Mass, Scalar Curvature,

Kähler Geometry, and All That

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Rutgers-Newark Colloquium
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Joint work with

Joint work with

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Fordham University

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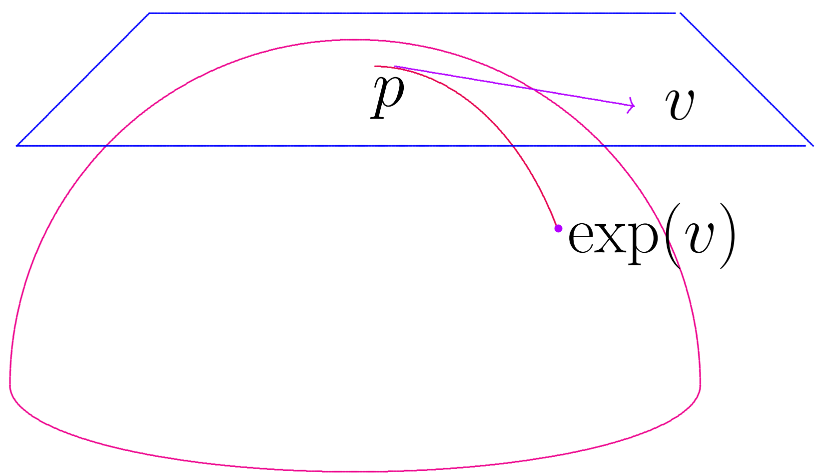

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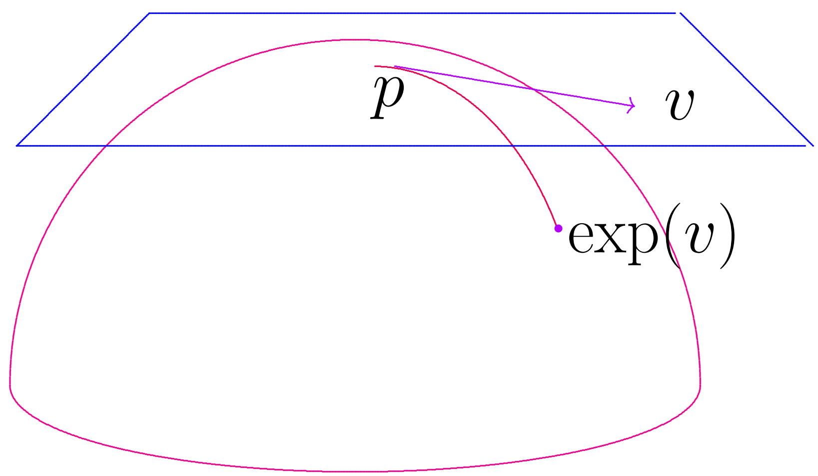

Small ball B_ε maps to the ε distance ball in M:

points reachable from p by paths of length $< \varepsilon$.

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$$s : M \to \mathbb{R}$$

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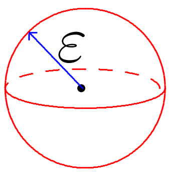
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{\text{Euclidean answer}} = 1 - s(p)\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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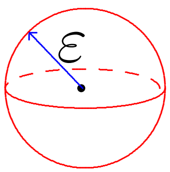

A metric g is called scalar-flat if it satisfies $s \equiv 0$.

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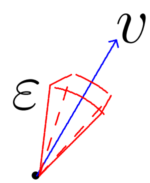
$$\frac{\mathrm{vol}_g(C_\varepsilon(p, v, \Omega))}{\text{Euclidean answer}} \approx 1 - r(v)\ \frac{n\varepsilon^2}{6(n+2)} + O(\varepsilon^3)$$



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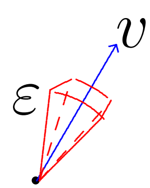

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Simple, natural problem:

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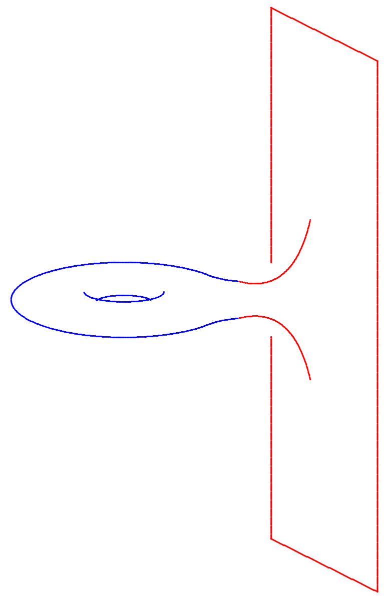
(M^n, g) complete non-compact Riemannian n-manifold.



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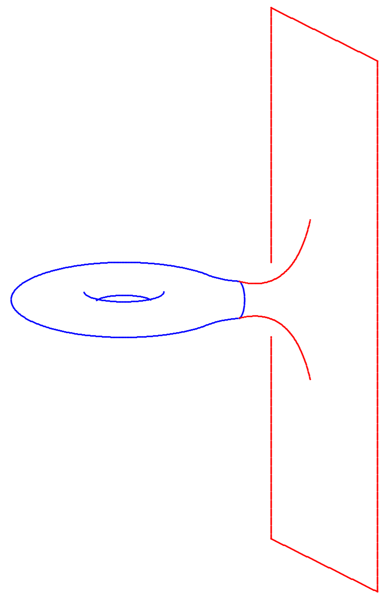
Suppose $\exists K \subset M$ compact



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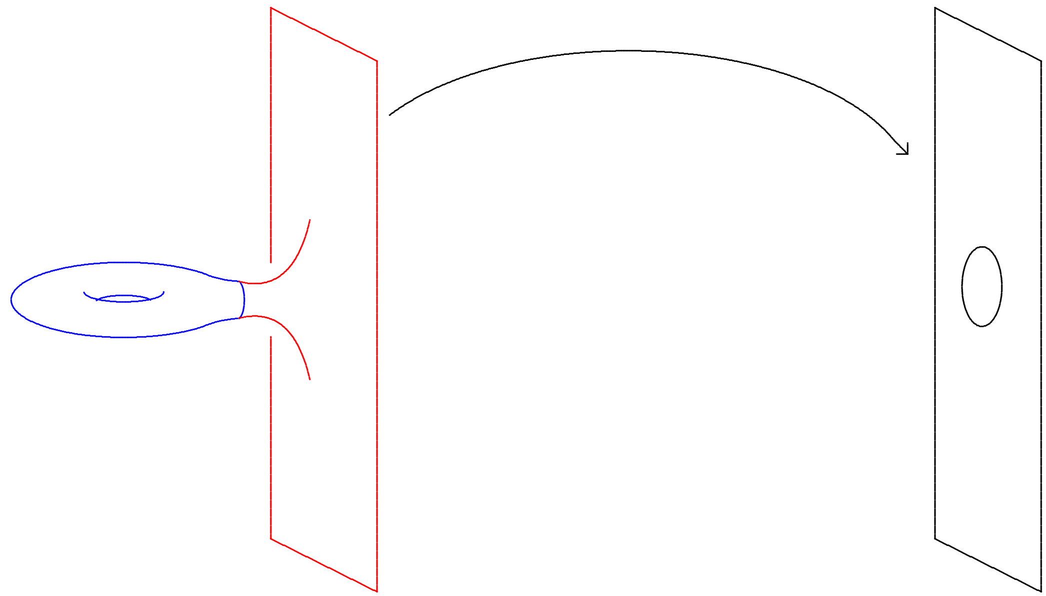

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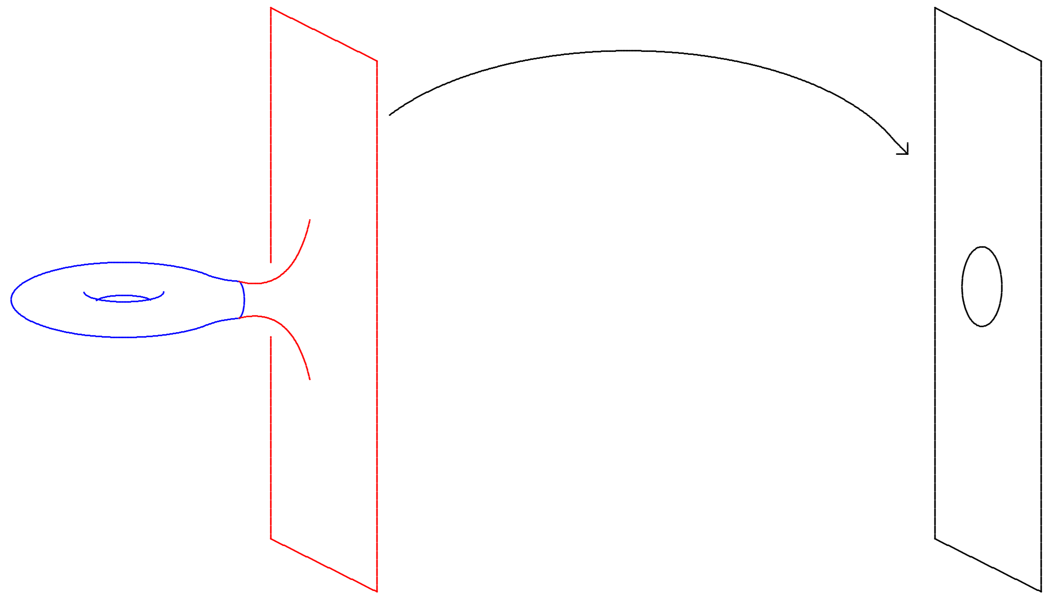

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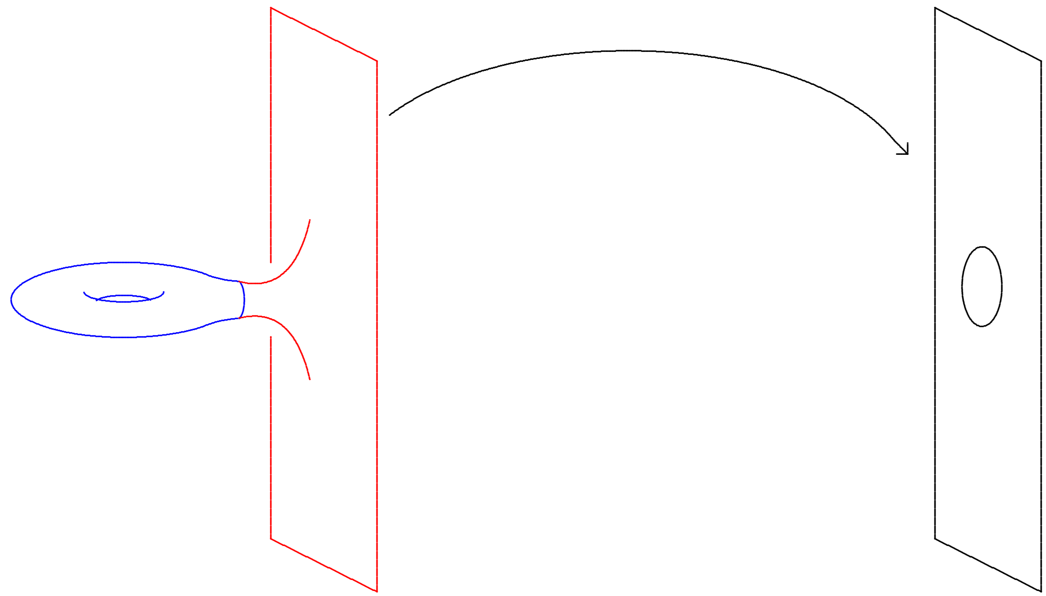

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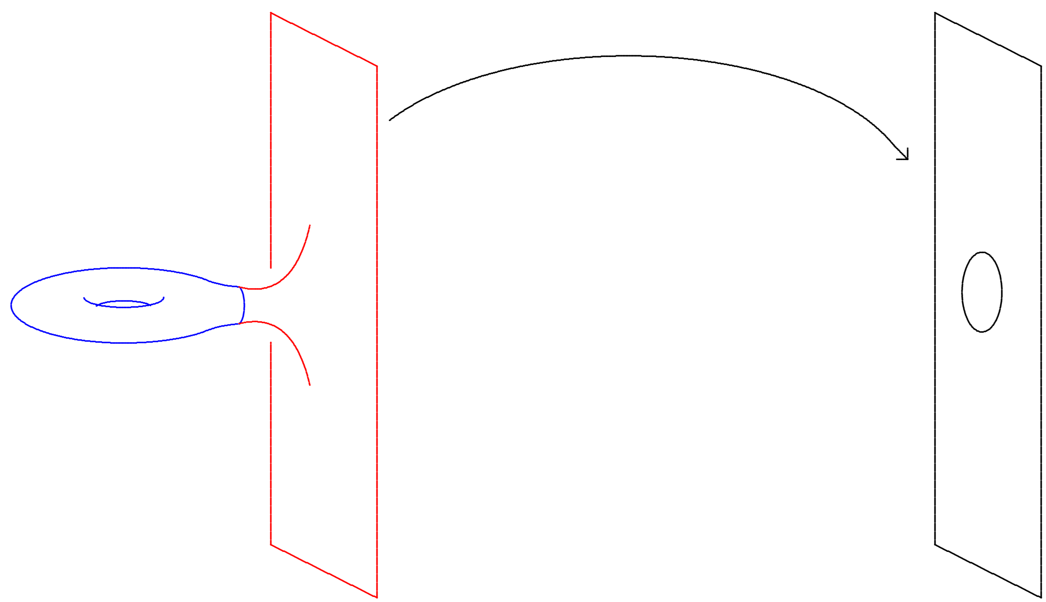

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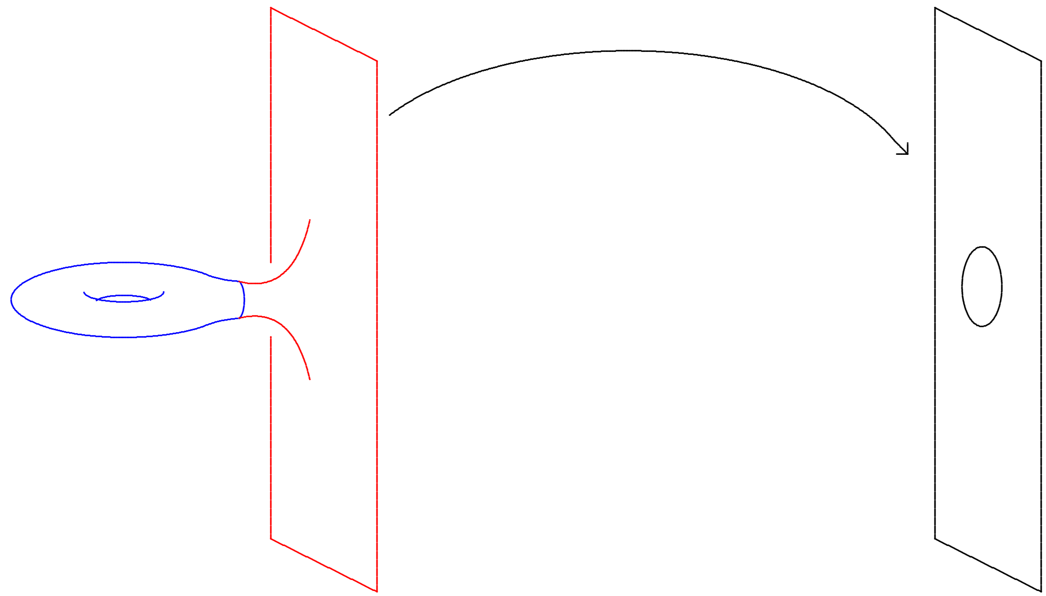

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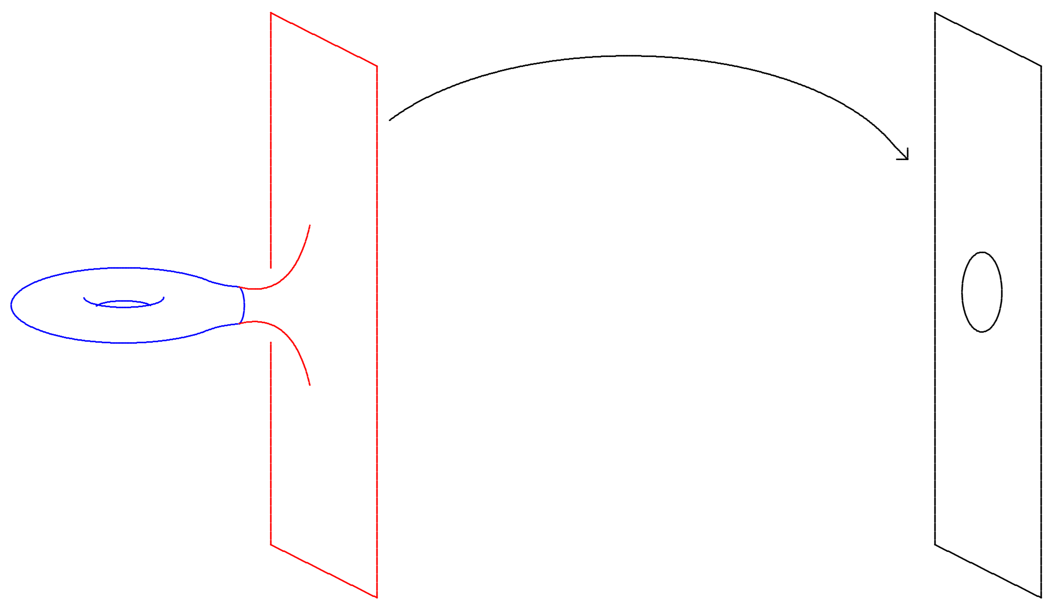

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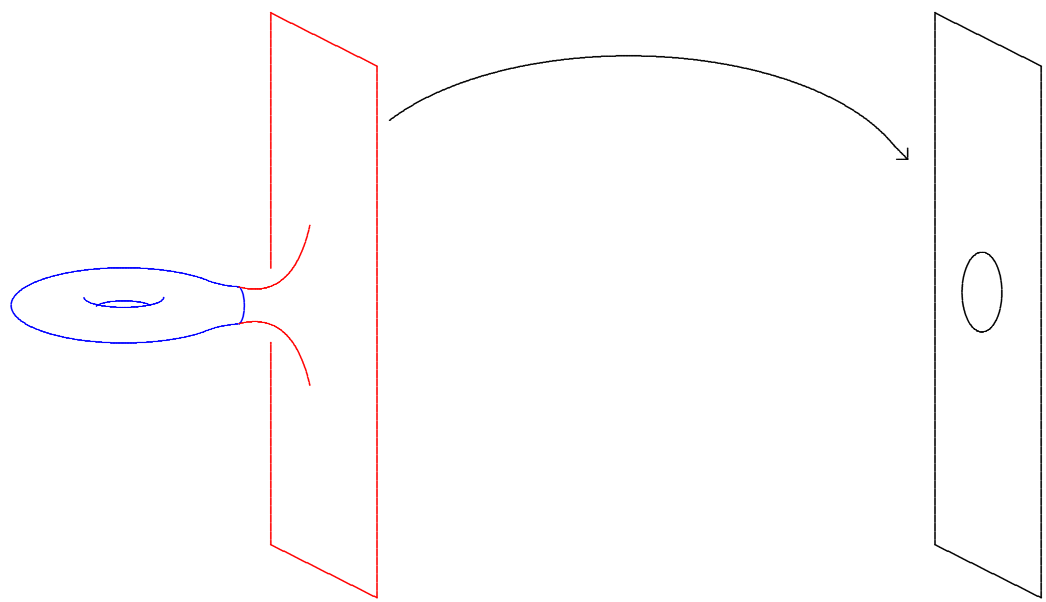

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Distance comparison...

