Model Estimation with AR(1) Endogenous Unobservable

1) Model:

The observables are y_{it} and x_{it}. The goal is to identify β in the relationship,

$$y_{it} = x_{it}'\beta + w_{it}'\gamma + \epsilon_{it} + \nu_{it}, \quad \nu_{it} \sim \text{iid}, \quad \nu_{it} \text{ is serially independent} \tag{1}$$

$$x_{it} = \gamma\epsilon_{it} + (1-\gamma)\, u_{it}, \quad u_{it} \sim \text{iid}, \quad u_{it} \text{ is serially dependent} \tag{2}$$

$$\epsilon_{it} = \rho\epsilon_{it-1} + \eta_{it}, \quad \eta_{it} \sim \text{iid}, \quad \eta_{it} \text{ is serially independent} \tag{3}$$

where w_{it} includes a constant.

2) Identification:

The usual argument for identification is based on the quasi-difference expression:

$$(y_{it} - \rho y_{it-1}) = (x_{it} - \rho x_{it-1})'\,\beta + (w_{it} - \rho w_{it-1})'\,\delta + \eta_{it} \tag{4}$$

Panel IV Approach: One approach is to rearrange Equation (4) as a panel regression:

$$y_{it} = y_{it-1}(\rho) + x_{it}'(\beta) + x_{it-1}'(-\rho\beta) + w_{it}'(\delta) + w_{it-1}'(-\rho\delta) + \eta_{it} \tag{5}$$

The only source of endogeneity in this regression is that x_{it} depends on η_{it}. This implies that β is identified by a regression of y_{it} on x_{it}, controlling for $(y_{it-1}, x_{it-1}, w_{it}, w_{it-1})$, and instrumented by $z_{it} = (x_{it-2})$ or $z_{it} = (x_{it-2}, y_{it-2})$ or $z_{it} = (x_{it-2}, y_{it-2}, w_{it-2})$.

GMM Approach: For any guess $\left(\hat{\beta}, \hat{\delta}, \hat{\rho}\right)$, we can define the guess of η_{it}:

$$\hat{\eta}_{it}\left(\hat{\beta}, \hat{\delta}, \hat{\rho}\right) \equiv (y_{it} - \hat{\rho} y_{it-1}) - (x_{it} - \hat{\rho} x_{it-1})'\,\hat{\beta} - (w_{it} - \rho w_{it-1})'\,\hat{\delta} \tag{6}$$

