SP4COM

Signal Processing for Communications

Multi-Antenna Interference Handling for Multi-User Multi-Cell

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Course Arrangements

- course assistants: ?
- theoretical part: exam, open notes and handouts
- practical part:

  (graded parts: copying forbidden, discussing solutions allowed)

  – 1 problem session (TD): not graded
  – 1 computer session (TP): graded $\Rightarrow$ remotely this year
  – 3 (?) homeworks (graded)
- emphasis on problem solving, knowing how to apply the theory
  $\Rightarrow$ TD/HW important
- copies of viewgraphs will be distributed throughout the term,
  also on the course web page :

  `https://moodle.eurecom.fr/course/view.php?id=87`
  or `http://my.eurecom.fr`

Course Context

This course = complements to other courses to complete the coverage.
Transmission systems to be considered:

- wireless systems:
  - cellular systems, relays
  - wireless LANs, WiFi, mmWave, Femto/Small Cells, HetNets
  - Fixed Wireless Access (wireless DSL)
  - satellite
  - broadcasting systems: FM, DAB (digital audio/radio), analog TV, DVB (digital video/TV)
  - other wireless: point-to-point, private, ad hoc, sensor arrays, IoT
  - Cognitive Radio, localization, full duplex radio
  - multiple antennas: from beamforming to MIMO, Massive MIMO, Interference Alignment

Course Context (2)

- voiceband modems (56K = V90) (voiceband of telephone line)
- Digital Subscriber Loop (xDSL, Asymmetric DSL, Symmetric DSL) (telephone line without bandwidth limitation)
  VDSL now uses MIMO techniques (G.Fast)
- ethernet, gigabit ethernet (coax cables)
- powerline communications (over 220V power net)
- optical communications
- Li-Fi (wireless over visible light)
- NFC (near field communications)
- underwater communications (marine biology,...)
- communications for control (drones, car, (VANETs), production,...)

Overview

- *Multi-carrier Systems:*
  OFDM (Orthogonal Frequency Division Multiplexing) systems, DMT (Discrete MultiTone) systems, cyclic prefixes, guard intervals, equalization techniques (TEQ and FEQ, per tone approaches), PAR issues, tone loading.

- *Synchronization:*
  Basic techniques for single-carrier and multi-carrier systems timing recovery, phase locked loops (PLLs), analog and digital approaches (interpolation). Carrier recovery.

- *Equalizer Design Issues:*
  FIR equalization, equalizer delay, equalization for continuous or packet transmission. Equalization options for time-varying systems.

Overview (2)

- *Kalman Filtering:*
  Discrete-time state-space systems, prediction filtering, smoothing.
- *Channel Estimation:*
  Multi-carrier systems and 2D channel interpolation. Channel modeling and channel prediction in wireless systems, channel sparsification, long-term statistics.
- *Sinusoids in Noise:*
  Modal analysis techniques.
- *Blind and Semi-Blind Channel Estimation:*
  Multichannel systems, SIMO systems, FIR equalization, blind equalization, MIMO systems

Overview (3)

- *Spatiotemporal Processing:*
  Multichannel systems, spatial processing, spatiotemporal processing, Single user systems, white and colored noise, multiuser systems. Options for utilizing spatial dimensions: interference cancellation, SDMA, spatial multiplexing.

- *CDMA systems:*
  Multiuser detection techniques, synchronous and asynchronous systems, frequency-flat and frequency selective systems, RAKE receivers.

- *xDSL Systems and Gigabit Ethernet:*
  An overview of single- and multi-carrier techniques. Equalization and synchronization approaches, echo cancellation for full duplex operation over twisted pairs, multirate filtering. Interference cancellation in cable bundles.

Overview (4)

- *Powerline Communications:*
  Channel and noise characterization, communication approaches.
- *DAB/DVB Broadcasting Systems:*
  System scenarii and the choice of design parameters.
- *Fixed-Point Implementation and Round-Off Error Analysis:*
  Examples from fixed and adaptive filtering.
- *Further Advanced Techniques:*
  Equalization and lattice reduction techniques, linear precoding, Tomlinson-Harashima and dirty paper precoding.

Overview (5)

- Interference single cell: Broadcast Channel (BC)
  - utility functions: SINR balancing, (weighted) sum rate (WSR)
  - BC with user selection: Dirty Paper Coding (DPC) vs beamforming (BF)
  - MIMO: role of receive (Rx) antennas
- Interference multi-cell/HetNets: Interference Channel (IC)
  - Degrees of Freedom (DoF) and Interference Alignment (IA), IA feasibility
  - IA forms: asymptotic symbol extension, decomposition, ergodic, signal scale, MIMO
  - multi-cell multi-user: Interfering Broadcast Channel (IBC)
  - max Weighted Sum Rate (WSR), min Weighted Sum MSE (WSMSE), UL/DL duality
  - Difference of Convex functions approach, relation to max Signal-to-Leakage-plus-Noise Ratio (SLNR)
  - Deterministic Annealing to find global maximum
  - FIR IA for Asynchronous FIR Frequency-Selective IBC

Overview (6)

- Max WSR with Partial CSIT
  - CSIT: perfect, partial, Line-of-Sight (LoS), pathwise and non-Kronecker covariance
  - Expected WSR, Expected WSMSE, Massive MIMO limit, large MIMO asymptotics
- CSIT acquisition and distributed designs
  - distributed global CSIT acquisition, netDoF
  - topology, rank reduced, decoupled Tx/Rx design, local CSIT
  - Massive MIMO, mmWave and covariance CSIT
  - pathwise CSIT, BF as Dual UL pathwise LMMSE Rx
  - distributed designs

Overview Part 1

- Lecture 1
  Spatial Receiver Structures (TDMA)
- Lecture 2
  Spatio-Temporal Receiver Structures (TDMA)
- Lecture 3
  Interference Cancellation (TDMA)
- Lecture 4
  CDMA Multi-User Detection/Interference Cancellation
- Lecture 5
  Downlink Processing, OFDM, SDMA, TX Diversity, Spatial Multiplexing
- Lecture 6
  Multi-cell multi-user systems (massive MIMO, interference alignment)

References Part 1

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Lecture 1:

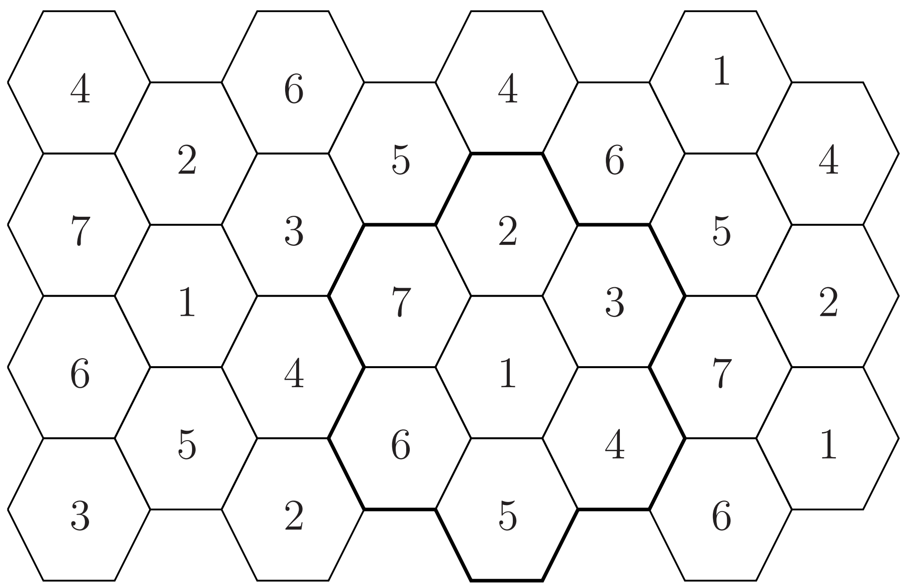
Spatial Receiver Structures (TDMA)

Overview

- cellular mobile systems context
- sources of multiple RX channels
- SIMO space-time channel models
- SIMO spatial processing
  - single-user, white noise
  - single-user, colored noise (interference cancellation)
  - multi-user detection, white and colored noise

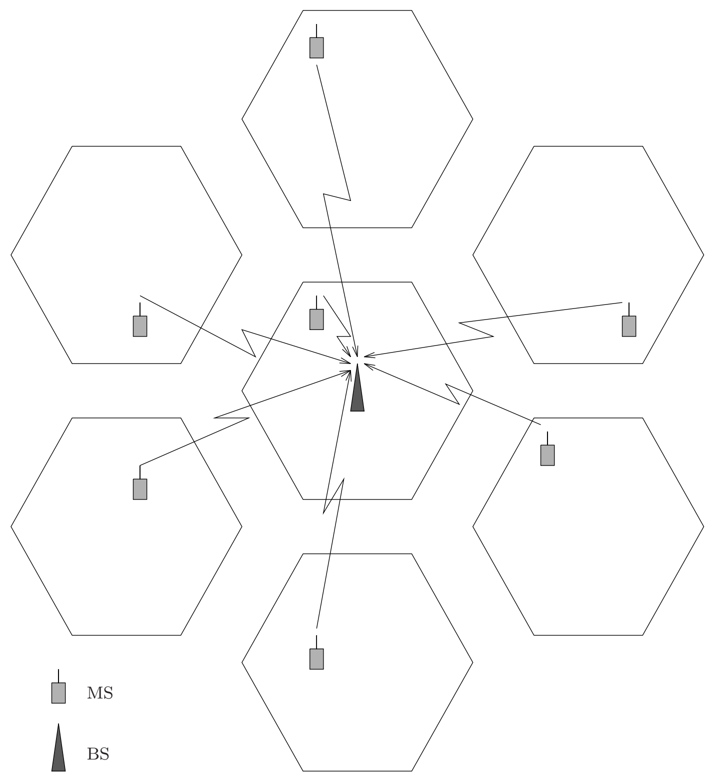
Cellular Systems

- *Frequency reuse*:
  Cells (served by one BS) are grouped into clusters.
  The surface is covered with a regular pattern of clusters.
  The available frequency channels are partitioned over the cells of a cluster and reused in the various clusters. Example with reuse factor of 7:



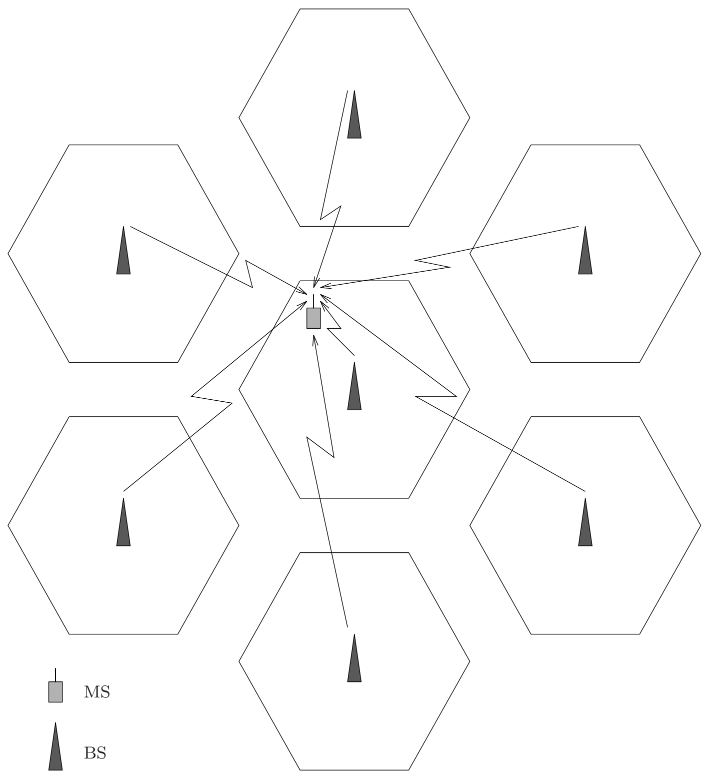
Uplink Co-Channel Interference (CCI)

- Uplink: from Mobile Station (MS) to Base Station (BS).



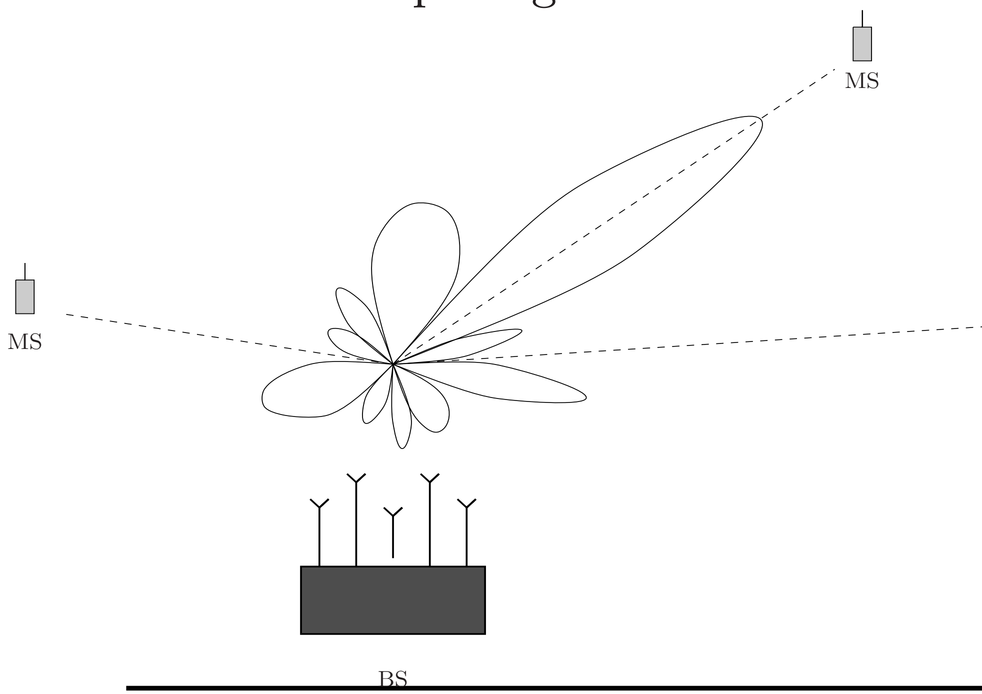
Downlink Co-Channel Interference (CCI)

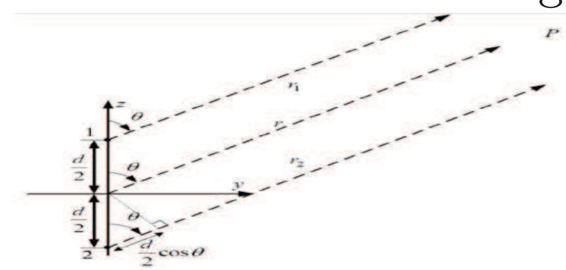
- Downlink: from Base Station (BS) to Mobile Station (MS).



BS Antenna Array Beam Pattern

- A *beam* is steered in the direction of the user of interest. Sidelobes in the beampattern are unavoidable (// FIR filter) but nulls should be put in the directions of interferers.
- e.g.: uniform linear array (ULA) response of the form $\mathbf{g}(\theta) = \left[\begin{array}{cccc} 1 & e^{-j2\pi\frac{d}{\lambda}\sin(\theta)} & \cdots & e^{-j(m-1)2\pi\frac{d}{\lambda}\sin(\theta)} \end{array}\right]^T$ where θ = angle between the direction of arrival (DOA) and the normal to the ULA, d is the spacing between the m antennas and λ is the wavelength.





[H]

Figure 2.4: Beam Replication on Different Transmit-antenna

Figure 2.4 shows one beam (with an angle of departure θ) replications on different transmit-antenna. We can now easily obtain path difference between any couple of these replications. Difference between between tow consecutive replications is $a = d/2cos(\theta)$. A given path difference leads to a a latency equal to the trajectory difference divided by light celerity: $\tau_i = a/c$.