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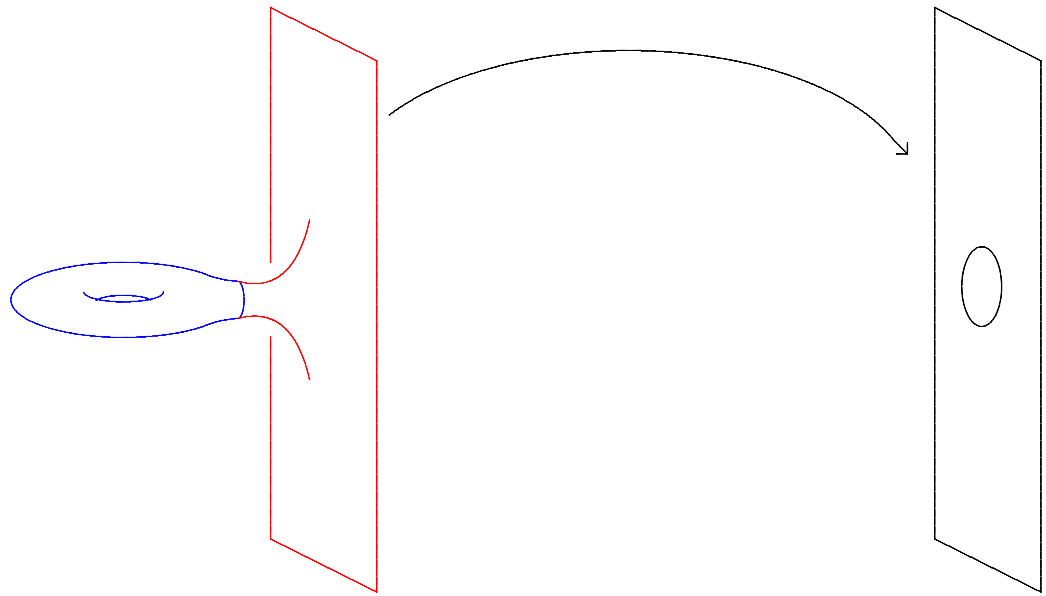

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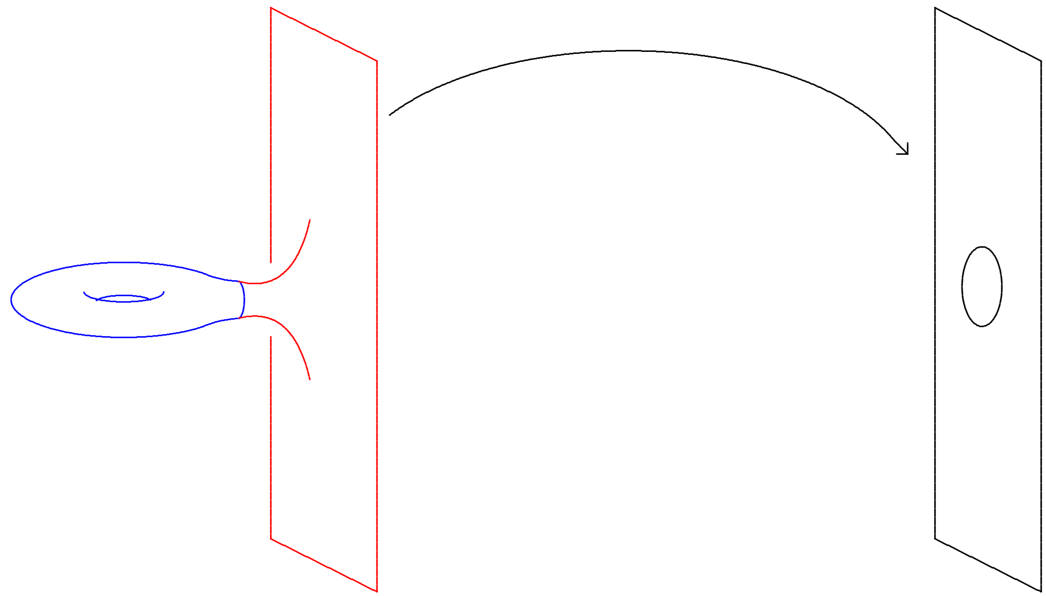

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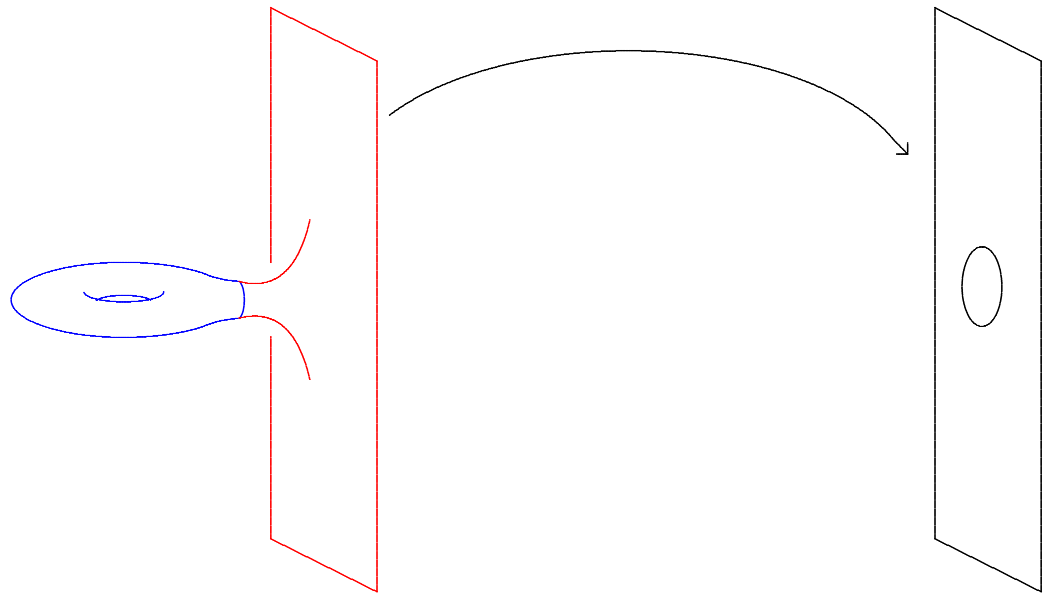

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Volume comparison...

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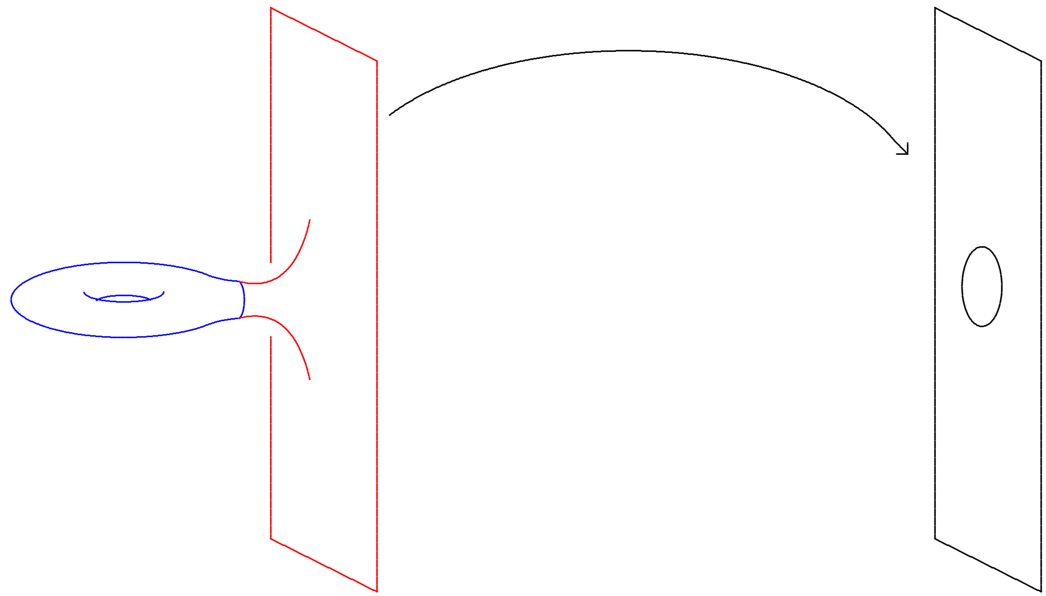

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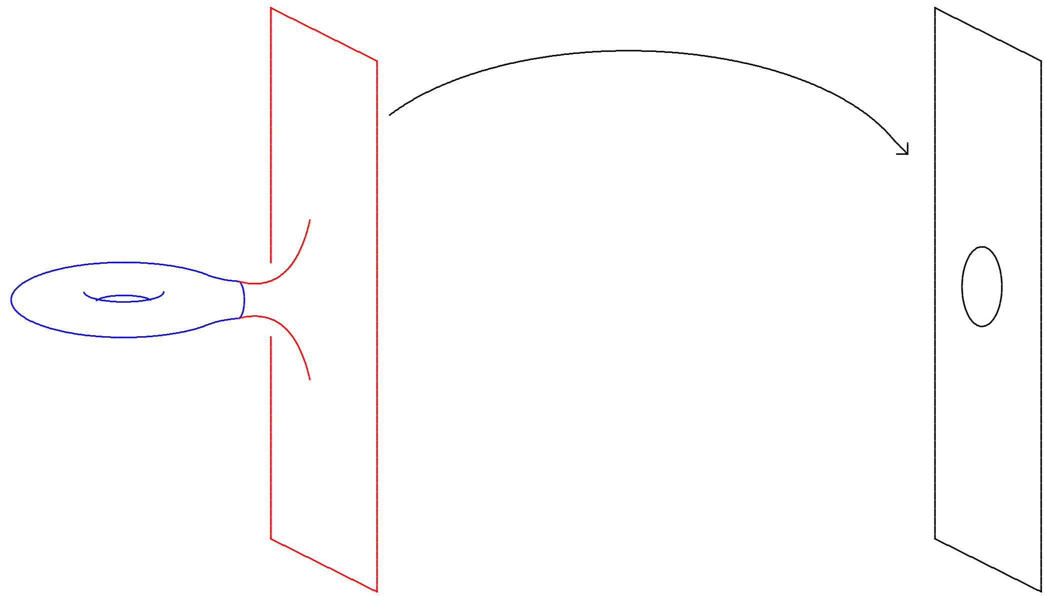

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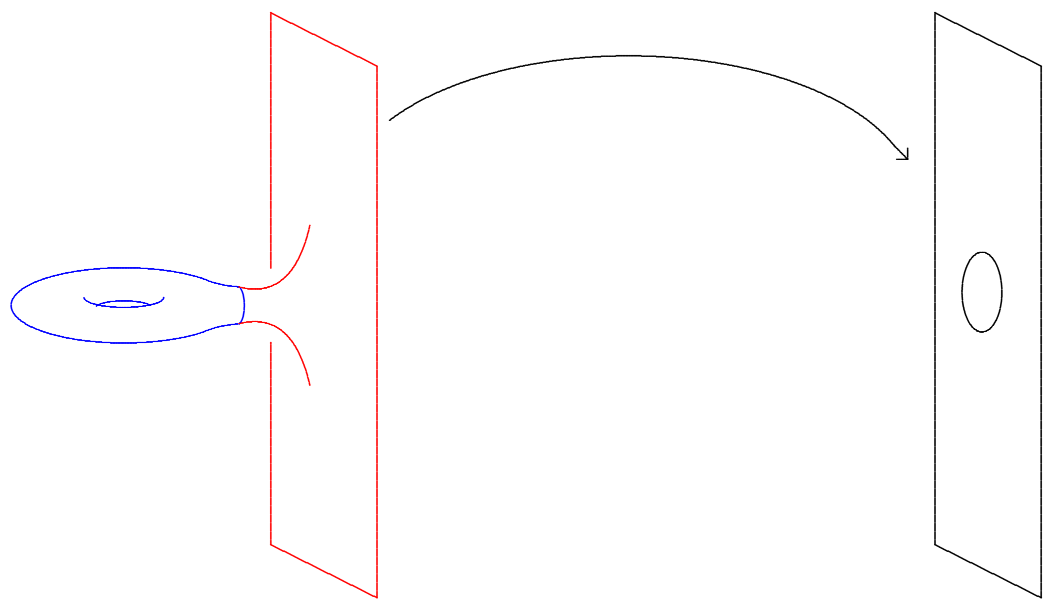

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at least if M spin, or if $n \leq 7$...

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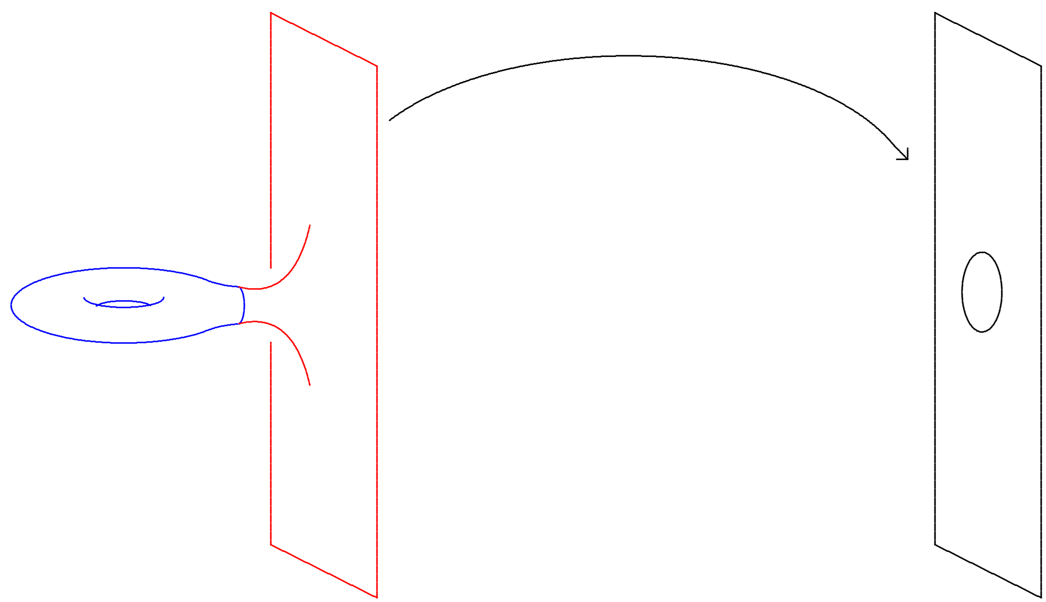

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Involves new idea from physics!

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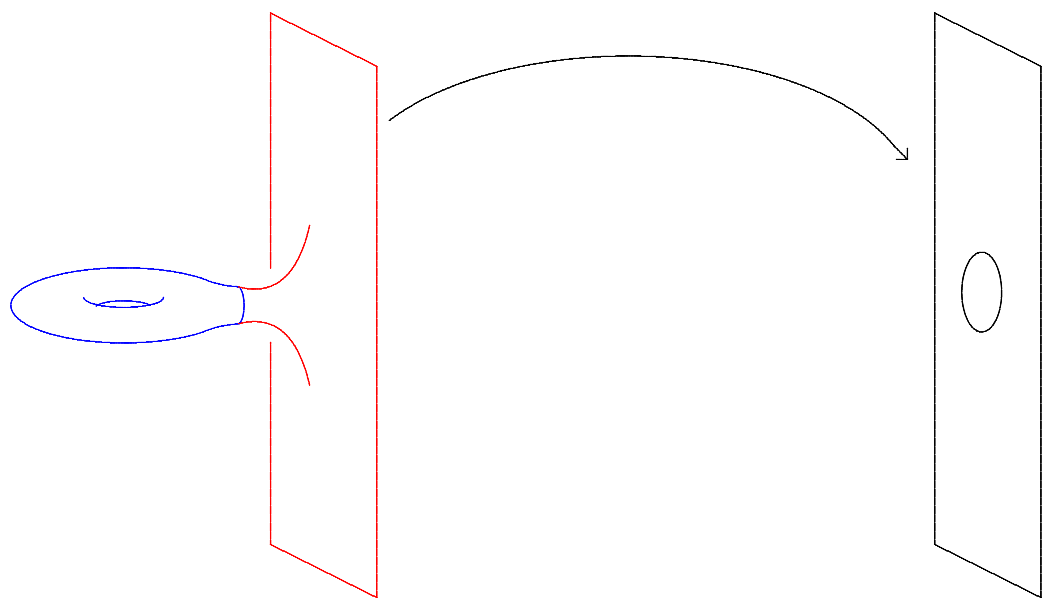

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Get result even with appropriate fall-off to Euclidean. . .

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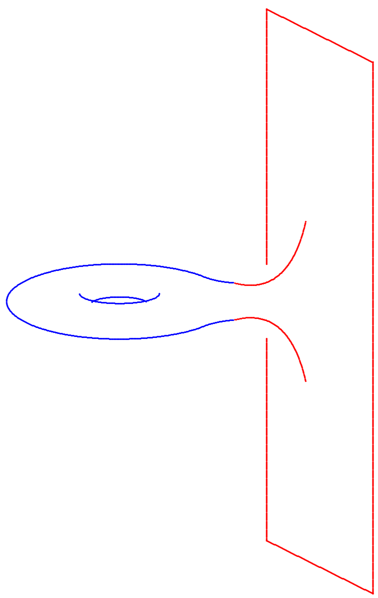

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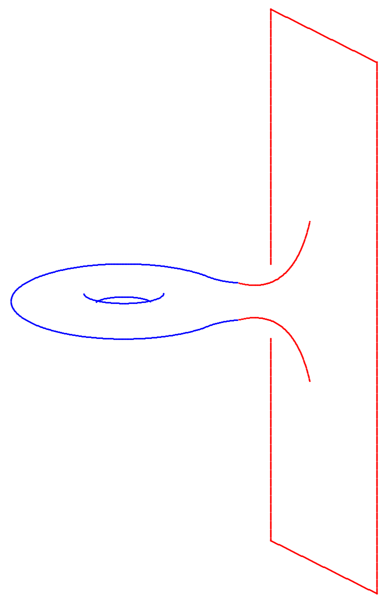

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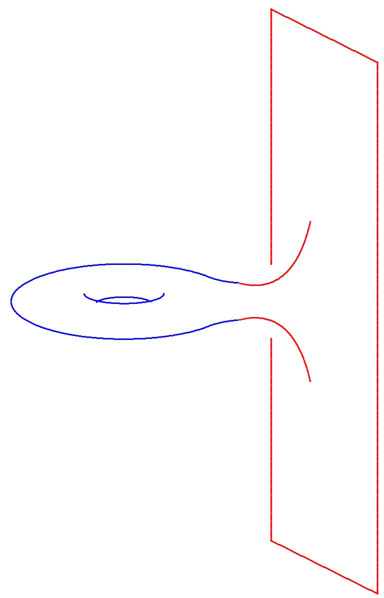

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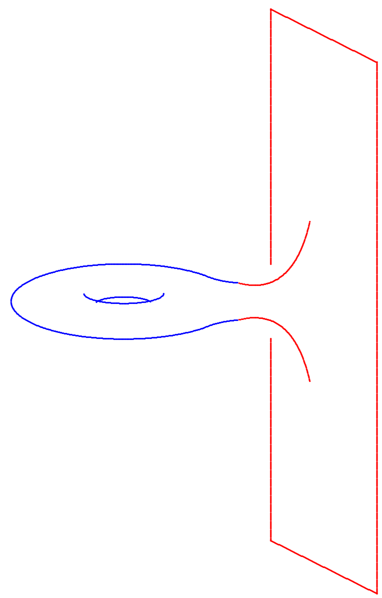

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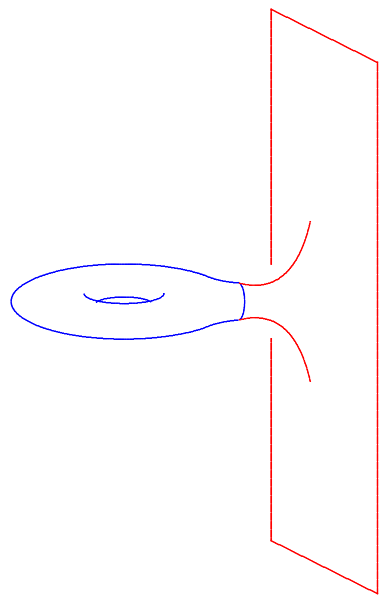

