TD: Spatial and SpatioTemporal Processing Solutions

Spatial Processing: Linear Interference Cancellation

1. Adaptation of the Spatial ICMF via LMS

(a)

$\mathbf{f}^o = R_{d\mathbf{X}} R_{\mathbf{XX}}^{-1} = (\mathbf{h}^H R_{\mathbf{VV}} \mathbf{h}^{\perp})\,(\mathbf{h}^{\perp H} R_{\mathbf{VV}} \mathbf{h}^{\perp})^{-1}$

$e[k] = \mathbf{h}^H \left(\mathbf{h}\, a[k] + \left[I_m - R_{\mathbf{VV}} \mathbf{h}^{\perp} (\mathbf{h}^{\perp H} R_{\mathbf{VV}} \mathbf{h}^{\perp})^{-1} \mathbf{h}^{\perp H}\right] \mathbf{v}[k]\right).$

$e[k] = \|\mathbf{h}\|^2\, a[k] + \mathbf{h}^{'H} P^{\perp}_{\mathbf{h}^{\perp'}}\, \mathbf{v}^{'}[k].$

When $R_{\mathbf{VV}} = \sigma_v^2\, I_m, \quad \|P^{\perp}_{\mathbf{h}^{\perp'}}\, \mathbf{h}^{'}\|^2 = \|P_{\mathbf{h}^{'}}\, \mathbf{h}^{'}\|^2 = \|\mathbf{h}^{'}\|^2 = \sigma_v^2 \|\mathbf{h}\|^2.$

(b)

$\epsilon[k] = d[k] - \mathbf{f}[k{-}1]\, \mathbf{x}[k], \;\; \mathbf{f}[k] \;=\; \mathbf{f}[k{-}1] + \mu\, \epsilon[k]\, \mathbf{x}^H[k]$

$\epsilon[k](\mathbf{f}[k]) = (1 - \mu \|\mathbf{x}[k]\|^2)\, \epsilon[k]$

(c)

From the update equation for $\widetilde{\mathbf{f}}[k]$ we can obtain an update equation for $\widetilde{\mathbf{f}}^H[k]\, \widetilde{\mathbf{f}}[k]$ and take expectation, which yields

$$R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k] = R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k{-}1] - \mu\, R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k{-}1]\, R_{\mathbf{XX}} - \mu\, R_{\mathbf{XX}}\, R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k{-}1] + \underbrace{\mu^2\, R_{\mathbf{XX}}\, R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}[k{-}1]\, R_{\mathbf{XX}}}_{\text{neglect}} + \mu^2\, \sigma_e^2\, R_{\mathbf{XX}}$$

which becomes in steady-state

$$R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}} = R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}} - \mu\, R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}\, R_{\mathbf{XX}} - \mu\, R_{\mathbf{XX}}\, R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}} + \mu^2\, \sigma_e^2\, R_{\mathbf{XX}}$$

from which we can easily solve for the desired expression for the Excess MSE $\mathrm{tr}\{R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}\, R_{\mathbf{XX}}\} = \mathrm{tr}\{R_{\mathbf{XX}}\, R_{\widetilde{\mathbf{f}}\widetilde{\mathbf{f}}}\}$.

$\epsilon[k] = \|\mathbf{h}\|^2\, a[k] + \mathbf{h}^{'H} P^{\perp}_{\mathbf{h}^{\perp'}}\, \mathbf{v}^{'}[k] + \widetilde{\mathbf{f}}[k{-}1]\, \mathbf{x}[k].$

$$\mathrm{SNR} = \frac{\|\mathbf{h}\|^4\, \sigma_a^2}{\|P^{\perp}_{\mathbf{h}^{\perp'}} \mathbf{h}^{'}\|^2 \,+ \frac{\mu}{2}\, \sigma_e^2\, \mathrm{tr}\{R_{\mathbf{XX}}\}} = \frac{\|\mathbf{h}\|^4\, \sigma_a^2}{\frac{\mu}{2}\, \mathrm{tr}\{R_{\mathbf{XX}}\}\, \|\mathbf{h}\|^4\, \sigma_a^2 + \|P^{\perp}_{\mathbf{h}^{\perp'}} \mathbf{h}^{'}\|^2\, (1 + \frac{\mu}{2}\, \mathrm{tr}\{R_{\mathbf{XX}}\})}$$

$$\leq \frac{1}{\frac{\mu}{2}\, \mathrm{tr}\{R_{\mathbf{XX}}\}}.$$

(d)

