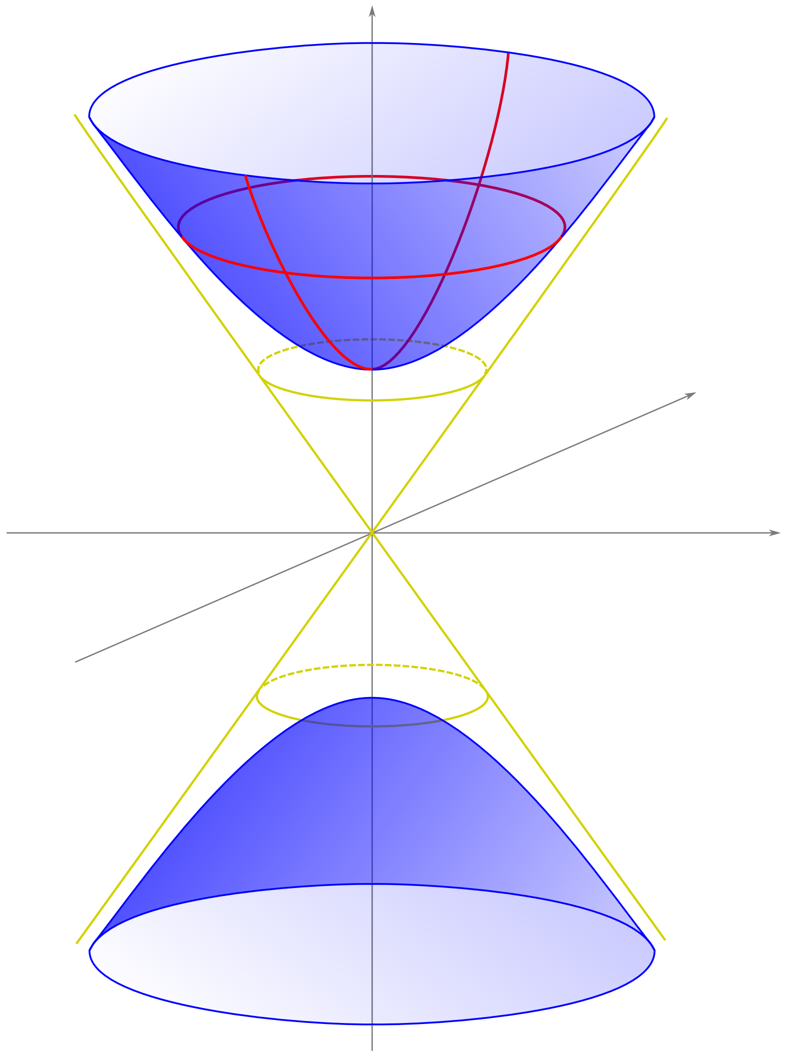
Hyperbolic geometry

Brice Loustau



A book, even if it is written with complete honesty, is always worthless from one *standpoint: for no one really needs to write a book, since there are many other things to do in the world.*

– Ludwig Wittgenstein[1]

[1] Letter to Ludwig von Ficker, 4 December 1919. English translation in [Luc96, pp. 82–98].

TODO

This document is the preliminary version of a textbook to be published soon.

Remaining chapters to write or complete:

- Chapter 5 (Relativity).
- Chapter 14 (More plane) and Chapter 15 (Tessellations).
- Chapter 16 (Graph embeddings) and Chapter 17 (Neural networks)
- Appendix A (Appendix)
- Chapter 12 (Recap)

Other additional material:

- More exercises and solutions.
- More figures (+ recreate the Wiki cross-ratio fig?)
- Cover
- Index of notations and index of subjects (see Lee)
- Appendix containing "review of basic notions" (see TODO.txt)

Changes in the existing text:

- Read everything over again, correct and improve.
- Check for spelling and typos

Typography:

- ~~Fix LaTeX warnings and bad boxes~~.
- Use KOMA-Script?
- Use the Springer style files (when everything else is finished)
- Make all matrices brackets?
- Figure out why bookmarksetupbold doesn't work

Misc:

- Decide whether to keep the Wittgenstein quote
- Check status of TODO.txt
- ~~Check placement of figures~~

Compiled on 19/06/2021 at 04:00

Preface

Hyperbolic geometry is a very special subject: it is the star of geometries, and geometry is the star of mathematics! Well, perhaps this is a bit of an exaggeration, yet a useful one to have in mind—few topics have such historical and conceptual weight.

The history of mathematics and science, indeed, speaks for the importance of hyperbolic geometry. The names of several of the greatest mathematicians are attached to it, such as Gauss and Poincaré, and its incredibly fertile development is related to the creation of projective spaces, Fuchsian groups, Minkowski spacetime, among other decisive notions for modern mathematics and physics, including the mathematical framework for Einstein's theory of relativity!

While the revolutionary discovery of hyperbolic geometry mainly took place in the 19th century, it continued to play a leading role in the mathematics of the 20th, culminating with Thurston's geometrization program and its completion in the early 21st century by Perelman, which solved the famous Poincaré conjecture. To this day, hyperbolic geometry and its avatars remain an intensely active field of research, both in mathematics and in applied sciences—it shows promise, for instance, in the emerging field of data science and machine learning.

* * *

Why did I write this textbook? It started as a set of lecture notes that I wrote for a Master course I taught at TU Darmstadt in the winter 2019–2020, building on notes for a similar course I held at Rutgers University in 2017. After a push of some colleagues and friends, I decided to upgrade them into a proper book. I hope that it fills a gap in the literature, as an ambitious first course on hyperbolic geometry—more details below.

Goal and intended audience. The goal of the book is to provide a first course on hyperbolic geometry with little or no prerequisites of differential geometry. It intends to be fairly thorough while staying self-contained and not too advanced. The book is suitable for a course at the early graduate (Master) level or advanced undergraduate (Bachelor) level in a mathematics curriculum. More broadly, it is meant to be useful to anyone looking to properly learn the basics (and more) of hyperbolic geometry, whether to pursue higher level education or research in a related area, or to apply it to other fields.

Contents. All the standard features of hyperbolic spaces are rigorously introduced and studied: the different models of any dimension (the hyperboloid model, Cayley–Klein and Beltrami–Klein models, Poincaré ball and half-space models); hyperbolic geodesics, distance, curvature, and isometries; the ideal boundary and the classification of isometries; hyperbolic triangles and trigonometry; tessellations of the hyperbolic plane, and more. Beyond hyperbolic geometry, readers will have the opportunity to learn many essential notions of geometry: the concept of curvature, Minkowski space and the Lorentz group, projective geometry and quadrics, Möbius transformations and conformal geometry, metric geometry and Gromov hyperbolicity… In addition, the book features a couple of "bonus chapters": on Einstein's theory of relativity and its connection to hyperbolic geometry, and on the applications of hyperbolic geometry to data science and machine learning.

Approach and style. The approach aims to be clean and rigorous, using the framework and style of modern mathematics, although historical aspects are occasionally mentioned. (For other subjects usually covered in mathematics textbooks, this would go without saying, but hyperbolic geometry is special given the historical weight of the "synthetic" approach.) After the first two introductory chapters, the book develops the formal concepts that allow the most effective definitions of hyperbolic spaces, such as pseudo-Euclidean spaces and projective spaces. Readers are therefore expected to be able to handle a certain level of abstraction.

Why learn hyperbolic geometry? I see at least three excellent reasons for students in mathematical sciences to learn hyperbolic geometry: (1) Since the 19th century, non-Euclidean geometry has become a standard framework in mathematics and physics. Hyperbolic geometry is the star of non-Euclidean geometries, and gives fundamental insight on all phenomena related to negative curvature. (2) A course in hyperbolic geometry is a great opportunity to learn a diversity of classical geometric notions that are useful across many areas of mathematics. In this book, you will (re-)discover bits of Riemannian geometry, relativity theory, real and complex projective geometry, and more. (3) In contemporary mathematical research, hyperbolic geometry is at the intersection of several important fields: geometry and topology, group theory, complex geometry, and others. I refer to [Can+97, §15] for a discussion of this. (4) Bonus reason! Hyperbolic geometry shows promising possibilities for data science and machine learning: this is the content of Part VII. It could therefore appeal to students who aspire to be data scientists or engineers.

What does this book not cover? This remains to be written.

Prerequisites. No prerequisites are assumed beyond a (solid) standard undergraduate curriculum in mathematics, including linear algebra and multivariable calculus. Students who have an additional background in geometry (such as differential geometry of curves and surfaces, Riemannian geometry, projective geometry, etc.) will nevertheless be able to put their prior knowledge to excellent use. In contrast to the course that I taught at Rutgers University, I elected not to include a mini-course on Riemannian geometry beyond the introductory

Chapter 2. As a consequence, the book focuses more on classical geometric aspects than differential ones, despite my personal inclination for the latter.

Exercises. Each chapter is concluded with a list of exercises. Some solutions and hints are included at the end of the book, but I recommend resisting the temptation to look at them as long as possible. It is essential that students really work on the exercises. No serious mathematics can truly be learned without asking and answering many questions and solving problems. Spending this time and effort cannot be spared, but it makes it more fun and often more effective to work with other students or a teacher.

Other references. There are already a number of excellent works on non-Euclidean geometry, hyperbolic geometry, and related topics. However, most books have a different focus or level from ours. (In fact, the absence of a similar textbook to this one is the reason I wrote my own lecture notes to begin with.) Below I attempt a brief review of the literature that I am aware of, especially the references that I recommend. Of course, this is not an exhaustive list.

Historical disclaimer. The historical considerations at the beginning of this preface are obviously very rough. The first chapter is supposed to provide some foundational background, but it does not try to meet any serious historical or epistemological standards—this book only claims to meet serious mathematical standards! Nonetheless, I wholeheartedly encourage curious students to explore the history and epistemology of mathematics, and hyperbolic geometry is a perfect topic for that (I will suggest some references in Chapter 1). I take this opportunity to express my gratitude to Alain Herreman, whom I was honored to have as a professor at the university of Rennes, and whose fantastic lectures on the history and epistemology of mathematics made an everlasting impact on me.

How did I write this book? This book was written with LaTeX, an extension of the TeX typesetting system. Technically, I run a `TeX Live` installation on the operating system `Debian GNU/Linux`, and I use the LaTeX editor `Kile`. To create figures, I have worked with the vector graphics editor `Inkscape`, the `Python` package `matplotlib`, and the LaTeX package `Tikz`. To keep track of my files, I use the version control system `Git`. (There is nothing original about these choices!) Most if not all the sofware that I use is *free* (as in "free speech", not as in "free beer") and open-source, not just because I support it philosophically, but also because it is often the best. Like many others, I am indebted to Donald Knuth, Richard Stallman, Linus Torvalds, and all other free-spirited enthusiasts and talented programmers who have made and continue to make free sofware a reality.

Acknowledgments. I am grateful to TU Darmstadt for a welcoming environment, especially to Karsten Große-Brauckmann for his kindness and many fruitful conversations. I am deeply grateful to Athanase Papadopoulos for his valuable advice and his suggestion to publish my lecture notes as a book—it would not have occurred to me otherwise! I also thank the Springer editor Elena Griniari for her kind encouragements and patience. As this is the first book that I write, I would like to thank all people, from the great mathematicians of the past

to all my teachers, comrades, and colleagues, who have played a part in my love of science and mathematics. Finally, I have been blessed with loving family, friends, and my amazing husband Benjamin; I am forever thankful for their unconditional love and support.

To all readers: learn and enjoy; write me! I sincerely hope that you find this book instructive and that you find joy and beauty in learning hyperbolic geometry. I absolutely appreciate all feedback: please contact me with any mathematical or non-mathematical questions, reports of mistakes and typos, suggestions for improvement, criticisms, and more. The best way to contact me is by e-mail: `brice@loustau.eu`. Thank you!

Brice Loustau
Heidelberg, June 2021

Contents

Part I: Preliminaries

You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of the parallels alone…I thought I would sacrifice myself for the sake of truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labors […] I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time.

– Farkas Bolyai to his son János in 1820, on Euclid's parallel postulate[2]

I have discovered such wonderful things that I was amazed, and it would be an everlasting piece of bad fortune if they were lost. When you, my dear Father, see them, you will understand; at present I can say nothing except this: that out of nothing I have created a strange new universe.

– János Bolyai's response to his father in 1823[3]

[2]Quoted from [Gra04].
[3]Quoted from [Gra04].

CHAPTER 1

From Euclid to hyperbolic geometry

In this first preliminary chapter, we propose a brief introduction to hyperbolic geometry that puts the emphasis on the axiomatic approach going back to Euclid's *Elements*.

Note that in the remainder of the book, we will neglect the historical (and the "synthetic") approach to hyperbolic geometry in favor of a modern and effective mathematical treatment. It will therefore be harmless to forget the contents of this chapter for the most part, although we will sporadically refer back to it for insight.

* * *

The discovery of non-Euclidean geometries in the 19th century was one of the most significant developments in the history of mathematics and had a profound impact on science and philosophy. This is well put by M. Greenberg [Gre93]:

> *Most people are unaware that in the early nineteenth century a revolution took place in the field of geometry that was as scientifically profound as the Copernican revolution in astronomy and, in its impact, as philosophically important as the Darwinian theory of evolution*[1]*. "The effect of the discovery of hyperbolic geometry on our ideas of truth and reality has been so profound," wrote the great Canadian geometer H. S. M. Coxeter, "that we can hardly imagine how shocking the possibility of a geometry different from Euclid's must have seemed in 1820." Today, however, we have all heard of the space-time geometry in Einstein's theory of relativity. [...]*

There are many excellent works discussing the fascinating discovery of non-Euclidean geometry and its deep historical, mathematical, and philosophical implications. For instance, I suggest checking out [Cox98; Gra11; Gre93; Mil82; Ros88; Sti96; Tru08].

The goal of this chapter is not to compete with these books—the outcome would be embarrassing. Instead, we will content with just enough background to convey a decent

[1] It is my impression that while these parallels are compelling, they should not be taken too seriously from an epistemological perspective. The Copernican revolution (1543) was part of the "Scientific Renaissance", and a classical—arguably "Euclidean"—theory. Hyperbolic geometry occurred in the romantic 19th century, which broke classical rules in art and science. A parallel with Einstein's theory of relativity would seem more pertinent to me.

sense of the "origin story" of hyperbolic geometry, without attempting to be historically precise or thorough. That being said, I absolutely encourage eager readers to explore these references and more; the subject is fascinating and great insight is to be gained.

1.1 Euclid's postulates

The long history leading up to the discovery of hyperbolic geometry originates in the *Elements* of Euclid[2]. This treatise of mathematics divided in 13 books was written in ca. 300 BC by the mathematician Euclid of Alexandria[3]. The *Elements* were the used as the main reference for geometry and more generally for mathematics until the 19th century. For this reason, it can hardly be disputed that it is the most important work of mathematics ever written.

Euclid's method is axiomatic and constructive. This approach is far from outdated, on the contrary: the foundation of contemporary mathematics, as it has been formalized since the first half of the 20th century with mathematical logic, is strikingly similar to Euclid's Elements. We discuss this more in § 1.3.

Euclid introduces 5 ***postulates*** (axioms):

(E1) There exists a line through any two points.
(E2) Any line may be extended indefinitely.
(E3) Any center and radius determines a unique circle.
(E4) All right angles are congruent. (See Figure 1.1.)
(E5) If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles. (See Figure 1.2.)

Remark 1.1. By ***line***, Euclid means *straight line segment*. He does not directly consider infinite lines, which is a very reasonable position. By two lines being ***parallel***, one must understand: they do not intersect, even when extended indefinitely.

Remark 1.2. Note that Euclid does not state uniqueness in the first postulate (E1). In particular, it does not exclude spherical geometry.

Remark 1.3. Let us interpret/explain the fourth postulate (E4): it means that if l_1 and l_2 are infinite lines intersecting at a point A at a right angle, and l_1' and l_2' are infinite lines intersecting at a point A' at a right angle, then there exists a rigid motion (an orientation-preserving isometry) which takes l_1 to l_1', l_2 to l_2', and A to A'.

[2]For a translation of the *Elements* in English, Thomas L. Heath's 1908 translation is the main reference. A second edition was published by Dover in 1956 [Euc56].

[3]Despite his epithet, Euclid was Greek, like his predecessors Thales and Pythagoras; he lived in Egypt under the reign of Ptolemy I, shortly after it was conquered by Alexander the Great. That being said, the contribution of the ancient Egyptians to Greek mathematics is often underestimated, as is shown in the wonderful article [Her18], which I recommend reading for a taste of a serious historical approach!

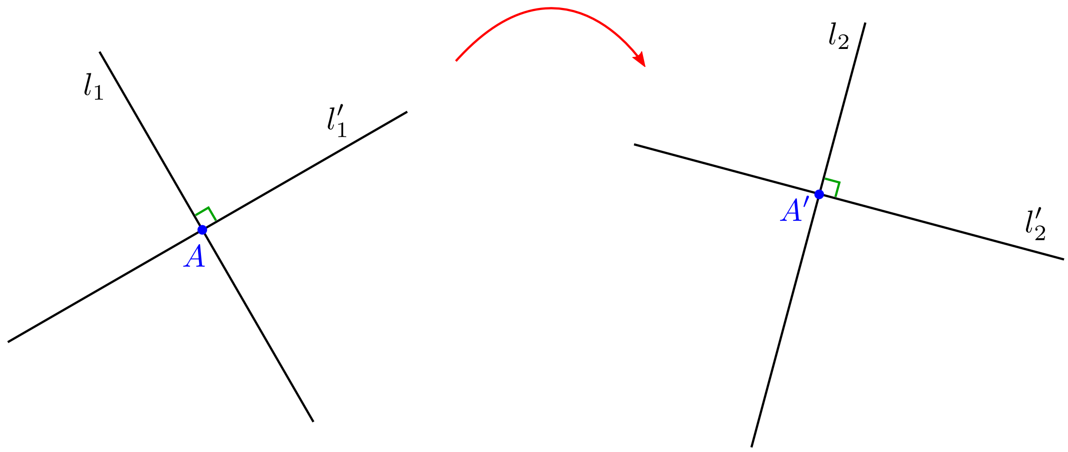


Figure 1.1: Euclid's fourth postulate: Any two right angles are congruent.

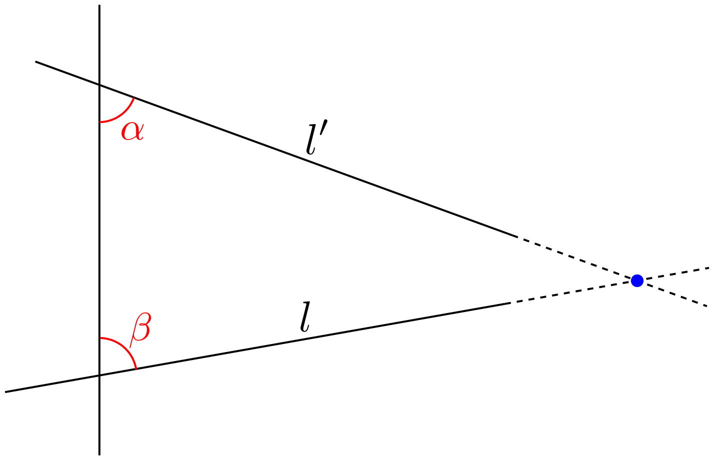


Figure 1.2: Euclid's fifth postulate: If $\alpha + \beta < \pi$, then l and l' intersect on the side of α and β.

Based on these five postulates (and five "common notions"), Euclid develops an extensive treatise of mathematics (geometry and number theory). It is divided in thirteen books, consisting of a collection of definitions, constructions, theorems, and proofs. For instance, the first book contains the Pythagorean theorem and the sum of the angles in a triangle, as well as many other constructions of plane geometry.

1.2 Discovery of non-Euclidean geometry

For centuries, mathematicians have questioned the fifth postulate, often called the ***Parallel postulate***. See Figure 1.2 for an illustration. This postulate sounds more convoluted than the first four. Could it not simply be derived from them?

Today, the parallel postulate often stated into the equivalent form:

(E5') Given a line and a point not on it, there exists a unique parallel through the point.

There are many other equivalent formulations of the fifth postulate, such as: the sum of the angles of any triangle is equal to two right angles.

Until the 19th century, mathematicians were unable to prove whether the fifth postulate was a consequence of the first four or not. The 19th century was the century of romanticism, which decided that classical rules should be broken. A breakthrough was achieved by Lobachevsky (and Gauss, Bolyai, Taurinus, Cayley, and others)[4]. He constructed a complete alternative to Euclidean geometry, starting from the assumption that the first four postulates are true, but the fifth postulate is false. Initially, the goal of this strategy was to reach a contradiction, which would prove that the fifth postulate does derive from the first four. However, it eventually became clear that this new geometry was as respectable and beautiful as Euclid's.

Remark 1.4. Spherical geometry also offers an alternative to non-Euclidean geometry. This is the geometry on a sphere, where straight lines are arcs of great circles. Note that it does not satisfy the first axiom of Euclid if we add uniqueness to straight lines through two points: consider antipodal points. In fact, antipodal points are especially problematic because there is an infinity of straight lines between them. One can remedy this issue by identifying any two antipodal points. The resulting surface is known as the real projective plane, equipped with the geometry inherited from the sphere. This geometry is called ***elliptic geometry***, and is the only other non-Euclidean geometry besides hyperbolic geometry. The fifth postulate for elliptic geometry reads: Any two lines intersect (i.e., there are no parallels). This case must be excluded to obtain hyperbolic geometry, therefore the fifth postulate for hyperbolic geometry reads:

(H5) Given a line and a point not on it, there exists at least two parallels through the point.

As an example, Lobachevsky developed the notion of ***angle of parallelism***: given a line l and a point A at distance a from l, the angle of parallelism α is the least angle such that the line l' as in Figure 1.3 is parallel to l (i.e. does not intersect l).

It is important to note that Lobachevsky, despite writing a considerable treatise of hyperbolic geometry *à la Euclid*, did not answer the question of whether Euclid's fifth postulate is independent of the first four. Indeed, it was still possible that Lobachevsky's geometry was inconsistent, and that he simply did not yet find a contradiction. The same can be said of the work of Gauss, Taurinus, and Bolyai.

The question was definitively settled by Beltrami in 1868 [Bel68a; Bel68b], who found a *model*—in fact several models—for the hyperbolic plane, in other words a "universe" where the axioms of hyperbolic geometry are satisfied.

The first model proposed by Beltrami is now known as the Beltrami–Klein model (or

[4]Hyperbolic geometry is still occasionally called ***Lobachevsky geometry***. Lobachevsky was the first to publish his extensive work in 1829, but by then other mathematicians had also discovered in part this new geometry: Gauss, Schweikart and Taurinus, Farkas and Janos Bolyai.

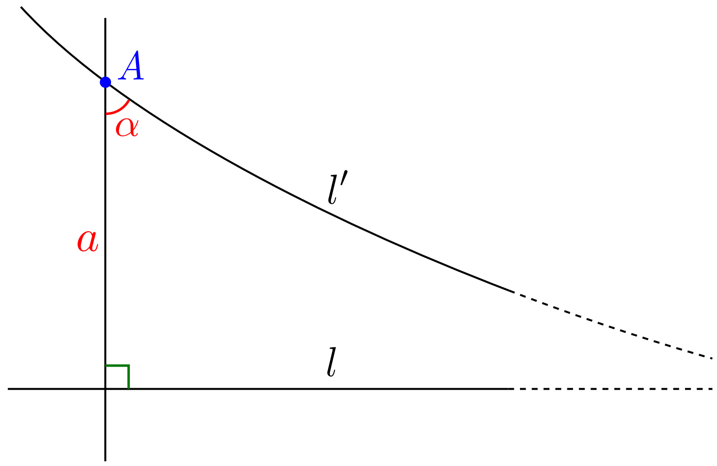


Figure 1.3: Angle of parallelism.

Cayley–Klein model, or simply Klein model[5]). We will study it in detail in Chapter 7, but its description is surprisingly simple: the hyperbolic plane is an open Euclidean disk; points in this model are points inside the disk, and lines are chords, i.e. straight line segments with (imaginary) endpoints on the boundary circle. See Figure 1.4. However, angles and distances are not as they appear to our Euclidean eyes. In particular, until we define angles, distances, and isometries, we cannot verify Euclid's axioms 3. and 4.

Beltrami also proposed a second pair of models, which are now known as the Poincaré disk and half-space models[6]. The disk model is again an open Euclidean disk, but this time lines are defined as circles of arcs that are orthogonal to the boundary circle. See Figure 1.5. Distances are also distorted in this model with respect to our Euclidean eyes, but not angles: it is a ***conformal*** model. We will study this model in Chapter 9.

1.3 Notions of mathematical logic

Let us reconsider the previous historical discussion in the eyes of mathematical logic. I warn the reader that what follows is a naive presentation: Euclid's system does not actually meet the requirements of a theory as it is defined by first-order logic. The axiomatic foundation of geometry has generated considerable work since the late 19th century; notable modern axiomatizations of Euclid's theory were proposed by Hilbert (1899), Birkhoff (1932), Tarski (1959).

A mathematical theory is based on a syntax, axioms, and rules of inference, from which

[5]Klein ([Kle71; Kle73]) showed the projective nature of Beltrami's model and gave the formula for the metric in terms of cross-ratios, inspired by work of Cayley [Cay59]. For a more detailed historical account, refer to [AP15].

[6]Poincaré rediscovered the disk and half-plane models in 1882 and revealed the connection between 2-dimensional hyperbolic geometry and complex geometry [Poi82].

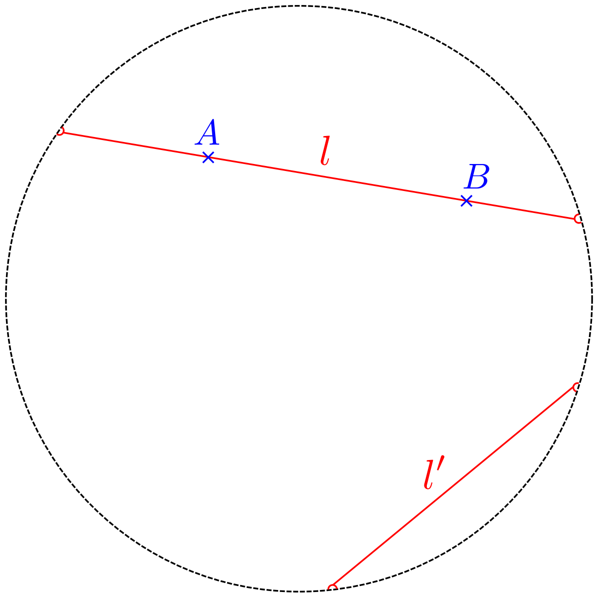


Figure 1.4: Points and lines in the Beltrami–Klein model.

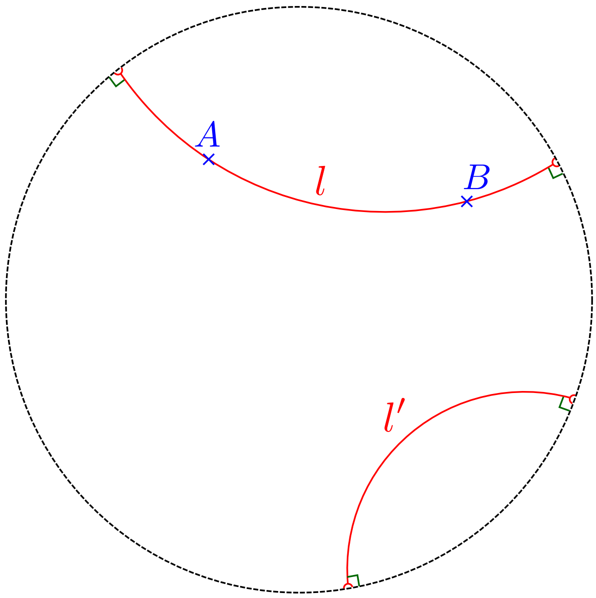


Figure 1.5: Points and lines in the Poincaré disk model.

theorems are derived (also cosmetically called lemmas, propositions, corollaries, etc). The majority of contemporary mathematics implicitly uses the theory of sets of Zermelo–Fraenkel, but other setups can also be relevant. Regardless, Euclid's treatise and its axiomatic approach appears singularly modern.

Ideally, the axioms that one chooses to base a mathematical theory should have the following qualities:

(1) ***Consistency***: No two axioms are incompatible; more generally, no contradiction can be derived from the axioms.
(2) ***Completeness***: Any mathematical statement that makes sense in the theory should be either provable or disprovable.
(3) ***Independence***: No axiom should be a consequence of the others.

Clearly, the most important quality is consistency: an inconsistent theory is worthless. Completeness is less essential, but a theory feels imperfect without it. In theory, independence is not an important quality, but it is the question of the independence of Euclid's axioms that led to the discovery of hyperbolic geometry!

Let us go back to discussing the independence of Euclid's fifth postulate. Beltrami's work shows that, assuming there exists a model for Euclidean geometry (the Euclidean plane!), one can construct a model where the axioms of hyperbolic geometry, namely (E1)–(E4) and (H5), are satisfied. In particular, he created a model for Lobachevsky's geometry. From the point of view of logic, Beltrami's model shows that hyperbolic geometry is consistent if Euclidean geometry is. This is a direct consequence of Gödel's completeness theorem:

Theorem 1.5 (Gödel's completeness theorem)**.** A theory is consistent if and only if it has a model.

Consequently, assuming Euclidean geometry is consistent, Euclid's fifth postulate cannot be a consequence of the first four.

Let us mention that in general, achieving/proving consistency and completeness of a theory in first-order logic is tragically elusive. The celebrated incompleteness theorems of Gödel say that

(1) It is impossible to prove the consistency of a theory; not unless one includes it in a larger theory that is assumed consistent.
(2) A theory is never complete.

However, Gödel's incompleteness theorems only apply to theories containing arithmetic. In particular, they do not apply to Euclidean geometry. It has been proven that (a modern axiomatization of) Euclidean geometry is in fact consistent and complete: refer to [Mat17] as a starting point to seek more information about this.

1.4 Exercises

Exercise 1.1. Beltrami–Klein disk and Poincaré disk

(1) Prove that Euclid's postulate (E1) holds in the Beltrami–Klein disk.

For now, we cannot really discuss postulates (E2), (E3), and (E4), because we have yet to define distances, angles, and isometries in this model, but we will see that they also hold.

(2) Show that Euclid's postulate (E5) does not hold in the Beltrami–Klein disk.

(3) Repeat the exercise with the Poincaré disk.

Exercise 1.2. Triangles in the Poincaré disk

We recall that the Poincaré disk model is *conformal*: the angles between two lines (or curves) from the point of view of hyperbolic geometry is the same as their Euclidean angle (i.e., the angle between the tangents).

(1) Draw a right-angled triangle in the Poincaré disk.

(2) Show that in the Poincaré disk, the sum of angles in a triangle is always less than π. Argue that over all nondegenerate triangles, the sum ranges in the interval $(0, \pi)$.

Exercise 1.3. Independence of Euclid's fifth postulate

Using Gödel's theorem, explain carefully why Beltrami's models for the hyperbolic plane show that hyperbolic geometry is no less consistent than Euclidean geometry. Conclude that if Euclidean geometry is consistent, then Euclid's fifth postulate is independent from the first four.

Remark: This exercise, just like the presentation of § 1.3, is naive: it implicitly assumes that Euclid's system meets the requirements of a theory as defined by first-order logic, where Gödel's theorem applies. This is not quite the case.

CHAPTER 2

The notion of curvature

In this second chapter, we propose a semi-formal introduction to the concept of curvature. This is another preliminary chapter, but it is important to study it because it includes definitions and propositions that will be used in subsequent chapters. In particular, a few useful notions of Riemannian geometry are introduced, without assuming any knowledge of differentiable manifolds.

Roughly speaking, curvature measures how much a geometric object—such as a curve, a surface, or a higher-dimensional object—deviates from being flat, in other words Euclidean. Exploring and developing this idea unveils substantial and beautiful mathematics. This is what this chapter attempts to do, although superficially in order to avoid getting overwhelmed by technical details or theoretical obstacles.

The main protagonist of this book, hyperbolic space, is the model geometric object of constant negative curvature. *One* of the goals of this course is to prove this fact in several ways, to derive some consequences, and beyond: to acquire a fairly deep understanding of the features of a negatively curved "world".

* * *

The flow of the chapter is quite straigthforward: we begin with the curvature of space curves in § 2.1, then surfaces § 2.2, and work our way towards more generality in § 2.3. We conclude with the model spaces of constant curvature in § 2.4 and a brief mention of curvature in metric spaces in § 2.5.

Prerequisites and references. Having some knowledge of differential geometry will make it far easier to read this chapter, but it is not a prerequisite. We only assume a solid background in Euclidean vector spaces and multivariable calculus, although we will recall many basic definitions. For readers looking to properly learn Riemannian geometry, I can recommend the great books of Jack Lee ([Lee18], preceded by [Lee11] and [Lee13]), I also like [GHL04] preceded by [Laf15]. For "elementary differential geometry" (curves and surfaces in Euclidean space), do Carmo [Car16] is a good option, but there are many others. Spivak's books [Spi99] are a thorough reference for differential geometry, though more advanced.

2.1 Curvature of space curves

Curves are the one-dimensional objects of differential geometry. A curve typically "lives" in an ambient space, such as the Euclidean plane, three-dimensional Euclidean space, or more general geometric objects (surfaces, higher dimensional manifolds, metric spaces, etc.). It is even possible to consider "abstract" curves, that do not live in any ambient space.

However, the curvature of a curve is a notion relative to the space in which it lives; abstract curves do not any have curvature in any reasonable sense[1]. In this section, we discuss the curvature of "space curves", i.e. curves in three-dimensional Euclidean space.

2.1.1 Basic definitions

Consider three-dimensional Euclidean space $E = \mathbb{R}^3$. More generally, we could take for E any Euclidean vector or affine space. Let us recall that a ***Euclidean vector space*** is a finite-dimensional vector space equipped with an inner product, and a ***Euclidean (affine) space*** is an affine space modelled on a Euclidean vector space.

Let γ be a ***smooth (parametrized) curve*** in E. By definition, this is a smooth map $\gamma\colon I \longrightarrow E$, where $I \subseteq \mathbb{R}$ is an interval of the real line (that is nonempty and not reduced to a point). In this book, we shall use ***smooth*** as an alias for "of class $\mathcal{C}^\infty$". The fact that we will always assume our curves and other differential geometric objects to be of class $\mathcal{C}^\infty$ is mainly a lazy habit; for instance it would just fine to work with curves of class $\mathcal{C}^2$ or $\mathcal{C}^3$.

The ***velocity*** of the curve γ at a time $t \in I$ is the derivative $\overrightarrow{\gamma'(t)} \in \mathbb{R}^3$. Note that when E is a Euclidean affine space, the velocity $\overrightarrow{\gamma'(t)}$ is not an element of E, but of the associated vector space. This being understood, we shall drop the arrow over $\gamma'(t)$, which is a useful notation but a bit cumbersome. Using the inner product of our vector space, one can measure the norm $\|\gamma'(t)\|$, called the ***speed*** of γ.

In many situations, one is not really interested in the parametrized curve γ itself, which is a map $I \longrightarrow E$, but only in its image, which is a subset of E. It is common, although not very rigorous, to say "the curve γ" to refer to either. One can always ***reparametrize*** a curve without changing its image: this consists in putting $\tilde{\gamma}(s) := \gamma(t)$ where $t = \varphi(s)$ is a change of variables given by a function $\varphi\colon J \longrightarrow I$ (which is assumed smooth, increasing, and bijective). Note that changing the parametrization of a curve does not change its image, but it does change its velocity and speed.

An example of quantity that is independent of parametrization is the ***length*** of a curve: by definition, this is the integral of its speed $l(\gamma) := \int_I \|\gamma'(t)\| \,\mathrm{d}t$. The fact that $l(\gamma)$ is left unchanged by a reparametrization is an immediate application of the change of variables theorem for integrals.

The parametrized curve γ is called ***regular*** if its velocity (equivalently its speed) never

[1] Rather, any sensible definition of curvature should imply that curves have vanishing intrinsic curvature. One way to convince oneself of this is to realize that any metric curve is locally isometric to a Euclidean line (the arclength parameter provides a local isometry to $\mathbb{R}$), which is flat by definition.

vanishes. It is always possible to parametrize a regular curve ***by arclength***: this means that γ has unit speed (constant speed equal to 1). In this situation, the parameter is usually denoted s and called ***arclength parameter*** . This name comes from the fact that the arclength parameter is unique up to addition of a constant, and given by $s = \int_{t_0}^{t} \|\gamma'(u)\| \, \mathrm{d}u$, in other words s is the length of the curve γ between a fixed time t_0 and the time t.

Example 2.1. A ***circle*** in the Euclidean plane $E = \mathbb{R}^2$ is the set of points at a distance $R > 0$ (the ***radius***) from some fixed point $\Omega = (x_0, y_0)$ (the ***center***). It can be parametrized by $\gamma(t) = \big(x_0 + R\cos(\omega(t - t_0)), y_0 + R\sin(\omega(t - t_0))\big)$ where ω and t_0 are real constants. This parametrization has constant speed $v = R\omega$, in fact any constant speed parametrization is of this form (up to reversing time).

More generally, in a Euclidean space E, the set of points at distance $R > 0$ from a point $\Omega \in E$ is a ***sphere***, and its intersection with any affine plane P going through Ω is a circle of center Ω and radius R. This circle is parametrized by $\gamma(t) = \Omega + R\cos(\omega(t - t_0))\vec{i} + \sin(\omega(t - t_0))\vec{j}$, where $(\vec{i}, \vec{j})$ is an orthonormal basis of the vector space underlying P.

2.1.2 Curvature

Let γ be a smooth regular curve. Without loss of generality, one can assume that γ is parametrized by arclength, in other words has unit speed. The following proposition is elementary but conceptually important:

Proposition 2.2. *A curve γ parametrized by arclength is a straight line if and only $\gamma'' = 0$.*

In the language of differential geometry, a straight line parametrized by arclength (or more generally by constant speed) is called a ***geodesic***. The lemma above thus says that geodesics are curves with vanishing acceleration, a characterization that holds in great generality (for Riemannian manifolds).

When γ is any regular curve, still parametrized by constant speed, its acceleration is always normal to the curve. Indeed, taking the derivative of the identity $\|\gamma'(s)\|^2 = 1$ yields $2\langle \gamma'(s), \gamma''(s)\rangle = 0$, and since $\gamma'(s)$ is non-null and tangent to the curve, $\gamma''(s)$ is normal.

Informally speaking, the direction of $\gamma''(s)$ shows in which direction the curve is turning, and its norm indicates how fast it is turning (this is consistent with the preceding lemma, at least). The next definition thus sounds reasonable:

Definition 2.3. Let γ be a smooth space curve parametrized by arclength. The ***curvature*** of γ is the function $\kappa(s) := \|\gamma''(s)\|$.

Remark 2.4. This notion of curvature is *extrinsic*, in the sense that depends on how the curve is embedded in Euclidean space. If one moves the curve in space, even without stretching it, its curvature will change. We shall see that for surfaces, it is possible to define an *intrinsic* curvature. One observation that holds in great generality is that curvature has to do with second-order derivatives.

While this definition of curvature is extremely simple analytically, it is possible to give a more geometric interpretation of it. First observe that, when γ is a circle, its curvature is equal to the inverse of the radius (this is an easy calculation, given the parametrization given in Example 2.1). This makes sense: the smaller the radius, the sharper the turn.

It is possible to extend this interpretation to any regular curve γ by introducing the notion of ***osculating circle***. By definition, the osculating circle at $\gamma(t_0)$ is the circle having best contact with γ at t_0. More precisely, it is the unique circle parametrized by constant speed such that $c(t_0) = \gamma(t_0)$, $c'(t_0) = \gamma'(t_0)$, and $\gamma''(t_0) = c''(t_0)$. We leave as an easy exercise to the reader to check that this circle is indeed uniquely defined. The radius $R(t_0)$ of the osculating circle is called the ***radius of curvature***. See Figure 2.1.

Remark 2.5. The osculating circle at $\gamma(t_0)$ is not well-defined when $\gamma''(t_0) = 0$, strictly speaking. Note however that the tangent line to the curve has second-order contact with the curve in this situation. One can therefore consider that this line is the osculating "circle", and that it has infinite radius $R(t_0) = +\infty$.

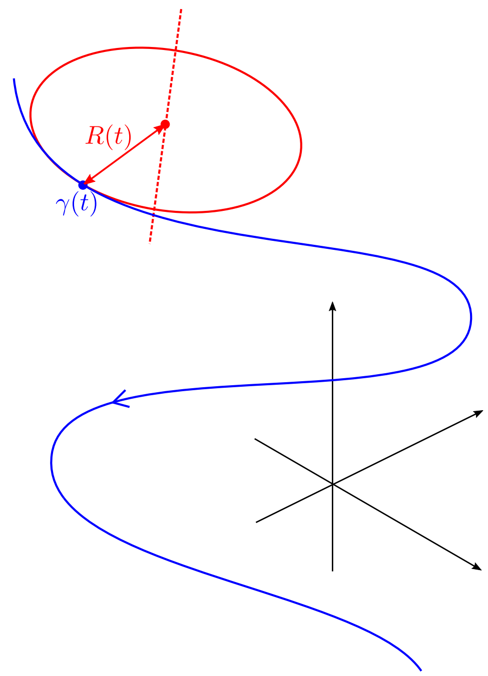


Figure 2.1: Osculating circle and radius of curvature.

Having set everything up, the following proposition should be fairly clear:

Proposition 2.6. *Let γ be a smooth regular curve. The curvature of γ is the inverse of its radius of curvature:*

$$\kappa(t) = \frac{1}{R(t)}$$

Proof. Since the circle of curvature does not "see" the parametrization of γ, we can assume that γ is parametrized by arclength. Let $s_0 \in I$. By definition, $\kappa(s_0) = \|\gamma''(s_0)\|$. Let $c(s)$ be the osculating circle to γ at s_0. We know that the curvature of a circle is equal to the inverse of its radius, therefore $\|c''(s_0)\| = \frac{1}{R(s_0)}$. Since $c''(s_0) = \gamma''(s_0)$, we conclude that $\kappa(s_0) = \frac{1}{R(s_0)}$. □

For completeness, we could further introduce the ***Frenet–Serret frame*** (T, N, B) , the notion of ***torsion*** , and mention the "fundamental theorem of space curves" (curves are determined by their curvature and torsion). This would not be require much more work but we shall not need these notions; out of interest, they are discussed in Exercise 2.1.

2.2 Curvature of surfaces

Let now $S \subseteq \mathbb{R}^3$ be surface. To be accurate, this means that S is a smooth 2-dimensional embedded submanifold of $\mathbb{R}^3$; there are several equivalent definitions of what this means, but let us not worry about these details.

Tangent plane. At any point $p \in S$, there is a tangent plane to the surface $\mathrm{T}_p S \subseteq \mathbb{R}^3$, which is an affine plane. There are many equivalent definitions of it; one possible way to think about tangent vectors $\vec{u} \in \mathrm{T}_p S$ is that they are the velocities of smooth curves $\gamma \colon I \to S$ (i.e. smooth curves $\gamma \colon I \to \mathbb{R}^3$ whose image is in S).

Unit normal. One can also define a ***unit normal*** $\vec{N}_p$ to the surface at p: it is a unit vector that is orthogonal to $\mathrm{T}_p S$. There are two choices for the unit vector $\vec{N}_p$. Locally, one can always make a consistent choice at all points near p (so that the map $p \mapsto \vec{N}_p$ is continuous). Globally, one can make a consistent choice for $\vec{N}$ if and only if the surface is orientable.

Geodesics. Among curves in S, the most special are ***geodesics***. Intuitively, a geodesic is easy to define: imagine that you have a little car toy, whose wheels are powered by a battery that never runs out. You can initially set the speed of rotation of the wheels, and never change it afterwards. Also, the car wheels are always straight, it never turns. If you release this car on a plane, its trajectory will be a straight line, parametrized by the car. More generally, if you release this car on a surface, it will define a geodesic. This is really the right way to think about geodesics: they are parametrized curves with constant intrinsic velocity, i.e. zero intrinsic acceleration. Of course, formalizing all this requires some work, which we skip. This description makes the following proposition intuitively obvious:

Proposition 2.7. *For any $\vec{v} \in \mathrm{T}_p S$, there exists a unique geodesic in S through p with initial tangent vector $\vec{v}$.*

We will denote this geodesic $\gamma_{\vec{v}}$. The following proposition gives a possible alternative definition for geodesics:

Proposition 2.8. *A curve γ on S is a geodesic if and only if it has constant speed and is locally length minimizing.*

Precisely, being locally length minimizing means that every $t_0 \in I$ and for every t_1 sufficiently close to t_0, the length of γ between t_0 and t_1 is minimal along all curves from t_0 to t_1.

We will make good use of the following proposition:

Proposition 2.9. *Let $f\colon S \longrightarrow S$ be an isometry (e.g., induced by an isometry of $\mathbb{R}^3$). Let $F \subseteq S$ denote a connected component of the fixed point set of f. If $v \in \mathrm{T}_p S$ is tangent to F, then the (image of) whole geodesic γ_v is contained in F.*

Proof. Consider the curve $\hat{\gamma}_v := f \circ \gamma_v$. Since f is an isometry, $\hat{\gamma}_v$ is also a geodesic. Moreover, since F is fixed by f, tangent vectors to F are fixed by $\mathrm{d}f$. It follows that $\hat{\gamma}_v'(0) = \mathrm{d}f(\gamma_v'(0)) = \mathrm{d}f(v) = v$. By uniqueness of the geodesic with initial velocity v, we conclude that $\hat{\gamma}_v = \gamma_v$. This shows that γ_v is contained in the fixed set of F, and one concludes by connectedness of γ_v. □

Remark 2.10. Proposition 2.9 holds more generally in any Riemannian manifold: the proof is the same.

Length of curves. Note that on S, one can measure the length of any curve $\gamma\colon I \longrightarrow S$: it is simply its length as a curve in $\mathbb{R}^3$.

Intrinsic metric (first fundamental form). Note that since the velocity of γ is always tangent to S, the length of curves in S only depends on the restriction of the inner product of $\mathbb{R}^3$ to the tangent planes to S. This data, the assignment $p \in S \mapsto g_p$ where $g_p x$ is the inner product on $\mathrm{T}_p S$, is called the ***intrinsic Riemannian metric*** on S, or ***first fundamental form***.

Remark 2.11 (Comment on the word "intrinsic"). Let $f\colon S \longrightarrow S'$ be an (Riemannian) isometry between surfaces in $\mathbb{R}^3$. By definition, this means that at any $p \in S$, the differential $\mathrm{d}f$ is a linear isometry between the Euclidean planes $(\mathrm{T}_p S, g_p)$ and $(\mathrm{T}_{f(p)} S, g'_{f(p)})$. This easily implies that f (locally) preserves lengths of curves, in particular f is a (local) metric isometry. A notion relative to surfaces is called ***intrinsic*** if, for any isometry $f\colon S \longrightarrow S'$, the notion on S' coincide with its transport from S to S' using f. One quickly see that the first fundamental form being intrinsic is essentially a tautology.

Extrinsic curvatures. We would like to define the ***extrinsic curvature*** of S at p in the direction $\vec{v} \in \mathrm{T}_p S$ as the curvature at p of the geodesic $\gamma_{\vec{v}}$ (see Remark 2.13). The only problem is that this does not have a sign, or rather it is always nonnegative. However, given a choice of unit normal $\vec{N}$, one can choose the sign as follows: we decide that the extrinsic curvature is positive if $\gamma''(0)$ and N have same direction ($\vec{N}$ points towards the center of the osculating circle), and is negative if they have opposite directions ($\vec{N}$ points away from the center of the osculating circle). For example, Figure 2.2 illustrates an negative extrinsic curvature. NB: It is a consequence of γ being a geodesic that its acceleration is always normal to the surface (by definition, a geodesic has vanishing intrinsic acceleration, which means that the orthogonal projection of the acceleration to the tangent space of the surface vanishes).

This definition of the extrinsic curvature, while theoretically right (see Remark 2.13), is not very practical because it is generally not easy to find the geodesic $\gamma_{\vec{v}}$ explicitly. Thankfully

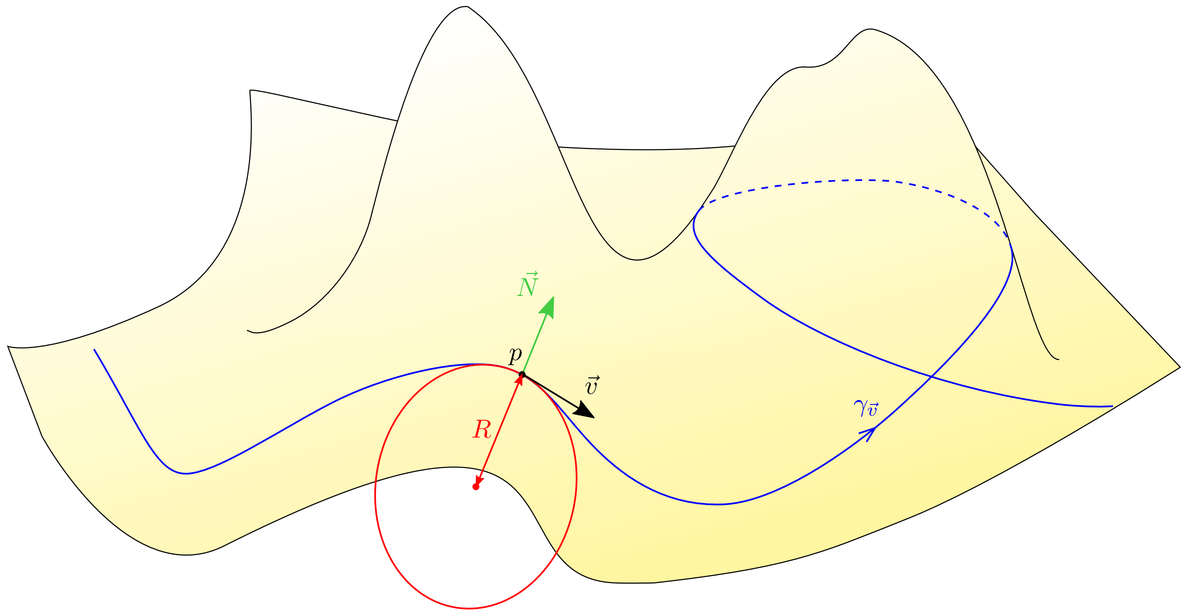


Figure 2.2: Given $p \in S$ and $\vec{v} \in \mathrm{T}_p S$, there is a uniquely defined geodesic $\gamma_{\vec{v}}$. In this example, $\gamma''_{\vec{v}}(0)$ and $\vec{N}$ have opposite directions (i.e. $\vec{N}$ points away from the center of the osculating circle), so the extrinsic curvature of S at p in the direction $\vec{v}$ is negative. It is therefore equal to $\rho(\vec{v}) := -\left\|\gamma''_{\vec{v}}(0)\right\| = -\frac{1}{R}$, where R is the radius of of the osculating circle.

there is a variation of this definition that allows straightforward calculations. Let γ be any curve in S with $\gamma'(0) = \vec{v}$. We cannot just take the curvature of γ, because that is not independent of the choice of γ. However, the quantity $\langle \vec{N}, \gamma''(0)\rangle$ is independent of γ (see Proposition 2.12) That quantity, usually called ***normal curvature of*** γ, clearly coincides with the extrinsic curvature for the geodesic $\gamma_{\vec{v}}$ since $\gamma''_{\vec{v}}(0)$ is collinear to $\vec{N}$. (Another special curve having this property is the curve $\gamma_{\vec{v},N}$ obtained by intersecting the affine plane through p spanned by $\vec{v}$ and $\vec{N}$ with S, parametrized by arclength.)

Proposition 2.12. *Let $p \in S$ and $v \in \mathrm{T}_p S$. The normal curvature $\left\langle \gamma''(0), \vec{N}\right\rangle$ is independent of the choice of the curve γ such that $\gamma(0) = p$ and $\gamma'(0) = \vec{v}$. We call it* ***extrinsic curvature*** *of S at p in the direction $\vec{v}$ and denote it $\rho_p(\vec{v})$. Moreover, it can be written:*

$$\rho_p(\vec{v}) = \left\langle \gamma''(0), \vec{N}\right\rangle = -\left\langle \nabla_{\vec{v}}\vec{N}, \vec{v}\right\rangle .$$

Proof. Since the curve γ is always in S, its velocity is always orthogonal to $\vec{N}$:

$$\langle \vec{N}_{\gamma(t)}, \gamma'(t)\rangle = 0 .$$

Differentiating this identity gives

$$\langle (\nabla_{\gamma'(t)}\vec{N})_{\gamma(t)}, \gamma'(t)\rangle + \langle \vec{N}_{\gamma(t)}, \gamma''(t)\rangle = 0 .$$

At $t = 0$, this reads $\left\langle \nabla_{\vec{v}} \vec{N}, \vec{v} \right\rangle + \langle \vec{N}_{\gamma(t)}, \gamma''(0) \rangle = 0$. □

Second fundamental form. Proposition 2.12 shows that $\rho_p(\vec{v})$ is a quadratic form of $\vec{v}$: there exists a symmetric bilinear form $B_p \colon \mathrm{T}_p S \times \mathrm{T}_p S \longrightarrow \mathbb{R}$ such that $\rho_p(\vec{v}) = B_p(\vec{v}, \vec{v})$, namely:

$$B_p(\vec{u}, \vec{v}) = -\left\langle \nabla_{\vec{u}} \vec{N}, \vec{v} \right\rangle \ .$$

B_p is the ***second fundamental form*** of S at p.

Remark 2.13. For advanced readers, let us mention that more generally, the second fundamental form can elegantly be defined as the (Riemannian) Hessian of the inclusion of a submanifold. This amounts to the definition above in terms of acceleration of geodesics. In general, the second fundamental form takes values in the normal bundle of the submanifold. Amusingly, the mean curvature is the trace of the Hessian (divided by the dimension), i.e. the "Laplacian" (also known as *tension field*) of the inclusion.

Principal curvatures, mean curvature, Gaussian curvature. By definition, the ***principal curvatures*** at p are the minimal and maximal values of the extrinsic curvatures at p in the directions of all unit vectors, attained in the respective ***principal directions of curvature***. The ***mean curvature*** $H_p \in \mathbb{R}$ (also sometimes called ***extrinsic curvature***) at p is defined as the average (half-sum) of principal curvatures, and the ***Gaussian curvature*** $K_p \in \mathbb{R}$ is the product of the principal curvatures.

A nice and immediate consequence of the spectral theorem is:

Theorem 2.14. The principal curvatures are the eigenvalues of the second fundamental form B (or rather, of the matrix of B taken in any orthonormal basis). The principal directions of curvature are orthogonal, and eigenvectors of B. The mean curvature is the trace of B, and the Gaussian curvature is the determinant of B.

A very important theorem is the *Theorema Egregium* (which roughly means "very important theorem"):

Theorem 2.15 (Theorema Egregium)**.** The Gaussian curvature is intrinsic.

In other words, if $f \colon S \longrightarrow S'$ is an isometry, then $K = K' \circ f$. We will not prove the Theorema Egregium. The other most important theorem of the theory of surfaces in $\mathbb{R}^3$ is the theorem of Gauss–Bonnet:

Theorem 2.16 (Gauss–Bonnet Theorem)**.** If S is a closed surface without boundary, then

$$\int_S K \, \mathrm{d}A = 2\pi \chi(S) \, .$$

The integer $\chi(S)$ is the Euler characteristic of S, a topological invariant. It is remarkable that the Gauss–Bonnet theorem relates the geometry and the topology of the surface.

Remark 2.17. The Gauss–Bonnet theorem holds more generally for abstract Riemannian surfaces, possibly with boundary. We will briefly see this general version in Chapter 13.

2.3 Curvature of Riemannian manifolds

2.3.1 Riemannian surfaces

Let S be a surface in $\mathbb{R}^3$. More generally, S can be any "abstract surface" (2-dimensional manifold), whatever that means—at the very least, S has a well-defined tangent space $\mathrm{T}_p S$ at any point $p \in S$. Suppose that, instead of taking the restriction of the inner product of $\mathbb{R}^3$ in $\mathrm{T}_p S$, we take any other inner product product. In other words, we choose a map g which assigns to a point p an inner product in $\mathrm{T}_p S$, and we require that g depends smoothly on p, whatever that means. (It means, for instance, that $g_p(X, Y)$ is a smooth function of $p \in S$ whenever X and Y are smooth vector fields.) Such a family of inner products on the tangent spaces of S is called a ***Riemannian metric*** on S.

An important class of examples is when $S = \Omega$ is an open subset of $\mathbb{R}^2$. In this case, $\mathrm{T}_p S$ is canonically identified to $\mathbb{R}^2$ for every $p \in S$. Therefore a Riemannian metric on S is simply a $\mathcal{C}^\infty$ map $g\colon \Omega \longrightarrow \mathrm{S}_2^+(\mathbb{R})$ where $\mathrm{S}_2^+(\mathbb{R})$ indicates the set of symmetric positive definite 2×2 matrices.

As an example, let us look at the Poincaré half-plane (we will study it in detail in Chapter 9). We take $\Omega = \{(x, y) \in \mathbb{R}^2, y > 0\}$ and $g_{(x,y)} = \frac{g_0}{y^2}$, where $g_0 = \langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$. In usual Riemannian geometry notations, this Riemannian metric is written

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} .$$

Remark 2.18 (Riemannian geometry notations). Professors or textbooks rarely take the time to carefully explain notations such as $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$; let me try to correct this bad habit.

The notation $g = \mathrm{d}s^2$ for the Riemannian metric is a customary abuse of notations. It relates to the fact that when s is an arclength parameter, $\mathrm{d}s$ is called the "line element", because it gives the length of the curve when integrated. Given any parametrization $t \mapsto \gamma(t)$ of a curve, the line element can be computed as $\mathrm{d}s = \|\vec{v}\| \, \mathrm{d}t$, where $\vec{v} = \gamma'(t)$ is the velocity. In other words, we have $\mathrm{d}s = \sqrt{g(\vec{v}, \vec{v})} \, \mathrm{d}t$. By abuse of notation, the quadratic form $\vec{v} \mapsto g(\vec{v}, \vec{v})$ and the associated symmetric bilinear form g are both denoted $\mathrm{d}s^2$.

Now let us explain the notations $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}x^2$, etc. Basically, the inner product g on $\mathbb{R}^2$ with matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ is denoted $g = \alpha \, \mathrm{d}x^2 + 2\beta \, \mathrm{d}x \, \mathrm{d}y + \gamma \, \mathrm{d}y^2$. But why?

Technically, $\mathrm{d}x$ is the derivative of the function $\Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \mapsto x$. It is the constant map on Ω with values in $\mathcal{L}(\mathbb{R}^2, \mathbb{R})$ which is always equal to e_1^*. Similarly, $\mathrm{d}y$ is the constant map equal to e_2^*. Recall that (e_1^*, e_2^*) denotes the dual basis of the canonical basis of $\mathbb{R}^2$: e_i^* is the linear form $u \mapsto u_i$.

Finally, $\mathrm{d}x \, \mathrm{d}y$ stands for the *symmetric product* of $\mathrm{d}x$ and $\mathrm{d}y$, while $\mathrm{d}x^2$ (resp. $\mathrm{d}y^2$) is the symmetric product of $\mathrm{d}x$ (resp. $\mathrm{d}y$) with itself. In general, the *symmetric product* of two linear forms α and β is the symmetric bilinear form defined by $(u, v) \mapsto \frac{\alpha(u)\beta(v)+\alpha(v)\beta(u)}{2}$.

When (S, g) is a surface equipped with a Riemannian metric, one can seamlessly develop all the same notions as before: the velocity of curves on S still make sense, so does their speed (using g), etc. In particular, geodesics are well-defined. Note however that unless we have an isometric embedding of (S, g) in a Euclidean space $\mathbb{R}^N$[2], we cannot define the Gaussian curvature like before. Nevertheless, it is possible to define the Gaussian curvature in a consistent way, so that whenever $S \longrightarrow S'$ is an isometry, $K = K' \circ f$. This can be done explicitly with formulas, but they are not very insightful.

One particular case of interest is when g is *conformally flat metric*, i.e. g is pointwise proportional to a Euclidean metric. On $\Omega \subseteq \mathbb{R}^2$, this means that $g = f g_0$ where $f = e^{2\varphi} : \Omega \longrightarrow \mathbb{R}$ is some smooth positive function, and $g_0 = \mathrm{d}x^2 + \mathrm{d}y^2$ is the Euclidean metric. In this case, the Gaussian curvature is computed by:

$$K = -e^{-2\varphi} \Delta\varphi \,.$$

(This is a particular case of the more general formula $K = e^{-2\varphi} (K_0 + \Delta_0 \varphi)$ relating the Gaussian curvatures of any two conformal metrics.)

In particular, for the Poincaré half-plane, we have $\varphi = -\log(y)$, from which we find $\Delta\varphi = \frac{1}{y^2}$, and $K = -1$. Thus we already have a "proof" that the Poincaré half-plane has constant curvature -1. We shall later see several other proofs of this fundamental feature of the hyperbolic plane.

2.3.2 Higher dimensional Riemannian manifolds

Sectional curvature

When M is a higher dimensional Riemannian manifold, the Gaussian curvature is generalized as the ***sectional curvature***. This depends on the choice of a point $p \in S$ and a 2-dimensional subspace $P \subseteq \mathrm{T}_p M$.

For instance, assume M is a submanifold of R^n. One can still equip M with a Riemannian metric by restricting the Euclidean inner product to each tangent space of M. Hence we can still measure lengths of curves, talk about geodesics, etc.

The definition of the sectional curvature is as follows. Consider the surface S_P obtained by taking all geodesics in M are tangent to P at p (their initial velocity belongs to P). Then S is a surface (for connoisseurs of Riemannian geometry: S is just $\exp_p(P)$. Just take its Gaussian curvature.

Riemann curvature tensor

The sectional curvature is very geometric, but as an mathematical object it is a bit complicated: one could say it is a real-valued function on the Grassmannian of 2-planes $\mathrm{Gr}_2 M$. It turns out that all sectional curvatures can be encoded in an object that has a concise definition and is easier to calculate: the Riemann curvature tensor. This object R is a quadrilinear map on the

[2]Such an isometric embedding always exists: this is the *Nash embedding theorem*.

tangent space: given 4 tangent vectors $v_1, v_2, v_3, v_4 \in \mathrm{T}_p M$, it assigns a number $R(v_1, v_2, v_3, v_4)$. This is a ***tensor***, i.e. is linear in $v_1, v_2, v_3, v_4 \in \mathrm{T}_p M$, moreover it has several symmetries, but let us not go into details.

The relation between the Riemann curvature tensor and the sectional curvature is that for any two vectors u and v at p, the sectional curvature of the plane spanned by u and v is

$$K(u, v) = \frac{\langle R(u, v)v, u \rangle}{\|u \wedge v\|^2}$$

where we denote $\|u \wedge v\|^2 = \|u\|^2 \|v\|^2 - \langle u, v \rangle^2$ (and $\langle \cdot, \cdot \rangle = g$ is the Riemannian metric).

It is "just" linear algebra to show that K determines R and conversely. For a differential geometer, R has an incredibly pleasant definition: it is exactly the lack of commutation of second derivatives in M. More precisely, it is the curvature of the Levi-Civita connection:

$$R(X, Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X}$$

But explaining this more precisely would take us beyond the scope of this course.

2.3.3 Taylor expansion of the metric

In a way, the curvature of a Riemannian manifold is precisely the measurement of how the Riemannian metric locally differs from the Euclidean metric to second order. This point of view is in fact faithful to Bernhard Riemann's original approach: he defines the curvature tensor in his 1854 habilitation [Rie13] via the formula:

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + O(r^3)$$

in normal coordinates. Let us give a more geometric characterization (we refer to [GLM19, Appendix A] for details).

Let $p \in M$ and consider two tangent vectors $u, v \in \mathrm{T}_p M$. Denote by γ_u and γ_v the geodesics from p with initial velocities u and v respectively. Then

$$d\left(\gamma_u(t), \gamma_v(t)\right)^2 = \|u - v\|^2 \, t^2 - \frac{1}{3} \langle R(u, v)v, u \rangle \, t^4 + O(t^5) \, .$$

as $t \longrightarrow 0$. In other words, with the sectional curvature:

$$d\left(\gamma_u(t), \gamma_v(t)\right)^2 = \|u - v\|^2 \, t^2 - \frac{1}{3} K(u, v) \|u \wedge v\|^2 \, t^4 + O(t^5) \, .$$

The important thing to note is that $d_{\mathrm{E}}\left(\gamma_u(t), \gamma_v(t)\right) := \|u - v\|^2 \, t^2$ is exact in a Euclidean space, therefore the next order term gives the deviation from the Euclidean distance. In particular, observe that if $K > 0$, then $d < d_{\mathrm{E}}$: the distance between geodesics is closer than in a Euclidean space; on the contrary, if $K < 0$ then $d > d_{\mathrm{E}}$: geodesics diverge faster than in a Euclidean space. See Figure 2.3 for an illustration.

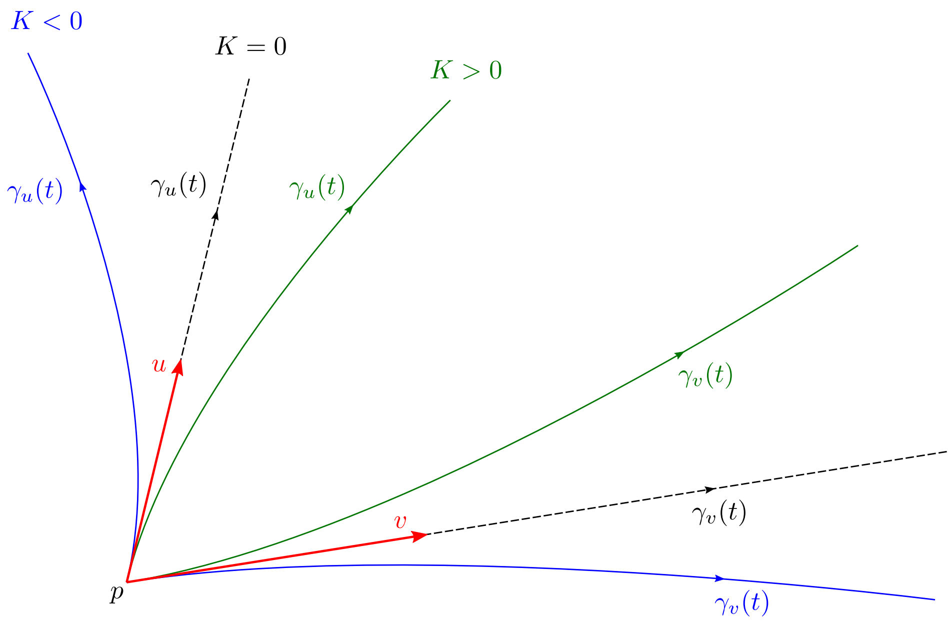


Figure 2.3: Geodesic deviation: the distance between geodesics $\gamma_u(t)$ and $\gamma_v(t)$ is controlled by the sectional curvature $K(u, v)$.

2.4 Model spaces of constant curvature

Using classical techniques of Riemannian geometry, one can show:

Theorem 2.19. Any two Riemannian manifolds of the same dimension and with same constant sectional curvature are locally isometric.

In other words, n-dimensional metrics of constant sectional curvature $k \in \mathbb{R}$ are locally unique. In a nutshell, the proof goes as follows: using Jacobi fields, one sees that for a Riemannian manifold of constant sectional curvature, the Riemannian metric's expression is forced to a have a fixed expression in normal coordinates. For more details, see [Lee18, Theorem 10.14, Corollary 10.15].

By definition, a ***model space*** or a ***space form*** of constant sectional curvature is a complete, simply-connected manifold of constant sectional curvature. We recall that a Riemannian manifold is called ***complete*** if lines (geodesics) can be extended indefinitely. Equivalently, it is complete as a metric space (Hopf-Rinow theorem).

Using the Cartan–Hadamard theorem, one can derive from the previous theorem:

Theorem 2.20. In any dimension, the space form of constant curvature $k \in \mathbb{R}$ is unique up to isometry.

Space forms are thus essentially unique; also, they exist! Depending on the sign of $k \in \mathbb{R}$, the space form $\mathbb{M}^n_k$ takes three different forms:

- For $k > 0$, the space form of constant curvature k is denoted $\mathbb{S}^n_R$ where $k = \frac{1}{R^2}$. The

usual model for it is the Euclidean sphere of squared radius R in $\mathbb{R}^{n+1}$.
- For $k = 0$, the space form of constant curvature k is Euclidean space $\mathbb{E}^n$. The usual model for it is $\mathbb{R}^n$ with its standard Euclidean structure.
- For $k < 0$, the space form of constant curvature k is hyperbolic space $\mathbb{H}^n_R$ where $k = -\frac{1}{R^2}$. One model for it is the pseudo-Euclidean sphere of "imaginary radius" $R\sqrt{-1}$ in Minkowski space $\mathbb{R}^{n,1}$, as we shall see in Chapter 4. However, we shall also see other useful models: the Beltrami–Klein model (Chapter 7), the Poincaré ball and half-space models (Chapter 9).

Combining the two previous theorems, we can state:

Theorem 2.21. Let M be a complete Riemannian manifold of constant sectional curvature $k \in \mathbb{R}$. Then M is covered by the space form $\mathbb{M}^n_k$. In other words, M is isometric to a quotient of the space form $\mathbb{M}^n_k$ by a free and wandering action of a discrete group of isometries.

Note that, after scaling the metric, one can assume M has constant sectional curvature 1, 0, or -1. In many ways, the latter case is the most interesting. A manifold with constant sectional curvature -1 is called a ***hyperbolic manifold***.

The previous theorem implies that any complete hyperbolic manifold is a quotient of hyperbolic space $\mathbb{H}^n$. This is remarkable because it shows that while Riemannian metrics are rather flexible objects, hyperbolic metrics are quite rigid. A consequence of this is that the study of hyperbolic manifolds is more algebraic, and less differential, that one could expect. This explains why a course in hyperbolic geometry belongs in the realm of classical geometry more than differential geometry, much like a course in Euclidean geometry.

2.5 Curvature of metric spaces

Metric spaces are clearly much more general than Riemannian manifolds. Can we extend the notion of curvature to metric spaces? In a nutshell, yes, there are several slightly different definitions of curvature in metric spaces that coincide for Riemannian manifolds. However:
- All such definitions build on the Riemannian case, or at least the model spaces of constant sectional curvature (called ***space forms***). Therefore, one should start by understanding curvature in Riemannian manifolds, or at least in space forms.
- There is a trade-off: the notion of curvature in metric spaces is not as precise as in Riemannian manifolds.

Despite these nuances, the notion of curvature in metric spaces is very useful. In particular, Gromov hyperbolic spaces offer the right frame to classify the isometries of hyperbolic space. We postpone this discussion until Chapter 11.

2.6 Exercises

Exercise 2.1. Frenet–Serret frame and torsion

Let $\gamma\colon I \longrightarrow E$ be a smooth regular curve in a 3-dimensional Euclidean space, parametrized by arclength. Assume that γ'' does not vanish.

(1) Let $T(s) := \gamma'(s)$ (***unit tangent***), $N(s) := \frac{\gamma''(s)}{\|\gamma''(s)\|}$ (***principal normal***), and $B(s) := T(s) \times N(s)$ (***unit binormal***). Show that $(T(s), N(s), B(s))$ is an orthonormal basis.

