Exam

For every answer you provide, try to give it in its simplest form, while answering correctly.
If you get stuck in a certain question, do not hesitate to try the other parts of the question or continue with the next question.
Results that are available in the course notes can be used and referenced and do not need to be rederived. You can answer in French or in English.

Multi-User MISO Downlink

1. Reduced-Order Zero-Forcing Beamforming and its Average SINR

Consider a MISO Broadcast Channel, i.e. the downlink in a single cell system with a base station (BS) with N antennas and K single antenna users. The received signal at user k can be written as

$$y_k = \underbrace{\mathbf{h}_k^H \mathbf{g}_k\, s_k}_{\text{signal}} + \underbrace{\sum_{i=1,\neq k}^{K} \mathbf{h}_k^H \mathbf{g}_i\, s_i}_{\text{interference}} + \underbrace{v_k}_{\text{noise}} \tag{1}$$

where s_i is the scalar zero mean white signal stream with variance p_i intended for user i, which is transmitted with the $N \times 1$ beamformer $\mathbf{g}_i$, and received through the $1 \times N$ MISO channel $\mathbf{h}_k^H$. The noise v_k is white circularly complex Gaussian with variance σ_k^2. All signal and noise terms are independent. Assume the beamformers to be normalized: $||\mathbf{g}_i|| = 1$. Let the channels be randomly distributed, namely circularly complex Gaussian with i.i.d. entries such that $\mathbf{h}_k \sim \mathcal{CN}(0, \frac{\alpha_k}{N}\mathbf{I}_N)$ where α_k represents the channel attenuation for user k.

(a) Show that the Signal to Interference plus Noise Ratio (SINR) in y_k is given by

$$\text{SINR}_k = \frac{|\mathbf{h}_k^H \mathbf{g}_k|^2\, p_k}{\sum_{i=1,\neq k}^{K} |\mathbf{h}_k^H \mathbf{g}_i|^2\, p_i \;+\; \sigma_k^2}\,. \tag{2}$$

Now, we have seen that optimal beamformers are of the MMSE type, which from the point of view of reception would be making a compromise between interference suppression and matched filtering to avoid noise amplification. To simplify the beamformer design, we shall consider a "reduced-order" (RO) zero-forcing (ZF) design in which we only ZF to a subset of users. Let

$$I_k = \{i_{k,1}, i_{k,2}, \ldots, i_{k,K_k}\} \;\subset\; I_0\backslash\{k\},\ \text{where } I_0 = \{1, 2, \ldots, K\}, \tag{3}$$

be the subset of users that the beamformer $\mathbf{g}_k$ for user k zero forces to. The size of this subset is $|I_k| = K_k$ where $0 \leq K_k \leq K-1$. The collection of channels of the users being zero forced to appears in the $N \times K_k$ matrix

$$\mathbf{H}_{I_k} = [\mathbf{h}_{i_{k,1}}\ \mathbf{h}_{i_{k,2}}\ \cdots \mathbf{h}_{i_{k,K_k}}]\,. \tag{4}$$

In the RO-ZF design, beamformer $\mathbf{g}_k^{ro}$ must satisfy the constraints

$$\mathbf{H}_{I_k}^H \mathbf{g}_k^{ro} = 0 \;, \quad ||\mathbf{g}_k|| = 1 \tag{5}$$

and this for every $k \in I_0$.

(b) By how many complex and by how many real parameters can a $N \times 1$ complex vector $\mathbf{g}_k$ be parameterized?
After satisfying the constraints in (5), how many real parameters are left to parameterize $\mathbf{g}_k^{ro}$?

(c) With beamformers of the form $\mathbf{g}_k^{ro}$, satisfying the constraints (5), how does the SINR expression in (2) change to a slightly modified expression SINR_k^{ro} ?

In the RO-ZF design, beamformer $\mathbf{g}_k^{ro}$ prefers to ignore the interference it causes to users in $\overline{I_k} = I_0 \setminus I_k$. Still, to optimize somewhat the resulting SINR_k^{ro}, we shall exploit the degrees of design freedom left in $\mathbf{g}_k^{ro}$ to maximize the signal strength $|\mathbf{h}_k^H \mathbf{g}_k^{ro}|^2$. We shall again do this via the Generalized Sidelobe Canceler (GSC) formulation. For this we shall need the orthogonal projection matrix onto the column space of a tall matrix $\mathbf{H}$, namely $\mathbf{P}_{\mathbf{H}} = \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$, and the projection onto its orthogonal complement $\mathbf{P}_{\mathbf{H}}^{\perp} = \mathbf{I}_N - \mathbf{P}_{\mathbf{H}}$. Remember that any projection matrix satisfies $\mathbf{P} = \mathbf{P}^H$ and $\mathbf{P}^2 = \mathbf{P}$. We shall also introduce the normalized version of a rectangular matrix $\mathbf{H}$, namely the semi-unitary $\overline{\mathbf{H}} = \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1/2}$.