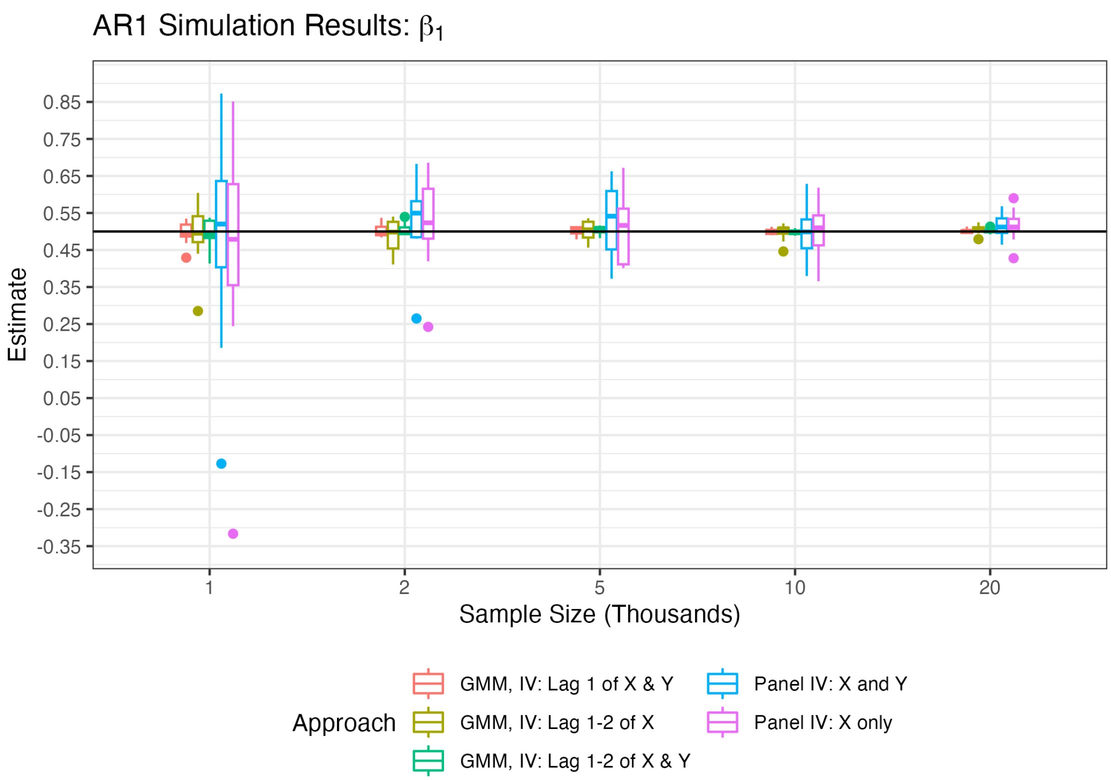
Then,

$$(\beta, \delta, \rho) \quad \text{solves} \quad \mathbb{E}\left[(w_{it}, z_{it})\ \hat{\eta}_{it}\left(\hat{\beta}, \hat{\delta}, \hat{\rho}\right)\right] = 0 \tag{7}$$

where z_{it} includes variables that are independent of η_{it}. The standard choices of instruments are $z_{it} = (x_{it-1}, y_{it-1})$ and $z_{it} = (x_{it-1}, x_{it-2})$. Note that the number of instruments z_{it} must include at least one more than the number of enodgenous variables in x_{it}. See the Appendix for further implementation details.

3) Simulation Exercise:

In order to compare the estimation approaches, Figure 1 simulates the model defined above. It sets $\beta = (0.5, -0.2)$, $\delta = 1$, $\rho = 0.5$, $\gamma = 0.5$, $\eta_{it} \sim \mathcal{N}(0,1)$, $u_{it} = u_{it-1} + \mathcal{N}(0,1)$, and $\nu_{it} = 0$. The length of the panel is $T = 3$, which is the minimum required. For various choices of N, it draws 10 random samples from the model, applies the estimators, and presents the box-plot of the distribution of estimates.



