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Research Statement

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My research lies primarily in differential geometry, especially Teichmüller–Thurston theory. Broadly speaking, this is the study of geometric structures and moduli spaces relative to surfaces. This field lies at the intersection of several important areas of mathematics, including complex analysis, low-dimensional geometry and topology, dynamical systems, Riemannian and symplectic geometry[1].

While this has been the main thread of my research, I have been working on several other topics, such as number theory and computational geometry. I also love programming and have been developing mathematical software. Last but not least, I have a strong dedication to mathematics education, and have produced several lecture notes and one book aimed at students in mathematics.

Keywords: Classical and higher Teichmüller–Thurston theory · Hyperbolic geometry · Symplectic geometry of moduli spaces · Hyper-Kähler structures · Harmonic maps · Minimal surfaces · Higgs bundles · Discrete differential geometry · Computational geometry · Mathematical programming · Mathematics education

Introduction

Let me attempt a very brief introduction to Teichmüller theory and the symplectic structure of moduli spaces. While this has been the main thread of my research from the start, my interests have since expanded to other topics, such as minimal surfaces, discrete geometry, number theory. I will nevertheless limit this introduction to the first subject, and delay explaining some of its connections to the others.

Let S be an orientable surface of negative Euler characteristic. The goal of classical Teichmüller theory is to study the space of all complex structures on S. This is a very rich and interesting theory, partly because a complex structure on S has several equivalent incarnations, such as:

- A complex-analytic local identification with $\mathbb{C}$ (this is the original point of view due to Riemann).
- An isometric local identification with the hyperbolic plane $\mathbb{H}^2$ (by the celebrated uniformization theorem).
- A discrete and faithful representation $\rho\colon \pi_1 S \to G$ where $G = \mathrm{PSL}_2(\mathbb{R})$ (the holonomy).

These viewpoints yield moduli spaces that are isomorphic in a nontrivial way: Teichmüller space $\mathcal{T}(S)$, The Fricke-Klein space of hyperbolic structures $\mathcal{F}(S)$, and a component of the character variety $\mathcal{X}(S, G) := \mathrm{Hom}(\pi_1 S, G)/G$.

The point of view best suited for studying the complex-analytic structure is the Teichmüller space $\mathcal{T}(S)$. Its naturality was fortified by Kodaira–Spencer and the algebraic approach of Grothendieck. The complex-analytic theory of Teichmüller space was then developed until the 1970s by Lars Ahlfors, Lipman Bers, and others.

Teichmüller space also has a symplectic structure, which together with its complex structure yields a Kähler metric. Historically, this was first discovered by Weil and Ahlfors. It was later understood that the symplectic structure is more natural on the Fricke-Klein space $\mathcal{F}(S)$ and the character variety $\mathcal{X}(S, G)$ [Gol84]. This realization was possible after the emphasis shifted to hyperbolic geometry and other geometric structures in the 1970s and 1980s, when Teichmüller theory was deeply influenced by the singular geometric vision of Thurston.

In recent years, several generalizations of Teichmüller spaces have been under intense investigation:

- Deformation spaces of geometric structures. Given a homogeneous space $X = G/H$, one can consider manifolds locally modelled on X/G and their moduli spaces.
- Higher Teichmüller spaces. This name is typically given to a component of the character variety $\mathcal{X}(S, G)$ whose geometric properties extend the case $G = \mathrm{PSL}_2(\mathbb{R})$.
- Moduli spaces of Higgs bundles. These are holomorphic objects, which depend on the choice of a complex structure on S, and are related to character varieties via harmonic maps and gauge theory.

These moduli spaces have rich interrelationships. The *nonabelian Hodge correspondence* [Sim92] between the character variety and Higgs bundles is a fascinating device that has generated a lot of attention in recent years.

The work of Goldman implies that all these moduli spaces have a natural symplectic geometry. In the 1980s, Wolpert painted a beautiful picture of the symplectic/Poisson geometry of $\mathcal{T}(S) \approx \mathcal{F}(S)$ [Wol82; Wol83; Wol85]. This has partially been extended to more general moduli spaces—partly by myself for the case of complex projective structures, see § 1.1. However, the existence of a (hyper-)Kähler geometry that would generalize the Weil–Petersson geometry of Teichmüller space is an important question that is yet to be understood.

[1] I am paraphrasing Papadopoulos's foreword in [Pap07].

1 Past research

1.1 Complex symplectic geometry of $\mathcal{CP}(S)$ – [Lou15b]

Complex projective structures are examples of geometric structures on surfaces. Their deformation space $\mathcal{CP}(S)$ is exceptionally interesting for several reasons, including: the fact that it fibers over Teichmüller space $\mathcal{T}(S)$ and extends Fricke-Klein space $\mathcal{F}(S)$ and that it is locally biholomorphic to the character variety $\mathcal{X}(S, \mathrm{PSL}(2, \mathbb{C}))$.

