

Digital Talent Scholarship 2022

Math for ML - Linear Algebra 3

Lead a sprint through Machine Learning with Tensorflow

The Gram-Schmidt process

Gram-Schmidt


$V = \{v_1, v_2, \dots, v_n\}$

$v_1 \quad e_1 = \frac{v_1}{|v_1|}$

$v_2 = (v_2 \cdot e_1) \frac{e_1}{|e_1|} + u_2$

$u_2 = v_2 - (v_2 \cdot e_1) e_1 \quad \frac{u_2}{|u_2|} = e_2$

$u_3 = v_3 - (v_3 \cdot e_1) e_1 - (v_3 \cdot e_2) e_2 \quad \frac{u_3}{|u_3|} = e_3$



The Gram-Schmidt process

Proof.

1. Misal $\mathbf{v}_1 = \mathbf{u}_1$
2. Membentuk vektor \mathbf{v}_2 yang ortogonal terhadap \mathbf{v}_1 dengan cara menghitung komponen dari \mathbf{u}_2 yang ortogonal terhadap ruang W_1 yang direntang oleh \mathbf{v}_1 , yaitu

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \text{proj}_{W_1} \mathbf{u}_2 \\ &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1\end{aligned}$$

The Gram-Schmidt process

Proof.

3. Membentuk vektor \mathbf{v}_3 yang ortogonal terhadap \mathbf{v}_1 dan \mathbf{v}_2 dengan cara menghitung komponen dari \mathbf{u}_3 yang ortogonal terhadap ruang W_2 yang direntang oleh \mathbf{v}_1 dan \mathbf{v}_2 , yaitu

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{u}_3 - \text{proj}_{W_2} \mathbf{u}_3 \\ &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2\end{aligned}$$

4. Proses dilanjutkan sampai \mathbf{v}_n , untuk menghasilkan himpunan ortogonal $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ yang terdiri dari n vektor bebas linear di V dan merupakan suatu basis ortogonal untuk V . Penormalan vektor-vektor di basis ortogonal akan menghasilkan basis ortonormal.

The Gram-Schmidt process

Example

Diberikan $V = R^3$ dengan hasil kali dalam Euclid. Terapkan algoritma **Gram-Schmidt** untuk mengortogonalkan basis

$$\{(1, -1, 1), (1, 0, 1), (1, 1, 2)\}$$

Normalisasikan vektor-vektor basis ortogonal yang diperoleh menjadi sebuah basis ortonormal.

The Gram-Schmidt process

Solution

Misal $\mathbf{u}_1 = (1, -1, 1)$, $\mathbf{u}_2 = (1, 0, 1)$, $\mathbf{u}_3 = (1, 1, 2)$

- Langkah 1

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, -1, 1)$$

- Langkah 2

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{u}_2 - \text{proj}_{W_1} \mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \\ &= (1, 0, 1) - \frac{1 \cdot 1 + 0 \cdot -1 + 1 \cdot 1}{3} (1, -1, 1) \\ &= (1, 0, 1) - \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3} \right) \\ &= \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3} \right)\end{aligned}$$

The Gram-Schmidt process

Solution

- *Langkah 3*

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{u}_3 - \text{proj}_{W_2} \mathbf{u}_3 \\ &= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 \\ &= (1, 1, 2) - \frac{2}{3} (1, -1, 1) - \frac{5}{2} \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3} \right) \\ &= \left(-\frac{1}{2}, 0, \frac{1}{2} \right)\end{aligned}$$

- *Dengan demikian, diperoleh basis ortogonal*

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ (1, -1, 1), \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3} \right), \left(-\frac{1}{2}, 0, \frac{1}{2} \right) \right\}$$

The Gram-Schmidt process

Solution

- Selanjutnya dapat diperoleh basis ortonormal $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ dengan

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, -1, 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right)$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(\frac{1}{3}, \frac{2}{3}, \frac{1}{3})}{\frac{\sqrt{6}}{3}} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right)$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{(-\frac{1}{2}, 0, \frac{1}{2})}{\frac{\sqrt{2}}{2}} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right)$$

Q & A

Agenda

1. Intro to eigenvalues dan eigenvectors
2. Konsep yang perlu dipahami
3. Next steps

Are your students cloud-ready?

Apa itu Eigen ?

**Eigen merupakan hasil translate-an
dari Bahasa German yang berarti
karakteristik**

Formula Sheet

Vector operations

$$\mathbf{r} + \mathbf{s} = \mathbf{s} + \mathbf{r}$$

$$2\mathbf{r} = \mathbf{r} + \mathbf{r}$$

$$\|\mathbf{r}\|^2 = \sum_i r_i^2$$

- dot or inner product:

$$\mathbf{r} \cdot \mathbf{s} = \sum_i r_i s_i$$

commutative $\mathbf{r} \cdot \mathbf{s} = \mathbf{s} \cdot \mathbf{r}$

distributive $\mathbf{r} \cdot (\mathbf{s} + \mathbf{t}) = \mathbf{r} \cdot \mathbf{s} + \mathbf{r} \cdot \mathbf{t}$

associative $\mathbf{r} \cdot (a\mathbf{s}) = a(\mathbf{r} \cdot \mathbf{s})$

$$\mathbf{r} \cdot \mathbf{r} = \|\mathbf{r}\|^2$$

$$\mathbf{r} \cdot \mathbf{s} = \|\mathbf{r}\| \|\mathbf{s}\| \cos \theta$$

- scalar and vector projection:

scalar projection: $\frac{\mathbf{r} \cdot \mathbf{s}}{\|\mathbf{r}\|}$

vector projection: $\frac{\mathbf{r} \cdot \mathbf{s}}{\mathbf{r} \cdot \mathbf{r}} \mathbf{r}$