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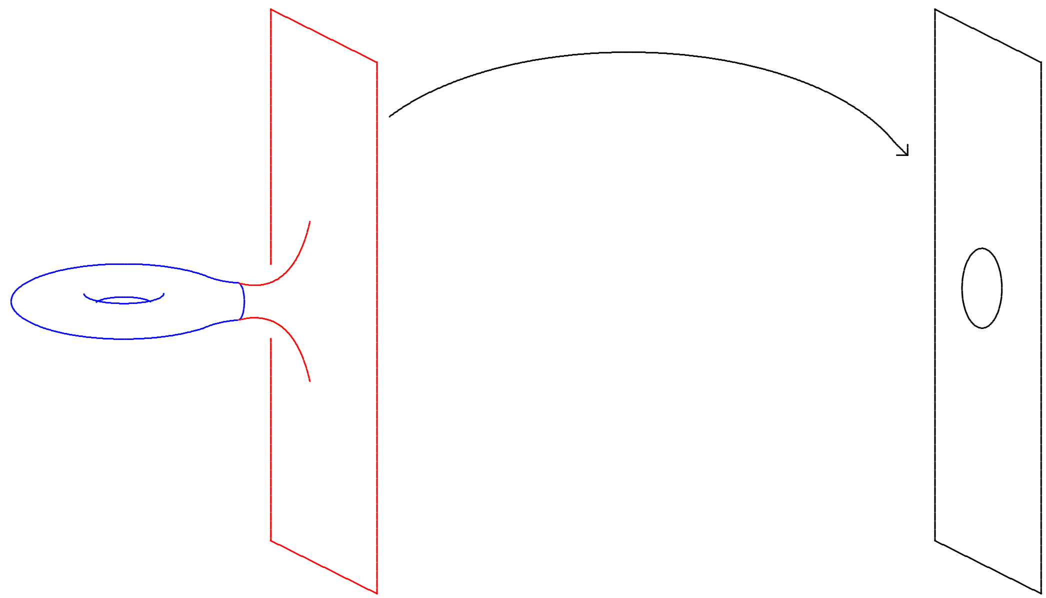

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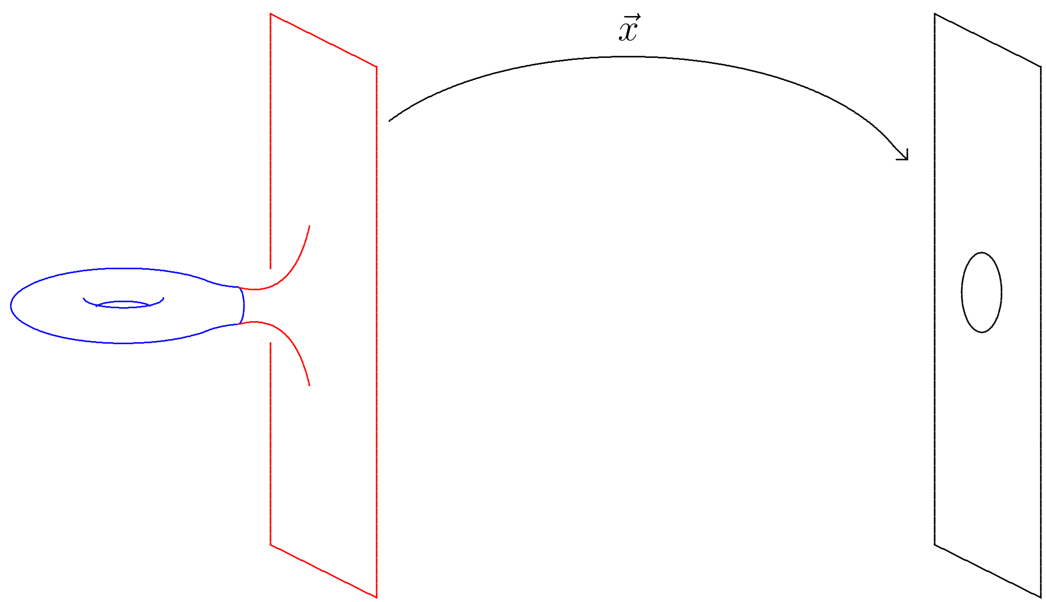

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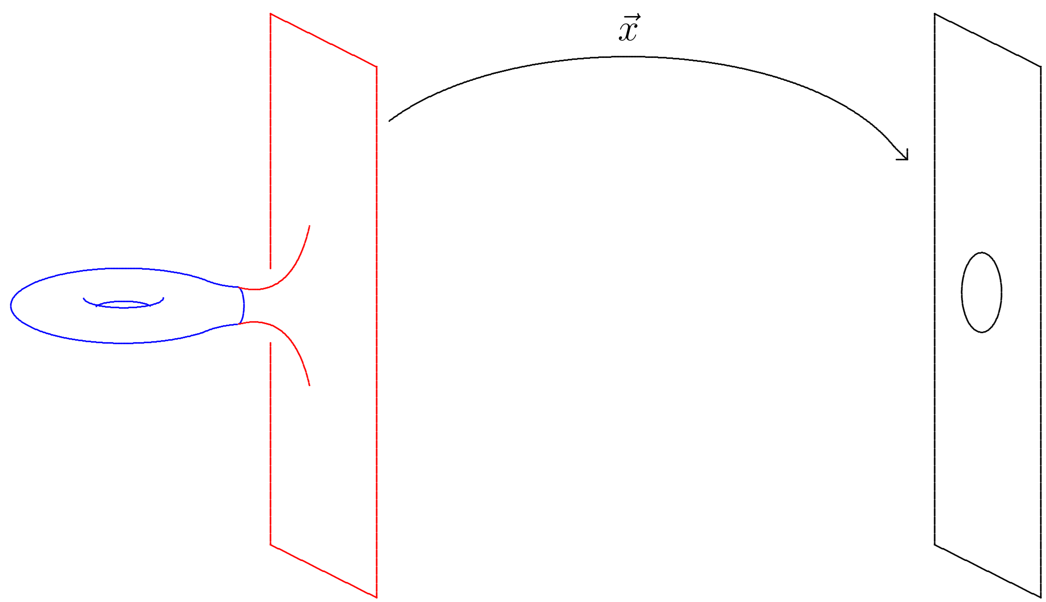

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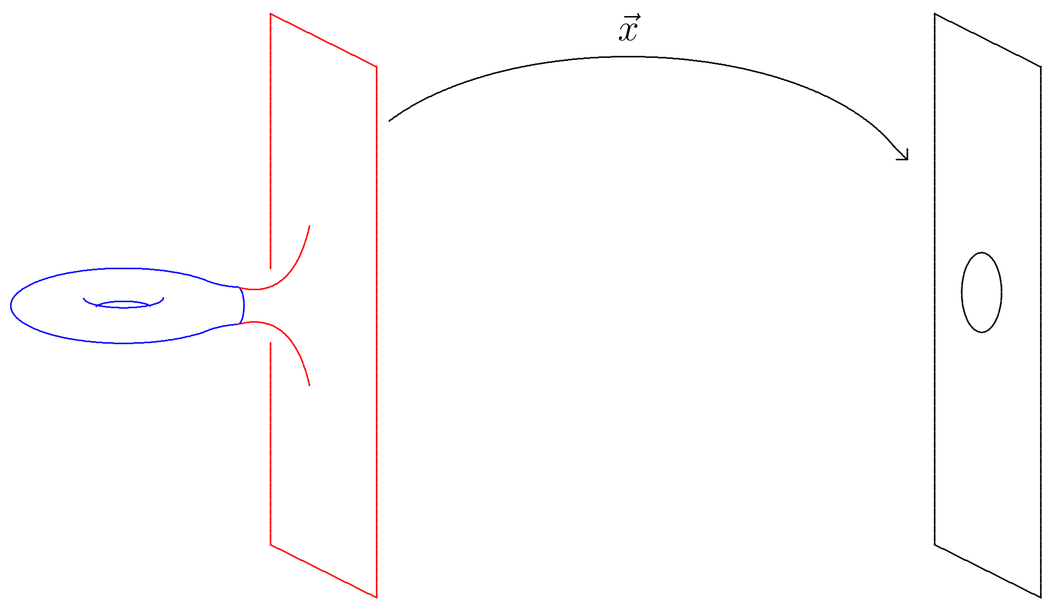

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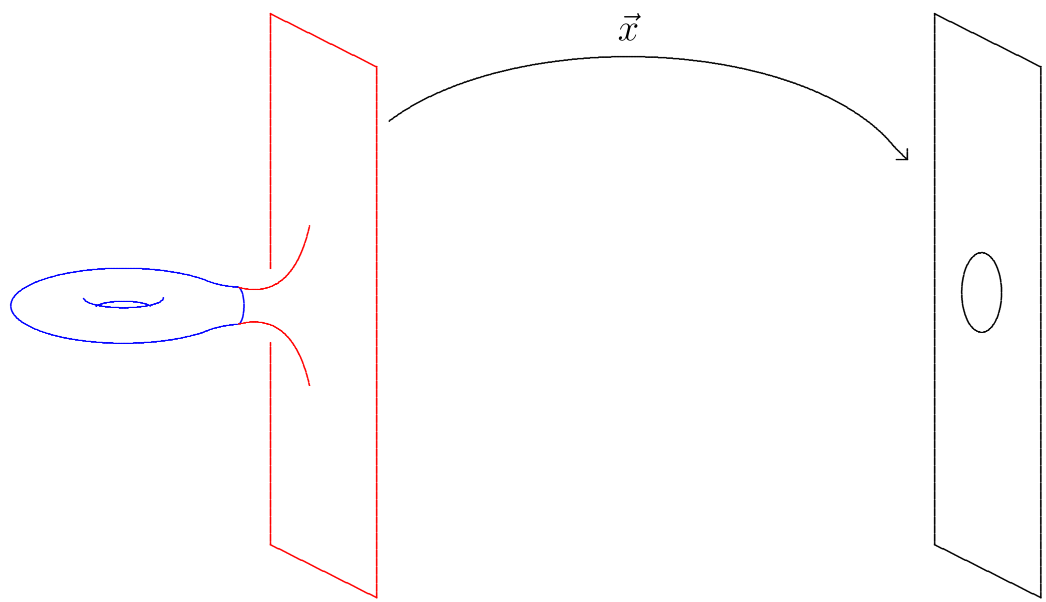

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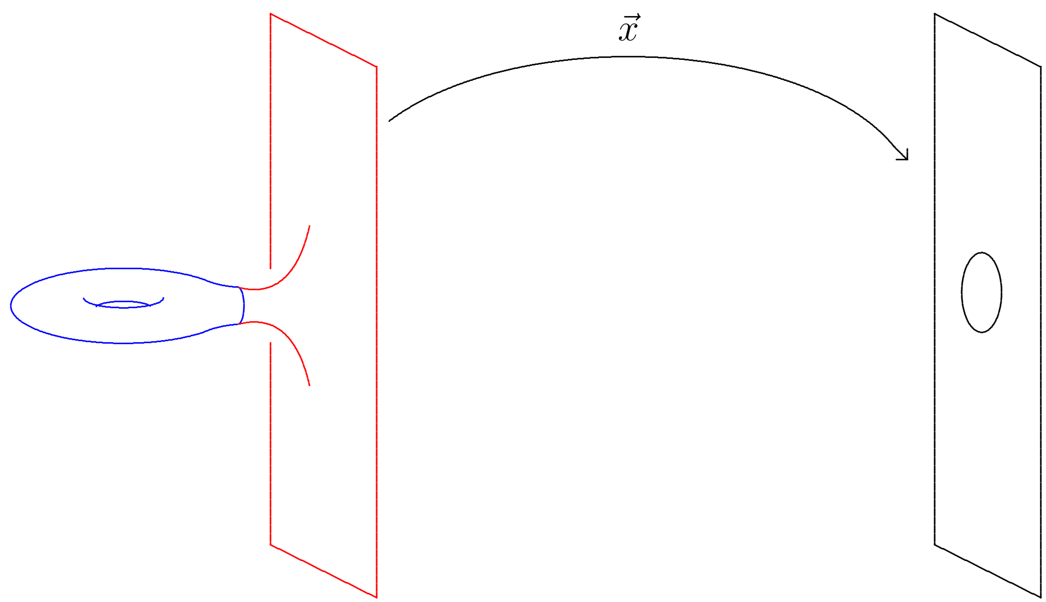

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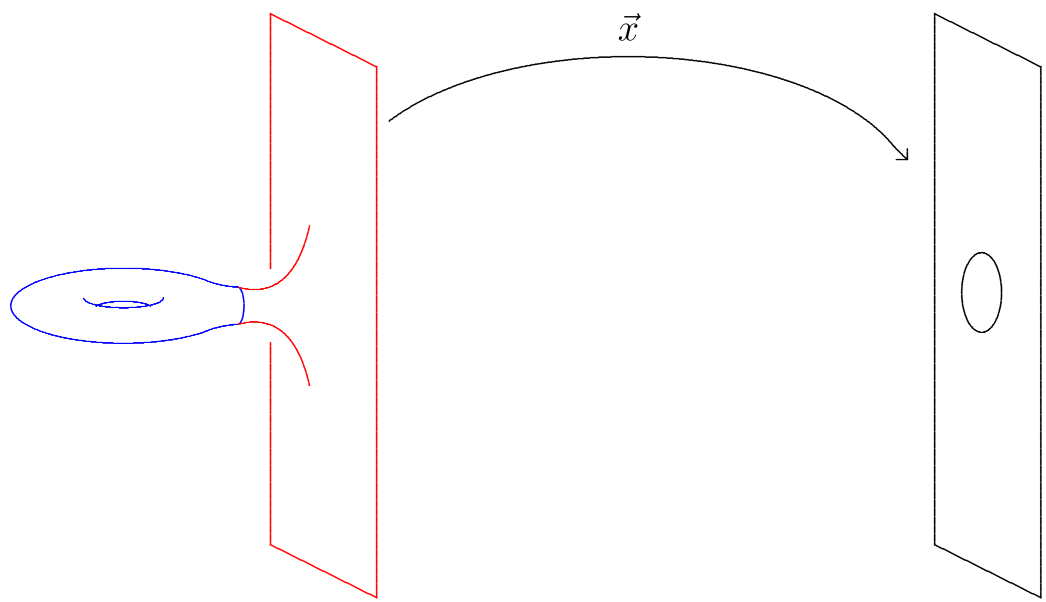

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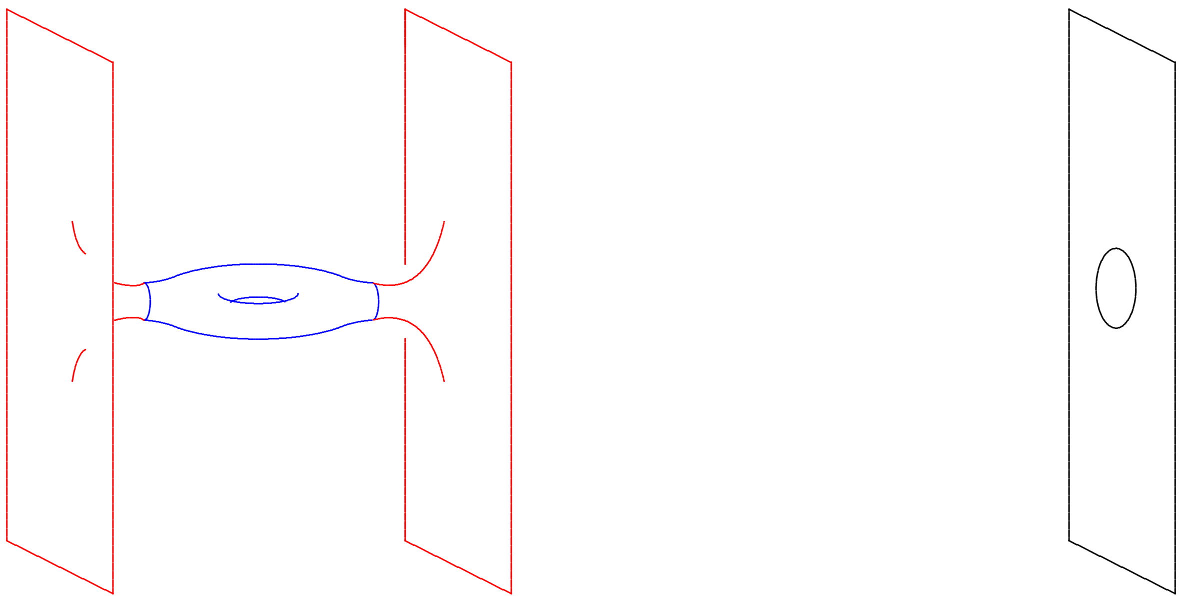

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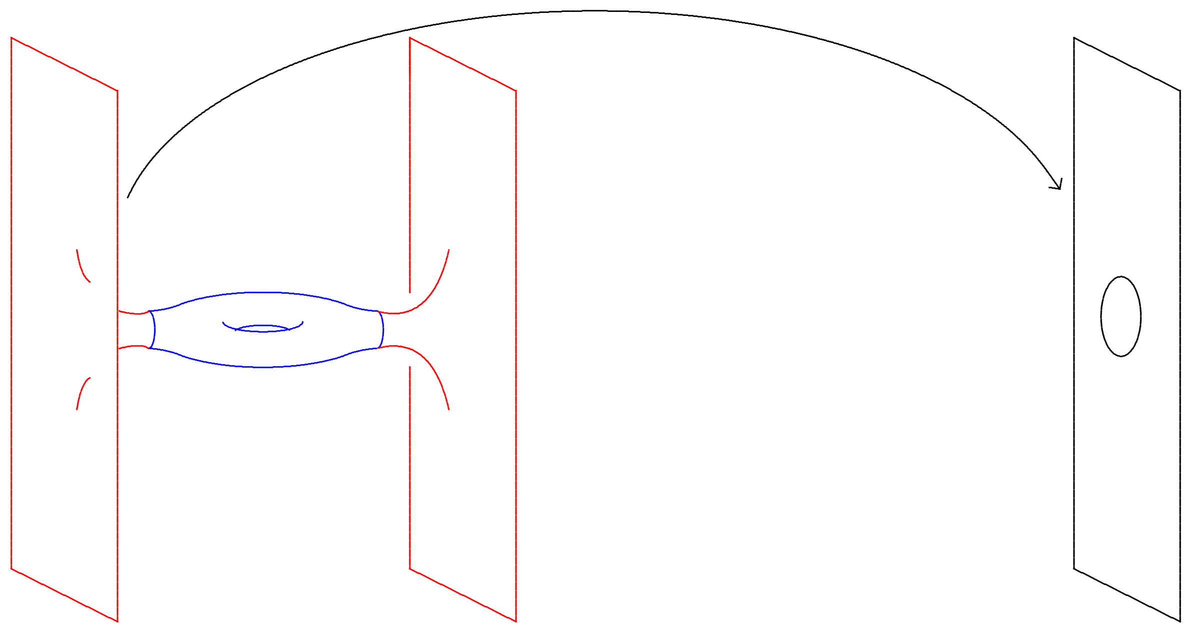

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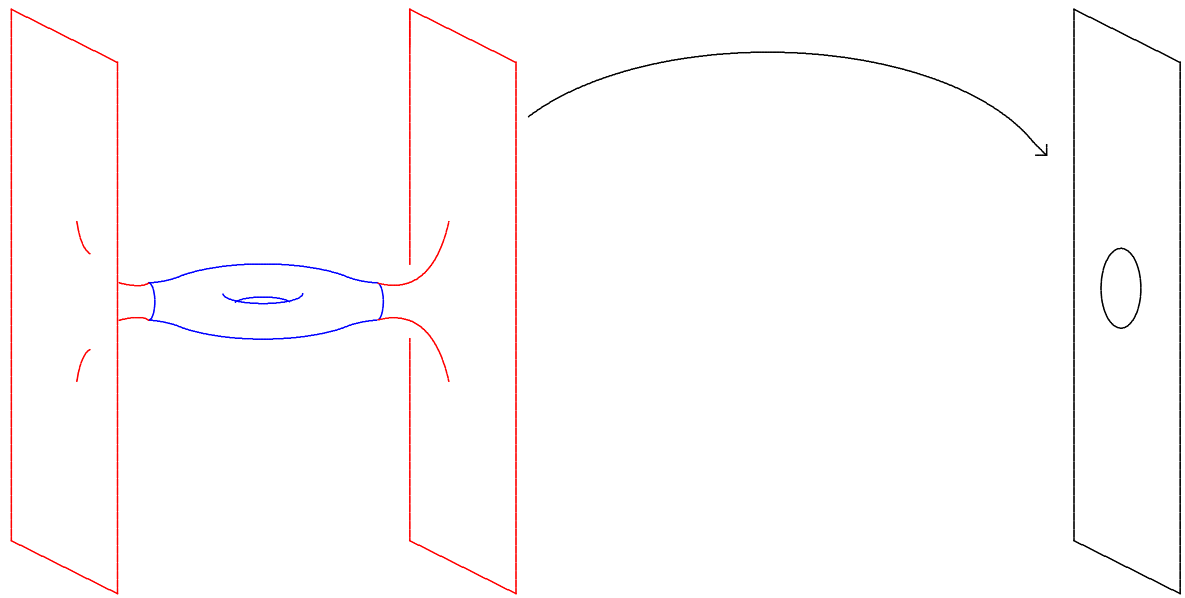