The signal compensation does not influence $\mathbf{f}^o$ since the signal part is absent from $\mathbf{x}[k]$ and (hence) does not influence the correlation between $\mathbf{x}[k]$ and $d[k]$.

$e[k] = \mathbf{h}'^H P^{\perp}_{\mathbf{h}^{\perp\prime}} \mathbf{v}'[k]$. MMSE $=\ \sigma_e^2 = \|P^{\perp}_{\mathbf{h}^{\perp\prime}} \mathbf{h}'\|^2$. SNR $= \dfrac{\|\mathbf{h}\|^4\,\sigma_a^2}{\|P^{\perp}_{\mathbf{h}^{\perp\prime}}\mathbf{h}'\|^2\,(1+\frac{\mu}{2}\,\mathrm{tr}\{R_{\mathbf{xx}}\})}$.

(e)

$\mathbf{w}[k](\mathbf{h}^o) = \mathbf{v}[k]$.

The LMS updates as

$$\mathbf{h}[k] \;=\; \mathbf{h}[k{-}1] - \nu\, \left.\frac{\partial \|\mathbf{w}[k](\mathbf{h})\|^2}{\partial \mathbf{h}^*}\right|_{\mathbf{h}=\mathbf{h}[k-1]}$$

which becomes

$$\mathbf{w}[k] = \mathbf{y}[k] - \mathbf{h}[k{-}1]\,a[k]\,, \quad \mathbf{h}[k] \;=\; \mathbf{h}[k{-}1] + \nu\,\mathbf{w}[k]\,a^*[k]\ .$$

The a priori error signal can be decomposed as $\mathbf{w}[k] = \mathbf{v}[k] + \widetilde{\mathbf{h}}[k-1]\,a[k]$ which, with the LMS recursion and the definition of $\widetilde{\mathbf{h}}[k]$, leads to

$$\widetilde{\mathbf{h}}[k] = (1-\nu\,|a[k]|^2)\,\widetilde{\mathbf{h}}[k{-}1] - \nu\,\mathbf{v}[k]\,a^*[k]\ .$$

The training symbols often have constant modulus so that we can also write

$$\widetilde{\mathbf{h}}[k] = (1-\nu\,\sigma_a^2)\,\widetilde{\mathbf{h}}[k{-}1] - \nu\,\mathbf{v}[k]\,a^*[k]\ .$$

The steady-state value for the channel error covariance matrix can now be found to be (up to first order in ν)

$$R_{\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}} = \frac{\nu}{2}\,R_{\mathbf{vv}}$$

or in other words $\widetilde{\mathbf{h}}[k]$ behaves like an additional noise $\sqrt{\frac{\nu}{2}}\,\mathbf{v}[k]$.

(f)

$d[k] = \mathbf{h}^H\,\mathbf{y}[k] = (\mathbf{h}^{oH} - \widetilde{\mathbf{h}}^H)\,(\mathbf{h}^o a[k] + \mathbf{v}[k]) \;\approx\; \|\mathbf{h}^o\|^2\,a[k] + \mathbf{h}^{oH}\,\mathbf{v}[k] - \widetilde{\mathbf{h}}^H\,\mathbf{h}^o\,a[k]$,

$\mathbf{x}[k] = \mathbf{h}^{\perp H}\,\mathbf{y}[k] \;\approx\; \mathbf{h}^{o\perp H}\,\mathbf{v}[k] - \widetilde{\mathbf{h}}^{\perp H}\,\mathbf{h}^o\,a[k] \;=\; \mathbf{h}^{o\perp H}\,\mathbf{v}[k] + \mathbf{h}^{o\perp H}\widetilde{\mathbf{h}}\,a[k]$.

