v}[k] = \mathbf{h} \mathbf{a}[k] + \mathbf{v}[k] = \begin{bmatrix} \mathbf{h}_i & \bar{\mathbf{h}}_i \end{bmatrix} \begin{bmatrix} a_i[k] \\ \bar{\mathbf{a}}_i[k] \end{bmatrix} + \mathbf{v}[k] \\ &= \mathbf{h}_i a_i[k] + \bar{\mathbf{h}}_i \bar{\mathbf{a}}_i[k] + \mathbf{v}[k] \end{aligned} \quad (1)$$

where we assume white Gaussian noise $\mathbf{v}[k] \sim \mathcal{CN}(0, \sigma_v^2 I_m)$ i.i.d. and white symbols $S_{\mathbf{a}\mathbf{a}}(z) = \sigma_a^2 I_p$. We shall focus on the reception of user i . Introducing a linear receiver \mathbf{f}_i , we get a symbol estimate

$$\hat{a}_i[k] = \mathbf{f}_i \mathbf{y}[k] = \underbrace{\mathbf{f}_i \mathbf{h}_i a_i[k]}_{\text{signal}} + \underbrace{\mathbf{f}_i \bar{\mathbf{h}}_i \bar{\mathbf{a}}_i[k]}_{\text{interference}} + \underbrace{\mathbf{f}_i \mathbf{v}[k]}_{\text{noise}}. \quad (2)$$

With some abuse of notation, we shall let UMMSE and ZFMMSE denote the MSE of the UMMSE and MMSE-ZF receivers.

- (a) Remember that $\mathbf{f}_i^{\text{UMMSE}} = (\mathbf{h}_i^H R_i^{-1} \mathbf{h}_i)^{-1} \mathbf{h}_i^H R_i^{-1}$ and recall (2). We have seen the first equality in

$$\text{UMMSE} = (\mathbf{h}_i^H R_i^{-1} \mathbf{h}_i)^{-1} = \text{UMMSE}^I + \text{UMMSE}^N \quad (3)$$

where UMMSE^I and UMMSE^N denote the contributions to the MSE of Interference and Noise respectively. Give expressions for UMMSE^I and UMMSE^N .

- (b) Using the Matrix Inversion Lemma on $R_i^{-1} = (\sigma_v^2 I_m + \sigma_a^2 \bar{\mathbf{h}}_i \bar{\mathbf{h}}_i^H)^{-1}$, show first that $R_i^{-1} \geq \sigma_v^{-2} P_{\bar{\mathbf{h}}_i}^\perp$ and second that hence $\text{UMMSE} \leq \text{ZFMMSE}$. Hence a fortiori $\text{UMMSE}^I \leq \text{ZFMMSE}$ and $\text{UMMSE}^N \leq \text{ZFMMSE}$: even though the (U)MMSE receiver allows for residual interference, this degree of freedom allows the noise enhancement to reduce so much that the sum of interference and noise powers is still lower than the noise power in the MMSE-ZF receiver.

- (c) Now take a step backwards and focus on a single user with $\mathbf{y}[k] = \mathbf{h} a[k] + \mathbf{v}[k]$ with colored noise so that the matched filter bound is $\text{MFB} = \sigma_a^2 \mathbf{h}^H R_{\mathbf{v}\mathbf{v}}^{-1} \mathbf{h}$. Assume now that the same symbol sequence is received in two $n \times 1$ subsystems

$$\begin{aligned} \mathbf{y}_1[k] &= \mathbf{h}_1 a[k] + \mathbf{v}_1[k] \\ \mathbf{y}_2[k] &= \mathbf{h}_2 a[k] + \mathbf{v}_2[k] \end{aligned} \quad (4)$$

where the noises $\mathbf{v}_1[k]$ and $\mathbf{v}_2[k]$ are uncorrelated and have covariance matrices $R_{\mathbf{v}_1\mathbf{v}_1}$ and $R_{\mathbf{v}_2\mathbf{v}_2}$ respectively. The two subsystems in (4) can be formulated jointly in the form $\mathbf{y}[k] = \mathbf{h} a[k] + \mathbf{v}[k]$, $(2n \times 1)$, for which $\mathbf{y}[k]$, \mathbf{h} and $R_{\mathbf{v}\mathbf{v}}$?