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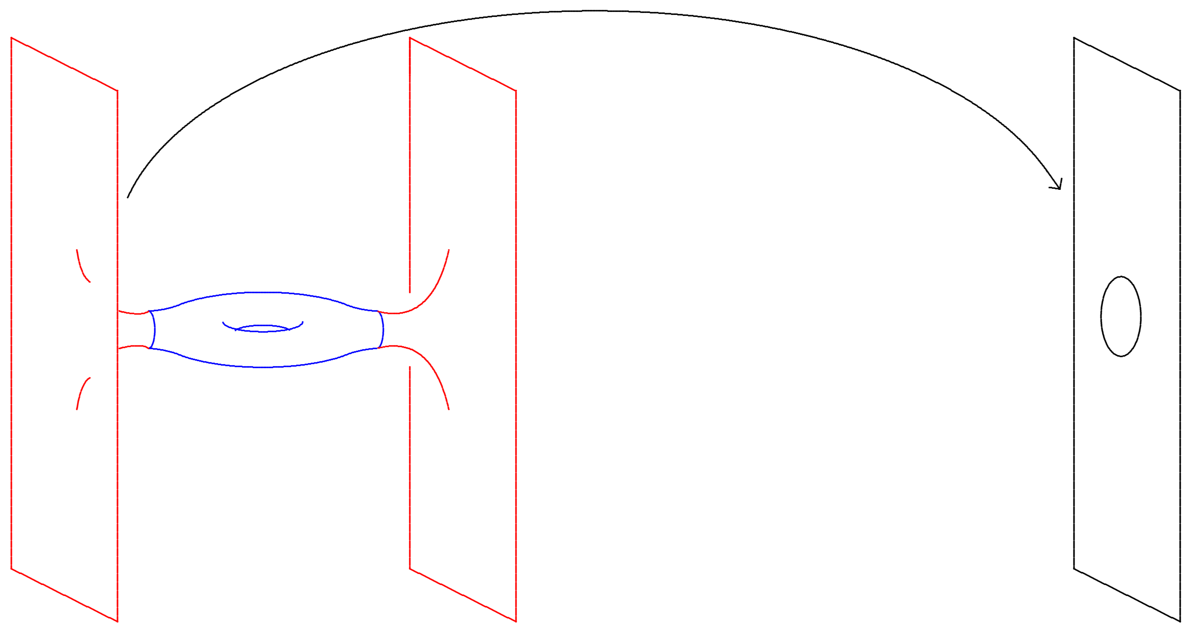

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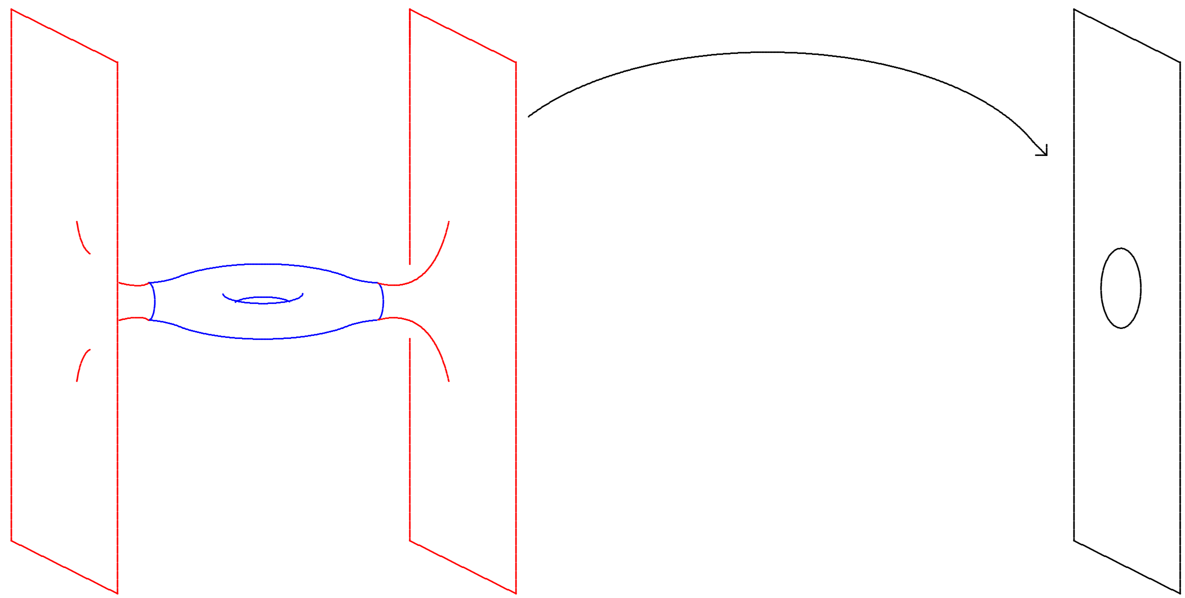

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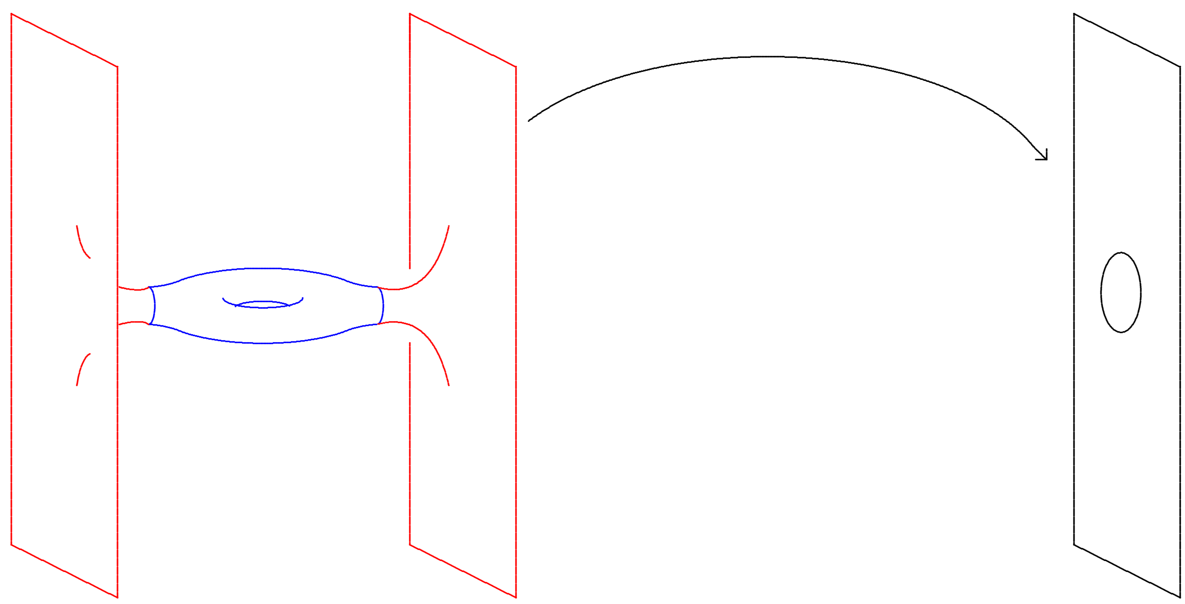

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Definition. *The* mass *(at a given end) of an* AE n*-manifold is defined to be*

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Seems to depend on choice of coordinates!

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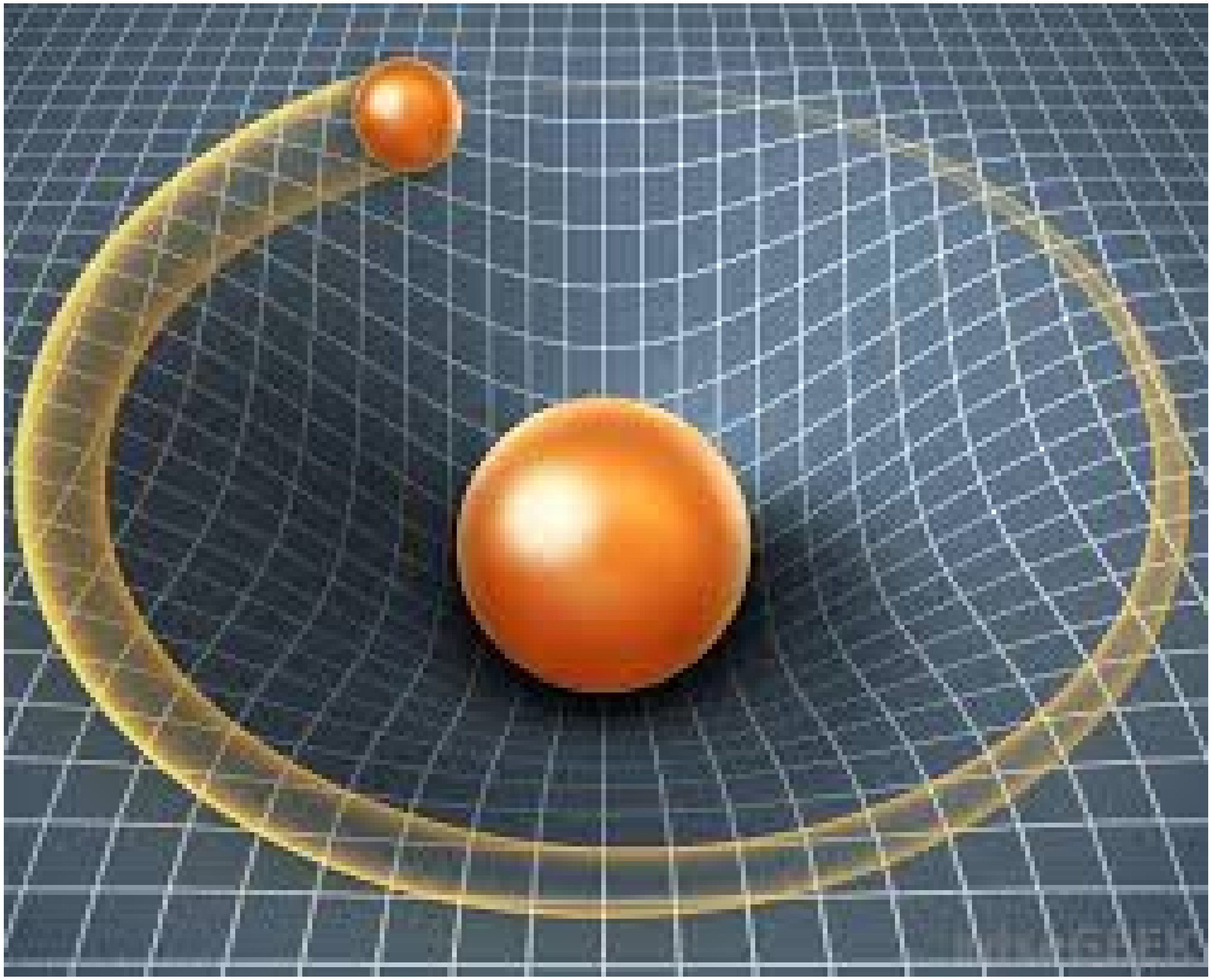

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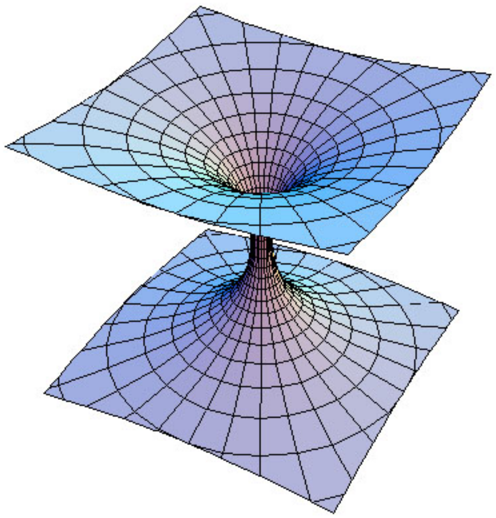

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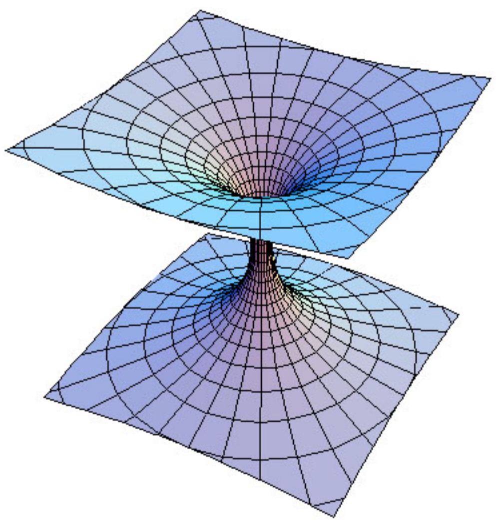

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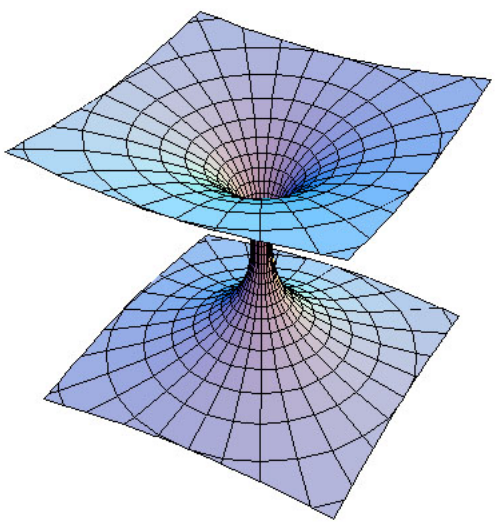

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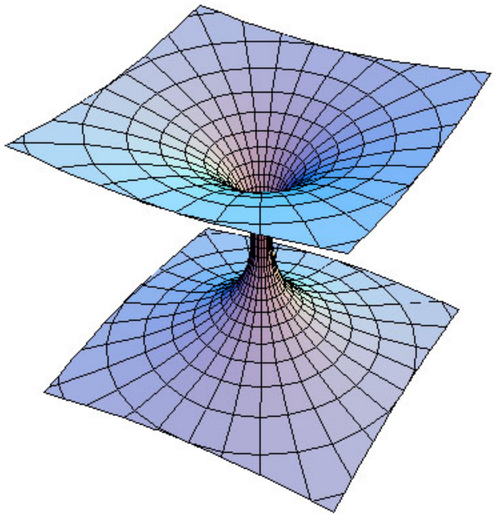

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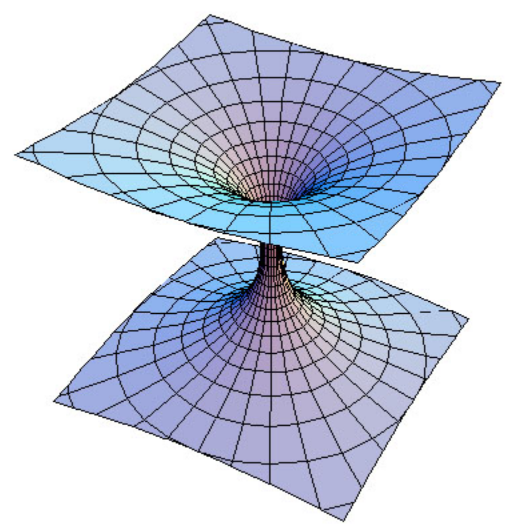

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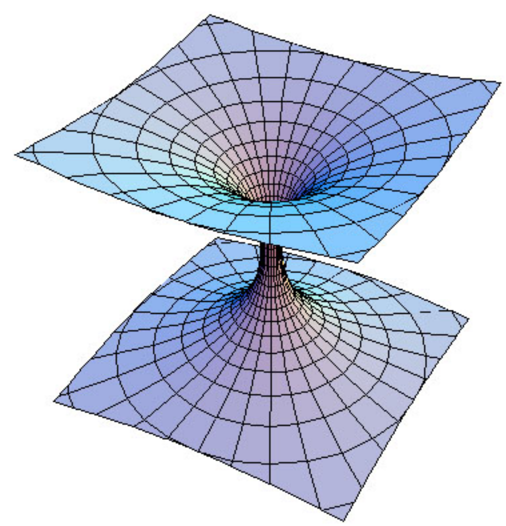

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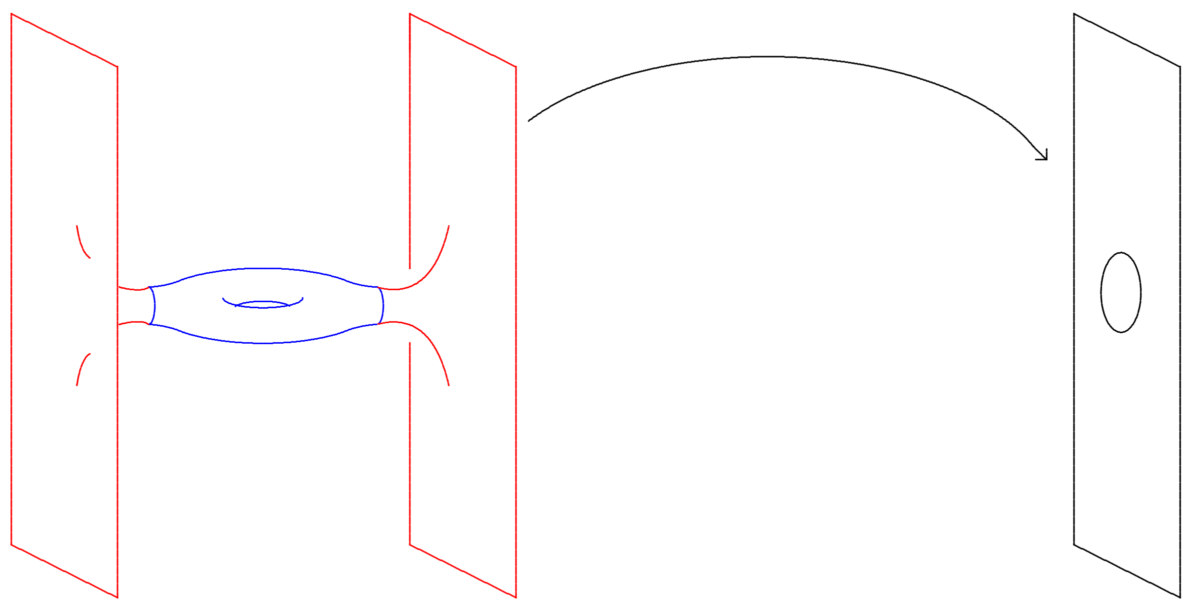

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A Generalization. . .

