EURECOM 13/02/2017
SSP Statistical Signal Processing

Exam

You can answer the questions in French or in English. If you get stuck in a particular part of a question, do not hesitate to try to answer the rest of the question or start the next question. Results that are available in the course notes can be quoted and do not need to be rederived.

Deterministic Parameter Estimation

1. ML Estimation of Roundtrip Delay Distribution.

Assume that for the roundtrip delay in a computer network, as considered in the homework, we now consider a truncated exponential distribution:

$$f(y|\lambda,\alpha,\beta) = \left\{ \begin{array}{ll} 0 & , y < \alpha \\ \gamma\, e^{-\lambda\, y} & , \alpha \leq y \leq \beta \\ 0 & , \beta < y \end{array} \right\} = \gamma\, e^{-\lambda\, y}\, 1_{[\alpha,\beta]}(y) \tag{1}$$

where γ is a normalization constant and

$$1_{\mathcal{A}}(y) = \left\{ \begin{array}{ll} 1 & , \quad y \in \mathcal{A} \\ 0 & , \quad y \notin \mathcal{A} \end{array} \right.$$

is the indicator function for the set $\mathcal{A}$.

(a) Determine the normalization constant γ as a function of λ, α and β.
In what follows, you need to substitute γ in $f(y|\lambda,\alpha,\beta)$ by this function of λ, α and β.

(b) We now collect n i.i.d. measurements y_i into the vector Y. Assume for the moment that $\lambda > 0$ is a given constant.
Find the likelihood function $l(\alpha,\beta|Y,\lambda)$ for α and β given Y and λ.
Note that $1_{[\alpha,\beta]}(y) = 1_{[\alpha,\infty)}(y)\, 1_{(-\infty,\beta]}(y)$.

(c) Maximize this likelihood function to determine the Maximum Likelihood (ML) estimate of α and β on the basis of (for given) Y and λ.

(d) Consider now also λ as unknown and determine its ML estimate.

In what follows, consider the special case in which $\alpha = 0$, $\beta = \infty$, i.e. the untruncated exponential distribution case. In this case, λ is the only remaining parameter in the distribution $f(y|\lambda)$.

(e) Determine the mean $m_y = \mathrm{E}\, y$ and the variance $\sigma_y^2 = \mathrm{E}\, y^2 - (\mathrm{E}\, y)^2$ as a function of λ.

(f) Determine the log likelihood function from n i.i.d. measurements Y, $L(\lambda|Y) = \ln f(Y|\lambda)$.

(g) Determine the Maximum Likelihood (ML) estimate $\widehat{\lambda}_{ML}$ and express it as a function of the sample mean $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

(h) Show that in this case the ML estimate can be interpreted as an application of the method of moments.

(i) Asymptotically ($n \gg 1$), the estimation error $\overline{y} - m_y$ will be very small. Hence develop $\widehat{\lambda}_{ML}$ up to first order in $\overline{y} - m_y$. From this asymptotic expression of $\widehat{\lambda}_{ML}$, obtain the asymptotic mean $m_{\widehat{\lambda}_{ML}}$ of $\widehat{\lambda}_{ML}$ and asymptotic variance (for large but finite n) $\sigma^2_{\widetilde{\lambda}_{ML}}$ of $\widetilde{\lambda}_{ML} = \lambda - \widehat{\lambda}_{ML}$. Express both $m_{\widehat{\lambda}_{ML}}$ and $\sigma^2_{\widetilde{\lambda}_{ML}}$ in terms of λ.
Is $\widehat{\lambda}_{ML}$ asymptotically unbiased?

(j) Determine the Fisher Information and the Cramer-Rao bound (CRB) for any unbiased estimator $\widehat{\lambda}$.
Is ML asymptotically efficient in this case?

2. Maximum Likelihood & Fisher Information

Consider the data model $y_i \sim \mathcal{N}(0, \sigma^2)$ i.i.d. for $i = 1, \ldots, n$ with unknown variance $\theta = \sigma^2$.
(See BLUE example in Lecture 5).

(a) Compute the Maximum Likelihood estimator $\widehat{\sigma^2}_{ML}$.