As the signal has a carrier frequency f_c, the final coefficient between those consecutive replication of this beam is so $e^{2\pi f_c d\frac{cos(\theta)}{2c}}$. By arranging different coefficient rowise into one single vector we can define transmit-side response vector as:

$$\mathbf{h}_t(\theta_i) = \begin{bmatrix} 1 \\ e^{2j\pi fc\tau} \\ e^{2j\pi fc\tau 2} \\ \vdots \\ e^{2j\pi fc\tau M} \end{bmatrix} \quad (2.4)$$

Beamforming Primer

- temporal processing with a FIR filter:

$$H(z) = \sum_{k=0}^{N-1} h_k\, z^{-k} = \alpha \prod_{i=1}^{N-1} (1 - z_i\, z^{-1})$$

If zeros on 1-circle: $z_i = e^{j2\pi f_i},\; H(e^{j2\pi f}) = \alpha \prod_{i=1}^{N-1} (1 - e^{j2\pi(f_i - f)})$:

$N-1$ zeros $f = f_i$

- Uniform Linear Array (ULA):

"far field" assumption $\Rightarrow$ planar wavefront

"narrowband" (NB) assumption: time for wave to travel across antenna array $= \frac{(N-1)d}{c} \ll \frac{1}{\mathrm{BW}}$

if inter-antenna spacing $d = \frac{\lambda}{2}$ $\Rightarrow$

NB: $\frac{N-1}{2}\,\frac{\lambda}{c} = \frac{N-1}{2}\,\frac{1}{f_c} \ll \frac{1}{\mathrm{BW}} \Leftrightarrow \frac{\mathrm{BW}}{f_c} \ll \frac{2}{N-1}$

- time difference of arrival of wave impinging on array at angle θ between 2 consecutive antennas: $\tau = \frac{d \sin\theta}{c}$

Beamforming Primer (2)

- NB $\Rightarrow$ modulated carrier $\approx$ pure sinusoid :
  $e^{j2\pi f_c(t-\tau)} = e^{-j2\pi f_c \tau} e^{j2\pi f_c t}$: delay = phase shift
- beamformer = linear filter combining antenna outputs, BF output

$$y(t) = \sum_{i=0}^{N-1} w_i\, s(t - i\tau) \overset{\text{NB}}{=} \left(\sum_{i=0}^{N-1} w_i\, e^{-j\, i\, 2\pi \frac{d}{\lambda} \sin\theta} \right) s(t) = f(\theta)\, s(t)$$

$f(\theta) = [w_0\, w_1 \cdots w_{N-1}]\, \mathbf{g}(\theta)$,
$\mathbf{g}(\theta)$ = antenna array response/manifold
$|f(\theta)|$ = antenna array diagram for given weights w_i

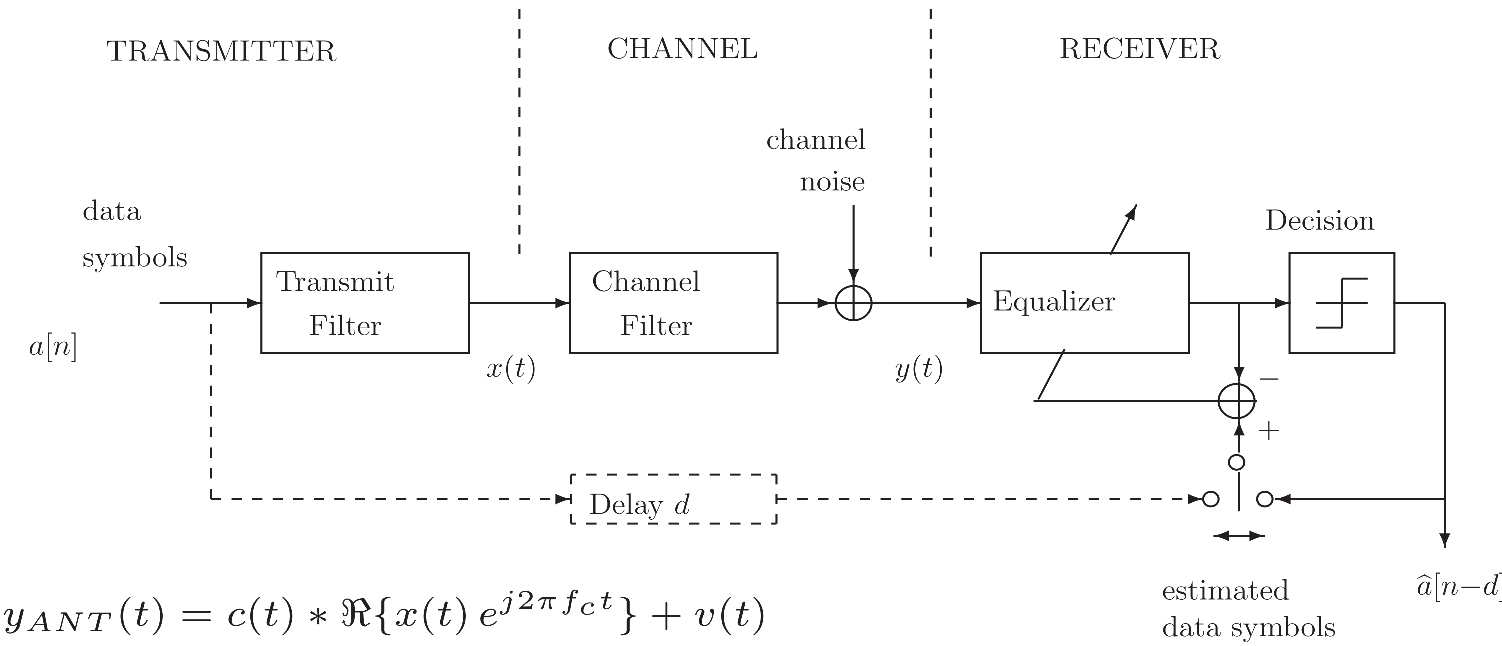
- If want to cancel directions: $f(\theta_i) = 0,\ i = 1, \ldots, N-1$
  then with $z_i = e^{j2\pi \frac{d}{\lambda} \sin\theta_i}$, $z = e^{j2\pi \frac{d}{\lambda} \sin\theta}$, and e.g. α is adjusted for $f(\theta_0) = 1$,

$$f(\theta) = \sum_{k=0}^{N-1} w_k\, z^{-k} = \alpha \prod_{i=1}^{N-1} (1 - z_i z^{-1}) \quad \Rightarrow \{w_i\}$$

- main lobe gain = $\max_\theta |f(\theta)| \sim N($ - # zeros$)$

Linear Modulation

- received signal before sampling: $y(t) = \sum_n a[n]\, h(t - nT) + v(t)$
- T = symbol period, $\frac{1}{T}$ = symbol rate
- symbols $a[n] \in \mathcal{A}$ symbol alphabet/constellation
- $h(t)$= convolved impulse responses of TX filter, channel, RX filter



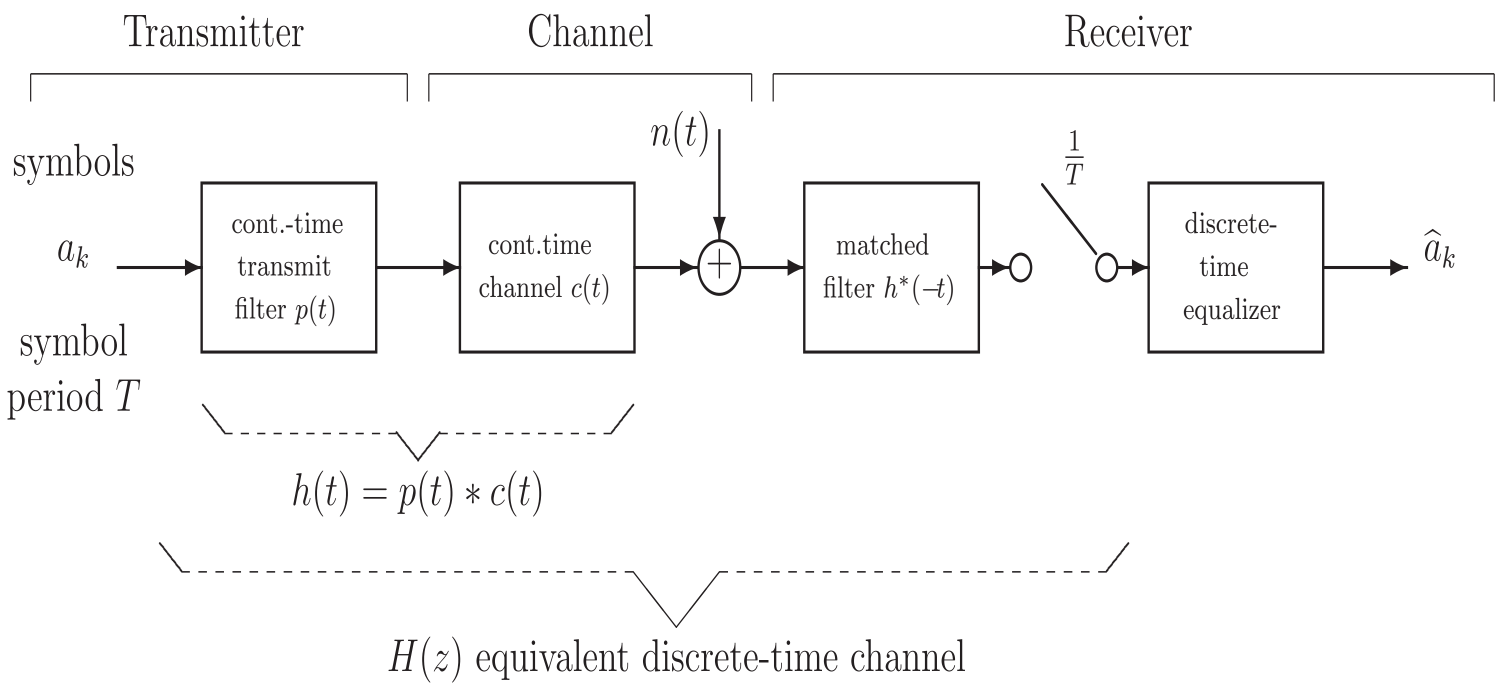
- $y_{ANT}(t) = c(t) * \Re\{x(t)\, e^{j2\pi f_c t}\} + v(t)$

  $y_{RX}(t) = LPF\{y_{ANT}(t)\, e^{-j2\pi f_c t}\}$, $x(t) = a[n] * p(t)$ (discrete/cont. time I/O)

  $= a[n] * \underbrace{LPF\{(c(t) * p(t))\, e^{-j2\pi f_c t}\}}_{h(t)} + v(t)$ (+ S&H DAC)

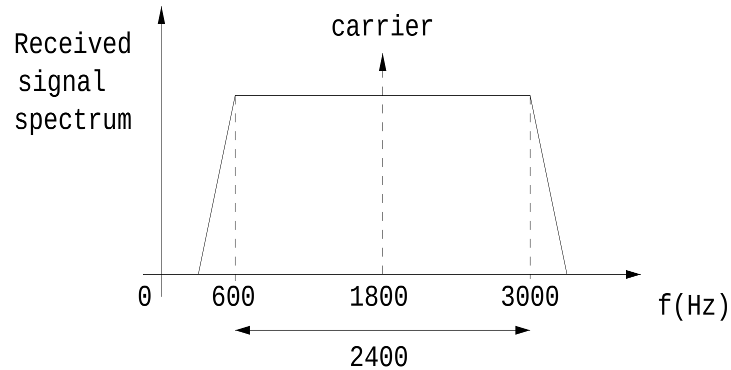
  $c(t) \;\Rightarrow\; h(t)$ complex due to up/down-modulation

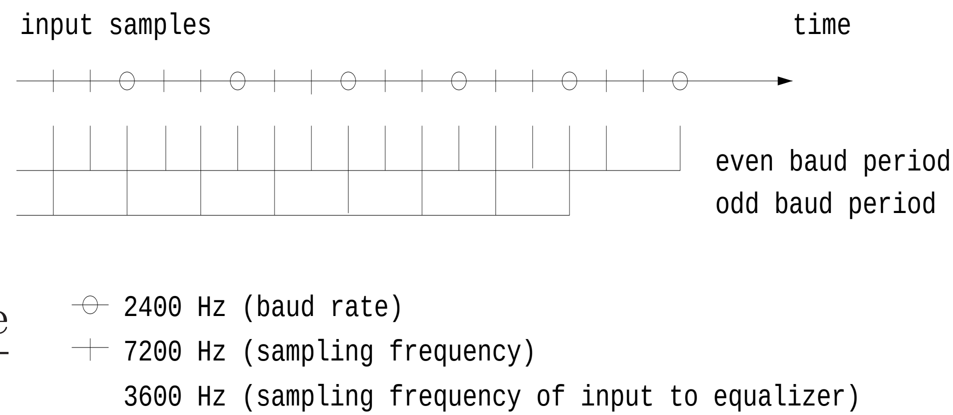
Ideal Reception



- assumes white noise
- output samples of matched filter at symbol rate $\frac{1}{T}$ form *sufficient statistics* for the detection of the symbols a_k
- matched filter and symbol rate sampling can be replaced by oversampling (satisfying Nyquist) and anti-aliasing filtering

Excess Bandwidth and Oversampling





- Tx filter (Nyquist pulse) passes a bandwidth > symbol (baud) rate
- oversample to satisfy Nyquist

$$\text{oversampling factor} = \frac{\text{sampling rate}}{\text{symbol rate}}$$

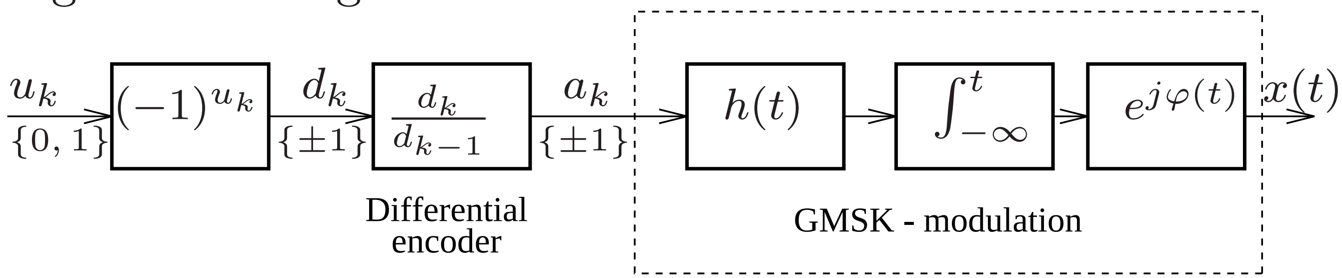
- example: CCITT V32
  9.6kbps voiceband modem
- example of oversampling factor $= \frac{3}{2}$, requires separate treatment of even and odd equalizer output samples (at symbol rate)

Signal Imperfections in Wireless Communications

- *nonlinearities*: DAC, ADC, RF amplifier (saturation issue ⇒ peak "power" constraint on Tx signal, PAPR: OFDM vs. GMSK, SC-CP, CDMA)
- *interference*:
  - CCI: cochannel interference (same frequency band)
  - ACI: adjacent channel interference (next frequency band)
- *noise*:
  - manmade noise: impulsive noise from machines
  - thermal noise in electronics
  - interstellar noise
- residual *frequency offset* due to modulator-demodulator mismatch and Doppler effect
- (sampling) *clock drift* w.r.t. symbol rate clock
- channel *fading* (variation in time due to movement)

Linearization of GMSK

- Gaussian Minimum Shift Keying: nonlinear modulation used in GSM
- Signal flow diagram for the GMSK modulation



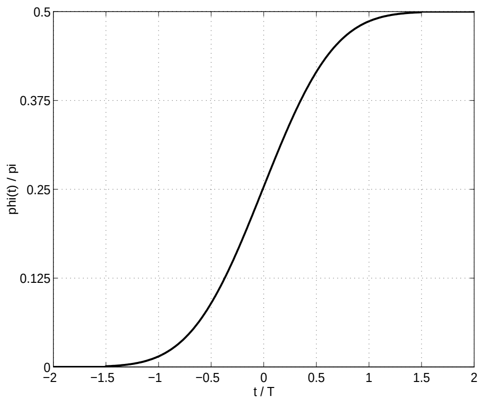
- GMSK signal: $x(t) = e^{j\varphi(t)}$ where

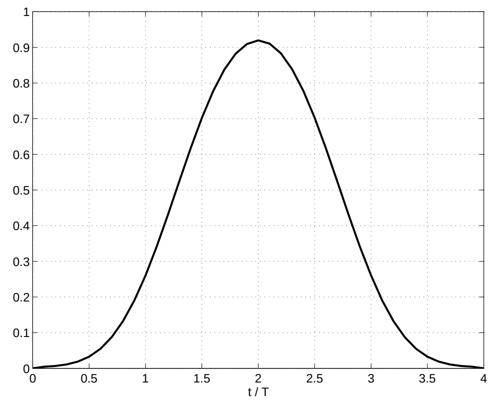
$$\varphi(t) = \tfrac{\pi}{2} \sum_k a_k \int_{-\infty}^{t} rect\left(\frac{u-kT}{T}\right) * h(u)du = \sum_k a_k \phi(t-kT)$$

$h(u) = \frac{1}{\sqrt{2\pi}\sigma T} e^{-\frac{u^2}{2\sigma^2 T^2}}$ where $\sigma = \frac{\sqrt{ln(2)}}{2\pi BT}$, B is the 3dB bandwidth, $BT = 0.3$.