$R_{d\mathbf{x}} = \mathbf{h}^{oH}\,R_{\mathbf{vv}}\,\mathbf{h}^{o\perp}$ is unperturbed due to the fact that $\widetilde{\mathbf{h}}$ has zero mean and is circular: $\mathrm{E}\,\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}^T = 0$ (since at least $\mathbf{v}[k]$ is circular, and possibly $a[k]$ too).

$R_{\mathbf{xx}} = \mathbf{h}^{o\perp H}\left(R_{\mathbf{vv}} + \sigma_a^2\,R_{\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}}\right)\mathbf{h}^{o\perp} = (1+\frac{\nu}{2}\sigma_a^2)\,\mathbf{h}^{o\perp H}\,R_{\mathbf{vv}}\,\mathbf{h}^{o\perp}$.

The LMMSE filter $\mathbf{f} = R_{d\mathbf{x}}R^{-1}_{\mathbf{xx}} = \frac{1}{1+\frac{\nu}{2}\sigma_a^2}\,(\mathbf{h}^{oH}\,R_{\mathbf{vv}}\,\mathbf{h}^{o\perp})(\mathbf{h}^{o\perp H}\,R_{\mathbf{vv}}\,\mathbf{h}^{o\perp})^{-1} = \frac{1}{1+\frac{\nu}{2}\sigma_a^2}\,\mathbf{f}^o$ is just scaled down w.r.t. its optimal value (problem of bias).

$e[k] = \|\mathbf{h}^o\|^2\,a[k] + \mathbf{h}'^H(I - \frac{1}{1+\frac{\nu}{2}\sigma_a^2}\,P_{\mathbf{h}^{\perp\prime}})\,\mathbf{v}'[k]$. MMSE $= \|\mathbf{h}^o\|^4\,\sigma_a^2 + \mathbf{h}'^H(I - \frac{1}{1+\frac{\nu}{2}\sigma_a^2}\,P_{\mathbf{h}^{\perp\prime}})^2\,\mathbf{h}'$.

Note for a projection matrix: $P^2 = P$ (idempotent). Let $\alpha = \frac{1}{1+\frac{\nu}{2}\sigma_a^2}$.

Increase in MSE:
$(-2\alpha+\alpha^2+1)\,\mathbf{h}^{'H}\,P_{\mathbf{h}^{\perp'}}\,\mathbf{h}^{'} = (1-\alpha)^2\,\mathbf{h}^{'H}\,P_{\mathbf{h}^{\perp'}}\,\mathbf{h}^{'} = (\frac{\frac{\nu}{2}\sigma_a^2}{1+\frac{\nu}{2}\sigma_a^2})^2\,\mathbf{h}^{'H}\,P_{\mathbf{h}^{\perp'}}\,\mathbf{h}^{'} \approx \frac{\nu^2}{4}\sigma_a^4\,\mathbf{h}^{'H}\,P_{\mathbf{h}^{\perp'}}\,\mathbf{h}^{'}$
When $R_{\mathbf{VV}} = \sigma_v^2 I_m$, the increase in MSE is zero since then $P_{\mathbf{h}^{\perp'}}\,\mathbf{h}^{'} = \frac{1}{\sigma_v}\,P_{\mathbf{h}^{\perp}}\,\mathbf{h} = 0$.

