

Model Estimation with AR(1) Endogenous Unobservable

1) Model:

The observables are y_{it} and x_{it} . The goal is to identify β in the relationship,

$$y_{it} = x'_{it}\beta + w'_{it}\gamma + \epsilon_{it} + \nu_{it}, \quad \nu_{it} \sim \text{iid}, \quad \nu_{it} \text{ is serially independent} \quad (1)$$

$$x_{it} = \gamma\epsilon_{it} + (1 - \gamma)u_{it}, \quad u_{it} \sim \text{iid}, \quad u_{it} \text{ is serially dependent} \quad (2)$$

$$\epsilon_{it} = \rho\epsilon_{it-1} + \eta_{it}, \quad \eta_{it} \sim \text{iid}, \quad \eta_{it} \text{ is serially independent} \quad (3)$$

where w_{it} includes a constant.

2) Identification:

The usual argument for identification is based on the quasi-difference expression:

$$(y_{it} - \rho y_{it-1}) = (x_{it} - \rho x_{it-1})' \beta + (w_{it} - \rho w_{it-1})' \delta + \eta_{it} \quad (4)$$

Panel IV Approach: One approach is to rearrange Equation (4) as a panel regression:

$$y_{it} = y_{it-1}(\rho) + x'_{it}(\beta) + x'_{it-1}(-\rho\beta) + w'_{it}(\delta) + w'_{it-1}(-\rho\delta) + \eta_{it} \quad (5)$$

The only source of endogeneity in this regression is that x_{it} depends on η_{it} . This implies that β is identified by a regression of y_{it} on x_{it} , controlling for $(y_{it-1}, x_{it-1}, w_{it}, w_{it-1})$, and instrumented by $z_{it} = (x_{it-2})$ or $z_{it} = (x_{it-2}, y_{it-2})$ or $z_{it} = (x_{it-2}, y_{it-2}, w_{it-2})$.

GMM Approach: For any guess $(\hat{\beta}, \hat{\delta}, \hat{\rho})$, we can define the guess of η_{it} :

$$\hat{\eta}_{it}(\hat{\beta}, \hat{\delta}, \hat{\rho}) \equiv (y_{it} - \hat{\rho}y_{it-1}) - (x_{it} - \hat{\rho}x_{it-1})' \hat{\beta} - (w_{it} - \rho w_{it-1})' \hat{\delta} \quad (6)$$

Then,

$$(\beta, \delta, \rho) \quad \text{solves} \quad \mathbb{E} \left[(w_{it}, z_{it})' \hat{\eta}_{it}(\hat{\beta}, \hat{\delta}, \hat{\rho}) \right] = 0 \quad (7)$$

where z_{it} includes variables that are independent of η_{it} . The standard choices of instruments are $z_{it} = (x_{it-1}, y_{it-1})$ and $z_{it} = (x_{it-1}, x_{it-2})$. Note that the number of instruments z_{it} must include at least one more than the number of endogenous variables in x_{it} . See the Appendix for further implementation details.

3) Simulation Exercise:

In order to compare the estimation approaches, Figure 1 simulates the model defined above. It sets $\beta = (0.5, -0.2)$, $\delta = 1$, $\rho = 0.5$, $\gamma = 0.5$, $\eta_{it} \sim \mathcal{N}(0, 1)$, $u_{it} = u_{it-1} + \mathcal{N}(0, 1)$, and $\nu_{it} = 0$. The length of the panel is $T = 3$, which is the minimum required. For various choices of N , it draws 10 random samples from the model, applies the estimators, and presents the box-plot of the distribution of estimates.