Linearization of GMSK (2)

- Phase impulse response and impulse response of GMSK

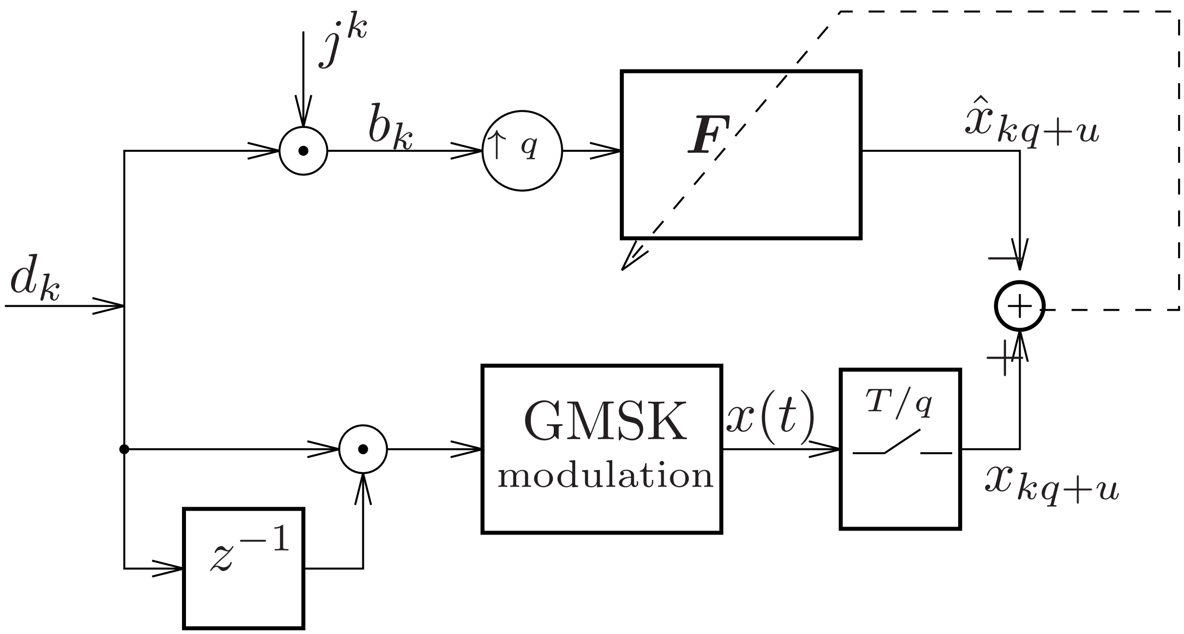




- From the phase impulse response $\phi(t)$ we conclude that one symbol a_k will have an influence on three symbol periods $(k-1, k, k+1)$. Hence some ISI gets introduced by the modulation itself.

Linearization of GMSK via LS

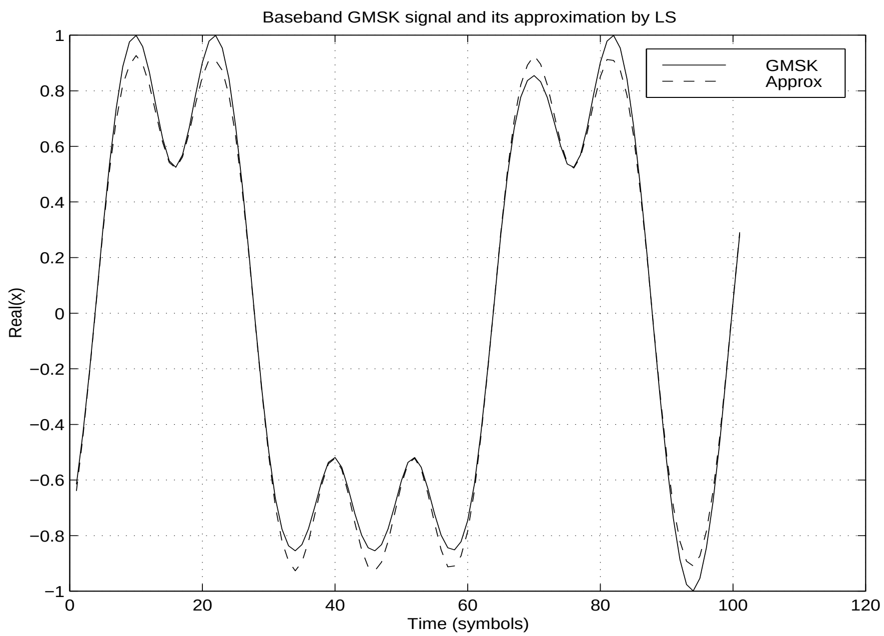
- A GMSK signal can be approximated by a linear filter with impulse response duration of about $L_f T = 4T$, fed by $b_k = j^k d_k$, a modulated version of the transmitted symbols ($j^k = e^{j\,2\pi\,\frac{1}{4}\,k}$).
  $b_k = \pm 1, k$ even, $b_k = \pm j, k$ odd.



- Least-squares estimation over a long stretch of signal.

Linearization of GMSK via LS (2)

- To have an idea of the quality of this estimation, we plot simultaneously the real part of the baseband GMSK signal $x(t)$ and the interpolated output of the discrete-time linear system $\hat{x}(t)$ assuming the Nyquist theorem is satisfied (*i.e.* $q = 6$).



- SNR = 25dB for the linearization approximation by LS (vs 20dB for the first term in Laurent's expansion).

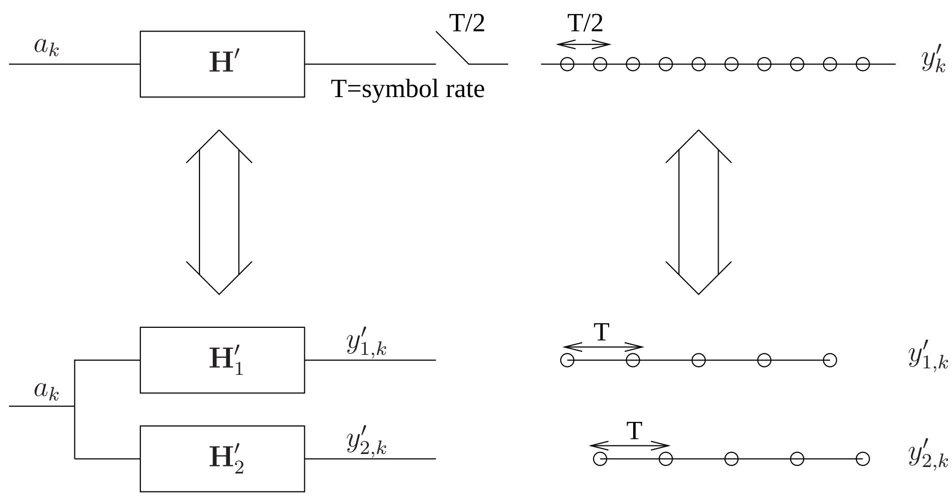
Sources of Multiple Channels (1)

- oversampling (w.r.t. symbol rate $\frac{1}{T}$) with factor r: case $r = 2$

$$y_l = y(t_0 + lT/2) = \sum_n a_n\, h(t_0 + lT/2 - nT)$$

$$l = 2k \Rightarrow y_{1,k} = y_{2k} = \sum_n a_n\, h(t_0 + kT - nT) = \sum_n a_n\, h_{1,k-n}$$

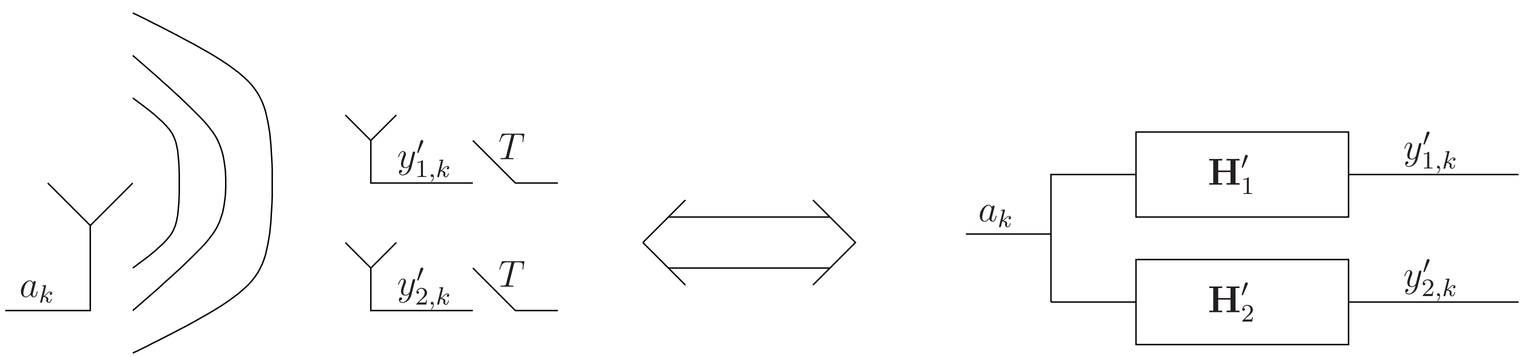
$$l = 2k{+}1 \Rightarrow y_{2,k} = y_{2k+1} = \sum_n a_n\, h(t_0 + \tfrac{T}{2} + kT - nT) = \sum_n a_n\, h_{2,k-n}$$



Discrete-time Channels at symbol rate

Sources of Multiple Channels (2)

- reception through multiple sensors:
  - antennas
  - polarizations
  - 6 EM components
    (only 4 degrees of freedom though [Marzetta:isit02])
  - preformed beams



Discrete-time Channels at symbol rate

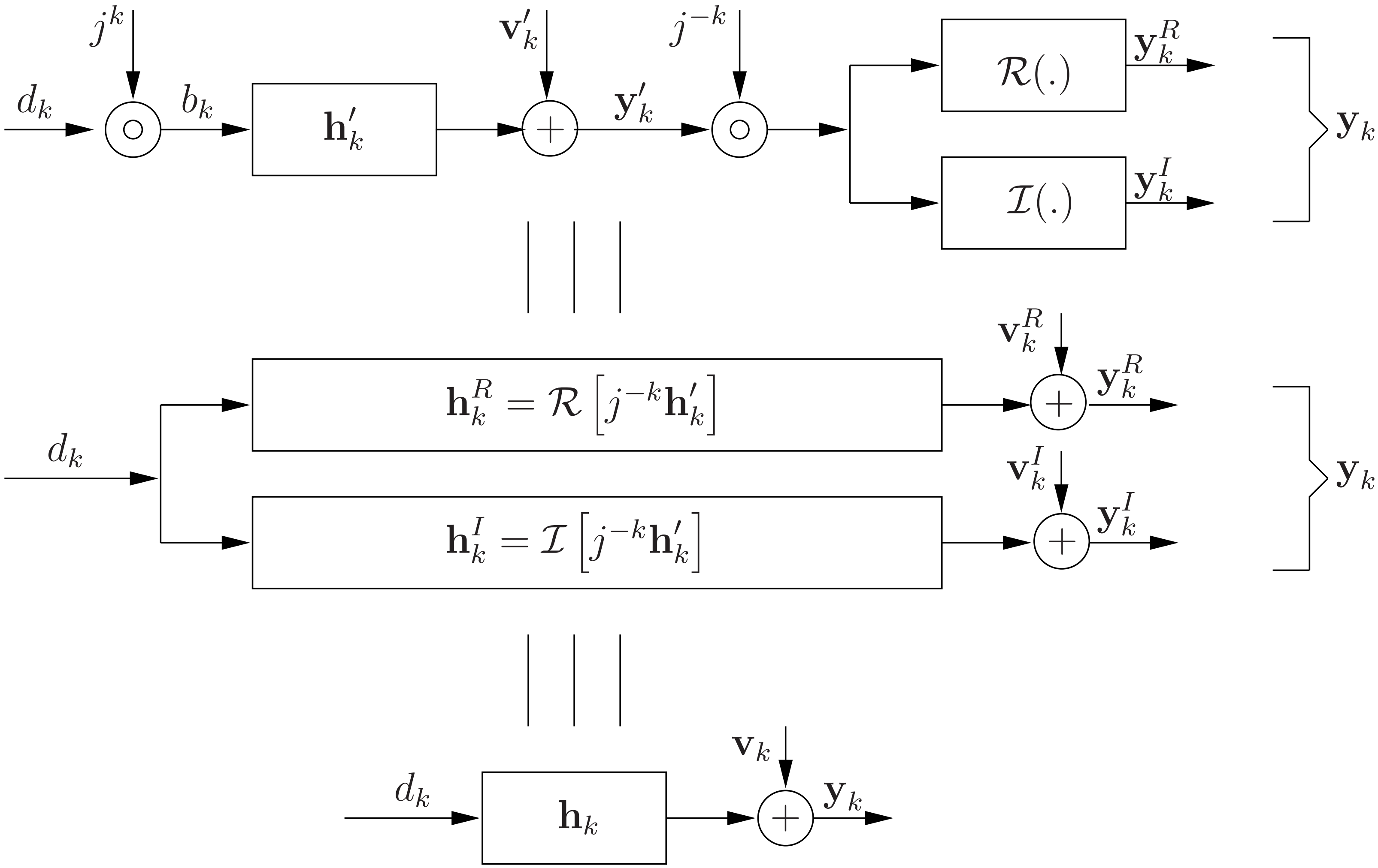
Sources of Multiple Channels (3)

- if the symbol constellation is real (BPSK, PAM), and the signal is transmitted through modulation-demodulation, then we can work with in-phase and in-quadrature components and get a signal representation with double the number of channels and only real quantities:

$$y_k = \sum_n a_n\, h_{k-n} \;\Rightarrow\; \left\{ \begin{array}{rcl} \mathcal{R}\{y_k\} & = & \sum_n a_k\, \mathcal{R}\{h_{k-n}\} \\ \mathcal{I}\{y_k\} & = & \sum_n a_k\, \mathcal{I}\{h_{k-n}\} \end{array} \right.$$

- ARQ protocols $\Rightarrow$ reception of multiple versions of the same data paquet

Case of Linearized GMSK

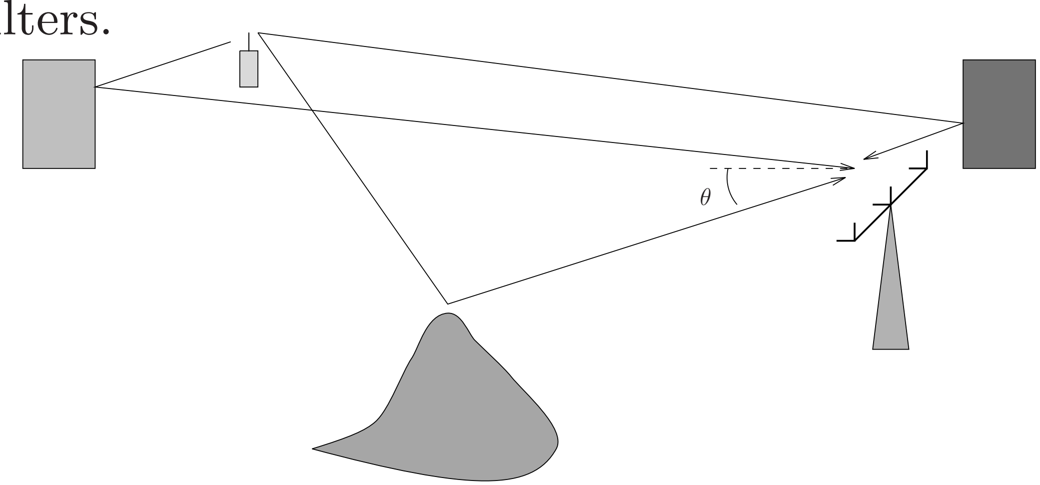


Multipath Channel Model

- multipath scattering (local to mobile, local to base, remote)
- received impulse response from MS at ith element of m-element antenna array:

$$h_i(t,\tau) = \sum_{l=1}^{M} g_i(\theta_l)\,\alpha_l(t)\,p(\tau-\tau_l) = \left(\sum_{l=1}^{M} g_i(\theta_l)\,\alpha_l(t)\,\delta(\tau-\tau_l)\right) * p(\tau)$$

where M = number of multipaths, $g_i(\theta_l)$ is the response of the ith antenna to the lth path from direction θ_l with delay τ_l and fading complex envelope $\alpha_l(t)$. $p(t)$ is the convolution of transmission and receiver filters.