(g)

$d[k] \;\approx\; \|\mathbf{h}^o\|^2\,a[k] + \mathbf{h}^{oH}\,\mathbf{v}[k] - \widetilde{\mathbf{h}}^H\,\mathbf{h}^o\,a[k], \quad \mathbf{x}[k] \;\approx\; \mathbf{h}^{o\perp H}\,\mathbf{v}[k] + \mathbf{h}^{o\perp H}\widetilde{\mathbf{h}}\,a[k].$

$R_{d\mathbf{X}} = \mathbf{h}^{oH}\,R_{\mathbf{VV}}\,\mathbf{h}^{o\perp} + \sigma_a^2\,\|\mathbf{h}^o\|^2\,\widetilde{\mathbf{h}}^H\mathbf{h}^{o\perp}.\;\; R_{\mathbf{XX}} = \mathbf{h}^{o\perp H}\left(R_{\mathbf{VV}} + \sigma_a^2\,\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}^H\right)\mathbf{h}^{o\perp} \approx \mathbf{h}^{o\perp H}\,R_{\mathbf{VV}}\,\mathbf{h}^{o\perp}.$

The LMMSE filter $\mathbf{f} = R_{d\mathbf{X}}R_{\mathbf{XX}}^{-1} \approx \mathbf{f}^o + \sigma_a^2\,\|\mathbf{h}^o\|^2\,\widetilde{\mathbf{h}}^H\mathbf{h}^{o\perp}\,(\mathbf{h}^{o\perp H}\,R_{\mathbf{VV}}\,\mathbf{h}^{o\perp})^{-1}$.

The error signal:

$$\begin{aligned} e[k] \;&=\; \underbrace{\|\mathbf{h}^o\|^2\,a[k] + (\mathbf{h}^{oH} - \mathbf{f}^o\,\mathbf{h}^{o\perp H})\,\mathbf{v}[k]}_{e^o[k]} \\ &\quad -\widetilde{\mathbf{h}}^H\,\mathbf{h}^o\,a[k] - \mathbf{f}^o\,\mathbf{h}^{o\perp H}\,\widetilde{\mathbf{h}}\,a[k] - \sigma_a^2\,\|\mathbf{h}^o\|^2\,\widetilde{\mathbf{h}}^H\,\mathbf{h}^{o\perp}\,(\mathbf{h}^{o\perp H}\,R_{\mathbf{VV}}\,\mathbf{h}^{o\perp})^{-1}\,\mathbf{h}^{o\perp H}\,\mathbf{v}[k] \end{aligned}.$$

To compute the MMSE, averaging over $\widetilde{\mathbf{h}}$, the perturbation terms are uncorrelated from $e^o[k]$ since $\widetilde{\mathbf{h}}$ is uncorrelated with all quantities in $e^o[k]$. The third perturbation term is uncorrelated with the first two since $\mathbf{v}[k]$ and $a[k]$ are uncorrelated. Finally, the first two perturbation terms are uncorrelated since $\widetilde{\mathbf{h}}$ is circular. We then get for the MMSE

$$\begin{aligned} \sigma_e^2 \;&=\; \sigma_{e^o}^2 + \tfrac{\nu}{2}\sigma_a^2\,\mathbf{h}^{oH}R_{\mathbf{VV}}\mathbf{h}^o + \tfrac{\nu}{2}\sigma_a^2\,\mathbf{f}^o\mathbf{h}^{o\perp H}R_{\mathbf{VV}}\mathbf{h}^{o\perp}\mathbf{f}^{oH} \\ &\quad+\; \sigma_a^4\|\mathbf{h}^o\|^4\;\underbrace{\mathrm{tr}\{\mathrm{E}\,(\mathbf{h}^{o\perp H}R_{\mathbf{VV}}\mathbf{h}^{o\perp})^{-1}\mathbf{h}^{o\perp H}\mathbf{v}[k]\mathbf{v}^H[k]\,\mathbf{h}^{o\perp}(\mathbf{h}^{o\perp H}R_{\mathbf{VV}}\mathbf{h}^{o\perp})^{-1}\mathbf{h}^{o\perp H}\widetilde{\mathbf{h}}\widetilde{\mathbf{h}}^H\mathbf{h}^{o\perp}\}}_{=\frac{\nu}{2}\,\mathrm{tr}\{I_{m-1}\} = \frac{\nu}{2}(m-1)\quad(+\mathcal{O}(\nu^2))} \end{aligned}$$

