Homework 3

Due: 04/06/2020 (HW to be turned in by email to {slock,roya.gholami}@eurecom.fr)

Homework policy: the homework is individual. Students are encouraged to discuss with fellow students to try to find the main structure of the solution for a problem, especially if they are totally stuck at the beginning of the problem. However, they should work out the details themselves and write down in their own words only what they understand themselves. For every answer you provide, try to give it in its simplest form, while answering correctly. Results that are available in the course notes can be used and referenced and do not need to be rederived.
You can answer in French or in English. Do not forget to answer all subquestions.
Word processing (Word, Latex,...) would be appreciated, or scanned notes.

Multi-User MISO Downlink

1. Reduced-Order Zero-Forcing Beamforming and its Average SINR

Consider a MISO Broadcast Channel, i.e. the downlink in a single cell system with a base station (BS) with N antennas and K single antenna users. The received signal at user k can be written as

$$y_k = \underbrace{\mathbf{h}_k^H \mathbf{g}_k \, s_k}_{\text{signal}} + \underbrace{\sum_{i=1,\neq k}^{K} \mathbf{h}_k^H \mathbf{g}_i \, s_i}_{\text{interference}} + \underbrace{v_k}_{\text{noise}} \tag{1}$$

where s_i is the scalar zero mean white signal stream with variance p_i intended for user i, which is transmitted with the $N \times 1$ beamformer $\mathbf{g}_i$, and received through the $1 \times N$ MISO channel $\mathbf{h}_k^H$. The noise v_k is white circularly complex Gaussian with variance σ_k^2. All signal and noise terms are independent. Assume the beamformers to be normalized: $||\mathbf{g}_i|| = 1$. Let the channels be randomly distributed, namely circularly complex Gaussian with i.i.d. entries such that $\mathbf{h}_k \sim \mathcal{CN}(0, \frac{\alpha_k}{N}\mathbf{I}_N)$ where α_k represents the channel attenuation for user k.

(a) Show that, for given channel $\mathbf{h}_k$, the Signal to Interference plus Noise Ratio (SINR) in y_k is given by

$$\text{SINR}_k = \frac{|\mathbf{h}_k^H \mathbf{g}_k|^2 \, p_k}{\sum_{i=1,\neq k}^{K} |\mathbf{h}_k^H \mathbf{g}_i|^2 \, p_i \; + \; \sigma_k^2} \, . \tag{2}$$

Now, we have seen that optimal beamformers are of the MMSE type, which from the point of view of reception would be making a compromise between interference suppression and matched filtering to avoid noise amplification. To simplify the beamformer design, we shall consider a "reduced-order" (RO) zero-forcing (ZF) design in which we only ZF to a subset of users. Let

$$I_k = \{i_{k,1}, i_{k,2}, \ldots, i_{k,K_k}\} \; \subset \; I_0 \backslash \{k\}, \text{ where } I_0 = \{1, 2, \ldots, K\}, \tag{3}$$

be the subset of users that the beamformer $\mathbf{g}_k$ for user k zero forces to. The size of this subset is $|I_k| = K_k$ where $0 \leq K_k \leq K-1$. The collection of channels of the users being zero forced to appears in the $N \times K_k$ matrix

$$\mathbf{H}_{I_k} = [\mathbf{h}_{i_{k,1}}\ \mathbf{h}_{i_{k,2}}\ \cdots \mathbf{h}_{i_{k,K_k}}]\ . \tag{4}$$

In the RO-ZF design, beamformer $\mathbf{g}_k^{ro}$ must satisfy the constraints

$$\mathbf{H}_{I_k}^H \mathbf{g}_k^{ro} = 0\ ,\quad ||\mathbf{g}_k^{ro}|| = 1 \tag{5}$$

and this for every $k \in I_0$.

(b) By how many complex and by how many real parameters can a $N \times 1$ complex vector $\mathbf{g}_k$ be parameterized?
After satisfying the constraints in (5), how many real parameters are left to parameterize $\mathbf{g}_k^{ro}$?

(c) With beamformers of the form $\mathbf{g}_k^{ro}$, satisfying the constraints (5), how does the SINR expression in (2) change to a slightly modified expression SINR_k^{ro} ?