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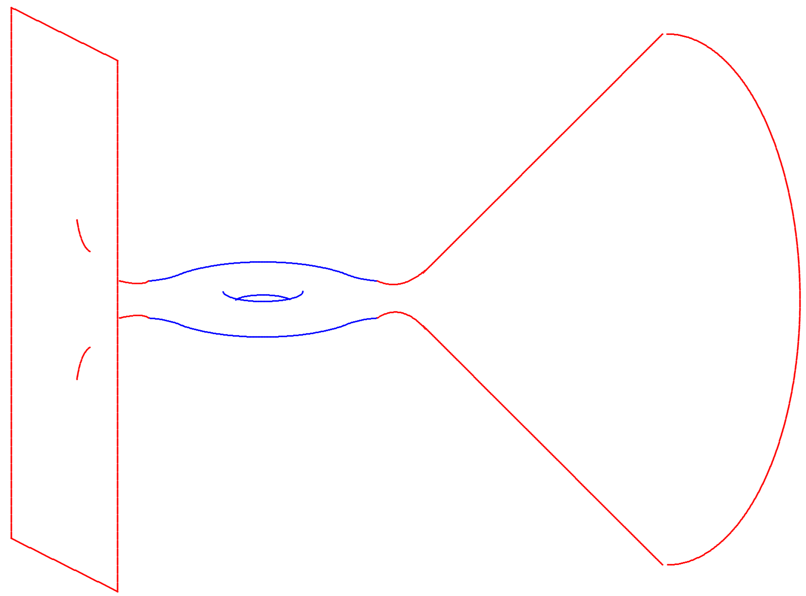

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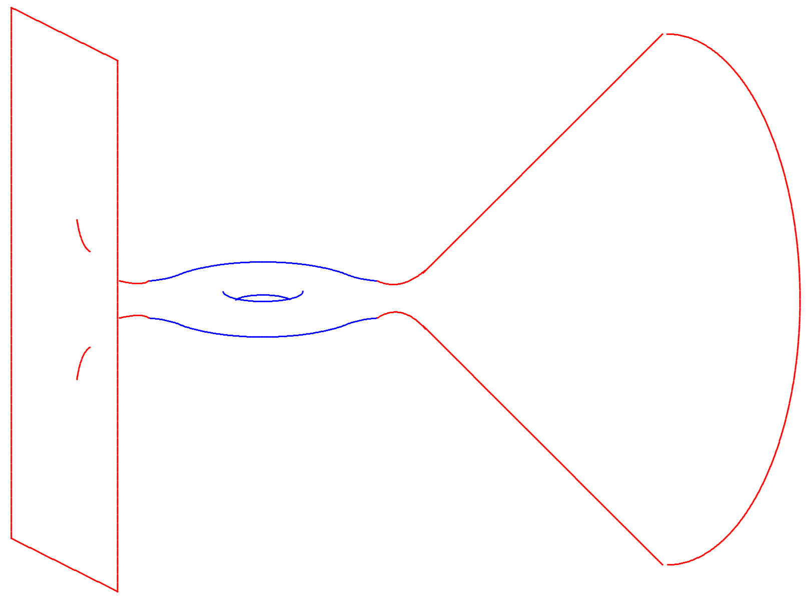

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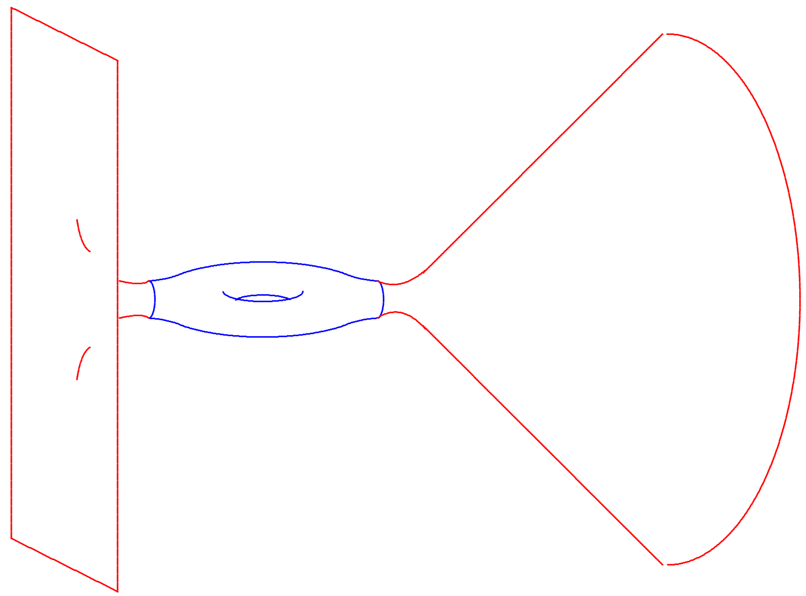

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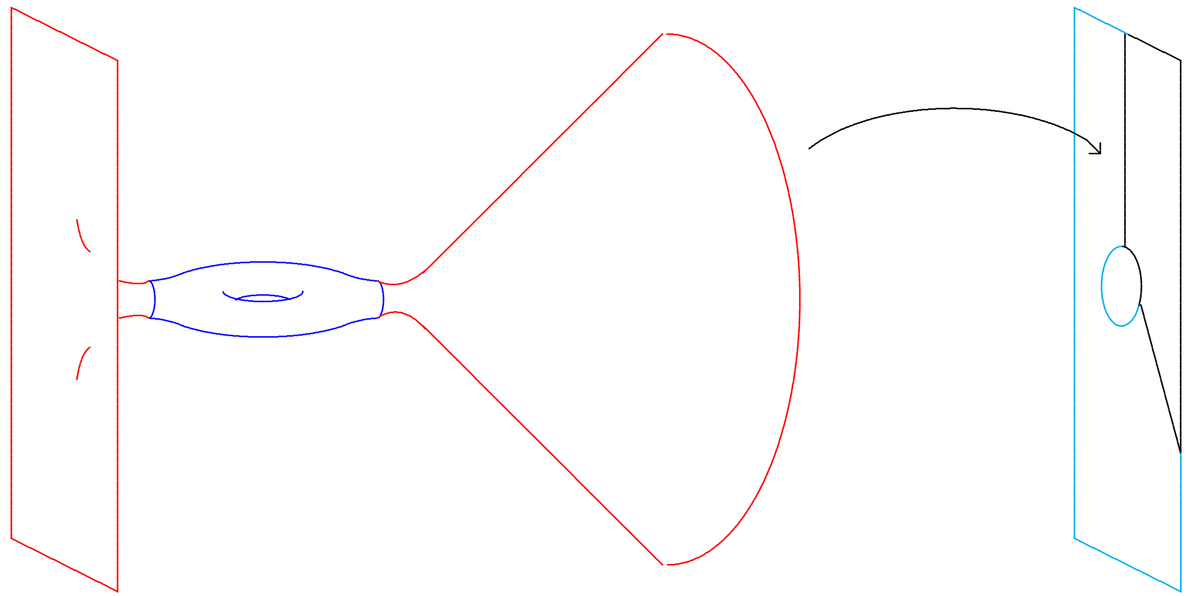

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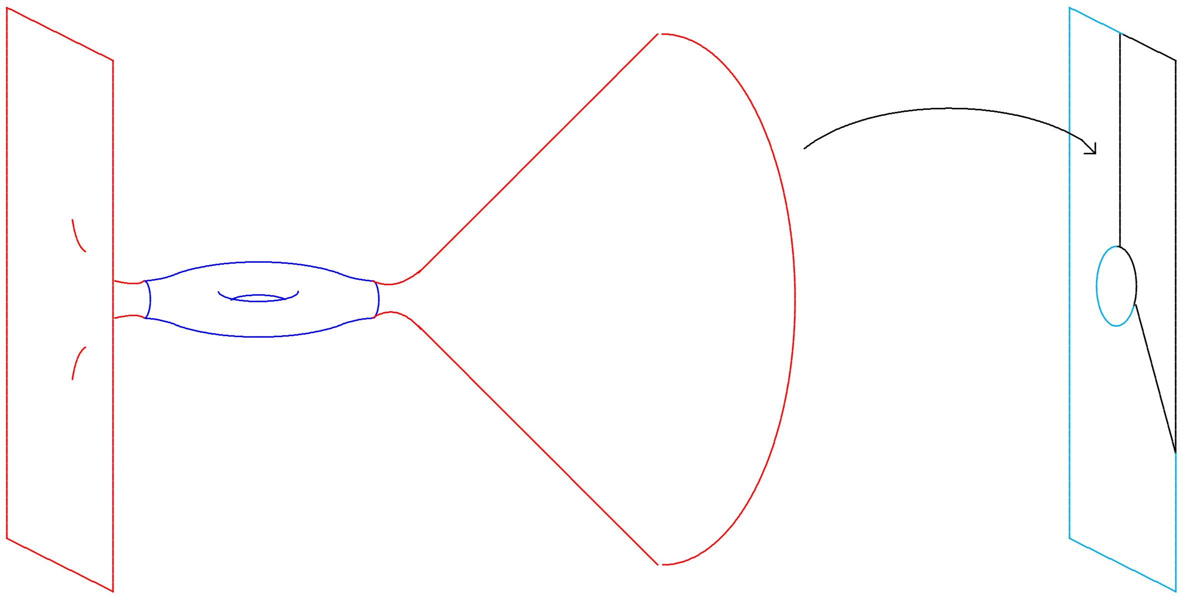

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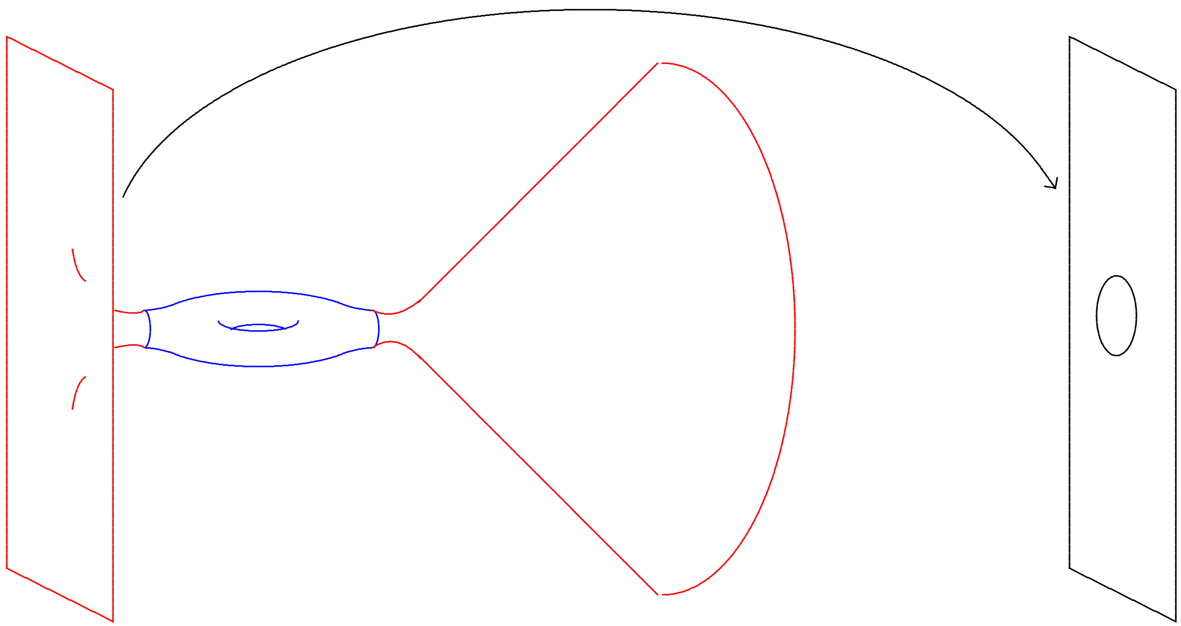

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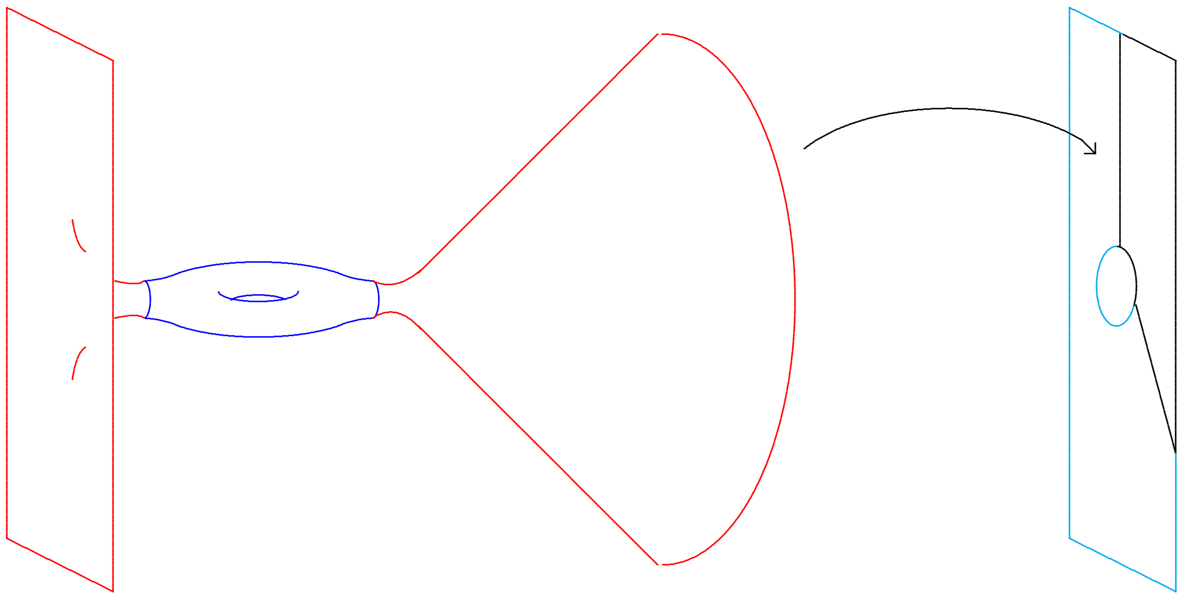

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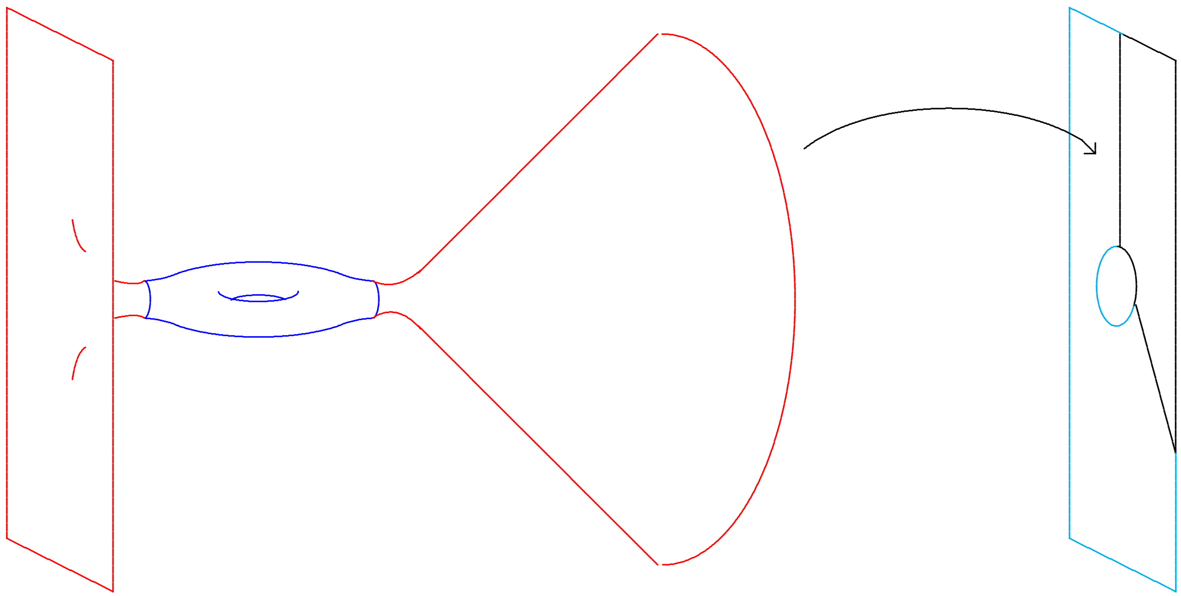

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Why consider ALE spaces?

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By contrast, any Ricci-flat AE manifold must be flat, by the Bishop-Gromov inequality…

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The G-H metrics are hyper-Kähler, and were soon rediscovered independently by Hitchin.

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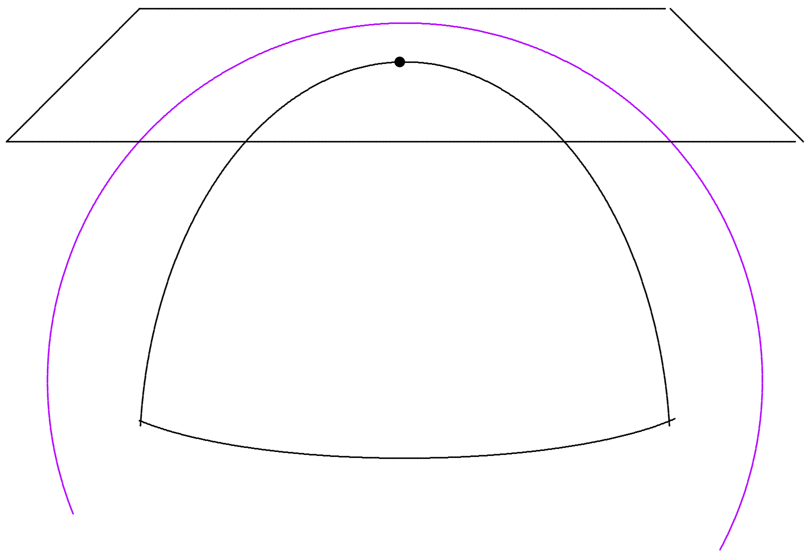

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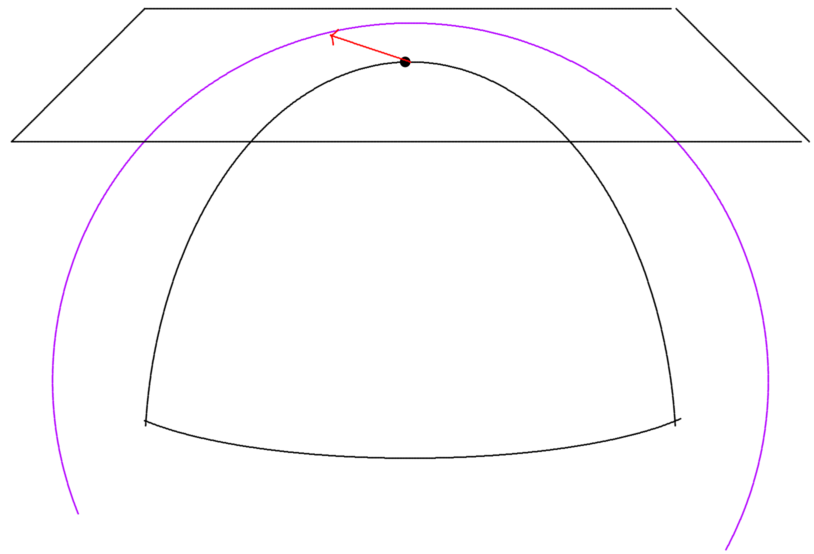

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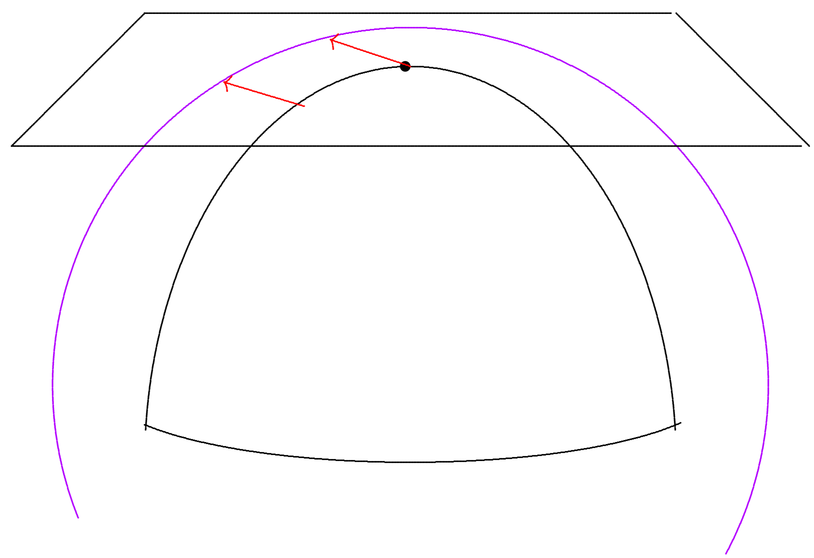