Multipath Channel Model (2)

- Narrowband assumption: inverse of signal bandwidth large compared to time to travel across array. As a result, the $g_i(\theta_l)$ reflect (mainly) the phase-shifts that the signal undergoes when impinging on the consecutive antenna elements from direction θ_l. The $g_i(\theta_i)$ also contain the variations in magnitude and phase of the individual amplifiers of the antenna receivers.

Multipath Channel Model (3)

- stacking the impulse responses in a column vector:

$$\mathbf{h}(t,\tau) \;=\; \sum_{l=1}^{M} \mathbf{g}(\theta_l)\,\alpha_l(t)\,p(\tau-\tau_l)$$

  $\mathbf{g}(\theta_l)$ is the array response vector in direction θ_l. The *array manifold* is the set of array response vectors $\mathbf{g}(\theta)$ for all possible angles θ. It depends strongly on the channel frequency. The array manifold is a key quantity in direction-of-arrival (DOA) estimation. It allows to determine the direction θ given an array response $\mathbf{g}(\theta)$.
- Due to the limited bandwidth of $p(t)$, delays τ_l are not infinitely resolvable. So paths should be grouped in groups of resolvable delays. With one delay several angles can be associated.
- Whether to use the spatiotemporal impulse response or the multipath parameters depends on the number of paths vs. the delay spread. Multipath channel model: recently more important for mmWave communications.

Multipath Channel Model (4)

- *slow* parameters: angles/DOAs θ_l, delays τ_l
- *fast* parameters: complex amplitudes $\alpha_l(t)$

  Doppler spectrum
- *diversity degree*: number of (resolvable) fast fading sources $\alpha_l(t)$
  = rank of channel impulse covariance matrix (averaged over fast parameters)
  = number of resolvable (in time or in space) paths

  increasing oversampling beyond Nyquist or number of antennas beyond resolving path angles does not yield increased diversity

Correlation and Covariance Matrices

- random vectors $\mathbf{x}$ and $\mathbf{y}$
- mean: $m_{\mathbf{x}} = \mathrm{E}\,\mathbf{x}\;,\quad m_{\mathbf{y}} = \mathrm{E}\,\mathbf{y}$ (E = Expectation)
- correlation matrix: $R_{\mathbf{x}\mathbf{y}} = \mathrm{E}\,\mathbf{x}\,\mathbf{y}^H\;,\quad R_{\mathbf{x}\mathbf{x}} = \mathrm{E}\,\mathbf{x}\,\mathbf{x}^H$
- covariance matrix:

  $C_{\mathbf{x}\mathbf{y}} = R_{\mathbf{x}-m_{\mathbf{x}},\mathbf{y}-m_{\mathbf{y}}} = \mathrm{E}\,(\mathbf{x}-m_{\mathbf{x}})(\mathbf{y}-m_{\mathbf{y}})^H = R_{\mathbf{x}\mathbf{y}} - m_{\mathbf{x}}\,m_{\mathbf{y}}^H$
- Euclidean norms: $\|\mathbf{x}\|^2 = \mathbf{x}^H\mathbf{x}\,,\ \|\mathbf{x}\|_R^2 = \mathbf{x}^H R\,\mathbf{x}$
- vector power (mean square value):

$$\begin{aligned}\mathrm{E}\,\|\mathbf{x}\|^2 &= \mathrm{tr}\left\{\mathrm{E}\,\|\mathbf{x}\|^2\right\} = \mathrm{E}\,\mathrm{tr}\left\{\|\mathbf{x}\|^2\right\} = \mathrm{E}\,\mathrm{tr}\left\{\mathbf{x}^H\mathbf{x}\right\}\\ &= \mathrm{E}\,\mathrm{tr}\left\{\mathbf{x}\,\mathbf{x}^H\right\} = \mathrm{tr}\left\{\mathrm{E}\,\mathbf{x}\,\mathbf{x}^H\right\} = \mathrm{tr}\left\{R_{\mathbf{x}\mathbf{x}}\right\}\end{aligned}$$

- LMMSE estimation:

  estimate $\widehat{\mathbf{x}} = R_{\mathbf{x}\mathbf{y}}R_{\mathbf{y}\mathbf{y}}^{-1}\,\mathbf{y}$, estimation error $\widetilde{\mathbf{x}} = \mathbf{x} - \widehat{\mathbf{x}}$

  $\mathrm{MSE} = \mathrm{E}\,\|\widetilde{\mathbf{x}}\|^2 = \mathrm{tr}\left\{R_{\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}}\right\} = \mathrm{tr}\left\{R_{\mathbf{x}\mathbf{x}} - R_{\mathbf{x}\mathbf{y}}R_{\mathbf{y}\mathbf{y}}^{-1}R_{\mathbf{y}\mathbf{x}}\right\}$

Math/Stat Background: Covariance Matrices

- *positive semidefinite* matrix C : notation $C \geq 0$:
  $\forall\, \mathbf{u} \in \mathcal{C}^m : \mathbf{u}^H\, C\, \mathbf{u} \geq 0$
  positive definite matrix C : notation $C > 0$:
  $\forall\, \mathbf{u} \in \mathcal{C}^m \setminus \{0\} : \mathbf{u}^H\, C\, \mathbf{u} > 0$

- covariance matrix

$$C_{\mathbf{x}\mathbf{x}} = \mathrm{E}\,(\mathbf{x}-m_{\mathbf{x}})\,(\mathbf{x}-m_{\mathbf{x}})^H = \int dx_1 \cdots \int dx_m f_{\mathbf{x}}(\mathbf{x})\, \underbrace{(\mathbf{x}-m_{\mathbf{x}})\,(\mathbf{x}-m_{\mathbf{x}})^H}_{\geq 0\;,\quad \text{rank } 1}$$

  where $C_{x_i x_j} = \mathrm{E}\,(x_i - m_{x_i})\left(x_j - m_{x_j}\right)^{*}$.

- for *circular (proper) complex random vectors*, $\mathrm{E}\,\mathbf{x}\,\mathbf{x}^T = 0$ (without complex conjugate)

- Observe $C_{\mathbf{x}\mathbf{x}} = C_{\mathbf{x}\mathbf{x}}^H \geq 0$ Hermitian and positive semidefinite, as weighted average of positive semidefinite matrices. Indeed,
  $\mathbf{u}^H\, C_{\mathbf{x}\mathbf{x}}\, \mathbf{u} = \mathbf{u}^H (\mathrm{E}\,(\mathbf{x}-m_{\mathbf{x}})\,(\mathbf{x}-m_{\mathbf{x}})^H)\mathbf{u} = \mathrm{E}|\mathbf{u}^H\,(\mathbf{x}-m_{\mathbf{x}})\,|^2 \geq 0$.

Math/Stat Background: Eigen Decomposition (Covariance) Matrix

- *eigenvalues* λ_i and corresponding *eigenvectors* V_i of C : $C\,V_i \;=\; \lambda_i\,V_i$

  fix *norm* $\|V_i\| = 1, \quad \|V_i\|^2 = V_i^H V_i$
- $(C - \lambda_i\,I_m)\,V_i = 0 \;\Rightarrow\; (C - \lambda_i\,I_m)$ *singular*

  λ_i solution of $\;\det(C - \lambda\,I_m) = 0$: *characteristic equation*
- $C = C^H \;\Rightarrow\; \lambda_i \in \mathcal{R}$, orthonormal basis: $V_i^H V_j = \delta_{ij}$, $i, j = 1, \ldots, n$

  Kronecker delta : $\delta_{ij} = \begin{cases} 1 & , \; i = j \\ 0 & , \; i \neq j \end{cases}$
- matrix $V = [V_1 \cdots V_m]$ $(m \times m)$:

  $\left[V^H V\right]_{ij} = V_i^H V_j = \delta_{ij} \;\Rightarrow\; V^H V = I_m \quad V =$ *unitary matrix*
- $I_m = V^H V \Rightarrow 1 = \det(I_m) = \det\left(V^H V\right) = \det\left(V^H\right)\,\det\left(V\right) = |\det V|^2 \;\Rightarrow\; |\det V| = 1$. We can multiply the V_i with factors $e^{j\theta_i}$ such that $\det V = 1$.

Math/Stat Background: Eigen Decomposition (Covariance) Matrix (2)

- $C \geq 0$ positive semidefinite: $\forall\, U \in \mathcal{C}^m : U^H C U \geq 0$
- take $U = V_i \,:\, U^H C U = V_i^H C V_i = \lambda_i V_i^H V_i = \lambda_i \geq 0$
- order the $\lambda_i \,:\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$
- If $\lambda_m = 0$, C is singular $\Rightarrow V_m^H C V_m = \mathrm{E}\,|V_m^H(\mathbf{x} - m_{\mathbf{x}})|^2 = 0$ mean and variance of $V_m^H(\mathbf{x} - m_{\mathbf{x}})$ are zero $\Rightarrow V_m^H(\mathbf{x} - m_{\mathbf{x}}) = 0$ in mean square. This means that at least one variable x_i is a linear combination of the other variables and 1. We shall in general exclude this possibility $\Rightarrow C > 0,\, \lambda_i > 0,\, i = 1, \ldots, m$
- $CV_i - V_i\lambda_i = 0$ are the columns of the matrix $CV - V\Lambda = 0$ where $\Lambda = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_m\}$. Using $V^{-1} = V^H$, we find

$$C \;=\; V\Lambda V^H \;=\; [V_1 \cdots V_m]\,\mathrm{diag}\,\{\lambda_1, \ldots, \lambda_m\}\,[V_1 \cdots V_m]^H \;=\; \sum_{i=1}^{m} \lambda_i V_i V_i^H$$

Math/Stat Background: Optimization w.r.t. Complex Variables

- Consider cost functions $f(.)$ that satisfy some constraints (the *real* $f(\mathbf{x},\mathbf{x}^*)$ depends on both variables $\mathbf{x}$ and $\mathbf{x}^*$ in a symmetric fashion) that will always be satisfied in problems considered in this course.

- When optimizing w.r.t. $\mathbf{x}$, treat $\mathbf{x}$ and $\mathbf{x}^*$ as if they are independent (unrelated) variables. Intuition? $\mathbf{x} = \mathbf{x}_R + j\,\mathbf{x}_I$ where $\mathbf{x}_R$, $\mathbf{x}_I$ are independent variables.

- Example: $\min\limits_{a} \|\mathbf{y} - \mathbf{h}\,a\|^2$.
  $\|\mathbf{y} - \mathbf{h}\,a\|^2 = (\mathbf{y} - \mathbf{h}\,a)^H(\mathbf{y} - \mathbf{h}\,a) = \mathbf{y}^H\mathbf{y} - \mathbf{y}^H\mathbf{h}\,a - a^*\,\mathbf{h}^H\mathbf{y} + a\,a^*\,\mathbf{h}^H\mathbf{h}$
  This is only linear in a or in a^* ! To end up with an equation in a, derive w.r.t. a^* instead of a:
  $$\frac{\partial\,\|\mathbf{y}-\mathbf{h}\,a\|^2}{\partial a^*} \;=\; -\mathbf{h}^H\mathbf{y} + a\,\mathbf{h}^H\mathbf{h} \;=\; 0 \;\Rightarrow\; a = \frac{\mathbf{h}^H\mathbf{y}}{\mathbf{h}^H\mathbf{h}}$$

- Minimum? Yes: Hessian $\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\|\mathbf{y} - \mathbf{h}\,a\|^2 \;=\; \frac{\partial}{\partial a^*}(\frac{\partial}{\partial a^*}\|\mathbf{y} - \mathbf{h}\,a\|^2)^* = \mathbf{h}^H\mathbf{h} > 0$ where the first equality follows from the symmetry of $f(\mathbf{x},\mathbf{x}^*)$ in its arguments.

Math/Stat Background: Optimization w.r.t. Complex Vectors

- $\mathbf{g}(\mathbf{x},\mathbf{x}^*)$: $1 \times n$ row vector function, its gradient w.r.t. $\mathbf{x}^*$:

$$\frac{\partial\, \mathbf{g}(\mathbf{x},\mathbf{x}^*)}{\partial \mathbf{x}^*} = \begin{bmatrix} \frac{\partial\, \mathbf{g}(\mathbf{x},\mathbf{x}^*)}{\partial x_1^*} \\ \vdots \\ \frac{\partial\, \mathbf{g}(\mathbf{x},\mathbf{x}^*)}{\partial x_m^*} \end{bmatrix} \qquad m \times n$$

  If $g(\mathbf{x},\mathbf{x}^*)$ is a scalar ($n = 1$), then $\frac{\partial\, g(\mathbf{x},\mathbf{x}^*)}{\partial \mathbf{x}^*}$ is a column vector of the same dimensions as $\mathbf{x}$.

- in particular: $\frac{\partial\, \mathbf{x}^H}{\partial \mathbf{x}^*} = \left[\frac{\partial\, x_j^*}{\partial x_i^*}\right] = [\delta_{ij}] = I_m$

- The gradient (linear) operator commutes with (other) linear operations. Let $\mathbf{y}$ be $m \times 1$

$$\frac{\partial}{\partial \mathbf{x}^*}\left(\mathbf{x}^H \mathbf{y}\right) = \left(\frac{\partial\, \mathbf{x}^H}{\partial \mathbf{x}^*}\right)\mathbf{y} = I_m\, \mathbf{y} = \mathbf{y}\,.$$

Math/Stat Background: Optimization w.r.t. Complex Vectors (2)

Example:

- training based channel estimation: $\min_{\mathbf{h}} \sum_k \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2$.

