

Exam #1

Monday, October 16 2017

Duration: 1	1H20
NAME:	
Please write	e clearly and properly. Justify your answers carefully.

Problem	Grade
1	
2	
3	
4	
Total	

Problem 1 (~ 4 points).				
Let $n \in \mathbb{N}$ and denote by U_n the set of n -th roots of unity in \mathbb{C} . Show that (U_n, \times) is a monoid. Does every element of U_n have an inverse?				

Problem 2 (∼ 4 points)**.**

Let (M, \otimes) and (N, \diamond) be two monoids with identity elements e_M and e_N respectively. Let $f: M \to N$ be a homomorphism. Denote by $K \subseteq M$ the set of preimages of e_N by f:

$K = \{x \in M \mid f(x) = e_N\}.$				
Show that K is closed in (M, \otimes) .				

Problem 3 (\sim 7 points).
Let $(M, *)$ be a monoid. Let e denote the identity element of M .
We recall that $x \in M$ is called idempotent when $x * x = x$. We recall that $x \in M$ is called invertible when there exists an element $y \in M$ which is an inverse of x .
(1) Let $x \in M$. Show that x is idempotent and invertible if and only if $x = e$.
(2) Let $n \in N$ and denote by $\mathcal{M}_n(\mathbb{R})$ the set of all $n \times n$ matrices with real coefficients. Let $M \in \mathcal{M}_n(\mathbb{R})$ such that $M^2 = M$ and $\det(M) \neq 0$. What can you say about M ?

(3) Let $M = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \in \mathcal{M}_2(\mathbb{R})$. Compute M^2 and $\det(M)$. Is this consistent with your previous answers?

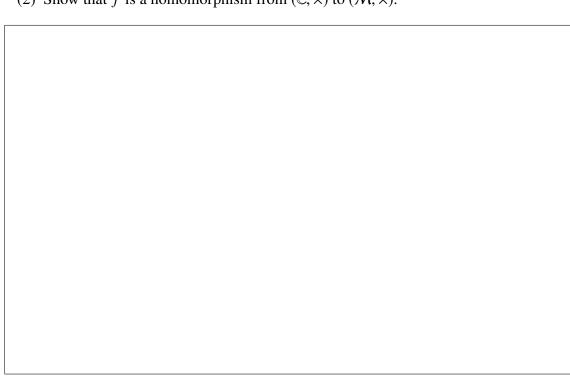
Problem 4 (\sim 7 points).

Let $\mathbb C$ denote the set of complex numbers and $\mathcal M=\mathcal M_2(\mathbb R)$ denote the set of 2×2 matrices with real coefficients. Consider the map

$$f: \mathbb{C} \to \mathcal{M}$$

$$z = a + ib \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

- (1) Show that f is a homomorphism from $(\mathbb{C}, +)$ to $(\mathcal{M}, +)$.
- - (2) Show that f is a homomorphism from (\mathbb{C}, \times) to (\mathcal{M}, \times) .



(3) Show that f is injective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.
(4) Show that f is not surjective.

(5) Let $C \subseteq \mathcal{M}$ denote the image of f , in other words $C = \{f(z), z \in \mathbb{C}\}$. Show that the map
$ ilde{f} \cdot \mathbb{C} o C$
$\tilde{f}: \mathbb{C} \to C$ $z \mapsto f(z)$
$\zeta \mapsto f(\zeta)$
is an isomorphism from $(\mathbb{C},+)$ to $(\mathcal{M},+)$ and from (\mathbb{C},\times) to (\mathcal{M},\times) .