Exam

For every answer, try to give it in its simplest form, while answering correctly.
If you get stuck in a certain question, do not hesitate to try the other parts of the question or continue with the next question.
Results that are available in the course notes can be used and referenced and do not need to be rederived. You can answer in French or in English.

Multi-User Spatial Channels

1. GSC Derivation of the MMSE-ZF Receiver in Presence of Interferers

Consider the multi-user spatial channel case with received signal

$$\begin{aligned} \mathbf{y}[k] &= [\mathbf{h}_1 \cdots \mathbf{h}_p] \begin{bmatrix} a_1[k] \\ \vdots \\ a_p[k] \end{bmatrix} + \mathbf{v}[k] = \mathbf{h}\,\mathbf{a}[k] + \mathbf{v}[k] = \begin{bmatrix} \mathbf{h}_i \; \overline{\mathbf{h}}_i \end{bmatrix} \begin{bmatrix} a_i[k] \\ \overline{\mathbf{a}}_i[k] \end{bmatrix} + \mathbf{v}[k] \\ &= \mathbf{h}_i\, a_i[k] + \overline{\mathbf{h}}_i\, \overline{\mathbf{a}}_i[k] + \mathbf{v}[k] \end{aligned} \tag{1}$$

where we assume white Gaussian noise $\mathbf{v}[k] \sim \mathcal{CN}(0, \sigma_v^2 I_m)$ i.i.d. and white symbols $S_{\mathbf{aa}}(z) = \sigma_a^2\, I_p$ that are independent of the noise. We shall focus on the reception of user i. Introducing a linear receiver $\mathbf{f}_i$, we get a symbol estimate

$$\widehat{a}_i[k] = \mathbf{f}_i\, \mathbf{y}[k] = \underbrace{\mathbf{f}_i\, \mathbf{h}_i\, a_i[k]}_{\text{signal}} + \underbrace{\mathbf{f}_i\, \overline{\mathbf{h}}_i\, \overline{\mathbf{a}}_i[k]}_{\text{interference}} + \underbrace{\mathbf{f}_i\, \mathbf{v}[k]}_{\text{noise}} . \tag{2}$$

For a $m \times p$ channel matrix $\mathbf{h}$ with linearly independent columns, introduce a matrix $\mathbf{h}^{\perp}$ with linearly independent columns such that we get the $m \times m$ matrices

$$[\mathbf{h}\; \mathbf{h}^{\perp}]^H [\underbrace{\mathbf{h}}_{m\times p} \;\; \underbrace{\mathbf{h}^{\perp}}_{m\times(m-p)}] = \begin{bmatrix} \mathbf{h}^H\mathbf{h} & 0 \\ 0 & \mathbf{h}^{\perp H}\mathbf{h}^{\perp} \end{bmatrix} \text{ with } \mathbf{h}^H\mathbf{h} \text{ and } \mathbf{h}^{\perp H}\mathbf{h}^{\perp} \text{ nonsingular.} \tag{3}$$

The square matrix $[\mathbf{h}\; \mathbf{h}^{\perp}]$ is invertible. The column space of $\mathbf{h}^{\perp}$ is the orthogonal complement of the column space of $\mathbf{h}$. The projection matrix onto the column space of e.g. $\mathbf{h}$ is defined as $P_{\mathbf{h}} = \mathbf{h}(\mathbf{h}^H\mathbf{h})^{-1}\mathbf{h}^H$ and the projection onto its orthogonal complement is defined as $P_{\mathbf{h}}^{\perp} = I_m - P_{\mathbf{h}}$.

(a) Show the following two identities (use (3) for one of them)

$$\left\{ \begin{array}{l} [\mathbf{h}\; \mathbf{h}^{\perp}] \left([\mathbf{h}\; \mathbf{h}^{\perp}]^H [\mathbf{h}\; \mathbf{h}^{\perp}] \right)^{-1} [\mathbf{h}\; \mathbf{h}^{\perp}]^H = I_m \\ [\mathbf{h}\; \mathbf{h}^{\perp}] \left([\mathbf{h}\; \mathbf{h}^{\perp}]^H [\mathbf{h}\; \mathbf{h}^{\perp}] \right)^{-1} [\mathbf{h}\; \mathbf{h}^{\perp}]^H = P_{\mathbf{h}} + P_{\mathbf{h}^{\perp}} \end{array} \right. \tag{4}$$

from which you conclude that $P_{\mathbf{h}^{\perp}} = P_{\mathbf{h}}^{\perp}$.