The signal part is $\|\mathbf{h}^o - \widetilde{\mathbf{h}}\|^2\,a[k] \approx \|\mathbf{h}^o\|^2\,a[k] - \mathbf{h}^{oH}\widetilde{\mathbf{h}}\,a[k] - \widetilde{\mathbf{h}}^H\mathbf{h}^o\,a[k]$.
The noise part of $e[k]$ is hence
$(\mathbf{h}^{oH} - \mathbf{f}^o\,\mathbf{h}^{o\perp H})\,(\mathbf{v}[k] + \widetilde{\mathbf{h}}\,a[k]) - \sigma_a^2\,\|\mathbf{h}^o\|^2\,\widetilde{\mathbf{h}}^H\,\mathbf{h}^{o\perp}\,(\mathbf{h}^{o\perp H}\,R_{\mathbf{VV}}\,\mathbf{h}^{o\perp})^{-1}\,\mathbf{h}^{o\perp H}\,\mathbf{v}[k]$.
The ICMF output SNR being the ratio of signal power to noise power, we obtain

$$\mathrm{SNR}^{out} = \frac{\|\mathbf{h}^o\|^4\,\sigma_a^2 + \nu\sigma_a^2\,\|\mathbf{h}^{'}\|^2}{\|P^{\perp}_{\mathbf{h}^{\perp'}}\mathbf{h}^{'}\|^2\,(1+\frac{\nu}{2}\sigma_a^2) + \|\mathbf{h}^o\|^4\,\sigma_a^2\,\frac{\nu}{2}\sigma_a^2\,(m-1)} \;\overset{(\mathrm{SNR}^{Rx}\to\infty)}{\underset{\equiv R_{\mathbf{VV}}\to 0}{\longrightarrow}}\; \frac{1}{\frac{\nu}{2}\sigma_a^2\,(m-1)}\;.$$

Spatial Diversity and SpatioTemporal Equalization

2. Negligible angular spreading: separable spatiotemporal channel case

(a) $\mathrm{MFB} = \frac{\sigma_a^2}{\sigma_v^2}\,\|\mathbf{H}\|^2 = \frac{\sigma_a^2}{\sigma_v^2}\,\|\mathbf{h}[0]\|^2\,\|H_1\|^2 = m\,\frac{\sigma_a^2}{\sigma_v^2}\,\|H_1\|^2 \;\sim\; m.$

(b)
$$\begin{aligned} \mathrm{SNR}_{MMSE\,ZF} \;&=\; \frac{\sigma_a^2}{\sigma_v^2}\,\frac{1}{\dfrac{1}{2\pi j}\displaystyle\oint\frac{dz}{z}(\mathbf{H}^\dagger(z)\mathbf{H}(z))^{-1}} = \frac{\sigma_a^2}{\sigma_v^2}\,\frac{1}{\dfrac{1}{2\pi j}\displaystyle\oint\frac{dz}{z}(H_1^\dagger(z)\mathbf{h}^H[0]\;\mathbf{h}[0]\,H_1(z))^{-1}} \\ &=\; m\,\frac{\sigma_a^2}{\sigma_v^2}\,\frac{1}{\dfrac{1}{2\pi j}\displaystyle\oint\frac{dz}{z}(H_1^\dagger(z)\,H_1(z))^{-1}} \le m\,\frac{\sigma_a^2}{\sigma_v^2}\,\frac{1}{2\pi j}\oint\frac{dz}{z}H_1^\dagger(z)\,H_1(z) = \mathrm{MFB} \end{aligned}$$