  $\sum_k \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2 = \sum_k (\mathbf{y}[k] - \mathbf{h}\, a[k])^H (\mathbf{y}[k] - \mathbf{h}\, a[k])$

  $= \sum_k \{\mathbf{y}^H[k]\mathbf{y}[k] - \mathbf{y}^H[k]\mathbf{h}\, a[k] - a^*[k]\, \mathbf{h}^H \mathbf{y}[k] + |a[k]|^2\, \mathbf{h}^H \mathbf{h}\}$

  This is only linear in $\mathbf{h}$ or in $\mathbf{h}^*$! To end up with an equation in $\mathbf{h}$, derive w.r.t. $\mathbf{h}^*$ instead of $\mathbf{h}$:

  $\frac{\partial}{\partial \mathbf{h}^*} \sum_k \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2 \;=\; -\sum_k a^*[k]\, \mathbf{y}[k] + \sum_k |a[k]|^2\, \mathbf{h} \;=\; 0$

  $\Rightarrow\;\mathbf{h} = \frac{1}{\sum_k |a[k]|^2} \sum_k \mathbf{y}[k]\, a^*[k]$

- Minimum? Yes: Hessian (matrix of second-order derivatives)

  $\frac{\partial}{\partial \mathbf{h}^*} (\frac{\partial}{\partial \mathbf{h}^*} \sum_k \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2)^H \;= (\sum_k |a[k]|^2)\, I_m > 0$

Spatial Case

- If channel *delay spread* $<$ symbol period $T \;\Rightarrow$ no Intersymbol Interference (ISI), sampled impulse response reduces to $\mathbf{h}[0]$
- Single user case. Received signal sampled at the symbol rate ($t = kT$)

$$\mathbf{y}[k] = \mathbf{h}\, a[k] + \mathbf{v}[k]$$

  $a[k]$ = transmitted symbol sequence (linear(izable) modulation),
  $\mathbf{v}[k]$ = noise plus interference samples
  $\mathbf{h} = \mathbf{h}[0]$ sometimes called the *spatial signature* $= \sum_{l=1}^{M} \alpha_l\, \mathbf{g}(\theta_l)$
- *diversity*: Maximum Likelihood symbol detection, if $\mathbf{v}(k)$ is spatially and temporally white, leads to an SNR improvement by $\sqrt{\displaystyle\sum_{i=1}^{m} |h_i|^2}$ which increases with m, becomes much less sensitive to the deep fades of the individual coefficients h_i.

Spatial Case: Channel Estimation

- using a *training sequence*: a sequence of known symbols $a[k]$

$$\widehat{\mathbf{h}} = \frac{\sum_k \mathbf{y}[k]\, a^*[k]}{\sum_k |a[k]|^2}$$

- *blind* estimation: without training sequence (increases throughput)
  - second-order statistics based:

    $$E\, \mathbf{y}[k]\mathbf{y}^H[k] = R_{\mathbf{yy}} = \sigma_a^2\, \mathbf{h}\mathbf{h}^H + \sigma_v^2 I_m$$

    hence $\mathbf{h}$ $\sim$ largest eigenvector of $R_{\mathbf{yy}} = \sum_{i=1}^m \lambda_i V_i V_i^H$.
    $V_1 \sim \mathbf{h}$, $\lambda_1 = \sigma_v^2 + \sigma_a^2 \|\mathbf{h}\|^2$, $V_i \perp \mathbf{h}$, $\lambda_i = \sigma_v^2$, $i = 2, \ldots, m$.
  - exploiting constellation property: *constant modulus*

    consider linear combiner: $z[k] = \mathbf{f}\, \mathbf{y}[k]$.

    $$\min_{\mathbf{f}} E\left(|z[k]|^p - 1\right)^2 \quad p = 1 \text{ or } 2 \text{ typically}$$
  - exploiting *finite alphabet* nature of $a[k]$: $a[k] \in \mathcal{A}$

    $$\min_{\mathbf{h}, a[k] \in \mathcal{A}} \sum_k \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2$$

Spatial Single-User Receivers, White Noise

- Assume white noise first. Gaussian noise: $\mathbf{v}[k] \sim \mathcal{CN}(0, \sigma_v^2 I_m)$ i.i.d.
  $\mathcal{A}$ = symbol constellation (e.g. QAM)

- Maximum Likelihood Sequence Estimation (MLSE):
  $$\min_{a[k]\in\mathcal{A}} \sum_k \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2 \Rightarrow \min_{a[k]\in\mathcal{A}} \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2 \,,\, \forall k$$
  $$\|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2 = \mathbf{y}[k]^H (I_m - \frac{\mathbf{h}\mathbf{h}^H}{\mathbf{h}^H\mathbf{h}})\mathbf{y}[k] + \|\mathbf{h}\|^2\, |a[k] - \widehat{a}[k]|^2$$
  $$\Rightarrow \widehat{\widehat{a}}[k] = dec(\widehat{a}[k])\,,\; \widehat{a}[k] = \frac{1}{\mathbf{h}^H\mathbf{h}}\mathbf{h}^H\, \mathbf{y}[k] \;= \arg \min_{a[k]\in\mathcal{C}} \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2$$
  $\widehat{\widehat{a}}$: "hard" decision ($\in \mathcal{A}$), $\widehat{a}$: "soft" decision ($\in \mathcal{C}$)

- SNR: $\widehat{a}[k] = a[k] + \frac{1}{\mathbf{h}^H\mathbf{h}}\mathbf{h}^H\, \mathbf{v}[k] \;\Rightarrow\;$ SNR $= \dfrac{\|\mathbf{h}\|^2\sigma_a^2}{\sigma_v^2}$

- $$\begin{aligned} \text{MFB} &= \text{Matched Filter Bound} \\ &= \text{SNR of MLSE when all other symbols known} = \frac{\|\mathbf{h}\|^2\sigma_a^2}{\sigma_v^2} \end{aligned}$$

Math/Stat: p-norm

- $\|\mathbf{h}\|_p = (\sum_{i=1}^{m} |h_i|^p)^{\frac{1}{p}}$

  Important instances:

  - $\|\mathbf{h}\| = \|\mathbf{h}\|_2 = \sqrt{\sum_{i=1}^{m} |h_i|^2}$
  - $\|\mathbf{h}\|_1 = \sum_{i=1}^{m} |h_i|$
  - $\|\mathbf{h}\|_\infty = \lim_{p\to\infty} (\sum_{i=1}^{m} |h_i|^p)^{\frac{1}{p}}$
  - different: $\|\mathbf{h}\|_0 = \lim_{p\to 0} \sum_{i=1}^{m} |h_i|^p = \#$ non-zero elements

- At high SNR: $P_e = \dfrac{c}{\mathrm{SNR}^d}$, $d =$ diversity order $= \#$ of (complex) constraints to impose on the channel response to lose the signal completely

Spatial Single-User Receivers, White Noise (2)

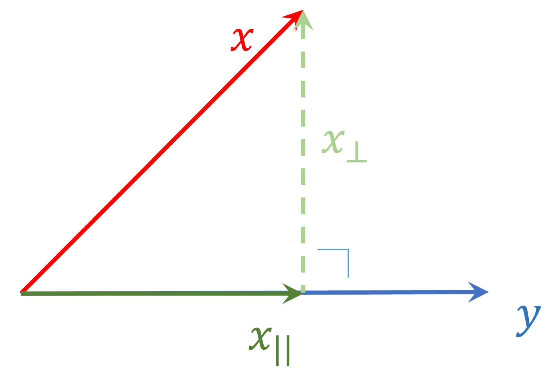
- linear combining receivers: $\widehat{a}[k] = \mathbf{f}\,\mathbf{y}[k]$, $\quad \mathbf{f}\ :\ 1 \times m$
- SNR: $\widehat{a}[k] = \mathbf{fh}\,a[k] + \mathbf{f}\,\mathbf{v}[k] \ \Rightarrow \mathrm{SNR} = \dfrac{|\mathbf{fh}|^2 \sigma_a^2}{\mathbf{ff}^H\, \sigma_v^2}$

  SNR is insensitive to a scale factor in $\mathbf{f}$.
- selection combining: $\mathbf{f} = \mathbf{e}_i^T$, $|h_i| = \max\limits_l |h_l| = \|\mathbf{h}\|_\infty$, $\mathrm{SNR} = \dfrac{\|\mathbf{h}\|_\infty^2 \sigma_a^2}{\sigma_v^2}$

  $\mathbf{e}_i^T = [\underbrace{0 \cdots 0}_{i-1}\ 1\ \underbrace{0 \cdots 0}_{m-i}]$
- equal gain combining: $\mathbf{f} = [1 \cdots 1]$, $\mathrm{SNR} = \dfrac{m\,|\overline{h}|^2\,\sigma_a^2}{\sigma_v^2}$, $\overline{h} = \mathbf{fh}/m$

  more exactly, if $h_i = |h_i| e^{j\phi_i}$: $\mathbf{f} = [e^{-j\phi_1} \cdots e^{-j\phi_m}]$, $\mathrm{SNR} = \dfrac{\|\mathbf{h}\|_1^2\,\sigma_a^2}{m\,\sigma_v^2}$,

  where $\|\mathbf{h}\|_p = (\sum\limits_{i=1}^m |h_i|^p)^{\frac{1}{p}}$, hence phase alignment: $\mathbf{fh} = \|\mathbf{h}\|_1 = \sum\limits_{i=1}^m |h_i|$



Math/Stat: An Optimization Problem in Inner Product Spaces

- Result from inner product spaces:
  vector space with inner product $< x, y >$, norm $\|x\|^2 =< x, x >$

  Result: $\min\limits_{x:<y,x>=1} \|x\|^2 \Rightarrow x = \dfrac{1}{\|y\|^2}\, y$

- Proof: $x = x_{||} + x_\perp$: component along y + component orthogonal to y
  $< y, x >= 1 =< y, x_{||} >$ but $\|x\|^2 = \underbrace{\|x_{||}\|^2}_{\text{constrained}} + \underbrace{\|x_\perp\|^2}_{\text{unconstrained}}$

  $\Rightarrow \;\; x_\perp = 0 \;\; \Rightarrow \;\; x = x_{||} = \alpha\, y$

  $< y, x >= 1 = \alpha < y, y >= \alpha \|y\|^2 \;\Rightarrow\; \alpha = \frac{1}{\|y\|^2} \;\Rightarrow\; x = \frac{1}{\|y\|^2}\, y$

Spatial Single-User Receivers, White Noise (3)

- maximum ratio combining (spatial matched filter):

$$\begin{aligned}
\arg\max_{\mathbf{f}} \mathrm{SNR}(\mathbf{f}) &= \arg\max_{\mathbf{f}} \frac{|\mathbf{f}\mathbf{h}|^2\sigma_a^2}{\mathbf{f}\mathbf{f}^H\,\sigma_v^2} = \arg\max_{\mathbf{f}} \frac{|<\mathbf{f}^H,\mathbf{h}>|^2}{\|\mathbf{f}^H\|^2} \\
&= \arg \min_{\mathbf{f}:<\mathbf{f}^H,\mathbf{h}>=1} \|\mathbf{f}^H\|^2 \Rightarrow \mathbf{f}^H = \frac{1}{\|\mathbf{h}\|^2}\,\mathbf{h} \\
&\Rightarrow \quad \mathbf{f} = \frac{1}{\|\mathbf{h}\|^2}\,\mathbf{h}^H \text{ or } \mathbf{f} = \mathbf{h}^H \text{ (scale unimportant)}
\end{aligned}$$

$$\mathrm{SNR}(\mathbf{f}=\mathbf{h}^H) = \frac{\|\mathbf{h}\|^2\sigma_a^2}{\sigma_v^2} = \mathrm{MFB}$$

spatial matched filter: $\widehat{a}[k] = \sum_{i=1}^{m} h_i^*\, y_i[k] = \sum_{i=1}^{m} h_i^*\,(h_i\, a[k] + v_i[k])$

$$= \sum_{i=1}^{m} |h_i|^2\, a[k] + \sum_{i=1}^{m} h_i^*\, v_i[k]$$

$h_i = |h_i|e^{j\phi_i} \Rightarrow h_i^* = |h_i|e^{-j\phi_i}$: compensate phase (align) and weigh

MF = F that maximizes SNR by phase compensation and weighting

Spatial Single-User Receivers, Colored Noise

- (circular) Gaussian noise, spatially colored, temporally white:

$$f(\mathbf{v}) = \frac{e^{-\mathbf{v}^H R_{\mathbf{vv}}^{-1} \mathbf{v}}}{\pi^m \det(R_{\mathbf{vv}})}$$

$$\mathbf{y}[k] = \mathbf{h}\, a[k] + \mathbf{v}[k] \;, \quad \mathbf{v}[k] \sim \mathcal{CN}(0, R_{\mathbf{vv}}), \text{ i.i.d. } , \begin{cases} \mathrm{E}\mathbf{v}[i]\mathbf{v}^H[k] = \delta_{ik}\, R_{\mathbf{vv}} \\ \mathrm{E}\mathbf{v}[i]\mathbf{v}^T[k] = 0 \end{cases}$$

- Maximum Likelihood Sequence Estimation (MLSE): $(\|\mathbf{x}\|^2_R = \mathbf{x}^H R \mathbf{x})$

$$\min_{a[k]\in\mathcal{A}} \sum_k \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2_{R_{\mathbf{vv}}^{-1}} \Rightarrow \min_{a[k]\in\mathcal{A}} \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2_{R_{\mathbf{vv}}^{-1}} \;, \forall k$$

$$\Rightarrow \;\widehat{\widehat{a}}[k] = dec(\widehat{a}[k])\;,\; \widehat{a}[k] = \frac{1}{\mathbf{h}^H R_{\mathbf{vv}}^{-1}\mathbf{h}} \mathbf{h}^H R_{\mathbf{vv}}^{-1} \mathbf{y}[k] \; = \arg\min_{a[k]} \|\mathbf{y}[k] - \mathbf{h}\, a[k]\|^2_{R_{\mathbf{vv}}^{-1}}$$

- SNR: $\widehat{a}[k] = a[k] + \frac{1}{\mathbf{h}^H R_{\mathbf{vv}}^{-1}\mathbf{h}} \mathbf{h}^H R_{\mathbf{vv}}^{-1} \mathbf{v}[k] \;\Rightarrow\; \text{SNR} = \sigma_a^2\; \mathbf{h}^H R_{\mathbf{vv}}^{-1} \mathbf{h}$

- $\text{MFB} = \sigma_a^2\; \|\mathbf{h}\|^2_{R_{\mathbf{vv}}^{-1}}$

Math/Stat: (Covariance) Matrix Square-Root

- "covariance" matrix R : $R = R^H > 0$
- $R^{1/2}$ is a matrix square-root of R if $R = R^{1/2}\, R^{H/2}$ where $R^{H/2}$ means $(R^{1/2})^H$; $R^{-1/2}$ means $(R^{1/2})^{-1}$
- non-uniqueness: if $R^{1/2}$ is a matrix square-root of R then so is $R^{1/2}\,Q$ for any unitary Q ($QQ^H = I$):
  $R^{1/2}\,Q\,(R^{1/2}\,Q)^H = R^{1/2}\,Q\,Q^H\,R^{H/2} = R^{1/2}\,R^{H/2} = R$
- the *"symmetric" square-root* : eigen decomposition $R = V\Lambda V^H$ then $R^{1/2} = V\Lambda^{1/2}V^H$) is symmetric (Hermitian in fact) (unique if require $\Lambda^{1/2} > 0$)
- the *Cholesky* factor: corresponds to the unique $R^{1/2}$ that is (lower) triangular with positive real diagonal elements
- LDU (UDL) factorization (unique): $R = L\,D\,L^H$ ($U\,D\,U^H$), L (U) = lower (upper) triangular with unit diagonal, D = diagonal (real and $D > 0$), then $R^{1/2} = L\,D^{1/2}$ (or $U\,D^{1/2}$) is a Cholesky factor.