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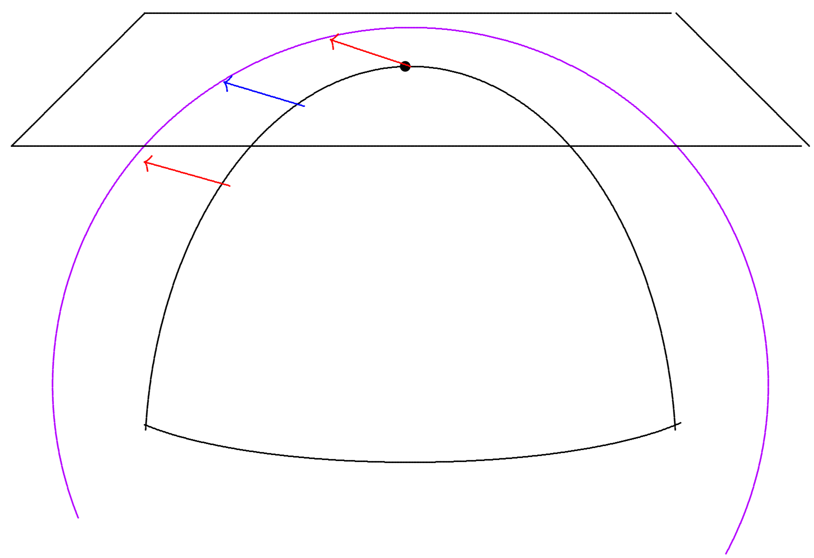

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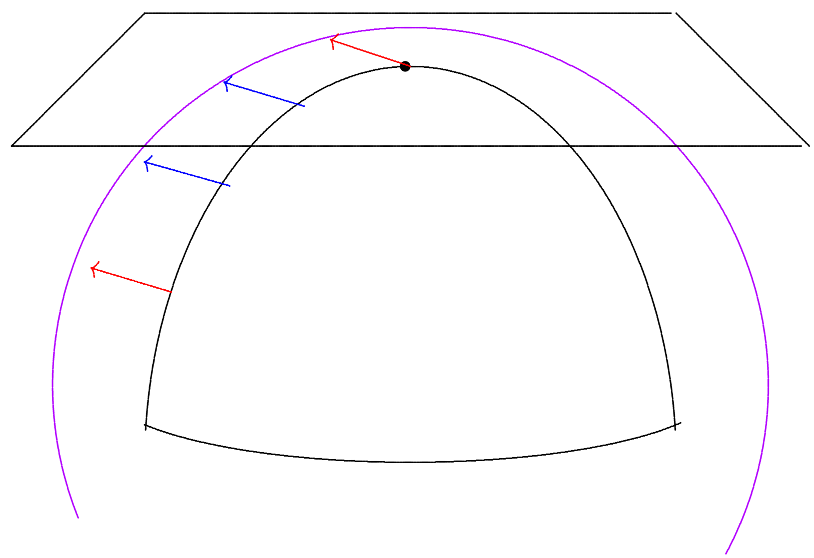

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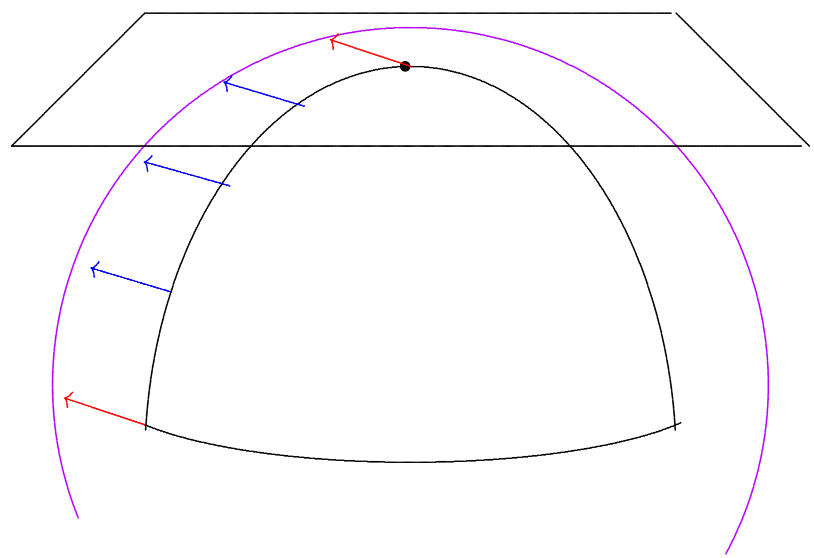

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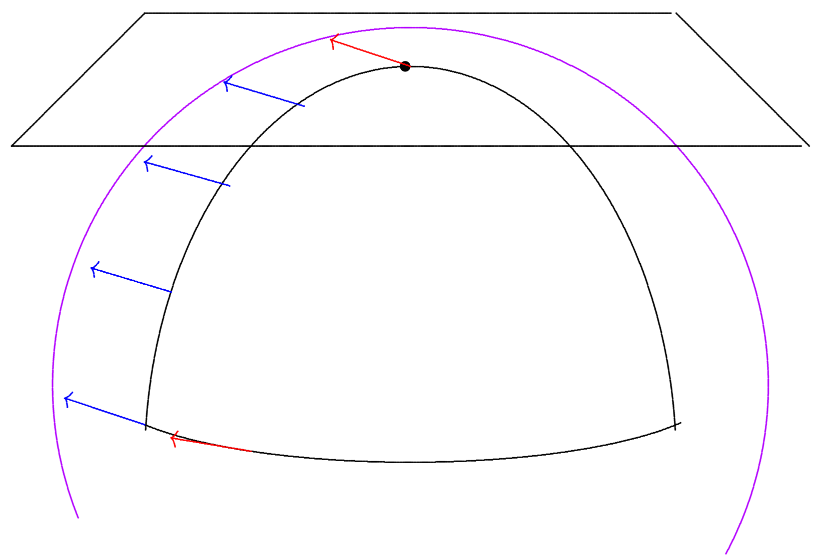

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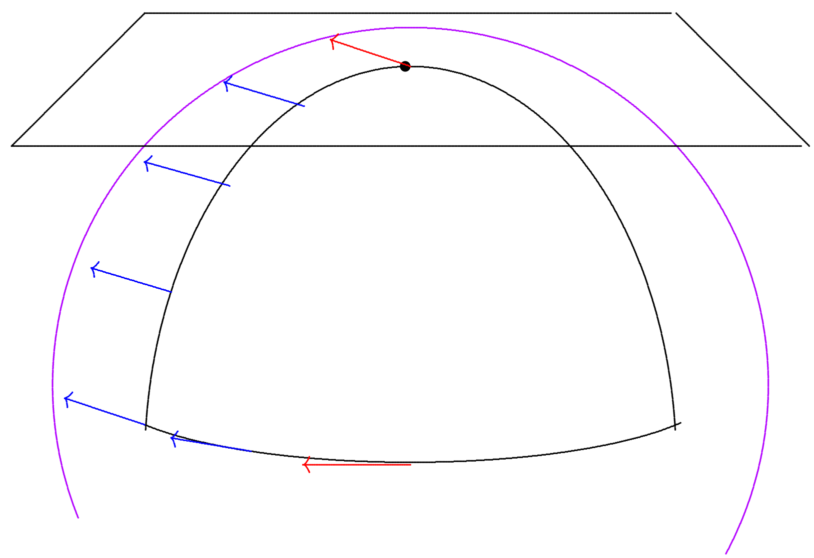

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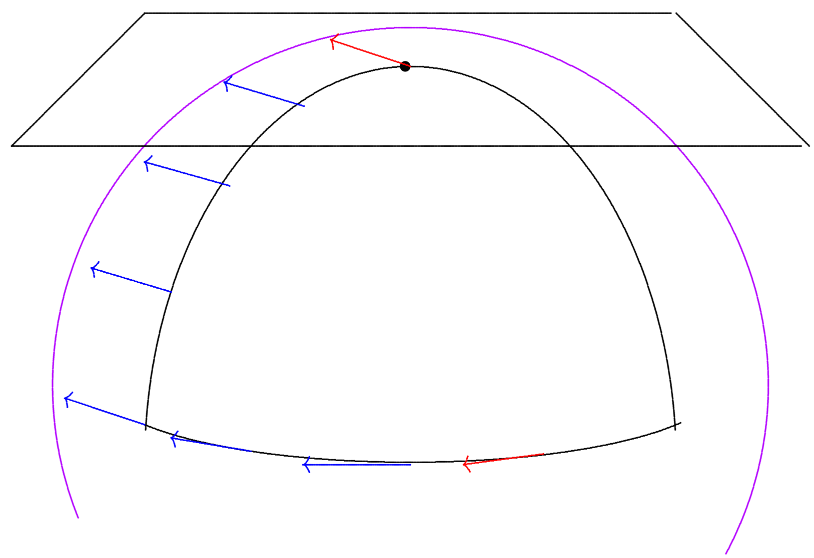

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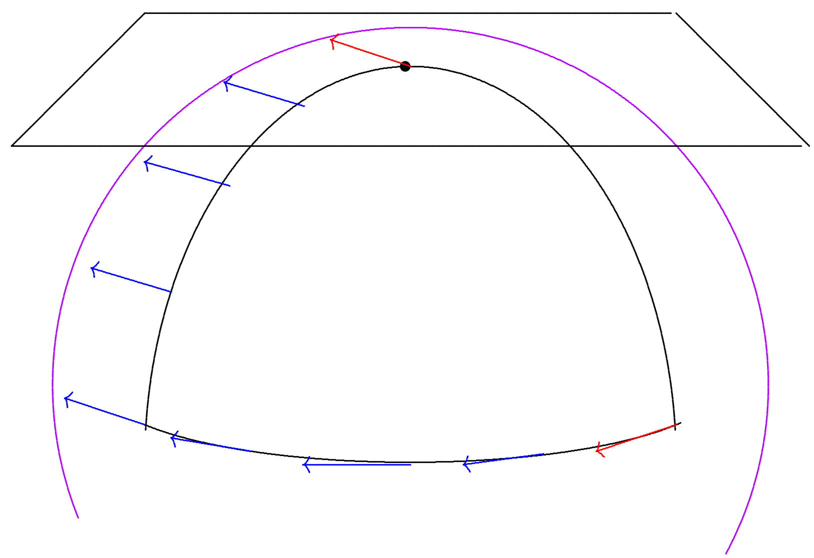

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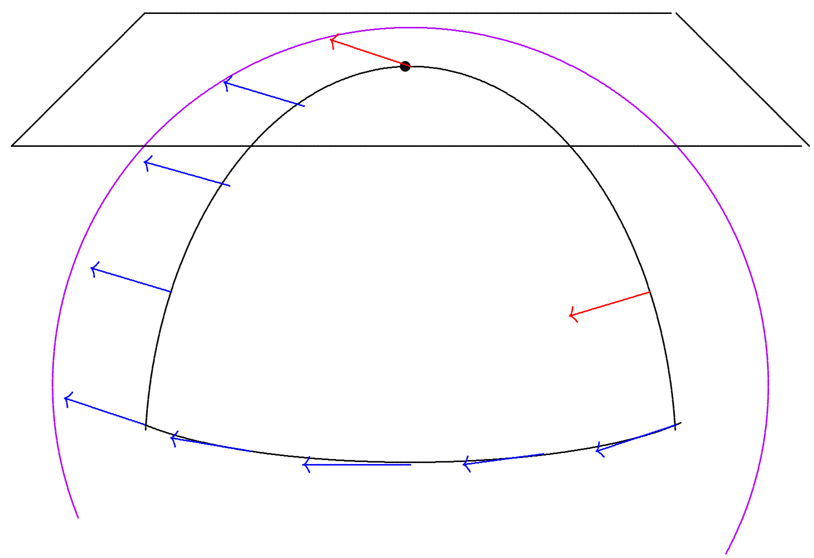

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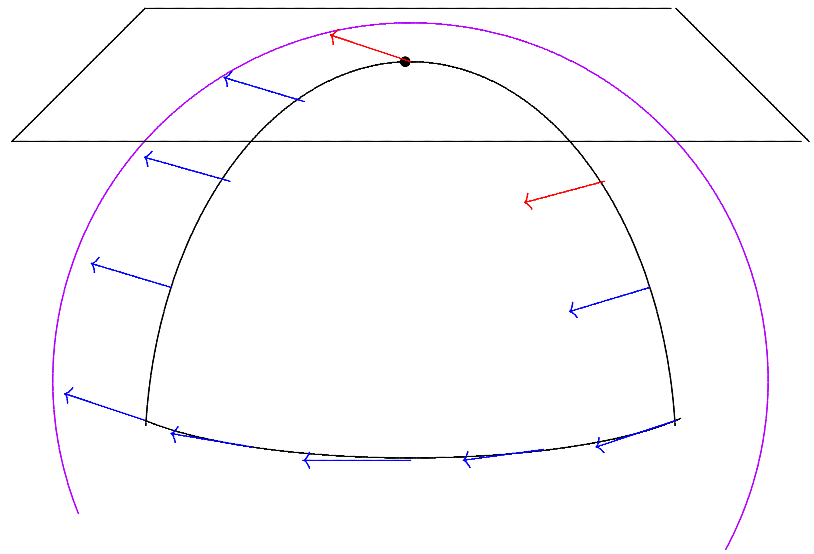

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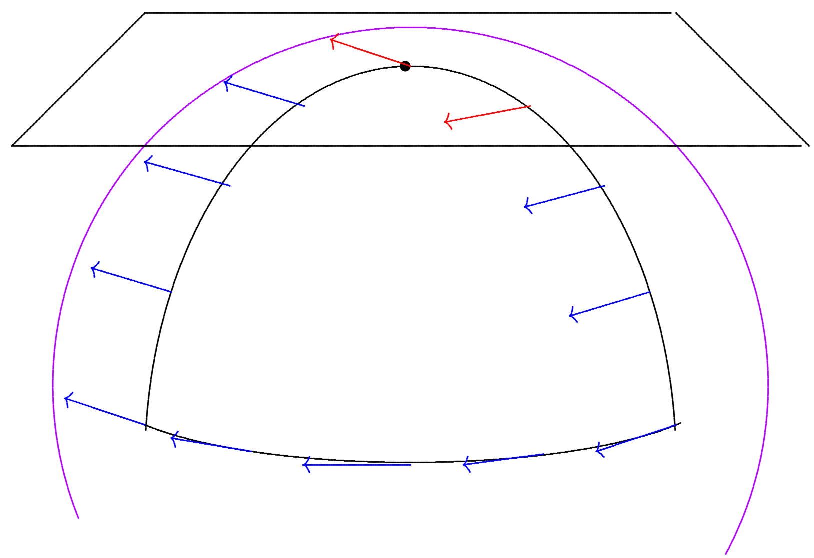
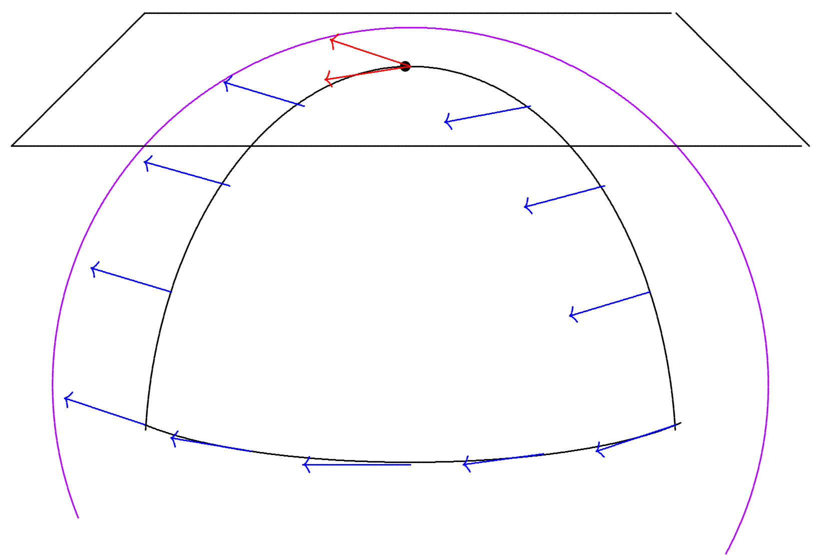


(M^n, g):

holonomy



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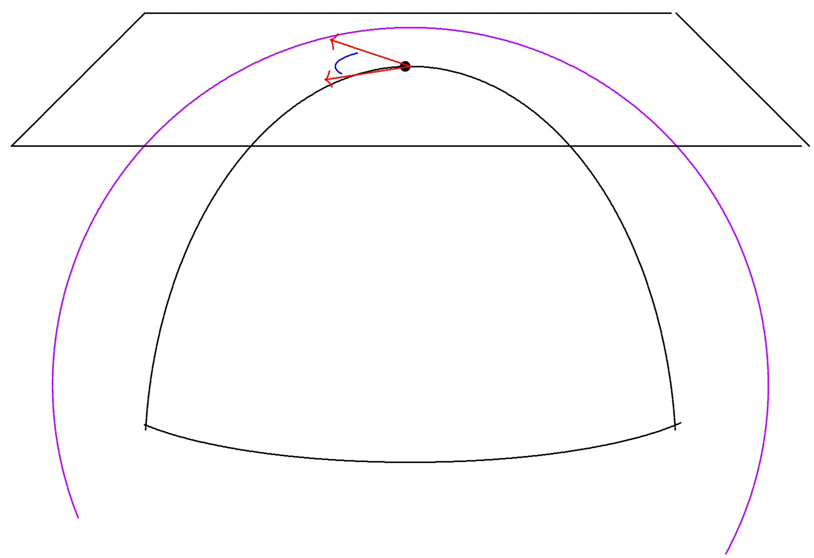
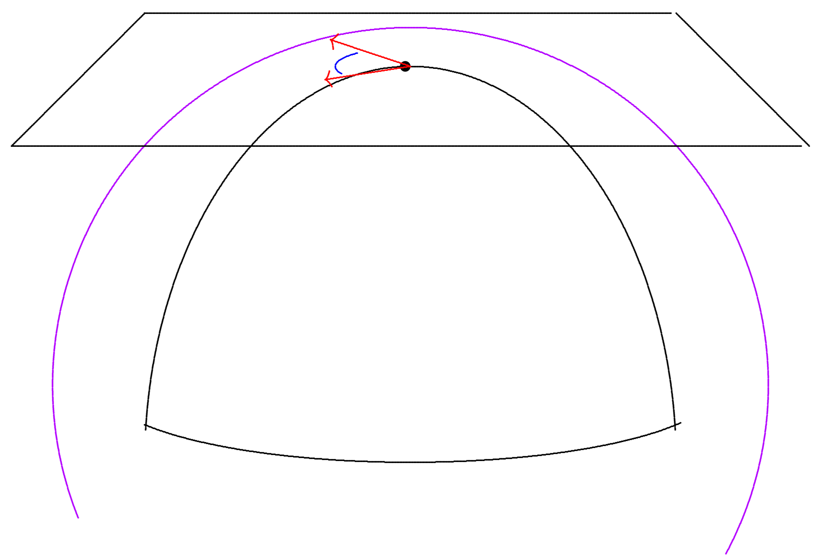
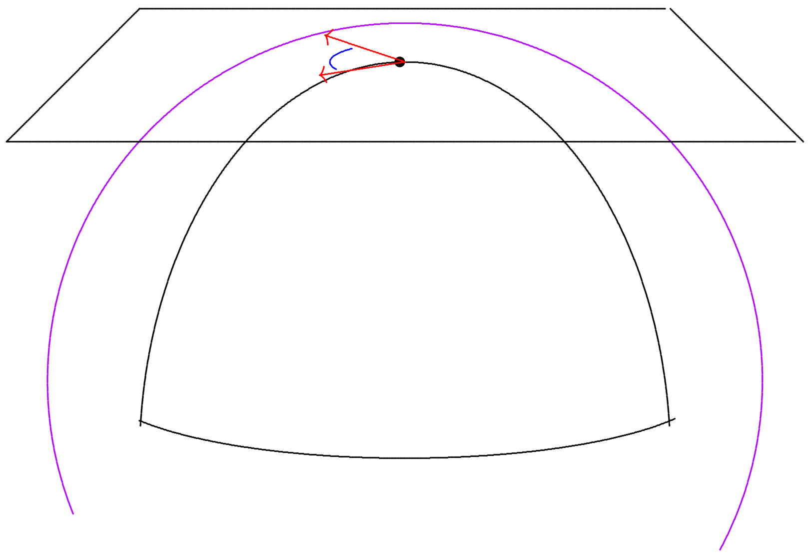