In [Lou15b], I carefully studied the complex symplectic geometry of $\mathcal{CP}(S)$. A striking feature of $\mathcal{CP}(S)$ is that the Schwarzian parametrization turns the projection $\mathcal{CP}(S) \to \mathcal{T}(S)$ into an affine bundle modelled on $\mathrm{T}^* \mathcal{T}(S)$. I proved that this parametrization is a symplectomorphism provided a Lagrangian section is chosen for the "zero section". I also obtained a characterization of such sections, showing that they include Bers and Schottky sections.

I also showed that Wolpert's beautiful picture of the symplectic geometry of $\mathcal{T}(S)$ extends to quasi-Fuchsian space, using the complex Fenchel-Nielsen coordinates introduced by Kourouniotis [Kou94] and Tan [Tan94].

Finally, I recovered and generalized several classical results of Teichmüller theory via symplectic geometry, such as McMullen's *quasi-Fuchsian reciprocity* [McM00].

1.2 Minimal surfaces in hyperbolic 3-manifolds and symplectic structures – [Lou15a]

In [Lou15a], I studied the symplectic structure of the almost-Fuchsian deformation space $\mathcal{AF}(S)$ and Taubes's moduli space $\mathcal{H}$ of minimal hyperbolic germs [Tau04].

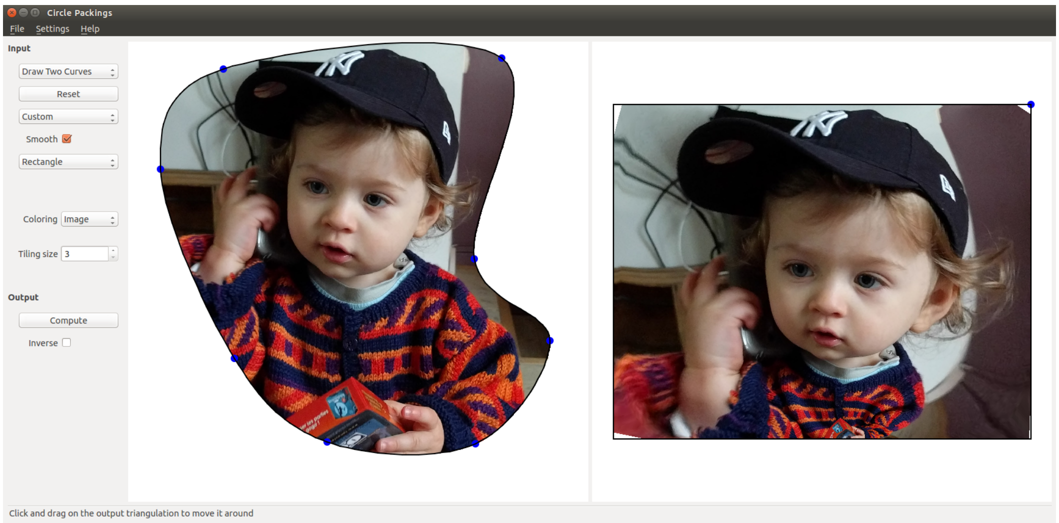
One particularly rich topic is the theory of minimal surfaces in hyperbolic 3-manifolds, which has deep connections to Teichmüller theory. In particular, taking the first and second fundamental forms of minimal surfaces leads to a parametrization of almost-Fuchsian space $\mathcal{AF}(S)$, which is a complex thickening of Fricke-Klein space $\mathcal{F}(S)$, as an open set in $\mathrm{T}^* \mathcal{T}(S)$. This is distinct from the Schwarzian parametrization mentioned in § 1.1, but coincides with the parametrization by Higgs bundles studied by Donaldson [Don03] and recently generalized by Labourie [Lab17].

The main result of [Lou15a] is that this parametrization is a symplectomorphism for the imaginary part of Goldman's symplectic form. I also proved that this symplectic structure agrees with Taubes's. The main ingredient in the proof is a variation of the notion of renormalized volume developed by Krasnov and Schlenker [KS12].

1.3 Computer software: Circle Packings (with B. Beeker)

I developed the mathematical computer software *Circle Packings* in collaboration with B. Beeker. With a graphical user interface, this software computes and displays circle packings relative to planar graphs and showing Riemann mappings in the plane.