Spatial Single-User Receivers, Colored Noise (2)

- linear combining receivers: $\widehat{a}[k] = \mathbf{f}\,\mathbf{y}[k]\;,\quad \mathbf{f}\;:\;1\times m$
- SNR: $\widehat{a}[k] = \mathbf{fh}\,a[k] + \mathbf{f}\,\mathbf{v}[k] \;\Rightarrow\; \text{SNR} = \dfrac{|\mathbf{fh}|^2\sigma_a^2}{\mathbf{f}R_{\mathbf{vv}}\mathbf{f}^H}$
- colored noise spatial matched filter: $(R_{\mathbf{vv}} = R_{\mathbf{vv}}^{1/2}R_{\mathbf{vv}}^{H/2},\; R_{\mathbf{vv}}^{1/2} = V\Lambda^{1/2}V^H)$

$$\max_{\mathbf{f}} \text{SNR}(\mathbf{f}) \;=\; \max_{\mathbf{f}} \frac{|< R_{\mathbf{vv}}^{H/2}\mathbf{f}^H, R_{\mathbf{vv}}^{-1/2}\mathbf{h} >|^2}{\|R_{\mathbf{vv}}^{H/2}\mathbf{f}^H\|^2} \qquad \text{h}' = R_{\mathbf{vv}}^{-1/2}\mathbf{h}$$

$$\overset{\mathbf{f}'=\mathbf{f}R_{\mathbf{vv}}^{1/2}}{\Rightarrow} \min_{\mathbf{f}' :< \mathbf{f}'^H, R_{\mathbf{vv}}^{-1/2}\mathbf{h}>=1} \|\mathbf{f}'^H\|^2 \;\Rightarrow\; \mathbf{f}'^H = \frac{1}{\|R_{\mathbf{vv}}^{-1/2}\mathbf{h}\|^2} R_{\mathbf{vv}}^{-1/2}\mathbf{h}$$

$$\Rightarrow \qquad \mathbf{f} = \mathbf{f}' R_{\mathbf{vv}}^{-1/2} = \frac{1}{\mathbf{h}^H R_{\mathbf{vv}}^{-1}\mathbf{h}}\,\mathbf{h}^H R_{\mathbf{vv}}^{-1} \text{ or } \mathbf{f} = \mathbf{h}^H R_{\mathbf{vv}}^{-1} \text{ (scale unimportant)}$$

$$\text{SNR}_{max} = \text{SNR}(\mathbf{f} = \mathbf{h}^H R_{\mathbf{vv}}^{-1}) = \sigma_a^2\;\mathbf{h}^H R_{\mathbf{vv}}^{-1}\mathbf{h} = \text{MFB}$$

- matched filter = max SNR receiver
- Noise whitening approach: $\mathbf{y}'[k] = R_{\mathbf{vv}}^{-1/2}\mathbf{y}[k] = R_{\mathbf{vv}}^{-1/2}\mathbf{h}\,a[k] + \mathbf{v}'[k]$

$R_{\mathbf{v}'\mathbf{v}'} = I,\; \sigma_{v'}^2 = 1,\; \mathbf{h}' = R_{\mathbf{vv}}^{-1/2}\mathbf{h}$

Spatial Single-User Receivers, Colored Noise (3)

- colored noise MF

$$\mathbf{h}'^H\mathbf{y}'[k] = \underbrace{\mathbf{h}^H}_{\text{channel MF}} \underbrace{R_{\mathbf{vv}}^{-H/2}}_{\text{whitening MF}} \underbrace{R_{\mathbf{vv}}^{-1/2}}_{\text{noise whitening}} \mathbf{y}[k]$$

Receiver properties

- symbol estimation error:

$$\widetilde{a}[k] = a[k] - \widehat{a}[k] = a[k] - \mathbf{f}\,\mathbf{y}[k] = (1-\mathbf{fh})\,a[k] - \mathbf{f}\,\mathbf{v}[k]$$

- receiver bias: $\mathrm{E}\,\widetilde{a}[k]|_{a[k]} = (1-\mathbf{fh})\,a[k]$
- unbiased linear receiver: $\mathbf{fh} = 1$ $\qquad \widehat{a}[k] = \underbrace{\mathbf{fh}}_{=1}\,a[k] + \mathbf{f}\,\mathbf{v}[k]$
- Mean Squared Error (MSE):
  $\mathrm{MSE} = \mathrm{E}\,|\widetilde{a}[k]|\|^2 = R_{aa} - \mathbf{f}R_{\mathbf{y}a} - R_{a\mathbf{y}}\mathbf{f}^H + \mathbf{f}R_{\mathbf{yy}}\mathbf{f}^H = |1-\mathbf{fh}|^2\sigma_a^2 + \mathbf{f}R_{\mathbf{vv}}\mathbf{f}^H$
- $\mathrm{SNR} = \dfrac{|\mathbf{fh}|^2\sigma_a^2}{\mathbf{f}R_{\mathbf{vv}}\mathbf{f}^H}$, for unbiased receivers: $\mathrm{SNR} = \dfrac{\sigma_a^2}{\mathrm{MSE}} = \dfrac{\sigma_a^2}{\sigma_{\widetilde{a}}^2}$

Math/Stat Background: Matrix Inversion Lemma (MIL)

- Let $A+BCD$ be a (low rank) modification of A and let A^{-1} (and C^{-1}) be available (and assume all inverses involved to exist). Then $(A+BCD)^{-1}$ can be found with little work:

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B+C^{-1})^{-1}DA^{-1}$$

Proof: $(A+BCD)(A^{-1} - A^{-1}B(DA^{-1}B+C^{-1})^{-1}DA^{-1}) = ... = I$

- Typically B = column vector, D = row vector, C= scalar $\Rightarrow$ $(A+BCD)^{-1}$ can be obtained from A^{-1} by just inverting a scalar, and multiplications.

- Often $(A+BCD)^{-1}$ appears premultiplied with D or postmultiplied with B, which simplify, e.g. $D(A+BCD)^{-1} = C^{-1}(DA^{-1}B+C^{-1})^{-1}DA^{-1}$ which, in the case of scalar C, is a scalar multiple of DA^{-1}, so $D(A+BCD)^{-1}$ and DA^{-1} are proportional.

Spatial Single-User Receivers, Colored Noise: (U)MMSE

- Minimum MSE receiver:

$$\begin{aligned}\mathbf{f}_{MMSE} &= R_{a\mathbf{y}}R_{\mathbf{yy}}^{-1} \;=\; \sigma_a^2\,\mathbf{h}^H R_{\mathbf{yy}}^{-1} \\ &= \sigma_a^2\mathbf{h}^H(R_{\mathbf{VV}}+\mathbf{h}\sigma_a^2\mathbf{h}^H)^{-1} \\ &= (\mathbf{h}^H R_{\mathbf{VV}}^{-1}\mathbf{h}+\sigma_a^{-2})^{-1}\,\mathbf{h}^H R_{\mathbf{VV}}^{-1}\end{aligned}$$

  using MIL: $(A+BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B+C^{-1})^{-1}DA^{-1}$

  and $\qquad D(A+BCD)^{-1} = C^{-1}(DA^{-1}B+C^{-1})^{-1}DA^{-1}$

  note: $\mathbf{f}_{MMSE}\mathbf{h} = \frac{\mathbf{h}^H R_{\mathbf{VV}}^{-1}\mathbf{h}}{\mathbf{h}^H R_{\mathbf{VV}}^{-1}\mathbf{h}+\sigma_a^{-2}} < 1 \quad : \quad \mathbf{f}_{MMSE}$ is a biased receiver

- since $\text{SNR} = \frac{\sigma_a^2}{\text{MSE}}$ for unbiased receivers,

  $\arg\max_{\mathbf{f}:\mathbf{fh}=1} \text{SNR}(\mathbf{f}) = \arg\min_{\mathbf{f}:\mathbf{fh}=1} \text{MSE}(\mathbf{f}) = \mathbf{f}_{UMMSE} = (\mathbf{h}^H R_{\mathbf{VV}}^{-1}\mathbf{h})^{-1}\,\mathbf{h}^H R_{\mathbf{VV}}^{-1}$

- since however $\mathbf{f}_{MMSE} \sim \mathbf{f}_{UMMSE} \sim \mathbf{h}^H R_{\mathbf{VV}}^{-1}$

  $\text{SNR}_{max} = \sigma_a^2\,\mathbf{h}^H R_{\mathbf{VV}}^{-1}\mathbf{h} = \text{SNR}(\mathbf{h}^H R_{\mathbf{VV}}^{-1}) = \text{SNR}(\mathbf{f}_{MMSE}) = \text{SNR}(\mathbf{f}_{UMMSE})$

  but $\text{SNR}_{max} = \frac{\sigma_a^2}{\text{MSE}(\mathbf{f})}$ only for $\mathbf{f} = \mathbf{f}_{UMMSE}$

Spatial Single-User Receivers, Colored Noise: LCMV/UMOE

- $\mathrm{MSE} = \sigma_a^2 - \mathbf{f}\mathbf{h}\,\sigma_a^2 - \sigma_a^2\mathbf{h}^H\mathbf{f}^H + \mathbf{f}R_{\mathbf{yy}}\mathbf{f}^H = \sigma_a^2(1-\mathbf{f}\mathbf{h}-(\mathbf{f}\mathbf{h})^H) + \mathbf{f}R_{\mathbf{yy}}\mathbf{f}^H$
- for unbiased receivers ($\mathbf{f}\mathbf{h} = 1$):
$$\mathrm{MSE} = \mathbf{f}R_{\mathbf{yy}}\mathbf{f}^H - \sigma_a^2 = \mathrm{OE} - \sigma_a^2 = \mathrm{V} - \sigma_a^2$$
OE = Output Energy, V = output Variance
- Linearly Constrained Minimum Variance (LCMV) receiver:
$$\mathbf{f}_{LCMV} = \arg\min_{\mathbf{f}:\mathbf{f}\mathbf{h}=1} \mathbf{f}R_{\mathbf{yy}}\mathbf{f}^H = (\mathbf{h}^H R_{\mathbf{yy}}^{-1}\mathbf{h})^{-1}\,\mathbf{h}^H R_{\mathbf{yy}}^{-1}$$
LCMV also called Unbiased Minimum OE (UMOE)
- can show:
$$\mathbf{f}_{LCMV} = (\mathbf{h}^H R_{\mathbf{yy}}^{-1}\mathbf{h})^{-1}\,\mathbf{h}^H R_{\mathbf{yy}}^{-1} = (\mathbf{h}^H R_{\mathbf{vv}}^{-1}\mathbf{h})^{-1}\,\mathbf{h}^H R_{\mathbf{vv}}^{-1} = \mathbf{f}_{UMMSE}$$
- under unbiasedness constraint: minimizing MSE $\equiv$ minimizing OE

Spatial Single-User Receivers, Colored Noise: GSC

- Generalized Sidelobe Canceler (GSC) formulation of LCMV/UMOE
- introduce $\mathbf{h}^{\perp}$ s.t.

$$[\mathbf{h}\;\mathbf{h}^{\perp}]^H[\underbrace{\mathbf{h}}_{m\times 1}\quad\underbrace{\mathbf{h}^{\perp}}_{m\times(m-1)}] = \begin{bmatrix}\mathbf{h}^H\mathbf{h} & 0\\ 0 & \mathbf{h}^{\perp H}\mathbf{h}^{\perp}\end{bmatrix} \text{ with } \mathbf{h}^{\perp H}\mathbf{h}^{\perp} \text{ nonsingular}$$

- reparameterize $\mathbf{f}$ with $\mathbf{f}_{\|}$, $\mathbf{f}_{\perp}$: $\quad \mathbf{f} = [\underbrace{\mathbf{f}_{\|}}_{1\times 1}\quad\underbrace{-\mathbf{f}_{\perp}}_{1\times(m-1)}]\underbrace{\begin{bmatrix}\mathbf{h}^H\\ \mathbf{h}^{\perp H}\end{bmatrix}}_{\text{invertible transformation}}$
- unbiasedness: $\mathbf{f}\mathbf{h} = 1 = \left[\mathbf{f}_{\|}\;\; -\mathbf{f}_{\perp}\right]\begin{bmatrix}\mathbf{h}^H\mathbf{h}\\ 0\end{bmatrix} = \mathbf{f}_{\|}\,\mathbf{h}^H\mathbf{h} \;\Rightarrow\; \mathbf{f}_{\|} = \frac{1}{\mathbf{h}^H\mathbf{h}}$

$\Rightarrow$ unbiased

$$\mathbf{f} = \underbrace{\frac{1}{\mathbf{h}^H\mathbf{h}}\mathbf{h}^H}_{\text{white noise } \mathbf{f}_{UMMSEw}} - \underbrace{\mathbf{f}_{\perp}}_{m-1 \text{ unconstrained parameters}} \mathbf{h}^{\perp H}$$

Spatial Single-User Receivers, Colored Noise: GSC (2)

- colored noise UMMSE : $\quad \mathrm{MSE} = \mathrm{E}\,|\widetilde{a}[k]|^2$

$$-\widetilde{a}[k] = -(1-\underbrace{\mathbf{f}\mathbf{h}}_{=1})\,a[k]+\mathbf{f}\,\mathbf{v}[k] = \underbrace{\frac{1}{\mathbf{h}^H\mathbf{h}}\mathbf{h}^H\,\mathbf{v}[k]}_{\mathbf{v}_1[k]} - \mathbf{f}_\perp\,\underbrace{\mathbf{h}^{\perp H}\mathbf{v}[k]}_{\mathbf{v}_2[k]} = \mathbf{v}_1[k]-\mathbf{f}_\perp\,\mathbf{v}_2[k]$$

$\Rightarrow\ \mathbf{f}_{\perp,UMMSE} = R_{\mathbf{v}_1\mathbf{v}_2}R^{-1}_{\mathbf{v}_2\mathbf{v}_2}$

- colored noise UMOE

$$\mathrm{OE} = \mathbf{f}R_{\mathbf{y}\mathbf{y}}\mathbf{f}^H = \mathrm{E}\,|\mathbf{f}\mathbf{y}[k]|^2 = \mathrm{E}\,|\frac{1}{\mathbf{h}^H\mathbf{h}}\mathbf{h}^H\,\mathbf{y}[k]-\mathbf{f}_\perp\,\mathbf{h}^{\perp H}\mathbf{y}[k]|^2 = \mathrm{E}\,|\mathbf{y}_1[k]-\mathbf{f}_\perp\,\mathbf{y}_2[k]|^2$$

$\Rightarrow\ \mathbf{f}_{\perp,UMOE} = R_{\mathbf{y}_1\mathbf{y}_2}R^{-1}_{\mathbf{y}_2\mathbf{y}_2}$

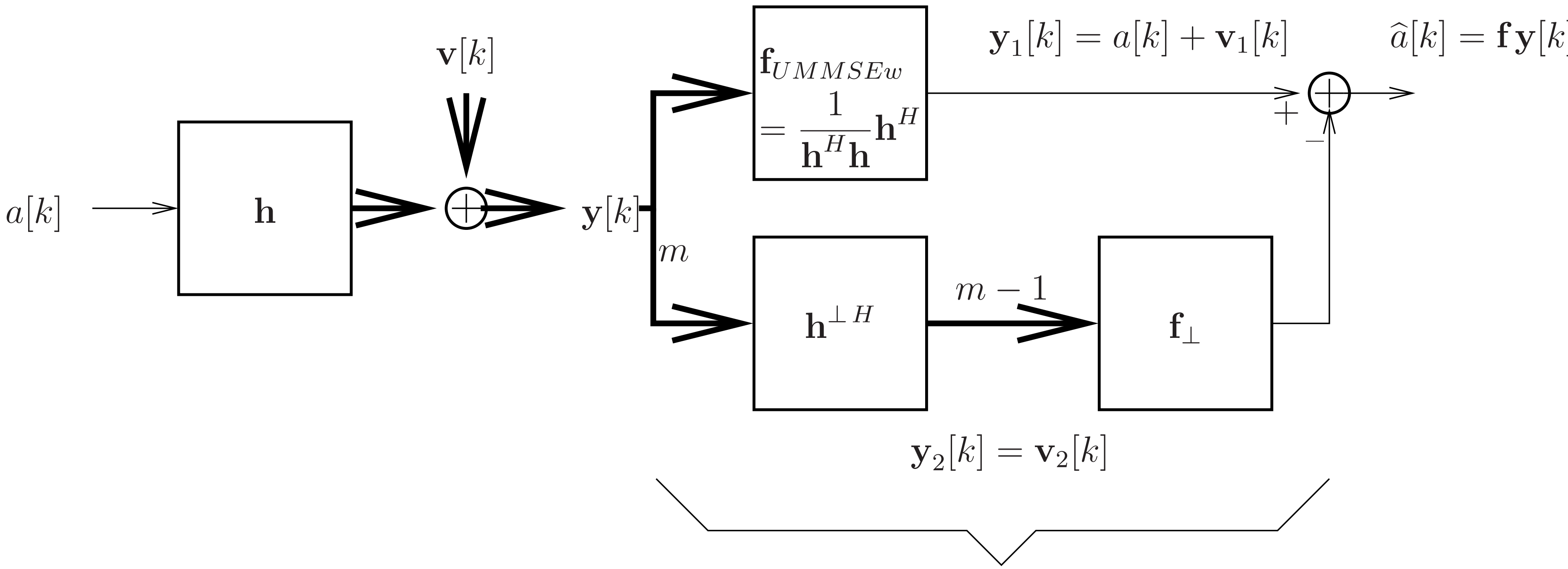
- however, $\mathbf{y}_1[k] = a[k] + \mathbf{v}_1[k]$, $\mathbf{y}_2[k] = \mathbf{v}_2[k]$