(M^n, g): holonomy $\subset \mathbf{O}(n)$



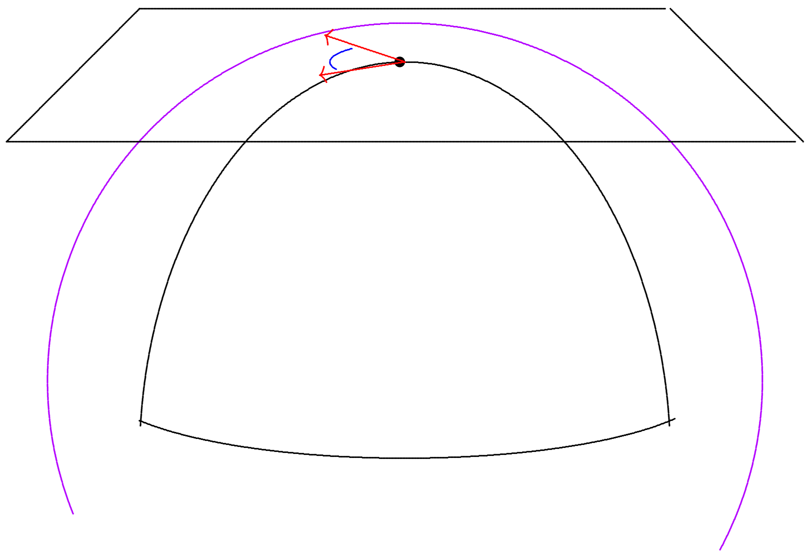
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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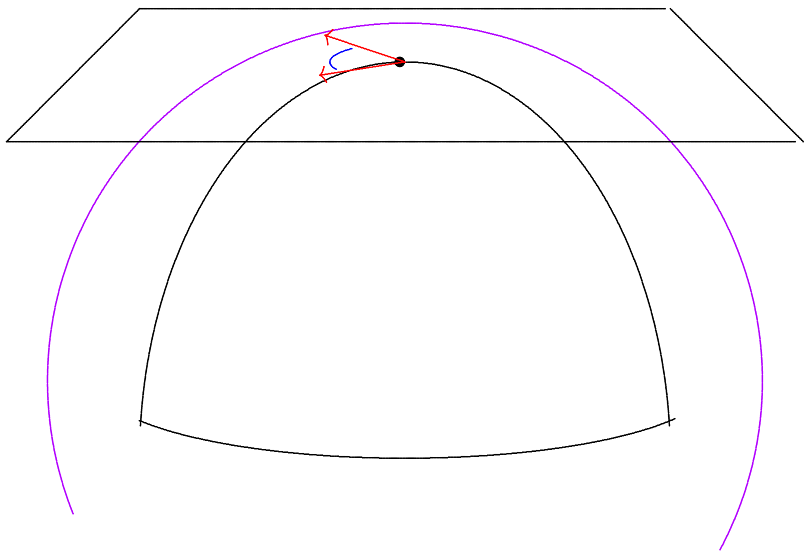

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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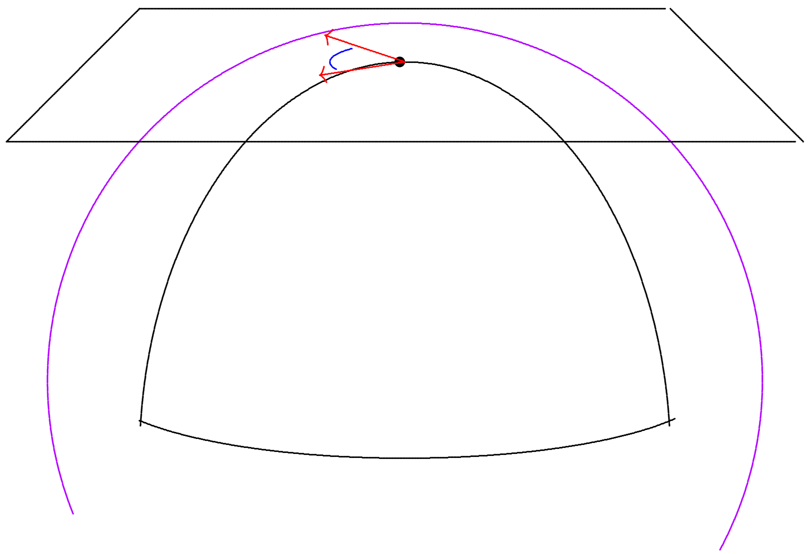

Makes tangent space a complex vector space!

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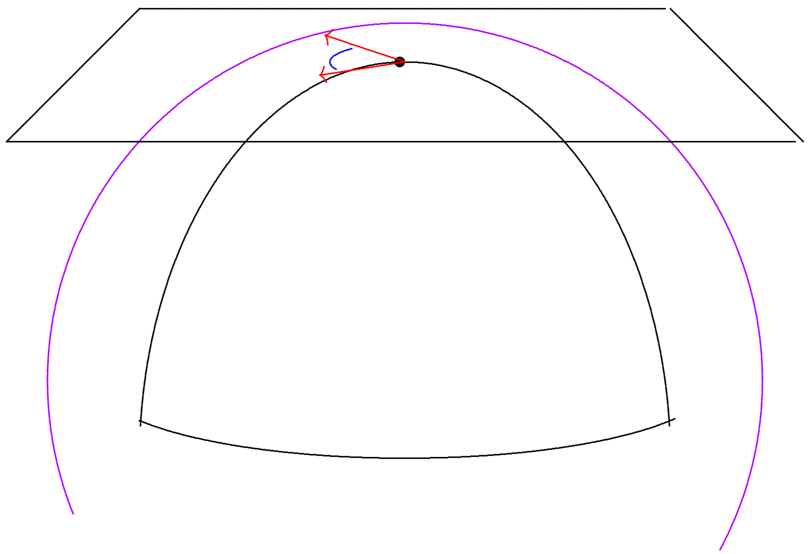

Makes tangent space a complex vector space!

$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

Kähler metrics:

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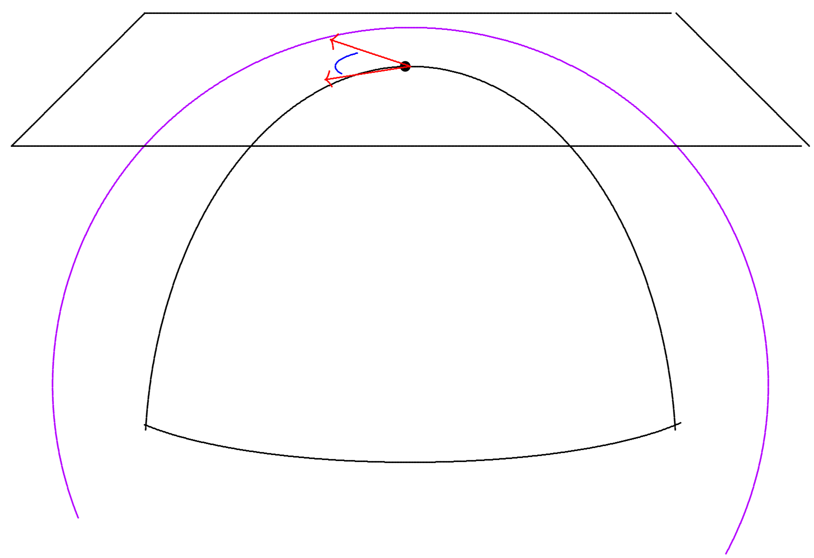

Makes tangent space a complex vector space!

Invariant under parallel transport!

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$\Longleftrightarrow$ (M^{2m}, g) is a complex manifold & $\exists$ J-invariant closed 2-form ω such that $g = \omega(\cdot, J\cdot)$.

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$$d\omega = 0$$

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$$[\omega] \in H^2(M)$$

"Kähler class"

Kähler metrics:

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$\iff$ In local complex coordinates $(z^1, \ldots, z^m)$,

$$g = -\sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

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$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k}\, dz^j \wedge d\bar{z}^k$$

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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Kähler magic:

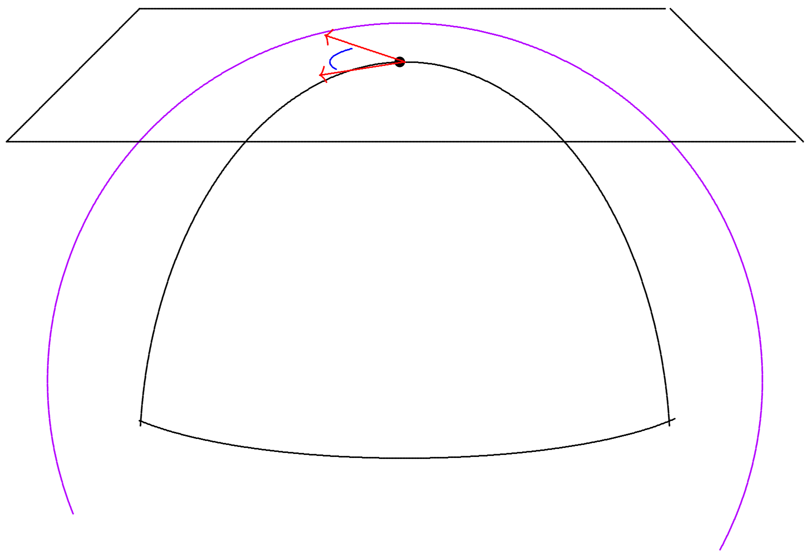
If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

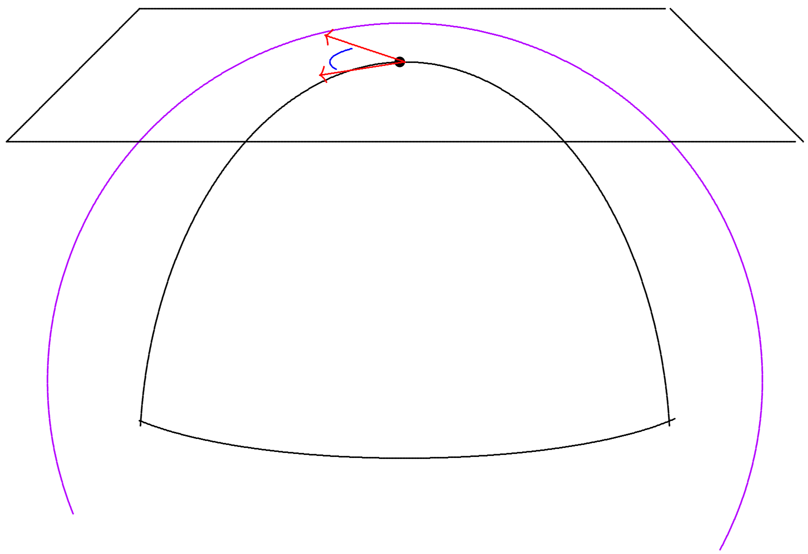
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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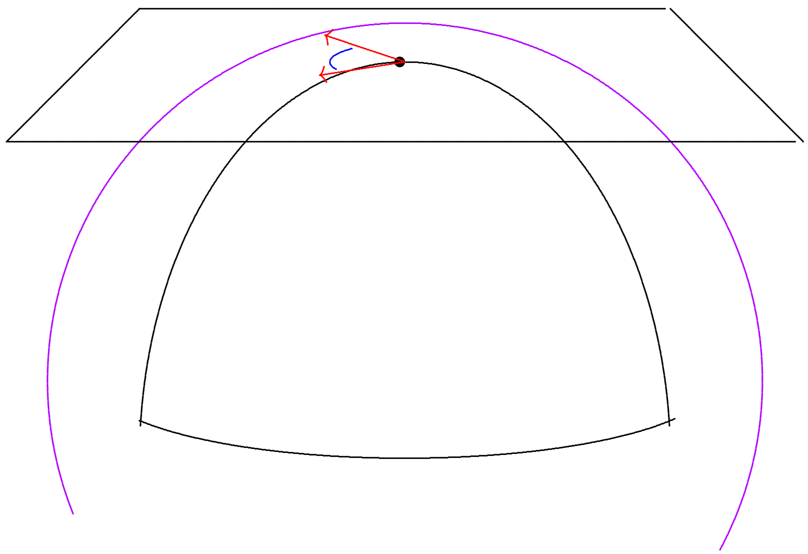
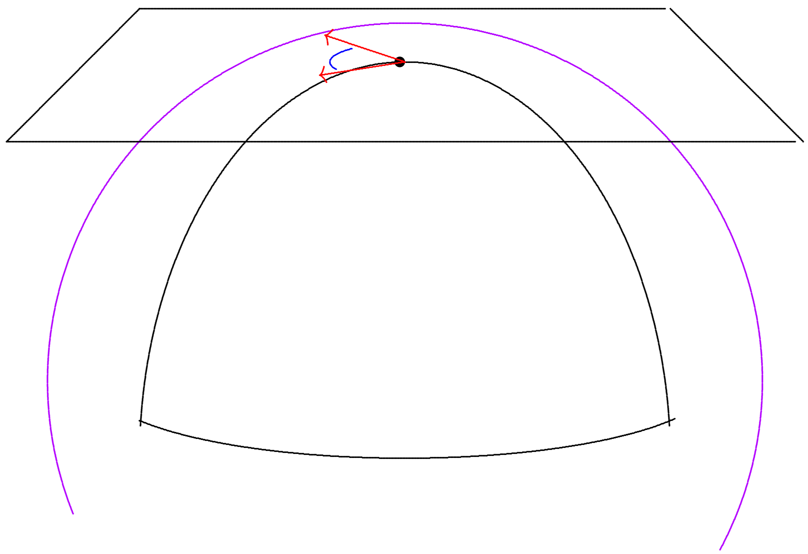


$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\qquad \{A \mid \det A = 1\}$

Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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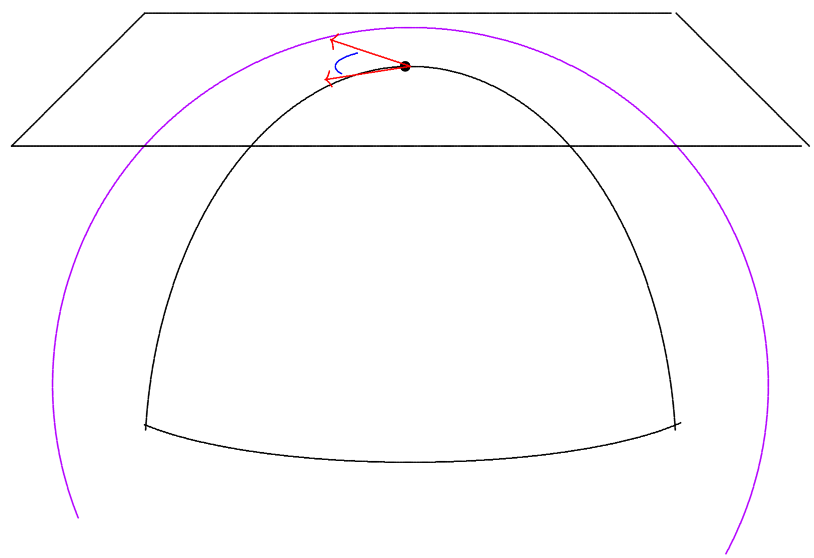
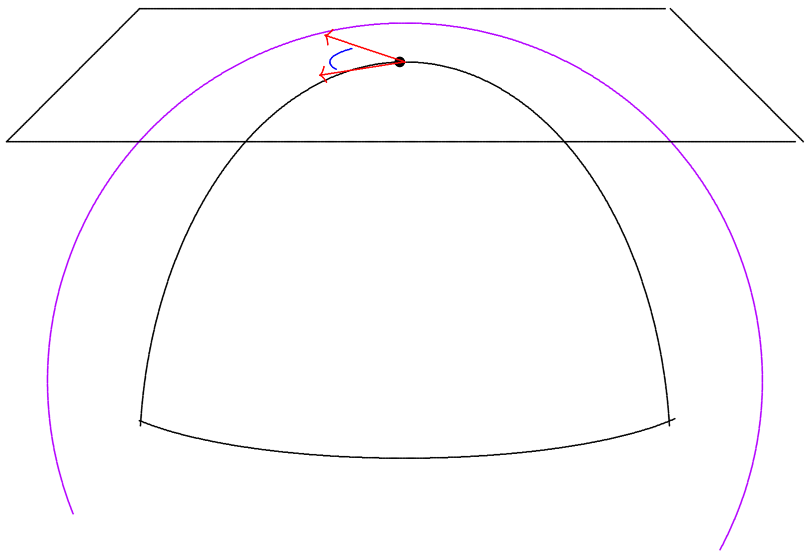


if M is simply connected.

Hyper-Kähler metrics:

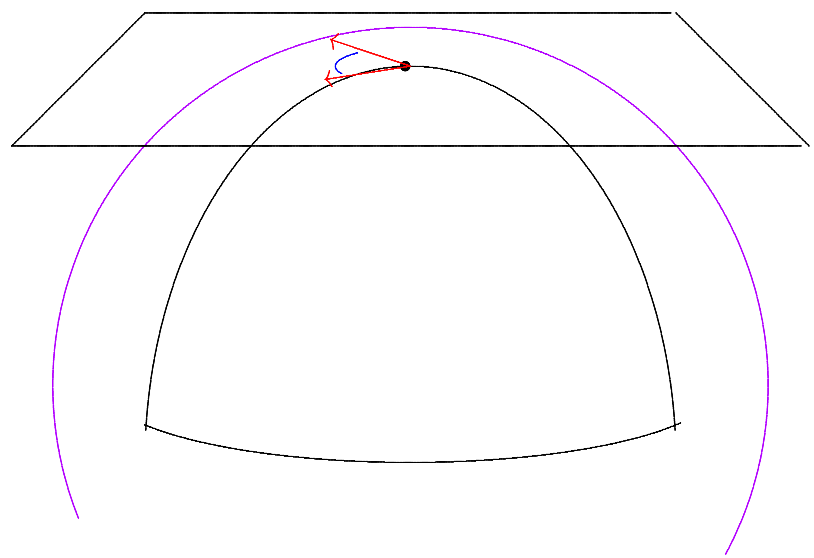
$(M^{4\ell}, g)$

holonomy



Hyper-Kähler metrics:

$(M^{4\ell}, g)$ hyper-Kähler $\iff$ holonomy $\subset \mathbf{Sp}(\ell)$



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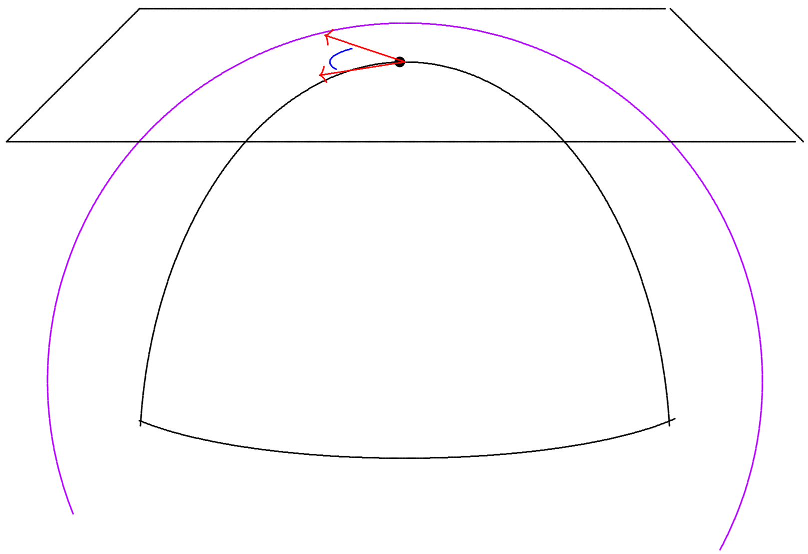

$$\mathbf{Sp}(\ell) := \mathbf{O}(4\ell) \cap \mathbf{GL}(\ell, \mathbb{H})$$

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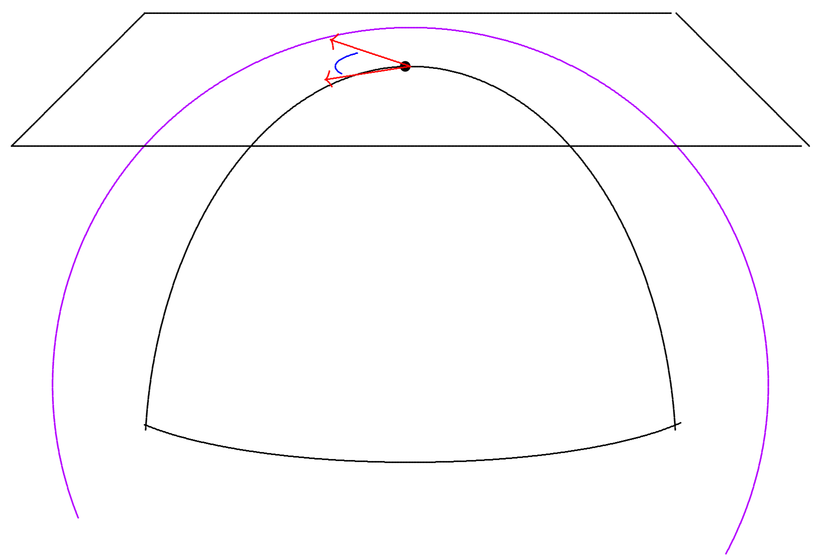

$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

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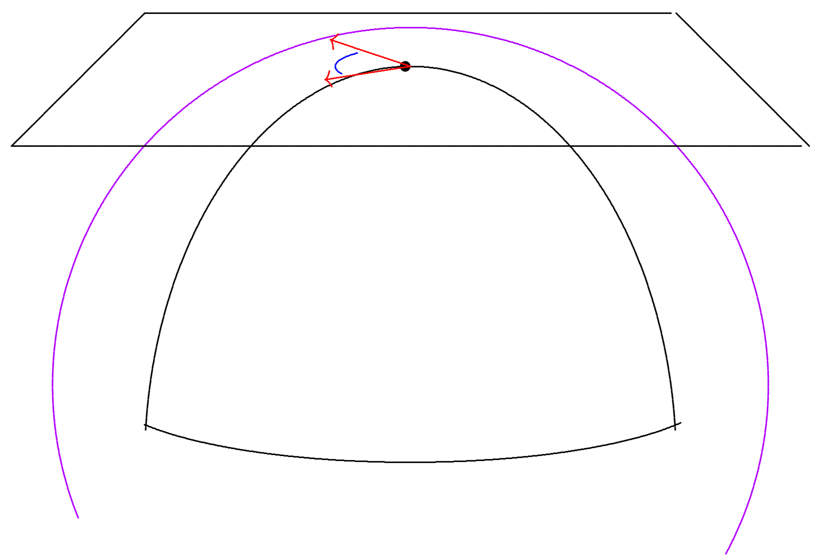

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

in many ways!