(d) Show that

$$\overline{\mathbf{H}}^H \overline{\mathbf{H}} = \mathbf{I} \tag{6}$$

which leads to the term "semi-unitary" for the tall rectangular matrix $\overline{\mathbf{H}}$.

(e) Show the two identities in

$$\mathbf{P}_{\mathbf{H}} = \mathbf{P}_{\overline{\mathbf{H}}} = \overline{\mathbf{H}}\,\overline{\mathbf{H}}^H \;. \tag{7}$$

(f) We introduce similarly the semi-unitary matrix $\overline{\mathbf{H}}^{\perp}$ of which the columns span the orthogonal complement of the column space of $\mathbf{H}$. Using the semi-unitarity of both $\overline{\mathbf{H}}$ and $\overline{\mathbf{H}}^{\perp}$ and their mutual orthogonality, show the following three identities

$$\begin{cases} [\overline{\mathbf{H}}\;\overline{\mathbf{H}}^{\perp}]^H [\overline{\mathbf{H}}\;\overline{\mathbf{H}}^{\perp}] = \mathbf{I}_N \\ [\overline{\mathbf{H}}\;\overline{\mathbf{H}}^{\perp}][\overline{\mathbf{H}}\;\overline{\mathbf{H}}^{\perp}]^H = \mathbf{I}_N \\ [\overline{\mathbf{H}}\;\overline{\mathbf{H}}^{\perp}][\overline{\mathbf{H}}\;\overline{\mathbf{H}}^{\perp}]^H = \mathbf{P}_{\overline{\mathbf{H}}} + \mathbf{P}_{\overline{\mathbf{H}}^{\perp}} \end{cases} \tag{8}$$

from which you conclude that $\mathbf{P}_{\overline{\mathbf{h}}^{\perp}} = \mathbf{P}_{\overline{\mathbf{h}}}^{\perp}$.

In what follows, we shall replace these generic matrices by the specific $\overline{\mathbf{H}}_{I_k}$ and $\overline{\mathbf{H}}_{I_k}^{\perp}$ Consider now a reparameterization of $\mathbf{g}_k^{ro}$ with $\mathbf{g}_{k,\|}$, $\mathbf{g}_{k,\perp}$ as follows

$$\mathbf{g}_k^{ro} = \underbrace{[\,\underbrace{\overline{\mathbf{H}}_{I_k}}_{N\times K_k} \quad \underbrace{\overline{\mathbf{H}}_{I_k}^{\perp}}_{N\times(N-K_k)}\,]}_{\text{unitary transformation}} \begin{bmatrix} \mathbf{g}_{k,\|} \\ \mathbf{g}_{k,\perp} \end{bmatrix} . \tag{9}$$

(g) What are the repercussions of "forcing the interference to zero" in (5) on $\mathbf{g}_{k,\|}$ and $\mathbf{g}_{k,\perp}$? What does the reparameterized $\mathbf{g}_k$ look like after taking this ZF repercussion into account?

The thus reparameterized $\mathbf{g}_k^{ro}$ represents all possible RO-ZF solutions. We shall optimized the remaining parameters in $\mathbf{g}_k^{ro}$ by maximizing the received SNR

$$\mathrm{SNR}_k^{ro} = \frac{|\mathbf{h}_k^H \mathbf{g}_k^{ro}|^2 \, p_k}{\sigma_k^2} \,. \tag{10}$$

under the normalization constraint in (5).

(h) Show that the normalization constraint in (5) now becomes

$$||\mathbf{g}_{k,\perp}||^2 = 1 \,. \tag{11}$$

(i) Hence, what does the optimization problem (10) under the normalization constraint (11) become for the remaining parameters in $\mathbf{g}_k^{ro}$? In other words, what are the free parameters, parameterizing which remaining cost function, under which constraint?

(j) Show that the resulting solution for the free parameters, when substituted in (9), leads to

$$\mathbf{g}_k^{ro} = \mathbf{P}^{\perp}_{\mathbf{H}_{I_k}} \mathbf{h}_k / ||\mathbf{P}^{\perp}_{\mathbf{H}_{I_k}} \mathbf{h}_k|| \tag{12}$$

which is the reduced-order MMSE-ZF beamformer.