$\Rightarrow\ R_{\mathbf{y}_2\mathbf{y}_2} = R_{\mathbf{v}_2\mathbf{v}_2}$ and $R_{\mathbf{y}_1\mathbf{y}_2} = R_{\mathbf{y}_1\mathbf{v}_2} = R_{\mathbf{v}_1\mathbf{v}_2}$

and $\mathbf{f}_{\perp,UMOE} = \mathbf{f}_{\perp,UMMSE}$ $\quad$ ($= 0$ for white noise)

- colored noise $\mathbf{f}_{UMOE} = \mathbf{f}_{UMMSE} = \mathbf{f}_{UMMSEw} - \mathbf{f}_{\perp,UMMSE}\mathbf{h}^{\perp H} = \mathbf{f}_{UMOEw} - \mathbf{f}_{\perp,UMOE}\mathbf{h}^{\perp H}$

Spatial Single-User Receivers, Colored Noise: GSC (3)



$$\mathbf{f} = \mathbf{f}_{UMMSEw} - \mathbf{f}_\perp\,\mathbf{h}^{\perp H}$$

Spatial Single-User Receivers, White Noise: Capon's Principle

- back to white noise case
- now suppose channel response $\mathbf{h}$ not known
- Capon's principle
  - first apply LCMV (UMOE) for a hypothesized channel $\widehat{\mathbf{h}}$, this leads to

$$\mathbf{f}_{LCMV} = (\widehat{\mathbf{h}}^H R_{\mathbf{yy}}^{-1} \widehat{\mathbf{h}})^{-1} \widehat{\mathbf{h}}^H R_{\mathbf{yy}}^{-1} \quad \text{and} \quad \text{MOE} = (\widehat{\mathbf{h}}^H R_{\mathbf{yy}}^{-1} \widehat{\mathbf{h}})^{-1}$$

  - then maximize the resulting MOE w.r.t. the hypothesized channel

$$\begin{aligned} \arg \max_{\widehat{\mathbf{h}}:\, \|\widehat{\mathbf{h}}\|=1} \; \min_{\mathbf{f}:\, \mathbf{f}\widehat{\mathbf{h}}=1} \text{OE} &= \arg \max_{\widehat{\mathbf{h}}:\, \|\widehat{\mathbf{h}}\|=1} (\widehat{\mathbf{h}}^H R_{\mathbf{yy}}^{-1} \widehat{\mathbf{h}})^{-1} \\ &= \arg \min_{\widehat{\mathbf{h}}:\, \|\widehat{\mathbf{h}}\|=1} \widehat{\mathbf{h}}^H R_{\mathbf{yy}}^{-1} \widehat{\mathbf{h}} \end{aligned}$$

$$\Rightarrow \quad \widehat{\mathbf{h}} = V_{min}(R_{\mathbf{yy}}^{-1}) = V_{max}(R_{\mathbf{yy}}) = \overline{\mathbf{h}} = \frac{1}{\|\mathbf{h}\|}\, \mathbf{h}$$

Spatial Multi-User Receivers, MLSE White Noise

- $$\mathbf{y}[k] = [\mathbf{h}_1 \cdots \mathbf{h}_p] \begin{bmatrix} a_1[k] \\ \vdots \\ a_p[k] \end{bmatrix} + \mathbf{v}[k] = \mathbf{h}\,\mathbf{a}[k] + \mathbf{v}[k] = \begin{bmatrix} \mathbf{h}_1 & \overline{\mathbf{h}}_1 \end{bmatrix} \begin{bmatrix} a_1[k] \\ \overline{\mathbf{a}}_1[k] \end{bmatrix} + \mathbf{v}[k]$$
- assume white Gaussian noise first: $\mathbf{v}[k] \sim \mathcal{CN}(0, \sigma_v^2 I_m)$ i.i.d.
- Maximum Likelihood Sequence Estimation (MLSE):

$$\min_{\mathbf{a}[k]\in\mathcal{A}^p} \sum_k \|\mathbf{y}[k] - \mathbf{h}\,\mathbf{a}[k]\|^2 \Rightarrow \min_{\mathbf{a}[k]\in\mathcal{A}^p} \|\mathbf{y}[k] - \mathbf{h}\,\mathbf{a}[k]\|^2 \text{ exhaustive search}$$

$$\Rightarrow \;\widehat{\widehat{a}}_1[k] = dec(\widehat{a}_1[k])\,, \quad \begin{aligned} \widehat{a}_1[k] &= \frac{1}{\mathbf{h}_1^H \mathbf{h}_1} \mathbf{h}_1^H \left(\mathbf{y}[k] - \overline{\mathbf{h}}_1 \widehat{\widehat{\overline{\mathbf{a}}}}_1[k]\right) \quad \text{DF} \\ &= \arg\min_{a_1[k]} \|\mathbf{y}[k] - \overline{\mathbf{h}}_1 \widehat{\widehat{\overline{\mathbf{a}}}}_1[k] - \mathbf{h}_1\, a_1[k]\|^2 \end{aligned}$$

- MFB = SNR of MLSE when all other symbols ($\overline{\mathbf{a}}_1[k]$) known

$$\widehat{a}_1[k] = a_1[k] + \frac{1}{\mathbf{h}_1^H \mathbf{h}_1} \mathbf{h}_1^H\,\mathbf{v}[k] \;\Rightarrow\; \text{MFB} = \text{SNR} = \frac{\|\mathbf{h}_1\|^2 \sigma_a^2}{\sigma_v^2}$$

"single-user MFB"

Spatial Multi-User Receivers, MLSE Colored Noise

- (circular) Gaussian noise, spatially colored, temporally white:

$$\mathbf{y}[k] = \mathbf{h}\,\mathbf{a}[k] + \mathbf{v}[k]\ ,\quad \mathbf{v}[k] \sim \mathcal{CN}(0, R_{\mathbf{vv}}),\ \text{i.i.d.}\ ,\ \begin{cases} \mathrm{E}\mathbf{v}[i]\mathbf{v}^H[k] = \delta_{ik}\, R_{\mathbf{vv}} \\ \mathrm{E}\mathbf{v}[i]\mathbf{v}^T[k] = 0 \end{cases}$$

- Maximum Likelihood Sequence Estimation (MLSE):

$$\min_{\mathbf{a}[k]\in\mathcal{A}^p} \sum_k \|\mathbf{y}[k] - \mathbf{h}\,\mathbf{a}[k]\|^2_{R^{-1}_{\mathbf{vv}}} \;\Rightarrow\; \min_{\mathbf{a}[k]\in\mathcal{A}^p} \|\mathbf{y}[k] - \mathbf{h}\,\mathbf{a}[k]\|^2_{R^{-1}_{\mathbf{vv}}}\ \text{exh. search}$$

$$\Rightarrow\ \widehat{\widehat{a}}_1[k] = dec(\widehat{a}_1[k])\,,\quad \begin{aligned} \widehat{a}_1[k] &= \frac{1}{\mathbf{h}_1^H R^{-1}_{\mathbf{vv}} \mathbf{h}_1} \mathbf{h}_1^H\, R^{-1}_{\mathbf{vv}} (\mathbf{y}[k] - \overline{\mathbf{h}}_1 \widehat{\widehat{\overline{\mathbf{a}}}}_1[k])\ \text{DF} \\ &= \arg\min_{a_1[k]} \|\mathbf{y}[k] - \overline{\mathbf{h}}_1 \widehat{\widehat{\overline{\mathbf{a}}}}_1[k] - \mathbf{h}_1\, a_1[k]\|^2_{R^{-1}_{\mathbf{vv}}} \end{aligned}$$

- MFB = SNR of MLSE when all other symbols ($\overline{\mathbf{a}}_1[k]$) known

$$\widehat{a}_1[k] = a_1[k] + \frac{1}{\mathbf{h}_1^H R^{-1}_{\mathbf{vv}} \mathbf{h}_1} \mathbf{h}_1^H\, R^{-1}_{\mathbf{vv}} \mathbf{v}[k] \;\Rightarrow\; \text{MFB} = \text{SNR} = \|\mathbf{h}_1\|^2_{R^{-1}_{\mathbf{vv}}}\, \sigma_a^2$$

"single-user MFB"

Spatial Multi-User Receivers, Colored Noise: S(I)NR

$$\begin{cases} \mathrm{E}\, \mathbf{a}[i]\mathbf{a}^H[k] = \delta_{ik}\, \sigma_a^2\, I_p \\ \mathrm{E}\, \mathbf{a}[i]\mathbf{a}^T[k] = 0 \text{ only if complex} \end{cases}$$

- linear combining receivers: $\mathbf{f} : p \times m$

$$\begin{bmatrix} \widehat{a}_1[k] \\ \vdots \\ \widehat{a}_p[k] \end{bmatrix} = \widehat{\mathbf{a}}[k] = \mathbf{f}\,\mathbf{y}[k] = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_p \end{bmatrix} \mathbf{y}[k] = \mathbf{f}\,\mathbf{h}\,\mathbf{a}[k] + \mathbf{f}\,\mathbf{v}[k]$$

- S(I)NR: $\widehat{a}_i[k] = \mathbf{f}_i\,\mathbf{y}[k] = \underbrace{\mathbf{f}_i\,\mathbf{h}_i\,a_i[k]}_{\text{signal}} + \underbrace{\mathbf{f}_i\,\overline{\mathbf{h}}_i\,\overline{\mathbf{a}}_i[k]}_{\text{interference}} + \underbrace{\mathbf{f}_i\,\mathbf{v}[k]}_{\text{noise}}$

- $\mathrm{SNR}_i = \dfrac{|\mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2}{\mathbf{f}_i R_{\mathbf{vv}}\mathbf{f}_i^H}$

- $$\begin{aligned} \mathrm{SINR}_i &= \frac{|\mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2}{\|\overline{\mathbf{h}}_i^H\mathbf{f}_i^H\|^2\sigma_a^2 + \mathbf{f}_i R_{\mathbf{vv}}\mathbf{f}_i^H} = \frac{|\mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2}{\mathbf{f}_i(\sigma_a^2\overline{\mathbf{h}}_i\overline{\mathbf{h}}_i^H + R_{\mathbf{vv}})\mathbf{f}_i^H} \\ &= \frac{|\mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2}{\mathbf{f}_i R_{\mathbf{yy}}\mathbf{f}_i^H - |\mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2} = \frac{|\mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2}{\mathrm{OE} - |\mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2} \end{aligned}$$

Spatial Multi-User Receivers, Colored Noise: MF

- the Matched Filter maximizes SNR, hence MF Rx same as in the single-user case:

$$\mathbf{f}_i^{MF} = \mathbf{h}_i^H R_{\mathbf{VV}}^{-1} \quad \Rightarrow \quad \mathbf{f}^{MF} = \mathbf{h}^H R_{\mathbf{VV}}^{-1}$$

  $\widehat{\mathbf{a}}^{MF}[k] = \mathbf{h}^H R_{\mathbf{VV}}^{-1}\,\mathbf{y}[k]$ is a sufficient statistic and will be the input for all multiuser detectors

- $$\begin{aligned} \mathrm{SINR}_i^{MF} &= \frac{|\mathbf{h}_i^H R_{\mathbf{VV}}^{-1}\mathbf{h}_i|^2\sigma_a^2}{\|\overline{\mathbf{h}}_i^H R_{\mathbf{VV}}^{-1}\mathbf{h}_i\|^2\sigma_a^2 + \mathbf{h}_i^H R_{\mathbf{VV}}^{-1}\mathbf{h}_i} \\ &= \frac{\mathrm{MFB}_i}{1 + \frac{\|\overline{\mathbf{h}}_i^H R_{\mathbf{VV}}^{-1}\mathbf{h}_i\|^2\sigma_a^2}{\mathbf{h}_i^H R_{\mathbf{VV}}^{-1}\mathbf{h}_i}} = \frac{\mathrm{MFB}_i}{1 + \mathrm{INR}_i^{MF}} \end{aligned}$$

Spatial Multi-User Receivers, Colored Noise

Receiver properties

- symbol estimation error:

$$
\begin{aligned}
\widetilde{\mathbf{a}}[k] &= \mathbf{a}[k] - \widehat{\mathbf{a}}[k] = \mathbf{a}[k] - \mathbf{f}\,\mathbf{y}[k] = (I_p - \mathbf{fh})\,\mathbf{a}[k] - \mathbf{f}\,\mathbf{v}[k] \\
&= (I_p - \mathrm{diag}(\mathbf{fh}))\,\mathbf{a}[k] - \overline{\mathrm{diag}}(\mathbf{fh})\,\mathbf{a}[k] - \mathbf{f}\,\mathbf{v}[k] \\
\widetilde{a}_i[k] &= (1 - \mathbf{f}_i\mathbf{h}_i)\,a_i[k] - \mathbf{f}_i\overline{\mathbf{h}}_i\,\overline{\mathbf{a}}_i[k] - \mathbf{f}_i\,\mathbf{v}[k]
\end{aligned}
$$

- receiver bias: $\mathrm{E}\,\widetilde{a}_i[k]|_{a_i[k]} = (1 - \mathbf{f}_i\mathbf{h}_i)\,a_i[k]$

- unbiased linear receiver: $\mathbf{f}_i\mathbf{h}_i = 1\,,\ \ i = 1,\ldots,p,$ $\qquad \mathrm{diag}(\mathbf{fh}) = I_p$

- zero-forcing (ZF) Rx: $\mathbf{fh} = I_p \ \Rightarrow\ \mathbf{f}_i\mathbf{h}_i = 1\,,\ i = 1,\ldots,p$: unbiased
  need $m \geq p$ for ZF !