(2) The frame of E with origin at $\gamma(s)$ and basis $(T(s), N(s), B(s))$ is called the ***Frenet–Serret frame***. What is the equation of the osculating circle in this frame?

(3) We pick a fixed orthonormal frame of E. Let $Q(s)$ be the matrix whose rows are given by the coordinates of $T(s)$, $N(s)$, and $B(s)$ respectively.
   (a) Argue that $Q \in \mathrm{O}(3, \mathbb{R})$, i.e. $Q(s)Q(s)^{\mathsf{T}} = I_3$.
   (b) Derive from the previous question that $Q'(s)Q(s)^{\mathsf{T}}$ is antisymmetric.
   (c) On the other hand, show that the first row of $Q'(s)Q(s)^{\mathsf{T}}$ is $\begin{bmatrix} 0 & \kappa(s) & 0 \end{bmatrix}$.
   (d) Derive from the two previous question that there exists a number $\tau(s)$, called the ***torsion*** of γ, such that $Q'(s)Q(s)^{\mathsf{T}} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}$.
   (e) Conclude that the ***Frenet–Serret formulas*** hold:

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa N + \tau B \\ B' &= -\tau B \end{aligned}$$

(4) Check that the ***helix*** $\gamma(t) = (a\cos t, a\sin t, bt)$ has constant curvature and torsion. (First draw the helix.)

(5) (*) Using the Picard–Lindelöf (aka Cauchy–Lipschitz) theorem, show that the curvature and torsion of a space curve determine it uniquely up to an affine isometry of E. (This is the "fundamental theorem of space curves".)

(6) Show that any curve with constant nonzero curvature and torsion is a helix.

Exercise 2.2. Mean curvature

Recall that for a surface $S \subseteq \mathbb{R}^3$, we defined the mean curvature H_p at a point $p \in S$ as the half-sum of the principal curvatures. Show that H_p could instead be defined as, quite literally, the mean extrinsinc curvature at p. (First you'll have to make sense of this statement!)

Exercise 2.3. The sphere

Let S^n_R denote the sphere of radius $R > 0$ centered at the origin in $\mathbb{R}^{n+1}$. We would like to understand geodesics and curvature in S^n_R. This exercise may seem basic, but it is very important: we will follow the same strategy for the hyperboloid in Minkowsi space.

(1) Show that any linear isometry of $\mathbb{R}^{n+1}$ induces a Riemannian isometry of S^n_R. Optional: show that the group of isometries of S^n_R is $\mathrm{O}(n+1)$.

(2) For now, we consider the sphere $S = S^2_R$ in $\mathbb{R}^3$.
   (a) Show that for any $p \in S$ and $v \in \mathrm{T}_p S$, there exists a plane $H \subseteq \mathbb{R}^3$ such that the reflection s_H through H leaves p and v invariant.
   (b) Show that geodesics on S are exactly the great circles (intersection of S with planes through the origin), parametrized with constant speed.
   (c) Show that we have the explicit expression:
$$\gamma_v(t) = \cos\left(\|v\|t\right) p + R\sin\left(\|v\|t\right) \frac{v}{\|v\|}\,.$$
   (d) Let $p, q \in S$. Show that their distance on S is given by $d(p,q) = R\,\angle(p,q)$ where $\angle(p,q)$ denotes the unoriented angle between p and q seen as vectors in $\mathbb{R}^3$.

(3) What is the exterior unit normal N at p? Show that the extrinsic curvature $\rho_p(v)$ is equal to $-\frac{1}{R}$ for any unit vector v. Conclude that the Gaussian curvature is $\frac{1}{R^2}$ at p, and hence everywhere.

(4) Let $n \geqslant 2$.
   (a) Show that (2) remains true with S^n_R instead of S and $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^3$, as long as by *plane* we mean a 2-dimensional subspace.
   (b) Let $P \subseteq \mathrm{T}_p S^n_R$ be a 2-plane. Denote $E_P \subseteq \mathbb{R}^{n+1}$ the subspace spanned by p and P. Show that the union of geodesics in S^n_R with initial velocity in P is the sphere S_P of radius R in E_P. *In the terminology of Riemannian geometry:* $\exp_p(P) = S_P$.
   (c) Conclude that S^n_R has constant sectional curvature $\frac{1}{R^2}$.

Exercise 2.4. The tractricoid

One of the obstacles to the discovery of the hyperbolic plane is that it cannot be smoothly completely embedded as a surface in $\mathbb{R}^3$. [3] However, it is possible to smoothly embed a piece of the hyperbolic plane in $\mathbb{R}^3$, as this exercise illustrates.

(1) Consider the ***tractrix*** curve in the xz-plane parametrized by:
$$\begin{aligned}\gamma\colon [0,+\infty) &\longrightarrow \mathbb{R}^3\\ t &\longmapsto \left(x(t) = \operatorname{sech} t, y(t) = 0, z(t) = t - \tanh t\right)\end{aligned}$$

[3] There are no complete surfaces of constant Gaussian curvature -1 of class $\mathcal{C}^2$ in $\mathbb{R}^3$ (Efimov's theorem, 1964 [Efi64], also see [Mil72]). Hilbert first proved it for class $\mathcal{C}^4$ in 1901. Surprisingly, there are $\mathcal{C}^1$ embeddings of the hyperbolic plane in $\mathbb{R}^3$. This is a corollary of the Nash-Kuiper $\mathcal{C}^1$ embedding theorem. See `http://www.math.cornell.edu/~dwh/papers/crochet/crochet.html` for illustrations of crocheted hyperbolic planes.

where $\operatorname{sech} = \frac{1}{\cosh}$ is the hyperbolic secant and $\tanh = \frac{\sinh}{\cosh}$ is the hyperbolic tangent. Draw the tractrix in the plane. Optional: Show that the tractrix is the path followed by a reluctant dog on a leash (in German, a tractrix is a ***Hundekurve***).

(2) The ***tractricoid*** (sometimes called ***pseudosphere***[4]) is the surface S in $\mathbb{R}^3$ obtained by rotating the tractrix defined above around the z-axis. Show that it has parametric equations:

$$\begin{aligned} x &= \operatorname{sech} t \cos\theta \\ y &= \operatorname{sech} t \sin\theta \\ z &= t - \tanh t\,. \end{aligned}$$

Show that rotations around the z-axis and reflections through vertical planes containing the z-axis are isometries of S. Draw a sketch of S.

(3) We denote $f(\theta,t) := (x(\theta,t), y(\theta,t), z(\theta,t))$. Consider the curves $c_t(\theta) = f(\theta,t)$ when t is fixed ("parallels") and $\gamma_\theta(t) = f(\theta,t)$ when θ is fixed ("meridians"). Draw such curves on S. Using a symmetry argument, show that the curves $\gamma_\theta(t)$ are geodesics up to parametrization.

(4) Consider a point $p = f(\theta_0, t_0)$ on the tractricoid. Our goal is to show that the Gaussian curvature of S at p is -1.
   (a) Explain why it is enough to show it when $\theta_0 = 0$.
   (b) Compute the velocities of c_{t_0} and γ_0 at p. Derive an expression of the unit normal vectors at p.
   (c) Compute the extrinsic curvatures of S at p in the unit directions tangent to c_{t_0} and γ_0.
   (d) Using a symmetry argument, explain why the principal directions of curvatures of S at p must be tangent to c_{t_0} or γ_0. Derive the value of the principal curvatures at p, conclude that S has Gaussian curvature -1 at p, and hence everywhere.

(5) Compute the arclength parameter of $\gamma(t)$. Show that the tractricoid is incomplete.

Exercise 2.5. The Poincaré disk

The ***Poincaré disk*** $\mathbb{D}$ is defined as the unit disk equipped with the Riemannian metric:

$$\mathrm{d}s^2 = \frac{4\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)}{(1 - x^2 - y^2)^2}$$

In this exercise, we denote $O \in \mathbb{D}$ the point which is at the origin in $\mathbb{R}^2$.

[4]Depending on authors, *pseudosphere* may refer to the tractricoid specifically, or to any surface in $\mathbb{R}^3$ of Gaussian curvature -1. I find the term more appropriate for level sets of the quadratic form in a pseudo-Euclidean vector space (this includes the hyperboloid model of the hyperbolic plane).

(1) Show that the Poincaré metric on $\mathbb{D}$ is conformal to the Euclidean metric. Is the Euclidean metric complete on $\mathbb{D}$?

(2) Show that any $f \in \mathrm{O}(2)$ induces an isometry of $\mathbb{D}$ that fixes O. Optional: show the converse.

(3) Show that any diameter of $\mathbb{D}$ (straight chord through the origin) is a geodesic. *Hint: consider the fixed points of a reflection* $f \in \mathrm{O}(2)$.

(4) Find a parametrization of geodesics through the origin. Find an expression of the distance between O and an arbitrary point in $\mathbb{D}$.

(5) Show that $\mathbb{D}$ is complete. *Use the Hopf-Rinow theorem.*

(6) Compute the curvature of $\mathbb{D}$.

Exercise 2.6. Euclid's postulates for Riemannian surfaces (*)

Give an interpretation of Euclid's postulates for Riemannian surfaces and discuss their implications.

This exercise is not easy, and best suited to students with a solid background of Riemannian geometry. Regardless, I recommend that you read the solution.

Part II: Minkowski space and the hyperboloid model

A four-dimensional continuum described by the "co-ordinates" x_1, x_2, x_3, x_4, was called "world" by Minkowski, who also termed a point-event a "world-point". From a "happening" in three-dimensional space, physics becomes, as it were, an "existence" in the four-dimensional "world". This four-dimensional "world" bears a close similarity to the three-dimensional "space" of (Euclidean) analytical geometry. [...] We can regard Minkowski's "world" in a formal manner as a four-dimensional Euclidean space (with an imaginary time coordinate) ; the Lorentz transformation corresponds to a "rotation" of the co-ordinate system in the four-dimensional "world."

– Albert Einstein[5]

The mathematical education of the young physicist [Albert Einstein] was not very solid, which I am in a good position to evaluate since he obtained it from me in Zurich some time ago.

– Hermann Minkowski[6]

[5] Einstein, *Relativity: The Special and the General Theory* [Ein15].
[6] Quoted from [New00].

CHAPTER 3

Minkowski space

In this chapter we review pseudo-Euclidean vector spaces and, in particular, Minkowski spaces. They will be the framework for the hyperboloid model discussed in the next chapter, and for the theory of relativity briefly presented in the subsequent chapter.

Historically, the development of Minkowski geometry is indeed closely related to the discovery of special relativity in the early 20th century. In special relativity, Minkowski space is the base model for spacetime: it is the solution of Einstein's equations in the vacuum. For our purposes, this connection to relativity can be ignored, although we retain some terminology from physics such as ***light cone***.

As a prerequisite for this chapter, we assume basic knowledge of Euclidean vector spaces, i.e. finite-dimensional vector spaces equipped with a positive definite symmetric bilinear form: orthogonality, linear isometries, etc.

3.1 Symmetric bilinear forms and orthogonality

Let V be a finite-dimensional real vector space.

Let $b\colon V\times V \to \mathbb{R}$ be a symmetric bilinear form, which we also denote $\langle\cdot,\cdot\rangle$. The associated quadratic form is the map $q\colon V \to \mathbb{R}$, $v \mapsto b(v,v)$. Recall that b is completely determined by q via the ***polarization identity*** $b(u,v) = \frac{1}{2}\left(q(u+v) - q(u) - q(v)\right)$.

Matrix. Given a basis $(e_1,\dots,e_n)$ of V, the matrix of b (or q) is $B = \left(b_{ij}\right)_{1\leqslant i,j\leqslant n}$ where $b_{ij} = b(e_i,e_j)$. For any $u,v \in V$, represented by column vectors U, V, we have:

$$b(u,v) = U^{\mathsf{T}} B\, V\,.$$

Orthogonality. Two vectors $u,v \in V$ are called ***orthogonal*** if $\langle u,v\rangle = 0$ (notation: $u \perp v$). If $A \subseteq V$, we say that $u \perp V$ if $u \perp v$ for all $v \in A$. The orthogonal of V is $A^{\perp} := \{x \in V : x \perp A\}$. It is a vector subspace of V.

Isotropic cone and kernel. A vector v is called ***isotropic*** (or ***null***) if $\langle v,v\rangle = 0$. The set of isotropic vectors is the ***isotropic cone***. It is a cone: it is invariant by scalar multiplication. The ***kernel*** of b is $\ker b := V^{\perp}$. It is a vector subspace of V. Using a basis as before, it is identified to the kernel of b. The isotropic cone always contains the kernel, it is easy to check that the converse is true if and only if b is positive semidefinite or negative semidefinite.

Remark 3.1. By definition, the isotropic cone is a ***quadric***: it is the set of solutions of a quadratic equation $q = 0$, i.e. a polynomial equation of degree 2 in n variables. Since q is a homogeneous polynomial of degree 2, the isotropic cone is in fact a projective quadric in $\mathbb{P}(V)$: we will discuss this more in Chapter 6.

Nondegeneracy. b is called ***nondegenerate*** if it has trivial kernel $\ker b = \{0\}$. Equivalently, the map

$$\begin{aligned}\tilde{b}\colon V &\longrightarrow V^*\\ v &\longmapsto \langle v, \cdot\rangle\end{aligned}$$

is an isomorphism.

Lemma 3.2. *For any subspace W, $\dim W^\perp \geqslant \operatorname{codim} W$. Furthermore, $W \oplus W^\perp = V$ if and only if $b_{|W}$ is nondegenerate.*

Proof. Consider the map

$$\begin{aligned}f\colon V &\longrightarrow W^*\\ v &\longmapsto \langle v, \cdot\rangle\end{aligned}$$

(in other words, $f(v)$ is the restriction of $\tilde{b}(v)$ to W). By the rank theorem, we have $\dim\ker f + \dim\operatorname{Im} f = \dim V$. Since $\ker f = W^\perp$ and $\dim\operatorname{Im} f \leqslant W$, we get $\dim\ker f \geqslant \dim V - \dim W$, as desired. Note that if b is nondegenerate, the argument is easier: $\tilde{b}$ is an isomorphism, so $\dim W^\perp = \dim W^\circ = \operatorname{codim} W$.

For the second statement, observe that $\ker b_{|W} = W \cap W^\perp$. Therefore $W \cap W^\perp = \{0\}$ if and only if $b_{|W}$ is nondegenerate. The conclusion then follows from the first statement. □

Index and signature. The ***positive index*** p of b is the maximal dimension of a vector subspace W such that b is positive definite in restriction to W. The ***negative index*** q or just ***index*** of b is the maximal dimension of a vector subspace W such that b is negative definite in restriction to W. The pair (p, q) is called the ***signature*** of b.

Proposition 3.3. *If W_+ is a maximal subspace where b is positive definite, then there exists a maximal subspace W^- where b is negative definite such that*

$$V = W_+ \oplus W_- \oplus \ker b$$

and these sums are b-orthogonal.

Proof. Let W_+ be a maximal subspace where b is positive definite and W' a maximal subspace where b is negative definite. Clearly, $W' \cap \ker b = 0$, moreover $W_+ \cap (W' + \ker b) = \{0\}$ (because b is positive definite on W_+ and negative semidefinite on $W' + \ker b$). This shows that W_+, W', and $\ker b$ are in direct sum. In particular, $p + q + \dim\ker b \leqslant n$. We win if we show that $W_- = W'$ can be chosen inside $W_+^\perp$ and that $p + q + \dim\ker b = n$.

First note that $W_+ \cap W_+^\perp = \{0\}$ (if $v \in W_+^\perp \cap W$, then $b(v,v) = 0$, which implies $v = 0$ since b is positive definite on W_+). Moreover, b is negative semidefinite on $W_+^\perp$, otherwise we could easily argue that the positive index of b is $> p$. Also note that $W_+^\perp$ contains $\ker b$ (the orthogonal of anything contains the kernel). Let W_- be a maximal subspace of $W_+^\perp$ where b is negative definite. Let us show that $W_+^\perp = W_- \oplus \ker b$. It is clear that $W_- \cap \ker b = \{0\}$ because b is negative definite on W_-. On the other hand, $W_- + \ker b = W_+^\perp$: assume $w \in W_+^\perp$ does not belong to $W_- + \ker b$. Without loss of generality, we can assume $w \perp W_-$: if not, take an orthonormal basis (e_j) of W_- for $-b$ (this exists because $-b$ is positive definite), it is easy to check that there is a unique choice of λ_j's so that $w + \sum \lambda_j e_j \perp W_-$. Since b cannot be negative definite in restriction to $W_- \oplus w$, there must exist $v \in W_-$ and $t \in \mathbb{R}$ such that $v + tw \neq 0$ and $b(v+tw, v+tw) = 0$ i.e. $b(v,v) + t^2 b(w,w) = 0$. If $t = 0$, we must have $b(v,v) = 0$ hence $v = 0$, which is included since $v + tw \neq 0$. If $t \neq 0$, we must have $b(w,w) = 0$, in other words w is isotropic. Since b is negative semidefinite on W_-, this implies that w is in $\ker\left(b_{|W_-}\right)$, which is easily argued to be the same as $\ker b$.

So far we have showed that

$$V = W_+ \oplus W_- \oplus \ker b \tag{3.1}$$

where W_+ is a maximal subspace where b is positive definite, W_- is a subspace where b is negative definite, and these sums are b-orthogonal. It remains to show that W_- is maximal, i.e. $\dim W_- = q$. By (3.1), we have $p + \dim W + \dim\ker b = n$. On the other hand, we have seen that $p + q + \dim\ker b \leqslant n$. Combining the two, we find $\dim W_- \geqslant q$, hence $\dim W_- = q$. □

Proposition 3.4 (Sylvester's law of inertia)**.** *There exists a basis of V such that*

$$b(e_i, e_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant p \\ -1 & \text{if } p+1 \leqslant i \leqslant p+q \\ 0 & \text{if } p+q < i \leqslant n \end{cases}$$

In other words, the matrix of b is the diagonal matrix:

$$B = \begin{bmatrix} \boxed{I_p} & 0 & 0 \\ 0 & \boxed{-I_q} & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

Proof. This is an immediate consequence of Proposition 3.3: first take an orthonormal basis of (W^+, b), then take an orthonormal basis of $(W^-, -b)$, finally take any basis of $\ker b$, and concatenate the three bases. □

Proposition 3.5. *If W is a subspace where b is positive definite (resp. negative definite), then $W \oplus W^\perp = V$, and the signature of b on $W^\perp$ is $(p - \dim W, q)$ (resp. $(p, q - \dim W)$).*

Proof. We have already seen that $W \oplus W^\perp = V$ in Lemma 3.2. It is enough to do the case where b is positive definite on W, for the other case, just apply the first case to $-b$. Apply

Proposition 3.3 to $W^\perp$: write $W^\perp = U_+ \oplus U_- \oplus \ker b_{|W^\perp}$ It follows:

$$V = (W \oplus U_+) \oplus U_- \oplus \ker b_{|W^\perp} .$$

Since b is positive definite on W, its kernel intersects W trivially, and one quickly sees that $\ker b_{|W^\perp} = \ker b$. Denote by (p_2, q_2) the signature of b on $W^\perp$, so that $p_2 = \dim U_+$ and $q_2 = \dim U_-$. From the previous equation we get $\dim W + p_2 + q_2 + \dim \ker b = n$. On the other hand, we know that $p + q + \dim \ker b = n$, and $p \geqslant \dim W + p_2$ since b is positive definite on $W \oplus U_+$ and $q \geqslant q_2$ since b is negative definite on U_-. we conclude that $p = \dim W + p_2$ and $q = q_2$. □

3.2 Pseudo-Euclidean and Minkowski spaces

Recall that an inner product on V is a positive definite symmetric bilinear form, and a ***Euclidean vector space*** is a finite-dimensional vector space with an inner product. We shall call ***pseudo-inner product*** any nondegenerate symmetric bilinear form, and ***pseudo-Euclidean vector space*** any finite-dimensional vector space with an pseudo-inner product.

As before, we denote b or $\langle \cdot, \cdot \rangle$ the inner product. Denote by (p, q) the signature of b.

Be cautious that b may be degenerate in restriction to a subspace! Give an example of this phenomenon.

Canonical pseudo-Euclidean vector space. By Sylvester's law of inertia, we have $p + q = n$, and by choosing a suitable basis we may assume that $V = \mathbb{R}^n$ and the pseudo-inner product is given by:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}$$

This canonical pseudo-Euclidean vector space with signature (p, q) is denoted $\mathbb{R}^{p,q}$.

When $q = 1$, the inner product on V is called ***Lorentzian***, which is a particular case of pseudo-Euclidean, and the pseudo-Euclidean space (V, b) is called a ***Minkowski space*** (or ***Minkowski spacetime***).

By the previous discussion, a Minkowski spacetime of dimension $n + 1$ can be identified to $\mathbb{R}^{n,1}$, after a suitable choice of basis. It is customary to call the last coordinate the time coordinate, and denote it t, but we will not do this, because we like to keep t for the time parameter in curves. Nevertheless, we will use the classical terminology:

Definition 3.6. A nonzero vector $v \in V$ is called:

- ***spacelike*** if $\langle v, v \rangle > 0$,
- ***timelike*** if $\langle v, v \rangle < 0$,
- ***lightlike*** if $\langle v, v \rangle = 0$ (i.e. v is isotropic).

The isotropic cone of b is also called the ***light cone***.

See Figure 3.1 for an illustration of the light cone in $\mathbb{R}^{2,1}$. Spacelike vectors are outside the cone, timelike vectors are inside the cone, and lightlike vectors on the cone.

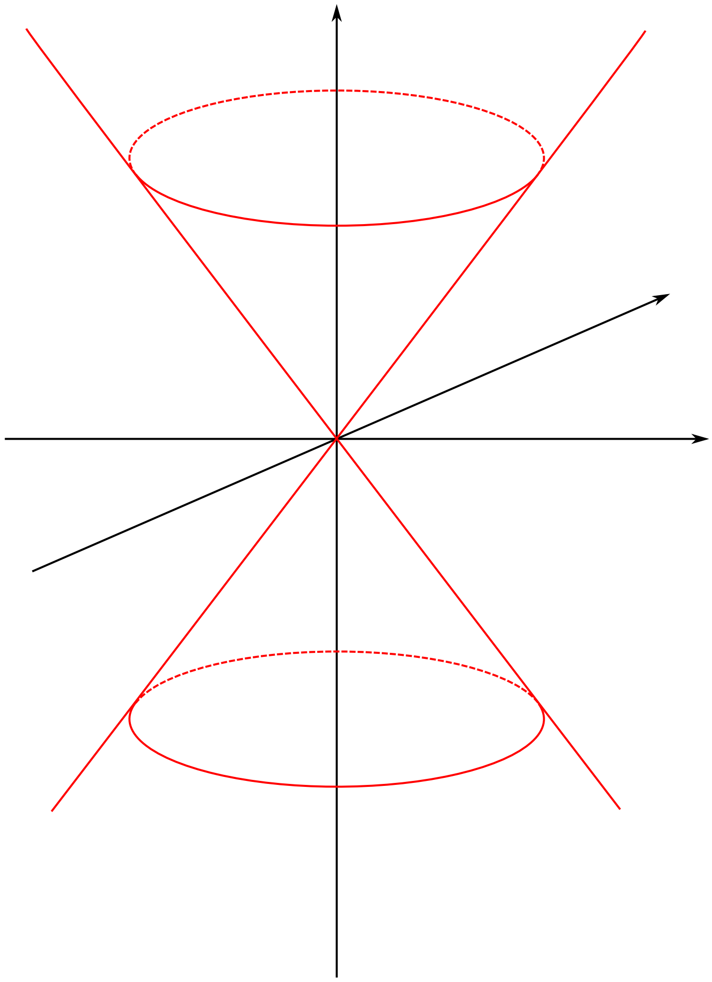


Figure 3.1: The light cone in Minkowski space $\mathbb{R}^{2,1}$.

Remark 3.7. This terminology may still be used for other pseudo-Euclidean spaces, because why not.

As an immediate application of Proposition 3.5, we have:

Proposition 3.8. *Let V be a Minkowski space. If $v \in V$ is timelike, then $v^\perp$ is a spacelike hyperplane, and $V = \mathbb{R}v \oplus v^\perp$.*

3.3 Distances and angles

Let V be a Minkowski space.

If one tries to define the norm of a vector with the usual formula $\|v\| = \sqrt{\langle v, v\rangle}$, then we have a problem when $\langle v, v\rangle < 0$. We can still define the norm of a spacelike (or lightlike) vector though. For a timelike vector, we could take $\|v\| = \sqrt{|\langle v, v\rangle|}$, but we have to be careful if we use this convention. In particular, there is no well-defined distance in V.

The same problem arises when trying to measure the lengths of curves. However, if $\gamma\colon I \longrightarrow V$ is a smooth curve that is ***spacelike***, i.e. such that $\langle \gamma'(t), \gamma'(t)\rangle > 0$ for all t, then it makes sense to define the length of γ as usual. For other curves, it is probably wise to avoid defining their length, although for timelike curves we can write a reasonable definition.

What about angles? Recall that in a Euclidean vector space, the angle between two vectors is defined (up to sign) by the identity

$$\langle u, v\rangle = \|u\|\,\|v\| \cos \angle(u, v)\,.$$

Note that this works thanks to the Cauchy–Schwarz inequality $\langle u, v\rangle^2 \leqslant \langle u, u\rangle\,\langle v, v\rangle$, ensuring that $\cos \angle(u, v)$ is ineed in $[-1, 1]$. In the case of a pseudo-Euclidean space, there is no such inequality unless b is definite on the plane spanned by u and v. However, for timelike vectors in a Minkowski space we have the opposite inequality:

Proposition 3.9. *Let u and v be timelike vectors in a Minkowski vector space. Then*

$$\langle u, v\rangle^2 \geqslant \langle u, u\rangle\,\langle v, v\rangle \tag{3.2}$$

Proof. If u and v are collinear, we easily see that there is equality in (3.2). Otherwise, consider the function $p(t) = \langle u + tv, u + tv\rangle$. This function is negative at $t = 0$, and it cannot have constant sign, otherwise b would be negative definite on the plane spanned by u and v, which is excluded because b has negative index 1. On the other hand, notice that $p(t)$ is a polynomial of degree 2 in t: $p(t) = t^2\langle v, v\rangle + 2t\langle u, v\rangle + \langle u, u\rangle$, and since it does not have constant sign, it must have nonnegative discriminant: $\langle u, v\rangle^2 - \langle u, u\rangle\,\langle v, v\rangle \geqslant 0$, as desired. □

As a consequence, in a Minkowski space, we can define the ***hyperbolic angle*** between two timelike vectors:

Definition 3.10. Let u and v be timelike vectors in a Minkowski vector space. The ***hyperbolic angle*** (or ***timelike angle***) between u and v is the unique nonnegative real number $\angle(u, v)$ such that:

$$\langle u, v\rangle^2 = \langle u, u\rangle\,\langle v, v\rangle \cosh^2 \angle(u, v)$$

3.4 Orthogonal group

Let (V, b) be a pseudo-Euclidean vector space.

Definition 3.11. The orthogonal group of b is its group of isometries, i.e. the group of automorphisms of V that preserve b:

$$\mathrm{O}(b) := \left\{ f \in \mathrm{GL}(V) \,:\, \forall (u, v) \in V^2\ \langle f(u), f(v)\rangle = \langle u, v\rangle \right\}$$

Elements of $O(b)$ are easily seen to have determinant ± 1 (use the matrix characterization). The subgroup $SO(b)$ consists of elements with determinant 1. Equivalently, $SO(b)$ consists of isometries that are (globally) orientation-preserving.

Example 3.12. Let $v \in V$ be a non-isotropic vector. Then $H = v^{\perp}$ is a hyperplane and $V = \mathbb{R}v \oplus H$. One can define the orthogonal reflection through H by $f(tv + h) = -tv + h$. Any reflection through a hyperplane is orientation-reversing. More generally, let W be a nondegenerate subspace. Then $V = W \oplus W^{\perp}$ by Lemma 3.2, and one can define the reflection through W similarly. It is orientation-preserving if and only if $\operatorname{codim} W$ is even.

We admit the following important theorem:

Theorem 3.13. Any element of $O(b)$ is a finite product of reflections through hyperplanes.

When $V = \mathbb{R}^{p,q}$, the orthogonal group is denoted $O(p, q)$. As a matrix group:

$$O(b) = \left\{ M \in GL(n, \mathbb{R}) \ : \ M^{\mathsf{T}} I_{p,q} M = I_{p,q} \right\}$$

where $I_{p,q}$ is the matrix of the pseudo-inner product.

When $q = 1$, i.e. $V = \mathbb{R}^{n,1}$ is a Minkowski space, the group $O(n, 1)$ is called the ***Lorentz group***.

Isometries of $\mathbb{R}^{p,q}$ can be independently be either space-orientation preserving/reversing and time-orientation preserving/reversing. Unfortunately, I was not able to find a good in general. But in the case of Minkowski space, one can give the following definition:

Definition 3.14. An isometry $f \in O(n, 1)$ is called ***time-orientation preserving*** [resp. ***time-orientation reversing***] if, for some (equivalently all) timelike vector v, $\langle f(u), v \rangle$ has the same sign (resp. opposite sign) as $\langle u, v \rangle$ for all $u \in V$ (equivalently, for some u not orthogonal to v).

One can proceed to declare that $f \in O(n, 1)$ is space orientation-preserving if and only if: f is globally orientation-preserving and time orientation-preserving, or globally orientation-reversing and time orientation-reversing. Of course, $f \in O(n, 1)$ is declared space orientation-reversing in the opposite scenario.

We denote $O^{+}(n, 1)$ the subgroup of isometries that are time orientation-preserving (physicists call it the ***orthochronous Lorentz group***). The subgroup $SO^{+}(n, 1)$ consists of isometries that are both time and space orientation-preserving. It turns out that $SO^{+}(n, 1)$ is connected: see Exercise 3.4. It follows that $SO^{+}(n, 1) = O_0(n, 1)$ is the identity component of $O(n, 1)$ and $O(n, 1)$ has four connected components, distinguished according to the space and time orientation-preserving/reversing quality of isometries.

We will see further properties of $O(n, 1)$ in Chapter 4, and propose a classification of isometries in Chapter 11.

3.5 Exercises

Exercise 3.1. Characterization of orthogonal decompositions

Let (V, φ) be a finite-dimensional vector space equipped with a symmetric bilinear form. Let $W \subseteq V$ be a subspace.

(1) Is $\dim W + \dim W^\perp \geqslant \dim V$ always true? Is $W + W^\perp = V$ always true?

(2) Recall the proof that $V = W \oplus W^\perp$ if and only if $\varphi_{|W}$ is nondegenerate.

Exercise 3.2. Orthogonal subspace to a timelike vector

Prove Proposition 3.8 (copied below) directly, without using the results of § 3.1.

Proposition. *Let V be a Minkowski space. If $v \in V$ is timelike, then $v^\perp$ is a spacelike hyperplane, and $V = \mathbb{R}v \oplus v^\perp$.*

Exercise 3.3. Time orientation-preserving criterion

Let M be a matrix in $\mathrm{O}(n, 1)$. Show that f is time orientation-preserving if and only if the bottom-right coefficient of M is positive.

Exercise 3.4. Lorentz boosts and structure of the Lorentz group

(1) Show that any element of $\mathrm{SO}^+(1, 1)$ can uniquely be written:

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

with $t \in \mathbb{R}$. Show that $\mathrm{SO}^+(1, 1)$ is connected.

(2) An element $f \in \mathrm{SO}^+(n, 1)$ is called a *Lorentz boost* if the set of fixed points of f contains a spacelike subspace of codimension 2. Show that in a suitable basis, a Lorentz boost looks like:

$$\begin{bmatrix} \boxed{I_{n-1}} & 0 \\ 0 & \boxed{\begin{matrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{matrix}} \end{bmatrix}$$

Argue that any Lorentz boost is in the connected component of the identity in $\mathrm{O}(n, 1)$.

(3) Show that for any two unit timelike vectors u and v, there exists a unique Lorentz boost f such that $f(u) = v$.

(4) Show that any matrix $M \in O^{+}(n,1)$ can uniquely be written as $M = QB$, where B is the matrix of a Lorentz boost and Q is a matrix of the form

$$\begin{bmatrix} \boxed{Q_1} & 0 \\ 0 & 1 \end{bmatrix}$$

with $Q_1 \in \mathrm{O}(n)$.

(5) Recall why $\mathrm{SO}(n)$ is connected (optional) and conclude that $\mathrm{SO}^{+}(n,1)$ is connected.

Exercise 3.5. Connected components of the Lorentz group and projective Lorentz group

We recall that the subgroup $\mathrm{SO}^{+}(n,1) \subseteq \mathrm{O}(n,1)$ is connected (see Exercise 3.4).

(1) Show that $\mathrm{SO}^{+}(n,1)$ is the identity component of $\mathrm{O}(n,1)$. Show that it is a normal subgroup. Show that the quotient $\mathrm{O}(n,1)/\mathrm{SO}^{+}(n,1)$ is isomorphic to the Klein four-group.

(2) Show that the the center of $\mathrm{O}(n,1)$ is equal to the subgroup of homotheties (scalar multiples of the identity) in $\mathrm{O}(n,1)$, that is, $Z(\mathrm{O}(n,1)) = \{\pm I_{n+1}\}$.

(3) Let $\mathrm{PO}(n,1) := \mathrm{O}(n,1)/Z(\mathrm{O}(n,1)) = \mathrm{O}(n,1)/\{\pm I_{n+1}\}$ denote the projective Lorentz group. Can you identify it to a subgroup of $\mathrm{O}(n,1)$?

CHAPTER 4

The hyperboloid model

In this chapter, we introduce the hyperboloid model for hyperbolic space, defined as a hypersurface in Minkowski space. This model is much analogous to the sphere in Euclidean geometry: the hyperboloid, a pseudosphere in Minkowski space, plays the role of the sphere in Euclidean space.

For many purposes, the hyperboloid is the best model of hyperbolic space: we shall see in particular that it is fairly easy and elegant to derive all the relevant geometric properties: the Riemannian metric, group of isometries, geodesics, distance function, and sectional curvature.

Historically, the idea of an imaginary sphere goes back to Lambert in 1766, and in 1826 Taurinus performs trigonometry calculations on a "sphere of imaginary radius". The connection with hyperbolic geometry and the other models was established by Poincaré in the 1880s, and the relation to Minkowski space followed the development of special relativity in the early 20th century. We refer to [Rey93, §14] for a more detailed historical account.

4.1 Description of the hyperboloid

4.1.1 Hyperboloid of dimension 2

Let $M = \mathbb{R}^{n,1}$ be Minkowski space. For the moment, let us take $n = 2$.

Consider the set

$$\mathcal{H} := \{v \in M \colon \langle v, v\rangle = -1\}\,.$$

By definition, this is a pseudosphere: it is a level set of the quadratic form in the pseudo-Euclidean space M. In other words, abusing notations, this is the "sphere" $\{\|v\|^2 = R^2\}$ in M, with $R = \sqrt{-1}$.

Let us use coordinates $v = (x, y, z)$ on M, so that the quadratic form of Minkowski space is $\langle v, v\rangle = x^2 + y^2 - z^2$. In these coordinates, $\mathcal{H}$ is the quadric defined by the equation

$$x^2 + y^2 - z^2 + 1 = 0\,.$$

Such a surface is called a ***hyperboloid of two sheets***.

It is easy to check that $\mathcal{H}$ is invariant by rotations around the z-axis and by reflections through vertical planes through the origin. (Note that this is a particular case of Theorem 4.7).

The intersection of $\mathcal{H}$ with horizontal planes $\{z = z_0\}$ is empty for $|z_0| < 1$, and is the circle $x^2 + y^2 = z_0^2 - 1$ for $|z_0| > 1$. In particular, it is clear that $\mathcal{H}$ has two connected components $\mathcal{H}^+$ and $\mathcal{H}^-$, called upper and lower sheets. On the other hand, its intersection with a vertical plane is a hyperbola. Indeed, by rotational symmetry, it is enough to consider the plane $y = 0$; it intersects the hyperboloid is the hyperbola $z^2 - x^2 = 1$.

Note that the upper arc of this hyperbola can be parametrized using the ***hyperbolic*** trig functions: $(x = \sinh t, z = \cosh t)$ (this is the explanation for the name of these functions). We shall see in § 4.4 that this parametrized curve is a geodesic. The hyperbola is asymptotic to its axes with equation $z^2 - x^2 = 0$, i.e. $z = \pm x$. The hyperboloid $\mathcal{H}$ itself is asymptotic to the cone $x^2 + y^2 - z^2 = 0$ (which can be obtained by rotating the hyperbola's axes), in other words to the light cone $\langle v, v\rangle = 0$. See Figure 4.1 for an illustration.

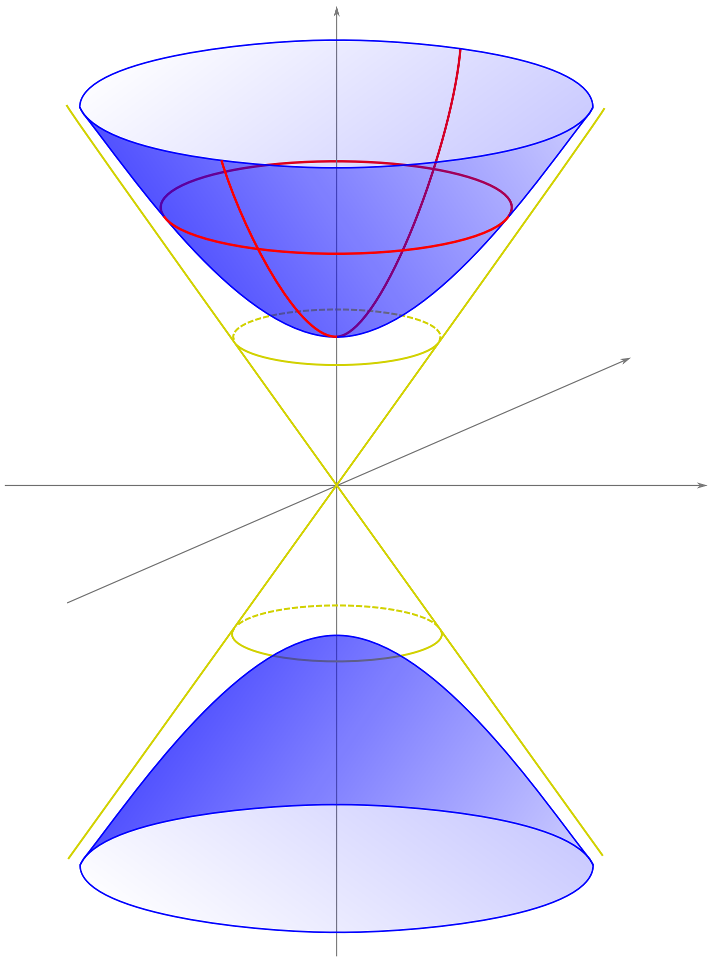


Figure 4.1: The hyperboloid $\mathcal{H}$ in $\mathbb{R}^{n,1}$.

In this chapter, we are interested in the upper sheet $\mathcal{H}^+$ of the hyperboloid.

4.1.2 Hyperboloid of dimension n

The previous story naturally generalizes to an arbitrary dimension $n \geqslant 1$. The (unit) hyperboloid of two sheets $\mathcal{H} \subseteq \mathbb{R}^{n,1}$ is still defined by

$$\mathcal{H} := \{v \in M \colon \langle v, v \rangle = -1\} .$$

In coordinates $v = (x_1, \ldots, x_{n+1})$ on Minkowski space, $\mathcal{H}$ is the quadric defined by the equation

$$x_1^2 + \cdots + x_n^2 - {x_{n+1}}^2 + 1 = 0 .$$

This quadric is invariant by rotations around the x_{n+1}-axis and by reflections through vertical planes through the origin. The intersection of $\mathcal{H}$ with horizontal hyperplanes $\{x_{n+1} = z_0\}$ is empty for $|z_0| < 1$, and is the sphere $x^2 + \cdots + x_n^2 = z_0^2 - 1$ for $|z_0| > 1$. Again, $\mathcal{H}$ has two connected components (sheets) $\mathcal{H}^+$ and $\mathcal{H}^-$, distinguished by the sign of x_{n+1}.

It is interesting to note that intersecting $\mathcal{H}$ with subspaces of M intersecting it yields lower dimensional hyperboloids:

Proposition 4.1. *Let W be a subspace of M intersecting $\mathcal{H}$. Then W is a Minkowski space, and $W \cap \mathcal{H}$ is the unit hyperboloid in W.*

Proof. Elementary: left as exercise. □

Again, the hyperboloid $\mathcal{H}$ is asymptotic to the light cone, which is the isotropic cone in Minkowski space. In coordinates:

$$x_1^2 + \cdots + x_n^2 - {x_{n+1}}^2 = 0 .$$

In the rest of this chapter, we use the notation $\mathcal{H}^+$ or $\mathbb{H}^n$ indistinctly to refer to the upper sheet of the hyperboloid equipped with the Riemannian metric defined below.

4.2 Riemannian metric

First we identify the tangent space:

Proposition 4.2. *The hyperboloid $\mathcal{H}$ is a smooth embedded surface in $\mathbb{R}^{n,1}$. Its linear tangent space $\mathrm{T}_p\,\mathcal{H}$ at a point $p \in \mathcal{H} \subseteq \mathbb{R}^{n,1}$ is the plane $p^\perp$.*

Remark 4.3. A couple of clarifications:

- Note that when we write $p^\perp$, we think of p as a vector in $\mathbb{R}^{n,1}$.
- We use the phrase ***linear tangent space*** to make it clear that it is a vector space. The ***affine tangent space*** at p is the affine plane through p in $\mathbb{R}^{n,1}$ with underlying vector space $\mathrm{T}_p\,\mathcal{H}$.

Proof. The hyperboloid is defined by the equation $q(p) = -1$, where $q(p) = \langle p, p\rangle$. The function q is $\mathcal{C}^\infty$ (it is a degree 2 polynomial), with derivative given by $\mathrm{d}q_p(h) = \langle p, h\rangle$. For any p, the derivative $\mathrm{d}q_p$ is not the zero linear form, since $\mathrm{d}q_p(p) = -1$. The map q is therefore a submersion, and it is a classical fact of differential geometry that any level set such as $q^{-1}(-1)$ is a smooth hypersurface. Moreover, the tangent space at p is the kernel of $\mathrm{d}q_p$, which is precisely $p^\perp$. □

We can apply Proposition 3.8 to see that $\mathrm{T}_v = p^\perp$ is spacelike. In other words, the restriction on the inner product of $\mathbb{R}^{n,1}$ is positive definite. We thus get:

Proposition 4.4. *The restriction of the inner product of $\mathbb{R}^{n,1}$ to $\mathcal{H}$ is a Riemannian metric.*

We now have a precise definition of the hyperboloid model:

Definition 4.5. The hyperboloid model of the hyperbolic plane is the upper sheet $\mathcal{H}^+$ equipped with the Riemannian metric induced from the Minkowski inner product.

Let us looks at the case $n = 2$ with coordinates (x, y, z) on $\mathbb{R}^{2,1}$: $\mathcal{H}^+$ is defined implicitly by the equation $x^2 + y^2 - z^2 = -1$ with $z > 0$, and the Riemannian metric is the restriction to $\mathcal{H}^+$ of the Minkowski metric

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2\,.$$

4.3 Isometries

We have seen in § 3.4 that the group of linear isometries of Minkowski space is $\mathrm{O}(n, 1)$. It is clear that these preserve the quadratic form $q(v) = \langle v, v\rangle$, therefore it preserves its level sets. In particular, $\mathcal{H}$ is invariant under the action of $\mathrm{O}(n, 1)$. Since the action of $\mathrm{O}(n, 1)$ on $\mathbb{R}^{n,1}$ is linear and preserves the inner product, its induced action on $\mathcal{H}$ is by Riemannian isometries. Of course, the action of an element $f \in \mathrm{O}(n, 1)$ is orientation-preserving if and only if $f \in \mathrm{SO}(n, 1)$. It is not hard to see that an element of $\mathrm{O}(n, 1)$ preserves $\mathcal{H}^+$ and $\mathcal{H}^-$ if it is time-orientation preserving, and exchanges them otherwise.

Theorem 4.6. The groups $\mathrm{O}^+(n, 1)$ and $\mathrm{SO}^+(n, 1)$ act isometrically on $\mathcal{H}^+$. Moreover:

(i) The action of $\mathrm{O}^+(n, 1)$ and $\mathrm{SO}^+(n, 1)$ on $\mathcal{H}^+$ is transitive.

(ii) For any $p \in \mathcal{H}^+$, the stabilizer $K_p \subseteq \mathrm{O}^+(n, 1)$ [resp. $K_p \subseteq \mathrm{SO}^+(n, 1)$] acts transitively on the set of [positive] orthonormal bases of $\mathrm{T}_p\,\mathcal{H}^+$. In particular, the action of K_p in $\mathrm{T}_p\,\mathcal{H}^+$ is transitive.

By definition, Theorem 4.6 shows that $\mathcal{H}^+$ is ***homogeneous*** and ***isotropic***. In particular, it satisfies Euclid's fourth postulate (E4).

Loosely speaking, Theorem 4.6 says $\mathcal{H}^+$ has a very big group of isometries. More precisely, it *contains* a very big group of isometries, namely $\mathrm{O}^+(n, 1)$; but in fact the full group of isometries cannot be bigger:

Theorem 4.7. The group of isometries of $\mathcal{H}^+$ is $\mathrm{O}^+(n,1)$, and the subgroup of orientation-preserving isometries is $\mathrm{SO}^+(n,1)$.

The proof of Theorem 4.6 and Theorem 4.7, as well as an expansion of the discussion preceding it, are treated in Exercise 4.1.

4.4 Geodesics

We are going to find the geodesics of the hyperboloid the same way we found the geodesics on the sphere in Exercise 2.3.

Let $p \in \mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ and $v \in \mathrm{T}_p\,\mathcal{H}^+$, i.e. $v \perp p$. Let us determine the geodesic γ_v. Denote by P the 2-plane in $\mathbb{R}^{n,1}$ containing p and v. Note that P has signature $(1,1)$ so it is nondegenerate, therefore the reflection r through P is a well-defined element of $\mathrm{O}(n,1)$ (see Example 3.12), moreover it fixes p so it must be in $\mathrm{O}^+(n,1)$. Call f the induced isometry of $\mathcal{H}^+$. The set of fixed points of f is $\mathcal{H}^+ \cap P$. Since $v \in P$, the geodesic γ_v must be contained by $\mathcal{H}^+ \cap P$ by Proposition 2.9. Note that $\mathcal{H}^+ \cap P$ is a 1-dimensional submanifold of $\mathcal{H}^+$ (a curve), so γ_v must simply be the constant speed parametrization of (an arc of) it.

Theorem 4.8. The geodesic γ_v in $\mathcal{H}$ with initial velocity $v \in \mathrm{T}_p\,\mathcal{H}^+$ is given by:

$$\gamma_v(t) = \cosh(\|v\|t)p + \sinh(\|v\|t)\frac{v}{\|v\|}\,. \tag{4.1}$$

Proof. By the previous discussion, it is enough to check that $\gamma(0) = p$, $\gamma'(0) = v$, γ has constant speed, and γ is contained in $\mathcal{H}^+ \cap P$. All these verifications are immediate. □

Remark 4.9. To be perfectly accurate, the previous argument only shows that γ_v is contained in $\mathcal{H}^+ \cap P$, so a priori the expression (4.1), call it $\tilde{\gamma}(t)$, only coincides with $\gamma_v(t)$ for t in some interval containing 0. However, by repeating the argument at another point $p_1 = \tilde{\gamma}(t_1)$ with $v_1 = \tilde{\gamma}'(t_1)$, we see that the curve $\tilde{\gamma}$ must also be the geodesic with these initial conditions. This proves that $\tilde{\gamma}$ is a geodesic for all t, hence it is a maximal geodesic.

Corollary 4.10. *The hyperboloid model $\mathcal{H}^+$ is complete.*

Proof. Theorem 4.8 shows that geodesics are defined for all time, i.e. $\mathcal{H}^+$ is geodesically complete. By the Hopf–Rinow theorem, this is equivalent to any of the well-known characterizations of complete Riemannian manifolds. □

Corollary 4.11. *Any two distinct points p and q in $\mathcal{H}^+$ are joined by a unique geodesic segment γ (up to parametrization), moreover γ is length-minimizing: $d(p,q) = L(\gamma)$.*

Proof. The discussion above shows that any geodesic through p and q must be contained in a 2-dimensional subspace $P \subseteq \mathbb{R}^{n,1}$. There is only one choice: it is the space spanned by p and q. This yields both existence and uniqueness of the geodesic, up to parametrization.

The fact that γ is length-minimizing is an immediate consequence of the standard fact in Riemannian geometry that there exists a length-minimizing geodesic between any two points in a complete Riemannian manifold: see [Lee18, Cor. 6.21]. □

Notice that Corollary 4.11 shows that the hyperboloid model satisfies Euclid's first postulate (E1), in its strictest interpretation, while Corollary 4.10 shows that it satisfies the second postulate (E2).

4.5 Distance

Theorem 4.12. The distance between any two points p and q in $\mathcal{H}^+$ is given by

$$d(p,q) = \angle(p,q) = \operatorname{arcosh}(-\langle p,q\rangle)$$

where $\angle(p,q)$ is the hyperbolic angle in $\mathbb{R}^{n,1}$.

Proof. By Corollary 4.11, it is enough to show that $d(p,q) = \angle(p,q)$ when $p = \gamma(t_0)$ and $q = \gamma(t_1)$ where γ is any geodesic. After reparametrizing, we can assume that $t_0 = 0$, $t_1 > 0$, and γ has unit speed. On the one hand, $d(p,q)$ is the length of γ between t_0 and t_1 since γ is the unique geodesic, that is $d(p,q) = t_1$. On the other hand, by Theorem 4.8, $\gamma(t) = \cosh(t)p + \sinh(t)v$ for some unit vector v, so we have $q = \cosh(t_1)p + \sinh(t_1)v$. Recall that the hyperbolic angle $\angle(p,q)$ is given by

$$\langle p,q\rangle^2 = \langle p,p\rangle \langle q,q\rangle \cosh^2 \angle(p,q)\,.$$

Here $\langle p,p\rangle = \langle q,q\rangle = -1$ and $\langle p,q\rangle = -\cosh t_1$, so we find $d(p,q) = t_1 = \operatorname{arcosh}(-\langle q,q\rangle)$, and $\cosh^2 t_1 = \cosh^2 \angle(p,q)$ yields $\angle(p,q) = t_1$. □

4.6 Curvature

The goal of this section is to prove:

Theorem 4.13. $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ has constant curvature sectional curvature -1.

This result holds in any dimension $n \geqslant 2$. Note that for $n = 1$, the hyperboloid $\mathcal{H}^+$ is still well-defined (it is an arc of hyperbola in $\mathbb{R}^{1,1}$), but the notion of sectional curvature is irrelevant for one-dimensional manifolds. First let us argue that it is enough to prove Theorem 4.13 in the case $n = 2$.

Consider a 2-plane $P \subseteq \mathrm{T}_p\,\mathcal{H}^+$. By definition, the sectional curvature $K_p(P)$ of $\mathcal{H}^+$ at p in the direction P is the Gaussian curvature at p of the surface $S_P \subseteq \mathcal{H}^+$ is the union of all geodesics whose initial tangent vector is in P (in the language of Riemannian geometry,

$S_P = \exp_p(P)$). We know from Theorem 4.8 that the geodesic with initial tangent vector v is $\mathcal{H}^+ \cap P_v$, where we have denoted P_v the 2-plane spanned by p and v. It follows that

$$\begin{aligned} S_P &= \bigcup_{v \in P} \mathcal{H}^+ \cap P_v \\ &= \mathcal{H}^+ \cap W \end{aligned}$$

where $W \subseteq \mathbb{R}^{n,1}$ is the 3-dimensional subspace spanned by p and P. Now, by Proposition 4.1, $S_P = \mathcal{H}^+ \cap W$ is the unit hyperboloid in the Minkowski space W, which has signature $(2, 1)$. Thus it is enough to show that cH^+ has sectional (i.e. Gaussian) curvature in the $n = 2$ case:

Theorem 4.14. $\mathcal{H}^+ \subseteq \mathbb{R}^{2,1}$ has constant Gaussian curvature -1.

In order to prove Theorem 4.14, we would like to use the fact that for surfaces in $\mathbb{R}^3$, the Gaussian curvature is equal to the product of the principal curvatures (i.e. the determinant of the second fundamental form in an orthonormal basis). For surfaces in Minkowski space $\mathbb{R}^{2,1}$, this result is still true, but with the opposite sign:

Lemma 4.15. *The Gaussian curvature of a surface $S \subseteq \mathbb{R}^{2,1}$ is equal to* minus *the product of the principal curvatures, i.e.* minus *the determinant of the second fundamental form in an orthonormal basis.*

The proof of this lemma requires some knowledge of Riemannian geometry, readers who have not taken a course in Riemannian geometry may skip what follows. Lemma 4.15 is a special case of the modified ***Gauss equation***:

Theorem 4.16. Let S be a spacelike hypersurface in a Lorentzian manifold M. The sectional curvature $\bar{K}$ of M and the sectional curvature K of S are related by:

$$\bar{K} = K + \det B\,. \tag{4.2}$$

Remark 4.17. More precisely, (4.2) means that for any orthonormal pair $\{X, Y\} \subseteq \mathrm{T}S$:

$$\bar{K}(X, Y) = K(X, Y) + B(X, X)B(Y, Y) - B(X, Y)^2\,.$$

We recall that in the Riemannian case, the Gauss equation is instead $\bar{K} = K - \det B$.

Proof. Choose a local unit normal N to S. Note that since S is a spacelike hypersurface, N must be timelike: $\langle N, N\rangle = -1$. As in the Riemannian case, the second fundamental form B may be defined by the formula

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N \tag{4.3}$$

where $\bar{\nabla}$ [resp. ∇] denotes the Levi-Civita connection of M [resp. S]. It is an elementary exercise which we leave to the reader to check that while this gives the formula

$$B(X, Y) = +\langle \bar{\nabla}_X N, Y\rangle$$

(instead of $B(X, Y) = -\langle \bar{\nabla}_X N, Y\rangle$), and that (4.3) and (4.6) yield the modified Gauss equation (4.2) (use the definition of sectional curvature with the Riemann curvature tensor). □

We can now prove Theorem 4.14:

Proof of Theorem 4.14 with extrinsic curvatures. By Lemma 4.15, we would like to show that the determinant of the second fundamental form of $\mathcal{H}^+ \subseteq \mathbb{R}^{2,1}$ is equal to 1 at any point p. Clearly, it is enough to show that the extrinsic curvature $\rho_p(v) = \langle \gamma_v''(0), N \rangle$ (which coincides with $B(v,v)$) is equal to 1 (or -1, depending on the choice of unit normal) for every unit vector $v \in \mathrm{T}_p M$. This is immediate to check with the explicit expression of γ_v given in (4.1). □

In the exercises, we will give two other nice proofs of the fact that $\mathcal{H}^+$ has constant sectional curvature -1:

- A classical proof using the Riemannian geometry notion of *Jacobi fields* is proposed in Exercise 4.3.
- A proof using distance between geodesics is proposed in Exercise 4.2.

4.7 Hyperbolic space of radius R

Instead of considering the "unit" hyperboloid $\mathcal{H} \subseteq \mathbb{R}^{n,1}$, we could instead have defined the hyperboloid $\mathcal{H}$ "of radius $R > 0$" by:

$$\mathcal{H}_R \coloneqq \{v \in M \colon \langle v, v \rangle = -R^2\} .$$

Everything we have seen about the unit hyperboloid $\mathcal{H} = \mathcal{H}_1$ works the same for $\mathcal{H}_R$, with some minor differences:

Riemannian metric. We still equip $\mathcal{H}_R^+$ with the metric induced from Minkowski space $\mathbb{R}^{n,1}$, which is positive definite.

Isometries. It is still the case that the group of isometries of $\mathcal{H}_R^+$ is the orthochronous Lorentz group $\mathrm{O}^+(n,1)$ and its group of orientation-preserving isometries is the special orthochronous Lorentz group $\mathrm{SO}^+(n,1)$.

Geodesics. Geodesics in $\mathcal{H}_R^+$ are still intersections on $\mathcal{H}_R^+$ with 2-planes in $\mathbb{R}^{n,1}$, but now the parametrization of the geodesic has to be written:

$$\gamma_v(t) = \cosh\left(\frac{\|v\|}{R} t\right) p + R \sinh\left(\frac{\|v\|}{R} t\right) \frac{v}{\|v\|} .$$

Distance. For the distance on $\mathcal{H}_R^+$, we now have

$$d(p,q) = R \angle(p,q)$$

where $\angle(p,q)$ is the hyperbolic angle in $\mathbb{R}^{n,1}$. This amounts to

$$d(p,q) = R \operatorname{arcosh}\left(\frac{-\langle p, q \rangle}{R^2}\right) .$$

Curvature. Following the same proof as before, the modified expression of geodesics leads to finding that $\mathcal{H}_R^+$ has constant sectional curvature $k = -\frac{1}{R^2}$. Let us recap the most important information:

Theorem 4.18. The upper sheet $\mathcal{H}_R^+$ of the hyperboloid of radius R is a complete and uniquely geodesic Riemannian manifold of constant sectional curvature $k = -\frac{1}{R^2}$.

We recall that in Riemannian geometry, one shows that such a model for the *space form of curvature* $k < 0$ is essentially unique: see the discussion of § 2.4 and in particular Theorem 2.20.

4.8 Exercises

Exercise 4.1. Isometries of the hyperboloid

The goal of this exercise is to dermine the group of isometries of hyperbolic space in the hyperboloid model, in particular to provide a careful proof of Theorem 4.7.

Let $M = \mathbb{R}^{n,1}$ be Minkowski space, denote $\mathcal{H}$ the hyperboloid of two sheets $\mathcal{H} = \{v \in M \colon \langle v, v\rangle = -1\}$, and $\mathcal{H}^+$ the upper sheet (with $x_{n+1} > 0$.)

(1) The goal of this question is to show that $\mathrm{O}^+(n,1)$ acts by isometries on $\mathcal{H}^+$.
   (a) Show that the action of $\mathrm{O}(n,1)$ on M leaves $\mathcal{H}$ invariant.
   (b) Show that $\mathrm{O}(n,1)$ acts on $\mathcal{H}$ by Riemannian isometries.
   (c) Show that $f \in \mathrm{O}(n,1)$ preserves $\mathcal{H}^+$ if and only if $f \in \mathrm{O}^+(n,1)$. Conclude that $\mathrm{O}^+(n,1) \subseteq \mathrm{Isom}(\mathcal{H}^+)$.
   (d) Optional: Show that $f \in \mathrm{O}^+(n,1)$ is orientation-preserving on $\mathcal{H}^+$ if and only if $f \in \mathrm{SO}^+(n,1)$. Conclude that $\mathrm{SO}^+(n,1) \subseteq \mathrm{Isom}^+(\mathcal{H}^+)$.

(2) The goal of this question is to show that, conversely, any isometry of $\mathcal{H}^+$ is induced by some element of $\mathrm{O}^+(n,1)$ acting on M.
   (a) Show that the action of $\mathrm{O}^+(n,1)$ on $\mathcal{H}^+$ is transitive. *Hint: use Exercise 3.4 (3).*
   (b) Derive from the previous question that it is enough to show that any isometry of $\mathcal{H}^+$ fixing some point is induced by some element of $\mathrm{O}(n,1)$ acting on M fixing that point.
   (c) Identify the subgroup K of $\mathrm{O}(n,1)$ fixing the point $v_0 = (0,\dots,0,1)$. Show that the induced action of K in $\mathrm{T}_{v_0}\mathcal{H}^+$ is transitive on the set of orthonormal bases of $\mathrm{T}_{v_0}\mathcal{H}^+$.
   (d) Let f be an isometry of $\mathcal{H}^+$ fixing v_0. Show that f is completely determined by its derivative at v_0.
   (e) Conclude that $\mathrm{Isom}(\mathcal{H}^+) = \mathrm{O}^+(n,1)$ and $\mathrm{Isom}^+(\mathcal{H}^+) = \mathrm{SO}^+(n,1)$.

Exercise 4.2. Distance between geodesics on the hyperboloid

We denote as usual $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ the upper sheet of the hyperboloid in Minkowski space. Let $p \in \mathcal{H}^+$ and let $v, w \in \mathrm{T}_p\mathcal{H}^+$ be an orthonormal pair of tangent vectors. It is a general fact of Riemannian geometry that the distance between the geodesics $\gamma_v(t)$ and $\gamma_w(t)$ satisfies

$$d\left(\gamma_v(t), \gamma_w(t)\right)^2 = 2t^2 - \frac{1}{3}K\, t^4 + O(t^5)$$

as $t \to 0$, where K denotes the sectional curvature of the plane spanned by v and w. (See § 2.3.3 for more information.)

(1) Show that $d\left(\gamma_v(t), \gamma_w(t)\right) = \operatorname{arcosh}\left(\cosh^2 t\right)$.

(2) Find the Taylor expansion of $\operatorname{arcosh}(\cosh^2 x)$ to order 3 as $x \to 0$.

(3) Conclude that $K = -1$.

(4) Show likewise that $\mathcal{H}^+_R$ has constant sectional curvature $-\frac{1}{R^2}$.

Exercise 4.3. Jacobi fields on the hyperboloid

We denote as usual $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ the upper sheet of the hyperboloid in Minkowski space.

(1) Let $v, w \in \mathrm{T}_p\,\mathcal{H}^+$ be an orthonormal pair. Let us define $\gamma\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathcal{H}^+$ by

$$\gamma(s,t) = \cosh(t)p + \sinh(t)\left[\cos(s)v + \sin(s)w\right] .$$

Show that:
 (i) $\gamma(s,\cdot)$ is a unit geodesic for all $s \in \mathbb{R}$,
 (ii) $\gamma(0,\cdot) = \gamma_v$.
Such a family γ is called a *variation of geodesics*.

(2) Let $J(t) = \frac{\partial}{\partial s}_{|s=0}\gamma(s,t)$. Check that $J(0) = 0$ and $J'(0) = w$. This is a *normal Jacobi field*.

(3) We admit the following fact: if $J(t)$ is a normal Jacobi field along a unit geodesic and satisfies $J''(t) + k(t)J(t) = 0$, then the sectional curvature of the plane spanned by $\gamma'(t)$ and $J(t)$ is equal to $k(t)$ for all $t > 0$[1]. Show that the plane spanned by v and w has curvature -1.

(4) Conclude that $\mathcal{H}^+$ has constant sectional curvature -1.

(5) Show similarly that the hyperboloid of radius R has constant sectional curvature $-\frac{1}{R^2}$.

Exercise 4.4. Horocycles on the hyperboloid

Let P be an affine plane in Minkowski space $\mathbb{R}^{2,1}$ whose underlying vector space $\vec{P}$ is the orthogonal of an isotropic vector n. The curve $\mathcal{H}^+ \cap P$ is called a *horocycle*.

(1) Show that $P = \{p \in \mathbb{R}^{2,1} \colon \langle p, n\rangle = c\}$ where c is a constant.

(2) Optional: Show that any two horocycles are congruent.

(3) Show that any horocycle is a parabola in $\mathbb{R}^{2,1}$.

(4) (*) Show that all the geodesics in $\mathcal{H}^+$ perpendicular to a given horocycle are asymptotic.

Exercise 4.5. Comparing hyperboloids

We denote $(\mathcal{H}^+_R, g_R)$ the upper sheet of the hyperboloid of radius R in $\mathbb{R}^{n,1}$ equipped with its Riemannian metric,

[1]Students who know Riemannian geometry should recall why this is true. It follows from the Jacobi equation $J''(t) + R(J(t),\gamma'(t))\gamma'(t) = 0$.

(1) Find a natural map $f\colon \mathcal{H}^+_R \to \mathcal{H}^+_1$.

(2) Compare g_R and f^*g_1. Recover the results of § 4.7.

Exercise 4.6. Euclid's fifth postulate for the hyperboloid

Does Euclid's fifth postulate hold for the hyperboloid model? Compute the angle of parallelism as a function of the distance a (see Figure 1.3).

CHAPTER 5

Relativity theory

In this "bonus chapter", we take an excursion to the amazing theory of relativity of Albert Einstein and explain its connection to hyperbolic geometry. In particular, we shall see that the relativistic addition of velocities can elegantly be expressed with a hyperbolic translation in the Klein model of hyperbolic space.

There were essentially two motivations for including this chapter:

- Hyperbolic geometry is tied—historically and mathematically—to relativity theory via Minkowski space and Lorentz transformations. This connection is a good enough pretext to take a dip into one of the most fascinating theories of Science!
- The relativistic addition of velocities can be seen as a hyperbolic analog of vector addition that can be used in Machine Learning for the construction of "hyperbolic neural networks" (see Chapter 17).

Note that this chapter does not fit in the linear flow of the book: first of all, it can safely be skipped; secondly, it refers (especially in § 5.2) to a few notions from later chapters (Chapter 7 and Chapter 11). First time learners of hyperbolic geometry, feel free to skip this chaper and return to it a later time!

5.1 Derivation of relativity theory

5.1.1 Introduction

Albert Einstein's discovery of relativity theory in the early 20th century was among the greatest and most shocking scientific revolutions in history. Contrary to classical physics, which proclaimed the immutable nature of space and time, relativity theory allows the measurement of distances and durations to depend on the observer. This idea seems so outrageous at first that it inspires a reaction akin to the amusing one of French poet Raymond Queneau:

> *I have read the articles in the papers about this German named Einstein and his relativity. It is in fashion these days and I am told that there is nothing to make of*

> *it... that it won't hold after looking at the facts; when it is 8am in the train station, it is not 7:55 in the train, even if the train is moving very fast.*[1]

Albeit counterintuitive from a physical standpoint, the theory of (special) relativity theory is actually remarkably simple to derive mathematically[2]. Essentially, it all follows from the hypothesis that the speed of light is a universal constant independent of the observer, a fact that had been observed by the Michel–Morley experiment in 1887 and confirmed to great accuracy in subsequent years.

From this hypothesis and a couple of basic physical principles it follows that the structure of spacetime (in the absence of matter and energy) is modelled on the Minkowski space $\mathbb{R}^{3,1}$ equipped with the group of Lorentz transformations $\mathcal{O}(3,1)$. It is then straightforward to derive the laws of motion, the concept of proper time, the celebrated formula $E = mc^2$, etc. General relativity, which we shall only briefly allude to, incorporates the concept that gravitation can be described by an alteration of the metric tensor, a beautiful insight of Albert Einstein which is the mathematical translation of the *equivalence principle.*

5.1.2 Setup

5.2 Relativistic addition of velocities

[1]This is my attempt at a translation of the original French quotation: *J'ai lu les articles dans les journaux sur cet Allemand qui se nomme Einstein et sa relativité. C'est à la mode en ce moment et il paraît qu'il n'y a rien à comprendre... que ça ne tiendra pas devant les faits ; quand il est 8 heures dans une gare, il n'est pas 8 heures moins cinq dans le train, même si ce train va très vite.*

[2]This is assuming, of course, a sufficient modern mathematical education. The point of view expressed here is personal, but I hope this chapter helps make a case for it.

Part III: Projective geometry and the Klein model

Projective geometry is all geometry.

– Arthur Cayley[3]

Projective geometry has opened up for us with the greatest facility new territories in our science, and has rightly been called the royal road to our particular field of knowledge.

– Felix Klein[4]

[3] Quoted from [New00].
[4] Quoted from [Bel86].

CHAPTER 6

Projective geometry

In this chapter we review fundamental notions of projective geometry, mostly from the modern point of view where a projective space is defined as the set of vector lines of a vector space. Projective geometry and hyperbolic geometry are two distinct subjects, so the reader may legitimately wonder what this chapter is doing in a course of hyperbolic geometry. We will need it for two reasons:

- The Klein model of hyperbolic space, covered in the next chapter, is an open set of a projective space. By nature, this model relies on projective geometry.
- We will see in Part IV that 2-dimensional and 3-dimensional hyperbolic geometry are intimately related to the geometry of the complex projective line $\mathbb{C}P^1$, with a central role played by the projective linear group $\mathrm{PGL}(2,\mathbb{C})$.

We deemed these reasons sufficient to justify the inclusion of this chapter, in addition to the opportunity to be introduced to another fascinating kind of geometry.

6.1 Projective spaces

6.1.1 Definition

Let $\mathbb{K}$ be a field. We will mostly be interested in the case $\mathbb{K} = \mathbb{R}$, but we will also benefit from including the case $\mathbb{K} = \mathbb{C}$. Consider a vector space V over $\mathbb{K}$.

Definition 6.1. The ***projective space of*** V, denoted $\mathbf{P}(V)$, is the set of vector lines in V.

Example 6.2. if $V = \{0\}$, then $\mathbf{P}(V)$ is empty. If V is 1-dimensional, then $\mathbf{P}(V)$ contains one element. When $V = \mathbb{K}^{n+1}$, the projective space $\mathbf{P}(\mathbb{K}^{n+1})$ is also denoted $\mathbb{K}P^n$.

For $x \in V - \{0\}$, let us denote $[x] = \mathbb{K}x$ the vector line containing x. Thus $[x]$ is an element of $\mathbf{P}(V)$.

Alternatively, one can define $\mathbf{P}(V)$ as a quotient of V. For this we put an equivalence relation $\sim$ on $(V - \{0\})/\sim$, namely collinearity. In other words: $x \sim y$ if $[x] = [y]$.

Definition 6.3. The projective space of V, denoted $\mathbf{P}(V)$, is the quotient set $(V - \{0\})/\sim$.

Note that this does not define exactly the same set as before: here the elements of $\mathbf{P}(V)$ are vector lines with zero removed. Nevertheless, there is an obvious identification between the two.

6.1.2 Topology

The advantage of the second definition is that when V has a topology, e.g. the usual topology when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and V is finite-dimensional, we can equip $\mathbf{P}(V)$ with the quotient topology. In order to understand this topology, it is useful to have yet another identification of $\mathbf{P}(V)$:

Proposition 6.4. *Assume $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and V is finite-dimensional. Equip V with any norm and let S denote the unit sphere. The inclusion $S \to V$ induces a homeomorphism $S/\mathrm{O}(1) \to \mathbf{P}(V)$ if $\mathbb{K} = \mathbb{R}$ or $S/\mathrm{U}(1) \to \mathbf{P}(V)$ if $\mathbb{K} = \mathbb{C}$, where we have denoted $\mathrm{O}(1) = \{\pm 1\}$ and $\mathrm{U}(1) = \{z \in \mathbb{C}^* : |z| = 1\}$.*

Corollary 6.5. *If V is a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then $\mathbf{P}(V)$ is a compact Hausdorff topological space.*

The proofs of Proposition 6.4 and Corollary 6.5 are elementary exercises of general topology, we leave them to the reader.

Example 6.6. The ***real projective plane*** $\mathbb{R}P^2$ is homeomorphic to $S^2/\pm 1$. This is a compact non-orientable surface.

Example 6.7. The complex projective line $\mathbb{C}P^1$ is homeomorphic to $S^3/\mathrm{U}(1)$. It turns out that this is homeomorphic to the 2-sphere S^2: this is known as the ***Hopf fibration***.