In what follows, we shall work with the $m\times(p{-}1)$ matrix $\overline{\mathbf{h}}_i$ rather than $\mathbf{h}$, and we assume $\mathbf{h}$ (and hence also $\overline{\mathbf{h}}_i$) to have linearly independent columns. Consider now a reparameterization of $\mathbf{f}_i$ with $\mathbf{f}_{i,\|}$, $\mathbf{f}_{i,\perp}$ as follows

$$\mathbf{f}_i = [\ \underbrace{\mathbf{f}_{i,\|}}_{1\times(p-1)} \quad \underbrace{\mathbf{f}_{i,\perp}}_{1\times(m-p+1)}\ ] \underbrace{\begin{bmatrix} \overline{\mathbf{h}}_i^H \\ \overline{\mathbf{h}}_i^{\perp H} \end{bmatrix}}_{\text{invertible transformation}} \quad . \tag{5}$$

We now invoke the zero-forcing (ZF) constraint: $\mathbf{f}_i\,\overline{\mathbf{h}}_i = 0$.

(b) What are the repercussions of "forcing the interference to zero" on $\mathbf{f}_{i,\|}$ and $\mathbf{f}_{i,\perp}$? What does the reparameterized $\mathbf{f}_i$ look like after taking this ZF repercussion into account?

(c) Now, remember that ZF implies furthermore unbiasedness. So which additional constraint does this unbiasedness impose on $\mathbf{f}_{i,\|}$ and $\mathbf{f}_{i,\perp}$?

Assume the "forcing the interference to zero" constraint to be enforced in what follows, and we shall keep in mind the unbiasedness constraint. The thus reparameterized $\mathbf{f}_i$ represents all possible ZF solutions.

(d) With the reparameterized "forcing the interference to zero" $\mathbf{f}_i$, equation (2) now looks like

$$\widehat{a}_i[k] = \mathbf{f}_i\,\mathbf{y}[k] = \mathbf{f}_i^{'}\left(\mathbf{h}_i^{'}\,a_i[k] + \mathbf{v}^{'}[k]\right) \ . \tag{6}$$

Is $\mathbf{f}_i^{'}$ equal to $\mathbf{f}_{i,\|}$ or to $\mathbf{f}_{i,\perp}$?
What is $\mathbf{h}_i^{'}$ equal to?
What does the unbiasedness constraint become in terms of $\mathbf{f}_{i,\|}$, $\mathbf{f}_{i,\perp}$ and $\mathbf{h}_i^{'}$?

(e) What is the covariance matrix $R_{\mathbf{v}'\mathbf{v}'}$ equal to?

(f) In signal model (6), the interference has disappeared. The MMSE-ZF receiver design for model (2) has become an UMMSE design for model (6). What is the UMMSE solution for $\mathbf{f}_i^{'}$ in terms of $\mathbf{h}_i^{'}$ and $R_{\mathbf{v}'\mathbf{v}'}$?

(g) By substituting the expressions for $\mathbf{h}_i^{'}$ and $R_{\mathbf{v}'\mathbf{v}'}$ into $\mathbf{f}_i^{'}$ and eventually into $\mathbf{f}_i$, and using the result from **(a)**, show that we obtain the following MMSE-ZF receiver:

$$\mathbf{f}_i = (\mathbf{h}_i^H P^{\perp}_{\overline{\mathbf{h}}_i}\,\mathbf{h}_i)^{-1}\,\mathbf{h}_i^H P^{\perp}_{\overline{\mathbf{h}}_i} \quad . \tag{7}$$

(h) Show that this receiver leads to the following MSE:

$$\mathrm{MSE}_i^{MMSE-ZF} = \frac{\sigma_v^2}{\|P^{\perp}_{\overline{\mathbf{h}}_i}\,\mathbf{h}_i\|^2} \quad . \tag{8}$$

Hint: remember the 2 properties of projection matrices.

(i) On the other hand we have seen in class that

$$\mathrm{MSE}_i^{MMSE-ZF} = \sigma_v^2\,\left[(\mathbf{h}^H\mathbf{h})^{-1}\right]_{ii} \quad . \tag{9}$$

Show that the expressions in (8) and (9) are equal.
Hint:

$$\left[(\mathbf{h}^H\mathbf{h})^{-1}\right]_{ii} = \left[\left([\mathbf{h}_i\ \overline{\mathbf{h}}_i]^H[\mathbf{h}_i\ \overline{\mathbf{h}}_i]\right)^{-1}\right]_{11} \quad .$$

<u>Relatively Short Questions</u>

In case the questions below admit a simple yes/no type of answer, you should add a bit of explanation (but not half a page).

