Stein's Unbiased Risk Estimator: SURE Principle

- Consider a simple additive white Gaussian noise model:

$$\boldsymbol{y} = \mathbf{z} + \boldsymbol{v}$$

  where $\boldsymbol{v} \sim \mathcal{N}(\boldsymbol{v}; 0, \sigma^2 \mathbf{I})$.
- Let $\widehat{\mathbf{z}}(\boldsymbol{y})$ be an estimator of $\mathbf{z}$. Then we get for the MSE

$$\begin{aligned}
\mathrm{MSE}_{\mathbf{z}} &= \mathrm{E}\,\|\widehat{\mathbf{z}} - \mathbf{z}\|^2 = \mathrm{E}\,\{\|\mathbf{z}\|^2 + \|\widehat{\mathbf{z}}\|^2 - 2\,\widehat{\mathbf{z}}^T \mathbf{z}\} \\
&\overset{(a)}{=} \mathrm{E}\,\{\|\mathbf{z}\|^2 + \|\widehat{\mathbf{z}}\|^2 - 2\,\widehat{\mathbf{z}}^T \boldsymbol{y} + 2\sigma^2\,\mathrm{tr}\{\tfrac{\partial \widehat{\mathbf{z}}^T}{\partial \boldsymbol{y}}\}\} \\
&= \mathrm{E}\,\{\|\mathbf{z}\|^2 - \|\boldsymbol{y}\|^2 + \|\widehat{\mathbf{z}} - \boldsymbol{y}\|^2 + 2\sigma^2\,\mathrm{tr}\{\tfrac{\partial \widehat{\mathbf{z}}^T}{\partial \boldsymbol{y}}\}\}
\end{aligned}$$

  where E is w.r.t. $\boldsymbol{v}$ ($\mathbf{z}$ is treated as deterministic) and (a) follows as a property of the Gaussian pdf [Eldar'09].
- By dropping expectation, we get an instantaneous unbiased estimate of the MSE and the corresponding SURE function (which is the part of $\widehat{\mathrm{MSE}}$ that depends on $\widehat{\mathbf{z}}$)

$$\widehat{\mathrm{MSE}}_{\mathbf{z}} = \|\mathbf{z}\|^2 - \|\boldsymbol{y}\|^2 + \mathrm{SURE}_{\mathbf{z}}\,, \quad \mathrm{SURE}_{\mathbf{z}} = \|\widehat{\mathbf{z}} - \boldsymbol{y}\|^2 + 2\sigma^2\,\mathrm{tr}\{\frac{\partial \widehat{\mathbf{z}}^T}{\partial \boldsymbol{y}}\}$$

  In $\mathrm{SURE}_{\mathbf{z}}$, the first term reflects the effect of bias in $\widehat{\mathbf{z}}$ whereas the second term reflects the variance of $\widehat{\mathbf{z}}$ and the noise effect in the first term due to replacing $\mathbf{z}$ by $\boldsymbol{y}$.