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SSP Statistical Signal Processing

Exam

You can answer the questions in French or in English. If you get stuck in a particular part of a question, do not hesitate to try to answer the rest of the question or start the next question. Results that are available in the course notes can be quoted and do not need to be rederived.

Parameter Estimation

1. Combination of Deterministic Estimates and Prior Information into a Bayesian Estimate

Assume that on the one hand we have two estimates $\widehat{\theta}_i(Y_i)\,,\ i=1,2$ of the parameter vector θ, treated as deterministic. These two estimates are based on independent data sets Y_1 and Y_2. Assume that these two estimates are Consistent Asymptotically Normal (CAN) so that for a large amount of data, they are distributed as $\widehat{\theta}_i \sim \mathcal{N}(\theta, C_i)\,,\ i=1,2$. In other words, we can consider that $\widehat{\theta}_i = \theta + V_i$ where $V_i \sim \mathcal{N}(0, C_i)\,,\ i=1,2$ and V_1 and V_2 are independent.

We wish to combine these two estimates from a deterministic parameter estimation point of view, together with prior information in the form of the Gaussian prior $\theta \sim \mathcal{N}(\theta_0, C_0)$ that is independent of V_1 and V_2. For that purpose, we shall consider as data $Y = \begin{bmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \end{bmatrix}$ for the estimation of θ.

(a) Elaborate on the model for the data Y. Is it a linear model of the form $Y = H\,\theta + V$? If yes, what are H and V?

(b) Give the MAP estimate $\widehat{\theta}(Y)$ of θ, which will be in terms of θ_0, $\widehat{\theta}_1$, $\widehat{\theta}_2$, C_0, C_1 and C_2. It is possible to give an expression for $\widehat{\theta}$ in which θ, $\widehat{\theta}_1$, $\widehat{\theta}_2$ on the one hand and C_0, C_1, C_2 on the other hand appear in a symmetric fashion (that means that the expression remains unchanged after a permutation of the indices $0,1,2$).

(c) Give the CRB for the estimation of θ in terms of the data Y and the prior information.

(d) Does the MAP estimate in (b) above attain the CRB in (c)?

(e) Is the MAP estimate in (b) unbiased?

(f) Give the MMSE estimate $\widehat{\theta}_{MMSE}$. Compare to $\widehat{\theta}_{MAP}$ in (b).

2. ML Estimation of Roundtrip Delay Distribution.

Assume that for the roundtrip delay in a computer network, as considered in the homework, we now consider a truncated exponential distribution:

$$f(y|\lambda,\alpha,\beta) = \left\{ \begin{array}{ll} 0 & , y < \alpha \\ \gamma\, e^{-\lambda y} & , \alpha \leq y \leq \beta \\ 0 & , \beta < y \end{array} \right\} = \gamma\, e^{-\lambda y}\, 1_{[\alpha,\beta]}(y) \tag{1}$$

where γ is a normalization constant and

$$1_{\mathcal{A}}(y) = \left\{ \begin{array}{ll} 1 & , \ y \in \mathcal{A} \\ 0 & , \ y \notin \mathcal{A} \end{array} \right.$$

is the indicator function for the set $\mathcal{A}$.

(a) Determine the normalization constant γ as a function of λ, α and β.

In what follows, you need to substitute γ in $f(y|\lambda,\alpha,\beta)$ by this function of λ, α and β.

(b) We now collect n i.i.d. measurements y_i into the vector Y. Assume for the moment that $\lambda > 0$ is a given constant.
Find the likelihood function $l(\alpha,\beta|Y,\lambda)$ for α and β given Y and λ.
Note that $1_{[\alpha,\beta]}(y) = 1_{[\alpha,\infty)}(y)\; 1_{(-\infty,\beta]}(y)$.

(c) Maximize this likelihood function to determine the Maximum Likelihood (ML) estimate of α and β on the basis of (for given) Y and λ.

(d) Consider now also λ as unknown and determine its ML estimate.

In what follows, consider the special case in which $\alpha = 0$, $\beta = \infty$, i.e. the untruncated exponential distribution case. In this case, λ is the only remaining parameter in the distribution $f(y|\lambda)$.

(e) Determine the mean $m_y = \mathrm{E}\; y$ and the variance $\sigma_y^2 = \mathrm{E}\; y^2 - (\mathrm{E}\; y)^2$ as a function of λ.