(b) How does the ML estimator $\widehat{\sigma^2}_{ML}$ compare to the BLUE estimator $\widehat{\sigma^2}_{BLUE}$?

(c) Is this ML estimator unbiased? (derive the bias)

(d) Compute the scalar FIM $J(\theta)$. Compute the CRB.

(e) Is this ML estimator consistent? Why?

(f) Is the ML estimator a UMVUE in this case? Why?

Next page please.

Spectrum Estimation

3. Spectral Resolution Issues

We consider the use of the Blackman-Tukey (BT) spectral estimator and more specifically with a Bartlett window of length $2M+1$. We want a spectral resolution in normalized frequency of at least 0.02 (assume the common rule of thumb for the resolution of two equiamplitude sinusoids). We furthermore want the BT spectral estimator to have a variance that is at least nine times lower than the variance of the periodogram. What is the minimum number N of samples that we need to have to reach these specifications?

4. Time-Frequency Signal Analysis Resolution Issues

(a) Assume we use the discrete-time Short-Time Fourier Transform (DFT) to do time-frequency analysis. Given a certain frequency resolution (Δf) and a certain temporal resolution (Δt), is it possible to find a unique sampling frequency f_s and number of frequency bins N that lead to these given frequency and temporal resolutions?

(b) Doing again time-frequency analysis using the DFT and given a sampling frequency $f_s = 16\text{kHz}$ and a temporal resolution $\Delta t = 1\text{ms}$, what is the frequency resolution Δf and what is the number of frequency bins N?

(c) If we now use the discrete-time Wavelet Transform (DTWT) and we use the same number of subbands as in (b), what are the frequency resolution Δf and the temporal resolution Δt in

(i) the highest subband?

(ii) the lowest subband?

Turn the page please.

Wiener and Adaptive Filtering

5. FIR Equalization of a 2-Tap Channel

Consider the output of a 2-tap channel:

$$y_k \;=\; \mathrm{C}(q)\, a_k + v_k \;\;, \quad \mathrm{C}(z) \;=\; c_0 + c_{N-1}\, z^{-(N-1)} \tag{2}$$

where a_k is the transmitted symbol sequence and v_k is the additive channel noise. We assume that a_k and v_k are white processes that are mutually uncorrelated.

Steepest-Descent Algorithm

(a) Show that the correlation sequence of y_k is of the form

$$r_{yy}(n) = \begin{cases} \alpha & , n = 0 \\ \beta & , n = \pm(N-1) \\ 0 & , \text{ elsewhere} \end{cases} \tag{3}$$

for certain α and β that you will specify in terms of c_0, c_{N-1}, σ_a^2 and σ_v^2.
Hint: it is easy to find $\mathrm{S}_{yy}(z)$, which is the z transform of $r_{yy}(n)$.

(b) This same process y_k is now used as the input to an FIR filter of length $N > 2$. Give the $N \times N$ covariance matrix R_{YY} of the input signal.

(c) Find the N eigenvectors V_i and corresponding eigenvalues λ_i of R_{YY}.
Hint: $N-2$ eigenvectors are of the form $[0 \;\, * \cdots * \;\, 0]^T$ while the other two eigenvectors are of the form $[* \; 0 \cdots 0 \; *]^T$ where $*$ denotes a non-zero (in general) scalar.

(d) What is the maximum stepsize μ for convergence?

(e) What is the stepsize value μ for fastest convergence?

(f) With this fastest stepsize, what is the slowest mode?

(g) With this fastest stepsize, how fast are the other modes?

FIR Equalization

(h) Consider now FIR Equalization (Wiener filtering) with y_k as input signal and $x_k = a_k$ as desired-response signal (0 delay equalization).
Compute the N coefficients of the optimal FIR equalizer (Wiener filter) H^o, in terms of c_0, c_{N-1}, σ_a^2 and σ_v^2.

(i) Compute the associated $MMSE$.

(j) Are there values of c_0, c_{N-1} for which you would recommend using another delay d such that $x_k = a_{k-d}$? For which values of c_0, c_{N-1} and which delay d?