2. Spatial Processing / Diversity / MIMO

(a) What is a MMSE-ZF linear receiver (Minimum Mean Squared Error Zero Forcing), compared to a basic ZF receiver?
Is there a relation between the number of antennas m and the number of users p for MMSE-ZF receivers to exist?

(b) In a multi-user (MU) setting, focusing on a given user of interest, can a linear MMSE receiver be determined/estimated if only the channel of the user of interest is known?

(c) Is spatial processing relevant in the context of OFDM?

(d) (Single-User SIMO.) Does receive antenna selection achieve the full multi-antenna receive diversity order m?

(e) (Single-User SIMO.) Assume we get at the output of a receiver SNR $= \frac{\sigma_a^2}{\sigma_v^2} \sum_{i=1}^{m} |h_i|^3$.

Does this receiver achieve full diversity order m?

(f) (Single-User MISO.) Transmit Diversity

- (i) What is the purpose of Transmit Diversity?
- (ii) What is the disadvantage of Delay Transmit Diversity?
- (iii) Do Transmit Diversity techniques require downlink channel knowledge at the transmitter?

(g) (Single-User MIMO.) Consider now a MIMO channel with N_t transmit antennas and N_r receive antennas. In an attempt to exploit the rich diversity of a MIMO channel, we are going to transmit a single stream from transmit antenna i to receive antenna j such that the channel gain $|h_{ij}|$ is maximum over all entries in the $N_r \times N_t$ MIMO channel response.

- (i) Does this approach allow to reach maximum diversity order?
- (ii) What is the maximum diversity order in this MIMO channel?
- (iii) Does this scheme enjoy spatial multiplexing gain?

(h) (Single-User MIMO.) Consider still the same $N_r \times N_t$ MIMO channel, with $N_t > N_r$.

- (i) How many streams can pass through this channel simultaneously? In other words, what is the maximum spatial multiplexing gain?
- (ii) In order to simplify transmission a bit, we are going to send N_r streams by using only the $N_r < N_t$ best transmit antennas. Does this scheme still allow full spatial multiplexing?
- (iii) How should the N_r "best" transmit antennas be chosen?
- (iv) Does this require (any) channel state information at the transmitter (CSIT)?

(v) Does this approach suffer any diversity loss compared to an optimal approach using all N_t transmit antennas?

(i) (OFDM.) What is the difference between OFDM and SC-CP (Single Carrier with Cyclic Prefix)?
Why is SC-CP used in the LTE uplink?

3. CDMA Uplink (UL)

(a) What is the MMSE-ZF receiver called in the context of CDMA? Why?

(b) Why is Polynomial Expansion (PE) well-suited for CDMA (at least in the flat channel case)?

(c) Which receiver structure(s) benefit from unequal user powers? Why?

(d) In the multipath propagation channel case, what is the matched filter receiver called in the context of CDMA? Why?

The next two questions refer to block $\underline{E}_{k,m}$ on slide 35 of Lecture 5.

(e) In $\underline{E}_{k,m}$, the 3 operations of path delay, spreading, pulse shape filtering occur in which order?

(f) Which of the following is the best description: the operations in $\underline{E}_{k,m}$

  (i) are done jointly on the different antenna signals
  (ii) are done separately on the different antenna signals
  (iii) are separate and identical on each antenna signal

4. CDMA Downlink (DL)

(a) Why are orthogonal codes used in the downlink and not in the uplink?

(b) If channel transfer function $h(z) = \alpha\, z^{-d}$, what would the optimal receiver look like?

(c) Why is a scrambler used?

(d) Does the scrambler affect the orthogonality of the Walsh-Hadamard codes?

(e) The "chip equalizer" receiver is the cascade "channel equalizer + descrambler + correlator". Is the MMSE chip equalizer an optimal (MMSE) overall linear receiver structure?

(f) In Lecture 4, slide 21, we find for the MMSE receiver:

$$\mathrm{SINR}_k = \left(\frac{1}{S_k^H R_{\boldsymbol{yy}}^{-1} S_k\, |c_k|^2 \sigma_d^2} - 1 \right)^{-1} = \frac{\sigma_d^2}{\mathrm{MMSE}_k} - 1\,. \tag{10}$$

Show the second equality. (This is the only question requiring a few lines.)
Hints: for an MMSE estimator the MMSE appears on slide 40 of Lecture 1.
Also, show that $S_k^H R_{\boldsymbol{yy}}^{-1} S_k\, |c_k|^2 \sigma_d^4$ is the power at the output of the MMSE receiver.