6.1.3 Projective subspaces and projective duality

An element of $\mathbf{P}(V)$ is called a ***point***. A ***projective subspace*** is a subset of the form $\mathbf{P}(W)$ where W is a subspace of V. When $\dim W = 2$, $\mathbf{P}(W)$ is called a ***(projective) line***; when $\dim W = 3$, $\mathbf{P}(W)$ is called a ***(projective) plane***, etc.

Let us denote V^* the dual space of V. For a subspace $W \subseteq V$, denote $W^\circ \subseteq V^*$ the ***annihilator*** of W: by definition, it consists of the linear forms who vanish on W. (Note: the notation $W^\perp$ is also used.) It is basic linear algebra to show that $\dim W^\circ = \operatorname{codim} W$, and the annihilator map is decreasing: if $W_1 \subset W_2$ then $W_2^\circ \subset W_1^\circ$.

Taking annihilators of subspaces induces a map between projective subspace of V and projective subspaces of V^*, namely

$$\mathbf{P}(W) \mapsto \mathbf{P}(W^\circ)\,.$$

This map is called ***projective duality***.

Proposition 6.8. *Let V be a finite-dimensional vector space. Projective duality is a bijective correspondence between projective subspaces of V and projective subspaces of V^*, and it is decreasing with respect to inclusion. Moreover, projective duality is involutive, in the sense that $\mathbf{P}(W^{\circ\circ}) = \mathbf{P}(W)$ under the identification $V^{**} \approx V$.*

The proof of Proposition 6.8 is elementary and left to the reader. (Exercise 6.1.)

Example 6.9. The most important example is when V is 3-dimensional, in other words in a projective plane $P = \mathbf{P}(V)$. In this case, projective duality defines a bijective correspondence between points [resp. lines] of P and lines [resp. points] of the projective dual.

6.1.4 Axioms of projective geometry

After defining lines and points, one can proceed to try and develop a projective geometry in the spirit of Euclid, starting with fundamental properties such as: there exists a unique line through any two points. In fact, there exists such a synthetic approach to projective geometry, which is in spirit more faithful to the historical development of projective geometry. Even though we will not follow this point of view, we mention it out of interest. We refer to [BR98] (or the German second edition [BR04]) for more details.

A ***projective space*** is a set P (the set of points), together with a set L of subsets of P (the set of lines), satisfying the axioms:

(P1) Each two distinct points belong to exactly one line.
(P2) If a, b, c, d are distinct points and the lines ab and cd meet, then so do the lines ac and bd.
(P3) Any line has at least three points on it.

Remark 6.10. The axiom (P2) is called *Veblen's axiom*. It amounts to saying that any two lines of a plane must intersect.

One can then define, for instance, a projective subspace by requiring that it is stable under taking the line through two points.

This axiomatic definition of projective geometry is almost equivalent to ours: the theorem of Veblen–Young says that if the dimension of a projective space is at least 3 (i.e. there exists two non-intersecting lines), then it is isomorphic to some projective space $\mathbf{P}(V)$ over a division ring $\mathbb{K}$. For dimensions 1 and 2, there are "exotic" examples.

6.1.5 Hyperplane at infinity

Let $\mathcal{P} = \mathrm{P}(V)$ be a projective space. Choose a projective hyperplane $\mathcal{H} = \mathbf{P}(H) \subset \mathcal{P}$. We shall call $\mathcal{H}$ the ***hyperplane at infinity***. Then $\mathcal{P} - \mathcal{H}$ may naturally be seen as an affine space. Indeed, choose an affine hyperplane $\bar{H} \subseteq V$ parallel to H (there is a one-parameter family of such hyperplanes; if V is Euclidean one can normalize the choice). Then any vector line in V not contained in H intersects $\bar{H}$ at a unique point; this yields a bijection $\mathcal{P} - \mathcal{H} \longrightarrow \bar{H}$. Hence the projective space $\mathcal{P}$ may be identified to the affine space $\bar{H}$, completed with the hyperplane at infinity $\mathcal{H}$.

Conversely, let $\bar{H}$ be an affine space and denote H the underlying vector space. Call ***point at infinity*** an equivalence class of parallel lines, equivalently a vector line in H, and denote $\partial_\infty \bar{H}$ the set of points at infinity. Then $\bar{H} \cup \partial_\infty \bar{H}$ can naturally be identified to a projective space $\mathcal{P}$ called the ***projective completion of*** $\bar{H}$, where $\partial_\infty \bar{H}$ is a projective hyperplane. Indeed, embed $\bar{H}$ as an affine hyperplane that does not go through the origin in a vector space V. Note that H is canonically identified to the vector hyperplane parallel to $\bar{H}$. The points of $\bar{H}$ are in bijection with the vector lines not contained in H, since each such line has a unique intersection with $\bar{H}$; while the points of $\partial_\infty \bar{H}$ are in bijection with the vector lines in H by definition. In other words, $\bar{H} \cup \partial_\infty \bar{H} \approx \mathbf{P}(V - H) \cup \mathbf{P}(H) = \mathbf{P}(V)$.

Example 6.11. Let E be the Euclidean affine plane. Observe that the projective completion of E is topologically a closed disk with diametrically opposed points identified. This is a well-known description of the projective plane as a topological surface.

Remark 6.12. In 3-dimensional Euclidean space, or more generally in any affine space, a ***central projection*** from a point O to an affine plane P not containing O sends each point M to the intersection of the line OM with P. The projection is not defined for points of the plane passing through O and parallel to P. The notion of projective space was originally introduced by extending the Euclidean space, that is, by adding points at infinity to it, in order to define the projection for every point except O. This description led to the modern definition of a projective space in terms of vector lines.

6.2 Coordinates

Let V be a vector space of dimension $n + 1$ and denote $\mathcal{P} = \mathbf{P}(V)$ the associated projective space. We choose a basis $(e_1, \dots, e_{n+1})$ of V, and denote $x = (x_1, \dots, x_{n+1})$ the associated system of coordinates.

6.2.1 Homogeneous coordinates

Recall that we denote $[x] \in \mathcal{P}$ the vector line through x. When using coordinates, we abbreviate $[x] = [(x_1, \dots, x_{n+1})]$ to $[x] = [x_1 : \dots : x_{n+1}]$. This notation is called ***homogeneous coordinates***[1].

Be wary that homogeneous coordinates are not coordinates in the usual sense: they are not unique. Indeed, $[x_1 : \dots : x_{n+1}] = [y_1 : \dots : y_{n+1}]$ whenever (x_j) and (y_j) are proportional. Also, note that $[0 : \dots : 0]$ is not allowed, because $[0]$ is not well-defined.

Example 6.13. Let $p = [a : b : c] \in \mathbb{R}P^2$. Then the projective line dual to p has Cartesian equation $ax + by + cz = 0$. We leave as an exercise to the reader to make this statement precise and prove it.

6.2.2 Affine charts

Given our coordinate system $(x_1, \dots, x_{n+1})$, we can choose the hyperplane at infinity $\mathcal{H} \subset \mathcal{P}$ defined by $\mathcal{H} = \mathbf{P}(H)$ where H is the vector hyperplane with equation $x_{n+1} = 0$. As a parallel affine hyperplane $\bar{H}$, choose the hyperplane with equation $x_{n+1} = 1$. Note that in this case,

[1] Homogeneous coordinates were introduced by the German mathematician Möbius in 1827.

$\mathcal{P} - \mathcal{H}$ consists of points of $\mathcal{P}$ with homogeneous coordinates $[X_1 \colon \dots \colon X_{n+1}]$ such that $X_{n+1} \neq 0$. The identification $\mathcal{P} - \mathcal{H} \to \bar{H}$ discussed in § 6.1.5 is given by

$$\begin{aligned} \mathcal{P} - \mathcal{H} &\to \bar{H} \\ [X_1 \colon \dots \colon X_{n+1}] &\mapsto \left(\frac{X_1}{X_{n+1}}, \dots \frac{X_n}{X_{n+1}}, 1 \right) . \end{aligned}$$

Clearly, we can identify H to $\mathbb{K}^n$ via the coordinates $(x_1, \dots, x_n)$, and the previous map is written

$$\begin{aligned} \varphi \colon \mathcal{P} - \mathcal{H} &\to \mathbb{K}^n \\ [X_1 \colon \dots \colon X_{n+1}] &\mapsto \left(x_1 = \frac{X_1}{X_{n+1}}, \dots x_n = \frac{X_n}{X_{n+1}} \right) . \end{aligned}$$

Assume $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. This map is a homeomorphism from $\mathcal{P} - \mathcal{H}$ to $\mathbb{K}^n$ and is called a ***affine patch*** or ***affine chart***.

Of course, we can similarly define affine charts φ_j for each $j \in \{1, \dots, n+1\}$, starting with the hyperplane H_j with equation $x_j \neq 0$. We leave it as an exercise to check that $\mathcal{P} - \mathcal{H}_j$ are open sets that cover $\mathcal{P}$, and that $\varphi_j \circ \varphi_i^{-1}$ is an analytic diffeomorphism for any $i, j \in \{1, \dots, n+1\}$. In other words, we have defined an analytic atlas on $\mathcal{P}$, which makes it an analytic manifold.

6.2.3 Projective frames

By definition, a ***projective frame*** in a projective space $\mathcal{P} = \mathbf{P}(V)$ is a $(n+2)$-tuple of points such that no projective hyperplane contains $n+1$ of them.

The point of this definition is that given a projective frame, one can make sense of homogeneous coordinates $[x_1 \colon \dots \colon x_{n+1}]$. Indeed, let $(p_1, \dots, p_{n+2})$ be a projective frame. Then, up to scalar multiplication by an element $\lambda \in \mathbb{K}^\times$, there exists a unique basis $(e_1, \dots, e_{n+1})$ of V such that $p_j = [e_j]$ for $1 \leqslant j \leqslant n+1$ and $p_{n+2} = [e_1 + \dots + e_{n+1}]$. Since this basis is unique up to scalar multiplication, the homogeneous coordinates $[x_1 \colon \dots \colon x_{n+1}]$ are well-defined.

We shall see in § 6.3 that a projective transformation is uniquely determined by the image of a projective frame.

6.3 Projective transformations

6.3.1 Projective linear maps

Let $\mathbf{P}(V)$ and $\mathbf{P}(W)$ be a projective space. Observe that any injective linear map $f \colon V \to W$ sends vector lines in V to vector lines in W, in other words it induces a map $\bar{f} \colon \mathbf{P}(V) \to \mathbf{P}(W)$. Explicitly: $\bar{f}([x]) = [f(x)]$. Such a map is called ***projective linear***. When $V = W$, $\bar{f}$ is also called a ***projective transformation*** or a ***homography***.

Remark 6.14. The definition of a projective linear map may be extended to include the case where f is not injective, but in this case the map $\bar{f}$ is only defined on the complement of the projective subspace $P = \mathbf{P}(\ker f)$.

Proposition 6.15. *If $f, g \colon V \to W$, are linear maps such that $\bar{f} = \bar{g}$, then there exists $\lambda \in \mathbb{K}^\times$ such that $f = \lambda g$.*

Proof. By definition, if $\bar{f} = \bar{g}$, then for every $x \in V - \{0\}$, $[f(x)] = [g(x)]$. In particular, $g(x) \in [g(x)] = [f(x)]$, therefore there exists $\lambda_x \in \mathbb{K}$ such that $g(x) = \lambda_x f(x)$. Using the linearity of f and g, it is easy to show that λ_x is independent of x. □

The linear map f is called a ***homogenization*** of the projective linear map $\bar{f}$. The content of Proposition 6.15 is that such a homogenization is unique up to scalar. When $V = W$, the map $f \mapsto \bar{f}$ defines an action of $\mathrm{GL}(V)$ on $\mathbf{P}(V)$, and Proposition 6.15 can be rephrased:

Proposition 6.16. *The kernel of the action of $\mathrm{GL}(V)$ on $\mathbf{P}(V)$ is the group of $\mathbb{K}^\times \operatorname{id}_V$ consisting of homotheties of V. More generally, the kernel of the action of a subgroup $G \subseteq \mathrm{GL}(V)$ is the subgroup of $G \cap \mathbb{K}^\times \operatorname{id}_V$ consisting of homotheties in G.*

6.3.2 The projective linear group

Let V be a $\mathbb{K}$-vector space. A ***homothety*** is an element of $\mathrm{GL}(V)$ of the form $\lambda \operatorname{id}_V$, with $\lambda \in \mathbb{K}^\times$. We denote $\mathbb{K}^\times \operatorname{id}_V \subseteq \mathrm{GL}(V)$ the subgroup of homotheties. For any subgroup $G \leqslant \mathrm{GL}(V)$, it is clear that the subgroup $G \cap \mathbb{K}^\times \operatorname{id}_V$ is a normal subgroup of G.

Definition 6.17. The ***projective group of*** G is the quotient group

$$PG := G \Big/ (G \cap \mathbb{K}^\times \operatorname{id}_V) .$$

In particular:

Definition 6.18. The ***projective linear group*** of V is the quotient group:

$$\mathrm{PGL}(V) := \mathrm{GL}(V) \big/ \mathbb{K}^\times \operatorname{id}_V .$$

The group $\mathrm{PGL}(\mathbb{K}^n)$ is also denoted $\mathrm{PGL}(n, \mathbb{K})$.

Remark 6.19. It is classical fact of linear algebra that for $G = \mathrm{GL}(V)$ or $G = \mathrm{SL}(V)$, the subgroup $G \cap \mathbb{K}^\times \operatorname{id}_V$ coincides with the center of G. In general though, it is only a subgroup: consider for instance an abelian group $G \leqslant \mathrm{GL}(V)$, such as the group of diagonal matrices. In a previous version of this text, I defined PG as the quotient of G by its center ZG, but Definition 6.18 is more adequate, especially in view of Proposition 6.20. There is nothing wrong with taking the quotient G/ZG, but one should probably not call it the projective group of G[2]. I thank Andy Sanders for helping me figure this out.

[2]For instance, you could call it the *inner group* of G, since the action of G on itself by inner automorphisms (conjugation) yields an isomorphism $G/ZG \xrightarrow{\sim} \operatorname{Inn} G$.

Note that given any action of a group G on a set X, one can turn it into a faithful action by replacing the group G by its quotient by the kernel of the action. That is the largest quotient of G acting faithfully on X. By Proposition 6.16, we thus have:

Proposition 6.20. *For any $G \leqslant \mathrm{GL}(V)$, the linear action of G on V induces a faithful action of PG on $P(V)$ by projective linear transformations. Moreover, PG is the largest quotient of G with this property.*

In particular:

Corollary 6.21. *The group of projective transformations of $\mathbf{P}(V)$ is canonically identified to* $\mathrm{PGL}(V)$.

Matrix representation. Let f be a projective transformation of $\mathbf{P}(V)$, i.e. f is an element of $\mathrm{PGL}(V)$. If V is given a basis $(e_1, \dots, e_{n+1})$, then f is represented by a matrix $M \in \mathrm{GL}(n, \mathbb{K})$, only defined up to scalar multiplication. Now, if x and y are points in $\mathbf{P}(V)$ represented by homogeneous coordinates $X = [X_1 : \dots : X_{n+1}]$ and $Y = [Y_1 : \dots : Y_{n+1}]$, then the relation $y = f(x)$ relates to $Y = MX$ in homogeneous coordinates, so up to scalar multiplication.

6.3.3 Collineations

We shall not use collineations, but let us mention them for completeness. By definition, a ***collineation*** between projective spaces is a map that preserves alignment of points. Equivalently, it is an order-preserving map with respect to inclusion of projective subspaces.

Clearly, projective linear maps are all collineations, but collineations are slightly more general since they include, for instance, any *automorphic collineation*: by definition this is a map that, in coordinates, is a field automorphism applied to the coordinates.

Note that for a projective space $\mathcal{P}$ of dimension 1, all the points are collinear, so any permutation of $\mathcal{P}$ is a collineation: this is not very interesting. However, for dimension at least 2, it can be shown that any collineation is a composition of a projective linear map and an automorphic collineation. Moreover, for $\mathbb{K} = \mathbb{R}$, there is no other automorphic collineation than the identity map, since $\mathbb{R}$ has no nontrivial field automorphisms. We thus have:

Theorem 6.22. Let $\mathcal{P}$ be a real projective space of dimension $\geqslant 2$. A map $f : \mathcal{P} \to \mathcal{P}$ is a collineation if and only if it is projective linear.

This result is known as the *(second) fundamental theorem of real projective geometry.*

Note however that in the case $\mathbb{K} = \mathbb{C}$, there is a nontrivial field automorphism: complex conjugation. In this case, in dimension $\geqslant 2$, collineations consist of projective linear maps and their composition with complex conjugation.

Remark 6.23. The point of collineations is that they can naturally be defined for projective spaces defined axiomatically (see § 6.1.4), so that they are the canonical notion of projective transformation in synthetic geometry. As far as we are concerned though, we will essentially view collineations as part of a dated approach to projective geometry, along with other notions such as *perspectivities*, which were historically introduced to describe perspective and projections in Euclidean geometry, before mathematicians realized that projective geometry was equivalent to the modern definition in terms of linear algebra.

6.3.4 Properties

There is a lot to say about the properties of projective transformations, including a classification in low dimensions, but this is beyond our scope. We discuss the important example of *central collineations* (also traditionally called *perspectivities*) in Exercise 6.4 (see Figure 6.1 for an illustration). Here let us only mention one important theorem, and reserve properties of projective linear maps related to cross-ratios in the next section.

Theorem 6.24. A projective linear map between projective spaces of the same dimension is uniquely determined by the image of a projective frame.

This theorem is sometimes referred to as the *first fundamental theorem of projective geometry*. With our definition of projective linear maps, the proof boils down to an elementary exercise of linear algebra. We leave it as an exercise for this chapter (Exercise 6.3).

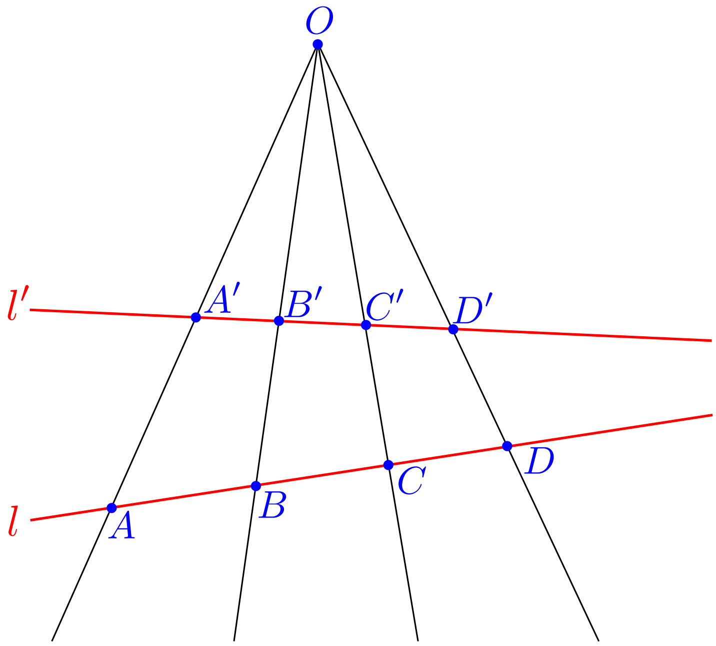


Figure 6.1: Central collineation

6.4 The projective line

Let us now examine more closely the 1-dimensional case: let $\mathcal{P} = \mathbf{P}(V)$ where V is a 2-dimensional vector space over a field $\mathbb{K}$. Such a one-dimensional projective space $\mathcal{P}$ is called a ***projective line***.

6.4.1 Coordinates

Choosing a basis of V amounts to choosing an isomorphism $V \approx \mathbb{K}^2$, which in turn induces an identification $\mathcal{P} \approx \mathbb{K}P^1$. A point of $\mathcal{P}$ is represented by homogeneous coordinates $[X\colon Y]$, where X and Y are elements of $\mathbb{K}$ that are not simultaneously 0.

Choose the hyperplane at infinity $Y = 0$: this contains a single point $[1\colon 0]$ (equal to $[X\colon 0]$ for any $X \neq 0$), which we denote ∞. Following § 6.2.2, we get an affine chart $\varphi = z\colon \mathbb{K}P^1 - \{\infty\} \to \mathbb{K}$ defined by $[X\colon Y] \mapsto \frac{X}{Y}$. We call this the ***standard affine chart*** (or ***standard affine coordinate***) on $\mathbb{K}P^1$. This allows us to identify the projective line $\mathbb{K}P^1$ with the extended line $\hat{\mathbb{K}} := \mathbb{K} \cup \{\infty\}$. In homogeneous coordinates, this identification is given by $[X\colon Y] \mapsto z = \frac{X}{Y}$, with the convention that that $\frac{X}{0} = \infty$ for $X \neq 0$.

When $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the extended line $\hat{\mathbb{K}}$ can be given the topology of the one-point compactification of $\mathbb{K}$, and it is an elementary exercise of topology to show that the identification $\mathbb{K}P^1 \approx \hat{\mathbb{K}}$ is a homeomorphism. For $\mathbb{K} = \mathbb{R}$, the extended line $\hat{\mathbb{R}}$ is a topological circle. For $\mathbb{K} = \mathbb{C}$, the extended line $\hat{\mathbb{C}}$ is a topological 2-sphere. By the discussion of § 6.2.2, $\mathbb{C}P^1 \approx \hat{\mathbb{C}}$ is a complex-analytic manifold, known as the ***Riemann sphere***.

Note that a projective frame of a projective line consists of 3 distinct points. The standard projective frame of $\mathbb{K}$ is given by the triple of points $[1\colon 0]$, $[0\colon 1]$, $[1\colon 1]$. In other words, using the standard affine coordinate z, this is the triple of points ∞, 0, 1.

6.4.2 Projective transformations

Following § 6.3, a projective linear automorphism f of $\mathbb{K}P^1$ coincides with an element of $\mathrm{PGL}(2, \mathbb{K})$. in other words, it is given by an invertible matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(2, \mathbb{K})$, unique up to scalar multiplication. In homogeneous coordinates, the map f is given by $f([X\colon Y]) = [aX + bY\colon cX + dY]$. In the standard affine chart z described above, this is rewritten:

$$f(z) = \frac{az + b}{cz + d}\,. \tag{6.1}$$

Note that we could have defined a map f from the extended line $\hat{\mathbb{K}}$ to itself by the expression (6.1) above, without any knowledge of projective transformations. Such maps are called ***linear fractional***. We have thus established that linear fractional transformations of $\hat{\mathbb{K}}$ are identified to projective linear transformations of $\mathbb{K}P^1$. This is the deep reason why the map $\mathrm{GL}(2, \mathbb{K}) \to \mathrm{Aut}(\hat{\mathbb{K}})$ defined by (6.1) is a group homomorphism, a fact that can otherwise be checked by direct computation.

Note that since a projective frame of $\mathbb{K}P^1$ consists of 3 distinct points, Theorem 6.24 says that given any triples of distinct points (p_1, p_2, p_3) and (q_1, q_2, q_3) in $\mathbb{K}P^1$, there exists a unique projective linear transformation f such that $f(p_j) = q_j$. Let us record this:

Theorem 6.25. The action of $\mathrm{PGL}(2,\mathbb{K})$ on $\mathbb{K}P^1$ is simply 3-transitive.

6.4.3 Cross-ratios

Let a, b, c, d be four distinct points in $\mathbb{K}P^1$. By Theorem 6.25, there exists a unique projective linear transformation f which sends the triple (a,b,c) to the standard projective frame $(\infty,0,1)$. By definition, $f(d)$ is the ***cross-ratio*** of the 4-tuple (a,b,c,d), denoted $[a,b,c,d] \coloneqq f(d)$.

Remark 6.26. Equivalently, the cross-ratio may be defined by declaring that the homogeneous coordinates of the point $[a,b,c,d]$ over the standard projective frame are equal to the homogeneous coordinates of d over the projective frame (a,b,c).

Proposition 6.27. *Under the identification $\mathbb{K}P^1 \approx \hat{\mathbb{K}}$ given by the standard affine chart, the cross-ratio of four distinct points is given by:*

$$[z_1,z_2,z_3,z_4] = \frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}\,.$$

Proof. Denote a, b, c, d the points of $\mathbb{K}P^1$ corresponding to z_1, z_2, z_3, z_4 respectively, so that if we have homogeneous coordinates $a=[a_1\colon a_2]$, etc, then $z_1=\frac{a_1}{a_2}$, etc. Let us find the cross-ratio by using the observation of Remark 6.26. In order to work in the projective frame (a,b,c), we need to remember that up to a multiplicative scalar, there exists a unique choice of λ_1, λ_2, λ_3 such that $\lambda_1(a_1,a_2)+\lambda_2(b_1,b_2)=\lambda_3(c_1,c_2)$. We may solve this for λ_1 and λ_2, this is just a 2×2 linear system of equations, which has the unique solution

$$\lambda_1 = \frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}\lambda_3$$
$$\lambda_2 = \frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}\lambda_3\,.$$

We take note that

$$\frac{\lambda_2}{\lambda_1} = \frac{a_1c_2-a_2c_1}{c_1b_2-c_2b_1}\,. \tag{6.2}$$

Call $e_1=\lambda_1(a_1,a_2)$ and $e_2=\lambda_2(b_1,b_2)$. By definition, saying that d has homogeneous coordinates $[k_1\colon k_2]$ over the projective frame (a,b,c) then means that

$$\lambda(d_1,d_2) = k_1e_1+k_2e_2 \tag{6.3}$$

for some scalar λ, which may be chosen $\lambda=1$ for the appropriate choice of (k_1,k_2). Again, the equation (6.3) can be solved for k_1 and k_2 as a 2×2 linear system of equations, one finds:

$$k_1 = \frac{d_1b_2-d_2b_1}{a_1b_2-a_2b_1}\,\frac{\lambda_3}{\lambda_1} \tag{6.4}$$
$$k_2 = \frac{a_1d_2-a_2d_1}{a_1b_2-b_1a_2}\,\frac{\lambda_3}{\lambda_2}\,.$$

Remark 6.26 says that the cross-ratio $[a, b, c, d]$ has homogeneous coordinates $[k_1 : k_2]$, so under the identification $\mathbb{K}P^1 \approx \hat{\mathbb{K}}$ it is given by $[z_1, z_2, z_3, z_4] = \frac{k_1}{k_2}$. With (6.4) we find

$$[z_1, z_2, z_3, z_4] = \frac{d_1 b_2 - d_2 b_1}{a_1 d_2 - a_2 d_1} \frac{\lambda_2}{\lambda_1}$$

and injecting (6.2) this is

$$[z_1, z_2, z_3, z_4] = \frac{(d_1 b_2 - d_2 b_1)(a_1 c_2 - a_2 c_1)}{(a_1 d_2 - a_2 d_1)(c_1 b_2 - c_2 b_1)} .$$

Dividing the numerator and denominator by $a_2 b_2 c_2 d_2$ yields

$$[z_1, z_2, z_3, z_4] = \frac{(z_4 - z_2)(z_1 - z_3)}{(z_1 - z_4)(z_3 - z_2)} .$$

□

A quicker (but less conceptual) proof of Proposition 6.27 is given in Exercise 6.5.

A fundamental property of cross-ratios is that they are invariant under projective transformations:

Theorem 6.28. For any four distinct points $a, b, c, d \in \mathbb{K}P^1$ and for any $f \in \mathrm{PGL}(2, \mathbb{K})$,

$$[f(a), f(b), f(c), f(d)] = [a, b, c, d] .$$

Proof. Let f_0 be the unique projective linear transformation that sends (a, b, c) to $(\infty, 0, 1)$. By definition of the cross-ratio, $[a, b, c, d] = f_0(d)$. Define $f_1 = f_0 \circ f^{-1}$ and observe that f_1 sends $(f(a), f(b), f(c))$ to $(\infty, 0, 1)$. By definition of the cross-ratio, $[f(a), f(b), f(c), f(d)] = f_1(f(d))$. Since $f_1(f(d)) = f_0(d) = [a, b, c, d]$, we are done. □

Remark 6.29. The reader should appreciate how the proof above is more concise and elegant than a direct proof using the expression of f as linear fractional transformation.

An important consequence is that on any projective line $\mathcal{P}$, the cross-ratio of four points is well-defined as an element of $\mathbb{K}P^1 \approx \hat{\mathbb{K}}$: it does not depend on the choice of a projective linear identification $\mathcal{P} \approx \mathbb{K}P^1$. In other words, the choice of any projective frame on $\mathcal{P}$ gives an identification $\mathcal{P} \approx \mathbb{K}P^1$ which allows one to define the cross-ratio of four distinct points, and the result is independent of the choice of the projective frame. In particular, let us record:

Proposition 6.30. *In a projective space $\mathcal{P}$ over a field $\mathbb{K}$, the cross-ratio of any four collinear points is a well-defined element of $\hat{\mathbb{K}}$.*

Remark 6.31. Note that in a projective plane (dimension 2), one can define the cross-ratio of any four concurrent lines by projective duality.

The next theorem is an immediate consequence of Theorem 6.28:

Theorem 6.32. Projective linear maps between projective spaces preserve the cross-ratios of 4-tuples of collinear points.

Example 6.33. In Figure 6.1, the cross-ratios $[A, B, C, D]$ and $[A', B', C', D']$ must be equal, since these 4-tuples of points differ by a central collineation, which is a projective transformation.

Example 6.34. Figure 6.2 features an application of Theorem 6.32 borrowed from Wikipedia [Wik19]: using cross-ratios to measure real-world dimensions from a photo. One can either derive the width of the adjacent street w from the widths AB and CD of the adjacent shops, using the points A, B, C, D; or from the width AB of only one adjacent shop using the points A, B, C, V. See Exercise 6.6 for details.

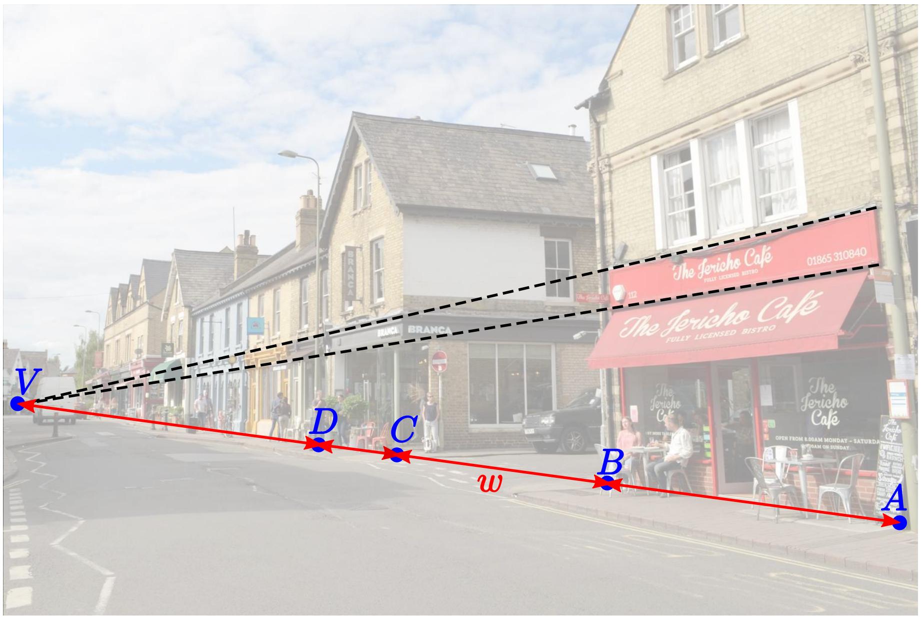


Figure 6.2: Use of cross-ratios to measure real-world dimensions.

6.5 Quadrics

6.5.1 Homogeneous functions

Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and let $f : V \to \mathbb{K}$ be a function, e.g. a polynomial function. f is called ***homogeneous of degree*** d if $f(\lambda v) = \lambda^d f(v)$ for all $\lambda \in \mathbb{K}$ and $v \in V$. When f is a polynomial function and $\mathbb{K}$ is infinite, it is an elementary exercise of algebra to show that f is homogeneous of degree d if and only if it is a sum of monomials that each has total degree d.

Example 6.35. Any linear form on V is homogeneous of degree 1, any quadratic form is homogeneous of degree 2.

Example 6.36. $f(x, y, z) = 2x^5 - y^2z^3 + 3x^3yz$ is a homogeneous polynomial of degree 5.

Note that when f is a homogeneous function (of any degree), the set of zeros of f in V is invariant by scalar multiplication. In other words, it is a union of vector lines. This is called a ***cone***. Any cone, as a set of vector lines, can naturally be seen as a subset of $\mathbf{P}(V)$. Thus we have established that the equation $f = 0$ defines a subset of $\mathbf{P}(V)$ when f is a homogeneous function.

Example 6.37. $f(x, y, z) = x^2 + y^2 - z^2$ is a homogeneous polynomial of degree 2 on $\mathbb{R}^3$. The equation $x^2 + y^2 - z^2 = 0$ defines a subset $\mathcal{C}$ of the projective plane $\mathbb{R}P^2$. This set $\mathcal{C}$ is the unit circle in the affine chart $z = 1$, more generally it is an ellipse in any affine chart given by a line at infinity which does not intersect $\mathcal{C}$.

Let us mention that, in algebraic geometry, a ***projective variety*** is defined as the set of the common zeros of a set of homogeneous polynomials.

Any polynomial function can be ***homogenized*** by adding an extra dimension. Indeed, if $P(X_1, \ldots, X_n)$ is any polynomial degree d in n variables, the expression

$$\hat{P}(X_1, \ldots, X_n, X_{n+1}) = X_{n+1}^d P\left(\frac{X_1}{X_{n+1}}, \ldots \frac{X_n}{X_{n+1}}\right)$$

defines a homogeneous polynomial of degree d in $n+1$ variables. Similarly, if $f\colon V \longrightarrow \mathbb{K}$ is a polynomial function of degree d, there exists a unique homogeneous function $\hat{f}\colon V \times \mathbb{K} \longrightarrow \mathbb{K}$ of degree d such that $f(v) = \hat{f}(v, 1)$.

Example 6.38. The homogenization of $f(x) = x^3 + ax + b$ is given by $\hat{f}(x, y) = x^3 + axy^2 + by^3$.

6.5.2 Projective quadrics

By definition, a ***projective quadric*** in a projective space $\mathbf{P}(V)$ over a field $\mathbb{K}$ is the subset $\mathcal{C} \subseteq \mathbf{P}(V)$ defined by the vanishing of a homogeneous polynomial of degree 2.

A homogeneous polynomial of degree 2 is the same thing as a quadratic form: this follows from the polarization identity $B(u, v) = \frac{1}{2}\left(f(u + v) - f(u) - f(v)\right)$.

Proposition 6.39. *Assume $\mathbb{K}$ has characteristic 0. Then any homogeneous polynomial of degree 2 on V is given by $f(v) = B(v, v)$, where B is a symmetric bilinear form on V.*

We can now make use of our knowledge of quadratic forms (see § 3.1). When $\mathbb{K} = \mathbb{R}$, it is a consequence of Sylvester's law of inertia (see Proposition 3.4) that projective quadrics are classified by the signature of the associated quadratic form:

Theorem 6.40. Let $\mathcal{C} \subseteq \mathbf{P}(V)$ be a projective quadric. There exists a pair of nonnegative integers (p, q) with $p + q \leqslant n$ (where $n = \dim V$) and a basis of V such that $\mathcal{C}$ is given by the equation:

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 0\,.$$

This is called the ***normal form*** of a quadric. The quadric is called ***nondegenerate*** if the associated quadratic form is nondegenerate, i.e. $p + q = n$.

Example 6.41. Quadrics in $\mathbb{R}P^2$ are called ***conics***. There are two normal forms of nondegenerate conics up to sign:

(1) $x^2 + y^2 + z^2 = 0$
(2) $x^2 + y^2 - z^2 = 0$

In the first case, the conic is the empty set. In the second case, choosing an affine chart, the conic is either an ellipse if the line at infinity does not intersect it (e.g. $z = 0$), or a hyperbola if the line at infinity intersects it along two points (e.g. $y = 0$), or a parabola if the line at infinity intersects it along one point (e.g. $z = x$).

Example 6.42. Quadrics in $\mathbb{R}P^3$ are called ***quadric surfaces***. There are three normal forms of nondegenerate quadric surfaces up to sign:

(1) $x^2 + y^2 + z^2 + t^2 = 0$
(2) $x^2 + y^2 + z^2 - t^2 = 0$
(3) $x^2 + y^2 - z^2 - t^2 = 0$

In the first case, the quadric is the empty set. In the second case, choosing an affine chart, the quadric is either an ellipsoid, an elliptic paraboloid or a hyperboloid of two sheets, depending on whether the plane at infinity intersects the quadric in the empty set, in a point, or in a nondegenerate conic respectively. In the third case, the quadric is either a hyperbolic paraboloid or a hyperboloid of one sheet, depending on whether the plane at infinity intersects it in two lines or in a nondegenerate conic.

Remark 6.43. In Example 6.42, the Gaussian curvature of the quadric is always positive in the second case, and always negative in the third case. The fact that the choice of the hyperplane at infinity does not change the sign of the Gaussian curvature is not a coincidence: see Exercise 6.9

Note that when $\mathbb{K} = \mathbb{C}$, quadratic forms on V are completely classified by their rank. In particular, there is only one nondegenerate quadric in $\mathbb{C}P^n$ up to projective transformations.

6.5.3 Projective completion of affine quadrics

By definition, an ***affine quadric*** in an affine space $\mathcal{V}$ is the zero set of a polynomial function of degree 2.

Example 6.44. When $\dim \mathcal{V} = 2$, an affine quadric in $\mathcal{V}$ is called a ***conic***. For instance, the equation $x^2 - 3y^2 + 2xy - 6x + y - 7 = 0$ defines a conic in $\mathbb{R}^2$.

Example 6.45. When $\dim \mathcal{V} = 3$, an affine quadric in $\mathcal{V}$ is called a ***quadric surface***. For instance, the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ defines a quadric in $\mathbb{R}^3$ known as an ***ellipsoid***.

Theorem 6.46. If $\hat{\mathcal{C}}$ is a projective quadric in a projective space $\mathcal{P}$, then for any choice of projective hyperplane at infinity $\mathcal{H}$, $\mathcal{C} := \hat{\mathcal{C}} - \mathcal{H}$ is an affine quadric in the affine space $\mathcal{V} = \mathcal{P} - \mathcal{H}$. Conversely, if $\mathcal{C}$ is an affine quadric in an affine space $\mathcal{V}$, then it can uniquely be completed to a projective quadric $\hat{\mathcal{C}}$ in the projective space $\mathcal{V} \cup \{\infty\}$.

The proof of Theorem 6.46 is instructive: it is important to remember that $\hat{C}$ is obtained from $\mathcal{C}$ by homogenization its equation, as described in § 6.5.1. $\hat{C}$ is called the ***projective completion*** of $\mathcal{C}$.

Proof. Let $\hat{\mathcal{C}}$ be a projective quadric in a projective space $\mathcal{P} = \mathbf{P}(V)$ and let $\mathcal{H} = \mathbf{P}(H)$ be a projective hyperplane. Choose coordinates $X_1, \ldots, X_{n+1}$ on V such that H is given by $X_{n+1} = 0$. The affine space $\mathcal{V} = \mathcal{P} - \mathcal{H}$ may then be identified to the affine hyperplane $\bar{H}$ with equation $X_{n+1} = 1$. Recall that the identification is given by the chart

$$\varphi\colon [X_1 : \ \ldots : X_{n+1}] \mapsto \left(x_1 = \frac{X_1}{X_{n+1}}, \ldots, x_n = \frac{X_n}{X_{n+1}}\right).$$

Let $\hat{f}\colon V \longrightarrow \mathbb{K}$ be the function defining $\hat{\mathcal{C}}$. By definition, there exists a homogeneous polynomial P of degree 2 in $n+1$ variables such that $\hat{f}(v) = P(X_1, \ldots, X_{n+1})$. For any $v \in \mathcal{V}$, we have $X_{n+1} \neq 0$, so the homogeneity of $\hat{f}$ implies that

$$\hat{f}(v) = X_{n+1}^2 P\left(\frac{X_1}{X_{n+1}}, \ldots, \frac{X_n}{X_{n+1}}, 1\right).$$

In other words, we have $\hat{f}(v) = X_{n+1}^2 f(x_1, \ldots, x_n)$, where $f(x_1, \ldots, x_n) = \hat{f}(x_1, \ldots, x_n, 1)$. Such a function f is a degree 2 polynomial in n variables, called the ***dehomogenization*** of $\hat{f}$. On $\mathcal{V}$, we thus have $\hat{f}(v) = 0$ if and only if $f(\varphi(v)) = 0$. This shows that in the affine chart φ, the set $\mathcal{C} = \hat{\mathcal{C}} - \mathcal{H}$ is the affine quadric with equation $f = 0$.

Conversely, let $\mathcal{C}$ be an affine quadric in an affine space $\mathcal{V}$, given by the equation $f = 0$ where f is a polynomial function of degree 2. Choose coordinates $(x_1, \ldots, x_n)$ on $\mathcal{V}$, identifying it with the affine hyperplane with equation $x_{n+1} = 1$ in $V = \mathbb{K}^{n+1}$. Then the homogenization $\hat{f}$ of f provides a homogeneous polynomial function of degree 2 on V, and the projective quadric $\hat{C}$ with equation $\hat{f} = 0$ is a completion of $\mathcal{C}$, since $\hat{\mathcal{C}}$ coincides with $\mathcal{C}$ in restriction to the affine patch $\{x_{n+1} = 1\}$. □

Example 6.47. The parabola $y = x^2$ is an affine conic in the Euclidean plane, its projective completion is the conic $yz = x^2$ in $\mathbb{R}P^2$. Choosing a line at infinity that does not intersect it such as $z = -y$, this conic becomes an ellipse in an associated affine chart, for instance $z = -y + 2$ yields $x^2 + (y-1)^2 = 1$. On the other hand, a line at infinity that intersects it twice, such as $z = y$, yields a hyperbola in an associated affine chart, e.g. $z = y + 2$ yields $x^2 - (y+1)^2 = -1$. This is an illustration of the fact that moving the line at infinity allows one to change from ellipses to hyperbolas, transitioning through parabolas. This can nicely be observed in $\mathbb{R}^3$ by taking intersections of a cone with different affine planes.

6.6 Exercises

Exercise 6.1. Projective duality

Let V be a finite-dimensional vector space.

(1) What is projective duality?

(2) If V is equipped with a pseudo-inner product, can you relate taking the orthogonal and taking the annihilator of subspaces of V?

(3) Let P be a projective plane. Prove that any two lines of P intersect at a unique point: first write a direct proof, then propose an alternate proof using projective duality.

Exercise 6.2. Axioms of projective geometry

Let V be a vector space over a field $\mathbb{K}$. Show that the projective space $\mathbf{P}(V)$ satisfies the axioms of projective geometry (see § 6.1.4).

Exercise 6.3. First fundamental theorem of projective geometry

Prove Theorem 6.24: *A projective linear map between projective spaces of the same dimension is uniquely determined by the image of a projective frame.*

Exercise 6.4. Central collineations

Let $\mathcal{P} = \mathbf{P}(V)$ be a projective space of dimension $\geqslant 2$. Given a point O and a projective hyperplane $\mathcal{H}$, a ***central collineation*** of center O and axis H is a projective transformation $f \colon \mathcal{P} \to \mathcal{P}$ such that $\mathcal{H}$ is fixed pointwise by f, and any line through O is preserved by f. Let $\hat{f}$ denote the element of $\mathrm{GL}(V)$ associated to f. (Is $\hat{f}$ well-defined?)

(1) Show that f is a central collineation if and only if $\hat{f}$ admits an eigenspace of codimension 1.

(2) A central collineation is called an ***elation*** if its center belongs to its axis, and a ***homology*** otherwise. Show that a central collineation f is a homology if and only if $\hat{f}$ is diagonalizable.

(3) Let f be a central collineation of the projective plane with center O. Let l be a line and let $l' = f(l)$. Show that for any point A on l, $A' = f(A)$ is the intersection of the lines l and OA. Comment Figure 6.3.

(4) In Figure 6.3, prove that $[A, B, C, D] = [A', B', C', D']$.

(5) (*) Show that every homography is the composition of a finite number of central collineations.

This result is sometimes known as the third fundamental theorem of projective geometry.

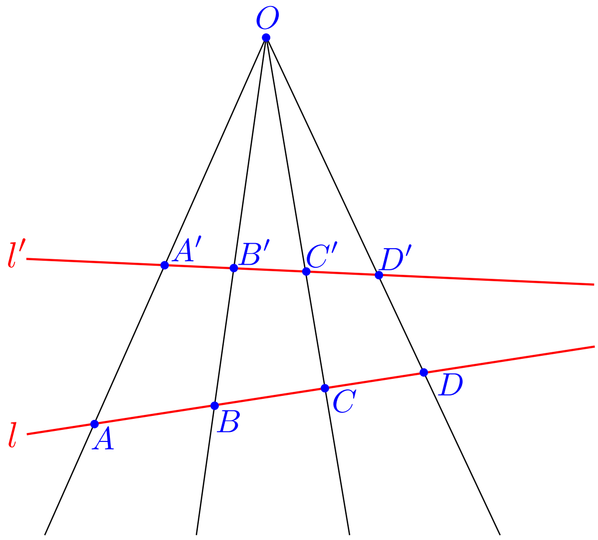


Figure 6.3: Central collineation in the projective plane.

Exercise 6.5. Formula for the cross-ratio

Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{K} \cup \{\infty\}$. Check that the map

$$f : z \mapsto \frac{(z - z_2)(z_1 - z_3)}{(z_1 - z)(z_3 - z_2)}$$

is a linear fractional transformation that maps z_1 to ∞, z_2 to 0, and z_3 to 1. Recover the formula for the cross-ratio.

Exercise 6.6. Cross-ratios and metrology

Consider the photo of Figure 6.4 (taken from Wikipedia). Denote by A, B, C, D, V the points in the real world, and by A', B', C', D', V' the points in the image. On the photo, one can measure the lengths (in pixels):

$$A'B' = 30\text{px} \quad B'C' = 20\text{px} \quad C'D' = 10\text{px} \quad D'V' = 60\text{px}$$

The goal is to determine the width $w = BC$ (in meters) of the side street.

(1) Justify the equality of cross-ratios $[A, B, C, D] = [A', B', C', D']$. Given the widths of the adjacent shops $AB = 7$m and $CD = 6$m, show that $w = 8$m.

(2) Justify the equality of cross-ratios $[A, B, C, V] = [A', B', C', V']$. Recover the result $w = 8$m using only the width of one adjacent shop $AB = 7$m.

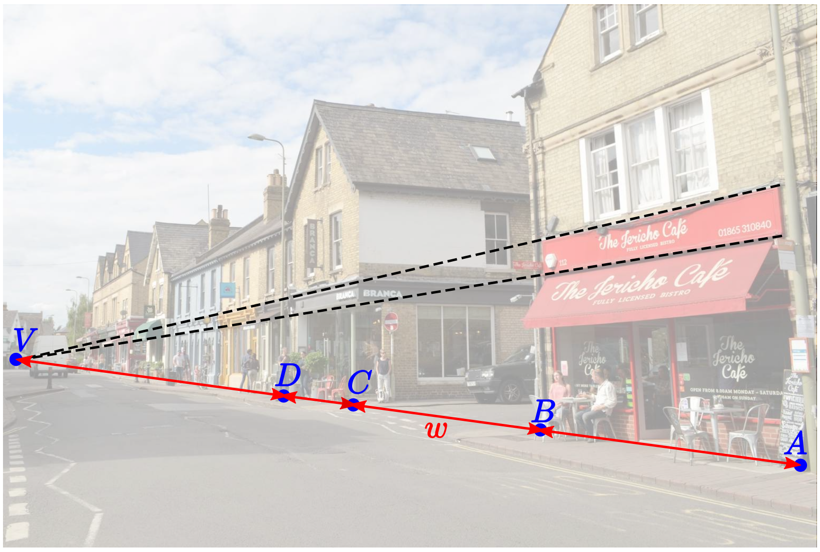


Figure 6.4: Use of cross-ratios to measure real-world dimensions.

Exercise 6.7. From a hyperboloid of two sheets to a sphere

Consider the hyperboloid $\mathcal{H}$ of two sheets with equation $x^2 + y^2 - z^2 = -1$ in $\mathbb{R}^3$.

(1) Show that by moving the plane at infinity $\partial_\infty \mathbb{R}^3$, the projective completion of $\hat{\mathcal{H}}$ can be seen as a sphere.

(2) Determine $\partial_\infty \mathcal{H}$ (the intersection of $\hat{\mathcal{H}}$ with the plane at infinity $\partial_\infty \mathbb{R}^3$). Can you describe why $\mathcal{H} \cup \partial_\infty \mathcal{H}$ is a topological sphere?

Exercise 6.8. Determinant quadric

Let $V = \mathcal{M}_{2\times 2}(\mathbb{K})$ denote the vector space of 2×2 matrices over a field $\mathbb{K}$.

(1) Show that the determinant function $\det\colon V \to \mathbb{K}$ is a quadratic form.

(2) Show that the set of non-invertible matrices defines a nondegenerate quadric in $\mathbf{P}(V)$. Find its normal form when $\mathbb{K} = \mathbb{R}$. *Optional: find an affine chart in which it is a hyperboloid of one sheet, and another where it is a hyperbolic paraboloid.*

(3) Show that $\mathrm{SL}(2, \mathbb{K})$ is an affine quadric in V. What is its projective completion? *Optional: when $\mathbb{K} = \mathbb{R}$, find an affine chart in which it is a hyperboloid of one sheet, and another where it is a hyperbolic paraboloid.*

Exercise 6.9. Gaussian curvature of quadric surface (*)

Show that the sign of the Gaussian curvature of a surface is a projective invariant. Determine the sign of the Gaussian curvature of the quadric surfaces in normal form.

CHAPTER 7

The Klein model

In this chapter we introduce and study the Klein model of hyperbolic space. This is a *projective* model: although it can simply be described as a disk (a ball in higher dimensions) with a special metric, it is best understood as a subset of the projective plane. In fact, the most natural definition of the Klein model makes it a special case of a *Cayley–Klein geometry*, which is a geometry that can be defined in the complement of a quadric in projective space. Remarkably, Euclidean geometry and elliptic geometry are also examples of Cayley–Klein geometries.

Historically, the Klein model was actually discovered by Eugenio Beltrami in 1868 ([Bel68a; Bel68b]), alongside what is now called the Poincaré models which we discuss in Chapter 9. While Beltrami described the model as a disk where chords are geodesics, Klein ([Kle71; Kle73]) showed its projective nature and gave the formula for the metric in terms of cross-ratios, inspired by work of Cayley [Cay59]. For a more detailed historical account, refer to [AP15].

7.1 Cayley–Klein geometries

In the complement of any quadric in projective space, one may define the Cayley–Klein "metric" using cross-ratios. Although we are mostly concerned with one case, namely the interior of an ellipsoid which will offer the Cayley–Klein model of hyperbolic space, it will be interesting to see that elliptic geometry can also be derived as a Cayley–Klein geometry, and even Euclidean geometry as a degenerate case. For a more extensive treatment, I recommend the paper [FS19]. Another good reference is the book [Ric11], which is very thorough.

7.1.1 The Cayley–Klein metric

Let $\mathcal{Q}$ be a quadric in a real projective space $\mathcal{P} = P(V)$ of dimension n. We denote $q\colon V \to \mathbb{R}$ a quadratic form defining $\mathcal{Q}$, i.e. so that $\mathcal{Q}$ is the cone $\{q = 0\}$, and $b\colon V \times V \to \mathbb{R}$ the associated symmetric bilinear form. In our setup, the quadric $\mathcal{Q}$ will be fixed. The following terminology is due to Cayley:

Definition 7.1. We shall call the quadric $\mathcal{Q} \subseteq \mathcal{P}$ the ***absolute***.

Example 7.2. When $\mathcal{Q}$ is of signature $(n, 1)$, it is called an ***ellipsoid***. By Sylvester's law of inertia, in suitable homogeneous coordinates $[X_1 : \ldots : X_{n+1}]$, $\mathcal{Q}$ is given by the equation

$$X_1^2 + \cdots + X_n^2 - X_{n+1}^2 = 0 .$$

Note that $\mathcal{Q}$ does not intersect the hyperplane $X_{n+1} = 0$, therefore $\mathcal{Q}$ is contained in the affine chart $\mathcal{P} - \{X_{n+1} = 0\}$, and its equation in the inhomogeneous coordinates $x_k = \frac{X_k}{X_{n+1}}$ is:

$$x_1^2 + \cdots + x_n^2 - 1 = 0 .$$

Thus we see in that $\mathcal{Q}$ is a sphere in such coordinates.

Typically, one could expect that $\mathcal{P} - \mathcal{Q}$ has two connected components determined by the sign of the quadratic form: $\Omega^+ := \{[x] : q(x) > 0\}$ and $\Omega^- := \{[x] : q(x) < 0\}$, e.g. the exterior and the interior of the ellipsoid.

Now let x, y be two points in $\mathcal{P} - \mathcal{Q}$, and consider the intersection of the line (xy) with the absolute $\mathcal{Q}$. In any affine chart, the points of intersection solve a polynomial equation of degree 2 in one variable, therefore there are three possibilities:

- There are two points of intersection I and J: the line (xy) is called ***hyperbolic***. For instance, this is the case when $\mathcal{Q}$ is an ellipsoid and $x, y \in \Omega^-$ are any two interior points as in Figure 7.1.
- There is one double point of intersection $I = J$: the line (xy) is called ***parabolic***.
- There are no points of intersection: the line (xy) is called ***elliptic***. In this case, one may still define I and J as complex points, that live in the complex projective space $\mathcal{P}^c := P(V \otimes \mathbb{C})$.

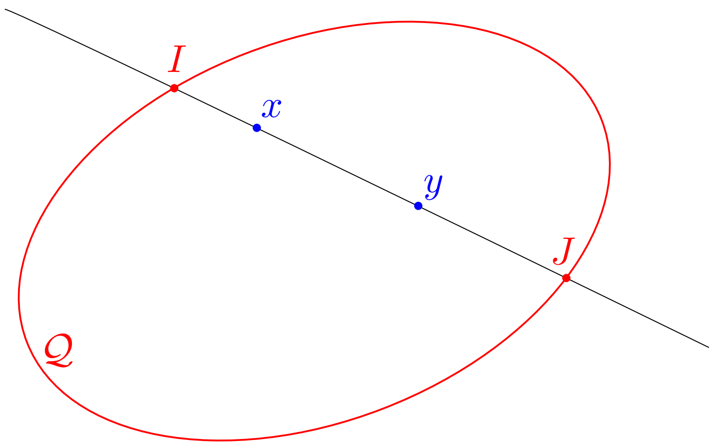


Figure 7.1: For any two x, y in the interior of the ellipsoid $\mathcal{Q}$, the projective line (xy) intersects $\mathcal{Q}$ in two distinct points I and J.

Remark 7.3. The case $(xy) \subseteq \mathcal{Q}$ is ruled out by the fact that $x, y \notin \mathcal{Q}$.

In all cases, one may take the cross-ratio:

$$c(x, y) := [x, y, J, I] .$$

This is a natural quantity to consider because it is a projective invariant, in particular it does not depend on the choice of coordinates on $\mathcal{P}$.

Proposition 7.4. *Let $l \subseteq \mathcal{P}$ be a projective line and consider the restriction c on $l - \mathcal{Q}$.*

- *If l is hyperbolic, then c is real-valued, and is positive on each component of $l - \mathcal{Q}$.*
- *If l is parabolic, then c is constant equal to* 1.
- *If l is elliptic, then c takes values in the unit circle* $\mathrm{U}(1) \subseteq \mathbb{C}$.

In all cases, for every $x, y, z \in l - \mathcal{Q}$:

$$c(x, y)c(y, z) = c(x, z) . \tag{7.1}$$

Proof. In the hyperbolic case and parabolic cases, c is real-valued by definition. Let us consider the hyperbolic case. The line l is a topological circle, therefore $l - \mathcal{Q} = l - \{I, J\}$ has two connected components. Choose any inhomogeneous coordinate on l, giving an identification $l \approx \hat{\mathbb{R}}$. Then the explicit formula for the cross-ratio (see Proposition 6.27) is:

$$c(x, y) = \frac{(J - x)(I - y)}{(J - y)(I - x)} . \tag{7.2}$$

We see with this formula that if x and y are in either component of $\hat{\mathbb{R}} - \{I, J\}$, then $c(x, y) > 0$.

For the parabolic case, since $I = J$, then $c(x, y) = [x, y, J, I] = 1$ by definition of the cross-ratio.

For the elliptic case, choose any inhomogeneous coordinate on l as before. We still have the formula (7.2), but now I and J are conjugate complex numbers: $J = \bar{I}$. Taking the modulus of (7.2) gives $c(x, y) = 1$.

Finally, the formula (7.1) is immediately checked using (7.2). □

In order to try and obtain a distance on (a connected component of) $\mathcal{P} - \mathcal{Q}$, it makes sense to take the logarithm of $c(x, y)$ in order to turn the multiplicative property (7.2) into an additive property.

Definition 7.5. The ***Cayley–Klein metric*** (or *Cayley–Klein pseudo-distance*) on $\mathcal{P} - \mathcal{Q}$ is the function defined by

$$d(x, y) := \frac{1}{2} |\ln c(x, y)|$$

where ln denotes the branch of the logarithm $\ln : \mathbb{C} - \{0\} \to \{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\pi, \pi]\}$.

Indeed, taking the logarithm of equation Equation 7.1 and using the identity

$$\ln(ab) = \ln(a) + \ln(b) \tag{7.3}$$

when $a = c(x, y)$ and $b = c(y, z)$, we find that $\ln c(x, z) = \ln c(x, y) + \ln c(y, z)$. The triangle inequality for real numbers then yields

$$d(x, z) \leqslant d(x, y) + d(y, z)$$

Hence we essentially proved that d satisfies the triangle inequality in restriction to any line. However we have to be a little more careful, because the complex logarithm does not always verify the identity (7.3), in general it only holds up to a multiple of $i\pi$.

The following proposition is trivial to prove by definition of the Cayley–Klein metric, but is nevertheless important to note:

Proposition 7.6. *The Cayley–Klein metric is natural when restricting to projective subspaces: let $\mathcal{P}' \subseteq \mathcal{P}$ be a projective subspace, then the Cayley–Klein metric of $\mathcal{P}' - \mathcal{Q}'$ (where $\mathcal{Q}' := \mathcal{Q} \cap \mathcal{P}'$) is equal to the restriction of the Cayley–Klein metric of $\mathcal{P} - \mathcal{Q}$.*

Remark 7.7. Whenever $\mathcal{P}' = P(W) \subseteq \mathcal{P} = P(V)$ is a projective subspace, the restricted quadric $\mathcal{Q}' := \mathcal{Q} \cap \mathcal{P}'$ is a quadric in $\mathcal{P}'$: the associated quadratic form is simply the restriction of q to W.

7.1.2 Isometries and geodesics

Assume that the symmetric bilinear from b is nondegenerate. Recall that the subgroup of $\mathrm{GL}(V)$ that preserves b is denoted $\mathrm{O}(b)$ (or $\mathrm{O}(q)$). Clearly, $\mathrm{O}(b)$ preserves the quadric $\hat{\mathcal{Q}} := \{q = 0\}$ in V, and the decomposition of V into cones $V = \hat{\Omega}^+ \sqcup \hat{\mathcal{Q}} \sqcup \hat{\Omega}^-$ where $\hat{\Omega}^+ := \{v \in V : q(v) > 0\}$ and $\hat{\Omega}^- := \{v \in V : q(v) < 0\}$. Going to the quotient, we have that $\mathrm{PO}(b) \subseteq \mathrm{PGL}(V)$ preserves the quadric $\mathcal{Q}$ in $\mathcal{P} = P(V)$, and the decomposition of $\mathcal{P} = \Omega^+ \sqcup \mathcal{Q} \sqcup \Omega^-$. Since projective transformations preserve the cross-ratio, we clearly have:

Theorem 7.8. The projective orthogonal group $\mathrm{PO}(b)$ acts on $\Omega^\pm$ by isometries with respect to the Cayley–Klein metric.

Remark 7.9. The Cayley–Klein metric is not an genuine distance in general, but Theorem 7.8 still makes sense: it means that the action of $\mathrm{PO}(b)$ on $\Omega^\pm$ preserves d.

Remark 7.10. It is not too hard to show that conversely, any isometry of the Cayley–Klein metric coincides with the action of an element of $\mathrm{PO}(b)$, at least still assuming that b is nondegenerate. We will only prove it in the hyperbolic case i.e. signature $(n, 1)$ (see Theorem 7.36), relying on the analogous result for the hyperboloid (Theorem 4.7). Note that in the degenerate cases, one cannot hope that the statement is literally true, as shows the Euclidean case (signature $(1, 0)$) where the Cayley–Klein metric is identically zero.

Another fact that almost comes for free is that lines are geodesics for the Cayley–Klein metric, more precisely:

Definition 7.11. A ***chord*** in $\Omega^\pm$ is the intersection of a line in $\mathcal{P}$ with $\Omega^\pm$.

Theorem 7.12. Chords are complete geodesics for the Cayley–Klein metric. More precisely:

- Hyperbolic chords are complete length-minimizing geodesics, in the sense that they can be parametrized as isometric curves $\gamma\colon \mathbb{R} \longrightarrow \Omega^{\pm}$ (i.e. such that $d(\gamma(s), \gamma(t)) = |s-t|$ for all $s, t \in \mathbb{R}$).
- Elliptic chords are complete geodesics, in the sense that they can be parametrized as locally isometric curves (i.e. such that $d(\gamma(s), \gamma(t)) = |s-t|$ for all s, t sufficiently close). Moreover, they are closed geodesics.
- Parabolic lines are degenerate geodesics in a sense that can be made precise, in particular the distance between any two points on a parabolic line is zero.

Remark 7.13. Note that any elliptic chord is equal to a whole projective line, so that it is a topological circle hence a closed geodesic (in particular it is not globally length-minimizing).

Proof. Let $c = l \cap \Omega^{\pm}$ be a chord. Assume that the line l is hyperbolic, so that $d(x, z) = d(x, y) + d(y, z)$ for any x, y, z that lie on c in this cyclic order. Let x_0 be any point on c and define $s(x) := \pm d(x_0, x)$ for any $x \in c$, where the sign is chosen so that $s(x) < 0$ when x is between I and x_0 and $s(x) > 0$ when x is between x_0 and J. It follows from the previous additive property that s is globally increasing along c, so that it gives a global coordinate on the chord. Moreover, one sees from (7.2) that $s(x) \longrightarrow \pm\infty$ when x approaches I or J, therefore $\gamma = s^{-1}$ is defined on $\mathbb{R}$. Finally, the fact that $d(\gamma(s), \gamma(t)) = |s - t|$ is again an immediate consequence of the additive property of the distance along c.

In the elliptic case, the additive property is only true if x, y, z are sufficiently close, but the proof is essentially the same.

In the parabolic case, it is clear that the distance between any two points on the line is zero. Let us leave the "sense that can be made precise" a mystery, but the example to have in mind is light-like geodesics in a pseudo-Riemannian manifolds. ◻

Remark 7.14. Again, it would be nice to prove that conversely, any geodesic for the Cayley–Klein metric is a projective line. We shall only do it in the hyperbolic case though (see § 7.2.4). As an exercise, the reader may prove the elliptic case (with the setup of § 7.1.5).

7.1.3 Cayley–Klein metrics in one dimension

Let us now examine the one-dimensional case more closely. Let $l = \mathcal{P} = P(V)$ be a projective line and let $Q \subseteq \mathcal{P}$ be a quadric as before, called the absolute, with associated quadratic form q and bilinear form b. As we have seen in § 7.1.1, Q consists of a pair of points I, J, possibly equal, possibly complex conjugate. Let us discuss these cases more precisely by looking at the signature of q.

Signature $(1, 1)$ *case.* (This is the case we are most interested in, which gives the Klein model.) By Sylvester's law of inertia, we can find coordinates (X_1, X_2) on V such that $q(X) = X_1^2 - X_2^2$. Therefore we see that $\mathcal{Q} = \{q = 0\}$ consists of two vector lines: $X_1 + X_2 = 0$ and $X_1 - X_2 = 0$, in other words Q consists of two points $I := [-1 \colon 1]$ and $J := [1 \colon 1]$. In the affine chart $\{X_2 \neq 0\}$ with coordinate $x = \frac{X_1}{X_2}$, this is $I = -1$ and $J = 1$. As expected, $\mathcal{P} - \mathcal{Q} \approx \hat{\mathbb{R}} - \{-1, 1\}$ consists of two connected components: $\Omega^- = \{|x| < 1\}$ and $\Omega^+ = \{|x| > 1\}$. Since the function c defined in (7.2) is positive on either connected components, the logarithm of c is the usual real logarithm, which satisfies (7.3). It follows that the Cayley–Klein metric (Definition 7.5) is a genuine distance on either connected components. Let us study it more precisely. The function c is given by

$$c(x,y) = \frac{(1-x)(-1-y)}{(1-y)(-1-x)}$$

so that

$$d(x,y) = \frac{1}{2}\left|\ln \frac{(1+x)(1-y)}{(1-x)(1+y)}\right| .$$

Let us consider the component Ω^-. There the factors $(1+x)$, $(1-y)$, $(1-x)$, $(1+y)$ are all positive, therefore

$$\begin{aligned} d(x,y) &= \left|\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)\right| \\ &= |\operatorname{artanh} x - \operatorname{artanh} y| \ . \end{aligned}$$

As we shall see in § 7.1.4, this expression makes it clear that Ω^- equipped with the Cayley–Klein metric is isometric to the hyperboloid model of hyperbolic space.

Remark 7.15. The component Ω^+ can be treated similarly. In fact, Ω^- and Ω^- are interchangeable: the fractional linear map $x \mapsto \frac{1}{x}$ is a projective transformation that exchanges the two. This symmetry is specific to dimension 1: the interior and exterior of higher-dimensional ellipsoids are not interchangeable: only the interior is convex.

Signature $(2,0)$ *or* $(0,2)$ *case.* Let us consider the $(2,0)$ case; the $(0,2)$ case is the same. Now in suitable coordinates we have $q(X) = X_1^2 + X_2^2$. The quadric Q is therefore empty, nevertheless we can define two imaginary points $I = [-i : 1]$ and $J = [i : 1]$, which correspond to the complex vector lines $X_1 + iX_2$ and $X_1 - iX_2$. In the affine chart $\{X_2 \neq 0\}$ with coordinate $x = \frac{X_1}{X_2}$, this is $I = -i$ and $J = +i$, which live in $\mathcal{P}^c \approx \mathbb{C}P^1$ instead of $\mathcal{P} \approx \mathbb{R}P^1$. Now the function c is given by

$$c(x,y) = \frac{(i-x)(-i-y)}{(i-y)(-i-x)}$$

so that

$$d(x,y) = \frac{1}{2}\left|\ln \frac{(x+i)(y-i)}{(x-i)(y+i)}\right| .$$

We would like as before to expand this expression using the identity $\ln(ab) = \ln a + \ln b$, but we have to be a little careful: for arbitrary nonzero complex numbers a and b this is only true up to a multiple of $2i\pi$. With this in mind, we continue:

$$\begin{aligned} d(x,y) &= \frac{1}{2}\left|\ln\left(\frac{y-i}{y+i}\right) - \ln\left(\frac{x-i}{x+i}\right) - 2ik\pi\right| \\ &= \left|\frac{i}{2}\ln\left(\frac{y-i}{y+i}\right) - \frac{i}{2}\ln\left(\frac{x-i}{x+i}\right) + k\pi\right| \end{aligned}$$

that is

$$d(x,y) = |\operatorname{arccot} y - \operatorname{arccot} x + k\pi| \tag{7.4}$$

where k is the unique integer that makes $\operatorname{arccot} y - \operatorname{arccot} x + k\pi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, in other words (7.4) defines $d(x,y)$ uniquely as an element of $[0, \frac{\pi}{2}]$. As we shall see in § 7.1.5, this expression makes it clear that $\mathcal{P}$ equipped with the Cayley–Klein metric is isometric to the elliptic space $S^1/\{\pm 1\}$.

Signature $(1,0)$ *or* $(0,1)$ *case.* Let us consider the $(1,0)$ case; the $(0,1)$ case is the same. Note that this is a degenerate case: the bilinear form b is degenerate, so is the quadric Q (by definition). Now in suitable coordinates we have $q(X) = X_2^2$. The quadric $\mathcal{Q}$ is therefore reduced to the point $I = J = [1:0]$, which correspond to the vector lines $X_2 = 0$. In the affine chart $\{X_2 \neq 0\}$ with coordinate $x = \frac{X_1}{X_2}$, this is $I = J = \infty$. Now the function c is constant equal to 1, therefore $d(x,y) = 0$ for any x, y. Clearly d is not a distance on $\mathbb{R}$; nevertheless one may interpret this case as the Euclidean one. Indeed, we have seen in the previous chapter (see § 6.1.5) that the complement of a hyperplane (here a point) in a projective space is naturally an affine space, and it admits a natural (although not completely canonical) Euclidean structure.

7.1.4 Cayley–Klein model of hyperbolic space

This is the most important subsection of § 7.1 for us, since it gives the Klein model of hyperbolic space.

Let $\mathcal{P} = P(V)$ be an n-dimensional real projective space and let the absolute $\mathcal{Q} \subseteq \mathcal{P}$ be a quadric of signature $(n,1)$. Such a quadric is called an ***ellipsoid***. As usual, we denote b and q the associated symmetric bilinear form and quadratic form.

Proposition 7.16. *The quadric $\mathcal{Q} \subseteq \mathcal{P}$ is a topological sphere of dimension $n-1$. $\mathcal{P} - \mathcal{Q}$ consists of two connected components: $\Omega^+ := \{[x] : q(x) > 0\}$ and $\Omega^- := \{[x] : q(x) < 0\}$. The component Ω^- is called the* ***interior*** *of the ellipsoid. It is a topological ball, it is convex, any line (xy) with $x, y \in \Omega^-$ is hyperbolic (it intersects $\mathcal{Q}$ in two distinct points).*

Let us introduce suitable coordinates to analyze the situation and prove Proposition 7.16 along the way. By Sylvester's law of inertia, in suitable homogeneous coordinates on V, the equation of $\mathcal{Q}$ is written

$$X_1^2 + \cdots + X_n^2 - X_{n+1}^2 = 0\,.$$

Note that $\mathcal{Q}$ does not intersect the hyperplane $X_{n+1} = 0$, therefore $\mathcal{Q}$ is contained in the affine chart $\mathcal{P} - \{X_{n+1} = 0\}$, and its equation in the inhomogeneous coordinates $x_k = \frac{X_k}{X_{n+1}}$ is:

$$x_1^2 + \cdots + x_n^2 - 1 = 0\,.$$

Thus we see that $\mathcal{Q}$ is an ellipsoid (it has the equation of a round sphere in the coordinates (x_i), but we do not have a Euclidean metric to distinguish between round spheres and other ellipsoids). The other claims of Proposition 7.16 are now immediate. In particular, the image of Ω^- in the affine chart $\mathcal{P} - \mathcal{Q} \to \{X_{n+1} = 1\} \approx \mathbb{R}^n$ is the unit ball, it is called the ***Beltrami–Klein disk*** (or ***Beltrami–Klein ball***).

Consider now the Cayley–Klein metric $d(x, y)$ on Ω^-. By the discussion carried out in in § 7.1.3, this is a genuine distance along any chord (intersection of Ω^- with a line) and it satisfies the additive property $d(x, y) + d(y, z) = d(x, z)$ whenever y is between x and z. It is tempting to say that d is a genuine distance on Ω^- and that the geodesics are the chords. Instead of proving it directly, we obtain this as a consequence of Theorem 7.18.