- (d) Show that for the two subsystems in (4) considered jointly, we have

$$\text{MFB} = \text{MFB}_1 + \text{MFB}_2 \quad . \quad (5)$$

- (e) Now we return to our multi-user in white noise problem (1). We introduce the two subsystems

$$\begin{aligned} \mathbf{y}_1[k] &= \bar{\mathbf{h}}_i^{\perp H} \mathbf{y}[k] \\ \mathbf{y}_2[k] &= \bar{\mathbf{h}}_i^H \mathbf{y}[k] \end{aligned} \quad (6)$$

in which we look at two orthogonally complementary parts of the signal. Also, we focus on user $a[k] = a_i[k]$. Considering the model in (4), give for (6) (and with $\mathbf{y}[k]$ from (1)) \mathbf{h}_1 , \mathbf{h}_2 , $R_{\mathbf{v}_1\mathbf{v}_1}$ and $R_{\mathbf{v}_2\mathbf{v}_2}$, and show that $\mathbf{v}_1[k]$ and $\mathbf{v}_2[k]$ are uncorrelated so that (5) holds.

- (f) Show that $\text{MFB}_1 = \text{SINR}^{MMSE-ZF}$ for user i in (1).

- (g) Show that for the joint system (4) $\text{MFB} = \text{SINR}^{UMMSE}$ and hence that

$$\text{SINR}^{UMMSE} = \text{SINR}^{MMSE-ZF} + \text{MFB}_2 \quad (\geq \text{SINR}^{MMSE-ZF}) \quad (7)$$

by manipulating $(\bar{\mathbf{h}}_i^H \bar{\mathbf{h}}_i + \frac{\sigma_v^2}{\sigma_a^2} I)^{-1}$ a bit (relating it to $(\bar{\mathbf{h}}_i \bar{\mathbf{h}}_i^H)^{-1}$).

Single-User Spatiotemporal Channels

2. Spatial Equalization of a Spatiotemporal Channel

Consider now a spatiotemporal channel with delay spread limited to $N = 2$. Hence we have

$$\mathbf{y}[k] = \mathbf{H}(q) a[k] + \mathbf{v}[k] = \mathbf{h}[0] a[k] + \mathbf{h}[1] a[k-1] + \mathbf{v}[k] \quad (8)$$

where the noise is circular and spatiotemporally white, $S_{\mathbf{v}\mathbf{v}}(z) = \sigma_v^2 I_m$, and the symbol sequence is circular and white, $S_{aa}(z) = \sigma_a^2$, and independent of the noise.

Consider spatial equalization with zero delay: $\hat{a}[k] = \mathbf{f}_1 \mathbf{y}[k]$ where the subscript 1 of the spatial filter \mathbf{f}_1 will become clear shortly.

- (a) Express the symbol estimation error $\tilde{a}[k] = a[k] - \hat{a}[k]$ in terms of $a[k]$, $a[k-1]$ and $\mathbf{v}[k]$.
- (b) What is the condition on the spatial equalizer \mathbf{f}_1 for it to be unbiased?
- (c) What are the conditions on the spatial equalizer \mathbf{f}_1 for it to be zero-forcing?
- (d) What is the condition on the number of subchannels m so that zero-forcing is possible?

We can reformulate the spatial equalization problem for this single-user spatiotemporal channel as a spatial filtering problem for a two-user spatial channel:

$$\begin{aligned} \mathbf{y}[k] &= \mathbf{h} \mathbf{a}[k] + \mathbf{v}[k], \quad \mathbf{h} = [\mathbf{h}[0] \quad \mathbf{h}[1]], \quad \mathbf{a}[k] = \begin{bmatrix} a[k] \\ a[k-1] \end{bmatrix} \\ \hat{a}[k] &= \mathbf{f}_1 \mathbf{y}[k] = \mathbf{e}_1^H \mathbf{f} \mathbf{y}[k], \quad \mathbf{e}_1^H = [1 \quad 0], \quad \mathbf{f}: p \times m, \quad p = 2 \end{aligned} \quad (9)$$

- (e) With $\text{SNR} = \frac{\sigma_a^2}{\sigma_v^2} \|\mathbf{h}[0]\|^2$, $\text{INR} = \frac{\sigma_a^2}{\sigma_v^2} \|\mathbf{h}[1]\|^2$, what is the SIR ratio?
- (f) By using the equivalent formulation in (9), express the MMSEZF solution for the spatial filter \mathbf{f}_1 , \mathbf{f}_1^{MMSEZF} , in terms of \mathbf{e}_1 and \mathbf{h} .
- (g) Express \mathbf{f}_1^{MMSEZF} in terms of $P_{\mathbf{h}[1]}^\perp \mathbf{h}[0]$.
- (h) Express SINR_1^{MMSEZF} in terms of $P_{\mathbf{h}[1]}^\perp \mathbf{h}[0]$.
- (i) If we introduce $\cos \theta = \frac{\mathbf{h}^H[1] \mathbf{h}[0]}{\|\mathbf{h}[1]\| \|\mathbf{h}[0]\|}$, express SINR_1^{MMSEZF} in terms of the MFB_1 and $\cos \theta$, $\sin \theta$ (which is real and satisfies $(\sin \theta)^2 + |\cos \theta|^2 = 1$).
- (j) Give the expression for \mathbf{f}_1^{UMMSE} , as a function of σ_a^2 , σ_v^2 , $\mathbf{h}[i]$, $\alpha_i = \|\mathbf{h}[i]\|$ and $\cos \theta$. Use the Matrix Inversion Lemma if necessary.
- (k) Give the expression of $\text{SINR}_1^{UMMSE} = \text{SINR}_1^{max}$, as a function of σ_a^2 , σ_v^2 , α_i and $\cos \theta$.
- (l) Compare the SINR_1 of the receivers MMSEZF, UMMSE and the MFB_1 , as a function of SNR_1 , of $\text{SIR} = \frac{\alpha_1^2}{\alpha_2^2}$ and of $\sin \theta$.
- (m) For which channels should we prefer to equalize with delay 1 (and obtain $\hat{a}[k-1] = \mathbf{f}_2 \mathbf{y}[k]$) instead of equalization with delay 0 as for \mathbf{f}_1 above?

