EURECOM 18/01/2024
SSP Statistical Signal Processing

Homework 3

Due: 02/02/2024 (HW3 to be turned in by email to Fangqing.Xiao@eurecom.fr)

Homework policy: the homework is individual. Students can discuss with fellow students to try to find the main structure of the solution for a problem, especially if they are totally stuck at the beginning of the problem. However, you should work out the details yourself and write down in your own words only what you understand yourself.

Wiener Filtering

1. Wiener Filtering vs Instantaneous LMMSE Estimation for Signal in Noise

We consider the signal in noise case, in which we estimate x_k from a noisy version $y_k = x_k + v_k$. We have seen that Wiener filter and MMSE are all determined in terms of the signal and noise spectra $S_{xx}(f)$ and $S_{vv}(f)$. In particular, the Wiener filter is

$$H(f) \;=\; \frac{S_{xx}(f)}{S_{xx}(f) + S_{vv}(f)} \;\in\; [0,1]$$

and the associated MMSE is

$$\mathrm{MMSE}_{WF} = \mathrm{E}\,\widetilde{x}_k^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\widetilde{x}\widetilde{x}}(f)\,df \;=\; \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{S_{xx}(f)\,S_{vv}(f)}{S_{xx}(f) + S_{vv}(f)}df$$

(a) Now consider the basic instantaneous LMMSE estimation from chapter 1. So here we estimate x_k as $\hat{x}_k = h\,y_k$ where we have seen that the LMMSE choice for h is $h = r_{xy}r_{yy}^{-1}$.

Express h in terms of $S_{xx}(f)$ and $S_{vv}(f)$.

Remember that e.g. $r_{xy} = r_{xy}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{xy}(f)\,df$.

(b) We have seen that the associated MMSE is

$$\mathrm{LMMSE} = r_{\widetilde{x}\widetilde{x}} = r_{xx} - r_{xy}r_{yy}^{-1}r_{yx}\ .$$

Express LMMSE in terms of $S_{xx}(f)$ and $S_{vv}(f)$.

(c) Can you find an inequality between MMSE_{WF} and LMMSE? Prove the inequality.

For what follows, consider the specific case in which the signal is lowpass and the noise is complementary highpass:

$$S_{xx}(f) \;=\; \begin{cases} \sigma^2 & ,\ |f| \le f_c \\ 0 & ,\ f_c \le |f| \le \frac{1}{2} \end{cases}$$

and

$$S_{vv}(f) = \begin{cases} 0 & , \ |f| \leq f_c \\ \sigma^2 & , \ f_c \leq |f| \leq \frac{1}{2} \end{cases}$$

(d) Compute the LMMSE coefficient h in this case.

(e) Compute the LMMSE $r_{\widetilde{x}\widetilde{x}}$ in this case.

(f) Compute the Wiener filter $H(f)$ in this case.

(g) Compute MMSE_{WF} in this case.

Adaptive Filtering

2. Convergence Analysis of LMS beyond the Averaging Theorem

Consider the LMS algorithm in the system identification set-up with fixed optimal parameters. Using the independence assumption, but not the averaging theorem, the correlation matrix of the parameter estimation error vector $\widetilde{H}_k$ is found to satisfy the recursion

$$\begin{aligned} C_k &= \mathrm{E}\left(\left[I - \mu Y_k Y_k^T\right] \widetilde{H}_{k-1}\widetilde{H}_{k-1}^T \left[I - \mu Y_k Y_k^T\right]\right) + \mu^2 \xi^o R_{YY} \\ &= \mathrm{E}_{Y_k}\left(\left[I - \mu Y_k Y_k^T\right] C_{k-1} \left[I - \mu Y_k Y_k^T\right]\right) + \mu^2 \xi^o R_{YY} \end{aligned} \quad . \qquad (1)$$

The remaining expectation operator E_{Y_k} is over the elements of the input vector $Y_k = [y_k\ y_{k-1} \cdots y_{k-N+1}]^T$. Assume now that the input signal y_k is Gaussian.

1. Use the following property for Gaussian random variables

$$\mathrm{E}\{y_1 y_2 y_3 y_4\} = \mathrm{E}\{y_1 y_2\}\,\mathrm{E}\{y_3 y_4\} + \mathrm{E}\{y_1 y_3\}\,\mathrm{E}\{y_2 y_4\} + \mathrm{E}\{y_1 y_4\}\,\mathrm{E}\{y_2 y_3\}$$

   to show that (1) becomes

$$\begin{aligned} C_k = \ & C_{k-1} - \mu C_{k-1} R_{YY} - \mu R_{YY} C_{k-1} + 2\mu^2 R_{YY} C_{k-1} R_{YY} \\ & + \mu^2 \left[\xi^o + \mathrm{tr}\{R_{YY} C_{k-1}\}\right] R_{YY} \ . \end{aligned} \qquad (2)$$

2. With the usual eigen decomposition $R_{YY} = V \Lambda V^T = \sum_{i=1}^{N} \lambda_i V_i V_i^T$, let $V_i^T C_k V_i = \mathrm{E}\, v_i^2(k)$ as usual. Introduce the vectors $D_k = [\,\mathrm{E}\, v_1^2(k) \cdots \mathrm{E}\, v_N^2(k)]^T$, $\lambda = [\lambda_1 \cdots \lambda_N]^T$. From (2), find a recursion for the vector D_k.
   (hint: the system matrix will be diagonal, except for a rank one term).

3. Assuming convergence, find the steady-state values D_∞ and ξ_∞^e.

4. What is the maximal stepsize for convergence?
   (hint: to have convergence, ξ_∞^e should be finite).
   Observe that the averaging approach led to a very optimistic value for the maximal stepsize.