Change of basis

Change from an original basis to a new, primed basis. The columns of the transformation matrix B are the new basis vectors in the original coordinate system. So

$$B\mathbf{r}' = \mathbf{r}$$

where \mathbf{r}' is the vector in the B -basis, and \mathbf{r} is the vector in the original basis. Or;

$$\mathbf{r}' = B^{-1}\mathbf{r}$$

If a matrix A is *orthonormal* (all the columns are of unit size and orthogonal to each other) then:

$$A^T = A^{-1}$$

Gram-Schmidt process for constructing an orthonormal basis

Start with n linearly independent basis vectors $\mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Then



$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$


$$\mathbf{u}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{e}_1) \mathbf{e}_1 \quad \text{so} \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$$

... and so on for \mathbf{u}_3 being the remnant part of \mathbf{v}_3 not composed of the preceding \mathbf{e} -vectors, etc. ...

**Ketika kita kita berbicara tentang
Problematika Eigen, maka kita berbicara
tentang properti karakteristik dari
sesuatu.**

Eigenstuff

matrix  $\mathbf{Ax} = \lambda \mathbf{x}$  eigenvector

 eigenvalue

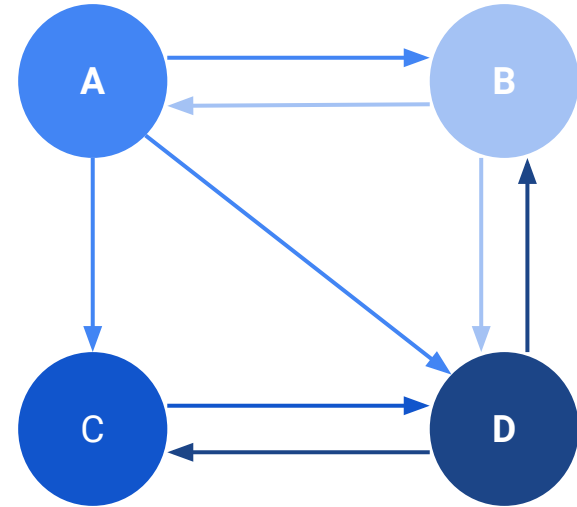
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

 identity matrix

What are the examples of Linear Algebra in Machine Learning?

Algoritma PageRank

- Algoritma ini dipublish pada tahun 1998 oleh Larry Page (Founder Google) dan teman-temannya.
- Digunakan oleh Google untuk membantu dalam mendapatkan keputusan order yang biasa kita lihat ketika searching.



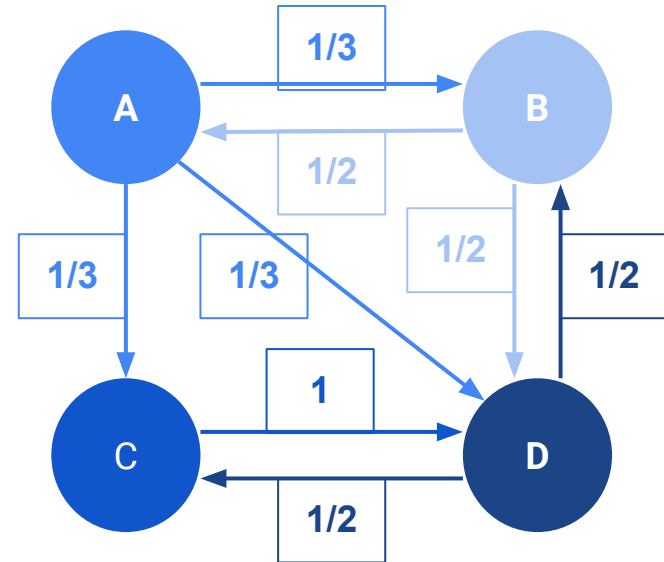
How PageRank works?

$$L_a = [0 \ 1/3 \ 1/3 \ 1/3]$$

$$L_b = [1/2 \ 0 \ 0 \ 1/2]$$

$$L_c = [0 \ 0 \ 0 \ 1]$$

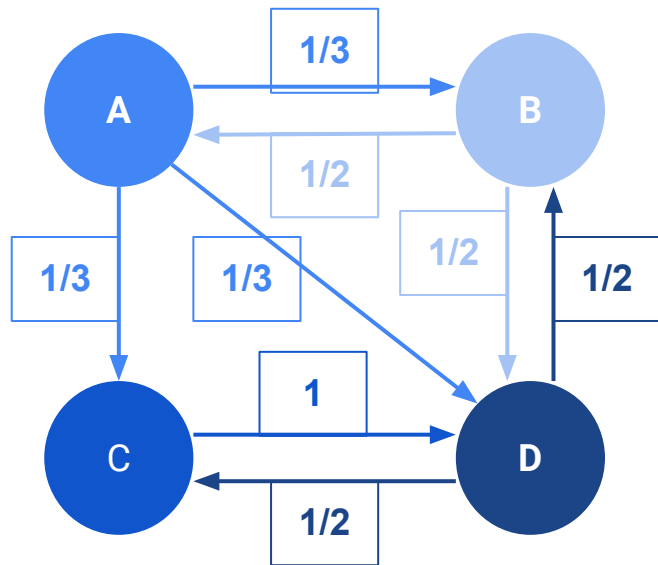
$$L_d = [0 \ 1/2 \ 1/2 \ 0]$$



How PageRank Works?

$$L = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix}$$

$$r_{i+1} = Lr_i$$





Terima Kasih