Relatively Short Questions

In case the questions below admit a simple yes/no type of answer, you should add a bit of explanation.

3. Spatial Processing / Diversity / MIMO

- (a) What is a MMSE-ZF linear receiver (Minimum Mean Squared Error Zero Forcing), compared to a basic ZF receiver?
Is there a relation between the number of antennas m and the number of users p for MMSE-ZF receivers to exist?
- (b) In a multi-user (MU) setting, focusing on a given user of interest, can a linear MMSE receiver be determined/estimated if only the channel of the user of interest is known?
- (c) Is spatial processing relevant in the context of OFDM?
- (d) (Single-User SIMO.) Does receive antenna selection achieve the full multi-antenna receive diversity order m ?
- (e) (Single-User SIMO.) Assume that at the output of a certain receiver, we get

$$\text{SNR} = \frac{\sigma_a^2}{\sigma_v^2} \left| \sum_{i=1}^m h_i^2 \right|. \quad (10)$$

Does this receiver achieve full diversity order m ?

- (f) (Single-User MIMO.) Consider now a MIMO channel with N_t transmit antennas and N_r receive antennas. In an attempt to exploit the rich diversity of a MIMO channel, we are going to transmit a single stream from transmit antenna i to receive antenna j such that the channel gain $|h_{ij}|$ is maximum over all entries in the $N_r \times N_t$ MIMO channel response.
 - (i) Does this approach allow to reach maximum diversity order?
 - (ii) What is the maximum diversity order in this MIMO channel?
 - (iii) Does this scheme enjoy spatial multiplexing gain?
- (g) (Single-User MIMO.) Consider still the same $N_r \times N_t$ MIMO channel, with $N_t > N_r$.
 - (i) How many streams can pass through this channel simultaneously? In other words, what is the maximum spatial multiplexing gain?
 - (ii) In order to simplify transmission a bit, we are going to send N_r streams by using only the $N_r < N_t$ best transmit antennas. Does this scheme still allow full spatial multiplexing?
 - (iii) How should the N_r "best" transmit antennas be chosen?
 - (iv) Does this require (any) channel state information at the transmitter (CSIT)?
 - (v) Does this approach suffer any diversity loss compared to an optimal approach using all N_t transmit antennas?

4. CDMA Uplink (UL)

- (a) What is the MMSE-ZF receiver called in the context of CDMA? Why?
- (b) Why is Polynomial Expansion (PE) well-suited for CDMA (at least in the flat channel case)?
- (c) Which receiver structure(s) benefit from unequal user powers? Why?
- (d) In the multipath propagation channel case, what is the matched filter receiver called in the context of CDMA? Why?

5. CDMA Downlink (DL)

- (a) Why are orthogonal codes used in the downlink and not in the uplink?
- (b) If channel transfer function $h(z) = \alpha z^{-d}$, what would the optimal receiver look like?
- (c) Why is a scrambler used?
- (d) Does the scrambler affect the orthogonality of the Walsh-Hadamard codes?
- (e) The "chip equalizer" receiver is the cascade "channel equalizer + descrambler + correlator". Is the MMSE chip equalizer an optimal (MMSE) overall linear receiver structure?
- (f) In lecture 4, slide 21, we find for the MMSE receiver:

$$\text{SINR}_k = \left(\frac{1}{S_k^H R_{\mathbf{y}\mathbf{y}}^{-1} S_k |c_k|^2 \sigma_d^2} - 1 \right)^{-1} = \frac{\sigma_d^2}{\text{MMSE}_k} - 1. \quad (11)$$

Show the second equality.

Hints: for an MMSE estimator the MMSE appears on slide 40 of Lecture 1.

Also, show that $S_k^H R_{\mathbf{y}\mathbf{y}}^{-1} S_k |c_k|^2 \sigma_d^4$ is the power at the output of the MMSE receiver.