Recall from Chapter 4 that $\mathcal{H} \subseteq V$ denotes the hyperboloid

$$\mathcal{H} := \{v \in M : \langle v, v\rangle = -1\} .$$

and $\mathcal{H}^+$ is the upper sheet $\mathcal{H}^+ = \mathcal{H} \cap \{X_{n+1} > 0\}$. In Chapter 4, we saw that the induced metric on $\mathcal{H}^+$ from (V, b) is a Riemannian metric which makes $\mathcal{H}^+$ a model of hyperbolic space. Now, observe that there is an obvious way to identify Ω^- and $\mathcal{H}^+$, since Ω^- is the set of timelike lines in V, and each such line intersects $\mathcal{H}^+$ exactly once:

Definition 7.17. We denote $\psi : \mathcal{H}^+ \to \Omega^-$ the bijective map

$$\begin{aligned} \psi : \mathcal{H}^+ &\to \Omega^- \\ v &\mapsto [v] . \end{aligned}$$

The ***stereographic projection*** of the hyperboloid is the bijective map $\xi : \mathcal{H}^+ \to B$, where B is the unit ball in the affine hyperplane $\{X_{n+1} = 1\} \approx \mathbb{R}^n$, obtained by post-composing ψ with the affine chart $\varphi : \mathcal{P} - \mathcal{Q} \to \{X_{n+1} = 1\}$. Its image B is the Beltrami–Klein disk (see Figure 7.2).

Theorem 7.18. The map ψ is an isometry with respect to the hyperbolic distance $d_{\mathcal{H}}$ on $\mathcal{H}^+$ and the Cayley–Klein metric d_{CK} on Ω^-.

Remark 7.19. Although we have not yet shown that the Cayley–Klein metric on Ω^- is a genuine distance, Theorem 7.18 means that $d_{\mathrm{CK}}(\psi(v), \psi(w)) = d_{\mathcal{H}}(v, w)$ for all $v, w \in \mathcal{H}^+$. (In addition, we know that ψ is bijective, so it deserves to be called a global isometry.) The fact that d_{CK} is a genuine distance is a corollary.

Proof. Let $v, w \in \mathcal{H}^+$, denote $x = \psi(v)$ and $y = \psi(w)$; we want to show that $d_{\mathrm{CK}}(x, y) = d_{\mathcal{H}}(v, w)$.

First we argue that it is enough to do the one-dimensional case, by simply restricting to the geodesic (vw) in $\mathcal{H}^+$, which is a one-dimensional hyperboloid. This amounts to intersecting $\mathcal{H}^+$ with a 2-dimensional vector subspace, therefore in $\mathcal{P}$ this corresponds to restricting to the projective line (xy). Indeed, on the one hand the Cayley–Klein metric is natural when restricting to projective subspaces (see Proposition 7.6); and on the other hand the hyperboloid is also natural when restricting to vector subspaces, (see Proposition 4.1): the restriction of the hyperbolic distance to a lower-dimensional hyperboloid is the hyperbolic distance on the

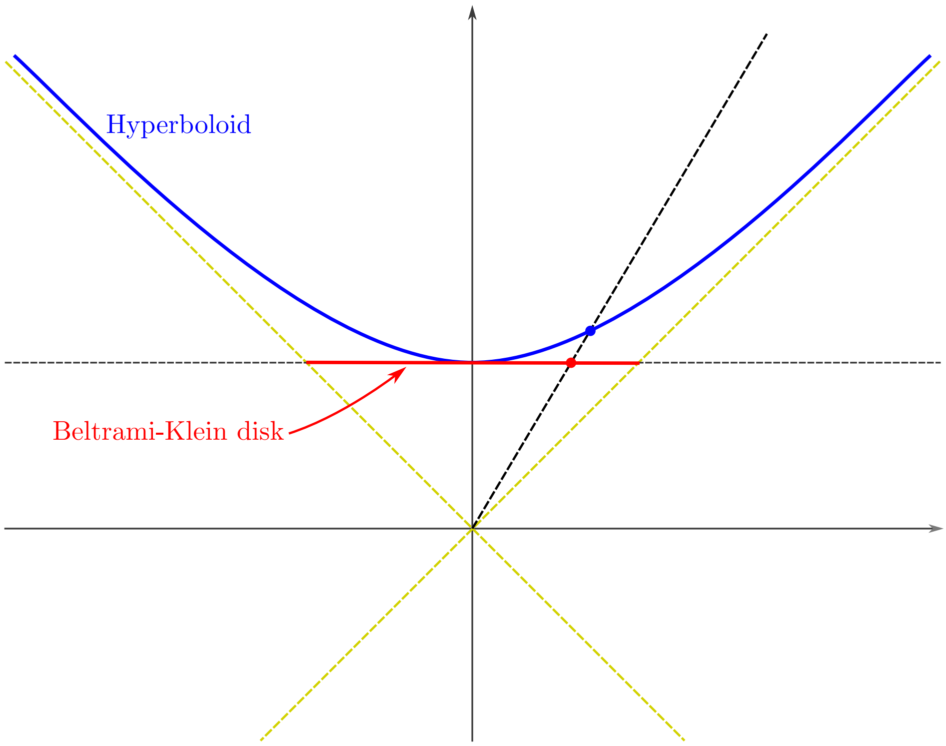


Figure 7.2: Stereographic projection of the hyperboloid to the Beltrami–Klein disk.

lower-dimensional hyperboloid. (In Riemannian geometry, one says that lower-dimensional hyperboloids are ***totally geodesic***.)

We thus now assume that $\mathcal{P}$ is a projective line, and we can reinvest the work of § 7.1.3. Choose coordinates such that the quadratic form is $q(X) = X_1^2 - X_2^2$, and denote $x = \frac{X_1}{X_2}$ the affine coordinate on $\mathcal{P} - \{X_2 = 0\}$. The quadric $\mathcal{Q}$ is the pair of points $I = -1$ and $J = 1$, the domain Ω^- is the interval $[-1, 1]$, and the Cayley–Klein metric is $d_{\mathrm{CK}}(x, y) = |\operatorname{artanh} x - \operatorname{artanh} y|$.

On the other hand, $\mathcal{H}^+$ is the upper arc of the hyperbola $X_1^2 - X_2^2 = -1$. It is parametrized by $\gamma(t) = (\sinh t, \cosh t)$, and in fact this is a unit geodesic (see Theorem 4.8). Let t_1 and t_2 be such that $v = \gamma(t_1)$ and $w = \gamma_{t_2}$. Since γ is a unit geodesic, we have $d_{\mathcal{H}}(v, w) = |t_1 - t_2|$. The points $x = \psi(v)$ is determined by $[x : 1] = [\sinh t_1 : \cosh t_1]$, so $x = \tanh t_1$. Similarly, $y = \tanh t_2$. Hence we find $d_{\mathrm{CK}}(x, y) = |\operatorname{artanh} x - \operatorname{artanh} y| = |t_1 - t_2| = d_{\mathcal{H}}(v, w)$ as desired. □

Corollary 7.20. *The Cayley–Klein metric on Ω^- may be written:*

$$d([u], [v]) = \operatorname{arcosh}\left(\frac{-b(u, v)}{\sqrt{q(u)q(v)}}\right) \tag{7.5}$$

Proof. The right-hand side of (7.5) is invariant by scaling u or v by positive numbers, therefore we may assume that $q(u) = q(v) = -1$. However in that case $\operatorname{arcosh}(-b(u, v))$ is the hyperbolic distance on $\mathcal{H}^+$ (see Theorem 4.12), so that (7.5) is precisely the statement of Theorem 7.18. □

As an immediate consequence of Theorem 7.18, we obtain:

Theorem 7.21. The Cayley–Klein metric is a distance on Ω^-. It is induced by a complete Riemannian metric of constant sectional curvature -1.

In other words, the previous theorem says that Ω^- equipped with the Cayley–Klein metric is a model of hyperbolic space. Being slightly pedantic, we call it the ***Cayley–Klein model*** or ***projective model***, to distinguish it from the ***Beltrami–Klein model*** which is the same model, except it is considered in an affine chart (where Ω^- becomes a disk).

Remark 7.22. Hilbert proposed a very elegant and elementary proof that the Cayley–Klein metric is a genuine distance that holds more generally on any proper convex set $\Omega \subsetneq \mathcal{P}$. This generalization of the Cayley–Klein metric is called the ***Hilbert metric***. Hilbert's proof is reproduced by Papadopoulos in [Pap14, §5.6], and I sincerely encourage you to go and read it. The key ingredient is the invariance of the cross-ratio under perspectivities (central collineations).

7.1.5 Cayley–Klein model of elliptic space

Consider now the case where b is of signature $(n+1, 0)$, in other words $(V, b) \approx \mathbb{R}^{n+1}$ is a Euclidean vector space. In this case, the quadric $\mathcal{Q}$ is empty, therefore the Cayley–Klein metric is defined on all $\mathcal{P}$. Notice that all lines are elliptic in this case.

Let $S = \{v \in V : q(v) = 1\}$ denote the unit sphere in (V, b). As in § 7.1.4, we would like to define the stereographic projection $\psi \colon S \to \mathcal{P}$, however note that each vector line in V intersects S twice, at two antipodal points $\pm v$. This is similar to the situation where each timelike line in Minkowski space intersects the hyperboloid $\mathcal{H}$ twice, except here there is no "upper sheet" of the sphere to resolve the issue. Instead, we have to define the stereographic projection as a map $\psi \colon S/\{\pm \mathrm{id}\} \to \mathcal{P}$, where $S/\{\pm \mathrm{id}\}$ is the set of pairs of antipodal points on the sphere.

Definition 7.23. We denote $\psi \colon S/\{\pm \mathrm{id}\} \to \mathcal{P}$ the bijective map

$$\begin{aligned} \psi \colon S/\{\pm \mathrm{id}\} &\to \mathcal{P} \\ \{\pm v\} &\mapsto [v]\,. \end{aligned}$$

Remark 7.24. In this setting, the ***stereographic projection*** of $S/\{\pm \mathrm{id}\}$ is the map $S/\{\pm \mathrm{id}\} \to \{X_{n+1} = 1\} \approx \mathbb{R}^n$, obtained by post-composing ψ with the affine chart $\mathcal{P} - \mathcal{Q} \to \{X_{n+1} = 1\}$. See Figure 7.3. Note that the stereographic projection is not defined on the $n-1$-dimensional sphere $S \cap \{X_{n+1} = 0\}$ (mod $\pm$ id), however it can be extended as a bijective map $S/\{\pm \mathrm{id}\} \to \{X_{n+1} = 1\} \cup \partial_\infty \{X_{n+1} = 1\} \approx \mathbb{R}P^n$.

Equip S with the Riemannian metric induced from the Euclidean metric on (V, b). As is well-known, this is a complete Riemannian metric of constant sectional curvature 1 (see

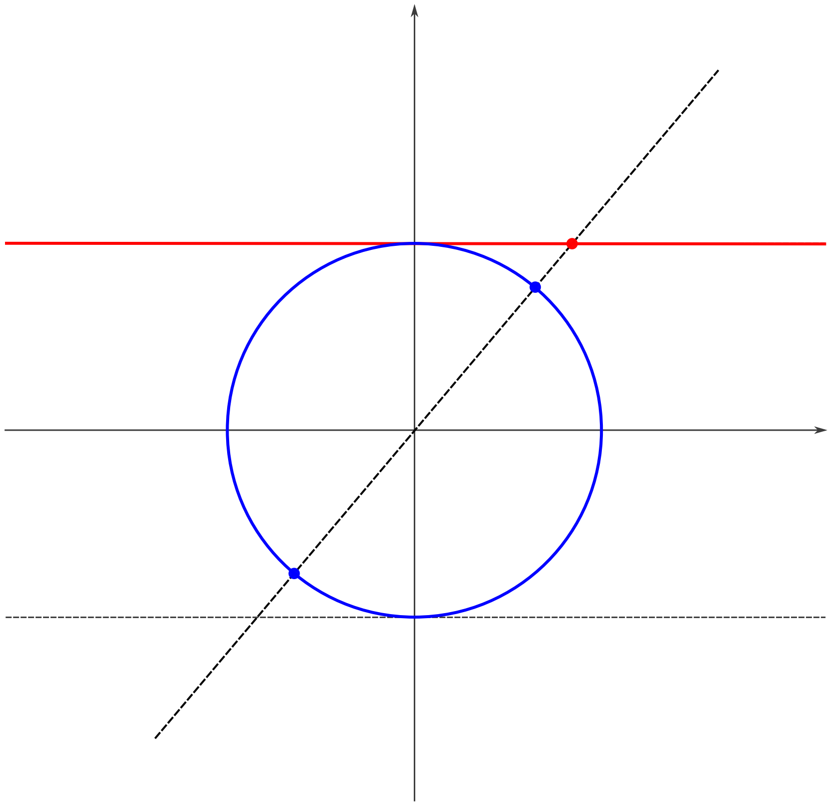


Figure 7.3: Stereographic projection of $S/\{\pm\,\mathrm{id}\}$.

Exercise 2.3). Since $v \mapsto -v$ is an isometry of S, the metric is well-defined on the quotient on $S/\pm 1$. We shall call the resulting Riemannian manifold $S/\{\pm\,\mathrm{id}\}$ the ***spherical model of elliptic space***.

Theorem 7.25. The map ψ is an isometry with respect to the spherical distance d_S on $S/\{\pm\,\mathrm{id}\}$ and the Cayley–Klein metric d_{CK} on $\mathcal{P}$.

Proof. The proof is completely analogous to that of Theorem 7.18; we leave it as an exercise to the reader (see Exercise 7.1). □

As an immediate consequence of the previous theorem, we obtain:

Theorem 7.26. The Cayley–Klein metric is a distance on $\mathcal{P}$. It is induced by a complete Riemannian metric of constant sectional curvature 1.

In other words, $\mathbb{R}P^n$ equipped with the Cayley–Klein metric is a model of elliptic space, which we naturally call the ***Cayley–Klein model*** or ***projective model***.

7.1.6 Cayley–Klein model of Euclidean space

Consider now the case where b is of signature $(1, 0)$, in particular it is degenerate. In suitable coordinates, the quadratic form is written

$$q(X) = X_{n+1}^2$$

therefore the degenerate quadric $\mathcal{Q}$ is the projective hyperplane $X_{n+1} = 0$. Note that in this case, all lines in $\mathcal{P} - \mathcal{Q}$ are parabolic, so the Cayley–Klein metric is constant equal to zero; it is not a Euclidean metric as one could hope. Nevertheless, we have already seen in the previous chapter (see § 6.1.5) that the complement $\mathcal{P} - \mathcal{Q}$ has a natural structure of an affine space. Like all affine spaces it admits a natural Euclidean structure, although it is not completely canonical: it depends on the choice of an inner product on the underlying vector space. In our case, this choice amounts to a Euclidean inner product b' on the kernel of b, so that $b + b'$ is a Euclidean inner product on V.

Let $\mathcal{E}^+ \subseteq V$ denote the affine hyperplane $X_{n+1} = 1$. Note that this is the upper sheet of the pseudosphere $\{q = 1\}$. The map ψ is now the bijective map $\psi \colon \mathcal{E}^+ \to \mathcal{P} - \mathcal{Q}$, given by $v \mapsto [v]$ as before. Note that it coincides with the inverse of the affine chart $\mathcal{P} - \mathcal{Q} \xrightarrow{\sim} \{X_{n+1} = 1\} \approx \mathbb{R}^n$, so the "stereographic projection" in this setting is the identity map $\mathcal{E}^+ \to \{X_{n+1} = 1\} \approx \mathbb{R}^n$. Equip $\mathcal{E}^+$ with the metric induced from $b + b'$. Then $\mathcal{E}^+$ is a complete Riemannian manifold of zero sectional curvature, in other words a model of Euclidean space. Using the stereographic projection ψ to transport the metric, we obtain that $\mathcal{P} - \mathcal{Q}$ is a model of Euclidean space, which we call the ***Cayley–Klein model*** or ***projective model***.

In summary, it is not the case that Euclidean geometry is obtained as a Cayley–Klein geometry, in the sense that the Euclidean metric is not a Cayley–Klein metric; nevertheless we can interpret the Cayley–Klein geometry associated to a degenerate quadric of signature $(1, 0)$ as a model of Euclidean geometry. In addition, in Exercise 7.2 it is shown that the Euclidean metric may be viewed as a degenerate elliptic metric.

7.1.7 Other Cayley–Klein geometries

Naturally, there are many more Cayley–Klein geometries, depending on the signature of quadric. Exploring these other examples is a fascinating program but beyond our scope, so we will be content with alluding to their existence.

Actually, as we saw in the Euclidean case, a Cayley–Klein geometry is not adequately defined by the Cayley–Klein metric in degenerate cases, nor even by the quadric alone. There are however more refined approaches to defining Cayley–Klein geometries. In [Ric11], a 2-dimensional Cayley–Klein geometry is defined by a "primal/dual" pair of conics, leading to seven types of Cayley–Klein geometries. A more general treatment is to define Cayley–Klein geometries as certain types of homogeneous spaces (spaces with a large group of symmetries), in the spirit of Klein's *Erlangen program*. This leads to nine 2-dimensional Cayley–Klein geometries: see e.g. [HOS00] (requires some knowledge of Lie theory!). I also recommend reading [FS19] for more geometric insights, especially on the Lorentzian geometries (Minkowski, de Sitter, anti de Sitter).

7.2 The Beltrami–Klein disk

7.2.1 Definition

Let us recap the setup of § 7.1.4. Let (V, b) be a Minkowski space of dimension $n+1$. By choosing a suitable basis, we can identify $(V, b) \approx \mathbb{R}^{n,1}$. We denote $(X_1, \dots, X_{n+1})$ the associated coordinates. We denote $\mathcal{Q} \subseteq \mathbb{R}P^n$ the projectivized light cone, it is the projective quadric associated to b, called an ellipsoid. The open set $\Omega^- \subseteq \mathbb{R}P^n$ is the set of timelike vector lines, it is the interior of the ellipsoid, and a convex set in $\mathbb{R}P^n$. The Cayley–Klein metric $d_{\mathrm{CK}}$is a distance in Ω^-, making it a model of hyperbolic space. The image of this model under the usual affine chart $\varphi\colon P(V) - P(\{X_{n+1} = 0\}) \xrightarrow{\sim} \{X_{n+1} = 1\} \approx \mathbb{R}^n$ is called the *Beltrami–Klein disk*:

Definition 7.27. The ***Beltrami–Klein disk*** (or ***ball***) (B, d) is the unit ball $B \subseteq \mathbb{R}^n$ with the distance d that is the image of the Cayley–Klein model $(\Omega^-, d_{\mathrm{CK}}) \subseteq \mathbb{R}P^n$ under the affine chart φ.

We shall give an explicit expression of the distance d in the next subsection. As a corollary of Theorem 7.18, we obtain:

Theorem 7.28. The Beltrami–Klein disk (B, d) is the isometric image of the hyperboloid model $(\mathcal{H}^+, d_{\mathcal{H}})$ under the stereographic projection $\xi\colon \mathcal{H}^+ \to B$.

We will derive many features of the Beltrami–Klein disk from the hyperboloid using Theorem 7.28, especially the Riemannian metric. For now, we have:

Corollary 7.29. *The Beltrami–Klein disk is a model of hyperbolic space.*

7.2.2 Distance

By definition, the distance d in the Klein disk B is the image (the push-forward) of the Cayley–Klein metric d_{CK}, which is defined in terms of cross-ratios. Since the cross-ratio can be computed directly in any affine chart (due to its invariance under projective transformations), this distance can be defined directly in the Klein model:

Proposition 7.30. *Let $x, y \in B$. Denote by I and J the intersections of the straight line $l = (xy) \subseteq \mathbb{R}^n$ with ∂B, so that I, x, y, J are aligned in this order. Choose any affine frame on l, identifying it with $\mathbb{R}$. Then*

$$\begin{aligned} d(x, y) &= \frac{1}{2} \ln [x, y, J, I] \\ &= \frac{1}{2} \ln \frac{|Jx||Iy|}{|Jy||Ix|} \end{aligned}$$

where we denote $|AB|$ the Euclidean distance between two points $A, B \in \mathbb{R}^n$.

More explicitly, the distance can also be written:

Proposition 7.31.

$$d(x, y) = \operatorname{arcosh}\left(\frac{1 - \langle x, y\rangle}{\sqrt{(1 - \|x\|^2)(1 - \|y\|^2)}}\right) \tag{7.6}$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the Euclidean inner product and norm in $\mathbb{R}^n$.

Proof. Although Proposition 7.30 is an immediate application of Corollary 7.20 (also see Exercise 7.4), it is a good exercise to write the direct proof.

Let K be the Euclidean midpoint of I and J: see Figure 7.4. Let us choose the pair of

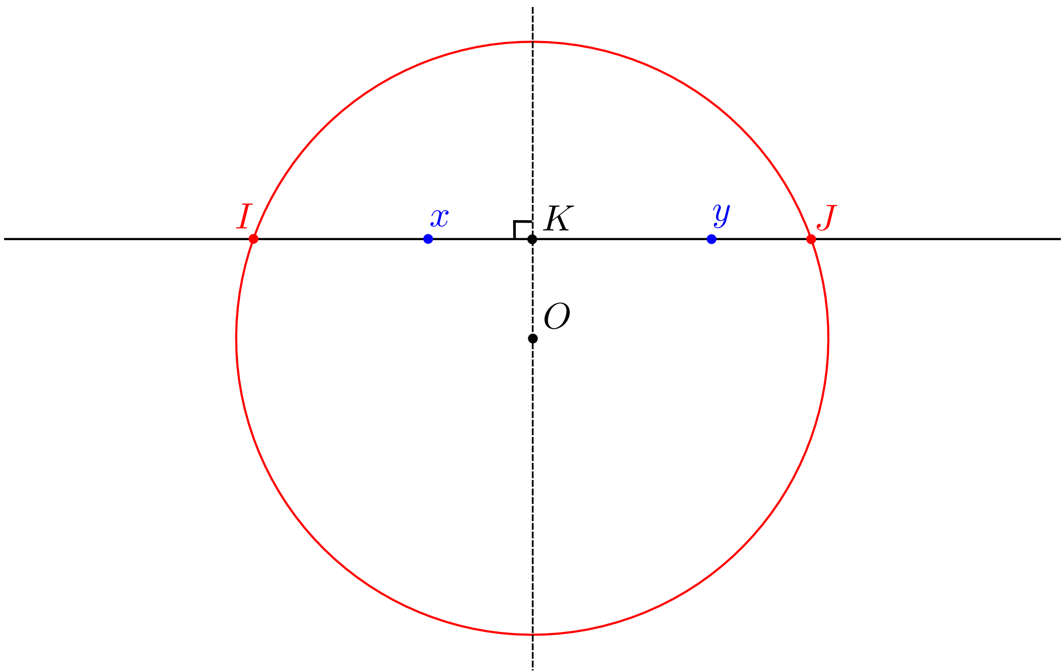


Figure 7.4: Calculating the distance $d(x, y)$ in the Beltrami–Klein disk.

points K and J as an affine chart on the line $l = (xy)$, giving an identification $(xy) \approx \mathbb{R}$: any point $m \in l$ is uniquely represented by a real coordinate λ such that $m = (1 - \lambda)K + \lambda J$. The coordinates of I, J, x, y are respectively:

$$\begin{aligned}
\lambda_I &= -1 \\
\lambda_J &= 1 \\
\lambda_x &= \frac{\overline{Kx}}{\sqrt{1 - \|K\|^2}} \\
\lambda_y &= \frac{\overline{Ky}}{\sqrt{1 - \|K\|^2}}
\end{aligned}$$

where we denote $\overline{Kx}$ the signed distance between K and x, same for $\overline{Ky}$ (for instance, in Figure 7.4, $\overline{Kx} = -|Kx|$ and $\overline{Ky} = +|Ky|$). The expressions for λ_x and λ_y above can be found by noticing that $KI = \sqrt{1 - \|K\|^2}$ by the Pythagorean theorem.

Since the cross-ratio can be computed in any affine coordinates on a line, we may use these coordinates to compute the distance between x and y:

$$\begin{aligned} d(x, y) &= \frac{1}{2} |\ln[x, y, J, I]| \\ &= \frac{1}{2} \left| \ln \frac{(1 - \lambda_x)(-1 - \lambda_y)}{(1 - \lambda_y)(-1 - \lambda_x)} \right| . \end{aligned}$$

Let us manipulate this expression in view of obtaining (7.6):

$$\begin{aligned} d(x, y) &= \frac{1}{2} \left| \ln \frac{1 + \frac{\lambda_x - \lambda_y}{1 - \lambda_x \lambda_y}}{1 - \frac{\lambda_x - \lambda_y}{1 - \lambda_x \lambda_y}} \right| \\ &= \left| \operatorname{artanh} \frac{\lambda_x - \lambda_y}{1 - \lambda_x \lambda_y} \right| \\ &= \operatorname{arcosh} \frac{1 - \lambda_x \lambda_y}{\sqrt{(1 - \lambda_x^2)(1 - \lambda_y^2)}} . \end{aligned} \tag{7.7}$$

For the last equality, we used the identity $\operatorname{artanh} |t| = \operatorname{arcosh} \frac{1}{\sqrt{1-t^2}}$ for $-1 < t < 1$.

To conclude, we compute:

$$\frac{1 - \lambda_x \lambda_y}{\sqrt{(1 - \lambda_x^2)(1 - \lambda_y^2)}} = \frac{1 - \|K\|^2 - \overline{Kx}\,\overline{Ky}}{(1 - \|K\|^2 - \overline{Kx}^2)(1 - \|K\|^2 - \overline{Ky}^2)} \tag{7.8}$$

By writing $x = K + (x - K)$ and $y = K + (y - K)$, we see that $\langle x, y \rangle = \|K\|^2 + \overline{Kx}\,\overline{Ky}$, $\|x\|^2 = \|K\|^2 + \overline{Kx}^2$, and $\|y\|^2 = \|K\|^2 + \overline{Ky}^2$, so that (7.8) is rewritten

$$\frac{1 - \lambda_x \lambda_y}{\sqrt{(1 - \lambda_x^2)(1 - \lambda_y^2)}} = \frac{1 - \langle x, y \rangle}{(1 - \|x\|^2)(1 - \|y\|^2)} . \tag{7.9}$$

Inserting (7.9) into (7.7) yields the desired result. □

7.2.3 Riemannian metric

The distance on the Beltrami–Klein disk is induced by a Riemannian metric, which can be computed as the pullback of the Riemannian metric on the hyperboloid under the inverse of the stereographic projection:

Proposition 7.32. *The Riemannian metric on the Beltrami–Klein disk $B \subseteq \mathbb{R}^n$ is given by*

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2}{1 - \|x\|^2} + \frac{(x_1\ \mathrm{d}x_1 + \cdots + x_n\ \mathrm{d}x_n)^2}{(1 - \|x\|^2)^2}$$

Proof. The inverse of the stereographic projection is the map

$$\begin{aligned} \xi^{-1} \colon B &\longrightarrow \mathcal{H}^+ \subseteq \mathbb{R}^{n,1} \\ x &\mapsto \frac{\hat{x}}{\|\hat{x}\|} \end{aligned}$$

where we have denoted $\hat{x} = (x, 1)$ and $\|\hat{x}\| = \sqrt{|q(\hat{x})|}$. In other words, this is:

$$\xi^{-1} \colon x \mapsto \frac{(x, 1)}{\sqrt{1 - \|x\|^2}}$$

where $\|x\|$ now denotes the Euclidean norm of $x \in \mathbb{R}^n$. Recall that the metric on the hyperboloid is induced by the Minkowski metric

$$\mathrm{d}s^2 = \mathrm{d}X_1^2 + \cdots + \mathrm{d}X_n^2 - \mathrm{d}X_{n+1}^2\,.$$

The pullback metric on B under ξ^{-1} is obtained by replacing each $\mathrm{d}X_k$ by its expression of terms of the $\mathrm{d}x_k$'s. More precisely, the map ξ^{-1} is written

$$\begin{aligned} X_k &= \frac{x_k}{\sqrt{1 - \|x\|^2}} \qquad \text{for } k \in \{1, \ldots, n\} \\ X_{n+1} &= \frac{1}{\sqrt{1 - \|x\|^2}} \end{aligned}$$

therefore we find:

$$\begin{aligned} \mathrm{d}X_k &= \frac{\mathrm{d}x_k}{\sqrt{1 - \|x\|^2}} + x_k (1 - \|x\|^2)^{-3/2} \left(\sum_j x_j\, \mathrm{d}x_j \right) \qquad \text{for } k \in \{1, \ldots, n\} \\ \mathrm{d}X_{n+1} &= (1 - \|x\|^2)^{-3/2} \left(\sum_j x_j\, \mathrm{d}x_j \right) \end{aligned}$$

Taking the squares (symmetric product of one-forms):

$$\begin{aligned} \mathrm{d}X_k^2 &= \frac{\mathrm{d}x_k^2}{1 - \|x\|^2} + \frac{x_k^2 \left(\sum_j x_j\, \mathrm{d}x_j\right)^2}{(1 - \|x\|^2)^3} + \frac{2x_k\, \mathrm{d}x_k \left(\sum_j x_j\, \mathrm{d}x_j\right)}{(1 - \|x\|^2)^2} \qquad \text{for } k \in \{1, \ldots, n\} \\ \mathrm{d}X_{n+1}^2 &= \frac{\left(\sum_j x_j\, \mathrm{d}x_j\right)^2}{(1 - \|x\|^2)^3} \end{aligned}$$

Combining these, we find

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}X_1^2 + \cdots + \mathrm{d}X_n^2 - \mathrm{d}X_{n+1}^2 \\
&= \frac{\sum_k \mathrm{d}x_k^2}{1-\|x\|^2} + \frac{\|x\|^2 \left(\sum_j x_j \,\mathrm{d}x_j\right)^2}{(1-\|x\|^2)^3} + \frac{2\left(\sum_j x_j \,\mathrm{d}x_j\right)^2}{(1-\|x\|^2)^2} - \frac{\left(\sum_j x_j \,\mathrm{d}x_j\right)^2}{(1-\|x\|^2)^3} \\
&= \frac{\sum_k \mathrm{d}x_k^2}{1-\|x\|^2} + \frac{\left(\sum_j x_j \,\mathrm{d}x_j\right)^2}{(1-\|x\|^2)^2}
\end{aligned}$$

as desired. □

Remark 7.33. We see from the expression that the Beltrami–Klein metric is not conformal to the Euclidean metric in B. More precisely, it is nowhere conformal except at the origin.

7.2.4 Geodesics

Since the stereographic projection $\mathcal{H}^+ \to B$ is a Riemannian isometry from the hyperboloid to the Beltrami–Klein disk, the (parametrized) geodesics on B are the images of the (parametrized) geodesics on $\mathcal{H}$. Ignoring the parametrization, recall that a geodesic on $\mathcal{H}$ is the intersection of $\mathcal{H}$ with a 2-plane in $\mathbb{R}^{n,1}$. In the projective model, this translates to the intersection of Ω^- with a projective line. In the Beltrami–Klein model, it thus translates to the intersection of B with a Euclidean straight line, in other words a chord.

Theorem 7.34. The (unparametrized) geodesics in the Beltrami–Klein model are the chords, i.e. Euclidean straight line segments joining two points of ∂B.

The curious reader will find an arclength parametrization of the geodesics (Exercise 7.8).

7.2.5 Isometries

We have seen in Theorem 7.8 that in the projective model, the projective orthogonal group $\mathrm{PO}(b)$ acts by isometries on Ω^- (and we promised to later prove that this is all the isometries). In the Beltrami–Klein disk, the action of $\mathrm{PO}(b)$ translates to the action of $\mathrm{PO}(n,1)$ on B by linear fractional transformations.

Example 7.35. Consider the Lorentz boost $M(t) \in \mathrm{SO}^+(2,1)$:

$$M(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}.$$

The projective linear action of $\pm M(t)$ on $\mathbb{R}P^2$ (preserving Ω^-) is given by

$$[X_1 : X_2 : X_3] \mapsto [X_1 : (\cosh t)X_2 + (\sinh t)X_3 : (\sinh t)X_2 + (\cosh t)X_3].$$

The linear fractional action of $\pm M(t)$ on $\mathbb{R}^2$ preserving the Beltrami–Klein disk $B \subseteq \mathbb{R}^2$ is given by:

$$(x_1, x_2) \mapsto \left(\frac{x_1}{(\sinh t)x_2 + \cosh t}, \frac{(\cosh t)x_2 + (\sinh t)}{(\sinh t)x_2 + (\cosh t)} \right).$$

By the discussion above, we have:

Theorem 7.36. The group of isometries of the Beltrami–Klein disk is $\mathrm{PO}(n,1)$ acting by linear fractional transformations. The subgroup of orientation-preserving isometries is $\mathrm{PSO}(n,1)$.

Remark 7.37. Given any $f \in \mathrm{O}(n,1)$, exactly one element of the pair $\{f,-f\}$ is in $\mathrm{O}^+(n,1)$. It follows that there is an obvious isomorphism $\mathrm{PO}(n,1) \approx \mathrm{O}^+(n,1)$, and $\mathrm{PSO}(n,1) \approx \mathrm{SO}^+(n,1)$. It follows that $\mathrm{PO}(n,1)$ has two connected components, $\mathrm{PSO}(n,1)$ (orientation-preserving isometries) and the other one (orientation-reversing isometries).

The only nontrivial part of Theorem 7.36 that remains to prove is that any isometry of the Beltrami–Klein disk is given by the action of some element of $\mathrm{PO}(n,1)$. This follows from Theorem 4.6 and the following proposition:

Proposition 7.38. *The action of* $\mathrm{O}^+(n,1)$ *on* $\mathcal{H}^+$ *translates to the action of* $\mathrm{PO}(n,1)$ *on the Beltrami–Klein disk. More precisely, the stereographic projection* $\xi\colon \mathcal{H}^+ \to B$ *conjugates the action of any* $f \in O^+(n,1)$ *on* $\mathcal{H}^+$ *(restricting the linear action on* $\mathbb{R}^{n,1}$*) to the projective linear (resp. fractional linear) action of* $\pm f$ *on* Ω^- *(resp. on* B*).*

The proof of Proposition 7.38 is essentially trivial and left to the reader: it is a matter of unraveling the definitions.

7.3 Exercises

Exercise 7.1. Cayley–Klein model of elliptic space

Let (V, b) be a Euclidean vector space. We denote S the unit sphere in V.

(1) Prove Theorem 7.25: *The stereographic projection $S/\{\pm \mathrm{id}\} \longrightarrow P(V)$ is an isometry with respect to the spherical distance on $S/\{\pm \mathrm{id}\}$ and the Cayley–Klein metric on $P(V)$.*

(2) Show that the Cayley–Klein metric on $P(V)$ may be written:

$$d([u],[v]) = \arccos\left(\frac{b(u,v)}{\sqrt{b(u,u)b(v,v)}}\right).$$

Exercise 7.2. Cayley–Klein model of Euclidean space

Let $\mathcal{P} = P(V)$ be a projective space of dimension n and let b be a symmetric bilinear form on V of signature $(1,0)$. Let q denote the associated quadratic form and $\mathcal{Q} \subseteq \mathcal{P}$ the associated quadric.

(1) Let b_0 be a Euclidean inner product on $\ker b$. Show that $b_\varepsilon := \varepsilon^2 b_0 + b$ is a Euclidean inner product on V. Write the Cayley–Klein metric d_ε on $P(V)$ associated to b_ε using Exercise 7.1 (2). Derive the following expression in a suitable affine chart $\mathcal{P} - \mathcal{Q} \xrightarrow{\sim} \mathbb{R}^n$:

$$d_\varepsilon(x,y) = \arccos\left(\frac{1+\varepsilon^2\langle x,y\rangle}{\sqrt{(1+\varepsilon^2\|x\|^2)(1+\varepsilon^2\|y\|^2)}}\right).$$

(2) Show that, when $\varepsilon \longrightarrow 0$, the Cayley–Klein metric d_ε converges to the constant function $d_0 = 0$. Is this expected?

(3) Show that, when $\varepsilon \longrightarrow 0$, the "blown-up" Cayley–Klein metric $\frac{1}{\varepsilon} d_\varepsilon$ converges to a Euclidean metric on $\mathcal{P} - \mathcal{Q}$, which can be identified to b_0. Is this expected?

Exercise 7.3. Hilbert metric

We have seen that the Cayley–Klein metric d is a distance in $\Omega \subseteq \mathbb{R}^n$ when Ω is the interior of an ellipsoid. Hilbert gave an elegant and elementary proof that applies more generally whenever Ω is a bounded convex open set. Your task is to go and read this proof in [Pap14, §5.6], and summarize it in a few lines.

Exercise 7.4. Beltrami–Klein distance and stereographic projection

(1) Recall the expression of the hyperbolic distance $d_{\mathcal{H}}$ on the hyperboloid $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ and the distance d_{BK} on the Beltrami–Klein disk $B \subseteq \mathbb{R}^n$.

(2) Compute the image of the distance $d_{\mathcal{H}}$ on B under the stereographic projection. Recover that the stereographic projection is an isometry from the hyperboloid to the Beltrami–Klein disk.

Exercise 7.5. Riemannian metric in the Beltrami–Klein disk

(1) Redo the calculation of the Riemannian metric in the Beltrami–Klein disk (preferably without looking at the lecture notes).

(2) Is the Beltrami–Klein metric conformal to the Euclidean metric in B?

Exercise 7.6. Distance to origin

Check that the distance from the origin to a point x in the Beltrami–Klein disk $B \subseteq \mathbb{R}^n$ is given by $d(O, x) = \operatorname{artanh}(\|x\|)$, using three different arguments:

(1) Using the expression of the Cayley–Klein metric in terms of cross-ratios.

(2) Using the explicit expression of the distance (see Proposition 7.31).

(3) Using the Riemannian metric.

Exercise 7.7. Circles in the Beltrami–Klein disk

A circle $C(x, R)$ in the 2-dimensional Beltrami–Klein disk (B, d) is the set of points at distance R from x. Show that any circle in the Beltrami–Klein disk is a Euclidean ellipse. Show an analogous result for higher-dimensional Beltrami–Klein disks.

Exercise 7.8. Geodesics in the Beltrami–Klein disk

Find the expression of any parametrized geodesic in the Beltrami–Klein disk.

Exercise 7.9. Isometries in the Beltrami–Klein disk

(1) Describe the action of $\mathrm{PO}(1, 1)$ on the 1-dimensional Beltrami–Klein disk.

(2) Consider the matrix

$$R(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $R(t) \in \mathrm{SO}(2, 1)$ and describe its action on the 2-dimensional Beltrami–Klein disk.

(3) Show that any element of $\mathrm{PSO}(2, 1)$ can be written $[L][R]$, for some Lorentz boost L and some $R = R(t)$. (We denote $[M]$ the element of PG associated to $M \in G$.) Recover the fact that $\mathrm{PSO}(2, 1)$ is connected.

Part IV: Möbius transformations and the Poincaré models

Mathematics is the art of giving the same name to different things.

– Henri Poincaré[1]

[1] [Poi08]. It is amusing to note that Poincaré fittingly writes *la mathématique* as a singular noun in the original French text, even though *mathématiques* is usually plural in French (as in English).

CHAPTER 8

Möbius transformations

In this chapter, we review Möbius transformations, which can be either defined as conformal self-maps of S^n or $\widehat{\mathbb{R}^n}$, or as products of inversions through spheres. These are extremely important maps in hyperbolic geometry because they are the isometries of hyperbolic space in the Poincaré models, as we shall see in the next chapter. In a nutshell, the Poincaré models are conformal domains of $\mathbb{R}^n$, therefore their isometries will be conformal maps of $\mathbb{R}^n$, which are Möbius transformations. We shall also see that Möbius transformations are crucial in understanding the relations between different models of hyperbolic space. As it turns out, Möbius transformations play an even more special role in 2- and 3-dimensional hyperbolic geometry, where they are a key part of a striking connection between hyperbolic geometry and complex geometry. This small miracle is essentially due to the coincidence of Möbius transformations of the sphere S^2 with projective automorphisms of the complex projective line $\mathbb{C}P^1$.

Möbius transformations are named after the 19th century German mathematician August Ferdinand Möbius. He is best known for the discovery of the Möbius strip, but also made important contributions to projective geometry (e.g., he introduced homogeneous coordinates), where Möbius transformations play an important part.

I recommend [Bea95, Chap. 3, Chap. 4] as a complementary treatment of Möbius transformations: the coverage is slightly less extensive than these notes, but it contains more details and proofs.

8.1 Conformal maps

8.1.1 Similarities

Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space. We recall that a linear map $f\colon V \to V$ is called a ***similarity*** if it satisfies one of the equivalent conditions:

(i) f multiplies all distances by a constant factor. Equivalently, there exists $k > 0$ such that $\|f(x)\| = k\|x\|$ for all $x \in V$.

(ii) f can be written as the composition of a linear isometry (an element of $\mathrm{O}(V)$) and a homothety (an element of $\mathbb{R}^* \operatorname{id}_V$).

Linear similarities form a subgroup of $\mathrm{GL}(V)$, which one may sensibly denote $\mathbb{R}^* \, \mathrm{O}(V)$.

Remark 8.1. More generally, similarities refer to the affine version of the definition above: they are the maps $V \to V$ that multiply all distances by a constant factor, equivalently they are affine maps whose linear part is a linear similarity.

On the other hand, a linear map $f \colon V \to V$ is called ***conformal*** (or ***angle-preserving***) if it preserves unoriented angles between vectors:

$$\forall u, v \in V \quad \angle(f(u), f(v)) = \angle(u, v) \, .$$

Remark 8.2. What is an angle? One may define the unoriented angle between two nonzero vectors $u, v \in V$ as the real number $\theta = \angle(u, v) \in [0, \pi]$ given by the formula

$$\langle u, v \rangle = \|u\| \, \|v\| \, \cos\theta \, .$$

Unless V is 2-dimensional, one cannot define a reasonable notion of oriented angles between vectors in V, so one may not define a notion of oriented angles-preserving map. It nevertheless makes sense to require that a map is angle-preserving and orientation-preserving.

Remark 8.3. By definition, if f is a linear conformal map, f must be injective: otherwise the angle $\angle(f(u), f(v))$ would not always be well-defined. Thus f is an element of $\mathrm{GL}(V)$.

It turns out that linear similarities and linear conformal maps are the same:

Proposition 8.4. *A linear map $f \colon V \to V$ is conformal if and only if it is a similarity.*

Proof. Elementary and left to reader. □

8.1.2 Conformal maps of $\mathbb{R}^n$

Let $V = \mathbb{R}^n$, or more generally any Euclidean vector space, and let $\Omega \subseteq V$ be an open set. Let $f \colon \Omega \to W$ be a differentiable map, where W is another Euclidean space. For our purposes we will take $W = V$, but the reader should easily be able to generalize to any W. Let us assume that $\mathrm{d}f$ is always injective, in other words f is an immersion. (In our case where $V = W$, this amounts to saying that $\mathrm{d}f$ is always bijective, i.e. f is a local embedding.)

Consider two regular curves $\gamma_1 \colon I_1 \to \Omega$ and $\gamma_2 \colon I_2 \to \Omega$. By ***regular*** we mean that γ_i is differentiable and γ_i' does not vanish. Let $p \in \Omega$ be a point of intersection of the two curves: assume $p = \gamma_1(t_1) = \gamma_2(t_2)$. One can define the (unoriented) angle between γ_1 and γ_2 as the angle between their tangent vectors:

$$\angle_p(\gamma_1, \gamma_2) \coloneqq \angle\left(\gamma_1'(t_1), \gamma_2'(t_2)\right) \, .$$

If f is an immersion, then the image curves $f \circ \gamma_1$ and $f \circ \gamma_2$ are regular curves in W that intersect at $f(p)$. One can again measure their angle of intersection. By definition, f is ***angle-preserving*** if, for any two regular curves γ_1 and γ_2 and for any point of intersection p,

$$\angle_{f(p)}(f \circ \gamma_1, f \circ \gamma_2) = \angle_p(\gamma_1, \gamma_2) \, .$$

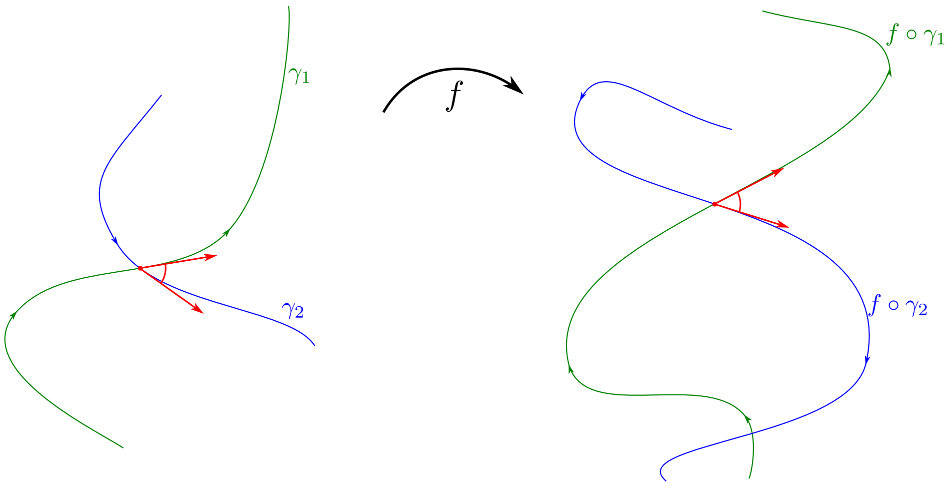


Figure 8.1: Angle-preserving map.

(See Figure 8.1.)

A synonym for *angle-preserving* is *conformal*:

Definition 8.5. A map $f\colon \Omega \subseteq V \to W$ is called ***conformal*** if it is an angle-preserving immersion.

The next characterization is left to the reader as an exercise (Exercise 8.1):

Proposition 8.6. *Let $f\colon \Omega \subseteq V \to W = V$. Then f is conformal if and only if f is differentiable and $\mathrm{d}f_x$ is a linear similarity for all $x \in \Omega$.*

In dimension 2, conformal maps are the same thing as locally injective holomorphic or anti-holomorphic maps:

Proposition 8.7. *Assume $V \approx \mathbb{C}$ be a 2-dimensional Euclidean vector space and $\Omega \subseteq V$ is an open connected subset. Then $f\colon \Omega \subseteq V \to V$ is conformal if and only if f is holomorphic or antiholomorphic and f' does not vanish.*

In higher dimensions, conformal maps are much more rigid, as shows the theorem of Liouville which will be given in § 8.2. Let us state a short version of this theorem here:

Theorem 8.8 (Liouville's conformal mapping theorem)**.** Let $f\colon \Omega \subseteq V \to V$ where $\dim V \geqslant 3$. Then f is conformal if and only if it is the restriction of a Möbius transformation of $\widehat{\mathbb{R}^n}$.

8.1.3 Conformal maps of Riemannian manifolds

The definitions of § 8.1.2 swiftly generalize to the Riemannian setting. Let (M, g) be a Riemannian manifold. We recall that the Riemannian metric g is the data of an inner product $\langle \cdot, \cdot \rangle$ in each tangent space $\mathrm{T}_x M$. In particular, one can measure the angle of intersection of two regular curves just like in $\mathbb{R}^n$, by taking the angle between tangent vectors. Once again, a conformal map as defined as an angle-preserving map:

Definition 8.9. Let (M, g) and (N, h) be Riemannian manifolds. A ***conformal map*** $f\colon \Omega \subseteq M \to N$ is an angle-preserving immersion.

By definition, two Riemannian metrics g_1 and g_2 on M are called ***conformal*** (or ***conformally equivalent***) if there exists a positive function $\lambda\colon M \to \mathbb{R}_{>0}$ such that $g_1 = \lambda g_2$. This means that any point $x \in \mathrm{T}_x M$, the inner products g_1 and g_2 define the same angle between any two vectors in $\mathrm{T}_x M$. A ***conformal structure*** on M consists of a conformal class of metrics. The next proposition is elementary:

Proposition 8.10. *A differentiable map $f\colon \Omega \subseteq M \to N$ is conformal if and only if the pullback metric f^*h is conformal to g.*

We leave the proof to Exercise 8.2 (elementary, yet a good exercise).

Definition 8.11. Let (M, g) be a Riemannian manifold. A ***conformal automorphism*** of M is a conformal diffeomorphism $f\colon M \to M$.

Remark 8.12. Definition 8.11 makes sense if M is only equipped with a conformal structure instead of a Riemannian metric.

Remark 8.13. By definition, any conformal map $M \to M$ is in particular a local diffeomorphism. If M is additionally compact and simply connected, then f must be a global diffeomorphism. Therefore, under these additional topological assumptions, the requirement that f is a diffeomorphism is superfluous in Definition 8.11. This is the case for instance when M is a topological sphere as in § 8.3.

8.2 Möbius transformations of $\widehat{\mathbb{R}^n}$

Let n be a positive integer. We denote $\widehat{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$. Topologically, $\widehat{\mathbb{R}^n}$ is the one-point compactification of $\mathbb{R}^n$ and is homeomorphic to S^n via, for instance, the famous stereographic projection. We shall soon see that the stereographic projection is in fact a conformal equivalence between $\widehat{\mathbb{R}^n}$ and S^n.

Let us say that $S \subseteq \widehat{\mathbb{R}^n}$ is a ***(hyper)sphere*** if either $S \subseteq \mathbb{R}^n$ is a (hyper)sphere, or $S = P \cup \{\infty\}$ where $P \subseteq \mathbb{R}^n$ is an affine (hyper)plane.

Definition 8.14. $S \subseteq \widehat{\mathbb{R}^n}$ be a hypersphere. The ***inversion through*** S is the map $f\colon \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ defined as follows:

- If $S = S(a, r)$ is a hypersphere in $\mathbb{R}^n$, then f is defined on $\mathbb{R}^n - \{a\}$ by the property that $x' = f(x)$ if and only if x and x' lie on a same half-line starting at a, and the Euclidean distances ax and ax' are related by:

$$ax \cdot ax' = r^2\,.$$

(See Figure 8.2.) f is continuously extended to $\widehat{\mathbb{R}^n}$ by $f(a) = \infty$ and $f(\infty) = a$.

- If $S = P \cup \{\infty\}$ where $P \subseteq \mathbb{R}^n$ is a hyperplane, then f is the Euclidean reflection through P on $\mathbb{R}^n$, extended to $\widehat{\mathbb{R}^n}$ by $f(\infty) = \infty$.

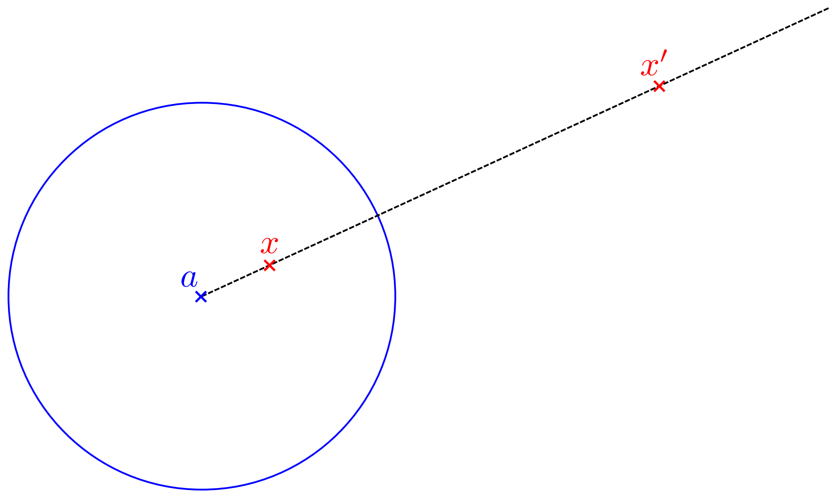


Figure 8.2: Inversion in a circle.

It is immediate to show from the definition that f is an involutive homeomorphism of $\widehat{\mathbb{R}^n}$, which fixes S and exchanges the two connected components of $\widehat{\mathbb{R}^n} - S$. It is also elementary to derive the analytic expression of the inversion in both cases (through a sphere or plane): see Exercise 8.4.

Definition 8.15. A map $f : \widehat{\mathbb{R}^n} \longrightarrow \widehat{\mathbb{R}^n}$ is called a ***Möbius transformation*** if it can be written as a finite product of inversions.

We will denote $\text{Möb}(\widehat{\mathbb{R}^n})$ the group of Möbius transformations of $\widehat{\mathbb{R}^n}$ and $\text{Möb}^+(\widehat{\mathbb{R}^n})$ the subgroup of orientation-preserving elements. It is easy to see that it is an index 2 subgroup: there is a short exact sequence

$$1 \longrightarrow \text{Möb}^+(\widehat{\mathbb{R}^n}) \longrightarrow \text{Möb}(\widehat{\mathbb{R}^n}) \longrightarrow \{\pm 1\} \longrightarrow 1$$

where $\text{Möb}(\widehat{\mathbb{R}^n}) \longrightarrow \{\pm 1\}$ is defined by assigning $+1$ [resp. -1] to an orientation-preserving (resp. reversing) Möbius transformation. Picking out any inversion $\tau \in \text{Möb}(\widehat{\mathbb{R}^n})$ yields a splitting of this short exact sequence via the isomorphism $\{\pm 1\} \xrightarrow{\sim} \{1, \tau\} \subseteq \text{Möb}(\widehat{\mathbb{R}^n})$.

Remark 8.16. It is quite common in the mathematics literature to impose that Möbius transformations are orientation-preserving, especially for $n = 2$. We do not make this restriction. We sometimes call $\text{Möb}(S^n)$ the ***full Möbius group*** and $\text{Möb}^+(S^n)$ the ***restricted Möbius group***.

Remark 8.17. We shall see in the next section that the Möbius group $\text{Möb}(\widehat{\mathbb{R}^n})$ is isomorphic to the Lie group $\text{PO}(n + 1, 1)$, whence $\text{Möb}^+(\widehat{\mathbb{R}^n})$ is identified to $\text{PSO}(n + 1, 1)$.

The central theorem of this section is:

Theorem 8.18. Let $n \geqslant 2$ and let $f\colon \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$. The following are equivalent:

(i) f is a Möbius transformation.

(ii) f preserves (unsigned) cross-ratios.

(iii) f is bijective and sphere-preserving, in the sense that it sends any sphere of lower dimension of $\widehat{\mathbb{R}^n}$ to a sphere.

(iv) f can be expressed as

$$f(x) = b + \frac{\alpha A(x-a)}{|x-a|^{\varepsilon}} \tag{8.1}$$

where $a, b \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, $A \in \mathrm{O}(n)$, and $\varepsilon \in \{0, 2\}$.

(v) f is a conformal automorphism.

Remark 8.19. To make sense of (ii), we need to define cross-ratios in $\widehat{\mathbb{R}^n}$. Let a, b, c, d be four distinct points in $\mathbb{R}^n$. Let us define the (unsigned) cross-ratio as

$$[a, b, c, d] = \frac{|ac||bd|}{|bc||ad|}$$

where we take the Euclidean distances. This expression can be extended when one of the points is ∞ by ignoring the factors containing it. Note that, when a, b, c, d are collinear, their cross-ratio coincides up to sign with their cross-ratio as four points on a real projective line as defined in § 6.4.3.

Remark 8.20. To be precise in (v), we should say what it means for an immersion $f\colon \widehat{\mathbb{R}^n} \to \widehat{\mathbb{R}^n}$ to be conformal at the point ∞. One way to define it is to say that for some/any inversion g through a sphere $S(a, r) \subseteq \mathbb{R}^n$, the composition $f \circ g$ is conformal at a. Similarly, at a point x_0 where $f(x_0) = \infty$, one can make sense of f being conformal at x_0 by requiring that for some/any inversion g through a sphere $S(a, r) \subseteq \mathbb{R}^n$, the composition $g \circ f$ is conformal at x_0.

We shall not prove theorem Theorem 8.18, but let us give a few insights. The proof of (i) $\Leftrightarrow$ (ii) is surprisingly simple, see [Bea95, Theorem 3.2.7]. The fact that inversions satisfy (iii), (iv), and (v) can be checked by direct computation. Clearly, these properties are stable under finite composition. Proving that conversely, (iii) implies (i) requires some tricks, but it is not too difficult. The fact that (iv) implies (i) may be seen as a variation of the important theorem of linear algebra that any orthogonal transformation is a finite product of reflections. It remains to show that (v) implies one of the other characterizations, which is the truly hard part of the theorem. When $n = 2$, the result can be proven using complex analysis (see § 8.5 for the derivation of the Möbius group in that case). When $n \geqslant 3$, the result is a special case of Liouville's theorem below.

Theorem 8.21 (Liouville's conformal mapping theorem)**.** Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ with $n \geqslant 3$. Then f is conformal if and only if f can be written as in (8.1).

Proving Liouville's theorem essentially amounts to solving a PDE, a higher-dimensional version of the Cauchy–Riemann equations. As can be expected, this is a hard task. We shall not provide a proof, which is more or less difficult depending on the regularity assumption on f: a proof avoiding functional analysis can be written for f of class $\mathcal{C}^3$, but the theorem is known to hold more generally for f in the Sobolev space $\mathrm{W}^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. We refer to [IM01] for a detailed account. Let us mention that, while $\mathrm{W}^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ does not include all differentiable functions, it is not hard to show that any conformal map is automatically $\mathrm{W}^{1,n}_{\mathrm{loc}}$: see [Dap]. I also recommend reading Danny Calegari's blog post [Cal13] for a sketch of proof with geometric insight.

8.3 Möbius transformations of S^n

8.3.1 Stereographic projection

There are several versions of the stereographic projection of a sphere to a plane. Let us consider the most standard one: given the unit sphere centered at the origin $S^n \subseteq \mathbb{R}^{n+1}$, we project the sphere S^n to the hyperplane $x_{n+1} = 0$ from the "North pole" $N = (0, \ldots, 0, 1)$. See Figure 8.3.

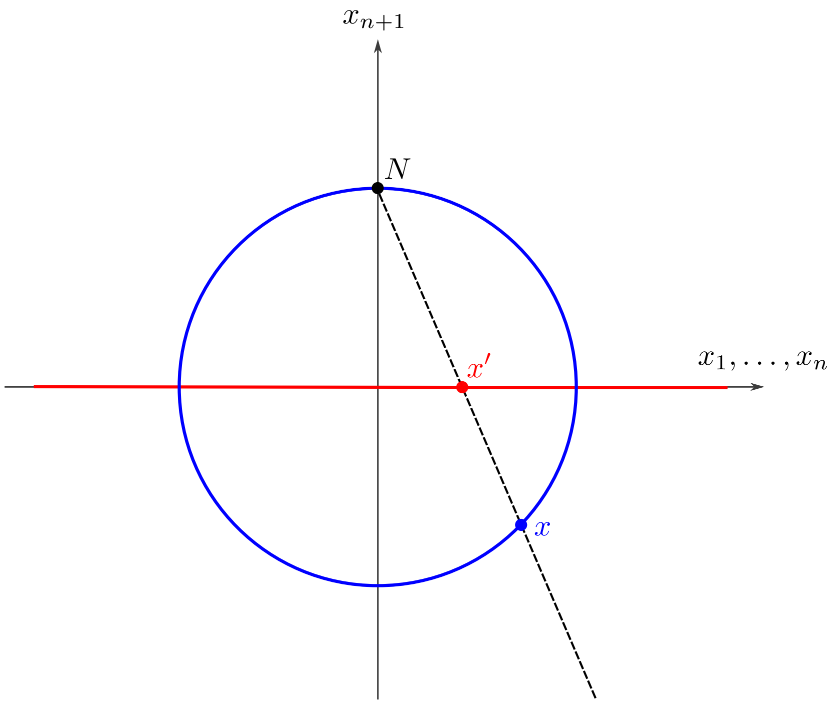


Figure 8.3: Standard stereographic projection $S^n - \{N\} \longrightarrow \mathbb{R}^n$.

The stereographic projection is a homeomorphism $s \colon S^n - \{N\} \longrightarrow \mathbb{R}^n$ that can be extended as a homeomorphism $S^n \longrightarrow \widehat{\mathbb{R}^n}$ by setting $s(N) = \infty$. It is elementary to derive its analytic expression: write $x' - N = \lambda(x - N)$ where $x' = s(x)$. Examining the last component gives

$0 - 1 = \lambda(x_{n+1} - 1)$, so $\lambda = \frac{1}{1-x_{n+1}}$. We thus find:

$$x'_k = \frac{x_k}{1 - x_{n+1}}$$

for all $k \in \{1, \ldots, n\}$. We easily recognize from this expression that the stereographic projection is the restriction to S^n of an inversion of $\widehat{\mathbb{R}^{n+1}}$ (figure out the details in Exercise 8.7):

Proposition 8.22. *The stereographic projection $s\colon S^n \to \widehat{\mathbb{R}^n}$ is the restriction to S^n of the inversion of $\widehat{\mathbb{R}^{n+1}}$ through the sphere $S(a, r)$ with $a = N$ and $r^2 = 2$.*

In particular, s is the restriction of a Möbius transformation, therefore it is conformal.

Corollary 8.23. *The stereographic projection $s\colon S^n \to \widehat{\mathbb{R}^n}$ is a conformal diffeomorphism.*

Remark 8.24. In Corollary 8.23, it is understood that the conformal structure of S^n is induced by $\mathbb{R}^{n+1}$. This coincides with the conformal structure on S^n underlying the spherical metric, since this metric is also induced by the Euclidean metric of $\mathbb{R}^{n+1}$.

8.3.2 Möbius transformations

Since the stereographic projection is the restriction of a Möbius transformation of $\widehat{\mathbb{R}^{n+1}}$, it is sphere-preserving: it sends spheres (of lower dimensions) in S^n to spheres. By definition, a map $S^n \to S^n$ is called an inversion if it is conjugate to an inversion of $\widehat{\mathbb{R}^n}$ by the stereographic projection. Thus inversions of S^n are conformal involutions and their fixed point sets are hyperspheres.

Let us define a Möbius transformation of S^n as a map $S^n \to S^n$ that can be written as a finite product of inversions. Using Theorem 8.18 we immediately obtain the characterization:

Theorem 8.25. Let $n \geqslant 2$ and let $f\colon S^n \to S^n$. The following are equivalent:

(i) f is a Möbius transformation (finite product of inversions).

(ii) f is conjugate to a Möbius transformation of $\widehat{\mathbb{R}^n}$ by the stereographic projection.

(iii) f is a sphere-preserving bijection.

(iv) f is a conformal automorphism.

Naturally, we denote $\mathrm{Möb}(S^n)$ the group of Möbius transformations of S^n and $\mathrm{Möb}^+(S^n)$ the index 2 subgroup of orientation-preserving elements. Clearly, the stereographic projection conjugates $\mathrm{Möb}(S^n)$ and the $\mathrm{Möb}(\mathbb{R}^n)$. In particular, they are isomorphic Lie groups.

8.3.3 Projective point of view

The projective point of view consists as seeing the sphere as a projective quadric. This will enable us to identify the Möbius group of S^n as the group of its projective automorphisms.

Consider Minkowski space $V = \mathbb{R}^{n+1,1}$. Recall that the projectivized light cone $P(\{\langle v, v\rangle = 0\})$ is a projective quadric $\mathcal{Q} \subseteq P(V)$ called an ellipsoid, whose equation in homogeneous coordinates is

$$X_1^2 + \cdots + X_{n+1}^2 - X_{n+2}^2 = 0\,.$$

In the affine chart $\varphi\colon P(\{X_{n+2} \neq 0\}) \xrightarrow{\sim} \{X_{n+2} = 1\} \approx \mathbb{R}^{n+1}$ with coordinates $x_k = \frac{X_k}{X_{n+2}}$, the equation of the ellipsoid is

$$x_1^2 + \cdots + x_{n+1}^2 - 1 = 0$$

so $\mathcal{Q}$ is identified to the unit sphere S^n in $\mathbb{R}^{n+1}$. Clearly, the Lorentz group $\mathrm{O}(n+1,1)$ acts on V preserving the light cone, therefore the projective Lorentz group $\mathrm{PO}(n+1,1)$ acts on $P(V)$ preserving the ellipsoid $\mathcal{Q}$.

Theorem 8.26. Let $n \geqslant 2$. The identification $S^n \approx \mathcal{Q}$ given by the inverse of the affine chart φ yields isomorphisms

$$\begin{aligned} \text{Möb}(S^n) &\approx \mathrm{PO}(n+1,1) \\ \text{Möb}^+(S^n) &\approx \mathrm{PSO}(n+1,1)\,. \end{aligned}$$

Remark 8.27. Recall that we also have $\mathrm{PO}(n+1,1) \approx \mathrm{O}^+(n+1,1)$ and $\mathrm{PSO}(n,1) \approx \mathrm{SO}^+(n+1,1)$ (the latter is called the restricted Lorentz group). Since $\mathrm{SO}^+(n,1)$ is connected (see § 3.4), it follows that $\text{Möb}^+(S^{n-1})$ is the identity component of $\text{Möb}(S^{n-1})$.

We do not give the detailed proof of Theorem 8.26, but here is the idea: both the Möbius transformations of S^{n-1} and its projective automorphisms can be characterized by the property that they are sphere-preserving, in the sense that they send spheres (of lower dimensions) to spheres. For the projective automorphisms, this characterization is a variation of Theorem 6.22. For Möbius transformations, it is part of Theorem 8.25. A variation of this proof using the cross-ratios preserving property might also be possible.

As a consequence of Theorem 8.26 and the discussion of § 8.3.2, we obtain:

Theorem 8.28. We have isomorphisms:

$$\begin{aligned} \text{Möb}(\widehat{\mathbb{R}^n}) &\approx \text{Möb}(S^n) \approx \mathrm{PO}(n+1,1) \approx \mathrm{O}^+(n+1,1) \\ \text{Möb}^+(\widehat{\mathbb{R}^n}) &\approx \text{Möb}^+(S^n) \approx \mathrm{PSO}(n+1,1) \approx \mathrm{SO}^+(n+1,1) \end{aligned}$$

Remark 8.29. In Theorem 8.28, we have isomorphisms of Lie groups: they are both homomorphisms of groups and diffeomorphisms of smooth finite-dimensional manifolds.

8.4 Möbius transformations of H^n and B^n

8.4.1 Möbius transformations of H^n

Consider the natural inclusion $\widehat{\mathbb{R}^{n-1}} \subseteq \widehat{\mathbb{R}^n}$ given by $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 0)$. Note that the complement $\widehat{\mathbb{R}^n} - \widehat{\mathbb{R}^{n-1}} = \mathbb{R}^n - \mathbb{R}^{n-1}$ consists of two half-spaces, which we denote H_+^n and H_-^n according to the sign of the last coordinate.

Clearly, any Möbius transformation of $\widehat{\mathbb{R}^n}$ that preserves $\widehat{\mathbb{R}^{n-1}}$ restricts to a Möbius transformation of $\widehat{\mathbb{R}^{n-1}}$. Moreover, it must either preserve or exchange H^n and H^-, since these are the connected components of $\widehat{\mathbb{R}^n} - \widehat{\mathbb{R}^{n-1}}$. Conversely, we have:

Theorem 8.30. Let $n \geqslant 2$. Any Möbius transformation of $\widehat{\mathbb{R}^{n-1}}$ uniquely extends to a Möbius transformation of $\widehat{\mathbb{R}^n}$ that preserves each of the two connected components of $\widehat{\mathbb{R}^n} - \widehat{\mathbb{R}^{n-1}}$.

Remark 8.31. If one does not insist that each of the two components of $\widehat{\mathbb{R}^n} - \widehat{\mathbb{R}^{n-1}}$ are preserved, then there are two possible extensions, which differ by the inversion through the hyperplane $\widehat{\mathbb{R}^{n-1}} \subseteq \widehat{\mathbb{R}^n}$.

Using the conformal equivalence $S^n \approx \widehat{\mathbb{R}^n}$ given by the stereographic projection, we obtain the equivalent form of the previous theorem:

Theorem 8.32. Let $n \geqslant 2$. Any Möbius transformation of $S^{n-1} \subseteq S^n$ uniquely extends to a Möbius transformation of S^n that preserves each of the two connected components of $S^n - S^{n-1}$.

Proof. We use the projective point of view explained in § 8.3.3. We see S^{n-1} as a projective quadric in $\mathbb{R}^{n,1}$. Consider the inclusion $\mathbb{R}^{n,1} \to \mathbb{R}^{n+1,1}$ given by $(x_1, \ldots, x_{n+1}) \mapsto (0, x_1, \ldots, x_{n+1})$. This induces an inclusion between the projective spaces, which restricts to an inclusion $S^n \to S^{n+1}$. Up to a change of coordinates, this is the same as the inclusion in the statement of theorem.

It is easy to check that the "obvious" inclusion of $\mathrm{PO}(n, 1)$ in $\mathrm{PO}(n+1, 1)$ given the diagonal embedding

$$\begin{aligned} \mathrm{O}(n,1) &\to \mathrm{O}(n+1,1) \\ M &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & \boxed{M} \end{bmatrix} \end{aligned}$$

provides an suitable extension of any $M \in \mathrm{O}(n, 1)$. Conversely, any suitable extension of M must be of the form

$$\hat{M} = \begin{bmatrix} x & 0 \\ v & \boxed{M} \end{bmatrix}$$

with $v \in \mathbb{R}^n$ and $x \in \mathbb{R}$, but the condition that $\hat{M} \in O(n + 1, 1)$ enforces $v = 0$ and $x^2 = 1$ (we leave this computation as an easy exercise). Finally, the fact that $\hat{M}$ preserves each component of $S^n - S^{n-1}$ rules out $x = -1$. ◻

Now let us examine the half-space $H^n := H^n_+$. The topological boundary of H^n in $\widehat{\mathbb{R}^n}$ is $\partial H^n = \widehat{\mathbb{R}^{n-1}}$. Observe that an inversion of $\widehat{\mathbb{R}^n}$ through a sphere S preserves H^n if and only if S is orthogonal to ∂H^n. Let us call ***Möbius transformation of*** H^n any map $f\colon H^n \to H^n$ that can be written as a product of such inversions. We have the characterization:

Theorem 8.33. Let $n \geqslant 2$ and $f\colon H^n \to H^n$. The following are equivalent:

(i) f is a Möbius transformation.
(ii) f is the restriction of a (unique) Möbius transformation of $\widehat{\mathbb{R}^n}$ that preserves H^n.
(iii) f is a conformal automorphism.

Proof. It is clear that (i) implies (ii) by definition. The converse is more tricky, we admit it. The fact that (ii) and (iii) are equivalent follows from Liouville's theorem when $n \geqslant 3$, and from direct analysis in the case $n = 2$ (see § 8.5). □

Note that in particular, any Möbius transformation H^n extends continuously to the boundary $\partial H^n = \widehat{\mathbb{R}^{n-1}}$, and the boundary map is a Möbius transformation of $\widehat{\mathbb{R}^{n-1}}$. The uniqueness of this boundary map is merely due to continuity, and its existence to the previous theorem. Conversely, given a Möbius transformation of $\widehat{\mathbb{R}^{n-1}}$, Theorem 8.30 guarantees that it extends to a Möbius transformation of H^n. Let us record this:

Theorem 8.34. Let $n \geqslant 2$. Any Möbius transformation of H^n extends continuously to the boundary $\partial H^n = \widehat{\mathbb{R}^{n-1}}$, and the boundary map is a Möbius transformation of $\widehat{\mathbb{R}^{n-1}}$. Conversely, any Möbius transformation $f \in \mathrm{Möb}(\widehat{\mathbb{R}^{n-1}})$ is the boundary map of a unique Möbius transformation $\hat{f} \in \mathrm{Möb}(H^n)$, called the ***Poincaré extension*** of f.