(f) Determine the log likelihood function from n i.i.d. measurements Y, $L(\lambda|Y) = \ln f(Y|\lambda)$.

(g) Determine the Maximum Likelihood (ML) estimate $\widehat{\lambda}_{ML}$ and express it as a function of the sample mean $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

(h) Show that in this case the ML estimate can be interpreted as an application of the method of moments.

(i) Asymptotically ($n \gg 1$), the estimation error $\overline{y} - m_y$ will be very small. Hence develop $\widehat{\lambda}_{ML}$ up to first order in $\overline{y} - m_y$. From this asymptotic expression of $\widehat{\lambda}_{ML}$, obtain the asymptotic mean $m_{\widehat{\lambda}_{ML}}$ of $\widehat{\lambda}_{ML}$ and asymptotic variance (for large but finite n) $\sigma^2_{\widetilde{\lambda}_{ML}}$ of $\widetilde{\lambda}_{ML} = \lambda - \widehat{\lambda}_{ML}$. Express both $m_{\widehat{\lambda}_{ML}}$ and $\sigma^2_{\widetilde{\lambda}_{ML}}$ in terms of λ.
Is $\widehat{\lambda}_{ML}$ asymptotically unbiased?

(j) Determine the Fisher Information and the Cramer-Rao bound (CRB) for any unbiased estimator $\widehat{\lambda}$.
Is ML asymptotically efficient in this case?

Wiener Filtering and Equalization

3. Zero-Forcing Linear Equalization of a Second-Order FIR Channel

Consider a discrete-time channel with the following transfer function

$$\mathrm{C}(z) = 1 - \frac{5}{2}\, z^{-1} - \frac{3}{2}\, z^{-2}\ .$$

(a) Compute the impulse response h_k of the (non-causal) zero-forcing (ZF) linear equalizer (LE).

(b) For the same channel, in which we assume the variance of the additive white noise to be σ_v^2 and the variance of the transmitted white symbols to be σ_x^2, compute the MSE of the ZF-LE and the corresponding SNR.

(c) Compute the Matched Filter Bound (MFB) and compare to the SNR for the ZF-LE.

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Wiener and Adaptive Filtering

4. FIR Equalization of a 2-Tap Channel

Consider the output of a 2-tap channel:

$$y_k = \mathrm{C}(q)\, a_k + v_k \;, \quad \mathrm{C}(z) = c_0 + c_{N-1}\, z^{-(N-1)} \tag{2}$$

where a_k is the transmitted symbol sequence and v_k is the additive channel noise. We assume that a_k and v_k are white processes that are mutually uncorrelated.

Steepest-Descent Algorithm

(a) Show that the correlation sequence of y_k is of the form

$$r_{yy}(n) = \begin{cases} \alpha & , n = 0 \\ \beta & , n = \pm(N-1) \\ 0 & , \text{ elsewhere} \end{cases} \tag{3}$$

for certain α and β that you will specify in terms of c_0, c_{N-1}, σ_a^2 and σ_v^2.
Hint: it is easy to find $\mathrm{S}_{yy}(z)$, which is the z transform of $r_{yy}(n)$.

(b) This same process y_k is now used as the input to an FIR filter of length $N > 2$. Give the $N \times N$ covariance matrix R_{YY} of the input signal.

(c) Find the N eigenvectors V_i and corresponding eigenvalues λ_i of R_{YY}.
Hint: $N-2$ eigenvectors are of the form $[0 \;\ast \cdots \ast\; 0]^T$ while the other two eigenvectors are of the form $[\ast\; 0 \cdots 0\; \ast]^T$ where $\ast$ denotes a non-zero (in general) scalar.

(d) What is the maximum stepsize μ for convergence?

(e) What is the stepsize value μ for fastest convergence?

(f) With this fastest stepsize, what is the slowest mode?

(g) With this fastest stepsize, how fast are the other modes?

FIR Equalization

(h) Consider now FIR Equalization (Wiener filtering) with y_k as input signal and $x_k = a_k$ as desired-response signal (0 delay equalization).
Compute the N coefficients of the optimal FIR equalizer (Wiener filter) H^o, in terms of c_0, c_{N-1}, σ_a^2 and σ_v^2.

(i) Compute the associated $MMSE$.

(j) Are there values of c_0, c_{N-1} for which you would recommend using another delay d such that $x_k = a_{k-d}$? For which values of c_0, c_{N-1} and which delay d?