-

$$
\begin{aligned}
\mathrm{MSE}_i &= \mathrm{E}\,|\widetilde{a}_i[k]\|^2 = R_{a_i a_i} - \mathbf{f}_i R_{\mathbf{y}a_i} - R_{a_i\mathbf{y}}\mathbf{f}_i^H + \mathbf{f}_i R_{\mathbf{yy}}\mathbf{f}_i^H \\
&= |1 - \mathbf{f}_i\mathbf{h}_i|^2\sigma_a^2 + \|\overline{\mathbf{h}}_i^H\,\mathbf{f}_i^H\|^2\sigma_a^2 + \mathbf{f}_i R_{\mathbf{vv}}\mathbf{f}_i^H \\
&= (|1 - \mathbf{f}_i\mathbf{h}_i|^2 - |\mathbf{f}_i\mathbf{h}_i|^2)\sigma_a^2 + \mathbf{f}_i R_{\mathbf{yy}}\mathbf{f}_i^H = (|1 - \mathbf{f}_i\mathbf{h}_i|^2 - |\mathbf{f}_i\mathbf{h}_i|^2)\sigma_a^2 + \mathrm{OE}_i
\end{aligned}
$$

Spatial Multi-User Receivers, Colored Noise (2)

- $\mathrm{SINR}_i = \dfrac{|\mathbf{f}_i \mathbf{h}_i|^2 \sigma_a^2}{\|\overline{\mathbf{h}}_i^H \mathbf{f}_i^H\|^2 \sigma_a^2 + \mathbf{f}_i R_{\mathbf{vv}} \mathbf{f}_i^H}$

- for unbiased Rx: $\mathrm{SINR}_i = \dfrac{\sigma_a^2}{\|\overline{\mathbf{h}}_i^H \mathbf{f}_i^H\|^2 \sigma_a^2 + \mathbf{f}_i R_{\mathbf{vv}} \mathbf{f}_i^H} = \dfrac{\sigma_a^2}{\mathrm{MSE}_i}$

- $$\begin{aligned} \mathrm{MSE} &= \sum_{i=1}^{p} \mathrm{MSE}_i = \mathrm{E}\,\|\widetilde{\mathbf{a}}[k]\|^2 = \mathrm{E}\,\widetilde{\mathbf{a}}^H[k]\widetilde{\mathbf{a}}[k] = \mathrm{E}\,\mathrm{tr}\,\widetilde{\mathbf{a}}^H[k]\widetilde{\mathbf{a}}[k] \\ &= \mathrm{E}\,\mathrm{tr}\,\widetilde{\mathbf{a}}[k]\widetilde{\mathbf{a}}^H[k] = \mathrm{tr}\mathrm{E}\widetilde{\mathbf{a}}[k]\widetilde{\mathbf{a}}^H[k] = \mathrm{tr}\, R_{\widetilde{\mathbf{a}}\widetilde{\mathbf{a}}} \end{aligned}$$

  in general, $\arg\min_{\mathbf{f}} \mathrm{MSE} = \arg\min_{\mathbf{f}} R_{\widetilde{\mathbf{a}}\widetilde{\mathbf{a}}}$

- $R_{\widetilde{\mathbf{a}}\widetilde{\mathbf{a}}} = \sigma_a^2 (I_p - \mathbf{fh})(I_p - \mathbf{fh})^H + \mathbf{f}\, R_{\mathbf{vv}}\, \mathbf{f}^H$

- for ZF Rx: $R_{\widetilde{\mathbf{a}}\widetilde{\mathbf{a}}} = \mathbf{f}\, R_{\mathbf{vv}}\, \mathbf{f}^H$, $\mathrm{SINR}_i^{ZF} = \mathrm{SNR}_i^{ZF}$

Spatial Multi-User Receivers: MMSE ZF

- Result from vector spaces with matrix valued inner product $< X, Y > = < Y, X >^H$, "norm" $\|X\|^2 = < X, X >$

  Result: $\min_{X:<Y,X>=I} \|X\|^2 \Rightarrow X = Y\,\|Y\|^{-2} = Y\ < Y, Y >^{-1}$

- MMSE ZF Rx, white noise: ZF: $\mathbf{fh} = I_p$, $p \times (m-p)$ degrees of freedom remaining in $\mathbf{f}$: fix them to min. MSE:

$$\arg \min_{\mathbf{f}:\mathbf{fh}=I_p} R_{\widetilde{\mathbf{a}}\widetilde{\mathbf{a}}} = \arg \min_{\mathbf{f}:\mathbf{h}^H\mathbf{f}^H=I_p} \mathbf{ff}^H = \arg \min_{\mathbf{f}:<\mathbf{h},\mathbf{f}^H>=I_p} \|\mathbf{f}^H\|^2$$

  with $< \mathbf{g}, \mathbf{h} > = \mathbf{g}^H\mathbf{h}$ for $\mathbf{g}, \mathbf{h} \in \mathcal{C}^{m\times p}$

- result $\Rightarrow$ $\mathbf{f}^H = \mathbf{h}\,\|\mathbf{h}\|^{-2} = \mathbf{h}\,(\mathbf{h}^H\mathbf{h})^{-1}$ $\Rightarrow$ $\mathbf{f}^{MMSEZF} = (\mathbf{h}^H\mathbf{h})^{-1}\mathbf{h}^H$

Spatial Multi-User Receivers: MMSE ZF (2)

- MMSE ZF Rx, colored noise: ZF:

$$\min_{\mathbf{f}:\mathbf{f}\mathbf{h}=I_p} R_{\widetilde{\mathbf{a}}\widetilde{\mathbf{a}}} = \min_{\mathbf{f}:\mathbf{h}^H\mathbf{f}^H=I_p} \mathbf{f}R_{\mathbf{v}\mathbf{v}}\mathbf{f}^H \overset{\mathbf{f}^{'}=\mathbf{f}R^{1/2}_{\mathbf{v}\mathbf{v}}}{\Rightarrow} \min_{\mathbf{f}^{'}:<R^{-1/2}_{\mathbf{v}\mathbf{v}}\mathbf{h},\mathbf{f}^{'H}>=I_p} \overbrace{\|R^{H/2}_{\mathbf{v}\mathbf{v}}\mathbf{f}^H\|^2}^{\|\mathbf{f}^{'H}\|^2}$$

- result

$$\Rightarrow\ \mathbf{f}^{'H} = R^{-1/2}_{\mathbf{v}\mathbf{v}}\mathbf{h}\,\|R^{-1/2}_{\mathbf{v}\mathbf{v}}\mathbf{h}\|^{-2}\ \Rightarrow\ \mathbf{f}^{MMSEZF} = (\mathbf{h}^H R^{-1}_{\mathbf{v}\mathbf{v}}\mathbf{h})^{-1}\mathbf{h}^H R^{-1}_{\mathbf{v}\mathbf{v}}$$

- note: $\widehat{\mathbf{a}}^{MMSEZF}[k] = \mathbf{f}^{MMSEZF}\mathbf{y}[k] = (\mathbf{h}^H R^{-1}_{\mathbf{v}\mathbf{v}}\mathbf{h})^{-1}\widehat{\mathbf{a}}^{MF}[k]$

- $R^{MMSEZF}_{\widetilde{\mathbf{a}}\widetilde{\mathbf{a}}} = (\mathbf{h}^H R^{-1}_{\mathbf{v}\mathbf{v}}\mathbf{h})^{-1}$

$$\Rightarrow \mathrm{MSE}^{MMSEZF}_i = [(\mathbf{h}^H R^{-1}_{\mathbf{v}\mathbf{v}}\mathbf{h})^{-1}]_{ii}\,,\ \mathrm{SINR}^{MMSEZF}_i = \frac{\sigma_a^2}{\mathrm{MSE}^{MMSEZF}_i}$$

- can show: $\mathbf{f}^{MMSEZF} = (\mathbf{h}^H R^{-1}_{\mathbf{v}\mathbf{v}}\mathbf{h})^{-1}\mathbf{h}^H R^{-1}_{\mathbf{v}\mathbf{v}} = (\mathbf{h}^H R^{-1}_{\mathbf{y}\mathbf{y}}\mathbf{h})^{-1}\mathbf{h}^H R^{-1}_{\mathbf{y}\mathbf{y}}$

ceivers: MMSE ZF

trix valued inner product

$\|X\|^2 = < X, X >$

$Y \|Y\|^{-2} = Y < Y, Y >^{-1}$

$\mathbf{h} = I_p$, $p \times (m-p)$ degrees of

o min. MSE:

$$\boxed{\mathbf{f}\mathbf{f}^H} = \arg \min_{\mathbf{f}:<\mathbf{h},\mathbf{f}^H>=I_p} \|\mathbf{f}^H\|^2$$

$$)^{-1} \Rightarrow \mathbf{f}^{MMSEZF} = (\mathbf{h}^H\mathbf{h})^{-1}\mathbf{h}^H$$

$$f^H = h\,(h^H h)^{-1}$$

$$\boxed{f^H} = [\,\underbrace{h}_{p} \mid \underbrace{h^\perp}_{m-p}\,]$$

$$\underbrace{\begin{bmatrix} f_{//} \\ f_\perp \end{bmatrix}}_{p} \begin{matrix} p \\ m-p \end{matrix}$$

$$f\,h = I = [f_{//}^H \; f_\perp^H] \begin{bmatrix} h^H \\ h_\perp^H \end{bmatrix} h = [f_{//}^H \; f_\perp^H] \begin{bmatrix} h^H h \\ 0_{m-p \times p} \end{bmatrix} = \underbrace{f_{//}^H}_{p\times p} \underbrace{h^H h}_{p \times p} = I_p$$

$$f_{//}^H = (h^H h)^{-1} = f_{//}$$

$$[f_{//}^H \; f_\perp^H] \begin{bmatrix} h^H \\ h_\perp^H \end{bmatrix} [h \;\; h_\perp] \begin{bmatrix} f_{//} \\ f_\perp \end{bmatrix} = (h^H h)^{-1} + f_\perp^H \underbrace{(h_\perp^H h_\perp)}_{>0} f_\perp$$

$$\begin{bmatrix} h^H h & 0 \\ 0 & h_\perp^H h_\perp \end{bmatrix}$$

$$\Rightarrow f_\perp = 0$$

$$\geq 0$$

Spatial Multi-User Receivers: (U)MMSE

- all users plus noise model:

$$\mathbf{y} = \mathbf{h}\,\mathbf{a} + \mathbf{v}$$
$$R_{\mathbf{yy}} = \sigma_a^2\,\mathbf{h}\mathbf{h}^H + R_{\mathbf{vv}}$$

- one user plus interferers and noise model:

$$\mathbf{y} = \mathbf{h}_i\,a_i + \overline{\mathbf{h}}_i\,\overline{\mathbf{a}}_i + \mathbf{v}$$
$$R_{\mathbf{yy}} = \sigma_a^2\,\mathbf{h}_i\mathbf{h}_i^H + R_i$$

- same situation as single-user case but with $R_{\mathbf{vv}}$ replaced by $R_i = \sigma_a^2\,\overline{\mathbf{h}}_i\overline{\mathbf{h}}_i^H + R_{\mathbf{vv}} = R_{\mathbf{yy}} - \sigma_a^2\,\mathbf{h}_i\mathbf{h}_i^H$ ("noise includes interference")

- UMMSE: for unbiased $\mathbf{f}_i$: $\text{SINR}_i = \dfrac{\sigma_a^2}{\text{MSE}_i} = \dfrac{\sigma_a^2}{\mathbf{f}_i R_i \mathbf{f}_i^H}$

Spatial Multi-User Receivers: (U)MMSE (2)

- UMMSE:
  $$\mathbf{f}_i^{UMMSE} = \mathbf{f}_i^{maxSINR} = (\mathbf{h}_i^H R_i^{-1} \mathbf{h}_i)^{-1} \mathbf{h}_i^H R_i^{-1} = (\mathbf{h}_i^H R_{\mathbf{yy}}^{-1} \mathbf{h}_i)^{-1} \mathbf{h}_i^H R_{\mathbf{yy}}^{-1}$$
- MMSE:
  $$\mathbf{f}^{MMSE} = R_{\mathbf{ay}} R_{\mathbf{yy}}^{-1} = \sigma_a^2 \mathbf{h}^H R_{\mathbf{yy}}^{-1} = (\mathbf{h}^H R_{\mathbf{vv}}^{-1} \mathbf{h} + \sigma_a^{-2} I_p)^{-1} \mathbf{h}^H R_{\mathbf{vv}}^{-1}$$
  $$\widehat{\mathbf{a}}^{MMSE}[k] = \mathbf{f}^{MMSE} \mathbf{y}[k] = (\mathbf{h}^H R_{\mathbf{vv}}^{-1} \mathbf{h} + \sigma_a^{-2} I_p)^{-1} \widehat{\mathbf{a}}^{MF}[k]$$
  $$\mathbf{f}_i^{MMSE} = R_{a_i \mathbf{y}} R_{\mathbf{yy}}^{-1} = \sigma_a^2 \mathbf{h}_i^H R_{\mathbf{yy}}^{-1} = (\mathbf{h}_i^H R_i^{-1} \mathbf{h}_i + \sigma_a^{-2})^{-1} \mathbf{h}_i^H R_i^{-1}$$
  $$\mathbf{f}_i^{UMMSE} = \frac{1}{\mathbf{f}_i^{MMSE} \mathbf{h}_i} \mathbf{f}_i^{MMSE} = \frac{1}{\sigma_a^2 \mathbf{h}_i^H R_{\mathbf{yy}}^{-1} \mathbf{h}_i} \mathbf{f}_i^{MMSE} = (\mathbf{h}_i^H R_i^{-1} \mathbf{h}_i)^{-1} \mathbf{h}_i^H R_i^{-1}$$
  $$\widehat{a}_i^{UMMSE}[k] = \sigma_a^{-2} (\mathbf{h}_i^H R_{\mathbf{yy}}^{-1} \mathbf{h}_i)^{-1} \, \widehat{a}_i^{MMSE}[k]$$

Spatial Multi-User Receivers: (U)MMSE (3)

- $\mathbf{f}_i$ unbiased $\Rightarrow$ $\text{MSE}_i = \mathbf{f}_i\, R_i\, \mathbf{f}_i^H$
- since $\mathbf{f}_i^{MMSE} \sim \mathbf{f}_i^{UMMSE} \sim \mathbf{h}_i^H R_{\mathbf{yy}}^{-1} \sim \mathbf{h}_i^H R_i^{-1} \not\sim \mathbf{h}_i^H R_{\mathbf{VV}}^{-1}$

$$\begin{aligned}\text{SINR}_i^{max} &= \sigma_a^2\, \mathbf{h}_i^H R_i^{-1} \mathbf{h}_i = \text{SINR}(\mathbf{h}_i^H R_{\mathbf{yy}}^{-1}) = \text{SINR}(\mathbf{h}_i^H R_i^{-1})\\ &= \text{SINR}(\mathbf{f}_i^{MMSE}) = \text{SINR}(\mathbf{f}_i^{UMMSE})\end{aligned}$$

but $\text{SINR}_i^{max} = \dfrac{\sigma_a^2}{\text{MSE}_i(\mathbf{f}_i)}$ only for $\mathbf{f}_i = \mathbf{f}_i^{UMMSE}$

- in general

$$\max\{\underbrace{\text{SINR}_i^{MF}}_{\max\ \text{SNR}}, \underbrace{\text{SINR}_i^{MMSEZF}}_{\max\ \text{SIR}}\} \leq \underbrace{\text{SINR}_i^{UMMSE}}_{\max\ \text{SINR}} \leq \text{MFB}_i$$

Spatial Multi-User Receivers: Nonlinear Rx

- for MMSE (ZF), $\widehat{\mathbf{a}}[k]$ found from

$$R\,\widehat{\mathbf{a}}[k] = \widehat{\mathbf{a}}^{MF}[k]\ ,\ R = \mathbf{h}^H R_{\mathbf{vv}}^{-1}\mathbf{h} + \epsilon\,\sigma_a^{-2} I_p \quad , \quad \epsilon = \begin{cases} 0 & ,\ \text{MMSE ZF} \\ 1 & ,\ \text{MMSE} \end{cases}$$

- Serial/Parallel Interference Cancellation (SIC/PIC):
  $R = \text{diag}(R) + \overline{\text{diag}}(R)$

$$\widehat{\mathbf{a}}[k] = \ dec((\text{diag}(R))^{-1}(\widehat{\mathbf{a}}^{MF}[k] - \overline{\text{diag}}(R)\ dec(\widehat{\mathbf{a}}[k])))$$

  iterative (multi-stage) versions possible/useful
  without $dec(.)$: iterative solutions for linear Rx

- Decision-Feedback (DF): $R = L^H D L\ ,\ \ L = I + \overline{\text{diag}}(L)$

$$\widehat{\mathbf{a}}[k] = \ dec(\quad \underbrace{D^{-1}L^{-H}}_{\text{linear anticausal IC}} \quad \widehat{\mathbf{a}}^{MF}[k] - \underbrace{\overline{\text{diag}}(L)\ dec(\widehat{\mathbf{a}}[k])}_{\text{nonlinear causal IC}}\ )$$

  one shot solution (no iterations)