Corollary 8.35. *Let $n \geqslant 2$. We have the isomorphisms:*

$$\mathrm{Möb}(H^n) \approx \mathrm{Möb}(\widehat{\mathbb{R}^{n-1}}) \approx \mathrm{PO}(n,1) \approx \mathrm{O}^+(n,1)$$
$$\mathrm{Möb}^+(H^n) \approx \mathrm{Möb}^+(\widehat{\mathbb{R}^{n-1}}) \approx \mathrm{PSO}(n,1) \approx \mathrm{SO}^+(n,1)$$

8.4.2 Cayley transform and Möbius transformations of B^n

Let $n \geqslant 2$ and consider the open unit ball $B^n \subseteq \mathbb{R}^n$. Its topological boundary is $\partial B^n = S^{n-1}$. The story we told for H^n and ∂H^n can be repeated for B^n and ∂B^n. Indeed, the two are conformally equivalent via a Möbius transformation of $\widehat{\mathbb{R}^n}$.

Consider the stereographic projection $s\colon S^{n-1} \to \widehat{\mathbb{R}^{n-1}}$ from the "South pole" P with coordinates $(0, \dots, 0, -1)$. Similarly to the stereographic projection from the North pole studied in § 8.3.1, s extends as an inversion of $\widehat{\mathbb{R}^n}$, namely the inversion through the sphere $S(a, r)$ with $a = P$ and $r^2 = 2$. We leave it as an easy exercise to the reader to argue that this inversion maps $\widehat{\mathbb{R}^{n-1}}$ to S^{n-1} and H^n to B^n, and conversely. However, this map is orientation-reversing, so instead let us consider the composition

$$c \coloneqq \tau \circ s$$

where τ is the inversion (reflection) through the hyperplane $\widehat{\mathbb{R}^{n-1}}$, which clearly preserves B^n. We thus have:

Theorem 8.36. The map c is an orientation-preserving Möbius transformation of $\widehat{\mathbb{R}^n}$ that restricts to a conformal equivalence $H^n \to B^n$, called the ***Cayley transform***.

It is straightforward to derive the expression of the Cayley transform:

$$x'_k = \frac{2x_k}{1 + \|x\|^2 + 2x_n} \quad \text{for } k \in \{1, \dots, n-1\} \qquad x'_n = \frac{\|x\|^2 - 1}{1 + \|x\|^2 + 2x_n}$$

We can now use the Cayley transform to transport the situation of H^n to B^n, following § 8.4.1 step by step.

Theorem 8.37. Let $n \geqslant 2$. Any Möbius transformation of S^{n-1} uniquely extends to a Möbius transformation of $\widehat{\mathbb{R}^n}$ that preserves each of the two connected components of $\widehat{\mathbb{R}^n} - S^{n-1}$.

An inversion of $\widehat{\mathbb{R}^n}$ through a sphere S preserves B^n if and only if S is orthogonal to $\partial B^n = S^{n-1}$ (be careful: this does not amount to saying that the center of S lies on S^{n-1}). Let us call ***Möbius transformation of*** B^n any map $f\colon B^n \longrightarrow B^n$ that can be written as a product of such inversions. We have the characterization:

Theorem 8.38. Let $n \geqslant 2$ and $f\colon B^n \longrightarrow B^n$. The following are equivalent:

(i) f is a Möbius transformation of B^n.
(ii) f is conjugate to a Möbius transformation of H^n by the Cayley transform.
(iii) f is the restriction of a (unique) Möbius transformation of $\widehat{\mathbb{R}^n}$ that preserves B^n.
(iv) f is a conformal automorphism.

In particular, any Möbius transformation of B^n extends continuously to the boundary $\partial B^n = S^{n-1}$, and the boundary map is a Möbius transformation of S^{n-1}. Conversely, given a Möbius transformation of S^{n-1}, it uniquely extends to a Möbius transformation of B^n:

Theorem 8.39. Let $n \geqslant 2$. Any Möbius transformation of B^n extends continuously to the boundary $\partial B^n = S^{n-1}$, and the boundary map is a Möbius transformation of S^{n-1}. Conversely, any Möbius transformation $f \in \text{Möb}(S^{n-1})$ is the boundary map of a unique Möbius transformation $\hat{f} \in \text{Möb}(B^n)$, called the ***Poincaré extension*** of f.

Corollary 8.40. *Let $n \geqslant 2$. We have the isomorphisms:*

$$\text{Möb}(B^n) \approx \text{Möb}(S^{n-1}) \approx \text{PO}(n,1) \approx \text{O}^+(n,1)$$
$$\text{Möb}^+(B^n) \approx \text{Möb}^+(S^{n-1}) \approx \text{PSO}(n,1) \approx \text{SO}^+(n,1)$$

8.5 Möbius transformations of $\hat{\mathbb{C}}$, $\mathbb{D}$, and $\mathbb{H}$

The 2-dimensional case is special, because the theorem of Liouville (Theorem 8.21) no longer holds for an arbitrary open set $\Omega \subseteq \mathbb{R}^2$. On the other hand, the possibility to use complex numbers and complex analysis opens new perspectives. By a fortunate coincidence, we will see that the conformal automorphisms of $\Omega \subseteq \widehat{\mathbb{R}^2} \approx \hat{\mathbb{C}}$ are indeed Möbius transformations when $\Omega = \widehat{\mathbb{R}^2}$, $\Omega = B^2$, and $\Omega = H^2$. As a result, the theory of Möbius transformations that we developed in the previous sections is still valid in the 2-dimensional case when working on these domains.

From now on, we identify $\mathbb{R}^2 = \mathbb{C}$ and $\widehat{\mathbb{R}^2} = \hat{\mathbb{C}}$, and we denote $\mathbb{D} = B^2$ the unit disk in $\mathbb{C}$ and $\mathbb{H} = H^2$ the upper half-plane.

8.5.1 Holomorphic and conformal maps on domains of $\mathbb{C}$

Let $f\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$ where Ω is an open set. We recall that f is called ***holomorphic*** if it is complex-differentiable at every $z_0 \in \Omega$. Equivalently, f is real-differentiable at z_0 and its derivative $\mathrm{d}f_{z_0}$ is $\mathbb{C}$-linear (this amounts to the so-called ***Cauchy–Riemann equations***). We also say that f is ***antiholomorphic*** if it is differentiable at every $z_0 \in \Omega$, and its derivative $\mathrm{d}f_{z_0}$ is $\mathbb{C}$-antilinear ($\mathrm{d}f(iv) = -i\,\mathrm{d}f(v)$). It is left as an easy exercise to the reader to show that f is antiholomorphic if and only if the complex conjugate $\bar{f}$ is holomorphic.

The relation between conformal maps and holomorphic maps in real dimension 2 is entirely explained by the following elementary lemma of linear algebra:

Lemma 8.41. *Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero linear map. Identify $\mathbb{R}^2 \approx \mathbb{C}$.*

If L is orientation-preserving ($\det L > 0$), then

$$L \text{ is a similarity} \quad \Leftrightarrow \quad L \text{ is } \mathbb{C}\text{-linear} \quad \Leftrightarrow \quad L(z) = az \quad (a \in \mathbb{C}^*)$$

If L is orientation-reversing ($\det L < 0$), then

$$L \text{ is a similarity} \quad \Leftrightarrow \quad L \text{ is } \mathbb{C}\text{-antilinear} \quad \Leftrightarrow \quad L(z) = a\bar{z} \quad (a \in \mathbb{C}^*)$$

Proof. Elementary and left to the reader as an exercise. □

Corollary 8.42. $f\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$*. We have:*

$$f \text{ is conformal} \quad \Leftrightarrow \quad f \text{ is } \pm\text{-holomorphic and } f' \text{ does not vanish}.$$

(We call ***±-holomorphic*** *a function that is holomorphic or antiholomorphic.)*

8.5.2 Möbius transformations of $\hat{\mathbb{C}}$

The extended complex line $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the ***Riemann sphere***, and it can be identified to the complex projective line $\mathbb{C}P^1$ via the standard affine chart $[z_1 : z_2] \mapsto \frac{z_1}{z_2}$. Under this identification, the group of projective automorphisms of $\mathbb{C}P^1$, which is the projective linear group $\mathrm{PGL}(2,\mathbb{C})$, acts on $\hat{\mathbb{C}}$ by fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$. For more details, see § 6.4.

Theorem 8.43. A map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an Möbius transformation if and only if it is fractional linear (orientation-preserving case) or its conjugate is fractional linear (orientation-reserving case).

Proof. See Exercise 8.9. □

The fact that Theorem 8.18 and Theorem 8.25 hold when $n = 2$, even though Liouville's theorem does not, follows from the next theorem, whose proof relies on complex analysis.

Theorem 8.44. Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The following are equivalent:
(i) f is an orientation-preserving Möbius transformation.
(ii) f is a fractional linear transformation.
(iii) f is orientation-preserving conformal automorphism.
(iv) f is a complex automorphism (a biholomorphism $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$).

Remark 8.45. We have seen that a map $f\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is orientation-preserving conformal if and only if f is holomorphic and f' does not vanish. We can extend this property to maps $f\colon \Omega \subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, but first we need to extend the notions of holomorphicity and conformality for such maps. We have already seen how to define conformality in Remark 8.20. One can similarly define holomorphicity by composing with the map $z \mapsto \frac{1}{z}$. More formally, $\hat{\mathbb{C}}$ (or $\mathbb{C}P^1$) can naturally be equipped with a structure of one-dimensional complex manifold (also known as Riemann surface), which is the right setting for holomorphicity.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) immediately follows from Theorem 8.43. The equivalence (i) $\Leftrightarrow$ (iv) essentially follows from the compatibility between the extension of the notion of holomorphicity and conformality at ∞ (see Remark 8.45). The skeptical or meticulous reader may take this equivalence as a definition of complex automorphism. Let us now prove that (ii) $\Leftrightarrow$ (iii).

If f is fractional linear, then it is easy to check that it is orientation-preserving and conformal. Essentially, this boils down to the fact that holomorphic maps (with non-vanishing derivative) are orientation-preserving and conformal. Even quicker, we can use (i): since f is a Möbius transformation, it is conformal.

Conversely, assume that f is an orientation-preserving conformal automorphism. Since fractional linear transformations act transitively on $\hat{\mathbb{C}}$, we may assume that $f(\infty) = \infty$ by precomposing f with a fractional linear transformation (for instance, take $z \mapsto \frac{wz+1}{z+w}$ where $w = f^{-1}(\infty)$). Thus f restricts to an entire function $\mathbb{C} \to \mathbb{C}$, moreover $\lim |f(z)| = +\infty$ when $|z| \to +\infty$. It is a classical exercise of complex analysis that this forces f to be polynomial. Indeed, the function $g\colon z \mapsto f(\frac{1}{z})$ has a pole at $z = 0$ (since $|g(z)| \to +\infty$ when $z \to 0$), therefore the Laurent series of g has finitely many nonzero coefficients of negative degree, i.e. the power series of f has finitely many nonzero coefficients. Since f is bijective it must have exactly one zero, therefore it has degree 1 by the fundamental theorem of algebra. Thus $f(z) = az + b$ for some $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, in particular f is fractional linear. □

Corollary 8.46. *The natural identifications $S^2 \approx \hat{\mathbb{C}} \approx \mathbb{C}P^1$ induce isomorphisms:*

$$\mathrm{M\ddot{o}b}^+(S^2) \approx \mathrm{Aut}(\hat{\mathbb{C}}) \approx \mathrm{Aut}(\mathbb{C}P^1)$$

where $\mathrm{Aut}(\hat{\mathbb{C}})$ is the group of complex (i.e. conformal orientation-preserving) automorphisms of the Riemann sphere $\hat{\mathbb{C}}$ and $\mathrm{Aut}(\mathbb{C}P^1)$ is the group of projective transformations of $\mathbb{C}P^1$.

Recall that we also have isomorphisms $\mathrm{Möb}^+(S^2) \approx \mathrm{PSO}(3,1)$ and $\mathrm{Aut}(\mathbb{C}P^1) \approx \mathrm{PGL}(2,\mathbb{C})$ (acting projective linearly on $\mathbb{C}P^1$ or fractional linearly on $\hat{\mathbb{C}}$), therefore we obtain the "accidental" isomorphism of Lie groups:

$$\mathrm{PSO}(3,1) \approx \mathrm{PGL}(2,\mathbb{C})\,.$$

Remark 8.47. Let us mention that there is also an accidental isomorphism of complex Lie groups $\mathrm{PGL}(2,\mathbb{C}) \approx \mathrm{SO}(3,\mathbb{C})$.

8.5.3 Möbius transformations of $\mathbb{D}$

Which subgroup of $\mathrm{PGL}(2,\mathbb{C})$ leaves the unit disk $\mathbb{D}$ invariant when acting on the Riemann sphere $\hat{\mathbb{C}}$? To answer this question, it is useful to work in $\mathbb{C}P^1$. In homogeneous coordinates, the disk $\mathbb{D}$ can be written: $\mathbb{D} = \{[z_1 : z_2] \mid |z_1|^2 - |z_2|^2 < 0\}$. Indeed, this is clearly equivalent to $|z|^2 < 1$ where $z = \frac{z_1}{z_2}$. Consider the Hermitian symmetric form on $\mathbb{C}^2$:

$$\begin{aligned} h\colon \mathbb{C}^2 \times \mathbb{C}^2 &\to \mathbb{C} \\ ((z_1,z_2),(z_1',z_2')) &\mapsto z_1\overline{z_1'} - z_2\overline{z_2'} \end{aligned}$$

and denote $q(z_1,z_2) = h((z_1,z_2),(z_1,z_2)) = |z_1|^2 - |z_2|^2$ the associated quadratic form. The signature of h as a Hermitian symmetric form is $(1,1)$, in fact its matrix in the canonical basis of $\mathbb{C}^2$ is

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,.$$

The subgroup of $\mathrm{GL}(2,\mathbb{C})$ leaving h invariant is denoted $\mathrm{U}(h)$ or simply $\mathrm{U}(1,1)$. Let us also introduce the group $\mathrm{SU}(1,1)$ of elements of $\mathrm{U}(1,1)$ with determinant 1. In terms of matrices (see Exercise 8.10):

$$\begin{aligned} \mathrm{U}(1,1) &= \{M \in \mathcal{M}_{2\times 2}(\mathbb{C}) \mid M^{\mathsf{T}} H \bar{M} = H\} \\ &= \{uA \mid |u| = 1, A \in \mathrm{SU}(1,1)\} \end{aligned}$$

$$\begin{aligned} \mathrm{SU}(1,1) &= \{M \in \mathrm{SL}(2,\mathbb{C}) \mid M^{\mathsf{T}} H \bar{M} = H\} \\ &= \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}) \mid |a|^2 - |b|^2 = 1 \right\} \end{aligned}$$

Clearly, the projective action of $\mathrm{U}(1,1)$ on $\mathbb{C}P^1$ preserves $\mathbb{D} = \mathbf{P}\{q < 0\}$, since $\mathrm{U}(1,1)$ preserves q by definition. Conversely, any projective transformation of $\mathbb{C}P^1$ preserving $\mathbb{D}$ is induced by some element of $\mathrm{U}(1,1)$, see Exercise 8.10. Also, note that since any element $M \in \mathrm{U}(1,1)$ may be written $M = uA$ with $A \in \mathrm{SU}(1,1)$, the projective action of M and A coincide, and the inclusion $\mathrm{SU}(1,1) \subseteq \mathrm{U}(1,1)$ induces an isomorphism $\mathrm{PSU}(1,1) \approx \mathrm{PU}(1,1)$.

Theorem 8.48. Let $f\colon \mathbb{D} \to \mathbb{D}$. The following are equivalent:

(i) f is an orientation-preserving Möbius transformation of $\mathbb{D}$.

(ii) f is a fractional linear transformation that preserves $\mathbb{D}$.
(iii) f is a fractional linear transformation induced by some element of $\mathrm{SU}(1,1)$.
(iv) f is an orientation-preserving conformal automorphism.
(v) f is a complex automorphism (a biholomorphism $\mathbb{D} \to \mathbb{D}$).

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from the general case Theorem 8.38 and from the characterization of Möbius transformations of $\hat{\mathbb{C}}$ as fractional linear transformations (Theorem 8.44). The equivalence (ii) $\Leftrightarrow$ (iii) follows from the discussion above the theorem. The equivalence (iv) $\Leftrightarrow$ (v) is Corollary 8.42.

Finally, let us show that (iii) $\Leftrightarrow$ (v). It is clear that (iii) $\Rightarrow$ (v), since fractional linear transformations are holomorphic and bijective. It remains to show that conversely, any complex biholomorphism of $\mathbb{D}$ is fractional linear. This is the hardest part of the theorem, which requires some basic knowledge of complex analysis.

So, let $f\colon \mathbb{D} \to \mathbb{D}$ be a biholomorphism. After composing f with a fractional linear transformation (that preserves $\mathbb{D}$), we can assume that $f(0) = 0$. Specifically, one may post-compose f with $z \mapsto \frac{z-a}{1-\bar{a}z}$ where $a = f(0)$. Conclude with the lemma of Schwarz (see e.g. [Ahl78, Theorem 13]) that $f(z) = uz$ for some $u \in \mathbb{C}$ with $|u| = 1$. [If you are unfamiliar with the lemma of Schwarz, the argument is essentially as follows: apply the maximum principle to the function $g(z) = \frac{f(z)}{z}$, which can be holomorphically extended at $z = 0$ by $g(0) = f'(0)$. By applying the maximum principle to g on the disk $D(0,r)$ with $r \to 1$, we obtain that $|g| \leqslant 1$ on $\mathbb{D}$. On the other hand, switching f and f^{-1} if necessary, we have $|g'(0)| \geqslant 1$. By the maximum principle, g is constant.] In particular, $f(z) = uz$ is fractional linear. □

Corollary 8.49. *We have isomorphisms:*

$$\mathrm{M\ddot{o}b}^+(B^2) \approx \mathrm{Aut}(\mathbb{D}) \approx \mathrm{PSU}(1,1)\,.$$

8.5.4 Möbius transformations of $\mathbb{H}$

We have seen that in general the Cayley transform is the conformal equivalence $c\colon H^n \to B^n$ which can be described as $c = \tau \circ s$, where τ is the reflection through the $x_n = 0$ hyperplane and s is the inversion through the sphere $S(a,r)$ where $a = (0,\dots,0,-1)$ and $r^2 = 2$. In the case $n = 2$, using the complex variable z, we find $\tau(z) = \bar{z}$ and $s(z) = -i\frac{\bar{z}-i}{\bar{z}+i}$, which gives us the expression of the Cayley transform and its inverse:

$$\begin{aligned} c\colon \mathbb{H} &\to \mathbb{D} & \qquad c^{-1}\colon \mathbb{D} &\to \mathbb{H} \\ z &\mapsto i\,\frac{z-i}{z+i} & z &\mapsto -i\,\frac{z+i}{z-i} \end{aligned}$$

Note that c is induced by the linear map $C\colon \mathbb{C}^2 \to \mathbb{C}^2$ with matrix

$$C = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}.$$

Our discussion from the previous subsection (Möbius transformations of $\mathbb{D}$) can be transported to $\mathbb{H}$ via the Cayley transform.

Consider the Hermitian form $\tilde{h} = C^* h$ associated to the matrix

$$\tilde{H} = C^{\mathsf{T}} H \bar{C} = \begin{bmatrix} 0 & 2i \\ -2i & 0\,. \end{bmatrix}$$

This is a Hermitian form on $\mathbb{C}^2$ of signature $(1, 1)$, with associated quadratic form

$$\begin{aligned} \tilde{q}(z_1, z_2) &= 2i(z_1\bar{z_2} - z_2\bar{z_1}) \\ &= -\operatorname{Im}(z_1\bar{z_2}) \end{aligned}$$

As expected, the locus $\{\tilde{q} < 0\}$ in $\mathbb{C}^2$ is the cone over $\mathbb{H} = \{\operatorname{Im}(z) > 0\} \subseteq \mathbb{C}P^1$, since $\operatorname{Im}(z) = \frac{\operatorname{Im}(z_1\bar{z_2})}{|z_2|^2}$ for $z = \frac{z_1}{z_2}$.

The subgroup of $\mathrm{GL}(2, \mathbb{C})$ preserving $\{\tilde{q} < 0\}$ is $\mathbb{C}^*\,\mathrm{U}(\tilde{h}) = \mathbb{C}^*\,\mathrm{SU}(\tilde{h})$, where $\mathrm{U}(\tilde{h})$ [resp. $\mathrm{SU}(\tilde{h})$] is the subgroup of $\mathrm{GL}(2, \mathbb{C})$ [resp. $\mathrm{SL}(2, \mathbb{C})$] preserving $\tilde{h}$. In terms of matrices:

$$\begin{aligned} \mathrm{SU}(\tilde{h}) &= \{M \in \mathrm{SL}(2, \mathbb{C}) \mid M^{\mathsf{T}} \tilde{H} \bar{M} = \tilde{H}\} \\ &= \mathrm{SL}(2, \mathbb{R}) \end{aligned}$$

Indeed, we leave it to the reader as an easy exercise to check that for $M \in \mathrm{SL}(2, \mathbb{C})$, $\bar{M}^{\mathsf{T}} \tilde{H} = \tilde{H} M^{-1}$ if and only if M has real coefficients.

Remark 8.50. Alternatively, we can write $\mathrm{SU}(\tilde{h}) = C^{-1}\,(\mathrm{SU}(1, 1))\,C$, and one can prove that $C^{-1}\,(\mathrm{SU}(1, 1))\,C = \mathrm{SL}(2, \mathbb{R})$ by direct computation.

By transporting Theorem 8.48 via the Cayley transform, we obtain:

Theorem 8.51. Let $f\colon \mathbb{H} \to \mathbb{H}$. The following are equivalent:

(i) f is an orientation-preserving Möbius transformation of $\mathbb{H}$.
(ii) f is a fractional linear transformation that preserves $\mathbb{H}$.
(iii) f is a fractional linear transformation induced by an element of $\mathrm{SL}(2, \mathbb{R})$.
(iv) f is an orientation-preserving conformal automorphism.
(v) f is a complex automorphism (a biholomorphism $\mathbb{H} \to \mathbb{H}$).

Corollary 8.52. *We have isomorphisms:*

$$\text{Möb}^+(H^2) \approx \mathrm{Aut}(\mathbb{H}) \approx \mathrm{PSL}(2, \mathbb{R})\,.$$

Since we also have isomorphisms $\text{Möb}^+(B^2) \approx \text{Möb}^+(H^2) \approx \text{Möb}^+(S^1) \approx \mathrm{PSO}(2, 1)$, we obtain the “accidental” isomorphisms of Lie groups:

$$\mathrm{PSU}(1, 1) \approx \mathrm{PSL}(2, \mathbb{R}) \approx \mathrm{PSO}(2, 1)\,.$$

8.6 Exercises

Exercise 8.1. Characterization of conformal maps of $\mathbb{R}^n$.

Let V, W be Euclidean vector spaces and $\Omega \subseteq V$ be an open set. Consider an immersion $f\colon \Omega \to V$.

(1) Let γ_1 and γ_2 be two regular curves in Ω that intersect at $p \in \Omega$. Denote v_i the tangent vector to γ_i at p. Show that $f \circ \gamma_1$ and $f \circ \gamma_2$ are two regular curves in W that intersect at $f(p)$, and that the tangent vector to γ_i at $f(p)$ is $\mathrm{d}f(v_i)$.

(2) Prove Proposition 8.6: *Let $f\colon \Omega \subseteq V \to W = V$. Then f is conformal if and only if f is differentiable and $\mathrm{d}f_x$ is a linear similarity for all $x \in \Omega$.*

(3) Prove Proposition 8.7: *$f\colon \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is conformal if and only if f is holomorphic or antiholomorphic and f' does not vanish.* (This question requires basic knowledge of holomorphic functions.)

Exercise 8.2. Characterization of conformal maps between Riemannian manifolds

Let (M, g) and (N, h) be Riemannian manifolds.

(1) Let $f\colon V \to W$ be a linear map between vector spaces. For any bilinear form b on W, we define the bilinear form f^*b on V by $f^*b(u, v) := b(f(u), f(v))$. Show that if b is an inner product, f^*b is an inner product if and only if f is injective.

(2) Let $f\colon (V, \langle\cdot,\cdot\rangle_V) \to (W, \langle\cdot,\cdot\rangle_W)$ be a linear map between Euclidean vector spaces. Show that f is angle-preserving if and only if there exists $\lambda \in \mathbb{R}_{>0}$ such that $f^*\langle\cdot,\cdot\rangle_W = \lambda\langle\cdot,\cdot\rangle_V$.

(3) Let $f\colon (M, g) \to (N, h)$ be a differentiable map between Riemannian manifolds. How do you define the pullback f^*h? Show that f is conformal if and only if f^*h is conformal to g.

Exercise 8.3. Full vs restricted Möbius group

Denote $\mathrm{M\ddot{o}b}^+(S^n)$ the restricted Möbius group of S^n, consisting of orientation-preserving Möbius transformations.

(1) Show that $\mathrm{M\ddot{o}b}^+(S^n)$ is an index 2 normal subgroup of $\mathrm{M\ddot{o}b}(S^n)$.

(2) Show that $\mathrm{M\ddot{o}b}^+(S^n)$ is the identity component of $\mathrm{M\ddot{o}b}(S^n)$.

(3) Show the same results for $\mathrm{M\ddot{o}b}^+(B^n) < \mathrm{M\ddot{o}b}(B^n)$ and $\mathrm{M\ddot{o}b}^+(\widehat{\mathbb{R}^n}) < \mathrm{M\ddot{o}b}(\widehat{\mathbb{R}^n})$.

Exercise 8.4. Inversions

(1) Let $S = S(a, r)$ be the sphere of center a and radius r in $\mathbb{R}^n$. What is its Cartesian equation? Show that the inversion through S has the expression:

$$f(x) = a + \frac{r^2}{\|x - a\|^2}(x - a) .$$

(2) Let $P \subseteq \mathbb{R}^n$ be an affine hyperplane. Denote v a nonzero normal vector and $\lambda \in \mathbb{R}$ such that $x_0 = \lambda v$ belongs to P (why is λ well-defined?). Show that the Cartesian equation of P is $\langle x - x_0, v\rangle = 0$. Show that the inversion through P has the expression:

$$f(x) = x - 2\langle x - x_0, v\rangle \frac{v}{\|v\|^2} .$$

(3) Show that the results of (2) may be obtained by taking the limit of (1) with $a = x_0 + tv$ and $r = t\|v\|$ when $t \longrightarrow +\infty$.

(4) Recover the result that any finite product of inversions may be written

$$f(x) = b + \frac{\alpha A(x - a)}{|x - a|^{\varepsilon}}$$

where $a, b \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, $A \in \mathrm{O}(n)$, and $\varepsilon \in \{0, 2\}$.

Exercise 8.5. More inversions

(1) Show that any translation $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ can be written as a product of two reflections. Could you expect such a result?

(2) Show that any linear similarity $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ can be written as a product of two inversions. Could you expect such a result?

Exercise 8.6. Möbius transformations vs Euclidean similarities

Show that the subgroup of $\mathrm{Möb}(\widehat{\mathbb{R}^n})$ fixing ∞ is isomorphic to the group of affine similarities of $\mathbb{R}^n$.

Exercise 8.7. Stereographic projection

(1) Recover the expression of the standard stereographic projection $s \colon S^n \longrightarrow \widehat{\mathbb{R}^n}$.

(2) Recover that the stereographic projection is the restriction to S^n of an inversion of $\widehat{\mathbb{R}^{n+1}}$. Derive that s is a conformal equivalence.

(3) Recover that s is conformal by direct computation: compute the pullback Riemannian metric s^*g on $S^n - \{N\}$, where g is the Euclidean metric on $\mathbb{R}^n$.

Exercise 8.8. Poincaré extension

(1) Find the Poincaré extension of an inversion of $\widehat{\mathbb{R}^n}$.

(2) Write a new proof of the existence of the Poincaré extension of a Möbius transformation. Can you extend your argument to also prove uniqueness?

Exercise 8.9. Möbius transformations of $\hat{\mathbb{C}}$

The goal of this exercise is to show Theorem 8.43: *A map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an Möbius transformation if and only if it is fractional linear (orientation-preserving case) or its conjugate is fractional linear (orientation-reserving case).*

(1) Argue that it is enough to show that f is an orientation-preserving Möbius transformation if and only if it is fractional linear.

(2) (a) Show that the inversion through the sphere $S(a, r)$ can be written $f(z) = a + \frac{r^2}{\bar{z}-\bar{a}}$.

(b) Show that the inversion through the line with normal vector v going through the point $z_0 = \lambda v$ can be written $f(z) = 2z_0 - \frac{v}{\bar{v}}\bar{z}$.

(c) Show that the composition of any two inversions is fractional linear. Conclude that any Möbius transformation of $\hat{\mathbb{C}}$ is fractional linear.

(3) (a) Show that any fractional linear transformation may be written as a composition of maps of the form: $z \mapsto z + b$ where $b \in \mathbb{C}$, $z \mapsto az$ where $a \in \mathbb{C}^*$, and $z \mapsto \frac{1}{z}$.

(b) Show that the three maps of the previous question may be written as a product of inversions.

(c) Conclude that any fractional linear transformation is a Möbius transformation of $\hat{\mathbb{C}}$.

Exercise 8.10. The group $\mathrm{PSU}(1, 1)$

(1) Recall the definition of $\mathrm{SU}(1, 1)$ and show that

$$\mathrm{SU}(1,1) = \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}) \mid |a|^2 - |b|^2 = 1 \right\}$$

(2) Show that $\mathrm{U}(1, 1) = \{uA \mid |u| = 1, A \in \mathrm{SU}(1, 1)\}$. Derive that $\mathrm{PU}(1, 1) \approx \mathrm{PSU}(1, 1)$.

(3) Show that the action of any element of $\mathrm{U}(1, 1)$ by fractional linear transformation can be written

$$z \mapsto u\frac{z-a}{1-\bar{a}z}$$

where $|u| = 1$ and $a \in \mathbb{D}$.

(4) Recover from the previous question that the action of $\mathrm{U}(1, 1)$ on $\hat{\mathbb{C}}$ preserves $\mathbb{D}$.

(5) Prove that conversely, a fractional linear transformation preserving $\mathbb{D}$ coincides with the action of an element of $\mathrm{U}(1, 1)$.

(6) Recall why $\text{Möb}^+(\mathbb{D}) \approx \text{Aut}(\mathbb{D}) \approx \text{PSU}(1,1)$.

Exercise 8.11. The group $\text{PSL}(2,\mathbb{R})$

(1) Recover by direct proof that the Cayley transform $c(z) = i\,\frac{z-i}{z+i}$ defines a biholomorphism from $\mathbb{H}$ to $\mathbb{D}$.

(2) Recover by direct proof that the fractional linear action of $M \in \text{SL}(2,\mathbb{C})$ on $\hat{C}$ preserves $\mathbb{H}$ if and only if M has real coefficients.

(3) Recover by direct proof that $\text{SL}(2,\mathbb{R}) = C^{-1}\,(\text{SU}(1,1))\,C$ where $C = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$. Recall the connection between this result and the previous question.

(4) Show that there are natural "inclusions"

$$\text{PSL}(2,\mathbb{R}) \hookrightarrow \text{PGL}(2,\mathbb{R}) \hookrightarrow \text{PGL}(2,\mathbb{C})$$
$$\text{PSL}(2,\mathbb{R}) \hookrightarrow \text{PSL}(2,\mathbb{C}) \xrightarrow{\sim} \text{PGL}(2,\mathbb{C})$$

How would you describe the difference between $\text{PSL}(2,\mathbb{R})$ and $\text{PGL}(2,\mathbb{R})$?

Exercise 8.12. The one-dimensional case

Throughout the chapter, we discussed conformal maps and Möbius transformations of $\widehat{\mathbb{R}^n}$, S^n, H^n, B^n for $n \geqslant 2$. What about the case $n = 1$? Work out as many details as possible about what still works and what breaks.

CHAPTER 9

The Poincaré models

In this chapter we present the Poincaré ball model and the Poincaré half-space model of hyperbolic geometry. These are conformal models, meaning that they can be defined as Euclidean domains equipped with a metric that is conformally equivalent to the Euclidean metric.

Alternatively, the Poincaré ball model may be obtained from the hyperboloid model studied in Chapter 4 via a stereographic projection, and the half-space model may be derived via a Möbius transformation called the Cayley transform. We will use these relations to showcase the essential features of these models.

Historically, both Poincaré models of the hyperbolic plane were discovered by Eugenio Beltrami in 1868 ([Bel68a; Bel68b]), alongside the Beltrami–Klein model which we discussed in Chapter 7[1]. Poincaré rediscovered the half-plane and disk models in 1882 and revealed the connection between 2-dimensional hyperbolic geometry and complex geometry, especially Fuchsian groups and automorphic functions [Poi82].

9.1 The Poincaré ball model

9.1.1 Stereographic projection of the hyperboloid

Let $n \geqslant 2$ be an integer (we could also allow $n = 1$ for most of this chapter). Embed $\mathbb{R}^n$ in Minkowski space $\mathbb{R}^{n,1}$ in the obvious way:

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{R}^{n,1} \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_n, 0)\,. \end{aligned}$$

Consider the point $S = (0, \ldots, 0, -1) \in \mathbb{R}^{n,1}$ (the "South pole"). Let us denote $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ the hyperboloid as in Chapter 4 and $B^n \subseteq \mathbb{R}^n$ the Euclidean unit ball. We call ***stereographic projection*** of the hyperboloid from the point S the map $s\colon \mathcal{H}^+ \to B$ such that for every $x \in \mathcal{H}^+$ and $x' \in B^n$, the points S, x', x are collinear if and only if $x' = s(x)$. See Figure 9.1.

[1]Beltrami also discovered the pseudosphere in [Bel68a], which we prefer to call tractricoid: see Exercise 2.4.

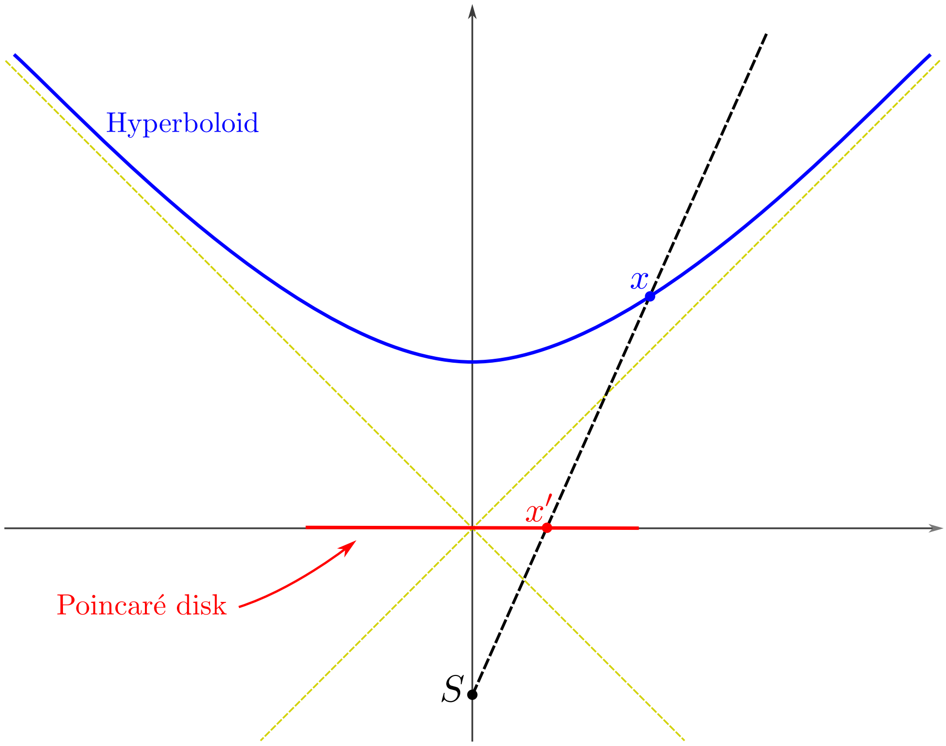


Figure 9.1: Stereographic projection of the hyperboloid to the Poincaré disk.

It is elementary to compute the analytic expression of the map s, and thereby prove that it is well-defined and bijective: writing $(x' - S) = \lambda(x - S)$ yields $\lambda = \frac{1}{1+x_{n+1}}$ by examining the last coordinate. Therefore we find:

$$x'_k = \frac{x_k}{1 + x_{n+1}}$$

for $k \in \{1, \dots, n\}$. In order to find the inverse, one can find the expression of λ in terms of x' by writing that $\langle x, x \rangle = -1$ (since $x \in \mathcal{H}^+$), which yields $\lambda = \frac{1-\|x'\|^2}{2}$. Therefore we find:

$$x_k = \frac{2x'_k}{1 - \|x'\|^2} \ (k \in \{1, \dots, n\}) \qquad x_{n+1} = \frac{1}{\lambda} - 1 = \frac{1 + \|x'\|^2}{1 - \|x'\|^2}\,.$$

In particular, we see that the stereographic projection s is a smooth (even real-analytic) diffeomorphis from the hyperboloid $\mathcal{H}^+$ to the ball B^n.

Definition 9.1. The ***Poincaré ball*** (or ***Poincaré disk***) (B^n, g_{B^n}) is the image of the hyperboloid $(\mathcal{H}^+, g_{\mathcal{H}^+})$ by the stereographic projection $s\colon \mathcal{H}^n \longrightarrow B^n$.

This definition means that we use the stereographic projection to transport the geometry of the hyperboloid to the unit ball. Technically, it is enough to transport the Riemannian metric, since all other geometric features follow: distance, geodesics, isometries, etc. It follows immediately from its definition that the Poincaré ball is a model of hyperbolic space:

Theorem 9.2. The Poincaré ball (B^n, g_{B^n}) is a complete, simply-connected Riemannian manifold of constant sectional curvature -1.

Remark 9.3. We have seen several different stereographic projections in this course. Their common feature is that they are all projections to a (hyper)plane by drawing lines from a single point. The stereographic projection of the hyperboloid to the Poincaré ball is especially similar to the stereographic projection of the hyperboloid to the Klein ball (see Figure 7.2). Nevertheless, the Poincaré ball and the Klein ball are significantly different models. See Exercise 9.3.

9.1.2 Riemannian metric

By definition, the hyperbolic metric (also called Poincaré metric) g_{B^n} is the pullback of the hyperbolic metric $g_{\mathcal{H}^+}$ on the hyperboloid by $s^{-1} \colon B^n \longrightarrow \mathcal{H}^+$. We leave it as an exercise (Exercise 9.1) to derive its explicit expression:

$$\mathrm{d}s^2 = 4\,\frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2}{(1 - \|x\|^2)^2}\,.$$

Remark 9.4. Of course, we could have defined the Poincaré ball by giving the Riemannian metric above, and then proved that it is isometric to the hyperboloid via stereographic projection.

We immediately note that $g_{B^n} = f g_0$, where g_0 is the Euclidean metric in B^n and $f(x) = \frac{4}{(1-\|x\|^2)^2}$ is a smooth function on B^n. This shows that the Poincaré metric is conformally equivalent to the Euclidean metric in B^n (see § 8.1.3). In short, we say that the Poincaré ball is a *conformal model* of hyperbolic space.

Remark 9.5. Note that $\lim_{\|x\|\to 1} f(x) = +\infty$: the conformal factor blows up as one approaches the boundary of the ball. This is expected because the hyperbolic metric in B^n is complete (unlike the Euclidean metric), therefore point of ∂B^n should be infinitely far away.

9.1.3 Distance

The distance function on the Poincaré ball can be computed directly as the pullback of the distance on the hyperboloid:

Proposition 9.6. *The distance in the Poincaré ball is given by*

$$d(x, y) = \operatorname{arcosh}\left(1 + \frac{2\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|^2)}\right).$$

Proof. Since the stereographic projection $s \colon \mathcal{H}^+ \longrightarrow B^n$ is a Riemannian isometry, it is also a metric isometry for the induced distances. Thus one can compute the distance on B^n as the pullback of the distance on $\mathcal{H}^n$: we have $d_{B^n} = \left(s^{-1}\right)^* d_{\mathcal{H}^n}$. Concretely:

$$\begin{aligned} d_{B^n}(x, y) &= d_{\mathcal{H}^n}\left(s^{-1}(x), s^{-1}(y)\right) \\ &= \operatorname{arcosh}\left(-\langle s^{-1}(x), s^{-1}(y)\rangle\right) \end{aligned}$$

The conclusion quickly follows from inputting the explicit expressions of $s^{-1}(x)$ and $s^{-1}(y)$, namely

$$s^{-1}(x) = \left(\frac{2x}{1 - \|x\|^2}, \frac{1 + \|x\|^2}{1 - \|x\|^2} \right)$$

and similarly for $s^{-1}(y)$, and writing the Minkowski inner product. □

Remarkably, the metric can be rewritten almost like a Cayley–Klein metric. Let $x, y \in B^n$ be any two distinct points. As we shall see in § 9.1.5, the geodesic through x and y is a Euclidean circle arc, which intersects the sphere ∂B^n orthogonally in two points. Call the two boundary points I and J as in Figure 9.2. We have seen in the previous chapter (Remark 8.19)

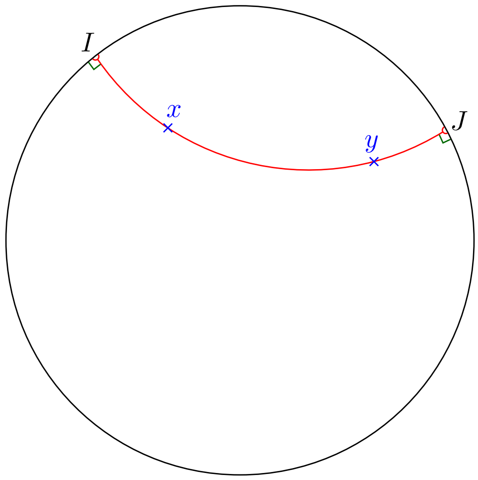


Figure 9.2: Geodesic in the Poincaré disk.

that one can define the (unsigned) cross-ratio of any 4-tuple of distinct points in $\mathbb{R}^n$. We claim:

Proposition 9.7. *The distance in the Poincaré ball is given by*

$$\begin{aligned} d(x, y) &= \ln[x, y, J, I] \\ &= \ln \frac{|Jx||Iy|}{|Jy||Ix|} \end{aligned} \tag{9.1}$$

It is a striking "coincidence" that the distance in the Poincaré ball can be written in such a similar fashion as the distance in the Klein ball (see Proposition 7.30)! Note however two differences: 1. There is a factor $\frac{1}{2}$ in the Cayley–Klein distance that does not appear here, and 2. The points I and J are different here, and the fours points I, x, y, J are not collinear in $\mathbb{R}^n$.

Proof. We shall see in § 9.1.4 that the isometries of the Poincaré ball are the Möbius transformations of the ball. Since Möbius transformations preserve cross-ratios (see Theorem 8.18), without loss of generality we can assume that $x = 0$ by choosing an isometry that maps x

to 0 (recall that isometries act transitively on hyperbolic space). Since geodesics through the origin are diameters (see § 9.1.5), the geodesic through x and y is a diameter $[I, J]$. We thus have $|I_x| = 1$, $|Jx| = 1$, $|Iy| = 1 + r$, $|Jy| = 1 - r$ where $r = \|y\|$. Therefore

$$\begin{aligned} \ln \frac{|Jx||Iy|}{|Jy||Ix|} &= \ln \frac{1+r}{1-r} \\ &= 2 \operatorname{artanh} r\,. \end{aligned}$$

On the other hand, by Proposition 9.6 we have

$$\begin{aligned} d(x, y) &= \operatorname{arcosh}\left(1 + \frac{2r^2}{1-r^2}\right) \\ &= 2 \operatorname{arcosh} \frac{1}{\sqrt{1-r^2}} \\ &= 2 \operatorname{artanh} r\,. \end{aligned}$$

We used the identities: $\operatorname{arcosh}(2x^2 - 1) = 2 \operatorname{arcosh} x$ and $\operatorname{arcosh} \frac{1}{\sqrt{1-x^2}} = \operatorname{artanh} x$. □

9.1.4 Isometries

In the previous chapter, we introduced Möbius transformations of the ball B^n.

Theorem 9.8. The group of isometries of the Poincaré ball is exactly the Möbius group of the ball:

$$\begin{aligned} \operatorname{Isom}(B^n, g_{B^n}) &= \operatorname{Möb}(B^n) \\ \operatorname{Isom}^+(B^n, g_{B^n}) &= \operatorname{Möb}^+(B^n) \end{aligned}$$

Proof. It is enough to prove $\operatorname{Isom}(B^n, g_{B^n}) = \operatorname{Möb}(B^n)$, the second identity follows immediately. Let us prove the mutual inclusion:

$\operatorname{Isom}(B^n, g_{B^n}) \subseteq \operatorname{Möb}(B^n)$: Since isometries are conformal, any $f \in \operatorname{Isom}(B^n, g)$ is a conformal automorphism of (B^n, g_{B^n}). Since g is conformally equivalent to the Euclidean metric g_0, f is also a conformal automorphism of (B^n, g_0). By Theorem 8.38, f is a Möbius transformation of B^n.

$\operatorname{Isom}(B^n, g_{B^n}) \supseteq \operatorname{Möb}(B^n)$: Since the Möbius group is generated by inversions, it is enough to prove that any inversion is an isometry. This can be checked by direct computation. Alternatively, since Möbius transformations preserves (unsigned) cross-ratios (Theorem 8.18), they preserve the distance (9.1). Conclude by remembering that distance-preserving maps and Riemannian isometries are the same. □

The next theorem follows immediately from Theorem 8.39.

Theorem 9.9. Any isometry of (B^n, g_{B^n}) uniquely extends continuously to $\partial B^n = S^{n-1}$, and the boundary map is a Möbius transformation of S^{n-1}. Conversely, any Möbius transformation $f \in \operatorname{Möb}(S^{n-1})$ extends to a unique isometry $\hat{f} \in \operatorname{Isom}(B^n, g_{B^n})$ called the ***Poincaré extension*** of f.

Corollary 9.10. *We have isomorphisms:*

$$\mathrm{Isom}(B^n, g_{B^n}) \approx \mathrm{M\ddot{o}b}(S^{n-1}) \approx \mathrm{PO}(n,1)$$
$$\mathrm{Isom}^+(B^n, g_{B^n}) \approx \mathrm{M\ddot{o}b}^+(S^{n-1}) \approx \mathrm{PO}^+(n,1)$$

In dimension 2, the Poincaré disk $B^2 = \mathbb{D}$ can be identified as a subset of $\hat{\mathbb{C}}$, and the orientation-preserving Möbius group of $\mathbb{H}$ is identified to $\mathrm{PSU}(1,1)$ acting by fractional linear transformations. This is also the group of complex automorphisms of $\mathbb{D}$. (See § 8.5.3 for details.)

Corollary 9.11. *The group of orientation-preserving isometries of the Poincaré disk is:*

$$\mathrm{Isom}^+(B^2, g_{B^2}) \approx \mathrm{Aut}(\mathbb{D}) \approx \mathrm{PSU}(1,1)\,.$$

In dimension 3, the boundary S^2 of Poincaré ball B^3 can be identified to $\hat{\mathbb{C}}$ by stereographic projection, or to $\mathbb{C}P^1$ by the standard affine chart. Any isometry of B^3 is uniquely determined by its extension to the boundary, which is a Möbius transformation of $S^2 \approx \hat{\mathbb{C}} \approx \mathbb{C}P^1$. We have seen in § 8.5.2 that the orientation-preserving Möbius group of S^2 is identified to $\mathrm{PSL}(2,\mathbb{C})$ acting by fractional linear transformations on $\hat{\mathbb{C}}$ or by projective transformations of $\mathbb{C}P^1$, and that this is also the group of complex automorphisms of $\hat{\mathbb{C}}$.

Corollary 9.12. *The group of orientation-preserving isometries of the* 3*-dimensional Poincaré ball is:*

$$\mathrm{Isom}^+(B^3, g_{B^3}) \approx \mathrm{Aut}(\hat{\mathbb{C}}) \approx \mathrm{PGL}(2,\mathbb{C})\,.$$

9.1.5 Geodesics

Theorem 9.13. The (unparametrized) geodesics of the Poincaré ball (B^n, g_{B^n}) are the intersections of B^n with circles in $\widehat{\mathbb{R}}^n$ that are orthogonal to $\partial B^n = S^{n-1}$.

Remark 9.14. A circle in $\widehat{\mathbb{R}}^n$ is either a Euclidean circle in $\mathbb{R}^n$, or $l \cup \{\infty\}$ where l is a straight line in $\mathbb{R}^n$. Therefore geodesics of the Poincaré ball are either arcs of Euclidean circles orthogonal to S^{n-1} (geodesics not going through the origin), or diameters (geodesics through the origin). See Figure 9.3 for a few geodesics in the Poincaré disk ($n = 2$).

Proof. It follows from our definition of the Poincaré ball that geodesics in B^n are the image of geodesics in $\mathcal{H}^+$ under the stereographic projection s.

First let us show that geodesics through the origin are diameters. Any such geodesic is the image of a geodesic in $\mathcal{H}^+$ through the point $(0,\dots,0,1)$, which is the intersection of $\mathcal{H}^+$ with a vertical 2-plane P. It is easy to see from the analytic expression of s that the image of $P \cap \mathcal{H}^+$ is $P \cap B^n$, which is a diameter.

Now let l be a geodesic of B^n that does not go through the origin. Let $x_0 \in l$. Since hyperbolic isometries act transitively, there exists $f \in \mathrm{Isom}(B^n, g_{B^n})$ such that $f(x_0) = 0$. Therefore $f(l) =: l'$ is a geodesic through the origin, so l' is a diameter. One can write

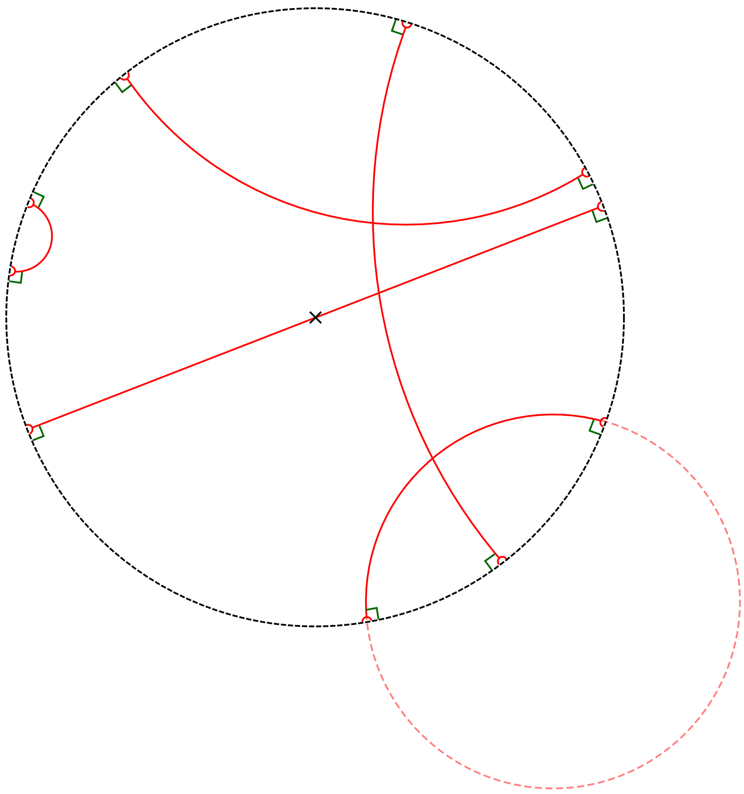


Figure 9.3: Geodesics in the Poincaré disk.

$l' = C \cap B^n$, where C is a circle of $\widehat{\mathbb{R}^n}$ orthogonal to S^{n-1}. Since f^{-1} is a Möbius transformation, it is conformal and sphere-preserving, therefore $f^{-1}(C)$ is circle of $\widehat{\mathbb{R}^n}$ orthogonal to S^{n-1}. We conclude that l is an arc of Euclidean circle orthogonal to S^{n-1}.

Conversely, let us argue that any diameter or arc of Euclidean circle orthogonal to S^{n-1} is a Poincaré geodesic. Consider such an arc l and denote its endpoints $I, J \in S^{n-1}$. Let l_0 be any geodesic through the origin, it is a diameter with endpoints $I_0, J_0 \in S^{n-1}$. There exists a Möbius transformation $f \in \text{Möb}(S^{n-1})$ such that $f(I_0) = I$ and $f(J_0) = J$. Indeed, it is not hard to argue with a little work that $\text{Möb}(S^{n-1})$ acts 2-transitively on S^{n-1} (when $n = 2$, it actually acts 3-transitively by Theorem 6.25). Let $\hat{f}$ be the Poincaré extension of f. Since $\hat{f}$ is a Möbius transformation, it sends l_0 to a circle of arc that intersects S^{n-1} orthogonally at I and J. We leave it as an exercise of Euclidean geometry to show that such an arc is unique, therefore $\hat{f}(l_0) = l$. On the other hand, $\hat{f}(l_0)$ is a geodesic since $\hat{f}$ is an isometry of the Poincaré ball. □

9.2 The Poincaré half-space model

9.2.1 Definition via the Cayley transform

We recall that the Cayley transform is a map $c\colon H^n \to B^n$, where $H^n \subseteq \mathbb{R}^n$ is the upper half-space. It is the restriction of an orientation-preserving Möbius transformation of $\widehat{\mathbb{R}^n}$, in particular c is a conformal equivalence between H^n and B^n. See § 8.4.2 for details and the analytic expression of the Cayley transform (also § 8.5.4 for $n = 2$).

Definition 9.15. The Poincaré upper half-plane (H^n, g_{H^n}) is the inverse image of the Poincaré ball (B^n, g_{B^n}) by the Cayley transform.

As before, we immediately obtain that the Poincaré upper half-plane is a model of hyperbolic space:

Theorem 9.16. The Poincaré half-space (H^n, g_{H^n}) is a complete, simply-connected Riemannian manifold of constant sectional curvature -1.

9.2.2 Riemannian metric

The Poincaré metric g_{H^n} can be computed as the pullback of g_{B^n} by the Cayley transform c. We leave the computation as an exercise to the reader (Exercise 9.1). One finds:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2}{{x_n}^2}\,.$$

We note once again that g_{H^n} is a conformal metric (i.e. conformally equivalent to the Euclidean metric g_0), with conformal factor $f(x) = \frac{1}{x_n^2}$. This was to be expected: we already know that g_{B^n} is a conformal metric in B^n, and the Cayley transform is a conformal map.

Remark 9.17. As expected, the Riemannian metric blows up when $x_n \to 0$, that is when x approaches $\partial H^n = \widehat{\mathbb{R}^{n-1}}$.

9.2.3 Distance

The Poincaré distance on H^n can be computed explicitly as $d_{H^n}(x, y) = d_{B^n}(c(x), x(y))$. Indeed, since the Cayley transform is a Riemannian isometry, it is also a metric isometry. After a few lines of calculations which we leave to the reader, one finds:

$$d(x, y) = \operatorname{arcosh}\left(1 + \frac{\|x - y\|^2}{2x_n y_n}\right) \tag{9.2}$$

Alternatively, one may again express the distance in terms of a cross-ratio:

$$\begin{aligned} d(x, y) &= \ln[x, y, J, I] \\ &= \ln \frac{|Jx||Iy|}{|Jy||Ix|}\,. \end{aligned}$$

Here, $I, J \in \widehat{\mathbb{R}^{n-1}}$ are now the ideal endpoints of the geodesic through x and y, which is a circle arc orthogonal to $\widehat{\mathbb{R}^{n-1}}$ (see § 9.2.5). The proof of this identity is quickly derived from the Poincaré ball case: since the Cayley transform is (the restriction of) a Möbius transformation of $\widehat{\mathbb{R}^n}$, it preserves cross-ratios.

9.2.4 Isometries

Theorem 9.18. The group of isometries of the Poincaré half-space is exactly the Möbius group of the upper half-space:

$$\begin{aligned}\mathrm{Isom}(H^n, g_{H^n}) &= \mathrm{Möb}(H^n)\\ \mathrm{Isom}^+(H^n, g_{H^n}) &= \mathrm{Möb}^+(H^n)\end{aligned}$$

Proof. Since (H^n, g_{H^n}) is the inverse image of (B^n, g_{B^n}) by the Cayley transform $c\colon H^n \longrightarrow B^n$, the group of isometries of (H^n, g_{H^n}) is conjugate to that of (B^n, g_{B^n}) by the Cayley transform: $\mathrm{Isom}(H^n, g_{H^n}) = c^{-1}\,(\mathrm{Isom}(B^n, g_{B^n}))\,c$. On the other hand, we know that $\mathrm{Isom}(B^n, g_{B^n}) = \mathrm{Möb}(B^n)$, and the Cayley transform conjugates $\mathrm{Möb}(H^n)$ and $\mathrm{Möb}(B^n)$. ◻

Corollary 9.19. *We have isomorphisms:*

$$\begin{aligned}\mathrm{Isom}(H^n, g_{H^n}) &\approx \mathrm{Möb}(\widehat{\mathbb{R}^{n-1}}) \approx \mathrm{PO}(n,1)\\ \mathrm{Isom}^+(H^n, g_{H^n}) &\approx \mathrm{Möb}^+(\widehat{\mathbb{R}^{n-1}}) \approx \mathrm{PO}^+(n,1)\end{aligned}$$

In dimension 2, the Poincaré half-plane $H^2 = \mathbb{H}$ can be identified as a subset of $\hat{\mathbb{C}}$, and the orientation-preserving Möbius group of $\mathbb{H}$ is identified to $\mathrm{PSL}(2,\mathbb{R})$ acting by fractional linear transformations. This is also the group of complex automorphisms of $\mathbb{H}$. (See § 8.5.4 for details.)

Corollary 9.20. *The group of orientation-preserving isometries of the Poincaré half-plane is:*

$$\mathrm{Isom}^+(H^2, g_{B^2}) \approx \mathrm{Aut}(\mathbb{H}) \approx \mathrm{PSL}(2,\mathbb{R})\,.$$

In dimension 3, the Poincaré half-space H^3 can be identified to $\mathbb{C} \times \mathbb{R}_{>0}$, and any isometry of H^3 is uniquely determined by its extension to the boundary $\partial H^3 = \hat{\mathbb{C}}$, which is a Möbius transformation of $\hat{\mathbb{C}}$. We have seen in § 8.5.2 that the orientation-preserving Möbius group of $\hat{\mathbb{C}}$ is identified to $\mathrm{PGL}(2,\mathbb{C})$ acting by fractional linear transformations, and that this is also the group of complex automorphisms of $\hat{\mathbb{C}}$.

Corollary 9.21. *The group of orientation-preserving isometries of the* 3*-dimensional Poincaré half-space is:*

$$\mathrm{Isom}^+(H^3, g_{H^3}) \approx \mathrm{Aut}(\hat{\mathbb{C}}) \approx \mathrm{PGL}(2,\mathbb{C})\,.$$

9.2.5 Geodesics

Theorem 9.22. The (unparametrized) geodesics of the Poincaré half-space (H^n, g_{H^n}) are the intersections of H^n with circles in $\widehat{\mathbb{R}}^n$ that are orthogonal to $\partial H^n = \widehat{\mathbb{R}^{n-1}}$.

Remark 9.23. A circle in $\widehat{\mathbb{R}^n}$ is either a Euclidean circle in $\mathbb{R}^n$, or $l \cup \{\infty\}$ where l is a straight line in $\mathbb{R}^n$. Therefore geodesics of the Poincaré half-space are either arcs of Euclidean circles orthogonal to $\widehat{\mathbb{R}^{n-1}}$, or vertical straight lines. See Figure 9.4 for a few geodesics in the Poincaré half-plane ($n = 2$).

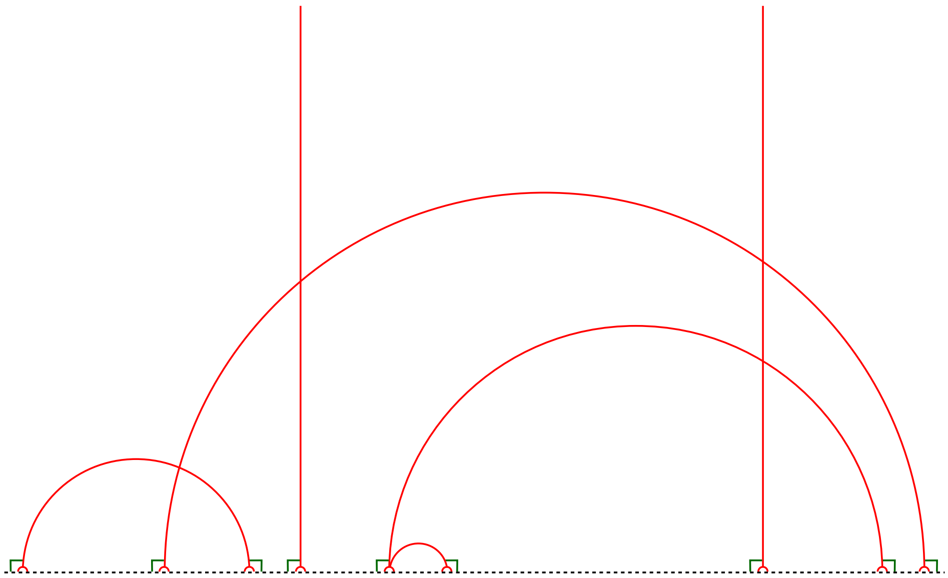


Figure 9.4: Geodesics in the Poincaré half-plane.

Proof. Geodesics in H^n are the inverse images of geodesics in B^n by the Cayley transform, and conversely. Since the Cayley transform is (the restriction of) a Möbius transformation of $\widehat{\mathbb{R}^n}$, it maps circles orthogonal to ∂H^n to circles orthogonal to ∂B^n, and conversely. □

9.3 Exercises

Exercise 9.1. Poincaré metric

Feel free to take $n = 2$ in this exercise. You can always do the general case afterwards.

(1) Recover the expression of the stereographic projection $s\colon \mathcal{H}^+ \longrightarrow B^n$.

(2) Recall the expressions of the Riemannian metrics $g_{\mathcal{H}^+}$ and g_{B^n} and recover the fact that s is a Riemannian isometry.

(3) Recover the expression of the Cayley transform $c\colon H^n \longrightarrow B^n$.

(4) Recall the expression of the metric g_{H^n} and recover that c is a Riemannian isometry.

Exercise 9.2. Curvature of the Poincaré metric

Let $\Omega \subseteq \mathbb{R}^n$ and let $g = e^{2\varphi} g_0$ be a conformal metric in Ω. Let u, v be an orthonormal pair of vectors in $\mathbb{R}^n$ and denote P the plane spanned by u and v. The following formula (reference: [Kap]) gives the sectional curvature of the metric g at a point $x \in \Omega$ in the direction of P:

$$K_P = -e^{-2\varphi} \left[\mathrm{D}^2\varphi(u,u) + \mathrm{D}^2\varphi(v,v) + \|\nabla\varphi\|^2 - \langle\nabla\varphi, u\rangle^2 - \langle\nabla\varphi, v\rangle^2 \right] .$$

(We have denoted $\nabla\varphi$ the gradient of φ.)

(1) Recover the curvature of the Poincaré metric in B^n by direct computation.

(2) Let $K < 0$. Can you find a metric of constant sectional curvature K in B^n?

(3) Same questions for H^n.

Exercise 9.3. Poincaré vs Klein ball

(1) Show that the natural identification between the Poincaré ball and the Beltrami–Klein ball is given by the map

$$\begin{aligned} \varphi\colon B^n_{\mathrm{P}} &\longrightarrow B^n_{\mathrm{K}} \\ x &\longmapsto \frac{2x}{1+\|x\|^2} . \end{aligned}$$

(2) Recover that φ is a Riemannian isometry by direct computation. *Feel free to take $n = 2$.*

Exercise 9.4. Poincaré vs Klein ball: the distance

(1) Let x, x' be two real numbers in $[0, 1)$ such that $x' = \frac{2x}{1+x^2}$. Show that $\frac{1+x'}{1-x'} = \left(\frac{1+x}{1-x}\right)^2$ and derive that $\operatorname{artanh} x' = 2 \operatorname{artanh} x$.

(2) Recover the fact that the map φ of Exercise 9.3 is a metric isometry, i.e. $d(\varphi(x), \varphi(y)) = d(x, y)$, in the case $y = 0$.

Exercise 9.5. Poincaré vs Klein ball: isometries

$\mathrm{PO}(n,1)$ acts by isometries on the Klein ball and the Poincare ball. Is this the same action on B^n? Show that the map φ of Exercise 9.3 conjugates the two actions.

Exercise 9.6. Hemisphere model

Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and denote S^n_+ the upper hemisphere (with $x_{n+1} > 0$). We also denote $S = (0,\dots,0,-1)$ the "South pole" of S^n. We recall that the Poincaré ball may be seen as the unit ball in $R^n \subseteq R^{n+1}$.

(1) Consider the stereographic projection $s\colon S^n \longrightarrow \widehat{\mathbb{R}^n}$. Find its analytic expression. Show that s restricts to a diffeomorphism $S^n_+ \longrightarrow B^n$.

(2) By definition, the ***hemisphere model*** $(S^n_+, g_{S^n_+})$ of hyperbolic space is the inverse image of the Poincaré ball (B^n, g_{B^n}) by the stereographic projection s. Prove that $g_{S^n_+}$ can be written:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2}{x_{n+1}^2}\,.$$

In what sense is the hemisphere model a conformal model?

Exercise 9.7. Relations between models

(1) Show that the different models of hyperbolic space are related as showed by the diagram in Figure 9.5.

(2) Show that geodesics in the hemisphere model are semi-circles that are orthogonal to the equator. Explain Figure 9.6.

(3) Recover that geodesics in the Poincaré half-space model are semi-circles that are orthogonal to the boundary.

Exercise 9.8. Matrix model of hyperbolic 3-space

Let H denote the set of 2×2 matrices with complex coefficients that are Hermitian symmetric:

$$H = \{A \in \mathcal{M}_{2\times 2}(\mathbb{C}) \mid A^* = A\}$$

where we denote $A^* = \bar{A}^{\mathsf{T}}$.

(1) Let $q(A) = -\det(A)$. Show that $q(A)$ is a quadratic form on H, with associated symmetric bilinear form $b(A,B) = -\frac{1}{2}\operatorname{tr}(A\,\mathrm{Comat}(B)^{\mathsf{T}})$.

(2) Show that (H,b) is isomorphic to $\mathbb{R}^{3,1}$ via

$$(x_1,x_2,x_3,x_4) \mapsto \begin{bmatrix} x_1+x_4 & x_2+ix_3 \\ x_2-ix_3 & x_1-x_4 \end{bmatrix}.$$

(3) Let $H_1 = H \cap \mathrm{SL}(2, \mathbb{C})$. Show that H_1 is a model of hyperbolic 3-space. What is the Riemannian metric?

(4) Show that $\mathrm{SL}(2, \mathbb{C})$ acts on H_1 by isometries via $M \cdot A = M A M^*$. What is the stabilizer of I_2? Recover that $\mathrm{Isom}^+(\mathbb{H}^3) \approx \mathrm{PSL}(2, \mathbb{C})$ and $\mathbb{H}^3 \approx \mathrm{PSL}(2, \mathbb{C})/\mathrm{PSU}(2)$.

Exercise 9.9. Hyperbolic subspace

Propose a definition of a hyperbolic subspace of a hyperbolic space $X = \mathbb{H}^n$, and describe the hyperbolic subspaces in all the different models of $\mathbb{H}^n$.

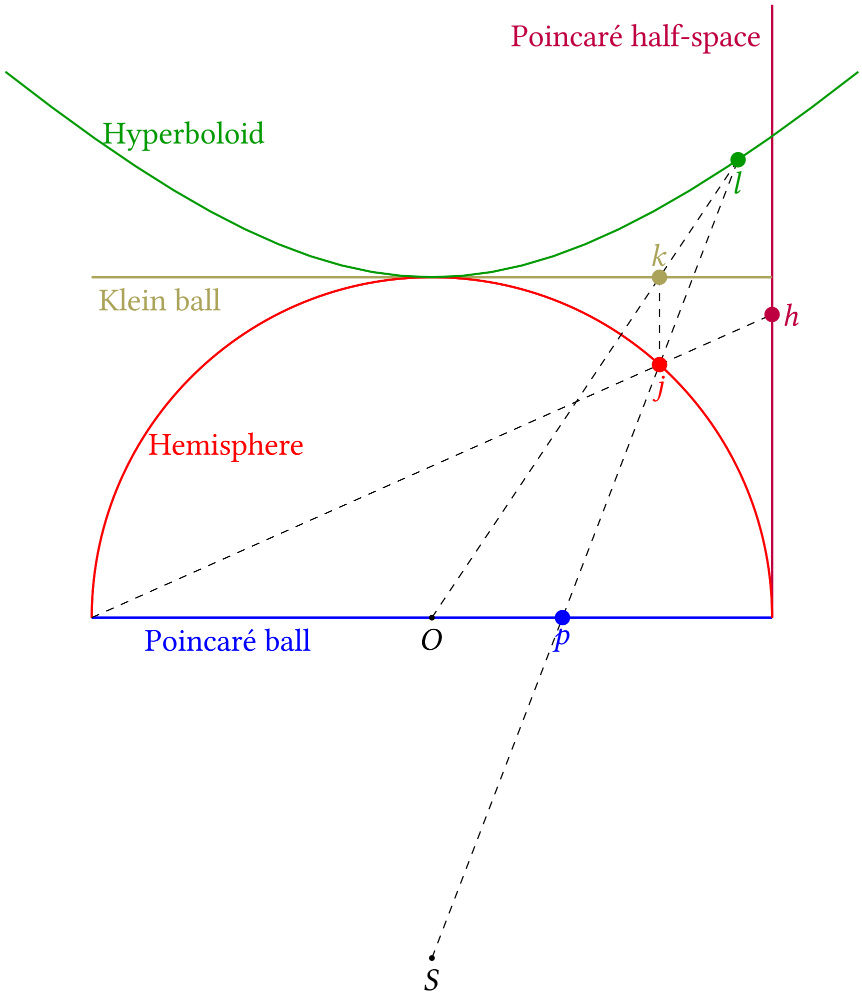


Figure 9.5: Relation between models of hyperbolic space.

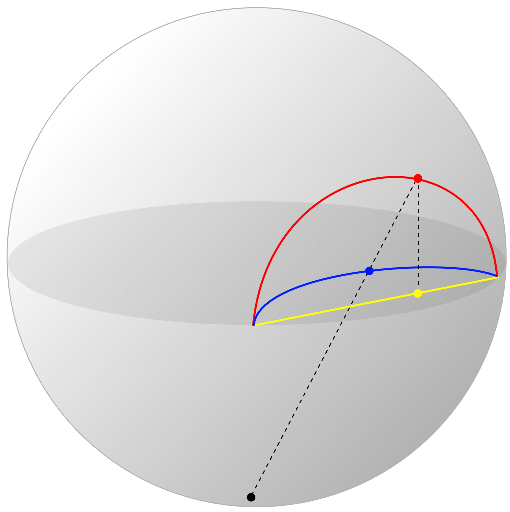


Figure 9.6: Geodesics in Poincaré ball, Klein ball, and hemisphere models.