In the massive MIMO regime, as $N \to \infty$, the scalar signal and interference powers appearing in SINR_k^{ro} under (c) converge to their expected value as a function of the random channels, due to the law of large numbers. I.e. we can write

$$\mathrm{SINR}_k^{ro} = \frac{P_{S,k}}{P_{I,k} + \sigma_k^2} \,. \tag{13}$$

(k) Show that

$$\mathbf{h}_k^H \mathbf{g}_k^{ro} = ||\mathbf{P}^{\perp}_{\mathbf{H}_{I_k}} \mathbf{h}_k|| \,. \tag{14}$$

(l) Hence show that

$$P_{S,k} = \mathrm{E}|\mathbf{h}_k^H \mathbf{g}_k^{ro}|^2 = \mathrm{E}_{\mathbf{H}_{I_k}} \mathrm{E}_{\mathbf{h}_k} \mathrm{tr}\{\mathbf{P}^{\perp}_{\mathbf{H}_{I_k}} \mathbf{h}_k \mathbf{h}_k^H\} \,. \tag{15}$$

(m) Continue this result to show that

$$\begin{aligned} P_{S,k} &= \tfrac{\alpha_k}{N} \mathrm{E}_{\mathbf{H}_{I_k}} \mathrm{tr}\{\mathbf{P}^{\perp}_{\mathbf{H}_{I_k}}\} = \tfrac{\alpha_k}{N} \mathrm{E}_{\mathbf{H}_{I_k}} \mathrm{tr}\{\overline{\mathbf{H}}^{\perp}_{I_k} \overline{\mathbf{H}}^{\perp H}_{I_k}\} = \tfrac{\alpha_k}{N} \mathrm{E}_{\mathbf{H}_{I_k}} \mathrm{tr}\{\overline{\mathbf{H}}^{\perp H}_{I_k} \overline{\mathbf{H}}^{\perp}_{I_k}\} \\ &= \tfrac{\alpha_k}{N} \mathrm{E}_{\mathbf{H}_{I_k}} \mathrm{tr}\{\mathbf{I}_{N-K_k}\} = \tfrac{\alpha_k}{N} \mathrm{tr}\{\mathbf{I}_{N-K_k}\} = \alpha_k (1 - \tfrac{K_k}{N}) \end{aligned} \tag{16}$$

which shows how the signal power decreases with increasing ZF order K_k.

(n) In the computation of the interference power $P_{I,k}$, show that

$$|\mathbf{h}_k^H \boldsymbol{g}_i^{ro}|^2 = \frac{|\mathbf{h}_k^H \mathbf{P}^{\perp}_{\mathbf{H}_{I_i}} \mathbf{h}_i|^2}{||\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}} \mathbf{h}_i||^2}, \text{ hence } \mathrm{E}|\mathbf{h}_k^H \boldsymbol{g}_i^{ro}|^2 = \frac{\mathrm{E}|\mathbf{h}_k^H \mathbf{P}^{\perp}_{\mathbf{H}_{I_i}} \mathbf{h}_i|^2}{P_{S,i}} \,. \tag{17}$$

(o) Next, show that for $k \notin I_i$,

$$\mathrm{E}|\mathbf{h}_k^H \mathbf{P}^\perp_{\mathbf{H}_{I_i}} \mathbf{h}_i|^2 = \mathrm{E}\left(\mathrm{tr}\{\mathbf{P}^\perp_{\mathbf{H}_{I_i}} \mathbf{h}_i \mathbf{h}_i^H \mathbf{P}^\perp_{\mathbf{H}_{I_i}} \mathbf{h}_k \mathbf{h}_k^H\}\right) = \frac{\alpha_k \alpha_i}{N^2} \mathrm{tr}\{\mathbf{P}^\perp_{\mathbf{H}_{I_i}}\} = \frac{\alpha_k \alpha_i}{N}(1 - \frac{K_i}{N}) \,. \tag{18}$$

(p) Finally show that we can write

$$\mathrm{SINR}_k^{ro} = \frac{p_k(1-\frac{K_k}{N})}{\frac{1}{N}\sum\limits_{i=1, k\notin I_i}^{K} p_i + \dfrac{\sigma_k^2}{\alpha_k}} \,. \tag{19}$$

(q) Show that for the full order ZF, we get from this

$$\mathrm{SINR}_k^{ZF} = \frac{\alpha_k \, p_k}{\sigma_k^2}\left(1 - \frac{K-1}{N}\right) \,. \tag{20}$$

(r) On the other hand we get for a Matched Filter, which is a zero$^{\mathrm{th}}$ order ZF ($K_k = 0$),

$$\mathrm{SINR}_k^{MF} = \frac{p_k}{\dfrac{P - p_k}{N} + \dfrac{\sigma_k^2}{\alpha_k}} \tag{21}$$

where P is the sum power of the BS. What does this give in the case of uniform powers $p_k = P/K$?