In the RO-ZF design, beamformer $\mathbf{g}_k^{ro}$ prefers to ignore the interference it causes to users in $\overline{I_k} = I_0 \setminus I_k$. Still, to optimize somewhat the resulting SINR_k^{ro}, we shall exploit the degrees of design freedom left in $\mathbf{g}_k^{ro}$ to maximize the signal strength $|\mathbf{h}_k^H \mathbf{g}_k^{ro}|^2$. We shall again do this via the Generalized Sidelobe Canceler (GSC) formulation. For this we shall need the orthogonal projection matrix onto the column space of a tall matrix $\mathbf{H}$, namely $\mathbf{P}_{\mathbf{H}} = \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$, and the projection onto its orthogonal complement $\mathbf{P}_{\mathbf{H}}^{\perp} = \mathbf{I}_N - \mathbf{P}_{\mathbf{H}}$. Remember that any projection matrix satisfies $\mathbf{P} = \mathbf{P}^H$ and $\mathbf{P}^2 = \mathbf{P}$. We shall also introduce the normalized version of a rectangular matrix $\mathbf{H}$, namely the semi-unitary $\overline{\mathbf{H}} = \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-H/2}$.

(d) Show that

$$\overline{\mathbf{H}}^H \overline{\mathbf{H}} = \mathbf{I} \tag{6}$$

which leads to the term "semi-unitary" for the tall rectangular matrix $\overline{\mathbf{H}}$.

(e) Show the two identities in

$$\mathbf{P}_{\mathbf{H}} = \mathbf{P}_{\overline{\mathbf{H}}} = \overline{\mathbf{H}}\,\overline{\mathbf{H}}^H\ . \tag{7}$$

(f) We introduce similarly the semi-unitary matrix $\overline{\mathbf{H}}^{\perp}$ of which the columns span the orthogonal complement of the column space of $\mathbf{H}$. Using the semi-unitarity of both $\overline{\mathbf{H}}$ and $\overline{\mathbf{H}}^{\perp}$ and their mutual orthogonality, show the following three identities

$$\begin{cases} [\overline{\mathbf{H}}\ \overline{\mathbf{H}}^{\perp}]^H [\overline{\mathbf{H}}\ \overline{\mathbf{H}}^{\perp}] = \mathbf{I}_N \\ [\overline{\mathbf{H}}\ \overline{\mathbf{H}}^{\perp}][\overline{\mathbf{H}}\ \overline{\mathbf{H}}^{\perp}]^H = \mathbf{I}_N \\ [\overline{\mathbf{H}}\ \overline{\mathbf{H}}^{\perp}][\overline{\mathbf{H}}\ \overline{\mathbf{H}}^{\perp}]^H = \mathbf{P}_{\overline{\mathbf{H}}} + \mathbf{P}_{\overline{\mathbf{H}}^{\perp}} \end{cases} \tag{8}$$

from which you conclude that $\mathbf{P}_{\overline{\mathbf{H}}^{\perp}} = \mathbf{P}_{\overline{\mathbf{H}}}^{\perp}$.

In what follows, we shall replace these generic matrices by the specific $\overline{\mathbf{H}}_{I_k}$ and $\overline{\mathbf{H}}_{I_k}^{\perp}$ Consider now a reparameterization of $\mathbf{g}_k^{ro}$ with $\mathbf{g}_{k,\|}$, $\mathbf{g}_{k,\perp}$ as follows

$$\mathbf{g}_k^{ro} = \underbrace{[\underbrace{\overline{\mathbf{H}}_{I_k}}_{N\times K_k} \quad \underbrace{\overline{\mathbf{H}}_{I_k}^{\perp}}_{N\times(N-K_k)}]}_{\text{unitary transformation}} \begin{bmatrix} \mathbf{g}_{k,\|} \\ \mathbf{g}_{k,\perp} \end{bmatrix} . \tag{9}$$

(g) What are the repercussions of "forcing the interference to zero" in (5) on $\mathbf{g}_{k,\|}$ and $\mathbf{g}_{k,\perp}$? What does the reparameterized $\mathbf{g}_k$ look like after taking this ZF repercussion into account?

The thus reparameterized $\mathbf{g}_k^{ro}$ represents all possible RO-ZF solutions. We shall optimize the remaining parameters in $\mathbf{g}_k^{ro}$ by maximizing the received SNR

$$\mathrm{SNR}_k^{ro} = \frac{|\mathbf{h}_k^H \mathbf{g}_k^{ro}|^2 \, p_k}{\sigma_k^2} . \tag{10}$$

under the normalization constraint in (5).

(h) Show that the normalization constraint in (5) now becomes

$$||\mathbf{g}_{k,\perp}||^2 = 1 . \tag{11}$$

(i) Hence, what does the optimization problem (10) under the normalization constraint (11) become for the remaining parameters in $\mathbf{g}_k^{ro}$? In other words, what are the free parameters, parameterizing which remaining cost function, under which constraint?

(j) Show that the resulting solution for the free parameters, when substituted in (9), leads to

$$\mathbf{g}_k^{ro} = \mathbf{P}_{\mathbf{H}_{I_k}}^{\perp} \mathbf{h}_k / ||\mathbf{P}_{\mathbf{H}_{I_k}}^{\perp} \mathbf{h}_k|| \tag{12}$$

which is the reduced-order MMSE-ZF beamformer.