Part V: Ideal boundary and classification of isometries

"The way I see the picture," said Gromov, "is that … we took two different, but sometimes overlapping, routes: Thurston concentrated on the most beautiful and difficult aspects of the field (hyperbolic 3-manifolds) and myself on the most general ones (hyperbolic groups)."

– Mikhail Gromov[2]

[2]Simons foundation, 2014. www.simonsfoundation.org/2014/12/22/mikhail-gromov/

CHAPTER 10

Ideal boundary of hyperbolic space

In this chapter, we introduce the ideal boundary of hyperbolic space and study some of its most important properties. For instance, we will see that any geodesic is uniquely determined by its pair of ideal endpoints. We will also discuss the related notions of Busemann functions and horospheres. In the next chapter, we will make critical use of the ideal boundary in order to classify isometries of hyperbolic space.

The ideal boundary is not strictly speaking part of hyperbolic space: its points are "at infinity". Nevertheless, it can be defined intrinsically from hyperbolic space, and offers a compactification of it that is geometrically meaningful.

Most of the notions of this chapter are naturally defined in a much more general framework, namely metric spaces of nonpositive curvature. Specifically, we shall use properties of hyperbolic space that hold more generally in CAT(0) metric spaces and/or Gromov hyperbolic metric spaces. For the reader interested to learn more about this point of view, I recommend [BH99]. Other excellent references include [BBI01; CDP90; GH90].

10.1 Metric properties of hyperbolic space

10.1.1 Basic properties

Throughout this chapter, let $(X, d) := \mathbb{H}^n$ denote the metric space that is n-dimensional hyperbolic space $\mathbb{H}^n$ with its distance function. (We take $n \geqslant 2$, although $n = 1$ is also acceptable.) We can alternatively use any of the models of hyperbolic space, since they are all isometric.

Let us point out that the notion of geodesic makes sense in a metric space: it is defined as map $\gamma \colon I \to X$, where $I \subseteq \mathbb{R}$ is an interval, such that for any sufficiently close $t_0, t_1 \in I$, $d(\gamma(t_0), \gamma(t_1)) = v|t_1 - t_0|$ for some constant $v > 0$ (the speed of the geodesic). When (X, d) is a manifold with the distance induced from a Riemannian metric, geodesics in (X, d) coincides with Riemannian geodesics (it is a fundamental theorem of Riemannian geometry that geodesics can be characterized as locally length-minimizing curves.)

Proposition 10.1. *Hyperbolic space $(X, d) = \mathbb{H}^n$ enjoys the following properties:*

(i) It is a complete metric space.

(ii) It is a proper metric space: any closed ball is compact.

(iii) For any two distinct points $x, y \in X$, there exists a unique geodesic γ from x to y up to reparametrization, moreover $d(x, y) = L(\gamma)$ (length of γ).

Remark 10.2. Property (iii) is sometimes called ***strong geodesic convexity***. It implies that (X, d) is ***uniquely geodesic***, which is the slightly weaker version: for any two distinct points $x, y \in X$, there exists a unique geodesic γ from x to y up to reparametrization such that $d(x, y) = L(\gamma)$. This implies in turn that X is a ***length space***: the distance between any two points is equal to the infimum of the lengths of rectifiable curves between them. Note that by definition, the Riemannian distance makes any Riemannian manifold a length space.

Proof. For (i), we use the famous Hopf-Rinow theorem of Riemannian geometry: a Riemannian manifold is complete as a metric space if and only if it is geodesically complete, i.e. all geodesics are defined on $\mathbb{R}$. We have seen that hyperbolic geodesics are defined on $\mathbb{R}$ in § 4.4 (see Corollary 4.10).

One way to prove (ii) is the following: let $B = \{x \in X \mid d(x, x_0) \leqslant r\}$ be a closed ball and consider the Riemannian exponential map $\exp_{x_0} : \mathrm{T}_{x_0} X \longrightarrow X$. By geodesic completeness, $\exp_{x_0}$ is globally well-defined on $\mathrm{T}_{x_0} X$. It follows immediately from (iii) and the definition of the Riemannian exponential that $B = \exp(B_{\mathrm{E}})$ where $B_{\mathrm{E}} = \{v \in \mathrm{T}_{x_0} X \mid \|v\| \leqslant r\}$. Of course, B_{E} is compact as a closed bounded set in a Euclidean space, therefore $B = \exp(B_{\mathrm{E}})$ is compact by continuity of $\exp_{x_0}$. Let us mention that a more intrinsic proof consists in arguing that any complete and locally compact length space is proper: see [BH99, Cor. 3.8 in Chap. I.3].

We have already proved (iii) in the hyperboloid model: see Corollary 4.11. $\square$

10.1.2 Convexity of the distance function

Consider the distance function on X: it is a map

$$d : X \times X \longrightarrow [0, +\infty) .$$

It is a general feature of CAT(0) metric spaces that the distance function is convex on $X \times X$. We shall not discuss CAT(0) metric spaces in general, because we are essentially interested in this particular property. Let us only mention that by definition, a CAT(0) metric space [1] is a space where geodesic triangles are thinner than Euclidean triangles with the same side lengths: see Figure 10.1. Any Hadamard manifold (complete, simply connected, with nonpositive sectional curvature) is a CAT(0) metric space. For a precise definition and a systematic treatment of CAT(k) spaces, we refer to [BH99].

The fact that the distance function is convex translates concretely as follows:

Theorem 10.3. Let γ_1 and γ_2 be any two geodesics in $X = \mathbb{H}^n$, not necessarily with same speed. The function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is convex on $\mathbb{R}$.

[1] Quoting [BH99]: The terminology "CAT(k)" was coined by M. Gromov [Gro87, p. 119]. The initials are in honour of E. Cartan, A.D. Alexandrov and V.A. Toponogov, each of whom considered similar conditions in varying degrees of generality.

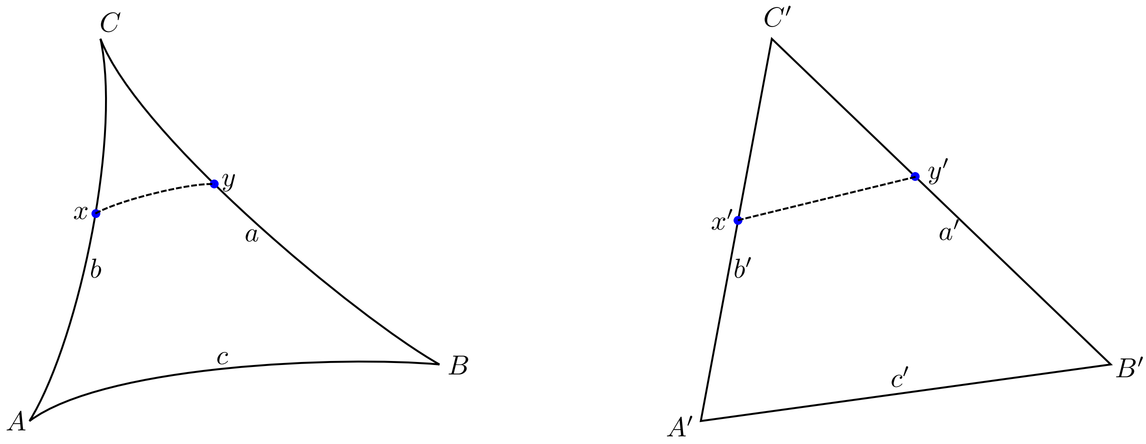


Figure 10.1: In a CAT(0) metric space, geodesic triangles are "slimmer" than Euclidean triangles with the same side lengths: in this schematic picture, we have $d_X(x, y) \leqslant d_{\mathbb{R}^2}(x', y')$.

We give a direct proof below of Theorem 10.3, using the explicit expression of the distance function in the hyperboloid model. (Another direct proof can be found in [Thu97, Theorem 2.5.8].) Let us nevertheless give a sketch of what a more intrinsic proof would look like. First of all, it is very straightforward to show that the distance function is convex in any CAT(0) metric space: see [BH99, Prop. 2.2 in Chap II.2][2]. Secondly, one can show [BH99, Ex 1.9d in Chap. II.1] that the CAT(0) condition is equivalent to the property that, for any geodesic triangle with side lengths a, b, c and opposite angles α, β, γ, we have:

$$c^2 \geqslant a^2 + b^2 - 2ab\cos\gamma\,.$$

(Note that the equality case is the law of cosines in Euclidean geometry.) When $X = \mathbb{H}^n$ is hyperbolic space, this inequality can be derived (see e.g. [Duc18, Prop. 3.4]) from the hyperbolic law of cosines (Theorem 13.8). Of course, writing the details of this proof involves significantly more work than our direct proof below, but this proof can be extended to show the much more general fact that any Riemannian manifold of sectional curvature $\leqslant k$ is locally CAT(k). This was originally proved by Cartan in 1928 [Car88] for $k = 0$ and Alexandrov [Ale51] in the general case. We refer to [BH99, Chap. II.1 Appendix] for details.

Proof of Theorem 10.3. We work in the hyperboloid model. We know (Theorem 4.8) that the geodesics γ_i ($i \in \{1, 2\}$) are of the form:

$$\gamma_i(t) = \cosh(\|v_i\| t) p_i + \sinh(\|v_i\| t) \frac{v_i}{\|v_i\|}$$

where p_i is a point on the hyperboloid, i.e. $p_i \in \mathbb{R}^{n,1}$ with $\langle p_i, p_i \rangle = -1$, and v_i is a tangent vector to the hyperboloid at p_i, i.e. $p_i \in \mathbb{R}^{n,1}$ with $\langle v_i, p_i \rangle = 0$.

[2] It is incorrectly assumed in [BH99] that the two geodesics have same (unit) speed, but the proof works without any changes for arbitrary geodesics.

The distance between $\gamma_1(t)$ and $\gamma_2(t)$ is given by:

$$d(t) = \operatorname{arcosh}\left(-\langle \gamma_1(t), \gamma_2(t)\rangle\right) .$$

It is straightforward to compute $d(t)$ and see that it is a $\mathcal{C}^\infty$ function of t, except possibly at $t = 0$ if $p_1 = p_2$. If $p_1 \neq p_2$, we can show that d is convex by proving that $d''(t) \geqslant 0$ for all t. It is sufficient to show that $d''(0) \geqslant 0$, since the other cases are obtained by reparametrizing the geodesics. As for the case $p_1 = p_2$, one can easily argue convexity by passing to the limit in the convexity inequality when $p_2 \longrightarrow p_1$. In summary, we can assume $p_1 \neq p_2$ and we want to show that $d''(0) \geqslant 0$.

By direct computation, one finds:

$$d''(0) = \sqrt{1+c^2}\,\frac{A-B}{c}$$

where

$$\begin{aligned} A &= \|v_1\|^2 + \|v_2\|^2 - 2\frac{\langle v_1, v_2\rangle}{\sqrt{1+c^2}} \\ B &= \frac{\left(\langle p_1, v_2\rangle + \langle v_1, p_2\rangle\right)^2}{c^2} \end{aligned}$$

and we have denoted $c^2 = \langle p_1, p_2\rangle^2 - 1$. (Note: these computations are guided by the fact that when $c \longrightarrow 0$, we approach the Euclidean scenario.) Thus it remains to show that $A \geqslant B$. Let us introduce the vectors:

$$\begin{aligned} u &= \frac{1}{c}\left(\langle p_1, p_2\rangle p_1 + p_2\right) \\ w_2 &= \frac{\langle p_1, v_2\rangle}{-1 - \langle p_1, p_2\rangle}(p_1 - p_2) - v_2 . \end{aligned}$$

It is immediate to check that $\langle u, p_1\rangle = \langle w_2, p_1\rangle = 0$, therefore these are two tangent vectors to the hyperboloid at p_1. Moreover, $\|u\| = 1$ and $\|w_2\| = \|v_2\|$. (Note: u is the initial velocity of the unit geodesic from p_1 to p_2, and w_2 is the inverse parallel transport of v_2 along that geodesic). It is straightforward to check that $B = \langle u, v_1 - w_2\rangle^2$, and we leave it as an exercise to show that $A \geqslant \|v_1 - w_2\|^2$ (with equality if and only if the vectors v_1, w_2, and u are collinear). We conclude that $A \geqslant B$ by the Cauchy-Schwarz inequality (in the tangent space $\mathrm{T}_{p_1}\mathcal{H}^+$, which is positive definite). □

Tracing the equality case in the proof above, we can improve the previous theorem:

Theorem 10.4. Given any two geodesics γ_1 and γ_2 in $X = \mathbb{H}^n$ (not necessarily with same speed), the function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is convex on $\mathbb{R}$. Moreover, it is strictly convex unless γ_1 and γ_2 are the same unoriented geodesic up to reparametrization.

Proof. In the proof of Theorem 10.3, we see that $d''(0) > 0$ unless $A = \|v_1 - w_2\|^2$, which occurs if and only if the vectors v_1, w_2, and u are collinear. Since u is the initial tangent vector of the unit geodesic γ from p_1 to p_2, the fact that v_1 is parallel to u means that $\gamma_1 = \gamma$ up to reparametrization. On the other hand, since w_2 is the inverse transport of v_2 along γ, and u is the inverse parallel transport of the tangent vector u_2 to γ at p_2, the fact that w_2 is parallel to u implies that v_2 is parallel to u_2. This means that $\gamma_2 = \gamma$ up to reparametrization. We conclude that $\gamma_1 = \gamma = \gamma_2$ up to reparametrization. □

Note that if γ_1 and γ_2 are two parametrizations of the same unoriented geodesics, one can write $\gamma_1(t) = \gamma_2(at+b)$, with $a \in \mathbb{R}-\{0\}$ and $b \in \mathbb{R}$. We then have $d(\gamma_1(t), \gamma_2(t)) = |(at+b)-t|$. This is a (piecewise) linear function of t, and it is constant if and only if $a = 1$. The case $a = 1$ means that γ_1 and γ_2 have same orientation and same speed, equivalently $\gamma_1(t) = \gamma_2(t - t_0)$ for some $t_0 \in \mathbb{R}$.

Corollary 10.5. *Let γ_1 and γ_2 be two complete geodesics in $X = \mathbb{H}^n$ such that $d(\gamma_1(t), \gamma_2(t))$ is bounded. Then $\gamma_1 = \gamma_2$ up to a reparametrization $t \mapsto t - t_0$.*

Proof. By Theorem 10.4, the function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is strictly convex unless γ_1 and γ_2 are the same unoriented geodesic up to reparametrization. Since a strictly convex function on $\mathbb{R}$ cannot be bounded, we conclude that γ_1 and γ_2 are the same unoriented geodesic up to reparametrization. Moreover, the discussion above shows that we must be in the case $a = 1$, otherwise $d(\gamma_1(t), \gamma_2(t))$ is again not bounded. □

Remark 10.6. Theorem 10.4 and Corollary 10.5 reflect the fact that $X = \mathbb{H}^n$ has negative curvature (bounded away from zero), in contrast to any CAT(0) space: for instance, two distinct parallel lines in the Euclidean plane furnish a counter-example to Corollary 10.5.

It is not hard to extend Theorem 10.3 to the case where one of the geodesics has zero speed, i.e. is a constant curve, although technically this is not called a geodesic.

Corollary 10.7. *For any fixed $y \in X = \mathbb{H}^n$, the function $x \mapsto d(x, y)$ is convex on X. In other words, the function $t \mapsto d(\gamma(t), y)$ is convex on $\mathbb{R}$ for any geodesic γ. Moreover, it is strictly convex unless γ goes through y.*

Proof. Let v be any tangent vector at y. For any $\varepsilon > 0$, the function $t \mapsto d(\gamma(t), \gamma_{\varepsilon v}(t))$ is convex by Theorem 10.3. By passing to the limit in the convexity inequality when $\varepsilon \to 0$, we obtain that $t \mapsto d(\gamma(t), y)$ is also convex.

Alternatively, we could write a direct proof from scratch using the explicit expression of $d(\gamma(t), y)$ in the hyperboloid model. The proof is then a simpler version of the proof of Theorem 10.3. It is also the best way to argue strict convexity. We leave out the details as an exercise. □

10.1.3 Gromov hyperbolicity

Let (X, d) be a geodesic metric space (there exists a length-minimizing geodesic between any two points). Consider a geodesic triangle, which consists of three vertices and three sides, i.e. length-minimizing geodesics between the vertices. Such a triangle is called ***δ-slim*** (where $\delta \geqslant 0$) if any side is contained in the δ-neighborhood of the union of the other two sides. See Figure 10.2.

Definition 10.8. A geodesic metric space (X, d) is called ***Gromov hyperbolic*** if there exists $\delta \geqslant 0$ such that any geodesic triangle is δ-slim.

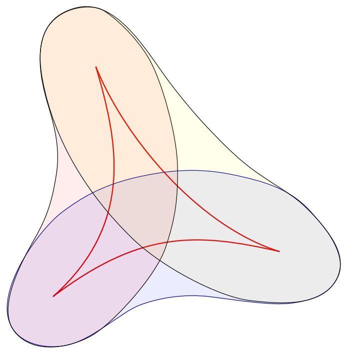


Figure 10.2: A δ-slim triangle: any of its side is contained in the δ-neighborhood of the union of the other two sides.

Example 10.9. Any geodesic metric space of bounded diameter is Gromov hyperbolic. The Euclidean plane is not Gromov hyperbolic: bigger and bigger triangles of the same aspect ratio require larger and larger δ's.

Remark 10.10. Contrary to the CAT(0) property or the convexity of the distance function, Gromov hyperbolicity only reflects negative curvature on a large scale, as opposed to an infinitesimal or local scale. One says that Gromov hyperbolicity is a ***coarse*** property.

Theorem 10.11. The hyperbolic space $X = \mathbb{H}^n$ is Gromov hyperbolic.

Proof. See Exercise 13.6. □

10.2 The ideal boundary

10.2.1 Visual boundary and ideal boundary

Let us call ***geodesic ray*** in $X = \mathbb{H}^n$ a unit geodesic defined on an interval of the form $[t_0, +\infty)$.

Definition 10.12. Two geodesic rays r_1 and r_2 are called ***asymptotic*** when the distance $d(r_1(t), r_2(t))$ is bounded when $t \to +\infty$.

Remark 10.13. The ***Hausdorff distance*** between two subsets $A, B \subseteq X$ is the infimum of all $\delta > 0$ such that A is contained in the δ-neighborhood of B and conversely. (This is not a proper distance, because it can be infinite and it is equal to zero whenever A and B have same closure.) It is easy to show that two geodesic rays are asymptotic if and only if they have finite Hausdorff distance.

Being asymptotic defines an equivalence relation $\sim$ on the set of all geodesic rays. Let us denote $r(+\infty)$ the equivalence class of a geodesic ray r. If γ is a complete geodesic, we also let $\gamma(+\infty)$ denote the equivalence class of the ray $t \in [0, +\infty) \mapsto \gamma(t)$, and $\gamma(-\infty)$ the equivalence class of the ray $t \in [0, +\infty) \mapsto \gamma(-t)$.

Definition 10.14. The ***ideal boundary*** (or ***Gromov boundary***, or ***boundary at infinity***) of $X = \mathbb{H}^n$ is the set of all equivalence classes of geodesic rays, denoted $\partial_\infty X$.

The Gromov boundary can be defined for any metric space, and enjoys some good properties when X is Gromov hyperbolic. On the other hand, we have the notion of visual boundary ("boundary at infinity in the vision of an observer"), which is best suited to CAT(0) spaces:

Definition 10.15. Let $x_0 \in X = \mathbb{H}^n$. The ***visual boundary*** $\partial_\infty^{x_0} X$ is the set of all equivalence classes of geodesic rays starting from x_0.

Given our definition of the ideal boundary and the visual boundary, it is clear that $\partial_\infty^{x_0} X$ is a subset of $\partial_\infty X$. In a general metric space, the two can be different, but in our case of interest $X = \mathbb{H}^n$ they are the same.

Lemma 10.16. *Let $x_0 \in X = \mathbb{H}^n$. Any geodesic ray r in X is asymptotic to a unique geodesic ray $\hat{r}$ starting from x_0.*

Proof. Let us first show uniqueness: assume that r_1 and r_2 are two geodesic rays defined on $[0, +\infty)$ with $r_1(0) = r_2(0) = x_0$, and are asymptotic. By Theorem 10.3, the function $f : t \in [0, +\infty) \mapsto d(r_1(t), r_2(t))$ is convex. Moreover, f is nonnegative, and by assumption f is bounded and $f(0) = 0$. Such a function must be constant equal to zero. This shows that $r_1 = r_2$.

Let us now show existence. Let r_n be the geodesic ray starting from x_0 that goes through $r(n)$. Since X is proper, any closed ball $B(x_0, R)$ is compact, and we can apply the Arzèla-Ascoli theorem to find a subsequence of r_n that converges uniformly on such balls to some limit $\hat{r}$. It is easy to argue that $\hat{r}$ is also a geodesic ray. It remains to show that $\hat{r}$ is asymptotic to r. Consider the geodesic triangle with vertices x_0, $r(0)$, and $r(n)$. The fact that it is δ-slim implies that the side $[x_0, r(n)]$ is contained in the δ'-neighborhood of the side $[r(0), r(n)]$ where $\delta' = \delta + d(x_0, r(0))$, and conversely. In other words, the segments $[x_0, r(n)]$ and $[r(0), r(n)]$ are within Hausdorff distance $\leqslant \delta'$. Passing to the limit when $n \to +\infty$, we obtain that the geodesic rays r and $\hat{r}$ are within Hausdorff distance $\leqslant \delta'$, therefore they are asymptotic. □

Remark 10.17. The uniqueness part of the proof works in any CAT(0) metric space. The existence part works in any proper Gromov hyperbolic space, but a different argument exists for complete CAT(0) metric spaces: see [BH99, Prop. 8.2].

Theorem 10.18. For any $x_0 \in \mathbb{H}^n$, we have an identification $\partial_\infty X \approx \partial_\infty^{x_0} X$. Moreover, $\partial_\infty^{x_0} X$ can be identified to the unit tangent space $\mathrm{T}^1_{x_0} X := \{u \in \mathrm{T}_{x_0} X \mid \|u\| = 1\}$.

Proof. It is clear that $\partial_\infty^{x_0} X$ is a subset of $\partial_\infty X$. In order to show that they are the same, we need to show that the map $\partial_\infty^{x_0} X \longrightarrow \partial_\infty X$ is surjective, which is to say that any geodesic ray r starting from some point $x \in X$ is asymptotic to some ray $\hat{r}$ starting from x_0. This follows from the existence part of Lemma 10.16.

For the second assertion, first observe that the uniqueness part of Lemma 10.16 says that the equivalence relation on geodesic rays starting from x_0 is trivial, in other words there is a unique geodesic ray representing each element of $\partial_\infty^{x_0} X$. Such a geodesic ray is uniquely determined by its initial tangent vector $u \in \mathrm{T}^1_{x_0} X$. □

10.2.2 Topology

Let $X = \mathbb{H}^n$ and let us denote $\bar{X}^\infty := X \sqcup \partial_\infty X$. There is a natural topology on $\bar{X}^\infty$ such that, for any geodesic ray r in X, $r(t) \longrightarrow r(+\infty)$ when $t \longrightarrow +\infty$. There are various ways to define this topology, here is one of them. Fix x_0 in X. For any $x \in \bar{X}^\infty$, there is a unique geodesic segment (when $x \in X$) or ray (when $x \in \partial_\infty X$), which we denote r_x, from x_0 to x. By definition, we say that $x_n \longrightarrow x$ in $\bar{X}^\infty$ when $r_{x_n} \longrightarrow r_x$ locally uniformly. We leave as an exercise to the reader to show that this is a well-defined topology on $\bar{X}^\infty$ and that it does not depend on the choice of x_0.

Theorem 10.19. Let $X = \mathbb{H}^n$ and consider $\bar{X}^\infty = X \sqcup \partial_\infty X$ with the topology defined above.

(i) The identifications $\partial_\infty X \approx \partial_\infty^{x_0} X \approx \mathrm{T}^1_{x_0} X$ are homeomorphisms. In particular, $\partial_\infty X$ is a topological $(n-1)$-sphere.
(ii) The inclusion $X \longrightarrow \bar{X}^\infty$ is a compactification of X: it is a homeomorphism to its image, which is dense, and $\bar{X}^\infty$ is compact. Topologically, $\bar{X}^\infty$ is a closed n-ball.

We leave the proof of Theorem 10.19 as an exercise for the most diligent readers.

Remark 10.20. There are various ways to compactify a topological space, the simplest being the one-point compactification. However, depending on the context, one may seek compactifications where the points at infinity retain some interesting information, so that the compactified space is insightful. The compactification of hyperbolic space (or more generally, a CAT(0) or a Gromov hyperbolic metric space) is an example of compactification that is geometrically meaningful. Other important examples include: the end compactification of a topological space, the Stone-Čech compactification of a topological space, and the projective compactification of a vector space. We have seen the latter in Chapter 6: embedding $\mathbb{K}^n$ as an affine hyperplane in $\mathbb{K}P^n$ is indeed a compactification (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

10.2.3 Essential properties

Let us put on the record a couple of essential properties of the ideal boundary, in addition to Theorem 10.18 and Theorem 10.19.

Theorem 10.21. Let $X = \mathbb{H}^n$. For any two distinct points $x, y \in \bar{X}^\infty$, there exists a unique geodesic from x to y.

Proof. When x and y are both in X, we already know that there exists a unique geodesic from x to y (see Proposition 10.1 (iii)). When $x \in X$ and $y \in \partial_\infty X$, the existence and uniqueness of a geodesic ray r starting from x such that $r(+\infty) = y$ is the content of Lemma 10.16. Finally, when x and y are both ideal points, the proof of the existence and uniqueness of a geodesic such that $\gamma(-\infty) = x$ and $\gamma(+\infty) = y$ can be conducted similarly to the proof of Lemma 10.16; we leave out the details. □

Considering the case where x and y are both ideal points, we immediately get:

Corollary 10.22. *Any complete geodesic $\gamma \colon \mathbb{R} \to X$ is uniquely determined by its pair of ideal points $\{\gamma(-\infty), \gamma(+\infty)\}$.*

The next theorem will be important in the next chapter:

Theorem 10.23. Let $X = \mathbb{H}^n$. Any isometry $f \colon X \to X$ uniquely extends to a continuous map $\hat{f} \colon \bar{X}^\infty \to \bar{X}^\infty$, and the restriction of $\hat{f}$ to $\partial_\infty X$ is a homeomorphism $\partial_\infty X \to \partial_\infty X$.

Proof. It is a straightforward exercise to check that the map $\hat{f}$ defined on $\partial_\infty X$ by $\hat{f}(r(+\infty)) := (f \circ r)(+\infty)$ is well-defined and extends f continuously. Moreover, $\widehat{f^{-1}} = \hat{f}^{-1}$, therefore $\hat{f}$ is a homeomorphism of $\bar{X}^\infty$, and it restricts to a homeomorphism of $\partial_\infty X$. □

Remark 10.24. As we shall see below, in the Poincaré ball model $X = B^n$, the ideal boundary is $\partial_\infty X = \partial B^n = S^{n-1}$. We already know from Theorem 9.9 that any isometry $f \colon X \to X$ uniquely extends to $\partial B^n = S^{n-1}$. This provides an alternative proof of Theorem 10.23. This proof is much more specific to $X = \mathbb{H}^n$ (as opposed to X being any Gromov hyperbolic metric space), but it also gives more information: Theorem 9.9 additionally tells us that the boundary map is a Möbius transformation of S^{n-1}, and uniquely determines f.

10.3 The ideal boundary in each model

One way to describe the ideal boundary of hyperbolic space in each of the different models is to choose our favorite base point in the model, and associate a natural "point at infinity" to each geodesic ray from that point, thus providing an identification of the visual boundary.

10.3.1 Ideal boundary of the hyperboloid model

Choose the base point $p_0 = (0, \ldots, 0, 1) \in \mathcal{H}^+$. Any geodesic ray starting from p_0 is of the form $r(t) = \cosh(t)p_0 + \sinh(t)v$, where v is a unit tangent vector at p_0. When $t \to +\infty$, $r(t) \sim e^t u$ where $u = p_0 + v$ is a lightlike vector. Thus the geodesic ray $r(t)$ is asymptotic to the lightlike line $l = \mathbb{R}u$. Conversely, any lightlike line l can be written $l = \mathbb{R}u$ where u is a lightlike vector of the form $u = (v_0, 1)$. Letting $v = (v_0, 0)$, we have that the geodesic ray $r(t)$ as above is asymptotic to l. In conclusion:

Theorem 10.25. The ideal boundary of the hyperboloid model $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ may be identified to the set of lightlike lines in $\mathbb{R}^{n,1}$.

Note that the set of lightlike lines in $\mathbb{R}^{n,1}$ is called the projectivized light cone, which we have encountered several times in this course. As a projective quadric, it is called an ellipsoid, and it is a topological sphere as expected.

10.3.2 Ideal boundary of the Klein model

We recall that there are two variations of the Klein model: the Cayley-Klein model, which is a projective model, and the Beltrami-Klein model, which is the Cayley-Klein model projected in an affine chart.

The Cayley-Klein model is the interior Ω^- of an ellipsoid $\mathcal{Q}$ in projective space $\mathcal{P} = \mathbb{R}P^n$, and geodesics are projective lines (or rather chords, i.e. projective lines restricted to Ω^-). It is clear that given any base point $x_0 \in \Omega^-$, each geodesic ray starting from x is uniquely determined by its intersection with $\mathcal{Q}$. In conclusion:

Theorem 10.26. The ideal boundary of the Cayley-Klein model $\Omega^- \subseteq \mathcal{P}$ is the ellipsoid $\mathcal{Q}$.

Remark 10.27. The ellipsoid $\mathcal{Q}$ is none other than the projectivized light cone of $\mathbb{R}^{n,1}$. The hyperboloid $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ and the Cayley-Klein model $\Omega^- \subseteq \mathbf{P}(\mathbb{R}^{n,1})$ thus have the same ideal boundary. Can you explain this "coincidence"? See Exercise 10.2.

Let us now turn to the Beltrami-Klein model. This is the unit ball $B^n \subseteq \mathbb{R}^n$ equipped with a Riemannian metric such that the geodesics in B^n are the chords (intersection of B^n with Euclidean straight lines in $\mathbb{R}^n$). Taking $x_0 = 0$ (the Euclidean center of B^n), a geodesic ray starting from x_0 is a Euclidean radius of B^n. Clearly, each such ray is uniquely determined by its intersection with $\partial B^n = S^{n-1}$. See Figure 10.3. In conclusion:

Theorem 10.28. The ideal boundary of the Beltrami-Klein ball $B^n \subseteq \mathbb{R}^n$ is the sphere $\partial B^n = S^{n-1}$.

10.3.3 Ideal boundary of the Poincaré models

The Poincaré ball is the unit ball $B^n \subseteq \mathbb{R}^n$ equipped with a Riemannian metric such that the geodesics in B^n are arcs of Euclidean circles orthogonal to the boundary $\partial B^n = S^{n-1}$, and Euclidean diameters of B^n. Taking $x_0 = 0$ (the Euclidean center of B^n), a geodesic ray

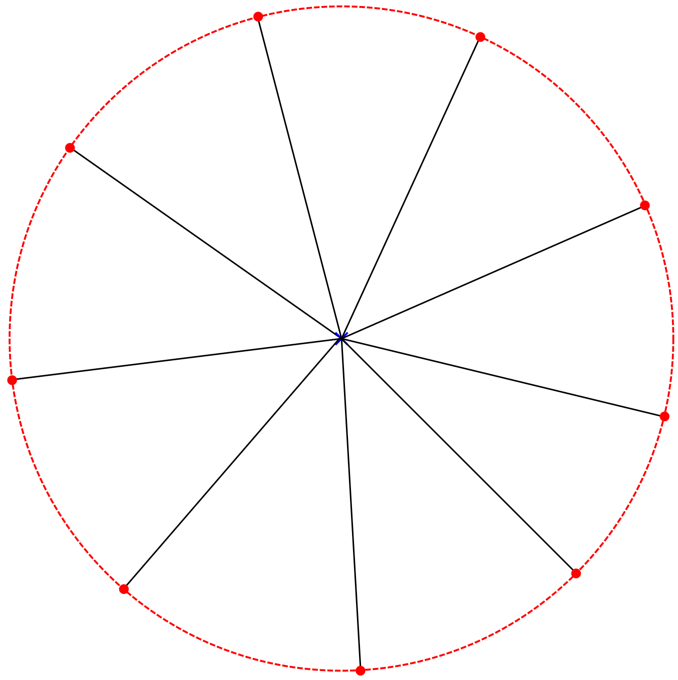


Figure 10.3: Visual boundary of the Beltrami-Klein disk (or the Poincaré disk) seen from the origin.

starting from x_0 is a Euclidean radius of B^n, just like in the Beltrami-Klein model (although the parametrization is different). Clearly, each such ray is uniquely determined by its intersection with $\partial B^n = S^{n-1}$ (again, see Figure 10.3). In conclusion:

Theorem 10.29. The ideal boundary of the Poincaré ball $B^n \subseteq \mathbb{R}^n$ is $\partial B^n = S^{n-1}$.

As for the Poincaré half-space $H^n \subseteq \mathbb{R}^n$, geodesics are Euclidean half-circles orthogonal to the boundary $\partial H^n = \widehat{\mathbb{R}^{n-1}}$. One can show again that each geodesic ray starting from some point $x_0 \in H^n$ is uniquely determined by its intersection with ∂H^n. One could either prove this directly,or derive it from the Poincaré ball case using the Cayley transform. In conclusion:

Theorem 10.30. The ideal boundary of the Poincaré half-space $H^n \subseteq \mathbb{R}^n$ is $\partial H^n = \widehat{\mathbb{R}^{n-1}}$.

10.4 Busemann functions and horospheres

10.4.1 Busemann functions

Let $X = \mathbb{H}^n$. For any geodesic ray $r\colon [0,+\infty) \to X$, let us define the ***Busemann function*** (relative to r) as:

$$\begin{aligned} B_r\colon X &\to \mathbb{R} \\ x &\mapsto \lim_{t\to+\infty} \left(d(x,r(t)) - t\right) \ . \end{aligned}$$

Proposition 10.31. *For any geodesic ray r, the Busemann function $B_r\colon X \to \mathbb{R}$ is well-defined, Lipschitz continuous with constant 1, and convex on X. Moreover B_{r_1} and B_{r_2} differ by an additive constant if and only if $r_1(+\infty) = r_2(+\infty)$.*

Proof. For any $x \in X$, the function $g\colon t \mapsto d(x,r(t)) - t$ is nonincreasing. Indeed, for $s \leqslant t$ we have $g(t) - g(s) = d(x,r(t)) - d(x,r(s)) - (t-s)$; by the triangle inequality $d(x,r(t)) - d(x,r(s)) \leqslant d(r(t),r(s)) = t-s$ so we obtain $g(t) - g(s) \leqslant 0$. Moreover, $g(t)$ is bounded below by $-d(x,r(0))$, since $t = d(r(0),r(t)) \leqslant d(r(0),x) + d(x,r(t))$. It follows that $g(t)$ converges when $t + \infty$ to some limit $B_r(x)$. By Dini's theorem, the convergence is locally uniform.

It follows from the triangle inequality that $|B_r(x) - B_r(y)| \leqslant d(x,y)$, i.e. B_r is Lipschitz continuous with constant 1. The convexity of B_r is immediately derived from the convexity of the distance function on $X = \mathbb{H}^n$ (Theorem 10.3).

If B_{r_1} and B_{r_2} differ by an additive constant, we may assume that $B_{r_1} = B_{r_2} =: B$ after reparametrizing r_1 or r_2. Let $t_0 \in [0,+\infty)$ and consider the closed convex set $C := \{B \leqslant -t_0\} \subseteq X$. Note that $B(r_1(t_0)) = -t_0$, therefore $r_1(t_0) \in C$. In fact, for $t \geqslant t_0$, $r_1(t_0)$ is the projection of $r_1(t)$ on C. Let us admit the previous point (see [BH99, Prop. 8.22 in Chap. II.8]) or leave it as an exercise. Similarly, $r_2(t_0)$ is the projection of $r_2(t)$ on $C$for $t \geqslant t_0$. It follows that $d(r_1(t),r_2(t)) \leqslant d(r_1(t_0),r_2(t_0))$ is bounded for $t \geqslant t_0$, hence r_1 and r_2 are asymptotic. Conversely, assume that r_1 and r_2 are asymptotic, and let us show that $B_{r_1} - B_{r_2}$ is constant. The function $t \mapsto d(r_1(t),r_2(t))$ is convex and bounded, therefore it has a finite limit when $t \to +\infty$. After reparametrizing of r_1 or r_2, we can assume that $\lim_{t\to+\infty} d(r_1(t),r_2(t)) = 0$. By the triangle inequality, $|B_{r_1}(x) - B_{r_2}(x)| \leqslant \lim_{t\to+\infty} d(r_1(t),r_2(t))$, so we conclude that $B_{r_1} = B_{r_2}$. □

Let now $\xi \in \partial_\infty X$ and let us define the ***Busemann function*** (relative to ξ) as:

$$\begin{aligned} B_\xi\colon X \times X &\to \mathbb{R} \\ (x,y) &\mapsto \lim_{t\to+\infty} \left(d(x,r(t)) - d(y,r(t))\right) \end{aligned}$$

where r is any geodesic ray with $r(+\infty) = \xi$. In other words, $B_\xi(x,y) = B_r(x) - B_r(y)$.

Proposition 10.32. *For any $\xi \in \partial_\infty X$, the Busemann function $B_\xi\colon X \times X \to \mathbb{R}$ is well-defined and continuous.*

Proof. As pointed out above, $B_\xi(x, y) = B_r(x) - B_r(y)$. It follows from the previous proposition that $B_\xi(x, y)$ is independent of the choice of geodesic ray r such that $r(+\infty) = \xi$. Moreover, B_ξ is clearly continuous since B_r is continuous. □

As an example, let us compute a Busemann function in the Poincaré half-space $X = H^n$.

Proposition 10.33. *In the Poincaré half-space $H^n \subseteq \mathbb{R}^n$, the Busemann function relative to the ideal point $\xi = \infty \in \partial H^n$ is:*

$$B_\xi(x, y) = \ln(y_n) - \ln(x_n) \ .$$

Proof. Let us choose a geodesic ray r in H^n such that $r(+\infty) = \xi$. Recall that geodesics having ∞ as an endpoint in the Poincaré half-space model are Euclidean vertical straight lines. We can take $r(t) = (0, \dots, 0, e^t)$. Indeed, $r(t)$ parametrizes the vertical straight line from 0 to ∞, and it is immediate to check that $r'(t) = (0, \dots, 0, e^t)$ has unit norm with respect to the Poincaré metric $\frac{\mathrm{d}x_1^2 + \dots \mathrm{d}x_n^2}{x_n^2}$, hence $r(t)$ is a geodesic ray.

The distance from a point $x = (x_1, \dots x_n)$ is given by (see (9.2)) $d(x, r(t)) = \operatorname{arcosh} A(t)$ where

$$\begin{aligned} A(t) &= 1 + \frac{x_1^2 + \dots + x_{n-1}^2 + (x_n - e^t)^2}{2x_n e^t} \\ &= \frac{x_1^2 + \dots + x_n^2 + e^{2t}}{2x_n e^t} = e^t a(t) \end{aligned}$$

with $a(t) = \frac{1}{2x_n}\left(1 + e^{-2t}\left(x_1^2 + \dots + x_n^2\right)\right)$. Since $\operatorname{arcosh}(A(t)) = \ln\left(A(t) + \sqrt{A(t)^2 - 1}\right)$, when $t \longrightarrow +\infty$ we have

$$\begin{aligned} d(x, r(t)) &\approx \ln\left(2A(t)\right) = t + \ln(2a(t)) \\ &\approx t + \ln\left(\frac{1}{x_n}\right) . \end{aligned}$$

We conclude that $B_r(x) = -\ln x_n$, and $B_\xi(x, y) = B_r(x) - B_r(y) = \ln(y_n) - \ln(x_n)$. □

10.4.2 Horospheres

Definition 10.34. A ***horosphere*** in $X = \mathbb{H}^n$ is a level set of a Busemann function B_r for some geodesic ray r. When $n = 2$, a horosphere is also called ***horocycle***.

One says that a horosphere given by a level set of B_r is *centered at* $\xi := r(+\infty)$. The next proposition follows immediately from the discussion of the previous subsection:

Proposition 10.35. *For any $\xi \in \partial_\infty X$ and any $x_0 \in X$, there exists a unique horosphere centered at ξ going through x_0; it is the set $\{x \in X \mid B_\xi(x, x_0) = 0\}$.*

The next proposition is also an immediate consequence of the discussion of the previous subsection:

Proposition 10.36. *Let $\xi \in \partial_\infty X$. Any geodesic γ with $\gamma(+\infty) = \xi$ intersects each horosphere centered at ξ exactly once.*

Proof. Let $x_0 = \gamma(t_0)$. We know that there exists a horosphere S centered at ξ going through x_0. Let us show that x_0 is the only intersection of S and γ. Let $r(t) = \gamma(t)$ for $t \in [t_0, +\infty)$. Since r is a geodesic ray with endpoint ξ, horospheres centered at ξ are level sets of the Busemann function B_r. Note that for any $x = \gamma(t_1)$ on the geodesic, $d(x, r(t)) = |t - t_1| - t$, we easily derive that $B_r(x) = -t_1$. In particular, S is the $-t_0$ level set of B_r, and it does not go through $x = r(t_1)$ unless $t_1 = t_0$. □

Proposition 10.37. *Let f be an isometry of X, and still denote f its extension to $\partial_\infty X$. For any $\xi \in \partial_\infty X$, f maps bijectively horospheres centered at ξ to horospheres centered at $f(\xi)$.*

Proof. Let r be a geodesic ray with $r(+\infty) = \xi$, then $f \circ r$ is a geodesic ray with $f \circ r(+\infty) = f(\xi)$. The fact that f is an isometry implies that $B_{f \circ r} = B \circ f^-1$. It follows that $S \subseteq X$ is a level set of B_r if and only if $f(S)$ is a level set of $B_{f \circ r}$. In other words, S is a horosphere centered at ξ if and only if $f(S)$ is a horosphere centered at $f(\xi)$. □

Now let us describe horospheres in the Poincaré models.

Theorem 10.38. In the Poincaré ball $X = (B^n, g_{B^n})$ or in the Poincaré half-space ball $X = (H^n, g_{H^n})$, the horospheres centered at any $\xi \in \partial_\infty X$ are the Euclidean hyperspheres of $\mathbb{R}^n$ contained in X that are tangent to $\partial_\infty X$ at ξ.

Figure 10.4 and Figure 10.5 feature a few horocycles in the Poincaré disk and in the Poincaré half-plane respectively.

Remark 10.39. In the Poincaré half-space $X = H^n$, recall that $\partial H^n = \widehat{\mathbb{R}^{n-1}}$. In the case where $\xi = \infty$, Theorem 10.38 must be understood as: Horospheres centered at ξ are horizontal hyperplanes, i.e. subsets $\{x_n = c\}$ with $c > 0$. See Figure 10.6.

Proof of Theorem 10.38. First we argue that it is enough to show the theorem for one particular ideal point $\xi_0 \in \partial_\infty X$. Recall that if X is the Poincaré ball or the Poincaré half-space, then any isometry $f \in \mathrm{Isom}(X)$ is uniquely determined by its extension to $\partial_\infty X$, which we abusively still denote f, and which is a Möbius transformation of $\partial_\infty X$. Since the Möbius group acts transitively on $\partial_\infty X$, if $\xi \in \partial_\infty X$ is any other ideal point, we can find an isometry $f \in \mathrm{Isom}(X)$ such that $f(\xi_0) = \xi$. By Proposition 10.37, f maps horospheres centered at ξ_0 to horospheres centered at ξ. On the other hand, f is a Möbius transformation of X, therefore it is sphere-preserving (see Theorem 8.18), and it also preserves tangency to $\partial_\infty X$. In conclusion, it is enough to show the theorem at ξ_0. Moreover it is enough to do the case $X = H^n$, because the case $X = B^n$ can then be derived using the Cayley transform.

Thus we take $X = H^n \subseteq \mathbb{R}^n$ and let us pick $\xi_0 = \infty \in \partial_\infty X$. In this case, the (generalized) Euclidean hyperspheres tangent to ξ_0 are the horizontal Euclidean hyperplanes in H^n. We want to show that such are the horospheres centered at ξ_0. For any $x \in H^n$, the horosphere through x is $S = \{y \in H^n \mid B_{\xi_0}(x, y) = 0\}$. By Proposition 10.33, we immediately find $S = \{y \in H^n \mid y_n = x_n\}$. In other words, S is the horizontal hyperplane through x. □

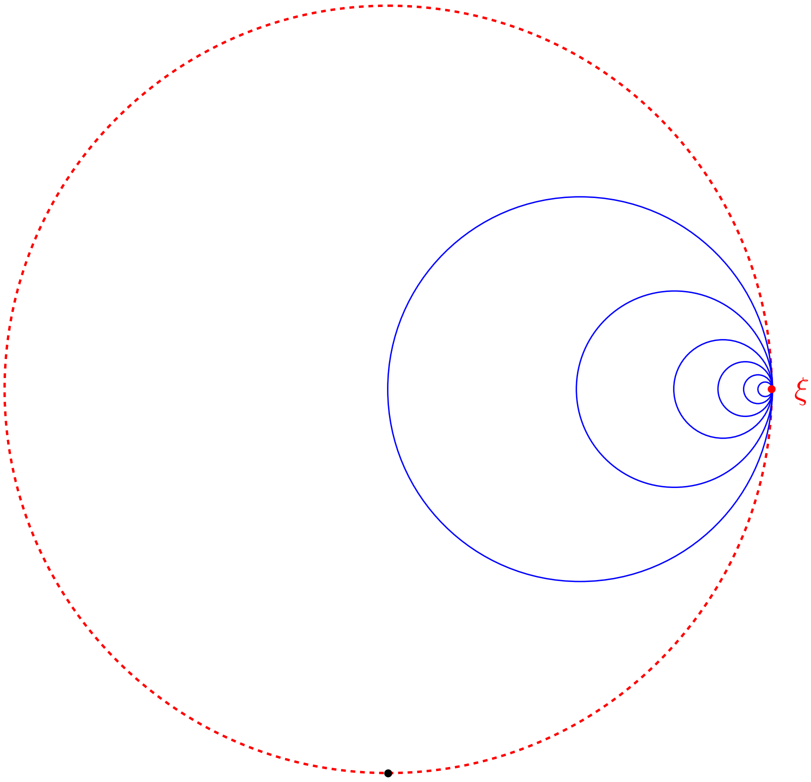


Figure 10.4: Horocycles in the Poincaré disk.

We leave as an exercise (rather, several exercises) to the curious reader to describe horospheres in the other models of hyperbolic space. In Chapter 4, there was an exercise that claims to describe horocycles on the hyperboloid when $n = 2$: see Exercise 4.4. Exercise 10.6 proposes to prove an analogous result in any dimension. As for the Klein models, a characterization is suggested in Exercise 10.7.

Let us conclude this chapter with the following important property of horospheres:

Theorem 10.40. Any horosphere $S \subseteq \mathbb{H}^n$ is a Euclidean space. In other words, any horosphere $S \subseteq \mathbb{H}^n$ is a complete simply-connected hypersurface with vanishing curvature. Equivalently, there exists an isometry $S \xrightarrow{\sim} \mathbb{R}^n$.

Proof. Since all models of $\mathbb{H}^n$ are isometric, it is enough to do the proof in the Poincaré half-space model. Moreover, since horospheres at some ideal point ξ_0 are mapped isometrically to horospheres at all other ideal points (see proof of Theorem 10.38), it is enough to consider horospheres at ξ_0.

Let us $\xi_0 = \infty$. We have seen that horospheres at ξ_0 are horizontal hyperplanes contained in H^n. Consider such a horosphere $S = \{x \in \mathbb{H}^n \mid x_n = c\}$ (where $c > 0$ is a constant). Recall

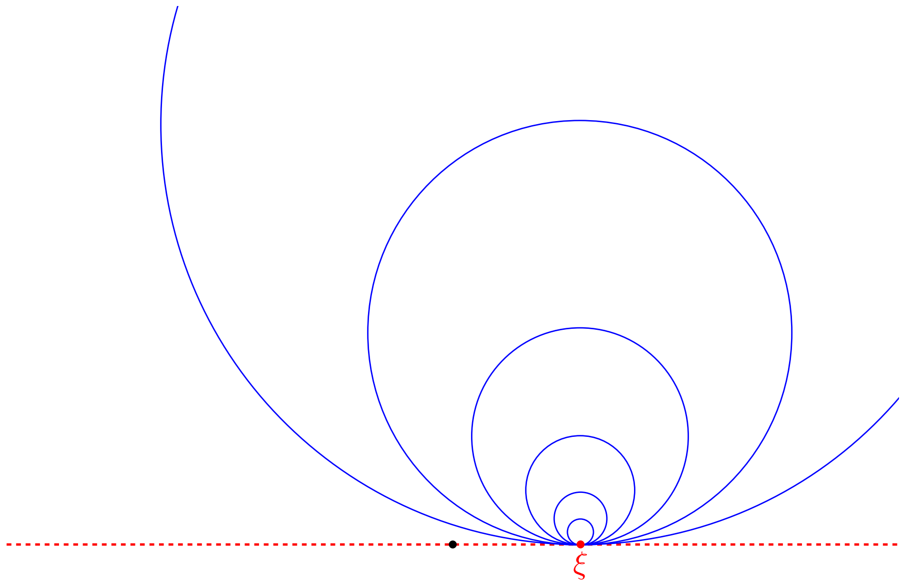


Figure 10.5: Horocycles in the Poincaré half-plane.

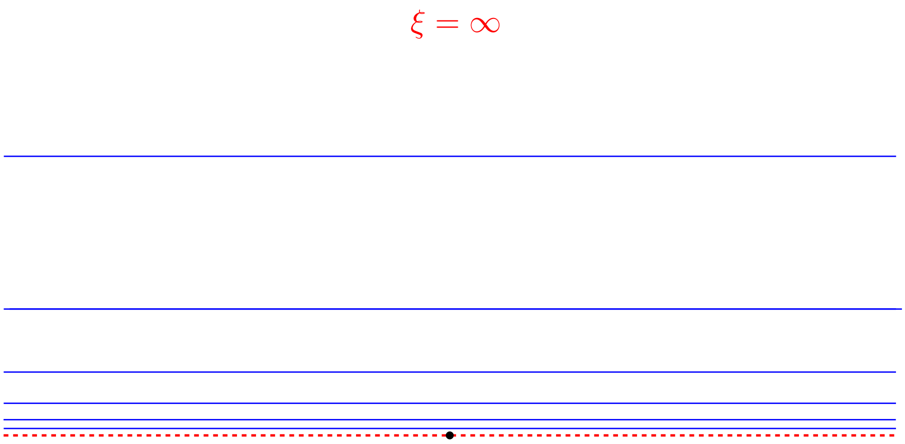


Figure 10.6: Horocycles centered at $\xi = \infty$ in the Poincaré half-plane.

that the hyperbolic metric in H^n is:

$$g_{H^n} = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2 + \mathrm{d}x_n^2}{x_n^2} .$$

Clearly $(x_1, \ldots, x_{n-1})$ offer a global system of coordinates on S, and the induced metric on S is simply:

$$g_S = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n-1}^2}{c^2} .$$

Up to the constant scaling factor $\frac{1}{c^2}$, this is the standard Euclidean metric g_0 on $\mathbb{R}^{n-1}$. Regardless, this is a complete Euclidean metric (in fact, g_S is isometric to g_0 via $x \mapsto x/c$). $\square$

10.5 Exercises

Exercise 10.1. (*) Quasi-isometric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f \colon X \to Y$ is called a ***quasi-isometry*** if:

(i) f is coarsely Lipschitz: there exists $A \geqslant 1, B \geqslant 0$ such that for all $x_1, x_2 \in X$:

$$\frac{1}{A} d_X(x_1, x_2) - B \leqslant d_Y(f(x_1), f(x_2)) \leqslant A d_X(x_1, x_2) + B \,.$$

(ii) f is coarsely surjective: there exists $C \geqslant 0$ such that for all $y \in Y$, there exists $x \in X$ such that $d(f(x), y) \leqslant C$.

When there exists a quasi-isometry $f \colon X \to Y$, one says that the metric spaces X and Y are ***quasi-isometric***.

(1) Show that any metric space of finite diameter is quasi-isometric to a point.

(2) Show that $\mathbb{R}^2$ and $\mathbb{H}^2$ are not quasi-isometric.

(3) Show that any quasi-isometry $f \colon \mathbb{H}^m \to \mathbb{H}^n$ extends to a homeomorphism $\partial_\infty \mathbb{H}^m \to \partial_\infty \mathbb{H}^n$. Conclude that $\mathbb{H}^m$ is quasi-isometric to $\mathbb{H}^n$ if and only if $m = n$.

Exercise 10.2. Ideal boundary of the hyperboloid and the Cayley–Klein models

We identified both the ideal boundary of the hyperboloid model $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$ and the ideal boundary of the Cayley–Klein model $\Omega^- \subseteq \mathbf{P}(\mathbb{R}^{n,1})$ as the projectivized light cone of $\mathbb{R}^{n,1}$. Can you explain this “coincidence”?

Exercise 10.3. Busemann function in the Poincaré disk

Let $X = (B^2, g_{B^2})$ be the Poincaré disk. We use the complex coordinate z on the unit disk $\mathbb{D} \approx B^2$.

(1) For any $\xi \in \partial_\infty X = \{z \in \mathbb{C} \mid |z| = 1\}$, check that the geodesic ray $r_\xi \colon [0, +\infty) \to X$ such that $r(0) = 0$ and $r(+\infty) = \xi$ has the expression: $r(t) = \tanh(t/2)\, \xi$.

(2) Show that the Busemann function B_r is given by

$$B_r(z) = -\ln\left(\frac{1 - |z|^2}{|z - \xi|^2}\right) \,.$$

(3) Recover the fact that horocycles centered at ξ are Euclidean circles tangent to $\partial_\infty X$ at ξ.

Exercise 10.4. Horospheres as limit of spheres

Let $x_0 \in \mathbb{H}^n$ and let $P \subseteq \mathrm{T}_{x_0} \mathbb{H}^n$ be a hyperplane.

(1) Show that for all $r > 0$, there exists exactly two hyperspheres $S_1(r)$ and $S_2(r)$ in $\mathbb{H}^n$ that go through x_0 and are tangent to P.

(2) Show that there exists exactly two horospheres S_1 and S_2 in $\mathbb{H}^n$ that go through x_0 and are tangent to P.

(3) Show that $\{\lim_{r\to+\infty} S_1(r), \lim_{r\to+\infty} S_2(r)\} = \{S_1, S_2\}$.

Exercise 10.5. Horospheres as hypersurfaces with asymptotic normal geodesics

(1) Let S be a horosphere centered at $\xi \in \partial_\infty \mathbb{H}^n$. Show that for any $x_0 \in S$, the geodesic going through x and with ideal endpoint ξ intersects S orthogonally. Show that it is also orthogonally transverse to any other horosphere centered at ξ.

(2) Show that a complete hypersurface $S \subseteq \mathbb{H}^n$ is a horosphere if and only if all geodesics that intersect S orthogonally share an ideal endpoint.

Exercise 10.6. Horospheres in the hyperboloid model

Show that in the hyperboloid model $\mathcal{H}^+ \subseteq \mathbb{R}^{n,1}$, horospheres are given by the intersection of $\mathcal{H}^+$ with hyperplanes of $\mathbb{R}^{n,1}$ whose normal lies in the light cone. Show that when $n = 2$, these are parabolas (also see Exercise 4.4).

Exercise 10.7. Horospheres in the Klein model

Show that in the Beltrami–Klein disk $B^2 \subseteq \mathbb{R}^2$, the horocycles centered at $\xi \in S^1$ are the Euclidean ellipses contained in B^2 that have a contact of order 4 with S^1 at ξ. Suggest and prove an analogous characterization in higher dimensions. Argue that this characterization also makes sense in the Cayley–Klein model.

Exercise 10.8. Isometries fixing an ideal point

Let $X = \mathbb{H}^n$ and $\xi \in \partial_\infty X$.

(1) Show that if $f \in \mathrm{Isom}(X)$ fixes ξ, then f maps any horosphere S centered at ξ to some other such horosphere S'. *Optional: in what case do we have $S' = S$?*

(2) Recall that any horosphere S is isometric to $\mathbb{R}^{n-1}$. Recall explicitly the isometric identification $S \approx \mathbb{R}^{n-1}$ when S is a horosphere centered at $\xi = \infty$ in the Poincaré half-space model. Show that f induces an affine similarity of $\mathbb{R}^{n-1}$.

(3) Recover the fact that the subgroup of the Möbius group of S^{n-1} fixing a point is isomorphic to the group of affine similarities of $\mathbb{R}^{n-1}$ (see Exercise 8.6).

CHAPTER 11

Isometries of hyperbolic space

In this chapter, we study the isometries of hyperbolic space and establish a classification thereof. We have already described the group of isometries in the different models (hyperboloid, Klein models, Poincaré models), and how it acts on each model; however we have yet to analyze the geometric behavior of isometries.

As an analogy, consider the group $\mathrm{E}^+(3) = \mathrm{Isom}^+(\mathbb{R}^3)$ of motions (orientation-preserving isometries) of Euclidean space. As a group, this is $\mathrm{E}^+(3) \approx \mathrm{SO}(3) \ltimes \mathbb{R}^3$, acting on $\mathbb{R}^3$ by affine transformations. But what do Euclidean isometries actually look like? As is well-known, they fall into distinct types: translations, rotations, and screw rotations (to include orientation-reversing isometries, there is also reflections, glide reflections, and rotation-reflections). This classification is easily generalized in any dimension.

The goal of this chapter is to present a similar classification of isometries of hyperbolic space $\mathbb{H}^n$. In order to do so, we will make a crucial use of the ideal boundary of hyperbolic space introduced in the previous chapter. Essentially, isometries can be classified according to their dynamics, which can be read off their extended action on the ideal boundary of hyperbolic space. Just like the notion of ideal boundary, this paradigm to classify isometries holds in a broad class of metric spaces. We attempt a presentation that is suggestive of this generality[1], but also discuss the specific features of the case of hyperbolic space.

After studying the isometries of hyperbolic space in arbitrary dimensions, we specialize to the 2- and 3-dimensional cases. We shall see that in the Poincaré half-space model, orientation-preserving isometries can be concretely described and characterized using matrices in $\mathrm{SL}(2,\mathbb{R})$ (in the 2-dimensional case) or $\mathrm{SL}(2,\mathbb{C})$ (in the 3-dimensional case).

11.1 Classification

Let (X, d) be a metric space and let $f \colon X \to X$ be an isometry. By definition, the ***displacement function of*** f is $d_f(x) := d(x, f(x))$, and the ***translation length*** of f is $l_f := \inf_{x \in X} d_f(x)$.

[1] To learn more about the classification of isometries in metric spaces of nonpositive curvature, I recommend [BH99] (for CAT(0) spaces) and [GH90] (for Gromov hyperbolic spaces).

Definition 11.1. An isometry $f: X \to X$ is called:

- ***elliptic*** if $l_f = 0$ is attained, i.e. f has a fixed point.
- ***hyperbolic*** (or ***loxodromic***) if $l_f > 0$ and is attained.
- ***parabolic*** if l_f is not attained.

Remark 11.2. A quick note about the terminology: for isometries of the second type, we will favor the term *hyperbolic* when X is a generic metric space, and *loxodromic* when $X = \mathbb{H}^n$ is hyperbolic space. There are two reasons to avoid using "hyperbolic" when $X = \mathbb{H}^n$: 1. Any isometry of $\mathbb{H}^n$ could reasonably be called a "hyperbolic isometry", just like any isometry of $\mathbb{R}^n$ is called a Euclidean isometry, and 2. It is common in the math literature to call "hyperbolic isometry" a subclass of loxodromic isometries, although we find this a poor choice of terminology (we will use instead the term "translation", see Definition 11.15).

Example 11.3. In Euclidean space $\mathbb{R}^n$, every isometry is either hyperbolic (translations, screw rotations, glide reflections) or elliptic (rotations, reflections, rotation-reflections). An isometry that is either elliptic or hyperbolic is called ***semisimple***, hence every Euclidean isometry is semisimple (i.e. there are no parabolics).

The main goal of this section is to present a characterization of elliptic, hyperbolic, and parabolic isometries of hyperbolic space, which we condense in the following three theorems. We postpone the definition of all the new terms appearing in these theorems (orbit, limit set, attracting/repelling/neutral fixed points, translation axis) until after their statement.

Theorem 11.4. Let $X = \mathbb{H}^n$ and let $f: X \to X$ be an isometry. The following are equivalent:

(i) f is elliptic.
(ii) Some/every orbit of f is bounded.
(iii) f has 0, 2, or infinitely many fixed points on $\partial_\infty X$, all of which are neutral fixed points.

Theorem 11.5. Let $X = \mathbb{H}^n$ and let $f: X \to X$ be an isometry. The following are equivalent:

(i) f is hyperbolic.
(ii) f has a translation axis.
(iii) f has exactly two fixed points on $\xi^-, \xi^+ \in \partial_\infty X$, one attracting ($\xi^+$) and one repelling ($\xi^-$).

Theorem 11.6. Let $X = \mathbb{H}^n$ and let $f: X \to X$ be an isometry. The following are equivalent:

(i) f is parabolic.
(ii) f preserves some/any horosphere S centered at some point $\xi \in \partial_\infty X$, and has no fixed points in S.
(iii) f has exactly one fixed point $\xi \in \partial_\infty X$.

Before proving these theorems let us define all the terms involved.

Orbits of f. By definition, an orbit of f is a subset of X of the form $\{f^n(x_0), n \in \mathbb{Z}\}$ for some $x_0 \in X$. Here we denote f^n the n-th iterate of f under composition, and f^{-n} is the inverse of f^n.

Fixed points of f at infinity. We have seen (Theorem 10.23) that any isometry $f: X \to X$

extends to the ideal boundary $\partial_\infty X$, and we still denote f the extension to the boundary. Therefore it makes sense to talk about fixed points of f on $\partial_\infty X$.

Attracting and repelling fixed points. A fixed point $\xi \in \partial_\infty f$ is called ***attracting*** if there exists a neighborhood U of ξ in $\partial_\infty X$ such that, for any neighborhood V of ξ, we have $f^n(U) \subseteq V$ for n sufficiently large. The fixed point ξ is called ***repelling*** if ξ is an attracting fixed points of f^{-1}. The fixed point ξ is called ***neutral*** if it is neither attracting nor repelling [2].

Remark 11.7. Assume that f has two fixed points ξ^+, ξ^-, with ξ^+ attracting and ξ^- repelling. If in the definition of attracting [resp. repelling] fixed point one may take for U any neighborhood of ξ^+ that avoids a neighborhood of ξ^- (resp. any neighborhood of ξ^- that avoids a neighborhood of ξ^+), one says that f ***has North-South dynamics*** on $\partial_\infty X$. We will see that any hyperbolic isometry of $X = \mathbb{H}^n$ has North-South dynamics on $\partial_\infty X$.

Translation axis. A geodesic in X is called a ***translation axis*** for an isometry X if f preserves the geodesic but does not fix it pointwise. Concretely, if $\gamma\colon \mathbb{R} \to X$ is such a geodesic, then there exists a real number $l \neq 0$ such that $f(\gamma(t)) = \gamma(t + l)$ for all $t \in \mathbb{R}$. We shall see that $|l| = l_f$ must be the translation length of f, and that f admits a translation axis if and only if it is a hyperbolic isometry. Moreover, the two fixed points $\xi^-, \xi^+ \in \partial_\infty X$ are the endpoints of its translation axis; in particular, the translation axis is unique by Theorem 10.21.

In order to prove Theorem 11.4, we shall use the notion of minimal bounding ball:

Definition 11.8. Let $A \subseteq X$ be a bounded set. A ***bounding ball*** for A is a closed ball $B \subseteq X$ containing A, and a ***minimal bounding ball*** is a bounding ball of minimal radius.

Lemma 11.9. *For any nonempty bounded subset $A \subseteq X = \mathbb{H}^n$, there exists a unique minimal bounding ball.*

Proof. Consider the function $R\colon X \to [0, +\infty)$ defined by $R(x) := \sup_{y \in A} d(x, y)^2$. Clearly, a minimum bounding ball is a closed ball whose center minimizes R. It is easy to see that R is a proper function on X, therefore it admits minimizers: this proves the existence of a minimum bounding ball.

Let us now prove uniqueness by arguing that R is a strictly convex function on X. Clearly, $R(x) = \sup_{y \in \bar{A}} d(x, y)^2$ is an equivalent definition of R, where $\bar{A}$ indicates the closure of A. By compactness of $\bar{A}$ (because X is a proper metric space: see Proposition 10.1), the supremum is attained in the definition of R. For any fixed $y \in X$, the function $x \mapsto d(x, y)^2$ is strictly convex on X: this can be proved by direct computation in the hyperboloid model; it is an easier version of Corollary 10.7. Therefore R is strictly convex on X as a maximum of strictly convex functions. Conclude by uniqueness of the minimizer of any strictly convex function. □

[2] There is a better definition of attracting, repelling, and neutral fixed points of f: these are respectively fixed points ξ where $f'(\xi)$ is < 1, > 1, or $= 1$. However defining the metric derivative $f'(\xi)$ requires more work, especially since we have not defined any metric on $\partial_\infty X$. To learn more on this, we refer to [GH90] or [DSU17]. Our definition of attracting and repelling fixed points is weaker in general, but equivalent in the case $X = \mathbb{H}^n$.

And another useful couple of lemmas, regarding fixed points and ideal fixed points of isometries of $X = \mathbb{H}^n$:

Lemma 11.10. *Let $X = \mathbb{H}^n$ and let $f \colon X \to X$ be an isometry. Then f has at least one fixed point or ideal fixed point.*

Proof. Recall that $\bar{X}^\infty = X \cup \partial_\infty X$ is a topological closed n-ball, as is illustrated by the Poincaré ball model for instance. The celebrated Brouwer fixed point theorem precisely says that any continuous map from a closed n-ball to itself has at least one fixed point. □

Lemma 11.11. *Let $X = \mathbb{H}^n$ and let $f \colon X \to X$ be an isometry. The fixed point set $F \subseteq X$ of f is either empty, or reduced to a point, or is a hyperbolic subspace of X. In other words, F is a subset of X that is stable under taking the complete geodesic through any two of its points.*

Proof. Assume F has at least two points, otherwise the lemma is vacuously true. Let x, y be two distinct points in F, and let $\gamma \colon \mathbb{R} \to X$ be the geodesic such that $\gamma(0) = x$ and $\gamma(1) = y$. Since f is an isometry, the curve $f \circ \gamma$ is also a geodesic in X. Moreover, $f \circ \gamma(0) = f(x) = x$ and $f \circ \gamma(1) = f(y) = y$. By uniqueness of the geodesic through x and y, we must have $f \circ \gamma = \gamma$. In other words, $\gamma(t) \in F$ for all $t \in \mathbb{R}$. □

Lemma 11.12. *Let $X = \mathbb{H}^n$ and let $f \colon X \to X$ be an isometry.*

(i) If $\xi_1, \xi_2 \in \partial_\infty X$ are two distinct ideal fixed points, then the geodesic with endpoints ξ_1 and ξ_2 is preserved by f.

(ii) If γ is any unit geodesic preserved by f, then $f(\gamma(t)) = \gamma(t + l)$ for all $t \in \mathbb{R}$, where $|l|$ is the translation length of f.

Proof. Since f is an isometry, $f \circ \gamma$ is a geodesic, moreover it has the same endpoints as γ therefore we have $f \circ \gamma$ parametrizes the same geodesic as γ by Theorem 10.21. Since $f \circ \gamma$ is a reparametrization of γ with same speed and orientation, we have $f(\gamma(t)) = \gamma(t + l)$ for some $l \in \mathbb{R}$. We must now show that $|l|$ is the translation length of f. Define the projection P to the geodesic parametrized by γ by putting that for any $x \in X$, $P(x)$ is the unique minimizer of the function $t \mapsto d(x, \gamma(t))^2$. Since this function is strictly convex, the map P is well-defined. Moreover, P is distance nonincreasing, we leave the proof of this claim as an exercise (there are several possible approaches, one may for instance compute the second derivative of P along any geodesic). It is straightforward to argue that $\pi(f(x)) = f(\pi(x))$, therefore we obtain $d(x, f(x)) \leqslant d(\pi(x), \pi(f(x)) = d(\pi(x), f(\pi(x))) = |l|$. This proves that l is the translation distance of f. □

We are now ready to prove the characterizations of elliptic isometries, hyperbolic, and parabolic isometries.

Proof of Theorem 11.4. It is obvious that f is an elliptic isometry if and only if f has a fixed point in X. In particular, f has a bounded orbit, since any fixed point is an orbit. More generally, if x_0 is a fixed point, then $d(f^n(x), x_0) = d(x, x_0)$ for any $n \in \mathbb{Z}$ by immediate induction, therefore the orbit of any point $x \in X$ is bounded. Conversely, assume that the orbit S of some point $x \in X$ is bounded. One can consider the unique minimal bounded ball B for S (see Lemma 11.9). Since $f(S) = S$, we have that $f(B) = B$, which means that the center of B must be fixed by f[3].

For the second characterization, first assume that f is elliptic. By Lemma 11.11, the fixed set F is either empty, or reduced to a point, or is a hyperbolic subspace. It follows that the intersection of $\partial_\infty X$ with the closure of F in $X \cup \partial_\infty X$ is either empty (when F is empty or reduced to a point), or consists of two points (when F is a geodesic), or infinitely many points (when F is a hyperbolic subspace of dimension $\geqslant 2$). Moreover, it is straightforward to prove that such points are neutral fixed points of f. To conclude that (ii) implies (iii), we show that f has no other fixed points in $\partial_\infty X$. Let $\xi \in \partial_\infty X$ be a fixed point. Since f is elliptic, it has a fixed point $x \in X$. The geodesic ray from x to ξ must be fixed by f, therefore ξ is indeed in the closure of F.

Let us finally show that (iii) implies that f is elliptic. First note that by Lemma 11.10, if f has no ideal fixed points, then f must have a fixed point in X. Now assume that f has two or more neutral ideal fixed points. For any two such fixed points ξ_1 and ξ_2, the geodesic with endpoints ξ_1 and ξ_2 is preserved by f by Lemma 11.12. Using the notations of Lemma 11.12, if $l \neq 0$ then the geodesic is a translation axis of f (by definition). However we shall see in the proof of Theorem 11.5 that an isometry that has a translation axis is hyperbolic, and has no neutral ideal fixed points. In conclusion we must have $l = 0$, in other words the geodesic is fixed pointwise, therefore f is elliptic. □

Proof of Theorem 11.5. Assume that f is hyperbolic. Let x_0 be a point where $l = \min d_f$ is attained. We claim that the geodesic through x_0 and $f(x_0)$ is a translation axis for f. Indeed, let γ be the unit parametrization of this geodesic so that $\gamma(0) = x_0$ and $\gamma(l) = f(x_0)$. Consider the geodesic $f \circ \gamma$. By definition of the translation length, we have $d(\gamma(t), f(\gamma(t))) \leqslant l$ for all $t \in \mathbb{R}$. On the other hand, $d(\gamma(t), f(\gamma(t))) = l$ when $t = 0$ and $t = l$. Since the function $t \mapsto d(\gamma(t), f(\gamma(t)))$ is convex and has two distinct minimizers, it must be constant. This implies that $f \circ \gamma$ and γ are the same geodesic up to parametrization (see Corollary 10.5), in fact we must have $f(\gamma(t)) = \gamma(t + l)$ for all $t \in \mathbb{R}$. This proves that γ is a translation axis for f. Note that such a translation axis is unique: if γ_2 is another translation axis, parametrized by unit speed so that $f(\gamma_2(t)) = \gamma_2(t + l)$ for all $t \in \mathbb{R}$ (same translation parameter by Lemma 11.12), then one argues similarly as before that $d(\gamma(t), \gamma_2(t))$ is bounded, and that γ and γ_2 parametrize the same geodesic. Conversely, if f admits a translation axis, then f is a hyperbolic isometry by Lemma 11.12.