I refer to the web page for more details: brice.loustau.eu/circlepackings. I also gave an online talk for a broad audience of non-mathematicians, it is available at brice.loustau.eu/ressources/HITS-20200907.mp4



1.4 Bi-Lagrangian structures (with A. Sanders) – [LS18]

In joint work with Andy Sanders [LS18], we studied real and complex bi-Lagrangian structures and applied this study to Teichmüller theory. Bi-Lagrangian structures lie at the intersection of symplectic, para-complex, and pseudo-

Riemannian geometry, and relate to affine structures via the Bott connection in a Lagrangian foliation.

The first segment of [LS18] attempts a clean exposition of bi-Lagrangian structures and their properties in the real setting and complex setting, where they had not been studied yet. We then show that the complexification of a Kähler manifold admits a natural complex bi-Lagrangian structure.

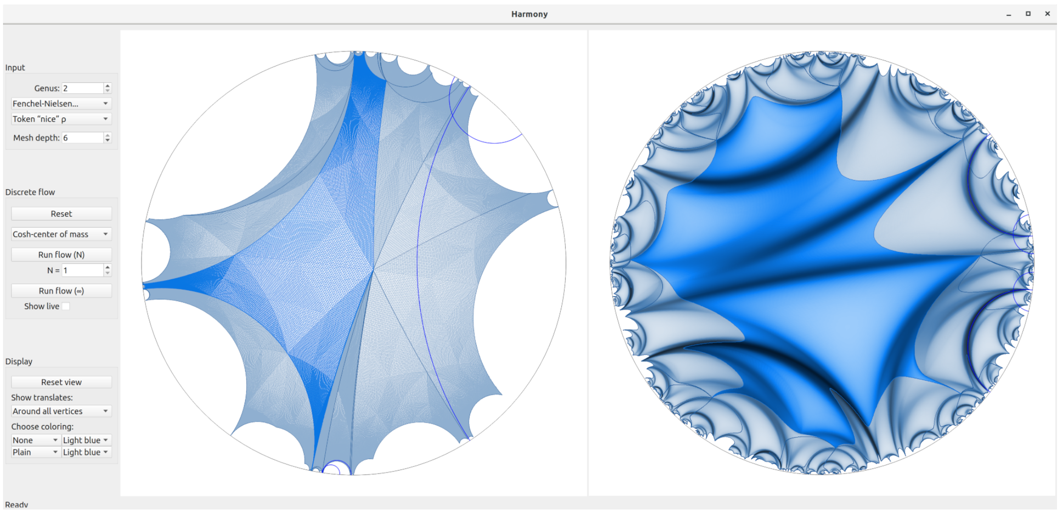
In the second segment of the paper, we turn to applications to Teichmüller theory. Pushing the philosophy of [Lou15a], we apply general machinery from symplectic geometry to obtain new results in Teichmüller theory or recover old ones. For instance, we show that quasi-Fuchsian space $\mathcal{QF}(S)$ has a natural complex bi-Lagrangian structure and show that the canonical transverse Lagrangian foliations are the well-known foliations by Bers slices.

1.5 Computing harmonic maps (with J. Gaster and L. Monsaingeon) – [GLM18; GLM19]

This is a long-term joint research project which has given birth to the two papers, soon a third (see § 2.4), and a computer software. The main goal is to effectively compute harmonic maps between Riemannian manifolds. Under the right assumptions, a harmonic map exists and is unique in any homotopy class [ES64; Har67]. The extension of this theorem to the equivariant setting is key for nonabelian Hodge theory.

In [GLM18], we study harmonic maps between Riemannian manifolds both in the smooth and in the discretized setting. We prove strong convexity properties of the energy functional, ensuring the convergence of the discrete heat flow to the discrete harmonic map. We also characterize harmonic maps via centers of mass, generalizing the mean value property of real-valued harmonic functions, and investigate center of mass methods. In [GLM19], we introduce *Laplacian systems of weights* on a triangulation of a Riemannian manifold, and carry out a careful analysis the convergence of the discrete theory to the smooth theory when taking finer and finer triangulations.

With J. Gaster, we have developed the computer software *Harmony* , illustrating the methods discussed above. We hope to expand it further in the future and bring new tools to experimentally investigate the nonabelian Hodge correspondence. See the web page for more details: brice.loustau.eu/harmony.



1.6 The sum of Lagrange numbers (with J. Gaster) – [GL20]

This short paper proves a striking result of number theory. Specifically, we show that the famous *Markov Uniqueness Conjecture* is equivalent to the identity $\sum_{n=1}^{\infty} (3 - L_n) = 4 - \varphi - \sqrt{2}$ where L_n is the nth Lagrange number and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The proof relies on and McShane's identity on a hyperbolic punctured torus and the connection between Markov numbers and hyperbolic geometry.