In the massive MIMO regime, as $N \to \infty$, the scalar signal and interference powers appearing in SINR_k^{ro} under (c) converge to their expected value as a function of the random channels, due to the law of large numbers. I.e. we can write

$$\mathrm{SINR}_k^{ro} = \frac{P_{S,k}}{P_{I,k} + \sigma_k^2} . \tag{13}$$

(k) Show that

$$\mathbf{h}_k^H \mathbf{g}_k^{ro} = ||\mathbf{P}_{\mathbf{H}_{I_k}}^{\perp} \mathbf{h}_k|| . \tag{14}$$

(l) Hence show that

$$P_{S,k}/p_k = \mathrm{E}|\mathbf{h}_k^H \mathbf{g}_k^{ro}|^2 = \mathrm{E}_{\mathbf{H}_{I_k}} \mathrm{E}_{\mathbf{h}_k} \mathrm{tr}\{\mathbf{P}_{\mathbf{H}_{I_k}}^{\perp} \mathbf{h}_k \mathbf{h}_k^H\} . \tag{15}$$

(m) Continue this result to show that

$$\begin{aligned} P_{S,k}/p_k &= \tfrac{\alpha_k}{N}\,\mathrm{E}_{\mathbf{H}_{I_k}}\mathrm{tr}\{\mathbf{P}^{\perp}_{\mathbf{H}_{I_k}}\} = \tfrac{\alpha_k}{N}\,\mathrm{E}_{\mathbf{H}_{I_k}}\mathrm{tr}\{\overline{\mathbf{H}}^{\perp}_{I_k}\overline{\mathbf{H}}^{\perp H}_{I_k}\} = \tfrac{\alpha_k}{N}\,\mathrm{E}_{\mathbf{H}_{I_k}}\mathrm{tr}\{\overline{\mathbf{H}}^{\perp H}_{I_k}\overline{\mathbf{H}}^{\perp}_{I_k}\} \\ &= \tfrac{\alpha_k}{N}\,\mathrm{E}_{\mathbf{H}_{I_k}}\mathrm{tr}\{\mathbf{I}_{N-K_k}\} = \tfrac{\alpha_k}{N}\,\mathrm{tr}\{\mathbf{I}_{N-K_k}\} = \alpha_k(1-\tfrac{K_k}{N}) \end{aligned} \tag{16}$$

which shows how the signal power decreases with increasing ZF order K_k.

(n) In the computation of the interference power $P_{I,k}$, show that

$$|\mathbf{h}_k^H\mathbf{g}_i^{ro}|^2 = \frac{|\mathbf{h}_k^H\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}}\mathbf{h}_i|^2}{||\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}}\mathbf{h}_i||^2},\ \text{hence}\ \mathrm{E}|\mathbf{h}_k^H\mathbf{g}_i^{ro}|^2 = \frac{\mathrm{E}|\mathbf{h}_k^H\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}}\mathbf{h}_i|^2}{P_{S,i}/p_i}\,. \tag{17}$$

(o) Next, show that for $k \notin I_i$,

$$\mathrm{E}|\mathbf{h}_k^H\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}}\mathbf{h}_i|^2 = \mathrm{E}\left(\mathrm{tr}\{\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}}\mathbf{h}_i\mathbf{h}_i^H\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}}\mathbf{h}_k\mathbf{h}_k^H\}\right) = \frac{\alpha_k\alpha_i}{N^2}\mathrm{tr}\{\mathbf{P}^{\perp}_{\mathbf{H}_{I_i}}\} = \frac{\alpha_k\alpha_i}{N}(1-\frac{K_i}{N})\,. \tag{18}$$

(p) Finally show that we can write

$$\mathrm{SINR}_k^{ro} = \frac{p_k(1-\frac{K_k}{N})}{\frac{1}{N}\displaystyle\sum_{i=1,k\notin I_i}^{K} p_i + \frac{\sigma_k^2}{\alpha_k}}\,. \tag{19}$$

(q) Show that for the full order ZF, we get from this

$$\mathrm{SINR}_k^{ZF} = \frac{\alpha_k\,p_k}{\sigma_k^2}\left(1-\frac{K-1}{N}\right)\,. \tag{20}$$

(r) On the other hand we get for a Matched Filter, which is a zero$^{\text{th}}$ order ZF ($K_k = 0$),

$$\mathrm{SINR}_k^{MF} = \frac{p_k}{\dfrac{P-p_k}{N} + \dfrac{\sigma_k^2}{\alpha_k}} \tag{21}$$

where P is the sum power of the BS. What does this give in the case of uniform powers $p_k = P/K$?