Relatively Short Questions

In case the questions below admit a simple yes/no type of answer, you should add a bit of explanation.

2. Spatial Processing / Diversity / MIMO

(a) What is a MMSE-ZF linear receiver (Minimum Mean Squared Error Zero Forcing), compared to a basic ZF receiver?
Is there a relation between the number of antennas m and the number of users p for MMSE-ZF receivers to exist?

(b) In a multi-user (MU) setting, focusing on a given user of interest, can a linear MMSE receiver be determined/estimated if only the channel of the user of interest is known?

(c) Is spatial processing relevant in the context of OFDM?

(d) (Single-User SIMO.) Does receive antenna selection achieve the full multi-antenna receive diversity order m?

(e) (Single-User SIMO.) Assume we get at the output of a receiver SNR $= \frac{\sigma_a^2}{\sigma_v^2} \left| \sum_{i=1}^{m} h_i^2 \right|$.
Does this receiver achieve full diversity order m?

(f) (Single-User MISO.) Transmit Diversity

- (i) What is the purpose of Transmit Diversity?
- (ii) What is the disadvantage of Delay Transmit Diversity?
- (iii) Do Transmit Diversity techniques require downlink channel knowledge at the transmitter?

(g) (Single-User MIMO.) Consider now a MIMO channel with N_t transmit antennas and N_r receive antennas. In an attempt to exploit the rich diversity of a MIMO channel, we are going to transmit a single stream from transmit antenna i to receive antenna j such that the channel gain $|h_{ij}|$ is maximum over all entries in the $N_r \times N_t$ MIMO channel response.

- (i) Does this approach allow to reach maximum diversity order?
- (ii) What is the maximum diversity order in this MIMO channel?
- (iii) Does this scheme enjoy spatial multiplexing gain?

(h) (Single-User MIMO.) Consider still the same $N_r \times N_t$ MIMO channel, with $N_t > N_r$.

- (i) How many streams can pass through this channel simultaneously? In other words, what is the maximum spatial multiplexing gain?
- (ii) In order to simplify transmission a bit, we are going to send N_r streams by using only the $N_r < N_t$ best transmit antennas. Does this scheme still allow full spatial multiplexing?
- (iii) How should the N_r "best" transmit antennas be chosen?
- (iv) Does this require (any) channel state information at the transmitter (CSIT)?
- (v) Does this approach suffer any diversity loss compared to an optimal approach using all N_t transmit antennas?

3. CDMA Uplink (UL)

(a) What is the MMSE-ZF receiver called in the context of CDMA? Why?

(b) Why is Polynomial Expansion (PE) well-suited for CDMA (at least in the flat channel case)?

(c) Which receiver structure(s) benefit from unequal user powers? Why?

(d) In the multipath propagation channel case, what is the matched filter receiver called in the context of CDMA? Why?

The next two questions refer to block $\underline{E}_{k,m}$ on slide 35 of Lecture 5.

(e) In $\underline{E}_{k,m}$, the 3 operations of path delay, spreading, pulse shape filtering occur in which order?

(f) Which of the following is the best description: the operations in $\underline{E}_{k,m}$

(i) are done jointly on the different antenna signals

(ii) are done separately on the different antenna signals

(iii) are separate and identical on each antenna signal

4. CDMA Downlink (DL)

(a) Why are orthogonal codes used in the downlink and not in the uplink?

(b) If channel transfer function $h(z) = \alpha\, z^{-d}$, what would the optimal receiver look like?

(c) Why is a scrambler used?

(d) Does the scrambler affect the orthogonality of the Walsh-Hadamard codes?

(e) The "chip equalizer" receiver is the cascade "channel equalizer + descrambler + correlator". Is the MMSE chip equalizer an optimal (MMSE) overall linear receiver structure?

(f) In lecture 4, slide 21, we find for the MMSE receiver:

$$\mathrm{SINR}_k = \left(\frac{1}{S_k^H R_{\boldsymbol{yy}}^{-1} S_k \, |c_k|^2 \sigma_d^2} - 1 \right)^{-1} = \frac{\sigma_d^2}{\mathrm{MMSE}_k} - 1 \, . \tag{22}$$

Show the second equality.
Hints: for an MMSE estimator the MMSE appears on slide 40 of Lecture 1.
Also, show that $S_k^H R_{\boldsymbol{yy}}^{-1} S_k \, |c_k|^2 \sigma_d^4$ is the power at the output of the MMSE receiver.