1.7 Harmonic maps from Kähler manifolds – [Lou20]

In this report, I attempt a clean presentation of the theory of harmonic maps from complex and Kähler manifolds to Riemannian manifolds. I explain the Bochner technique adapted to Kähler manifolds by Siu and Sampson and its main consequences, such as the strong rigidity results of Siu. I also recount the applications to symmetric spaces of noncompact type and their relation to Mostow rigidity. Finally, I explain the key role of this theory for the nonabelian Hodge correspondence.

2 Ongoing and future research

2.1 Complex geometry of the Higgs bundles universal moduli space (with A. Sanders and N. Tholozan)

In this project, we study the complex structure of the universal moduli space of Higgs bundles $\mathcal{M}$, defined as a bundle over Teichmüller space $\mathcal{T}(S)$. We show that it has a natural complex structure that can be computed explicitly, and showcase many properties. For instance, the subbundle $\mathcal{U}$ corresponding to unitary connections admits a natural Kähler metric, and $\mathcal{M}$ inherits an interesting structure as the quotient bundle from $\mathrm{T}^*\mathcal{U} \to \mathrm{T}^*\mathcal{T}(S)$, whose total space is hyper-Kähler. This construction is promising for understanding how the Kähler geometry of Teichmüller space $\mathcal{T}(S)$ might extend to higher Teichmüller-Thurston spaces.

2.2 Hyper-Kähler geometry of minimal hyperbolic germs (with F. Bonsante, A. Sanders, and A. Seppi)

I mentioned Taubes's moduli space $\mathcal{H}$ of minimal hyperbolic germs in § 1.2. It can be described as the space of minimal equivariant immersions $\tilde{S} \to \mathbb{H}^3$, and contains the almost Fuchsian space $\mathcal{AF}(S)$. In the remarkable paper [Don03], Donaldson constructs a hyper-Kähler structure in $\mathcal{AF}(S)$ and shows that it coincides with the hyper-Kähler extension of Teichmüller space of Feix and Kaledin.

In this project, we show that the hyper-Kähler structure extends in $\mathcal{H}$ outside of the singular locus of the Jacobi operator. Moreover, the area functional on $\mathcal{H}$ is a Kähler potential for the metric, whose signature is controlled by the index of the Jacobi operator. We additionally prove that the natural maps from $\mathcal{H}$ to $\mathrm{T}^*\mathcal{T}(S)$ and to $\mathcal{X}(\pi_1 S, \mathrm{PSL}(2,\mathbb{C}))$ are complex symplectomorphisms for the appropriate complex symplectic structures.

2.3 Symplectic properties of Wick rotations (with C. Scarinci)

In this upcoming work, we examine the symplectic and paracomplex geometry of the deformation space of globally hyperbolic Einstein metrics on a 3-manifold, when one varies the cosmological constant Λ. This project uses the bi-Lagrangian structures that I studied with A. Sanders (§ 1.4). We also introduce Λ-grafting, which generalizes both earthquakes and cataclysms in Teichmüller theory, and symplectic properties of Wick rotations.

2.4 Discrete Bochner formula on Riemannian manifolds (with J. Gaster and L. Monsaingeon)

In this upcoming work, we establish a Bochner formula for maps defined on a discretized Riemannian manifold. Putting this result into application, we generalize the results of [GLM19] concerning the convergence of the discrete heat flow, and we propose a discretized solution to the Schoen conjecture recently solved by Benoist–Hulin [BH17] and Markovic [Mar17].

2.5 Geometry Labs United

I am part of the team that is setting up the *Geometry Labs United* at Heidelberg University, under the lead of Anna Wienhard. The goal is to provide an interface and equipment for researchers and students to engage in projects revolving around mathematics and visualization. It is inspired by the GLU project initiated in the United States.

2.6 Other future projects

I have started other collaborations that will hopefully lead to results and publications in the upcoming years. Let me briefly mention some of them:

- With A. Sanders, we started a project building on [LS18] to construct and study the Fedosov deformation quantization of quasi-Fuchsian space.
- I would like to complement a theorem of Schlenker–Yarmola [SY18] regarding the existence of circle packings on complex projective surfaces, to answer a conjecture of Kojima–Mizushima–Tan [KMT06].
- With J. Gaster and D. Zierau, we showed remarkable properties of cross-ratios of torsion points on elliptic curves in relation to uniformization of punctured spheres. These results (unpublished) are contained in Zierau's Master thesis, which I supervised, and could be expanded in a future publication.
- With J. Gaster, we have ideas to further explore the Markov Uniqueness Conjecture via hyperbolic geometry.
- With L. Ruffoni, we have a project to extend complex grafting and complex Fenchel-Nielsen coordinates on a maximal open set in the deformation space of complex projective structures $\mathcal{CP}(S)$.
- With T. Zhang, we have a project to extend some aspects of the Labourie–Loftin parametrization of convex projective structures to higher dimensions.

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