Now let us prove that when f is hyperbolic, the induced map (still denoted f) on the ideal boundary $\partial_\infty X$ has North-South dynamics. We know that $\partial_\infty X$ is a compact Hausdorff topological space, and that f has exactly two fixed points $\xi^-, \xi^+ \in \partial_\infty X$ (the endpoints of

[3] The idea of this proof goes back to Cartan [Car88], who used the center of mass in place of the center of the minimal bounding sphere to show the existence of a fixed point for the action of any compact group of isometries of a complete and simply connected manifold of nonpositive sectional curvature. This is known as the Cartan fixed point theorem.

its axis). In such a situation, it is enough to show that for any $\xi \in \partial_\infty X - \{\xi^-\}$, the point ξ^- is not an accumulation point of the sequence $(f^n(\xi))_{n\in\mathbb{N}}$: this is an exercise of general topology that we leave to the diligent reader. By contradiction, assume that there exists $\xi \neq \xi-$ and a sequence of integers $n_k \longrightarrow +\infty$ such that $\lim_{k\to+\infty} f^{n_k}(\xi) = \xi^-$. Let r be the geodesic ray from x_0 to ξ, where x_0 is some point on the axis of f, and denote $y_0 = r(l_f)$. Clearly, $f^n(\xi)$ is the endpoint of the geodesic ray from $f^n(x_0)$ that goes through $f^n(y_0)$ at time $t = l_f$. On the other hand, ξ^- is the endpoint of the geodesic ray from $f^n(x_0)$ that goes through $f^{n-1}(x_0)$ at time $t = l_f$. The topology on $\partial_\infty X$ implies that if $\lim_{k\to+\infty} f^{n_k}(\xi) = \xi^-$, then $d(f^{n_k}(y_0), f^{n_k-1}(x_0)) \longrightarrow 0$. However this distance is constant equal to $d(y_0, f^{-1}(x_0))$, hence the contradiction.

Finally, let us show that if f has two ideal fixed points ξ^-, ξ^+ on the boundary, which are not neutral, then f has a translation axis. By Lemma 11.12, the geodesic with endpoints ξ^- and ξ^+ is preserved by f, and either entirely consists of fixed points, in which case f is elliptic, or is a translation axis for f. In the first case, we have seen that ξ^- and ξ^+ are neutral fixed points, so it is excluded. □

Proof of Theorem 11.6. Let f be a parabolic isometry. Since f has no fixed points in X, f must have at least one ideal fixed point $\xi \in \partial_\infty X$. There can be no other ideal fixed point, for otherwise f would be elliptic or hyperbolic by Lemma 11.12. Conversely, if f has a unique ideal fixed point, then f must be parabolic because Theorem 11.4 and Theorem 11.5 rule out f being elliptic or hyperbolic.

By Proposition 10.37, f must send any horosphere S centered at ξ to another such horosphere S'; let us show that if $S' \neq S$ then f cannot be parabolic[4]. Let $x_0 \in S$ be a point that minimizes $d(x, f(x))$ for all $x \in S$. Such a minimizer exists: indeed, consider a minimizing sequence $(x_n)_{n\in\mathbb{N}}$. By compactness of $S \cup \{\xi\}$, one can extract a converging subsequence in $S \cup \{\xi\}$. The limit cannot be ξ, since $d(x_n, f(x_n)) \longrightarrow +\infty$ whenever $x_n \in S \longrightarrow \xi$, we leave this claim as an exercise. Let γ be the geodesic through x_0 with endpoint ξ. Call S_t is the horosphere centered at ξ going through $\gamma(t)$, so that $S_0 = S$ and $S_d = S'$ where $d = d(x_, f(x_0))$. Repeat the same procedure as before to find a minimizer $x_t \in S_t$ of $d(x, f(x))$ for all $x \in S_t$. Since $f^n(S_0) = S_{nd}$ for all $n \in \mathbb{Z}$, we may find a global minimum of $t \in \mathbb{R} \mapsto d(x_t, f(x_t))$ in the interval $[0, d]$. It is straightforward to conclude that this is a minimum of $d(x, f(x))$ over all $x \in X$. This proves that the translation distance of f is attained, so that f cannot be parabolic.

It remains to show that if an isometry f preserves some horosphere S and has no fixed points in S, then f is parabolic. First of all, it is clear that f fixes the center $\xi \in \partial_\infty X$ of S, since ξ is the only ideal point in the closure of S. Secondly, it is immediate from the definition of a horosphere that f must actually preserve any horosphere centered at ξ. If f was elliptic, it would have a fixed point $x_0 \in X$. The whole geodesic through x_0 and with endpoint ξ would then have to be pointwise fixed. This geodesic intersects each horosphere centered at ξ (once), therefore we would find a fixed point of f in S, contrary to the assumption. If f

[4]Refer to [BH99, Chap. II.8, Prop. 8.25] for an alternative proof that uses the convexity of the displacement function.

was hyperbolic, by Theorem 11.5 it would have another endpoint $\xi' \in \partial_\infty X$, and the geodesic with endpoints ξ and ξ' would be its axis. Let γ be a unit parametrization of this geodesic, so that $f(\gamma(t)) = \gamma(t+l)$ for all $t \in \mathbb{R}$, where $|l|$ is the translation length of f. Such a geodesic intersects S at a unique point $x_0 = \gamma(t_0)$, therefore $f(x_0) = \gamma(t_0 + l)$ cannot belong to S, contrary to the assumption that f preserves S. □

11.2 Description

The classification established in the previous section is fundamental, but let us characterize in more detail the elliptic, loxodromic, and parabolic isometries of hyperbolic space $\mathbb{H}^n$. In the next section, we shall give even more explicit descriptions when $n = 2$ and $n = 3$.

For many purposes, it is good enough to understand isometries up to conjugation, in other words to classify conjugacy classes of isometries. Indeed, one can easily derive the properties of an isometry from that of a conjugate: for instance, if f is a loxodromic isometry with axis L, then gfg^{-1} is a loxodromic with axis $g(L)$ and same translation length, etc.

11.2.1 Elliptic isometries

Let f be an elliptic isometry of $\mathbb{H}^n$. We have seen that the set of fixed points F of f is a hyperbolic subspace of $\mathbb{H}^n$, in other words F is a copy (totally geodesic embedding) of $\mathbb{H}^k$ inside $\mathbb{H}^n$. Note that we allow $k = 0$ (F is reduced to a point) and $k = 1$ (F is a geodesic).

Let x_0 be any point in F. The fact that $\mathbb{H}^n$ is uniquely geodesic implies that f is completely determined by its derivative $\mathrm{d}f_{x_0}$. Indeed, for any $x \in \mathbb{H}^n$, we have $f(x) = \gamma_{\mathrm{d}f_{x_0}(u)}(1)$, where $x = \gamma_u(1)$ (in other words f is conjugate to $\mathrm{d}f_{x_0}$ by the Riemannian exponential map $\exp_{x_0}$). The linear map $\mathrm{d}f_{x_0}$ is a linear isometry of the Euclidean vector space $\mathrm{T}_{x_0} M$, and its +1-eigenspace (a.k.a fixed point set) is the tangent subspace to F. Thus the "interesting" part of the action of f resides in the behavior of $\mathrm{d}f$ in the orthogonal complement. Let us record these simple observations:

Theorem 11.13. Any elliptic isometry of $\mathbb{H}^n$ is uniquely determined by:

(1) Its set of fixed points $F \subseteq \mathbb{H}^n$, which is a hyperbolic subspace.
(2) For some $x_0 \in F$, a Euclidean isometry of $\mathrm{T}_{x_0} \mathbb{H}^n$, whose +1-eigenspace is $\mathrm{T}_{x_0} F$.

In this description, the splitting of $\mathrm{d}_{x_0}\mathbb{H}^n$ as $V \oplus V^\perp$ corresponds to two orthogonally transverse hyperbolic subspaces of $\mathbb{H}^n$ through x_0, the first (F) being fixed by f, and the second being preserved by f with x_0 as the unique fixed point.

Alternatively to this infinitesimal approach, one can realize that f is adequately described by its set of fixed points F and by a Euclidean isometry by looking at its action on horospheres orthogonally transverse to F. Indeed, it is immediate that any such horosphere S must be preserved by f (why?). Moreover, we recall the important fact that the hyperbolic metric restricts to a Euclidean metric on any horosphere (Theorem 10.40). Therefore f acts by Euclidean isometries on S.

Example 11.14. Consider an elliptic isometry $f \in \mathrm{Isom}^+(\mathbb{H}^3)$ in the Poincaré upper half-space model H^3, whose set of fixed points is the geodesic F with endpoints 0 and ∞. Then f preserves each horosphere centered at ∞, which is a horizontal plane in H^3, and is orthogonal to F. Per the above discussion, f acts in such a plane by Euclidean rotations. In the coordinates $(z = x_1 + ix_2, x_3) \in H^3$, the map f is written $f(z, x_3) = (e^{i\theta}z, x_3)$ for some real number θ. Note that horospheres centered at 0 are also orthogonal to F and preserved by f, as expected. See Figure 11.1 and Figure 11.2.

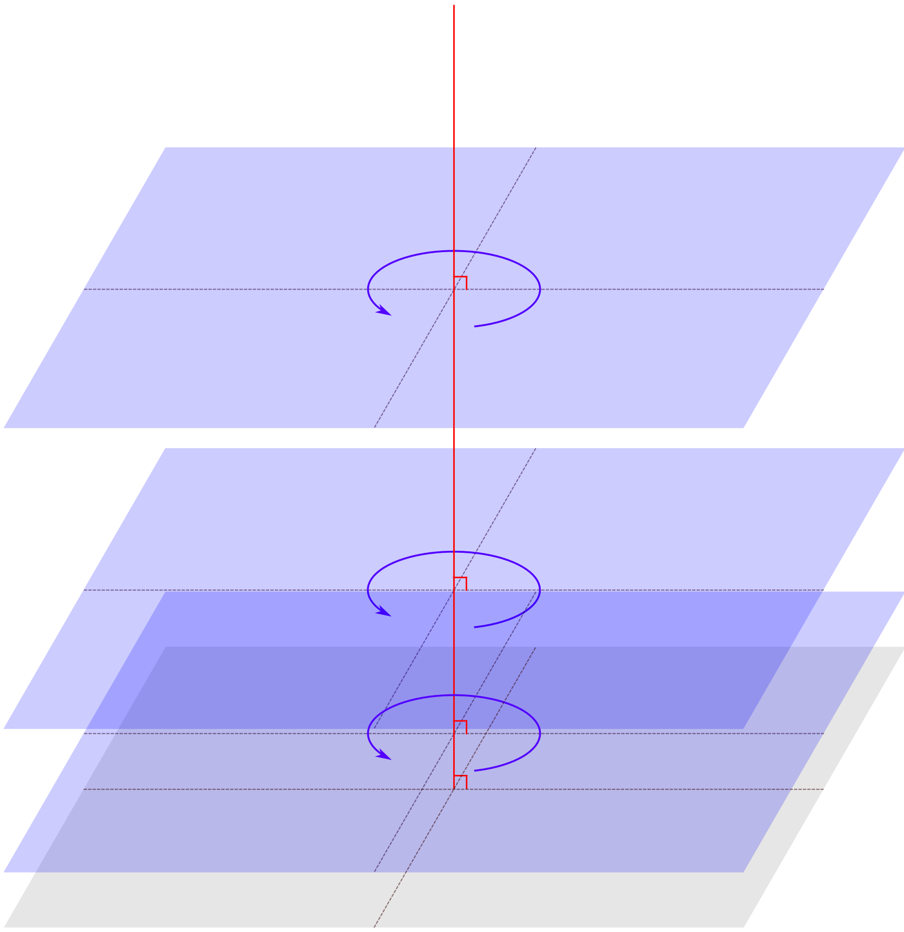


Figure 11.1: An elliptic isometry of the Poincaré half-space and its action by Euclidean rotations on horospheres centered at $\xi = \infty$.

11.2.2 Loxodromic isometries

Let us now turn to loxodromic isometries of hyperbolic space. We called such isometries *hyperbolic* in a general metric space X (see Definition 11.1), but the term *loxodromic* should be preferred when $X = \mathbb{H}^n$.

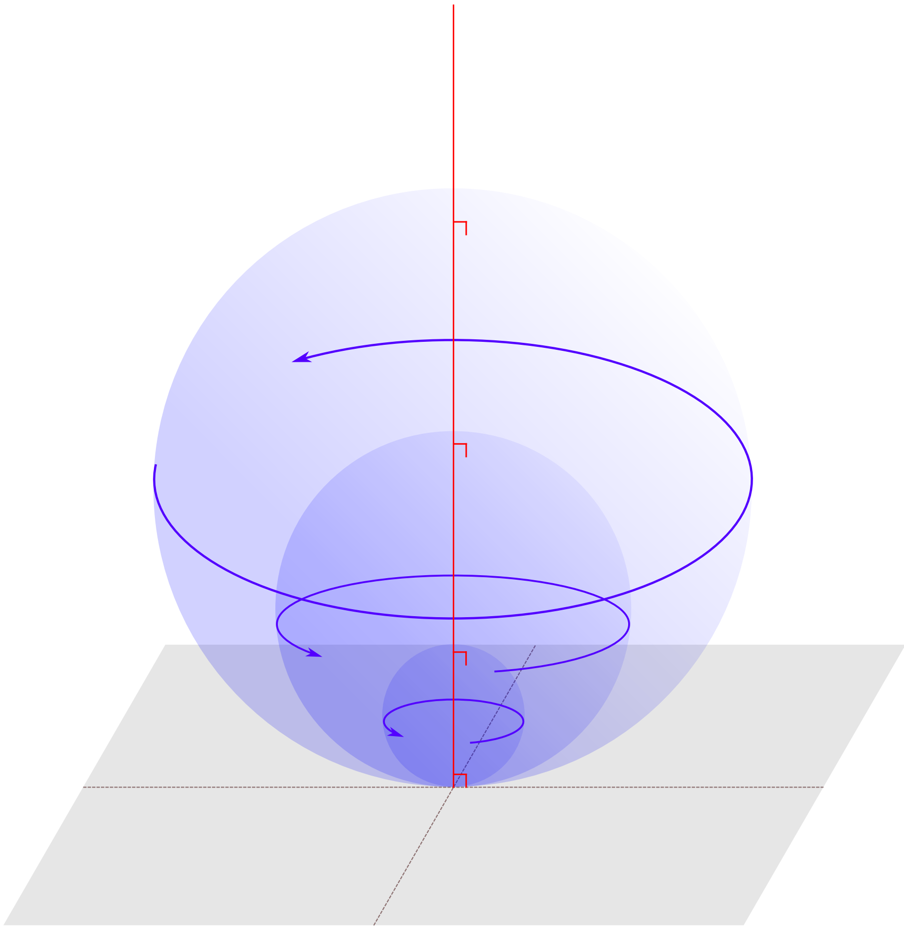


Figure 11.2: The same elliptic isometry as in Figure 11.1, acting on horospheres centered at $\xi = 0$.

Translations

Translations are the "nicest" loxodromic isometries.

Definition 11.15. A loxodromic isometry $f\colon \mathbb{H}^n \to \mathbb{H}^n$ is called a ***translation*** if it preserves some/any equidistant curve from its axis.

The fact that "some" and "any" are equivalent in the definition above will be apparent in the proof of Proposition 11.18.

Remark 11.16. It is quite common in the literature to use the term *hyperbolic isometry* instead of *translation*. Such authors will also typically exclude translations from loxodromic isometries. I recommend not using this terminology (see Remark 11.2), or at least saying "purely hyperbolic" for translations and "purely loxodromic" for other loxodromic elements, to avoid any confusion.

Example 11.17. For any $\lambda > 0$, the map $z \mapsto \lambda z$ defines a translation in the Poincaré half-plane. In fact, the next characterization of translations shows that any translation is conjugate to a map of this form.

Proposition 11.18. *Let $f: \mathbb{H}^n \to \mathbb{H}^n$. The following are equivalent:*

(i) f is a loxodromic isometry and preserves some/any equidistant curve from its axis.

(ii) f is conjugate to the transformation of the Poincaré half-space given by $x \in H^n \mapsto e^l x$, where l is the translation length of f.

Proof. This is the content of Exercise 11.4. □

A translation is uniquely determined by its axis and translation length:

Theorem 11.19. For any oriented geodesic line $L \subseteq \mathbb{H}^n$ and $l > 0$, there exists a unique translation with axis L and translation length l.

Proof. Denote by f_0 the transformation of the Poincaré half-space given by $x \in H^n \mapsto e^l x$ (as in Proposition 11.18). This is a translation with axis L_0, the geodesic line with endpoints 0 and ∞, and with translation length l. Let $\varphi: \mathbb{H}^n \to H^n$ be any isometry that sends the endpoints of L to L_0, preserving orientation (why does this exist?). Then $\varphi f \varphi^{-1}$ is a translation with axis L and translation length l . This shows existence.

For uniqueness, assume that f_1 and f_2 are two translations with same axis L and translation length l. $g := f_2 \circ f_1^{-1}$ fixes L pointwise, so that g is an elliptic transformation whose set of fixed points contains L. In particular, g preserves the horospheres centered at ∞, which are the horizontal hyperplanes in $H^n \subseteq \mathbb{R}^n$. On the other hand, g must preserve the equidistant lines from L_0, which are the Euclidean straight half-lines starting from 0. Since any such half-line intersects any aforementioned horosphere exactly once, □

General loxodromic transformations

A general loxodromic transformation is determined by the data of an axis, a translation length, and a Euclidean isometry. More precisely:

Theorem 11.20. Let $f: H^n \to H^n$. The following are equivalent:

(i) f is a loxodromic isometry with axis L and translation length l.

(ii) $f = t \circ r$ where t is the translation with axis L and translation length l, and r is an elliptic isometry whose set of fixed points contains L.

Remark 11.21. The decomposition $f = t \circ r$ in Theorem 11.20 is unique, since t is uniquely determined by L and l (Theorem 11.19).

Proof. Let f be a loxodromic isometry with axis L and translation length l and let t be the unique translation with axis L and length t. It is immediate that $r := f \circ t^{-1}$ is an isometry that fixes L pointwise, therefore r is an elliptic isometry.

Conversely, assume $f = r \circ t$ where r and t are as before. Clearly, f is translates by l in restriction to L. By Theorem 11.5, since f has an axis, it is a loxodromic isometry. More precisely, f is a loxodromic isometry with translation length l by Lemma 11.12. □

We shall see examples of loxodromic isometries in § 11.3 and § 11.4, e.g. Figure 11.6.

11.2.3 Parabolic isometries

A parabolic isometry is determined by the choice of an ideal fixed point and a Euclidean isometry without fixed points. More precisely:

Theorem 11.22. Any parabolic isometry with ideal fixed point $\xi \in \partial_\infty \mathbb{H}^n$ acts as a Euclidean isometry in any horosphere with center ξ. Conversely, given a Euclidean isometry f_0 in some horosphere S_0 centered at ξ, without any fixed points, there exist a unique parabolic isometry whose restriction to S_0 coincides with f_0.

Proof. Let f be a parabolic isometry with fixed point $\xi \in \partial_\infty X$. By Theorem 11.6, f preserves any horosphere with center ξ. Recall that any horosphere with its induced metric is isometric to Euclidean space (Theorem 10.40). It follows that f must act as a Euclidean isometry in any horosphere with center ξ.

Conversely, let us show that any Euclidean isometry f_0 of some horosphere S_0 centered at ξ uniquely extends as a parabolic isometry. For any $x \in \mathbb{H}^n$ and $t \in \mathbb{R}$, let $\varphi_t(x)$ denote the point through which the unit geodesic starting from x and with endpoint ξ goes at time t. Such a geodesic is orthogonally transverse to all horospheres centered at ξ (see Exercise 10.5). Moreover, for any horosphere S centered at ξ, there exists a unique $t \in \mathbb{R}$ such that $\varphi_t(S_0) = S$ (t is the signed distance between S_0 and S). One can show that any parabolic isometry f with fixed point ξ commutes with φ_t for any $t \in \mathbb{R}$, let us leave this claim as an exercise. It easily follows that f is uniquely determined by its restriction to any horosphere S centered at ξ. □

11.3 Isometries of $\mathbb{H}^2$

We shall now describe isometries even more concretely in dimensions 2 and 3. For simplicity, we shall only consider orientation-preserving isometries. We recall that in the Poincaré models of hyperbolic space, isometries can be described as Möbius transformations; moreover in dimensions 2 and 3 these are identified to fractional linear transformations.

11.3.1 Isometries of the Poincaré half-plane

Let us favor the Poincaré half-plane model $\mathbb{H} \subseteq \mathbb{C}$. The group of orientation-preserving isometries of $\mathbb{H}^2$ is identified to $\mathrm{PSL}(2, \mathbb{R})$, acting on $\mathbb{H}$ by fractional linear transformations.

Let us briefly recall how this works: any matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

induces an isometry of $\mathbb{H}$ given by

$$f_M : z \mapsto \frac{az + b}{cz + d} .$$

The assignment $M \to f_M$ is a group homomorphism from $\mathrm{SL}(2, \mathbb{R})$ to $\mathrm{Isom}(\mathbb{H})$, whose image is $\mathrm{Isom}^+(\mathbb{H})$ and whose kernel is $\{-I_2, I_2\}$, so that it induces an isomorphism $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I_2\} \xrightarrow{\sim} \mathrm{Isom}^+(\mathbb{H})$.

As a consequence of this discussion, the trace of an orientation-preserving isometry of $\mathbb{H}$ is well-defined *up to sign.*

11.3.2 Elliptic isometries

Theorem 11.23. Let $f : \mathbb{H} \to \mathbb{H}$ be an orientation-preserving isometry, represented by $M \in \mathrm{SL}(2, \mathbb{R})$. Assume $f \neq \mathrm{id}$. The following are equivalent:

(i) f is an elliptic isometry.

(ii) f has a unique fixed point in $\mathbb{H}$.

(iii) $\mathrm{tr}\, M \in (-2, 2)$.

(iv) M is conjugate in $\mathrm{SL}(2, \mathbb{R})$ to $\pm R_\theta$ for some $\theta \in \mathbb{R} - 2\pi\mathbb{Z}$, where $R_\theta = \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$.

(v) f is conjugate in $\mathrm{Isom}^+(\mathbb{H})$ to f_θ for some $\theta \in \mathbb{R} - 2\pi\mathbb{Z}$, where $f_\theta(z) = \frac{(\cos(\frac{\theta}{2}))z + \sin(\frac{\theta}{2})}{-(\sin(\frac{\theta}{2}))z + \cos(\frac{\theta}{2})}$.

Before proving this theorem, let us establish an elementary yet useful lemma.

Lemma 11.24. *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$ *and denote* $f : z \mapsto \frac{az+b}{cz+d}$ *the associated fractional linear transformation of* $\hat{\mathbb{C}}$.

- *If* $(\mathrm{tr}\, M)^2 > 4$, *then* f *has two fixed points, both of which lie in* $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$.
- *If* $(\mathrm{tr}\, M)^2 < 4$, *then* f *has two fixed points, one in* $\mathbb{H}$ *and the other is its complex conjugate.*
- *If* $(\mathrm{tr}\, M)^2 = 4$, *then either* f *is the identity, or* f *has a unique fixed point, which lies in* $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$.

Proof. This is a nice exercise: see Exercise 11.5. □

Proof of Theorem 11.23. The fact that (i) ⇔ (ii) ⇔ (iii) immediately follows from Lemma 11.24. The implication (iv) ⇒ (iii) is trivial, as is (iv) ⇔ (v).

Finally, let us prove (ii) ⇒ (v). Assume f has a unique fixed point $z_0 \in \mathbb{H}$. Since $\mathrm{Isom}^+(\mathbb{H})$ acts transitively on $\mathbb{H}$, there exists $g \in \mathrm{Isom}^+(\mathbb{H})$ such that $g(z_0) = i$. Then $f_1 := gfg^{-1}$ is a fractional linear transformation of $\mathbb{H}$ that fixes i. It is elementary to check by direct computation that $z \mapsto \frac{a_1 z + b_1}{c_1 z + d_1}$ fixes i if and only if $d_1 = a_1$ and $b_1 = -c_1$. Since $a_1 d_1 - b_1 c_1 = 1 = a_1^2 + c_1^2$, there exists $\theta \in \mathbb{R}$ such that $a_1 = \cos(\frac{\theta}{2})$ and $c_1 = -\sin(\frac{\theta}{2})$. We conclude that $f_1 = f_\theta$. ◻

Remark 11.25. We can alternatively write a proof of (iii) $\Rightarrow$ (iv) using only linear algebra. The characteristic polynomial of $M \in \mathrm{SL}(2,\mathbb{R})$ is $\chi_M(\lambda) = \lambda^2 - (\operatorname{tr} M)\lambda + 1$, with discriminant $\Delta = (\operatorname{tr} M)^2 - 4$. If $\operatorname{tr} M \in (-2,2)$, then χ_M has two non-real complex conjugate roots, and since their product is 1 they must be $\lambda = e^{\pm i\frac{\theta}{2}}$ for some $\theta \in \mathbb{R} - 2\pi\mathbb{Z}$. It follows that M has two distinct eigenvalues $\lambda = e^{\pm i\frac{\theta}{2}}$, therefore M is conjugate in $\mathrm{SL}(2,\mathbb{C})$ to $D_\theta = \operatorname{diag}(e^{i\frac{\theta}{2}}, e^{-i\frac{\theta}{2}})$. On the other hand, the matrix R_θ is also conjugate to D_θ in $\mathrm{SL}(2,\mathbb{C})$. We therefore find that M is conjugate to R_θ in $\mathrm{SL}(2,\mathbb{C})$. Conclude with the standard–albeit non-trivial–fact of linear algebra that two matrices in $\mathrm{SL}(2,\mathbb{R})$ are conjugate in $\mathrm{SL}(2,\mathbb{C})$ if and only if they are conjugate in $\mathrm{SL}(2,\mathbb{R})$.

A representation of the "standard" elliptic isometry f_θ is shown in Figure 11.3.

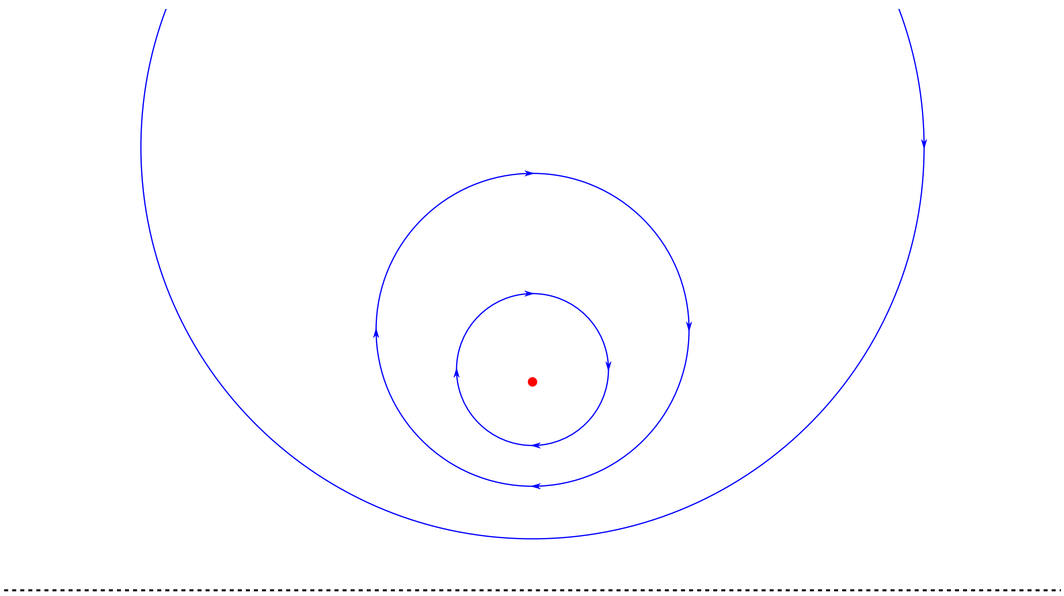


Figure 11.3: An elliptic isometry of the Poincaré half-plane.

Corollary 11.26. *The conjugacy class of an elliptic element of* $\mathrm{Isom}^+(\mathbb{H})$ *is uniquely determined by its trace (which is a real number defined up to sign).*

11.3.3 Loxodromic isometries

Theorem 11.27. Let $f\colon \mathbb{H} \longrightarrow \mathbb{H}$ be an orientation-preserving isometry, represented by $M \in \mathrm{SL}(2,\mathbb{R})$. The following are equivalent:

(i) f is a loxodromic isometry.
(ii) f has no fixed points in $\mathbb{H}$, and two distinct fixed points in $\partial_\infty \mathbb{H} = \hat{\mathbb{R}}$.
(iii) $\operatorname{tr} M \in \mathbb{R} - [-2, 2]$.
(iv) M is conjugate in $\mathrm{SL}(2, \mathbb{R})$ to $\pm T_l$ for some $l \in \mathbb{R} - \{0\}$, where $T_l = \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}$.
(v) f is conjugate in $\mathrm{Isom}^+(\mathbb{H})$ to f_l for some $l \in \mathbb{R} - \{0\}$, where $f_l(z) = e^l z$.
(vi) f is a translation.

The absolute value of the number l in (iv) and (v) is the translation length of f.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is general: see Theorem 11.5. Lemma 11.24 shows that (ii) $\Leftrightarrow$ (iii). The equivalence (iii) $\Leftrightarrow$ (iv) is elementary linear algebra (it is an easier version of Remark 11.25, since M is diagonalizable over $\mathbb{R}$). The equivalence (iv) $\Leftrightarrow$ (v) is immediate. To prove that (v) implies (vi), it suffices to check that f_l is a translation, since the conjugate of any translation is a translation. The fact that f_l is a translation is a special case of Proposition 11.18. Finally, (vi) $\Rightarrow$ (i) is trivial. ◻

Remark 11.28. We emphasize that there are no "purely loxodromic" isometries of $\mathbb{H}^2$: this is just a way to rephrase (i) $\Leftrightarrow$ (vi).

A representation of the "standard" translation f_l is shown in Figure 11.4.

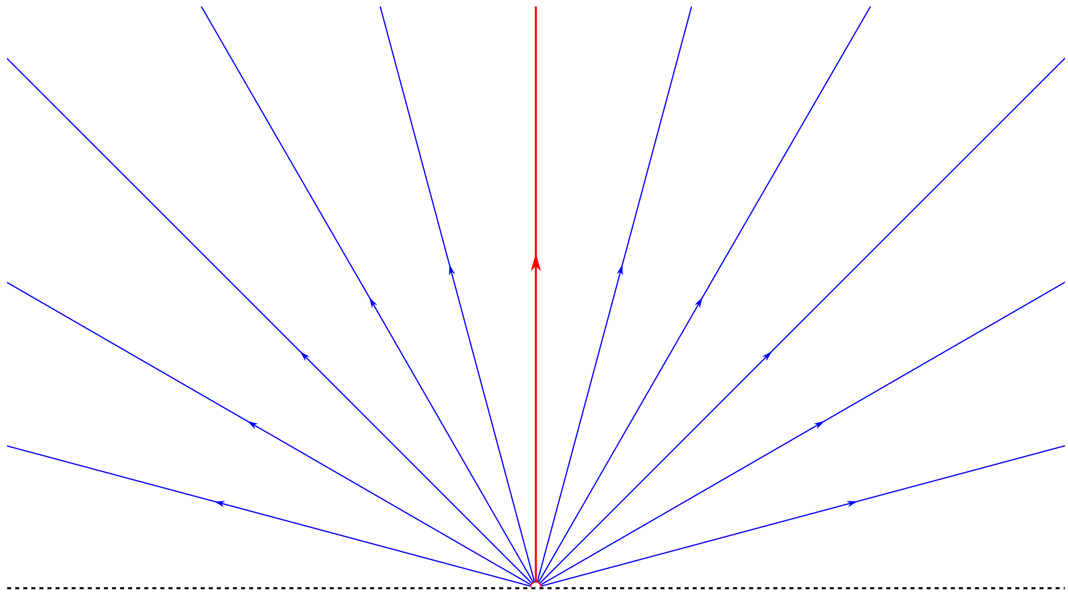


Figure 11.4: A translation of the Poincaré half-plane.

Corollary 11.29. *The conjugacy class of a loxodromic element of* $\mathrm{Isom}^+(\mathbb{H})$ *is uniquely determined by its trace (which is a real number defined up to sign).*

11.3.4 Parabolic isometries

Theorem 11.30. Let $f\colon \mathbb{H} \to \mathbb{H}$ be an orientation-preserving isometry, represented by $M \in \mathrm{SL}(2,\mathbb{R})$. Assume $f \neq \mathrm{id}$. The following are equivalent:

(i) f is a parabolic isometry.

(ii) f has no fixed points in $\mathbb{H}$, and one fixed point in $\partial_\infty\mathbb{H} = \hat{\mathbb{R}}$.

(iii) $\operatorname{tr} M = \pm 2$.

(iv) M is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to $\pm P$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(v) f is conjugate in $\mathrm{Isom}^+(\mathbb{H})$ to $z \mapsto z + 1$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is general: see Theorem 11.6. Lemma 11.24 shows that (ii) $\Leftrightarrow$ (iii). The fact that (iii) implies (iv) is elementary linear algebra: one quickly shows that M has $+1$ as a repeated eigenvalue (or -1, but in that case consider $-M$), moreover M cannot be diagonalizable otherwise we would have $M = I_2$ and $f = \mathrm{id}$, therefore the Jordan normal form of M must be P. The converse (iv) $\Rightarrow$ (iii) is trivial. Finally, the equivalence (iv) $\Leftrightarrow$ (v) is trivial. □

We emphasize that there is only one conjugacy class of parabolic isometries of $\mathbb{H}^2$:

Corollary 11.31. *Any parabolic element of* $\mathrm{Isom}^+(\mathbb{H})$ *has trace* ± 2, *and is conjugate to* $z \mapsto z+1$. *Conversely, any* $f \in \mathrm{Isom}^+(\mathbb{H})$ *of trace* ± 2 *is parabolic,* provided $f \neq \mathrm{id}$.

A representation of the "standard" parabolic isometry $z \mapsto z + 1$ is shown in Figure 11.5.

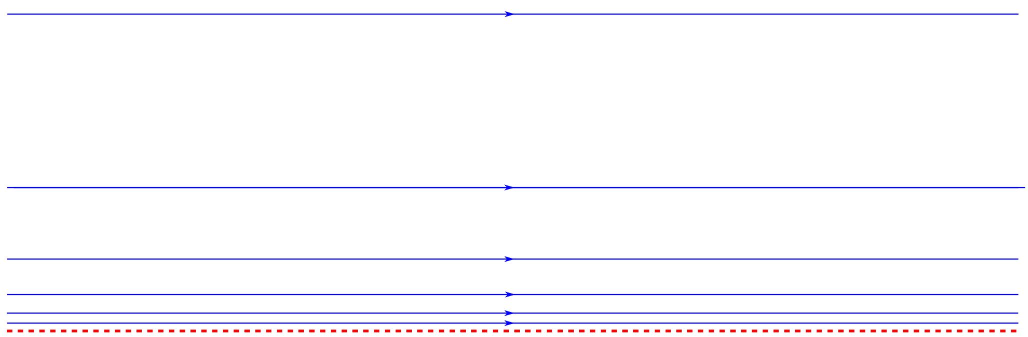


Figure 11.5: A parabolic isometry of the Poincaré half-plane.

11.3.5 Conjugacy classes and trace

As a consequence of Corollary 11.26, Corollary 11.29, and Corollary 11.31, we obtain:

Theorem 11.32. The conjugacy class of an element of $f \in \mathrm{Isom}^+(\mathbb{H}) - \{\mathrm{id}\}$ is uniquely determined by its trace (which is a real number defined up to sign).

More precisely, summarizing previous results:

- If $\mathrm{tr}(f) = \pm 2\cos(\frac{\theta}{2}) \in [2, 2]$, then f is elliptic and conjugate to $z \mapsto \frac{(\cos(\frac{\theta}{2}))z+\sin(\frac{\theta}{2})}{-(\sin(\frac{\theta}{2}))z+\cos(\frac{\theta}{2})}$.
- If $\mathrm{tr}(f) = \pm 2\cosh(l/2) \in \mathbb{R} - [2, 2]$, then f is a translation and conjugate to $z \mapsto e^l z$.
- If $\mathrm{tr}(f) = \pm 2$ and $f \neq \mathrm{id}$, then f is parabolic and conjugate to $z \mapsto z + 1$.

Remark 11.33. Although it is unambiguous from the definition that $f = \mathrm{id}$ is an elliptic isometry, it is quite special: it has the same trace as parabolic isometries. Moreover, it can be approached by translations as well as by non-trivial rotations. Informally speaking, $f = \mathrm{id}$ is at the junction between elliptic, loxodromic, and parabolic isometries.

11.4 Isometries of $\mathbb{H}^3$

11.4.1 Isometries of the Poincaré half-space

Let us favor the Poincaré half-space model $H^3 = \mathbb{C} \times [0, \infty) \subseteq \mathbb{R}^3$. We shall use coordinates $(z = x_1 + ix_2, x_3)$. The group of orientation-preserving isometries of $\mathbb{H}^3$ is identified to $\mathrm{PGL}(2, \mathbb{C})$, acting on $\partial_\infty H^3 = \hat{\mathbb{C}}$ by fractional linear transformations, and acting in H^3 by via the Poincaré extension. Let us recall how this works: any matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{C})$$

induces a orientation-preserving Möbius transformation of $\hat{\mathbb{C}}$ given by

$$f_M \colon z \mapsto \frac{az + b}{cz + d} .$$

Any such Möbius transformation uniquely extends as a Möbius transformation of H^3 (this is called the Poincaré extension, see Theorem 8.30), which we still denote f_M, and which is an isometry of H^3. Conversely, any orientation-preserving isometry of H^3 is a Möbius transformation, and is uniquely determined by its continuous extension to $\partial_\infty H^3 = \hat{\mathbb{C}}$, which is an orientation-preserving Möbius transformation of $\hat{\mathbb{C}}$. The latter coincides with a fractional linear transformation as above.

The assignment $M \to f_M$ is a group homomorphism from $\mathrm{GL}(2, \mathbb{C})$ to $\mathrm{Isom}(H^3)$, whose image is $\mathrm{Isom}^+(H^3)$ and whose kernel is the group of homotheties $\mathbb{C}^* I_2$, so that it induces an isomorphism $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* I_2 \xrightarrow{\sim} \mathrm{Isom}^+(H^3)$.

Instead of $\mathrm{PGL}(2,\mathbb{C})$, in this section we will favor $\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\{-I_2, I_2\}$, which is basically the same group (there is a natural isomorphism $\mathrm{PSL}(2,\mathbb{C}) \xrightarrow{\sim} \mathrm{PGL}(2,\mathbb{C})$). Essentially, any matrix in $\mathrm{GL}(2,\mathbb{C})$ can be multiplied by some $\lambda \in \mathbb{C}^*$ so that the resulting matrix has determinant 1, and the associated fractional linear transformations are the same. More precisely, the story above can be repeated for $\mathrm{SL}(2,\mathbb{C})$: the assignment $M \to f_M$ is a group homomorphism from $\mathrm{SL}(2,\mathbb{C})$ to $\mathrm{Isom}(H^3)$, whose image is still $\mathrm{Isom}^+(H^3)$ and whose kernel is $\mathbb{C}^* I_2 \cap \mathrm{SL}(2,\mathbb{C}) = \{-I_2, I_2\}$, so that it induces an isomorphism $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{-I_2, I_2\} \xrightarrow{\sim} \mathrm{Isom}^+(H^3)$.

The benefits of $\mathrm{SL}(2,\mathbb{C})$ over $\mathrm{GL}(2,\mathbb{C})$ is that not only it will be useful in this section to assume that all matrices have determinant 1, it is especially convenient that we can associate a matrix $M \in \mathrm{SL}(2,\mathbb{C})$ unique up to sign to any $f \in \mathrm{Isom}^+(H^3)$. In particular, the trace of $f \in \mathrm{Isom}^+(H^3)$ is well-defined complex number up to sign.

11.4.2 Elliptic isometries

Theorem 11.34. Let $f\colon H^3 \to H^3$ be an orientation-preserving isometry, represented by $M \in \mathrm{SL}(2,\mathbb{C})$. Assume $f \neq \mathrm{id}$. The following are equivalent:

(i) f is an elliptic isometry.

(ii) The set of fixed points of f in H^3 is a geodesic.

(iii) $\operatorname{tr} M \in (-2,2) \subseteq \mathbb{R} \subseteq \mathbb{C}$.

(iv) M is conjugate in $\mathrm{SL}(2,\mathbb{C})$ to $\pm R_\theta$ for some $\theta \in \mathbb{R} - 2\pi\mathbb{Z}$, where $R_\theta = \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{bmatrix}$.

(v) f is conjugate in $\mathrm{Isom}^+(\mathbb{H})$ to f_θ for some $\theta \in \mathbb{R} - 2\pi\mathbb{Z}$, where f_θ is given by $(z, x_3) \mapsto (e^{i\theta} z, x_3)$.

Before writing the proof of this theorem, we show the useful lemma:

Lemma 11.35. *Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C})$ and denote $f\colon z \mapsto \frac{az+b}{cz+d}$ the associated fractional linear transformation of $\hat{\mathbb{C}}$.*

- *If $\operatorname{tr} M \in \mathbb{C} - (-2,2)$, then f has exactly two fixed points in $\hat{\mathbb{C}}$, one attracting and one repelling.*
- *If $\operatorname{tr} M \in (-2,2)$, then f has exactly two fixed points in $\hat{\mathbb{C}}$, both neutral.*
- *If $(\operatorname{tr} M)^2 = 4$, then either f is the identity, or f has a unique fixed point in $\hat{\mathbb{C}}$, which is neutral.*

Proof. Firstly, one readily shows that a fixed point $z_0 \in \mathbb{C}$ is attracting [resp. repelling, resp. neutral] in the sense defined in § 11.1 if and only if the “multiplier” $|f'(z_0)|$ is < 1 [resp. > 1, resp. $= 1$]. For $z_0 = \infty$, take $|g'(0)|$ instead, where $g(z) = 1/f(1/z)$.

If $c \neq 0$, a fixed point of f is a root of the quadratic polynomial $cz^2 + (d-a)z - b$, with discriminant $\Delta = (\operatorname{tr} M)^2 - 4$. The derivative of f is $f'(z) = \frac{1}{(cz+d)^2}$, and at the two fixed points we have $f'(z) = \frac{4}{(\operatorname{tr} M \pm \sqrt{\Delta})^2}$. Notice that the product $f'(z_1) f'(z_2)$ is equal to 1, so that z_1 and

z_2 are either distinct and attracting/repelling, or distinct and neutral, or equal and neutral. The conclusion quickly follows.

If $c = 0$, one must have $d = \frac{1}{a} \neq 0$, and the fixed points of f solve $a(az + b) = z$. If $a \neq 1$, f has two fixed points: $z_1 = \frac{ab}{a^2-1}$ and $z_2 = \infty$, with multipliers a^2 and $1/a^2$. Therefore z_1 and z_2 are either attracting and repelling, or repelling and attracting, or both neutral, depending on whether $|a| > 1$, $|a| < 1$, or $|a| = 1$. The first two cases correspond to $\operatorname{tr} M = a + \frac{1}{a} \in \mathbb{C} - (-2, 2)$, and the third case to $\operatorname{tr} M = a + \frac{1}{a} \in (-2, 2)$. Finally, if $a = d = 1$: we have $\operatorname{tr} M = 2$, and either $b = 0$ and f is the identity map, or $b \neq 0$ and f admits ∞ as a unique fixed point with multiplier 1. □

Proof of Theorem 11.34. Assume that f is elliptic. By Lemma 11.11, the set of fixed points F of f in H^3 is either a point, or a geodesic, or a 2-dimensional hyperbolic subspace. If f had a unique fixed point $p_0 \in H^3$, then f could not have any ideal fixed point $z \in \hat{\mathbb{C}}$, for otherwise the geodesic through p_0 with endpoint z would be fixed. However Lemma 11.35 shows that f has at least one ideal fixed point. If F was a two-dimensional hyperbolic subspace, then f would have infinitely many fixed points in $\hat{\mathbb{C}}$; again this is ruled out by Lemma 11.35. Thus we proved that (i) implies (ii), and the converse is trivial.

The equivalence (i) ⇔ (iii) is an immediate consequence of Theorem 11.4 and Lemma 11.35. The equivalence (iii) ⇔ (iv) is elementary linear algebra.

Finally, let us prove (iv) ⇔ (v). It is clear that M is conjugate to R_θ if and only if the restriction of f to $\hat{\mathbb{C}}$ is conjugate to $z \mapsto e^{i\theta}$. It remains to show that $f_\theta \colon (z, x_3) \mapsto (e^{i\theta} z, x_3)$ is the Poincaré extension of $z \mapsto e^{i\theta}$. It is enough to realize that f_θ is an isometry of H^3 (it clearly preserves the Riemannian metric), and that its continuous extension to $\hat{\mathbb{C}}$ is indeed $z \mapsto e^{i\theta}$. □

The "standard" elliptic isometry f_θ is shown in Figure 11.1 and Figure 11.2. I also recommend checking out the website [Nel20] for cool animations of elliptic, "hyperbolic", loxodromic, and parabolic isometries in the Poincaré half-space model.

Corollary 11.36. *The conjugacy class of an elliptic element of* $\operatorname{Isom}^+(H^3)$ *is uniquely determined by its trace (which is a complex number defined up to sign).*

11.4.3 Loxodromic isometries

Theorem 11.37. Let $f \colon H^3 \to H^3$ be an orientation-preserving isometry, represented by $M \in \mathrm{SL}(2, \mathbb{C})$. The following are equivalent:

(i) f is a loxodromic isometry.

(ii) f has no fixed points in H^3, and two distinct fixed points in $\partial_\infty H^3 = \hat{\mathbb{C}}$, one attracting and one repelling.

(iii) $\operatorname{tr} M \in \mathbb{C} - [-2, 2]$.

(iv) M is conjugate in $\mathrm{SL}(2, \mathbb{R})$ to $\pm T_l$ for some $l \in \mathbb{C} - i\mathbb{R}$, where $T_l = \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}$.

(v) f is conjugate in $\mathrm{Isom}^+(H^3)$ to f_l for some $l \in \mathbb{C} - i\mathbb{R}$, where $f_l(z, x_3) = (e^l z, x_3)$.

The absolute value of the complex number l in (iv) and (v) is the translation length of f.

Proof. The fact that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is an application of Theorem 11.20 and Lemma 11.35. The equivalence (iii) $\Leftrightarrow$ (iv) is elementary linear algebra. Finally, the proof of (iv) $\Leftrightarrow$ (v) is the same as in Theorem 11.34. □

Among loxodromic isometries, translations are special:

Corollary 11.38. *Let $f\colon H^3 \to H^3$ be an orientation-preserving isometry, represented by $M \in \mathrm{SL}(2, \mathbb{C})$. The following are equivalent:*

(i) f is a translation.

(ii) $\mathrm{tr}\, M \in \mathbb{R} - [-2, 2] \subseteq \mathbb{C}$.

(iii) M is conjugate in $\mathrm{SL}(2, \mathbb{C})$ to $\pm T_l$ for some $l \in \mathbb{R} - \{0\}$, where $T_l = \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}$.

(iv) f is conjugate in $\mathrm{Isom}^+(H^3)$ to f_l for some $l \in \mathbb{R} - \{0\}$, where $f_l(z, x_3) = (e^l z, e^{\mathrm{Re}(l)} x_3)$.

The absolute value of the real number l in (iv) and (v) is the translation length of f.

A representation of the “standard” loxodromic isometry f_l ($l \in \mathbb{C} - (i\mathbb{R} \cup \mathbb{R})$) and the “standard” translation f_l ($l \in \mathbb{R} - \{0\}$) are shown in Figure 11.6 and Figure 11.7. I also recommend checking out the website [Nel20].

Corollary 11.39. *The conjugacy class of a loxodromic element of $\mathrm{Isom}^+(H^3)$ is uniquely determined by its trace (which is a complex number defined up to sign).*

11.4.4 Parabolic isometries

Theorem 11.40. Let $f \in \mathrm{Isom}^+(H^3)$, represented by $M \in \mathrm{SL}(2, \mathbb{C})$. Assume $f \neq \mathrm{id}$. The following are equivalent:

(i) f is a parabolic isometry.

(ii) f has no fixed points in H^3, and one fixed point in $\partial_\infty H^3 = \hat{\mathbb{C}}$.

(iii) $\mathrm{tr}\, M = \pm 2$.

(iv) M is conjugate in $\mathrm{SL}(2, \mathbb{C})$ to $\pm P$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(v) f is conjugate in $\mathrm{Isom}^+(H^3)$ to $(z, x_3) \mapsto (z + 1, x_3)$.

Proof. The fact that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is an application of Theorem 11.6 and Lemma 11.35. The proof of (iii) $\Leftrightarrow$ (iv) is elementary linear algebra, it is the same as in Theorem 11.30. Finally, the proof of (iv) $\Leftrightarrow$ (v) is the same as in Theorem 11.34. □

As in the two-dimensional case, there is only one conjugacy class of orientation-preserving parabolic isometries of $\mathbb{H}^3$:

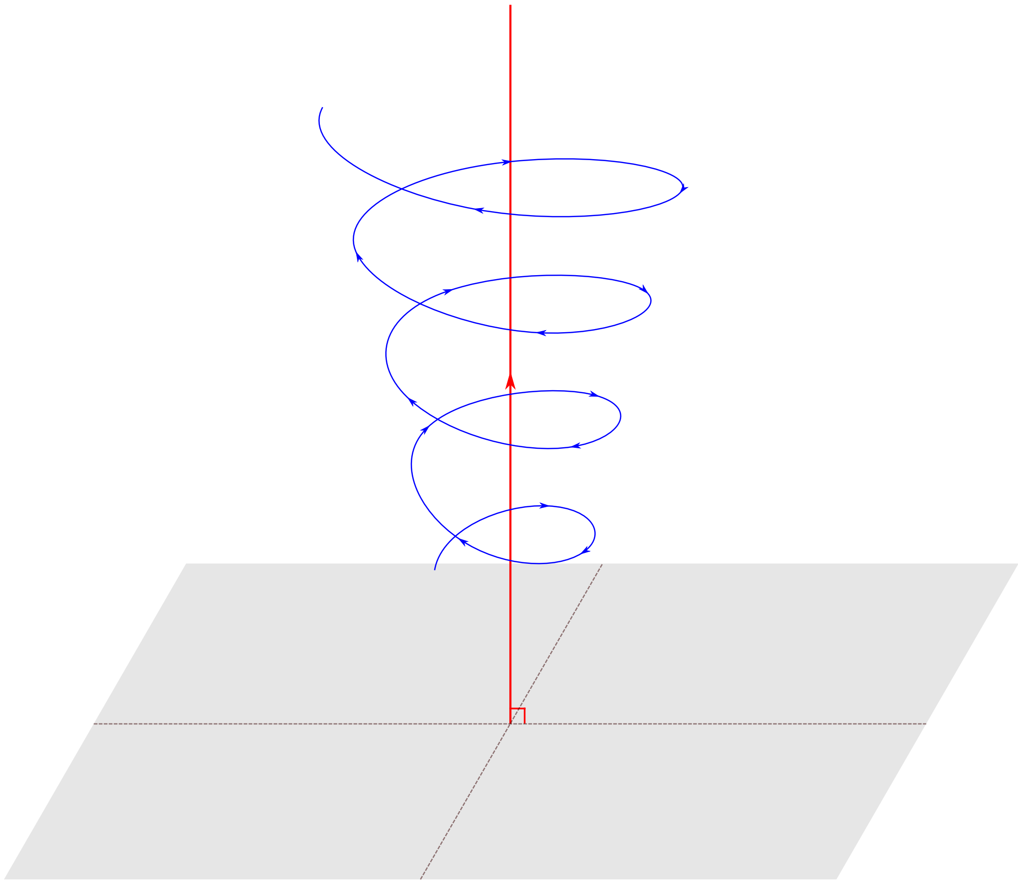


Figure 11.6: A loxodromic isometry of the Poincaré half-plane.

Corollary 11.41. *Any parabolic element of* $\mathrm{Isom}^+(H^3)$ *has trace* ± 2*, and is conjugate to* $(z, x_3) \mapsto (z+1, x_3)$*. Conversely, any* $f \in \mathrm{Isom}^+(H^3)$ *of trace* ± 2 *is parabolic,* provided $f \neq \mathrm{id}$.

A representation of the "standard" parabolic isometry $(z, x_3) \mapsto (z+1, x_3)$ is shown in Figure 11.8.

11.4.5 Conjugacy classes and trace

As a consequence of Corollary 11.36, Corollary 11.39, and Corollary 11.41, we obtain:

Theorem 11.42. The conjugacy class of an element of $f \in \mathrm{Isom}^+(H^3) - \{\mathrm{id}\}$ is uniquely determined by its trace (which is a real number defined up to sign).

More precisely, summarizing previous results:

- If $\mathrm{tr}(f) = \pm 2\cos(\frac{\theta}{2}) \in [2, 2]$, then f is elliptic and conjugate to $(z, x_3) \mapsto (e^{i\theta} z, x_3)$.

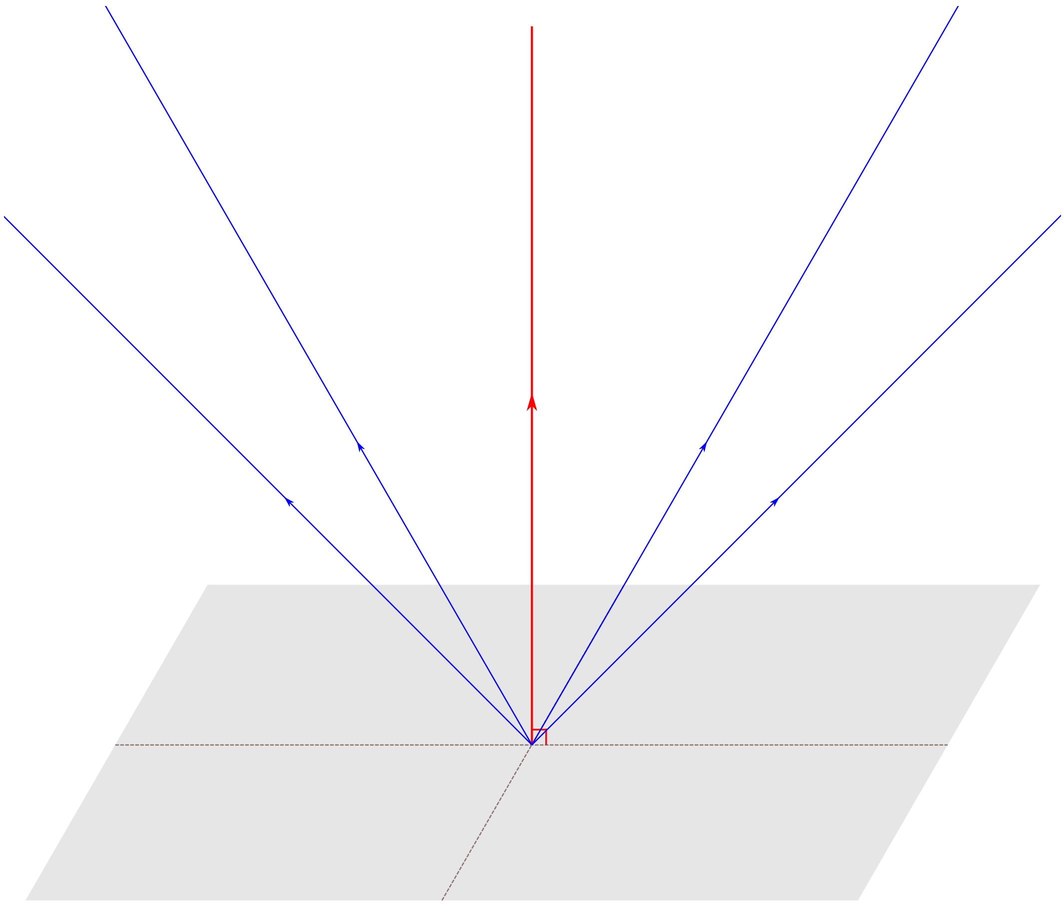


Figure 11.7: A translation of the Poincaré half-space.

- If $\operatorname{tr}(f) = \pm 2\cosh(l/2) \in \mathbb{C} - [2, 2]$, then f is a translation and conjugate to $(z, x_3) \mapsto (e^l z, x_3)$.
- If $\operatorname{tr}(f) = \pm 2$ and $f \neq \mathrm{id}$, then f is parabolic and conjugate to $(z, x_3) \mapsto (e^l z, e^{\operatorname{Re}(l)} x_3)$.

Remark 11.43. Remark 11.33 also holds for H^3.

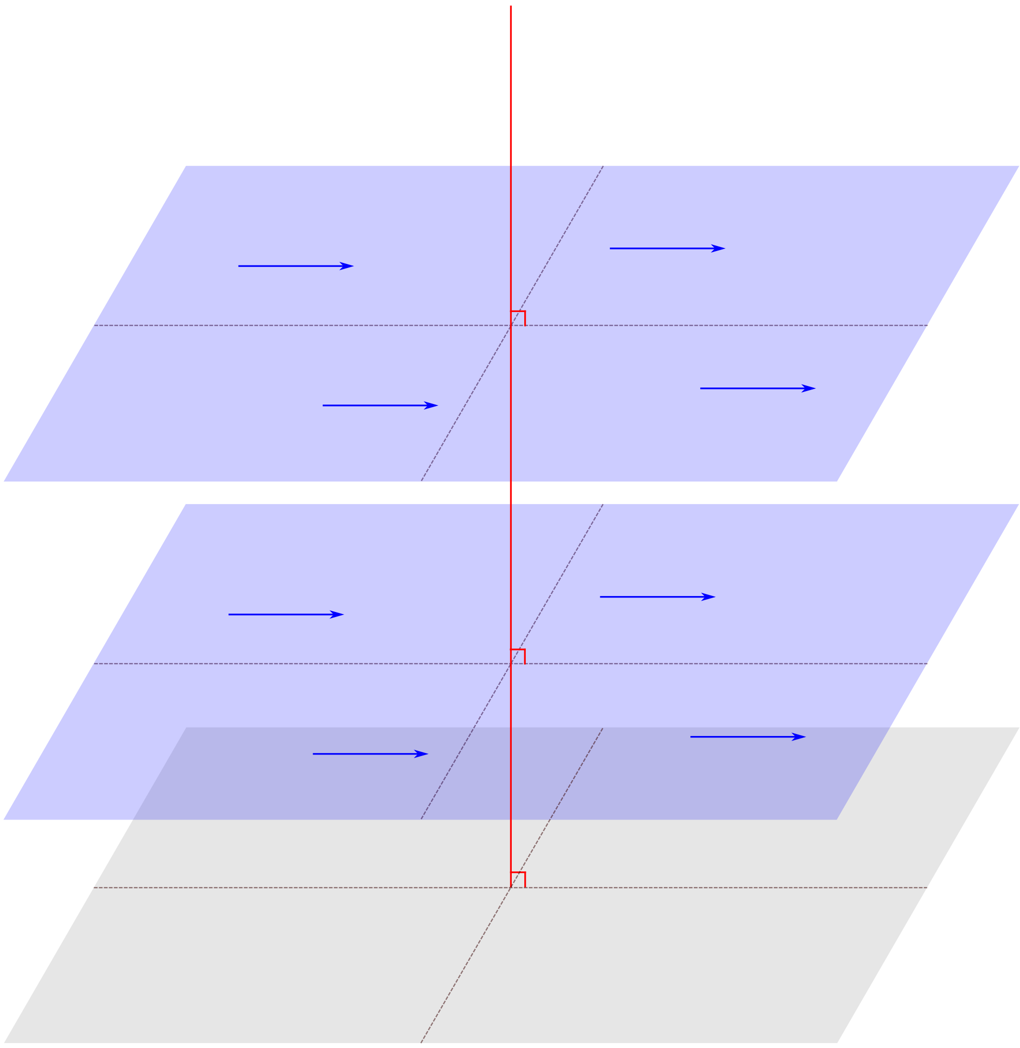


Figure 11.8: A parabolic isometry of the Poincaré half-space.

11.5 Exercises

Exercise 11.1. Characterization of translation length (*borrowed from [BH99, Chap. II.6].*)

Let X be a metric space and let $f: X \to X$.

(1) Show that for any $x \in X$, the sequence $\frac{1}{n}d(x, f^n(x))$ converges in $[0, +\infty)$. *Hint: First show that $d(x, f^n(x))$ is a sub-additive function of n. Then show that $\frac{g(n)}{n}$ converges for any sub-additive function $g: \mathbb{N} \to \mathbb{R}$.*

(2) Show that $\lim_{n\to+\infty} \frac{1}{n}d(x, f^n(x))$ is independent of x.

(3) Show that if f is semi-simple (elliptic or hyperbolic), then $l_f = \lim_{n\to+\infty} \frac{1}{n}d(x, f^n(x))$.

Exercise 11.2. Parabolic fixed point

Let f be a parabolic isometry of $X = \mathbb{H}^n$. Denote $\xi \in \partial_\infty X$ its ideal endpoint.

(1) Show that for any $x \in X \cup \partial_\infty X$, $\lim_{n\to+\infty} f^n(x) = \xi$. Is ξ an attracting fixed point?

(2) Show that for any compact set $K \subseteq \partial_\infty X - \{\xi\}$ and for any neighborhood U of ξ in $\partial_\infty X$, $f^n(K) \subseteq U$ for n sufficiently large. Is ξ an attracting fixed point?

Exercise 11.3. Translation length of a parabolic

Let f be a parabolic isometry of $X = \mathbb{H}^n$. Show that f has zero translation length.

Exercise 11.4. Equidistant curves and translations

(1) Let $L \subseteq \mathbb{H}^n$ be a geodesic line. How would you define an equidistant curve from L? Show that for any $x_0 \in \mathbb{H}^n$, there exists a unique equidistant curve from L.

(2) Let L be the geodesic line with ideal endpoints 0 and ∞ in the Poincaré half-space H^n. Show that the equidistant curves from L are the Euclidean straight half-lines starting from 0.

(3) Prove Proposition 11.18: $f: \mathbb{H}^n \to \mathbb{H}^n$ is a translation if and only if there exists an isometry $\varphi: \mathbb{H}^n \to H^n$ such that $\varphi f \varphi^- 1$ is $x \in H^n \mapsto e^l x$, where l is the translation length of f.

Exercise 11.5. Fixed points and trace

Recall Lemma 11.24: Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$ and denote $f: z \mapsto \frac{az+b}{cz+d}$ the associated fractional linear transformation of $\hat{\mathbb{C}}$.

- If $(\operatorname{tr} M)^2 > 4$, then f has two fixed points, both of which lie in $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$.
- If $(\operatorname{tr} M)^2 < 4$, then f has two fixed points, one in $\mathbb{H}$ and the other is its complex conjugate.
- If $(\operatorname{tr} M)^2 = 4$, then either f is the identity, or f has a unique fixed point, which lies in $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$.

(1) Prove the lemma by direct computation, solving the equation $\frac{az+b}{cz+d} = z$.

(2) Consider the projective transformation $\hat{f} \colon \mathbb{C}P^1 \to \mathbb{C}P^1$ associated to M. Explain why the fixed points of $\hat{f}$ are the eigenlines of M. Recover the lemma.

Exercise 11.6. Limits of loxodromics

(1) Recall the "standard form" of orientation-preserving elliptic, loxodromic, and parabolic isometries of $\mathbb{H}^3$ in the Poincaré half-space model.

(2) Using the previous question, show that any elliptic element of $\operatorname{Isom}^+(\mathbb{H}^3)$ can be obtained as a limit of loxodromic elements.

(3) Prove more generally that any elliptic isometry of $\mathbb{H}^n$ can be obtained as a limit of loxodromic isometries.

(4) Going back to $\mathbb{H}^3$, write a different proof using matrices. Prove in fact that loxodromic elements are dense in $\operatorname{Isom}^+(\mathbb{H}^3)$.

Exercise 11.7. A baby character variety

Let us work in the Poincaré half-space model $\mathbb{H} \subseteq \mathbb{C}$ of the hyperbolic plane $\mathbb{H}^2$. We denote $G = \operatorname{Isom}^+(\mathbb{H})$ the group of orientation-preserving isometries, which can be identified to $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I_2\}$ equipped with the quotient topology.

(1) Show that $f_0 = \mathrm{id} \in G$ is in the closure of the conjugacy class $C \subseteq \operatorname{Isom}^+(\mathbb{H})$ of some/any parabolic isometry.

(2) Let G act on itself by conjugation. Derive from the previous question that the quotient $\mathcal{R}$ is not Hausdorff.

(3) (*) We recall that an element of G is called ***semisimple*** (or *completely reducible*, or *polystable*, depending on context) if it is not parabolic. Let $\mathcal{X} \subseteq \mathcal{R}$ denote the subset of conjugacy classes of semisimple elements. Show that $\mathcal{X}$ is Hausdorff.

Exercise 11.8. Trace relations

We let $G = \mathrm{SL}(2,\mathbb{C})$ in this exercise.

(1) Show that for any $A, B \in G$, $\operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) = \operatorname{tr} A \operatorname{tr} B$.

(2) Show that the trace of any element of the subgroup of G generated by A and B can be expressed as a polynomial in $\operatorname{tr} A$, $\operatorname{tr} B$, and $\operatorname{tr} AB$ with integer coefficients.

(3) *Optional.* Show that any polynomial function of $(A, B) \in G \times G$ that is invariant by conjugation (that is, invariant by $(A, B) \mapsto (gAg^{-1}, gBg^{-1})$ for all $g \in G$) can be expressed as a polynomial function of $\operatorname{tr} A$, $\operatorname{tr} B$, and $\operatorname{tr} AB$.

Exercise 11.9. Classification in $\mathrm{O}^{+}(n, 1)$

Recall that $\operatorname{Isom}(\mathbb{H}^n) \approx \mathrm{O}^{+}(n, 1)$, e.g. via the hyperboloid model. Using linear algebra, find a characterization of elliptic, loxodromic, and parabolic elements of $\mathrm{O}^{+}(n, 1)$.

Part VI: Plane hyperbolic geometry

[...] the way in which I have proceeded does not lead to the desired goal, the goal that you declare you have reached, but instead to a doubt of the validity of [Euclidean] geometry. I have certainly achieved results which most people would look upon as proof, but which in my eyes prove almost nothing; if, for example, one can prove that there exists a right triangle whose area is greater than any given number, then I am able to establish the entire system of [Euclidean] geometry with complete rigor. Most people would certainly set forth this theorem as an axiom; I do not do so, though certainly it may be possible that, no matter how far apart one chooses the vertices of a triangle, the triangle's area still stays within a finite bound. I am in possession of several theorems of this sort, but none of them satisfy me.

– Carl Friedrich Gauß[5]

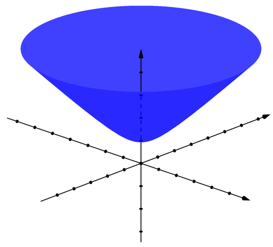
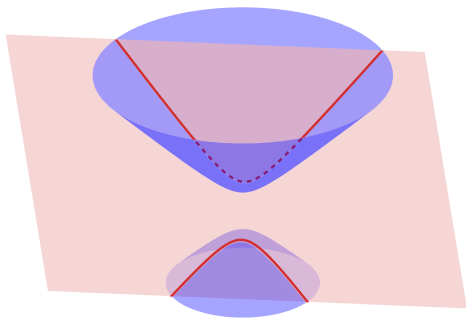
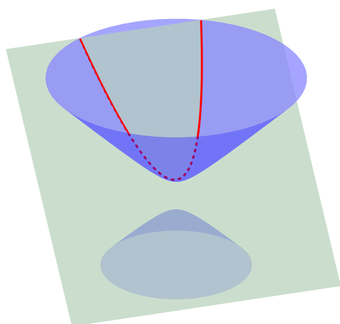
[5] 1799, Answer to a letter from Farkas Bolyai in which Bolyai claimed to have proved Euclid's fifth postulate.

CHAPTER 12

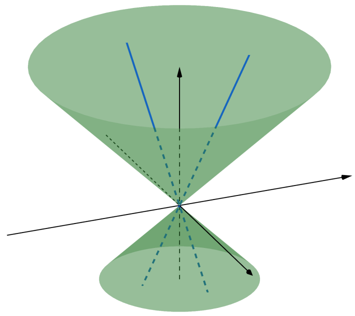
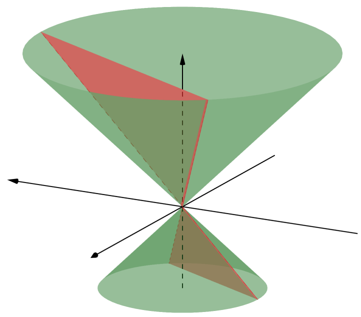
Recap of 2D models

This is not a chapter.