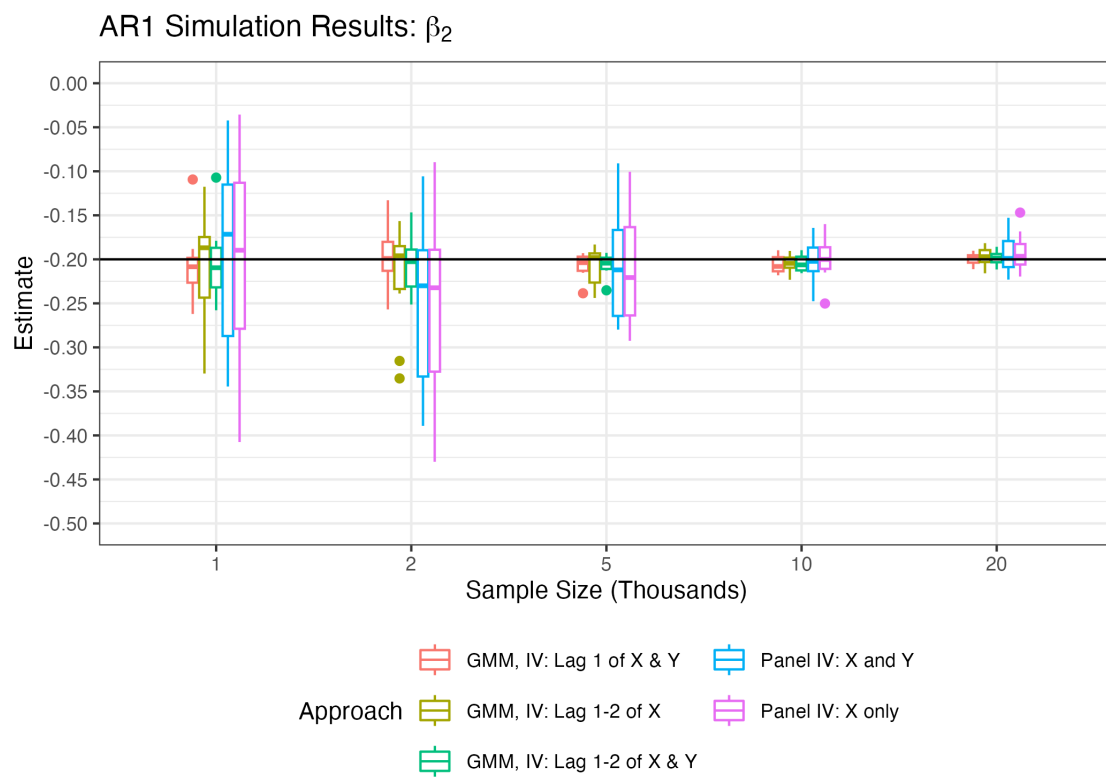
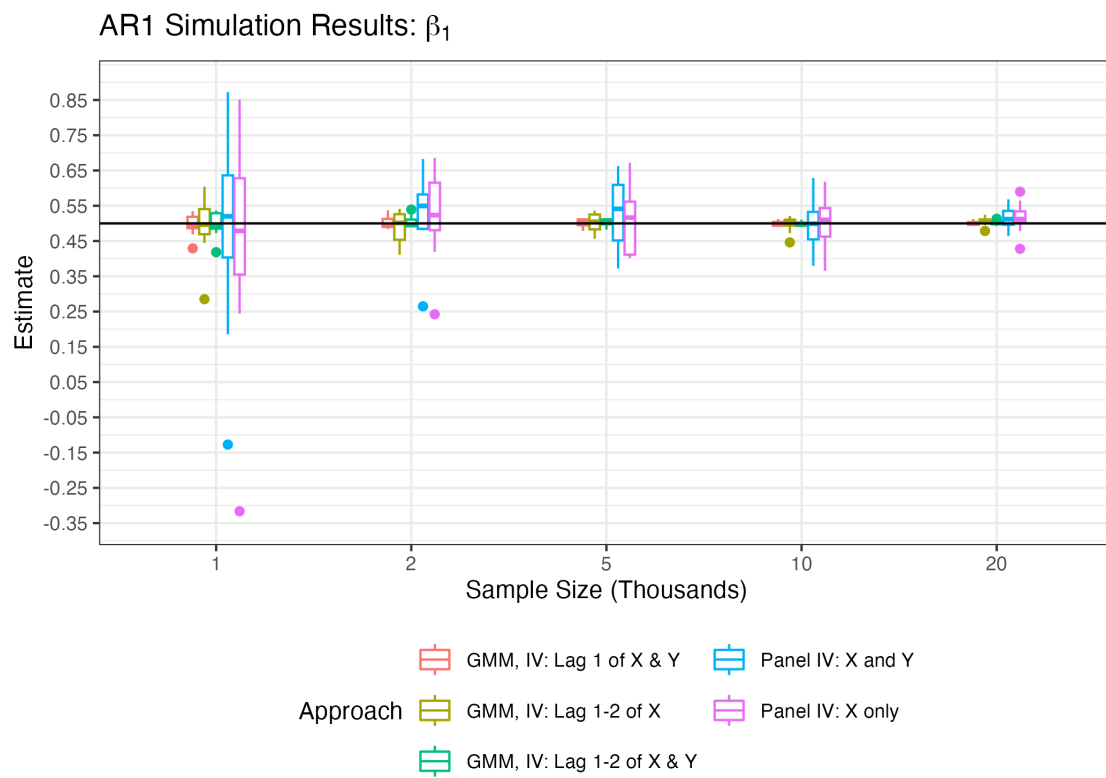