So, $\mathrm{SNR}_{MMSE\,ZF} \;\sim\; m.$

(c) We can state $\mathbf{H}^{\perp}(z)$ has the same column space as $\mathbf{h}^{\perp}[0]$. In particular: $\mathbf{h}^{\perp H}[0]\,\mathbf{H}(z) = \mathbf{h}^{\perp H}[0]\,\mathbf{h}[0]\,H_1(z) = 0$. So we can choose $\mathbf{h}^{\perp}[0]$, a purely spatial filter (and not spatiotemporal), as blocking filter in the bottom branch of the ICMF. For the matched filter in the top branch, remark that $\mathbf{H}^{\dagger}(z) = H_1^{\dagger}(z)\,\mathbf{h}^H[0]$. We can start with the spatial MF $\mathbf{h}^H[0]$, then perform interference cancellation at its output, and then continue with the temporal MF $H_1^{\dagger}(z)$ or any other form of temporal equalization. Due to the fact that the spatial MF and spatial blocking filter do not introduce (additional) delay spread, the interference cancelling filter probably requires little delay spread and a spatial filter may already be quite useful.

(d) $m-1$ interferers can be cancelled, the dimension of the output of the blocking filter. This dimension is in fact independent of the interferers' channels being separable or not. But if they are separable, the interferers can be cancelled with a purely spatial filter only.

3. Significant angular spreading: case of a paraunitary spatiotemporal channel

(a)
$$\begin{aligned}\mathbf{H}^{\dagger}(z)\,\mathbf{H}(z) &= (\sum_{k=0}^{N-1}\mathbf{h}^H[k]\,z^k)(\sum_{i=0}^{N-1}\mathbf{h}[i]\,z^{-i}) = \sum_{k=0}^{N-1}\sum_{i=0}^{N-1}\mathbf{h}^H[k]\,\mathbf{h}[i]\,z^{k-i}) \\ &= \sum_{k=0}^{N-1}\sum_{i=0}^{N-1}\|\mathbf{h}[i]\|^2\,\delta_{ki}\,z^{k-i} = \sum_{i=0}^{N-1}\|\mathbf{h}[i]\|^2 = \|\mathbf{H}\|^2\end{aligned}$$

which is a constant (the channel energy). $\mathbf{H}(z)$ is said to be paraunitary or lossless.

(b) $\|\mathbf{H}(e^{j2\pi f})\|^2 = \mathbf{H}^{\dagger}(z)\,\mathbf{H}(z)\Big|_{z=e^{j2\pi f}} = \|\mathbf{H}\|^2$ is constant with frequency f? The channel is also called allpass since all frequencies are passed with the same gain.

(c) $\text{MFB} = \frac{\sigma_a^2}{\sigma_v^2}\,\|\mathbf{H}\|^2$.

$$\text{SNR}_{MMSE\,ZF} = \frac{\sigma_a^2}{\sigma_v^2}\frac{1}{\frac{1}{2\pi j}\oint\frac{dz}{z}(\mathbf{H}^{\dagger}(z)\mathbf{H}(z))^{-1}} = \frac{\sigma_a^2}{\sigma_v^2}\frac{1}{\frac{1}{\|\mathbf{H}\|^2}} = \text{MFB}.$$

So for an allpass channel, a MMSE ZF equalizer (which becomes in fact simply the MF) is the optimal receiver! That is because after the MF (which equalizes the phase distortion), the channel is perfectly equalized, and the MF is the starting point of any reasonable receiver.

(d) $\text{MFB}^{(1)} = \frac{\sigma_a^2}{\sigma_v^2}\,\|\mathbf{H}\|^2 = \text{MFB}^{(2)}$.

(e) $\text{SNR}^{(1)}_{MMSEZF} \le \text{MFB}^{(1)} = \text{MFB}^{(2)} = \text{SNR}^{(2)}_{MMSEZF}$.

(f) $$\frac{\text{MFB}^{(2)}_{(m)}}{\text{MFB}^{(2)}_{(1)}} = \frac{\|\mathbf{H}\|^2}{\|H_1\|^2} = \frac{m\,\|H_1\|^2}{\|H_1\|^2} = m.$$

(g) $$\frac{\text{SNR}^{(2)}_{MMSEZF,(m)}}{\text{SNR}^{(2)}_{MMSEZF,(1)}} = \frac{\text{MFB}^{(2)}_{(m)}}{\text{MFB}^{(2)}_{(1)}} = m.$$