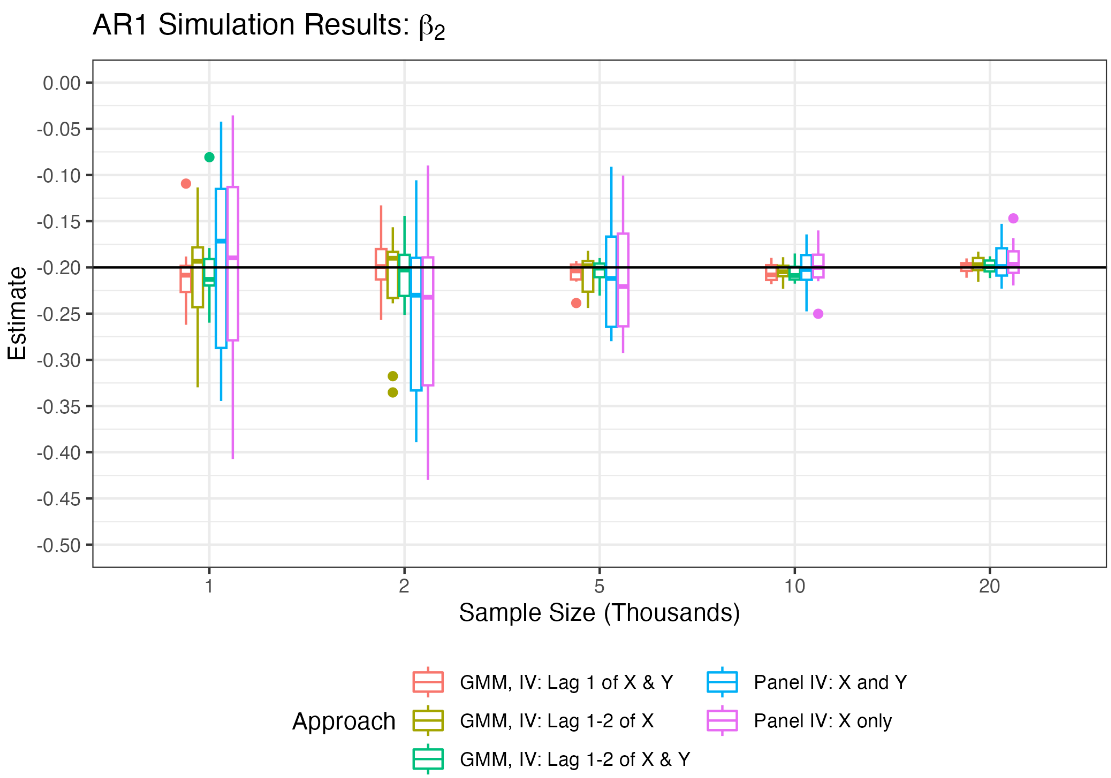


Figure 1: Simulation Exercise

Appendix: GMM Implementation Algorithm

Denote $g_{it}\left(\hat{\beta},\hat{\delta},\hat{\rho}\right) \equiv (w_{it}, z_{it})\ \hat{\eta}_{it}\left(\hat{\beta},\hat{\delta},\hat{\rho}\right)$.

1. Set the weighting matrix to $W = I$. Solve $(\beta, \delta) = \min_{\hat{\beta},\hat{\delta}} \mathbb{E}\left[g_{it}\left(\hat{\beta},\hat{\delta},0\right)\right]' W \mathbb{E}\left[g_{it}\left(\hat{\beta},\hat{\delta},0\right)\right]$. This gives an initial solution given $\rho = 0$.

2. Solve $(\beta, \delta, \rho) = \min_{\hat{\beta},\hat{\delta},\hat{\rho}} \mathbb{E}\left[g_{it}\left(\hat{\beta},\hat{\delta},\hat{\rho}\right)\right]' W \mathbb{E}\left[g_{it}\left(\hat{\beta},\hat{\delta},\hat{\rho}\right)\right]$, taking the most recent solution as the initialization point.

3. Set the weighting matrix to $W = \hat{\Omega}^{-1}$, where $\hat{\Omega} = \mathbb{E}\left[g_{it}\left(\hat{\beta},\hat{\delta},\hat{\rho}\right) g_{it}\left(\hat{\beta},\hat{\delta},\hat{\rho}\right)'\right]$, evaluated at the most recent solution.

4. Repeats steps 2 then 3, until numerical convergence is achieved. The solutions are $(\beta, \delta, \rho), \Omega$.

5. Compute $G \equiv \mathbb{E}\left[\nabla g_{it}(\beta, \delta, \rho)\right]$, where G is a $(|w| + |z|) \times (|\beta| + |\delta| + |\rho|)$ dimensional matrix with elements,

$$
\begin{aligned}
\frac{\partial}{\partial \beta^{(j)}} g_{it} &= -(w_{it}, z_{it})\ \left(x_{it}^{(j)} - \rho x_{it-1}^{(j)}\right) \\
\frac{\partial}{\partial \rho} g_{it} &= -y_{it-1} + x_{it-1}'\hat{\beta} + w_{it-1}'\hat{\delta} \\
\frac{\partial}{\partial \delta^{(1)}} g_{it} &= -(w_{it}, z_{it})\ (1 - \rho) \\
\frac{\partial}{\partial \delta^{(j)}} g_{it} &= -(w_{it}, z_{it})\ \left(w_{it}^{(j)} - \rho w_{it-1}^{(j)}\right),\ j > 1
\end{aligned}
$$

6. Compute $\mathrm{SE}(\beta, \delta, \rho) = \mathrm{diag}\left(\mathrm{Var}(\beta, \delta, \rho)\right)^{1/2}$, where $\mathrm{Var}(\beta, \delta, \rho) = \left(G'\Omega^{-1}G\right)^{-1}/(NT)$.