Figure 1: Simulation Exercise

Appendix: GMM Implementation Algorithm

Denote $g_{it}(\hat{\beta}, \hat{\delta}, \hat{\rho}) \equiv (w_{it}, z_{it})' \hat{\eta}_{it}(\hat{\beta}, \hat{\delta}, \hat{\rho})$.

1. Solve for an initial guess of (β, δ, ρ) using the fast moment-matching estimator discussed below.
2. Set the weighting matrix to $W = \hat{\Omega}^{-1}$, where $\hat{\Omega} = \mathbb{E} \left[g_{it}(\hat{\beta}, \hat{\delta}, \hat{\rho}) g_{it}(\hat{\beta}, \hat{\delta}, \hat{\rho})' \right]$, evaluated at the most recent solution.
3. Repeats steps 2 then 3, until numerical convergence is achieved. The solutions are $(\beta, \delta, \rho), \Omega$.
4. Compute $G \equiv \mathbb{E} [\nabla g_{it}(\beta, \delta, \rho)]$, where G is a $(|w| + |z|) \times (|\beta| + |\delta| + |\rho|)$ dimensional matrix with elements,

$$\begin{aligned} \frac{\partial}{\partial \beta^{(j)}} g_{it} &= - (w_{it}, z_{it})' \left(x_{it}^{(j)} - \rho x_{it-1}^{(j)} \right) \\ \frac{\partial}{\partial \rho} g_{it} &= -y_{it-1} + x'_{it-1} \hat{\beta} + w'_{it-1} \hat{\delta} \\ \frac{\partial}{\partial \delta^{(1)}} g_{it} &= - (w_{it}, z_{it})' (1 - \rho) \\ \frac{\partial}{\partial \delta^{(j)}} g_{it} &= - (w_{it}, z_{it})' \left(w_{it}^{(j)} - \rho w_{it-1}^{(j)} \right), \quad j > 1 \end{aligned}$$

5. Compute $\text{SE}(\beta, \delta, \rho) = \text{diag}(\text{Var}(\beta, \delta, \rho))^{1/2}$, where $\text{Var}(\beta, \delta, \rho) = (G' \Omega^{-1} G)^{-1} / (NT)$.

Appendix: Fast Moment-matching Estimator

Recall from equation (4):

$$y_{it} = y_{it-1}(\rho) + x'_{it}(\beta) + x'_{it-1}(-\rho\beta) + w'_{it}(\delta) + w'_{it-1}(-\rho\delta) + \eta_{it}$$

We can form a moment using the covariance between y_{it} and any instrument z_{it} :

$$\begin{aligned} 0 &= (1) \text{Cov}(z_{it}, y_{it}) + (-\rho) \text{Cov}(z_{it}, y_{it-1}) \\ &+ \sum_{k=1, \dots, |x_{it}|} (-\beta^{(k)}) \text{Cov}(z_{it}, x_{it}^{(k)}) + \sum_{k=1, \dots, |x_{it}|} (\rho \beta^{(k)}) \text{Cov}(z_{it}, x_{it-1}^{(k)}) \\ &+ \sum_{m=2, \dots, |w_{it}|} (-\delta^{(m)}) \text{Cov}(z_{it}, w_{it}^{(m)}) + \sum_{m=2, \dots, |w_{it}|} (\rho \delta^{(m)}) \text{Cov}(z_{it}, w_{it-1}^{(m)}) \end{aligned}$$

where δ_1 does not appear because it corresponds to the intercept. This is one equation in $|x_{it}| + |w_{it}|$ unknowns.

With $J \geq |x_{it}| + |w_{it}|$ instruments, we can solve this set of equations for ρ, β, δ (except for the intercept δ_1). The covariances only have to be computed once; they do not have to be updated for each parameter guess, which means we can solve for the parameters very quickly. We can then solve for δ_1 such that the unconditional expectation of y_{it} equals the model prediction from the right-hand side. Thus, we can provide a moment-matching estimate of all parameters very quickly.