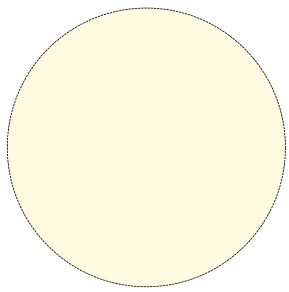
12.1 Hyperboloid model

Name	**Hyperboloid**
Definition	$\mathcal{H}^+ = \mathcal{H} \cap \{z > 0\}$ is the upper sheet of the hyperboloid $\mathcal{H} = \{(x, y, z) \mid x^2 + y^2 - z^2 = -1\}$ $= \{p \in \mathbb{R}^{2,1} \mid \langle p, p \rangle = -1\} \subseteq \mathbb{R}^{2,1}$ 
Riem. metric	$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ (restricted to the tangent plane to $\mathcal{H}^+$, given by $\mathrm{T}_p\,\mathcal{H}^+ = \{p\}^\perp$)
Distance	$d(p, q) = \angle(p, q)$ (hyperbolic angle) $= \operatorname{arcosh}(-\langle p, q \rangle)$
Geodesics	Hyperbolas $\gamma = \mathcal{H}^+ \cap P$ where P is a vector plane in $\mathbb{R}^{2,1}$ They are nicely parametrized: $\gamma(t) = \cosh(\lVert v \rVert t)p + \sinh(\lVert v \rVert t)v$ 
Isometries	$\mathrm{O}^+(2, 1)$ acting linearly on $\mathbb{R}^{2,1}$ (in restriction to $\mathcal{H}^+$) Orientation-preserving isometries: $\mathrm{SO}^+(2, 1)$ $(= \mathrm{O}_0(2, 1))$
Curvature	$K \equiv -1$
Ideal boundary	The hyperboloid model is not best suited to see the ideal boundary. It can be described as the set of future-directed isotropic half-lines in $\mathbb{R}^{2,1}$, which is essentially the same as the projectivized light cone (see § 12.2).
Horocycles	Parabolas $C = \mathcal{H}^+ \cap P$ where P is an affine plane with isotropic normal (See Exercise 4.4, Exercise 10.6) 

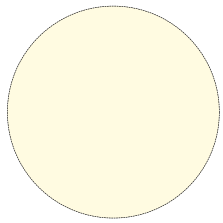
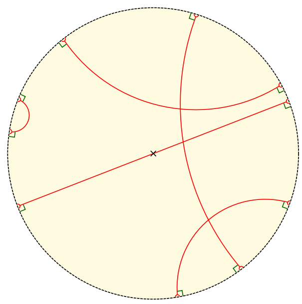
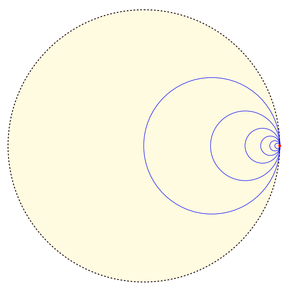
12.2 Cayley–Klein model

Name	**Cayley–Klein model** or **Projective model**
Definition	$\Omega^- = \mathbf{P}\left(\{q < 0\}\right) \subseteq \mathbf{P}\left(\mathbb{R}^{2,1}\right)$ i.e. $\Omega^- = \{\text{lines inside the light cone}\}$ 
Riem. metric	(See § 12.3 for expression in affine chart)
Distance	$d(p,q) = \frac{1}{2}\lvert\ln[p,q,J,I]\rvert$ i.e. $d([u],[v]) = \operatorname{arcosh}\left(\frac{-b(u,v)}{\sqrt{q(u)q(v)}}\right)$
Geodesics	Projective lines $l \subseteq \mathbf{P}(\mathbb{R}^{2,1})$ intersected with Ω^- (i.e. *chords* in Ω^-) 
Isometries	$\mathrm{PO}(2,1)$ acting projective linearly on $\mathbf{P}(\mathbb{R}^{2,1})$ (in restriction to Ω^-) Orientation-preserving isometries: $\mathrm{PSO}(2,1)$
Curvature	$K \equiv -1$
Ideal boundary	$\partial\Omega^- = \mathbf{P}\left(\{q = 0\}\right) \subseteq \mathbf{P}\left(\mathbb{R}^{2,1}\right)$ (projectivized light cone) Note: this is a circle (more precisely, a projective ellipse)
Horocycles	Projective ellipses that are tangent to $\partial\Omega^-$ to order 4 (See Exercise 10.7)

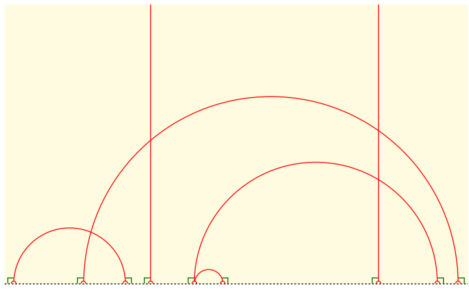
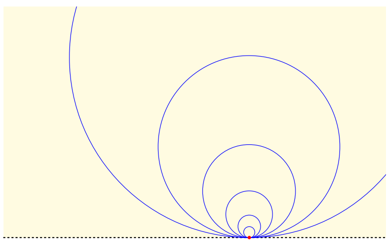
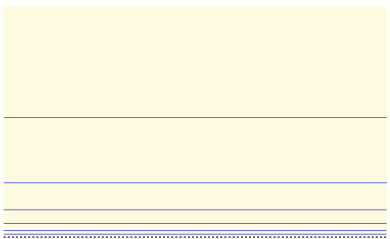
12.3 Beltrami–Klein model

Name	**Beltrami–Klein disk** or **Klein disk**
Definition	$=$ $=$ 
Riem. metric	$\mathrm{d}s^2 =$
Distance	$d(z_1, z_2) =$
Geodesics	
Isometries	
Curvature	$K \equiv -1$
Ideal boundary	
Horocycles	

12.4 Poincaré disk model

Name	Poincaré disk
Definition	$\mathbb{D} = \left\{(x, y) \mid x^2 + y^2 < 1\right\} \subseteq \mathbb{R}^2$ $= \{z \in \mathbb{C} \mid \lvert z\rvert < 1\} \subseteq \mathbb{C}$ 
Riem. metric	$ds^2 = 4\dfrac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} = 4\dfrac{\lvert dz\rvert^2}{(1 - \lvert z\rvert^2)^2}$
Distance	$d(z_1, z_2) = \operatorname{arcosh}\left(1 + \dfrac{2\lvert z_1 - z_2\rvert^2}{(1 - \lvert z_1\rvert^2)(1 - \lvert z_2\rvert^2)}\right) = \ln[z_1, z_2, J, I]$
Geodesics	Circle arcs $\perp$ to $\partial\mathbb{D}$ (including diameters) 
Isometries	$\mathrm{PSU}(1, 1) = \mathrm{PU}(1, 1)$ acting by fractional linear transformations: $z \mapsto \dfrac{az + b}{\bar{b}z + \bar{a}}$ with $\lvert a\rvert^2 - \lvert b\rvert^2 = 1$ i.e. $z \mapsto u\dfrac{z - a}{1 - \bar{a}z}$ with $\lvert u\rvert = 1$, $\lvert a\rvert < 1$ (for orientation-reversing isometries, replace z by $\bar{z}$)
Curvature	$K \equiv -1$
Ideal boundary	$\partial_\infty\mathbb{D} = \partial\mathbb{D} = \{\lvert z\rvert = 1\}$
Horocycles	Euclidean circles tangent to $\partial\mathbb{D}$ 

12.5 Poincaré half-plane model

Name	**Poincaré half-plane**
Definition	$\mathbb{H} = \{(x,y) \mid y > 0\} \subseteq \mathbb{R}^2$ $= \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \subseteq \mathbb{C}$
Riem. metric	$ds^2 = \dfrac{dx^2 + dy^2}{y^2} = \dfrac{\lvert dz\rvert^2}{(\operatorname{Im} z)^2}$
Distance	$d(z_1, z_2) = \operatorname{arcosh}\left(1 + \dfrac{\lvert z_1 - z_2\rvert^2}{2y_1y_2}\right) = \ln[z_1, z_2, J, I]$
Geodesics	Circle arcs $\perp$ to $\partial\mathbb{H}$ (including vertical lines) 
Isometries	$\mathrm{PSL}(2,\mathbb{R})$ acting by fractional linear transformations: $z \mapsto \dfrac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$, $ad - bc = 1$ (for orientation-reversing isometries, replace z by $-\bar{z}$)
Curvature	$K \equiv -1$
Ideal boundary	$\partial_\infty\mathbb{H} = \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$
Horocycles	  Euclidean circles tangent to $\partial_\infty\mathbb{H}$ (including horizontal lines)

12.6 Exercises

Exercise 12.1. Comparison of 2D models

Discuss the advantages and disadvantages of each of the 2-dimensional models. Do you have a favorite?

CHAPTER 13

Hyperbolic trigonometry

Trigonometry, in the literal sense[1], is the study of measurements in triangles, especially the relations between side lengths and angles at the vertices. Such relations are fundamental because not only they are inherent to the geometry of the "universe" (e.g. Euclidean, spherical, or hyperbolic space), they completely characterize it.

After reviewing the basics of triangles in the hyperbolic plane, we shall see that relations between sides and angles are incarnated by the *hyperbolic law of cosines.* Two direct applications of this formula set hyperbolic geometry uniquely apart from Euclidean geometry: the fact that two triangles with the same angles are congruent, and the notion of angle of parallelism. Next, we turn to the strikingly simple relation between the area of a hyperbolic triangle and the sum of its interior angles. This is a trivial consequence of the Gauss—Bonnet theorem, but we present an elegant alternative proof, also due to Gauss. We conclude the chapter by showing that $\mathbb{H}^2$ is a hyperbolic metric space in the sense of Gromov, a feature that we have used in Chapter 10. It will appear in these discussions that the notion of *ideal triangle*, i.e. triangle with vertices are "at infinity", is very useful in hyperbolic geometry.

A chapter on hyperbolic trigonometry could very well be the first in a course of hyperbolic geometry. It is therefore somewhat amusing (or suspicious!) that it arrives so late in our presentation[2]. This is a consequence of my decision to go for a "clean and modern" presentation of hyperbolic geometry, which assumes notions of Riemannian geometry, Minkowski spaces, projective geometry and Möbius transformations. Among the benefits of this approach, beyond the fact that all the theorems of this chapter will be given concise and rigorous proofs, it will be elegant and effective to juggle the different models of the hyperbolic plane. For instance, the hyperbolic law of cosines is easily derived from the hyperboloid model, while the dual law of cosines can be understood in the Klein model via projective duality, and the Poincaré models are best suited to study ideal triangles and compute areas.

[1]The word *trigonometry* is derived from the Greek $\tau\rho\tilde{\iota}\gamma o\nu o\nu$ (*trigōnon*), triangle, and $\mu\acute{\epsilon}\tau\rho o\nu$ (*metron*), measure.

[2]This is the last chapter of the course that I taught at TU Darmstadt, although I initially planned for two additional chapters: the next would contain more plane hyperbolic geometry, including tessellations of $\mathbb{H}^2$, and the final chapter would discuss hyperbolic structures on surfaces.

13.1 Hyperbolic triangles

In the whole chapter, let $\mathbb{H}^2$ denote the hyperbolic plane, not favoring a particular model unless otherwise stated.

Basic definitions

By definition, a ***hyperbolic triangle*** consists of three points typically denoted A, B, C, the ***vertices***, and the three geodesic segments between them, denoted AB, BC, CA, the ***sides*** (or ***edges***). We allow ***degenerate*** triangles, where the three vertices are collinear (lie on a geodesic), including the cases where two or three vertices are equal. We typically denote the ***side lengths*** by $a = d(B,C)$, $b = d(C,A)$, $c = d(A,B)$, and the ***interior angles*** by $\hat{A}$, $\hat{B}$, $\hat{C}$, i.e. the unoriented angles between the sides[3]. See Figure 13.1.

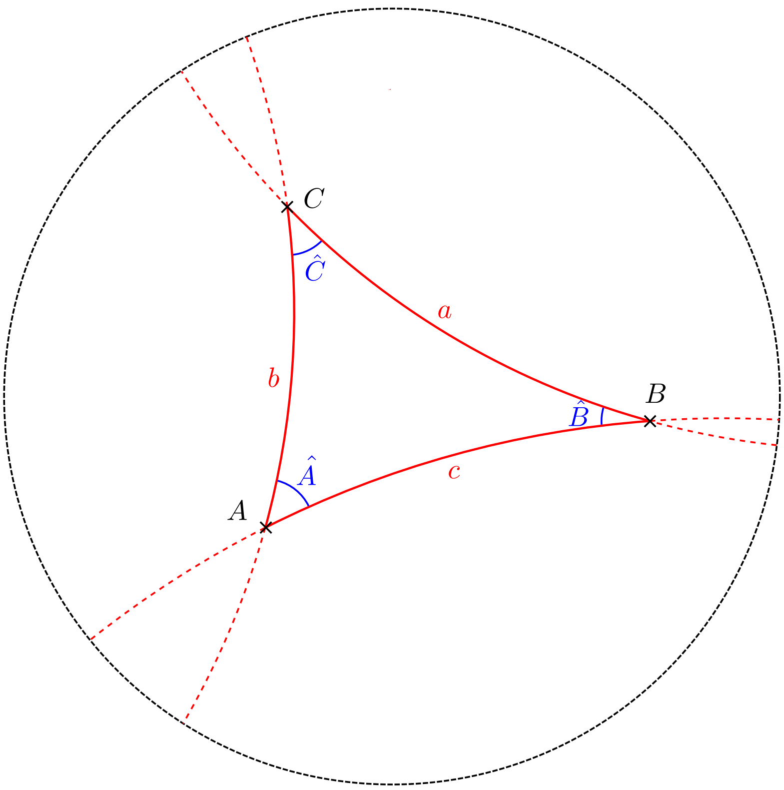


Figure 13.1: A typical hyperbolic triangle in the Poincaré disk model.

[3]Let us recall that the angle between two intersecting geodesics, in fact between any two intersecting curves, is defined as the angle between their tangent vectors at the intersection. This works in any Riemannian manifold, as we saw in § 8.1.

Remark 13.1. If two of the vertices are equal, say $A = B$, then the angles $\hat{A}$ and $\hat{B}$ are undefined (and $\hat{C} = 0$, unless $C = A = B$). In the rest of this chapter, any identity involving $\hat{A}$ implicitly assumes that it only applies to triangles where A is distinct from B and C.

As in the Euclidean plane, we call ***right triangle*** a triangle that has a right angle, ***isoceles triangle*** a triangle that has two equal side lengths, etc.

Congruent triangles

Two triangles are called ***congruent*** if there exists an isometry that takes one to the other. It is enough to require that the isometry maps the vertices of the first triangle to the vertices of the second; such an isometry automatically maps the sides to the sides. It is clear that congruence is an equivalence relation on the set of hyperbolic triangles.

Theorem 13.2. Given $a, b, c \in [0, +\infty)$, there exists a hyperbolic triangle with side lengths a, b, c if and only if the triangle inequalities $a \leqslant b + c$, $b \leqslant c + a$, $c \leqslant a + b$ are satisfied. Moreover, any two hyperbolic triangles are congruent if and only if they have the same side lengths.

Proof. It is clear that if there exists a triangle with side lengths a, b, c, then the triangle inequalities are satisfied. This is because $\mathbb{H}^2$ is a genuine metric space, as is any connected Riemannian manifold with the induced distance.

Conversely, assume that a, b, c satisfy the triangle inequalities. Let us show both the existence of a triangle ABC with side lengths a, b, c and its uniqueness up to congruence at the same time. We shall work in the Poincaré disk model $\mathbb{D} \subseteq \mathbb{C}$ of the hyperbolic plane. First choose the position of A and B in $\mathbb{D}$ so that $d(A, B) = c$. After applying a translation, we can assume that A is the origin $0 \in \mathbb{D}$, and after applying a rotation we can assume that B lies on a the ray $[0, 1)$. It is clear that under these conditions, the position of B is completely determined by the condition $d(A, B) = c$.

Now let us look for the position of C. After applying the reflection $z \mapsto \bar{z}$ if necessary, we can assume that $\mathrm{Im}(C) \geqslant 0$. Let us show that the position of C is now completely determined by $d(A, C) = b$ and $d(B, C) = a$. In other words, we need to show that the circles $C(A, b)$ and $C(B, a)$ have a unique point of intersection C with $\mathrm{Im}(C) \geqslant 0$. These two circles are Euclidean circles by Lemma 13.3, whose centers lie on the same line as A and B.

There is a limited number of configurations of two circles in the Euclidean plane: either one is contained in the interior of the other, or they are each contained in the exterior of the other, or they intersect in two (possibly equal) points. In our situation, $a \leqslant b + c$ and $b \leqslant a + c$ rule out the first configuration, and $c \leqslant a + b$ rules out the second configuration. Therefore we must be in the third configuration where the circles intersect. Moreover, their two (possibly equal) points of intersection are symmetric with respect to the line $(-1, 1)$, due to the invariance of our configuration under the isometry $z \mapsto \bar{z}$. The conclusion follows. □

Lemma 13.3. *Let $C = C(A, R)$ denote the circle with center A and radius $R \geqslant 0$ in the Poincaré disk, i.e. the set of points in $\mathbb{D}$ at distance R from A. Then C is a Euclidean circle.*

Remark 13.4. Be careful: the Euclidean center of C is different from A. That is, unless $A = 0$. Also, the Euclidean radius of C is different from R.

Proof of Lemma 13.3. If $A = 0$, it is easy to see from the expression of the hyperbolic distance that C is a Euclidean circle centered at 0 (and with Euclidean radius $r = \tanh(R/2)$). If $A \neq 0$, one can always use an isometry to send A to 0 (isometries act transitively). The conclusion follows from the fact that isometries of $\mathbb{D}$ are Möbius transformations, and Möbius transformations map Euclidean circles to Euclidean circles. □

Triangles with ideal vertices

It is convenient to allow hyperbolic triangles to have one or more ideal vertices. For instance, if $A \in \partial_\infty \mathbb{H}^2$ is an ideal point and $B, C \in \mathbb{H}^2$ are "interior" points, the triangle ABC still consists of the three vertices A, B, C and the three sides AB, BC, CA; however now the sides AB and CA are semi-infinite geodesic lines (i.e. geodesic rays) with ideal endpoint A. A triangle with 1 ideal vertex is called ***1/3 ideal***. Similarly, we have obvious definitions of triangles having 2 ideal vertices, called ***2/3 ideal triangles***, and 3 ideal vertices, called ***ideal triangles***. An ideal triangle is shown in Figure 13.2 (and another in Figure 13.3).

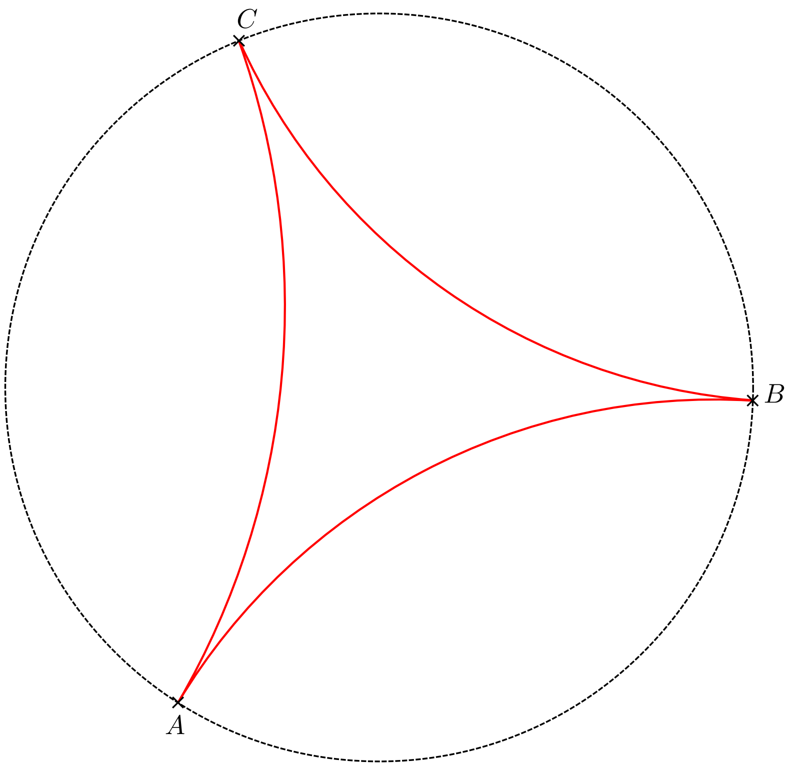


Figure 13.2: An ideal triangle in the Poincaré disk model.

Clearly, there is only one sensible (i.e. continuous) to extend the notion of side length and interior angles for triangles with one or more ideal vertices: the sides adjacent to an ideal vertex have side length $+\infty$, and the interior angle at an ideal vertex is zero. Indeed, in the Poincaré disk model, recall that angles in $\mathbb{H}^2$ are equal to Euclidean angles (the Poincaré disk model is conformal). At an ideal vertex, the two adjacent sides are both orthogonal to the boundary, therefore the Euclidean angle between them is zero.

13.2 The hyperbolic law of cosines

Review: the Euclidean case

Before we jump to the hyperbolic law of cosines, let us quickly review the Euclidean case. A good starting point is the celebrated Pythagorean theorem: a triangle ABC has a right angle at C if and only if we have the identity $c^2 = a^2 + b^2$. The Euclidean law of cosines is a generalization:

Theorem 13.5 (Euclidean law of cosines)**.** For any triangle ABC, with angles denoted $\hat{A}$, $\hat{B}$, $\hat{C}$ and opposite side lengths a, b, c, we have:

$$c^2 = a^2 + b^2 - 2ab\cos\hat{C}\ .$$

Proof. There are many proofs of the Euclidean law of cosines. A modern proof with vector calculus is elementary: starting with $c = \|\overrightarrow{AB}\|$, we have

$$\begin{aligned} c^2 &= \|\overrightarrow{AC} + \overrightarrow{CB}\|^2 \\ &= \|\overrightarrow{AC}\|^2 + \|\overrightarrow{CB}\|^2 + 2\langle \overrightarrow{AC}, \overrightarrow{CB}\rangle\ . \end{aligned}$$

The conclusion follows, since $\langle \overrightarrow{AC}, \overrightarrow{CB}\rangle = -\langle \overrightarrow{CA}, \overrightarrow{CB}\rangle = -ba\cos\hat{C}$. □

Remark 13.6. The Euclidean law of cosines is also known as *Al-Kashi's theorem* (for instance, this is the name that I learned as a high-schooler in France in the early 2000s), after the Persian mathematician Jamshid al-Kashi who proved the theorem in 1427[4]. It must be noted an equivalent version of this theorem is proved in Euclid's *Elements*[5] (3rd century BC), although without using trigonometric functions.

Next we have the law of sines. First, for a triangle ABC with a right angle at C, we have

$$\sin\hat{A} = \frac{a}{c} \qquad \sin\hat{B} = \frac{b}{c}$$

[4] It is contained in Al Kashi's main mathematical work, *Miftāḥ al-Ḥisab* (*Key to Arithmetic*). This work, which consists of five books, is recently being translated to English with commentary: [AH19].

[5] [Euc56, Book 2, Propositions 12 and 13].

so that $\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{1}{c}$. More generally, the ***law of sines*** says that for any triangle ABC, we have

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c} .$$

We leave the proof of the law of sines as an exercise of elementary Euclidean geometry.

Remark 13.7. It can be useful to memorize the additional equalities:

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{b} = \frac{1}{2R} = \frac{2S}{abc}$$

where R is the radius of the circumscribed circle and S is the area of the triangle.

Hyperbolic law of cosines, dual law of cosines, and law of sines

Let us go back to triangles the hyperbolic plane $\mathbb{H}^2$.

Theorem 13.8. For any hyperbolic triangle ABC with $C \neq A$ and $C \neq B$, we have the ***hyperbolic law of cosines***:

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \hat{C} . \tag{13.1}$$

Proof. It is easiest to prove the hyperbolic law of cosines in the hyperboloid model $\mathcal{H}^+ \subseteq \mathbb{R}^{2,1}$. Let γ_u [resp. γ_v] be the unit geodesic from C to A [resp. from C to B]. Our notation indicates that u [resp. v] is the initial tangent vector to the geodesic. Since $\gamma_u(t)$ is a length-minimizing geodesic parametrized by arclength, it reaches A when $t = d(C, A)$, in other words: $A = \gamma_u(b)$. For the same reason, $B = \gamma_v(a)$. Given the explicit expression of geodesics in the hyperboloid model (see Theorem 4.8), namely $\gamma_u(t) = (\cosh t)C + (\sinh t)u$, we find that:

$$\begin{aligned} A &= (\cosh b)C + (\sinh b)u \\ B &= (\cosh a)C + (\sinh a)v . \end{aligned}$$

On the other hand, we have $\cosh c = \cosh d(A, B) = -\langle A, B\rangle$ where $\langle \cdot, \cdot \rangle$ indicates the inner product in Minkowski space $\mathbb{R}^{2,1}$ (see Theorem 4.12). Substituting the expressions of A and B above, we find:

$$\cosh c = -\left\langle (\cosh b)C + (\sinh b)u \;,\; (\cosh a)C + (\sinh a)v \right\rangle .$$

Now simply expand this inner product, noticing that: $\langle C, C\rangle = -1$ since C is on the hyperboloid, $\langle C, u\rangle = \langle C, v\rangle = 0$ since u and v are tangent vectors at C, and $\langle u, v\rangle = \cos \hat{C}$ by definition of the angle at $\hat{C}$. What comes out is the desired identity. □

Next we have the dual hyperbolic law of cosines and the hyperbolic law of sines:

Theorem 13.9. For any hyperbolic triangle ABC, we have the ***dual hyperbolic law of cosines***:

$$\cos\hat{C} = -\cos\hat{A}\cos\hat{B} + \sin\hat{A}\sin\hat{B}\cosh c\,.$$

And the ***hyperbolic law of sines***:

$$\frac{\sin\hat{A}}{\sinh a} = \frac{\sin\hat{B}}{\sinh b} = \frac{\sin\hat{C}}{\sinh c}\,.$$

Proof. The dual hyperbolic law of cosines can be derived from the three hyperbolic law of cosines in the triangle ABC and basic calculus; we leave out the details. Alternatively, one can derive it from the hyperbolic law of cosines (13.1) through projective duality in the Cayley–Klein model, see remark below. The reader may also refer to [Thu97, Chap. 2.4] for a different proof.

To prove the hyperbolic law of sines, first assume that the triangle ABC has a right angle at C. By the law of cosines, we have

$$\cosh b = \cosh a\cosh c - \sinh a\sinh c\cos\hat{B}\,.$$

Substituting $\cosh c = \cosh a\cosh b$ (by the law of cosines) and $\cos\hat{B} = \cosh b\sin\hat{A}$ (by the dual law of cosines), we find that $1 = \cosh^2 a - \sinh a\sinh c\sin\hat{A}$, therefore

$$\sin\hat{A}\sinh c = \sinh a\,.$$

Now for a generic triangle ABC, let H be the orthogonal projection of C on the geodesic line AB. Applying the previous identity in the triangles AHC and BHC, we find

$$\sin\hat{A}\sinh b = \sinh h = \sin\hat{B}\sinh a$$

where $h = d(H,C)$. In particular, we have $\frac{\sin\hat{A}}{\sinh a} = \frac{\sin\hat{B}}{\sinh b}$ as desired. The same argument can be repeated after relabeling the vertices A, B, C, so the second equality follows. □

Remark 13.10. The most elegant proof of the dual hyperbolic law of cosines is through projective duality in the Cayley–Klein model: essentially, distances between points in $\mathbb{H}^2$ corresponds to angles between lines under projective duality, and the dual law of cosines is nothing more than the law of cosines in the projective dual. Making this argument rigorous is a great exercise, but it turns out to be a bit tricky. The subtlety is that projective duality sends lines contained in $\mathbb{H}^2$ (i.e. secant to the quadric $\mathcal{Q}$) to points *outside* the dual conic $\mathcal{Q}^*$ (such points are sometimes called ***ultra-ideal***). Nevertheless, the Cayley–Klein metric is still defined outside of $\mathcal{Q}^*$, and the fact that it is purely imaginary (before taking the absolute value) accounts for the presence of regular cosines and sines in the dual law instead of hyperbolic cosines and sines[6].

[6]Check out [MvG] for an equivalent explanation, which is more detailed but not completely clean in my opinion: distances between points and angles between lines should not be defined independently; the point is to show the relation between them.

Consequences

It is easy to derive many formulas in hyperbolic triangles from the law of cosines, the dual law of cosines, and the law of sines. For instance, the ***hyperbolic Pythagorean theorem*** reads: if ABC has a right angle at C, then $\cosh c = \cosh a \cosh b$.

Remark 13.11. Observe that the second-order expansion of the hyperbolic Pythagorean theorem $\cosh c = \cosh a \cosh b$ is $c^2 = a^2 + b^2$, i.e. the Euclidean Pythagorean theorem. This is not a coincidence: informally speaking, small hyperbolic triangles look almost Euclidean. More generally, hyperbolic geometry limits to Euclidean geometry on a small scale. It is a good exercise to make a precise interpretation of this statement.

Still assuming that ABC has a right angle at C, the sine and cosine of the angle at A can be computed as:

$$\sin \hat{A} = \frac{\sinh a}{\sinh c}$$
$$\cos \hat{A} = \frac{\tanh b}{\tanh c}\,.$$

Let us spare all these silly calculations.

One very interesting consequence of the hyperbolic law of cosines is the following:

Theorem 13.12. The congruence class of a hyperbolic triangle with distinct vertices is uniquely determined by its interior angles.

Proof. It follows from the dual law of cosines that the three side lengths a, b, c, are uniquely determined by the three angles $\hat{A}$, $\hat{B}$, $\hat{C}$. Conclude with Theorem 13.2. □

We leave it as an exercise that conversely, given any three numbers $\alpha, \beta, \gamma \geqslant 0$ such that $\alpha + \beta + \gamma < \pi$, there exists a hyperbolic triangle whose interior angles are equal to α, β, γ: see Exercise 13.2.

Remark 13.13. It is important to realize that Theorem 13.12 this is drastically different from the Euclidean situation, where two homothetic triangles have same interior angles but different side lengths. In other words, Euclidean triangles can be conformally equivalent without being isometric. By contrast, any conformal automorphism of the hyperbolic plane is an isometry, as we have seen in the Poincaré models.

Another application of the hyperbolic law of cosines is the easy computation of the angle of parallelism (see Figure 1.3):

Theorem 13.14. Let l be a line in the hyperbolic plane and A be a point at distance $a > 0$ from l. The angle of parallelism at A is the angle $\Pi(a) \in (0, \pi/2)$ given by

$$\sin \Pi(a) = \frac{1}{\cosh a}\,. \tag{13.2}$$

Proof. To avoid confusion below, let us rename c the distance between A and l. Let B be the nearest-point projection of A on l and let $C \in \partial_\infty \mathbb{H}^2$ be an ideal endpoint of l. Clearly, the angle of parallelism at A is the angle $\hat{A}$ in the triangle ABC. By the dual law of cosines, which extends to triangles with one or more ideal vertices by continuity, we have $\cos \hat{C} = -\cos \hat{A} \cos \hat{B} + \sin \hat{A} \sin \hat{B} \cosh c$. We find $1 = \sin \hat{B} \cosh c$ and the conclusion follows. □

Remark 13.15. The formula (13.2) can also be written

$$\Pi(a) = \frac{\pi}{2} - gd(a)$$

where $gd(x) = \int_0^x \frac{\mathrm{d}t}{\cosh t}$ is the *Gudermannian function.*

13.3 Area of hyperbolic triangles

The goal of this section is to prove the following theorem:

Theorem 13.16. Let ABC be a hyperbolic triangle with three distinct vertices, one or more possibly ideal. Denote by $\mathrm{Area}(ABC)$ the hyperbolic area enclosed by the triangle. We have the identity:

$$\mathrm{Area}(ABC) = \pi - (\hat{A} + \hat{B} + \hat{C})\,. \tag{13.3}$$

Proof with the Gauss–Bonnet theorem

Theorem 13.16 is an immediate consequence of the Gauss–Bonnet theorem. The Gauss–Bonnet theorem is a deep theorem of Riemannian geometry that we shall not discuss; nevertheless, we mention this proof out of interest.

Theorem 13.17 (Gauss–Bonnet theorem)**.** Let (S, g) be a compact 2-dimensional Riemannian manifold with boundary. Then

$$\int_S K_g \,\mathrm{d}\sigma_g + \int_{\partial S} k_g \,\mathrm{d}s = 2\pi\chi(S)$$

where K_g denotes the Gaussian curvature in S, $\mathrm{d}\sigma_g$ the area element in S, k_g the geodesic curvature along ∂S, $\mathrm{d}s$ the line element along ∂S, and $\chi(S)$ the Euler characteristic of S.

Let us not explain precisely all these terms, and only mention what they are when S is the interior of a hyperbolic triangle:

- The Gaussian curvature (a.k.a sectional curvature) K_g is constant equal to -1 inside S, since it is an open subset of the hyperbolic plane.
- The area element $\mathrm{d}\sigma_g$ is the hyperbolic area element $\mathrm{d}A$.
- The geodesic curvature k_g vanishes along the sides of the triangle, because the sides are geodesic by assumption. However, when the boundary ∂S is only piecewise smooth, one must add to $\int_{\partial S} k_g \,\mathrm{d}s$ the exterior angle at each point of discontinuity. Thus in our situation, we have $\int_{\partial S} k_g \,\mathrm{d}s = (\pi - \hat{A}) + (\pi - \hat{B}) + (\pi - \hat{C})$, that is $\int_{\partial S} k_g \,\mathrm{d}s = 3\pi - (\hat{A} + \hat{B} + \hat{C})$.

- The Euler characteristic of the triangle is $\chi(S) = 1$: that is +3 (vertices) −3 (edges) +1 (face).

Putting all this together, the Gauss–Bonnet formula reads:

$$\int_{ABC} (-1)\, \mathrm{d}A + (3\pi - (\hat{A} + \hat{B} + \hat{C})) = 2\pi$$

and the formula (13.3) follows.

Ideal triangles

Before turning to an alternative proof, let us examine the case of ideal triangles.

Theorem 13.18. All ideal triangles are congruent, and have area π.

Proof. The fact that all ideal triangles are congruent is an immediate consequence of the fact that isometries of $\mathbb{H}^2$ act 3-transitively on the ideal boundary. Indeed, we have seen that the projective linear group $\mathrm{PGL}_2(\mathbb{R})$ acts 3-transitively on $\mathbb{R}P^1$ (Theorem 6.25), in other words it acts 3-transitively on $\hat{\mathbb{R}}$ by fractional linear transformation. The Poincaré extension of any such transformation of $\hat{\mathbb{R}} \approx \partial_\infty \mathbb{H}^2$ is an isometry of $\mathbb{H}^2$ (in the Poincaré disk or half-plane model), so we are done.

Since all ideal triangles are isometric, they all have the same area, so we can pick our favorite to check that its area is equal to π. Let us choose the ideal triangle with vertices $A = -1$, $B = 1$, $C = \infty$ in the Poincaré half-plane: see Figure 13.3. Computing its area is now elementary calculus:

$$\begin{aligned}
\mathrm{Area}(ABC) &= \int_{ABC} \mathrm{d}A \\
&= \int_{x=-1}^{1} \int_{y=\sqrt{1-x^2}}^{+\infty} \frac{\mathrm{d}x\, \mathrm{d}y}{y^2} \\
&= \int_{x=-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} .
\end{aligned}$$

The change of variables $x = \sin\theta$ yields

$$\begin{aligned}
\mathrm{Area}(ABC) &= \int_{\theta=-\pi/2}^{\pi/2} \mathrm{d}\theta \\
&= \pi .
\end{aligned}$$

□

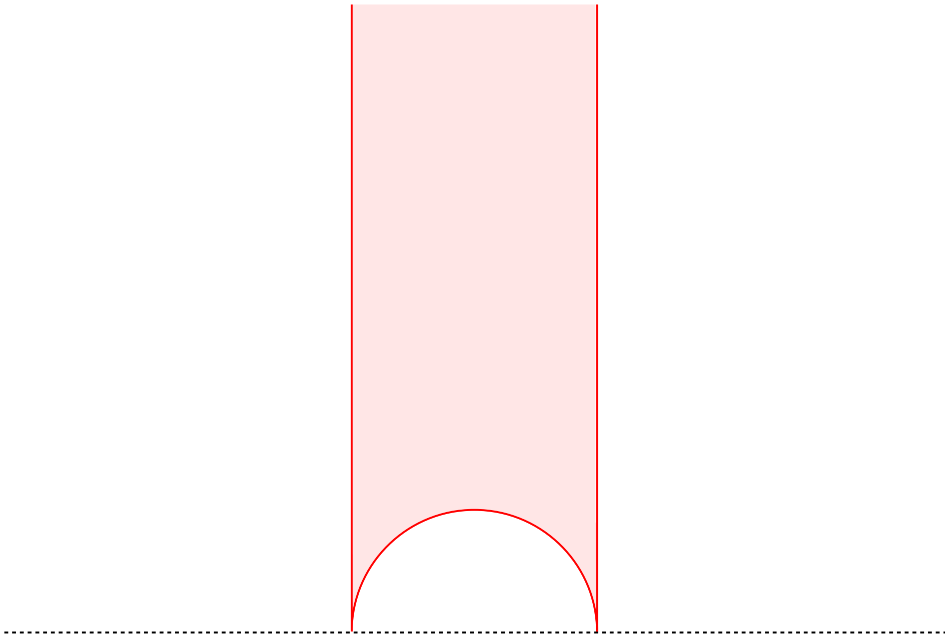


Figure 13.3: The ideal triangle with vertices -1, 1, ∞ in the Poincaré disk model.

Gauss's proof

We now propose an alternative and elegant proof of Theorem 13.16, also due to Gauss (according to [Thu97]). What follows is based on [Thu97, Prop. 2.4.13].

First consider a 2/3-ideal triangle. The congruence class of such a triangle is completely determined by the angle at the interior vertex. Indeed, after applying a translation, we can assume that the interior vertex is the origin in the Poincaré disk. It is then clear that any two such triangles with same interior angle are related by a rotation. Denote by $A(\theta)$ the area of any 2/3-ideal triangle whose angle at the interior vertex is $\pi - \theta$. Per our discussion, $A(\theta)$ is a well-defined function of $\theta \in (0, \pi)$.

Gauss's clever observation is that $A(\theta)$ is an additive function of θ: we have $A(\theta_1 + \theta_2) = A(\theta_1) + A(\theta_2)$ whenever $\theta_1, \theta_2, \theta_1 + \theta_2 \in (0, \pi)$. To see this, consider Figure 13.4. The triangles BOA, BOB', and $A'OB'$ have areas $A(\theta_1) = \mathcal{A}_1$, $A(\theta_2) = \mathcal{A}_2 + \mathcal{A}_3$, and $A(\theta_1 + \theta_2) = \mathcal{A}_3 + \mathcal{A}_4$ respectively. On the other hand, the half-turn (π-rotation) through Ω takes the triangle ΩAB to $\Omega A'B'$, so we have $\mathcal{A}_4 = \mathcal{A}_1 + \mathcal{A}_2$. Therefore $A(\theta_1 + \theta_2) = \mathcal{A}_3 + (\mathcal{A}_1 + \mathcal{A}_2) = A(\theta_1) + A(\theta_2)$.

The function $\theta \in (0, \pi) \mapsto A(\theta)$ being additive and continuous, it must be linear. Moreover, it extends continuously at π by $A(\pi) = \pi$ as a consequence of Theorem 13.18. This forces $A(\theta) = \theta$ for all $\theta \in [0, \pi]$. Hence we have proved Theorem 13.16 for 2/3-ideal triangles.

The case of 1/3-ideal triangles easily follows by a cut-and-paste procedure: any such triangle can be written as the difference of two 2/3-ideal triangles. The case of triangles with no ideal vertices is derived from the 1/3-ideal case with the same trick, writing any triangle with no ideal vertices as the difference of two 1/3-ideal triangles. We leave it to the reader to draw the appropriate sketches.

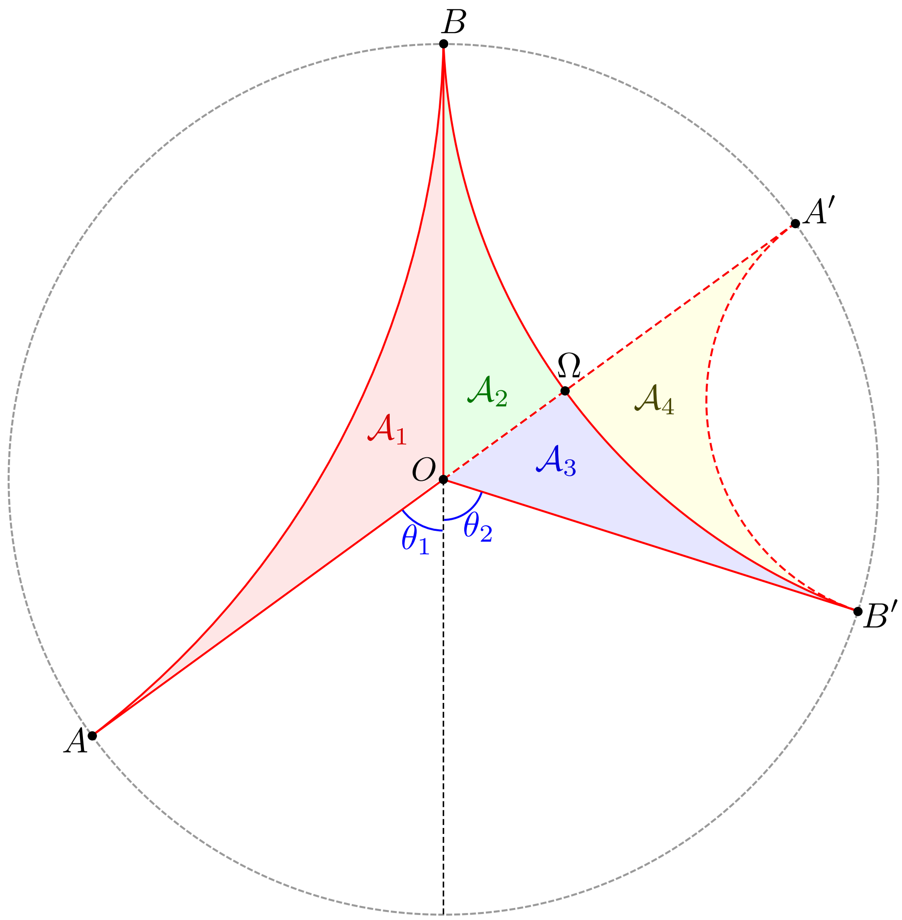


Figure 13.4: Gauss's trick to compute the area of 2/3-ideal triangles.

13.4 Gromov hyperbolicity of the hyperbolic plane

We conclude this chapter by showing that the hyperbolic plane $\mathbb{H}^2$ is a hyperbolic space in the sense of Gromov, which is a property regarding hyperbolic triangles. It readily follows that hyperbolic space $\mathbb{H}^n$ is also Gromov hyperbolic for all $n \geqslant 2$.

We have already discussed Gromov hyperbolicity in general metric spaces in Chapter 10 (see § 10.1.3), where we mentioned that the notion of ideal boundary is well-suited to such spaces (e.g., we used it for Lemma 10.16).

By definition, $\mathbb{H}^2$ being Gromov hyperbolic means that there exists $\delta \geqslant 0$ such that all hyperbolic triangles are ***δ-slim***: any point on one side of the triangle is within distance $\leqslant \delta$ of some point on another side. In other words, any side is contained in the δ-neighborhood of the union of the two other sides: see Figure 10.2. In the case of $\mathbb{H}^2$, taking $\delta = 1$ is sufficient; in fact the best δ can be computed as $\delta = \operatorname{arsinh}(1) \approx 0.88137\ldots$

Theorem 13.19. The hyperbolic plane $\mathbb{H}^2$ is hyperbolic in the sense of Gromov. The smallest constant $\delta \geqslant 0$ such that all hyperbolic triangles are δ-hyperbolic is $\delta = \operatorname{arsinh}(1)$.

Proof. A detailed proof is proposed in Exercise 13.6. □

13.5 Exercises

Exercise 13.1. Congruent triangles with ideal vertices

We have seen (Theorem 13.2) that two hyperbolic triangles are congruent if and only if they have the same side lengths. State and prove a generalization for triangles having one or more ideal vertices.

Exercise 13.2. Congruent triangles and angles

Show that for any three numbers $\alpha, \beta, \gamma \geqslant 0$ such that $\alpha + \beta + \gamma < \pi$, there exists a hyperbolic triangle whose interior angles are equal to α, β, γ. Show that moreover, any two such triangles are congruent. Is this true for Euclidean triangles?

Exercise 13.3. Inscribed and Circumscribed circles

(1) Show that not all hyperbolic triangles admit a circumscribed circle.

(2) In Chapter 11, we saw that any bounded set in $\mathbb{H}^n$ has a well-defined *minimum bounding ball*, which some authors call "circumball". Is that not a contradiction with the previous question?

(3) Show that any hyperbolic triangle admits a uniquely defined inscribed circle.

(4) Show that there exists a finite upper bound for the radii of the inscribed circle of all hyperbolic triangles.

Exercise 13.4. Unified law of cosines

For $R \in \mathbb{C} - 0$, define the generalized cosine and sine functions by:

$$\cos_R(x) = \cos\left(\frac{x}{R}\right)$$
$$\sin_R(x) = R\sin\left(\frac{x}{R}\right) .$$

Consider the "unified law of cosines for curvature $k = \frac{1}{R^2}$":

$$\cos_R c = \cos_R a \cos_R b + \frac{1}{R^2} \sin_R a \sin_R b \cos \hat{C} .$$

(1) Check that in the case $k = -1$, i.e. $R = \pm i$, one recovers the hyperbolic law of cosines.

(2) Prove the hyperbolic law of cosines in the hyperbolic space of constant curvature $k < 0$.

(3) Predict the spherical law of cosines. Prove it.

(4) Show that the Euclidean law of cosines may be obtained asymptotically from the unified law of cosines when $k \longrightarrow 0$.

(5) *Optional.* Can you come up with a heuristic explanation for the existence of a unified law of cosines that works in any constant curvature?

Exercise 13.5. Area of hyperbolic polygons

How would you define a hyperbolic polygon? Find a formula for the area of any hyperbolic polygon, and prove it.

Exercise 13.6. Gromov hyperbolicity of hyperbolic space

Let $n \geqslant 2$. The goal of this exercise is to show that hyperbolic space $\mathbb{H}^n$ is Gromov hyperbolic (see Definition 10.8): there exists $\delta > 0$ such that any triangle in $\mathbb{H}^n$ is δ-slim.

(1) Argue that it is enough to do the case $n = 2$.

(2) Argue that it is enough to show that some ideal triangle is δ-slim.

(3) Consider the ideal triangle with vertices $A = 0$, $B = \infty$, and $C = 1$ in the Poincaré half-plane. What are the sides (AB), (BC), and (CA) of this triangle? Draw a picture.

(4) Let $p = (0, y) \in (AB)$. Show that the distance from p to (BC) is achieved at $p' = (1, \sqrt{1 + y^2})$. Derive that $d(p, (BC)) = \operatorname{arsinh}(1/y)$.

(5) Find an isometry that maps $A \mapsto B$, $B \mapsto C$, $C \mapsto A$. Show that $d(p, (CA)) = \operatorname{arsinh}(y)$.

(6) Conclude that $d(p, (BC) \cup (CA)) \leqslant \delta$ where $\delta = \operatorname{arsinh}(1)$ and conclude the exercise.

(7) Is the constant $\delta = \operatorname{arsinh}(1)$ optimal?

CHAPTER 14

More plane hyperbolic geometry

14.1 Curves

14.1.1 Circles

14.1.2 Horocycles

14.1.3 Equidistant curves

14.2 Polygons

14.3 Plane hyperbolic geometry in the Klein model

14.4 More

CHAPTER 15

Tessellations of the hyperbolic plane

Part VII: Hyperbolic geometry and data science

The greatest mathematicians, as Archimedes, Newton, and Gauss, always united theory and applications in equal measure.

– Felix Klein[1]

[1] [Kle67].

CHAPTER 16

Graph embeddings in hyperbolic space

CHAPTER 17

Hyperbolic neural networks

APPENDIX A

Basic notions

A.1 Algebra

A.1.1 Group actions

A.1.2 Linear algebra

Assume known: vector spaces, dimension, linear maps, endomorphisms, bases, matrices

Inner product spaces

orthogonal group, spectral theorem, Gram-Schmidt

Affine spaces

Euclidean spaces Affine frame

Complexification

A.2 Analysis

A.2.1 Multivariable calculus

A.2.2 Complex functions

A.2.3 Hyperbolic functions

A.3 Geometry

Metric spaces

Notions of topology

Elementary Riemannian geometry

I think this should be a part of Chapter 2 maybe.

Exercises solutions and hints

Chapter 1

Exercise 1.2. (2) First prove it for a triangle that has a vertex at the center of the disk. Then play a game of cut and paste.

Chapter 2

Exercise 2.1. (4) Straightforward calculations lead to $\kappa = \frac{|a|}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$. The helix with parameters $a = 2$ and $b = 3$ is shown in Figure A.1.

(5) Isometries of E act transitively on affine frames, therefore one can assume that $\gamma(0)$ is fixed as well as the Frenet–Serret frame (T, N, B) at $t = 0$. The Frenet–Serret formulas define a linear system of ODEs for (T, N, B), therefore it has a unique solution by Picard–Lindelöf given fixed initial conditions. Conclude by noting that the curve γ is recovered from integrating T. A more detailed proof can be found anywhere, e.g. [Car16, Chap. 1-5], [Pre10, Thm. 2.3.6], or [Spi99, Vol. II, Chap. 1].

(6) Note that the formulas giving κ and τ in terms of a and b can easily be inverted: $a = \frac{\pm\kappa}{\kappa^2+\tau^2}$ and $b = \frac{\tau}{\kappa^2+\tau^2}$. By (4), the helix with parameters a and b has constant curvature κ and torsion τ. By the previous question, any other curve with such curvature and torsion is an image of that helix by an isometry of E.

Exercise 2.2. Let $U_p \subseteq \mathrm{T}_p S$ denote the unit circle, consisting of unit tangent vectors at p. Given an orthonormal basis of $\mathrm{T}_p M$, U_p can be identified to the unit circle in $\mathbb{R}^2$, parametrized by the angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Let us denote $\vec{v}(\theta)$ the unit tangent vector with angle θ and $\rho_p(\theta) := \rho_p\big(\vec{v}(\theta)\big)$ its extrinsic curvature. We claim that $H_p = \frac{1}{2\pi}\int_0^{2\pi} \rho_p(\theta)\, \mathrm{d}\theta$.

Indeed, the average of any symmetric bilinear form on $\mathbb{R}^2$ on the unit circle is equal to one half of its trace. We leave the proof of this fact as a subsidiary exercise to the reader, with the hint: use the spectral theorem.

Exercise 2.3. (1) We recall that by definition, a Riemannian isometry is a map whose derivative at any point is a linear isometry between tangent spaces.

(2) (b) Use Proposition 2.9.

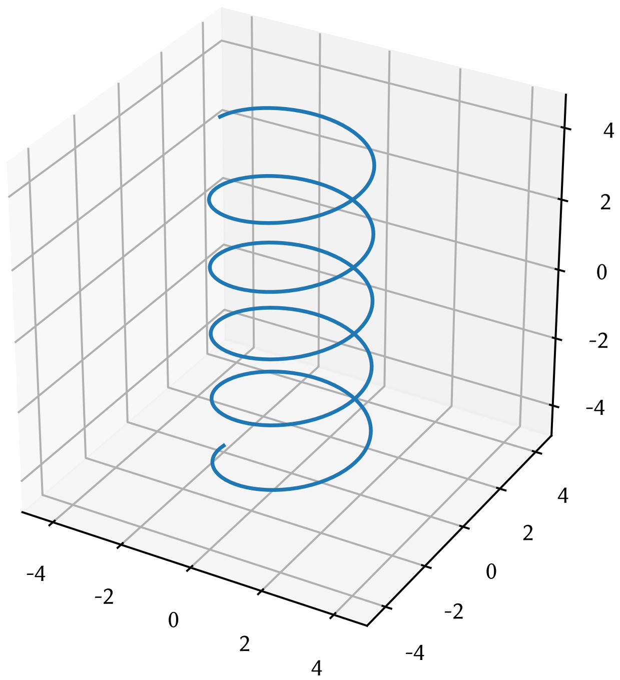


Figure A.1: The helix with $a = 2$ and $b = 3$

Note: This figure and others were created with Python using the `matplotlib` library [Hun07].

Exercise 2.4. (1) See Figure A.2 for a picture of the tractrix.

(3) See Figure A.3 for a picture of the tractricoid with meridians and parallels. It is a general fact that if a regular curve is the set of fixed points of an isometry, then this curve must be a geodesic, up to reparametrization: otherwise, it is easy to see that uniqueness of geodesics with a given initial velocity would be violated. In our case, consider the reflection through a vertical plane.

(4) (a) Since there exists an isometry of $\mathbb{R}^3$ preserving S and taking $p_\theta := f(\theta, t_0)$ to $p_0 := f(0, t_0)$, namely the rotation of angle θ around the z-axis, S must have same Gaussian curvature at p_θ and p_0. Indeed, this isometry transports everything from p_θ to p_0: geodesics, normal to S, etc, so S must have same principal curvatures at p_θ and p_0, and have same Gaussian curvature. Note that this is an illustration, in an easy case, of the Theorema Egregium.

(b) $\gamma_0'(t) = (-\operatorname{sech} t \tanh t, 0, \tanh^2 t) =: u$ and $c_t'(0) = (0, \operatorname{sech} t, 0) =: v$. To get a vector normal to S at p, we can take the cross-product of u and v, and renormalize to get a unit vector. One finds $\vec{N} = \pm(\tanh t, 0, \operatorname{sech} t)$. We take $+$ for the "exterior" normal.

(c) Calculations yield $\gamma_0''(t) = (\operatorname{sech} t(1 - 2\operatorname{sech} t), 0, 2\operatorname{sech}^2 t \tanh t)$ so we find the normal curvature $\langle \gamma_0''(t), \vec{N} \rangle = \operatorname{sech} t \tanh t$. This is the extrinsic curvature $\rho_p(u)$ where $u = \gamma_0'(t)$; in order to get the extrinsic curvature $\rho_p(u_1)$ where $u_1 = \frac{u}{\|u\|}$ is the unit tangent, we have to divide by $\|u\|^2 = \tanh^2 t$, because ρ_p is quadratic. We obtain $\rho_p(u_1) = \frac{1}{\sinh t}$. Similar

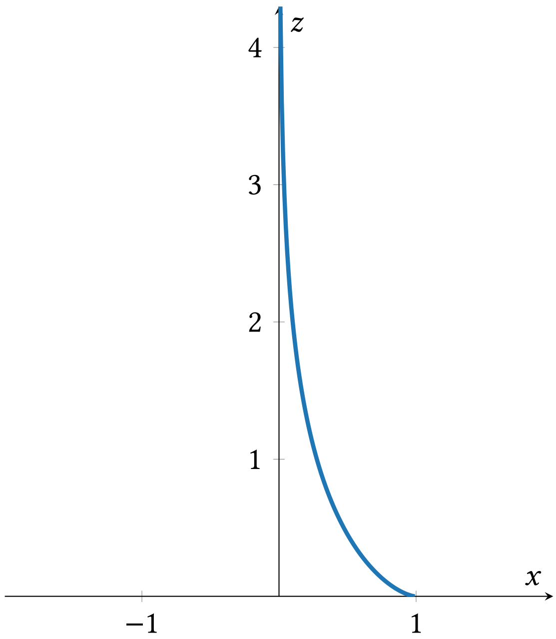


Figure A.2: The tractrix

Note: This figure was created with LaTeXusing the package `Tikz`.

calculations yield $\rho_p(v_1) = -\sinh t$, where v_1 is the unit tangent to c_t at p.

(d) For the symmetry argument: consider the reflection through the vertical plane containing the curve γ_0. Show that up to sign, it preserves the unit vectors giving the principal directions of curvature.

(5) The arclength is easily computed as $\mathrm{d}s = \tanh t$, which gives $s = \ln(\cosh t)$. In particular, the arclength parameter stays bounded when $t \longrightarrow 0$. This shows that the geodesic γ, or rather, its arclength parametrization, is incomplete. Thus S is not geodesically complete, equivalently it is not a complete metric space by the Hopf-Rinow theorem. Note that if we try to extend the tractricoid by allowing t to take negative values, then the resulting surface is singular at points where $t = 0$.

Exercise 2.6. Let us interpret Euclid's postulates in the realm of surfaces equipped with a Riemannian metric. Note that this is not only anachronistic, but also too restrictive: Euclid's postulates could be interpreted in much more generality. Nevertheless, it is an interesting exercise.

Let (S, g) be a Riemannian surface. In this setting, a *line* must be understood as a geodesic.

First postulate. The first postulate of Euclid reads: there exists a geodesic segment between any two points in (S, g). Note that if we add uniqueness, this excludes S having closed geodesics and self-intersecting geodesics. In particular, S must be simply connected. If we only require uniqueness of the maximal geodesic through two points, then the projective plane $S^2/\{\pm 1\}$, model of elliptic geometry, is acceptable.

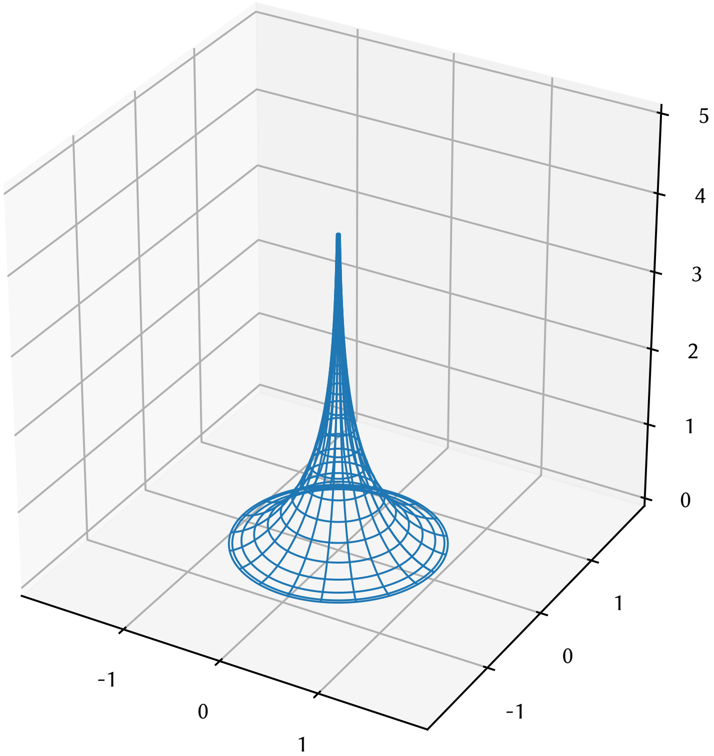


Figure A.3: The tractricoid

Second postulate. The second postulate is precisely saying that (S, g) is geodesically complete. By the Hopf-Rinow theorem, this is equivalent to (S, g) being complete as a Riemannian manifold. Note that this implies the first postulate, without uniqueness.

At this point, any Hadamard 2-manifold is acceptable. That is a simply-connected, complete 2-manifold of nonpositive sectional curvature.

Third postulate. In this setting, the third postulate is trivially true: given a point p and a radius $r > 0$, the circle $C(p, R)$ is uniquely defined as the set of points at distance r from p. Although one way to interpret the postulate is that this circle is nonempty, or is a topological circle.

Fourth postulate. This is arguably the most important postulate. Firstly, it implies that (S, g) is *homogeneous*, i.e. that the group G of isometries of (S, g) acts transitively on S. The additional requirement on right angles is equivalent to (S, g) being *isotropic*: for any $p \in G$, G acts (via derivatives of its elements) transitively in $\mathrm{T}_p\, G$. Every complete isotropic Riemannian manifold is homogeneous, making the first requirement unnecessary.

At this stage, I think that up to isometry, there are no other models of Euclid's axioms than the space forms of constant curvature: the sphere S^2 and its analogs S^2_R of constant curvature $k = \frac{1}{R^2}$ for any $R > 0$, the Euclidean plane $\mathbb{R}^2$, and the hyperbolic plane $\mathbb{H}^2$ and its analogs $\mathbb{H}^2_R$ of constant curvature $k = -\frac{1}{R^2}$ for any $R > 0$. More precisely, depending on a more or less restrictive interpretation of the first postulate, we may exclude or include the spheres S^2_R and/or their quotients $S^2_k/\{\pm 1\}$.

Fifth postulate. Given a geodesic and a point not on it, there exists a unique geodesic through the point which does not intersect the first. This postulate excludes the hyperbolic planes $\mathbb{H}^2_R$ and, regardless of the interpretation of the first postulate, the spheres S^2_R and/or their quotients $S^2_R/\{\pm 1\}$.

We can therefore wrap up:

Theorem. *Let (S, g) be a smooth connected surface equipped with a Riemannian metric. Then*

(i) (S, g) satisfies the first four postulates of Euclid if and only if it isometric to either $\mathbb{R}^2$, or $\mathbb{H}^2_R$ for some $R > 0$. Depending on the interpretation of the first postulate, the spheres S^2_k and/or their quotients $S^2_R/\{\pm 1\}$ should also be included.

(ii) (S, g) satisfies the five postulates of Euclid if and only if it is isometric to $\mathbb{R}^2$.

Chapter 3

Exercise 3.4. (3) First show that the subspace spanned by u and v has signature $(1, 1)$.

Exercise 3.5. (1) Use the determinant function $\det\colon \mathrm{O}(n, 1) \to \{\pm 1\}$ and the "time signature" function $\sigma\colon \mathrm{O}(n, 1) \to \{\pm 1\}$. First argue that they are both continuous (for the time signature, use the criterion of Exercise 3.3).

Chapter 4

Exercise 4.1. (1) (b) We recall that by definition, a Riemannian isometry is a map whose derivative at any point is a linear isometry between tangent spaces.

(2) (c) Hint: identify this action to the action of $\mathrm{O}(n)$ in $\mathbb{R}^n$. (d) Hint: Recall that $\mathcal{H}^+$ is uniquely goedesic: for any $v \in \mathcal{H}^+$, there exists a unique geodesic from v_0 to v.

Exercise 4.2. (1) This is an immediate computation after recalling that the distance on the hyperboloid is $d(p, q) = \operatorname{arcosh}(-\langle p, q\rangle)$.

(2) $\operatorname{arcosh}(\cosh^2 x) = \sqrt{2}x + \frac{1}{6\sqrt{2}}x^3 + O(x^4)$.

(3) It follows from (1) and (2) that $d(\gamma_v(t), \gamma_w(t))^2 = 2t^2 - \frac{1}{3}t^4 + O(t^5)$, hence $K(v, w) = -1$. Since this holds for any p and any orthonormal pair $v, w \in \mathrm{T}_p\,\mathcal{H}^+$, we proved that $\mathcal{H}^+$ has constant sectional curvature $K = -1$.

(4) We now find $d(\gamma_v(t), \gamma_w(t)) = R \operatorname{arcosh}\left(\frac{\cosh^2 t}{R}\right)$. It follows from the Taylor expansion found in (2) that $d(\gamma_v(t), \gamma_w(t))^2 = 2t^2 - \frac{1}{3R^2}t^4 + O(t^5)$, hence $K(v, w) = -\frac{1}{R^2}$.

Chapter 6

Exercise 6.9. Hint: Show that for a surface $S \subseteq \mathbb{R}^3 \subseteq \mathbb{R}P^3$, the sign of any extrinsic curvature $\rho_p(v)$ is invariant under orientation-preserving projective linear transformations of $\mathbb{R}P^3$.

Chapter 7

Exercise 7.6. (1) By definition of the Cayley–Klein metric,

$$\begin{aligned} d(x,y) &= \frac{1}{2}\left|\ln([0, \pm\|x\|, -1, 1])\right| \\ &= \frac{1}{2}\left|\ln \frac{1 \mp \|x\|}{1 \pm \|x\|}\right| \\ &= \operatorname{artanh}(\|x\|)\,. \end{aligned}$$

(2)

$$\begin{aligned} d(x,y) &= \operatorname{arcosh}\left(\frac{1 - \langle x, 0\rangle}{\sqrt{(1-0)(1-\|x\|^2)}}\right) \\ &= \operatorname{arcosh}\left(\frac{1}{\sqrt{1-\|x\|^2}}\right) \\ &= \operatorname{artanh}(\|x\|)\,. \end{aligned}$$

(3) Let $\gamma(t) = tx$ for $t \in [0,1]$. Since the image of γ is a minimizing geodesic, $d(0,x) = L(\gamma) = \int_0^1 \|\gamma'(t)\|\,\mathrm{d}t$. Here we have $\gamma'(t) = tx$ and the expression of the Riemannian metric on the Beltrami–Klein disk gives (after a couple lines of calculations) $\|\gamma'(t)\| = \frac{\|x\|}{1-t^2\|x\|^2}$. Note that this is $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{artanh}(t\|x\|)$, so we find $d(0,x) = \operatorname{artanh}(\|x\|)$.

Exercise 7.7. Show that: (i) The result is true when $x_0 = 0$, (ii) $\mathrm{PO}(2,1)$ acts transitively on circles of radius R, and (iii) $\mathrm{PO}(2,1)$ sends ellipses to ellipses. Of course, you could also try a direct proof, let me know if you succeed that.

Chapter 8

Exercise 8.7. (3) You should find that the pullback metric on $S^n \subseteq \mathbb{R}^{n+1}$ is $\frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2}{(1-x_{n+1})^2}$. Clearly, this is conformal to the Euclidean metric of $\mathbb{R}^{n+1}$, which is the spherical metric in restriction to S^n.

Exercise 8.9. (3) (b) For a translation $z \mapsto z + b$, take two reflections having b as a normal vector. For a similarity $z \mapsto az$ with $a \in \mathbb{C}^*$, first write it as the composition of the rotation

$z \mapsto e^{i\theta} z$ and the homothety $z \mapsto \rho z$, where $a = \rho e^{i\theta}$. For the rotation, try two reflections whose axes intersect at the origin. For the homothety, try two inversions through spheres centered at the origin. Finally, write $z \mapsto \frac{1}{z}$ as the composition of the inversion through the unit circle and the reflection through the real axis.

Exercise 8.10. (5) Hint: Show that if a fractional linear transformation preserves $\mathbb{D}$, then it also preserves the unit circle $\partial\mathbb{D} = \{|z| = 1\}$. Then show that if the fractional linear action of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ preserves $|z| = 1$, then $a\bar{b} - c\bar{d} = 0$ and $|a|^2 - |c|^2 = |d|^2 - |b|^2$. Conclude that, after multiplying M by a constant, it belongs to $\mathrm{U}(1, 1)$.

Exercise 8.12. *Still works:* Defining inversions, Möbius transformations as product of inversions, both for $\widehat{\mathbb{R}}$ and S^1. It is still true that the $\mathrm{Möb}(\widehat{\mathbb{R}}) \approx \mathrm{Möb}(S^1) \approx \mathrm{PO}(1, 1)$ and $\mathrm{Möb}^+(\widehat{\mathbb{R}}) \approx \mathrm{Möb}^+(S^1) \approx \mathrm{PSO}(2, 1)$. In this case, one can also identify $\mathrm{Möb}^+(\widehat{\mathbb{R}})$ to $\mathrm{PGL}_2^+(\mathbb{R})$, acting by fractional linear transformations on $\widehat{\mathbb{R}} \approx \mathbb{R}P^1$. What also works is the Poincaré extension from dimension 1 to 2: any Möbius transformation of $\mathrm{Möb}(\widehat{\mathbb{R}})$ [resp. S^1] extends to a unique transformation of H^2 [resp. B^2]. In fact we see directly that $\mathrm{PGL}_2^+(\mathbb{R})$ acts both on $\mathrm{Möb}(\widehat{\mathbb{R}})$ and $\mathbb{H}$ by fractional linear transformations; similarly $\mathrm{PSU}(1, 1)$ acts both on $S^1 \subseteq \mathbb{C}$ and $\mathbb{D} \subseteq \mathbb{C}$ by fractional linear transformations.

Breaks down: Any diffeomorphism $\mathbb{R} \to \mathbb{R}$ is conformal, therefore the Liouville theorem does not hold: these are not all Möbius transformations. It is also not true that Möbius transformations can be characterized as sphere-preserving, because in this case lower dimensional spheres are pairs of points, so any injective map is sphere-preserving. The Poincaré extension from dimension 0 to 1 also fails: it is not true that any Möbius transformation of $\widehat{\mathbb{R}}$ is uniquely determined by its restriction to $\widehat{\mathbb{R}^0} = \{0, \infty\}$. This is because while it is still true that the subgroup of $\mathrm{PO}(2, 1)$ preserving $\widehat{\mathbb{R}^0}$ is $\mathrm{PO}(1, 1)$, the latter does not act faithfully on $\widehat{\mathbb{R}^0}$; in other words, what fails is that $\mathrm{Möb}(\widehat{\mathbb{R}^n}) = \mathrm{PO}(n + 1, 1)$ is not correct for $n = 0$.

Chapter 9

Exercise 9.9. If we define a hyperbolic space *à la Euclid*, axiomatically, then we could define a hyperbolic subspace of X as a subset $X' \subseteq X$ where the axioms still hold. In order for this to make sense, we should assume that X' is stable under taking the line through two points. It turns out that this condition is sufficient.

Let us take instead the modern definition of a hyperbolic space as a complete, simply-connected Riemannian manifold of constant sectional curvature -1. A hyperbolic subspace is a complete and totally geodesic submanifold $X' \subseteq X$. Equivalently, X' is a subset of X stable under taking the complete geodesic through any two points. Equivalently, X' is a totally geodesically embedded copy of $\mathbb{H}^k$ in $\mathbb{H}^n$ for some $k \leqslant n$.

Hyperbolic subspaces have very natural incarnations in the different models. In the hyperboloid model, a hyperbolic subspace is the intersection with a subspace of Minkowski space (see Proposition 4.1). In the Cayley–Klein model, it is the intersection with a projective subspace. In the Beltrami–Klein ball, it is the intersection with an affine subspace. In the Poincaré models, it is the intersection with a half-sphere orthogonal to the boundary. We leave it to the reader to prove all these descriptions.

Chapter 10

Exercise 10.1. (2) Hint: Compare the distance between two geodesics from the same point in $\mathbb{R}^2$ versus in $\mathbb{H}^2$.

(3) Hint 1: Show that any quasi-isometric (i.e. coarsely surjective) map $r\colon [0,+\infty) \longrightarrow \mathbb{H}^n$ is at finite distance from a geodesic ray. Hint 2: Show that if there exists a quasi-isometry $X \longrightarrow Y$, then there exists a quasi-isometry $Y \longrightarrow X$.

Exercise 10.2.. Hint: start by recalling the relation between the hyperboloid model and the Cayley–Klein model.

Chapter 11

Exercise 11.1. (1) By definition, a function $g\colon \mathbb{N} \longrightarrow \mathbb{R}$ is subadditive if $g(x+y) \leqslant g(x) + g(y)$ for all $x, y \in \mathbb{N}$. For such a function, $\lim_{n\to+\infty} \frac{g(n)}{n}$ always exists. Indeed, for a fixed integer $d > 0$, the Euclidean division of n by d is written $n = qd + r$ with $0 \leqslant r < d$. The subadditivity condition implies that $\frac{g(n)}{n} \leqslant \frac{g(d)}{d} + \frac{g(r)}{n}$, hence $\limsup_{n\to+\infty} \frac{g(n)}{n} \leqslant \frac{g(d)}{d}$. Therefore $\limsup_{n\to+\infty} \frac{g(n)}{n} \leqslant \liminf_{d\to+\infty} \frac{g(d)}{d}$.

Exercise 11.8. (1) Hint: Derive from the Cayley-Hamilton theorem that $B + B^{-1} = \mathrm{tr}(B)I$.

(2) Hint: Start by words of length 1, 2, 3, etc. (in the generators A, B, A^{-1}, and B^{-1}).

Exercise 11.9. For instance, try to prove the following classification:

- $M \in \mathrm{O}^+(n,1)$ is elliptic if and only if M has a timelike eigenvector. In this case, all complex eigenvalues of M have unit modulus.
- $M \in \mathrm{O}^+(n,1)$ is loxodromic if and only if M has a complex eigenvalue λ of modulus $\neq 1$. In this case, λ and λ^{-1} are the only complex eigenvalues of M of modulus $\neq 1$.
- $M \in \mathrm{O}^+(n,1)$ is elliptic if and only if all complex eigenvalues of M have unit modulus, and M has no timelike eigenvector.

For additional guidance, you can check out [Thu97, Problem 2.5.24].

Index of notations

Index of terminology

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