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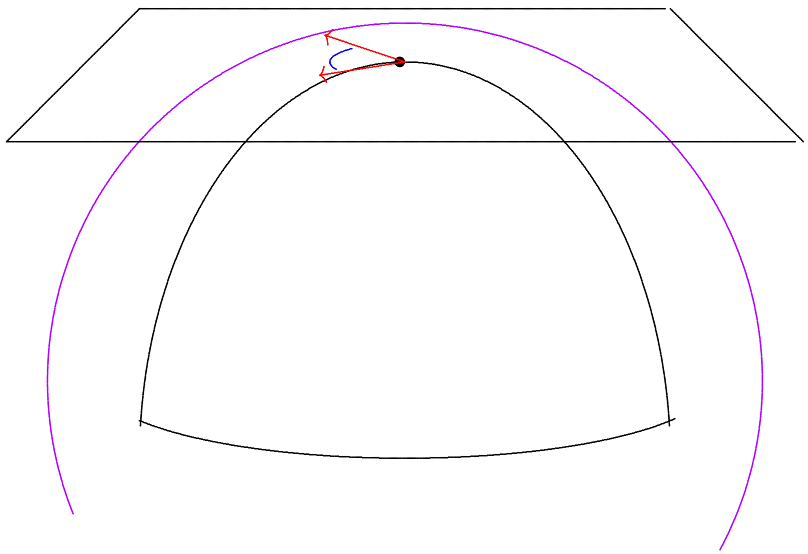

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in many ways! (For example, permute i, j, $k\ldots$)

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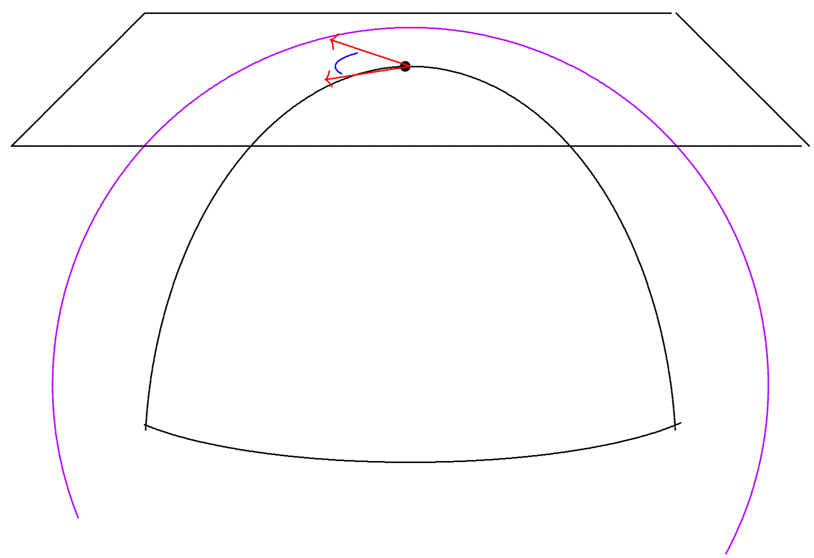

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

Ricci-flat and Kähler,

for many different complex structures!

Hyper-Kähler metrics:

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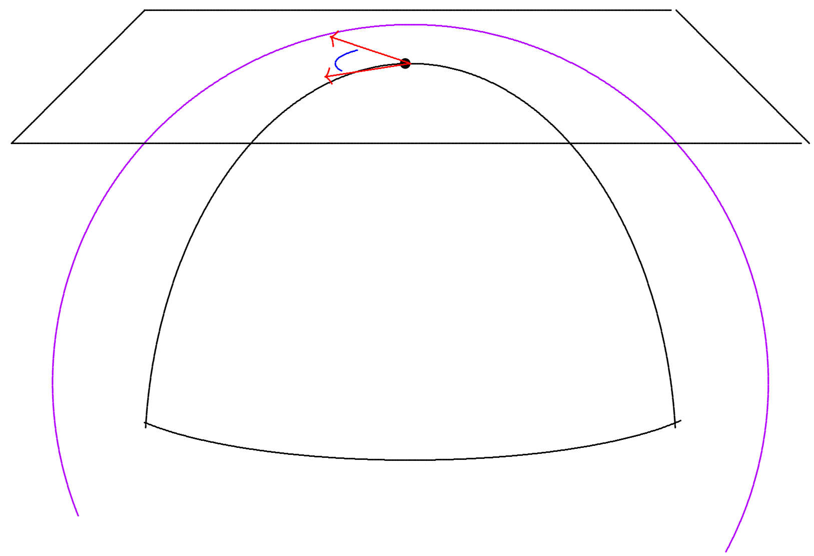
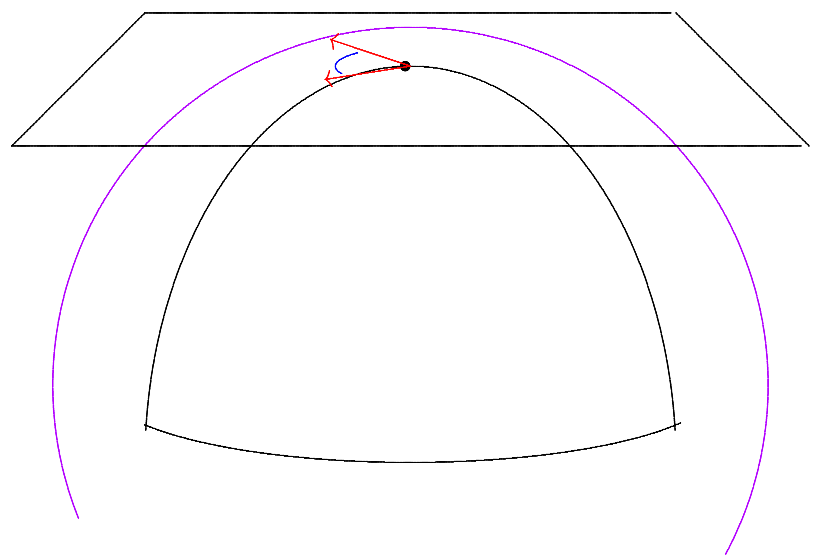


$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

Hyper-Kähler metrics:

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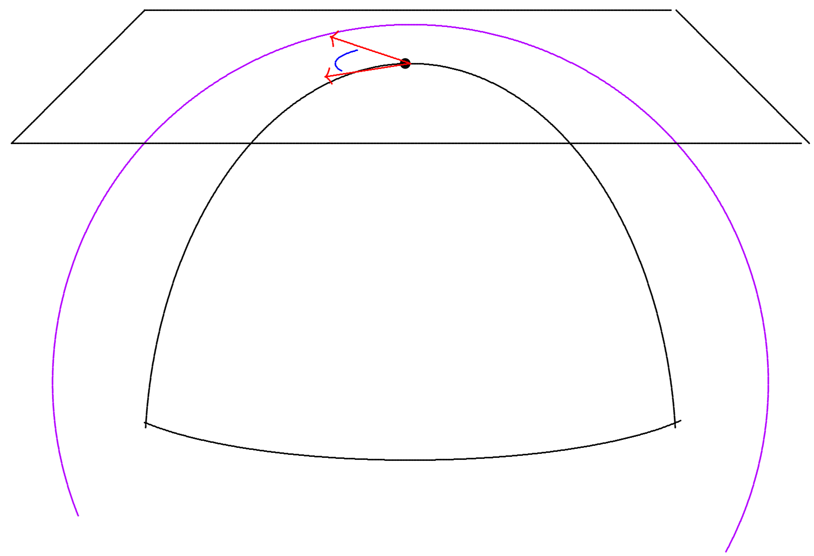
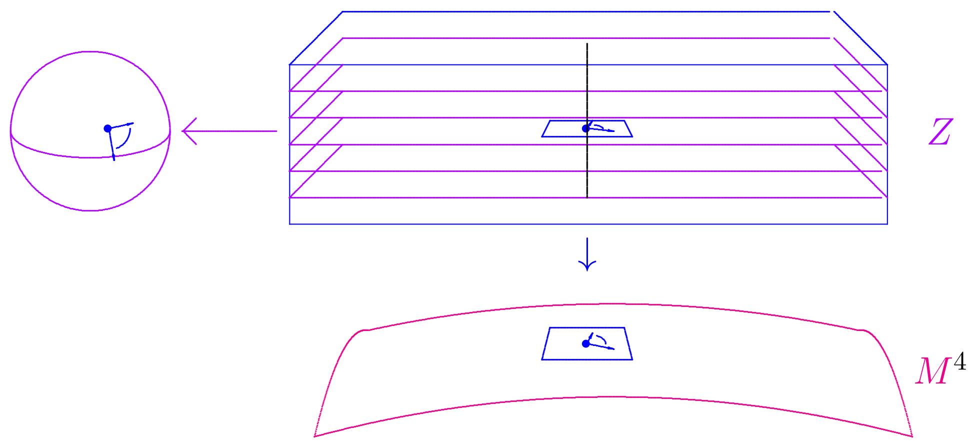


$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

When (M^4, g) simply connected:

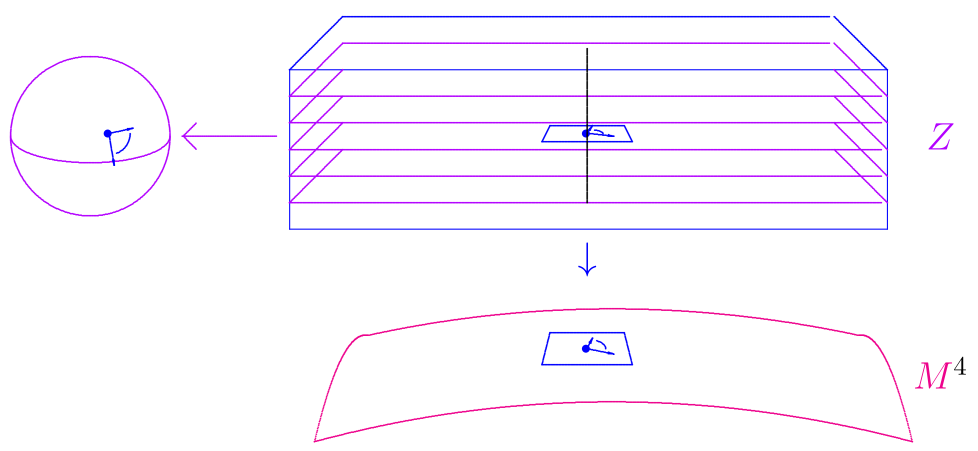
hyper-Kähler $\Longleftrightarrow$ Ricci-flat Kähler.

All these complex structures can be repackaged



All these complex structures can be repackaged as

Penrose Twistor Space (Z, J),



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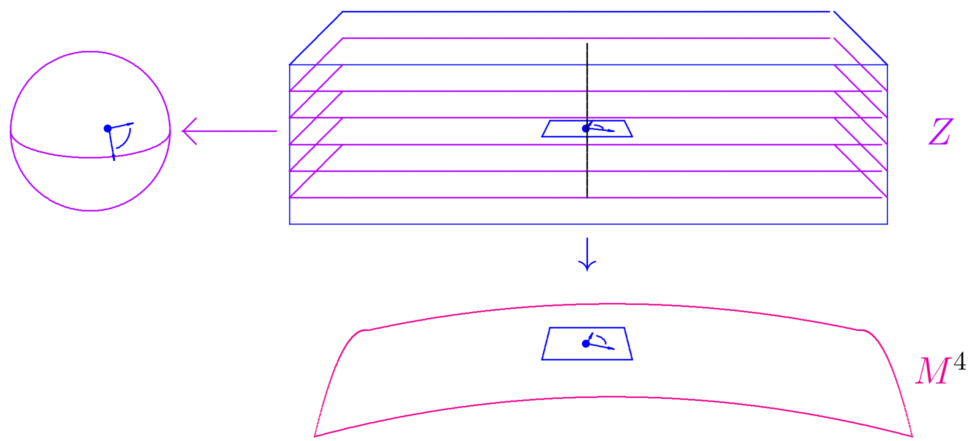

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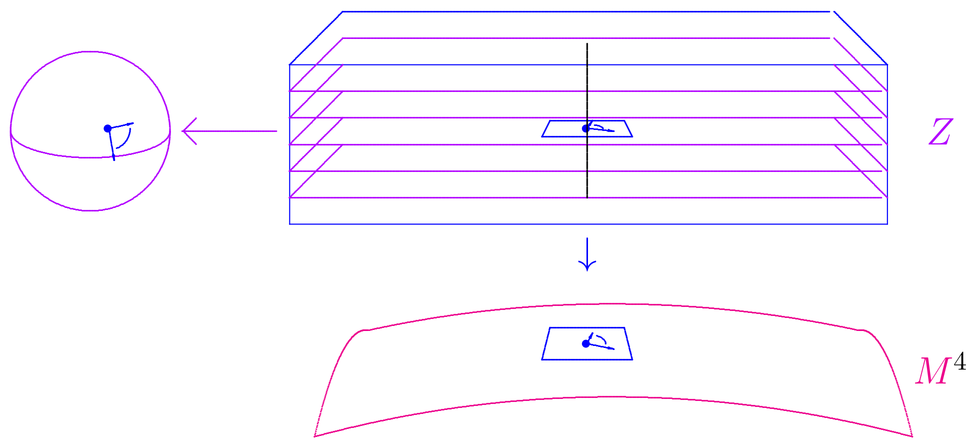

Riemannian non-linear graviton construction.

Key examples:

Term ALE coined by Gibbons & Hawking, 1979.

They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.

Their examples have just one end, with

$$\Gamma \cong \mathbb{Z}_\ell \subset \mathbf{SU}(2) \subset \mathbf{O}(4).$$

The G-H metrics are hyper-Kähler, and were soon rediscovered independently by Hitchin.

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Hitchin conjectured that similar metrics would exist for each finite $\Gamma \subset \mathbf{SU}(2)$.

This conjecture was proved by Kronheimer, 1986.

Felix Klein, 1884: $\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$

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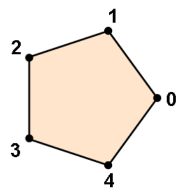

$$\mathbb{Z}_{k+1} \longleftrightarrow xy + z^{k+1} = 0$$

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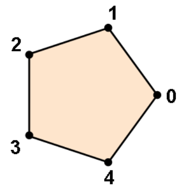

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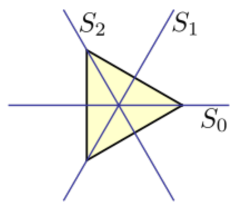

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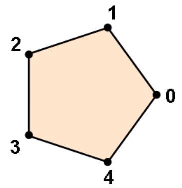

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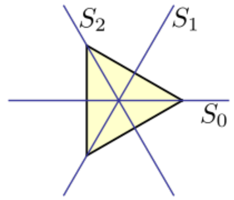

$\mathrm{Dih}^*_{k-2} \longleftrightarrow x^2 + z(y^2 + z^{k-2}) = 0$

$T^* \longleftrightarrow x^2 + y^3 + z^4 = 0$

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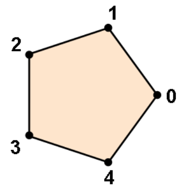

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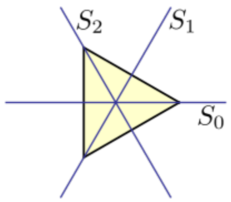

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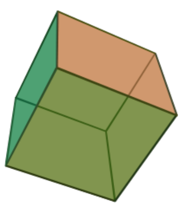
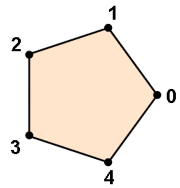
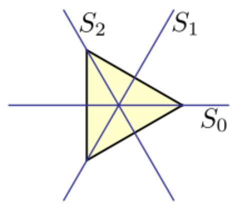


$O^* \longleftrightarrow x^2 + y^3 + yz^3 = 0$

Felix Klein, 1884: $\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$



$\mathbb{Z}_{k+1} \quad \longleftrightarrow \quad xy + z^{k+1} = 0$



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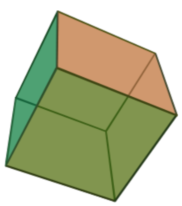

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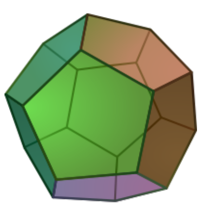
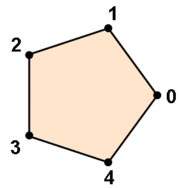


$I^* \quad \longleftrightarrow \quad x^2 + y^3 + z^5 = 0$



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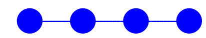

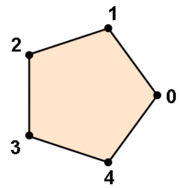

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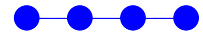

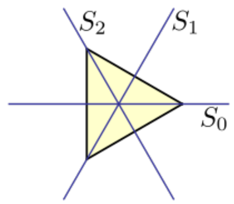

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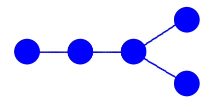

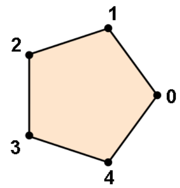

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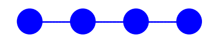

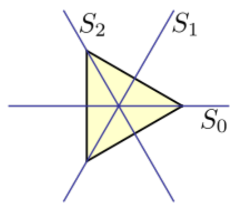

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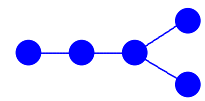

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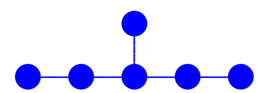

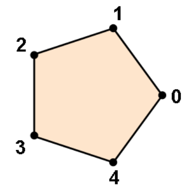

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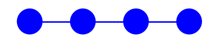

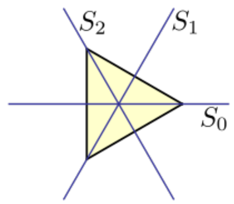

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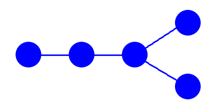

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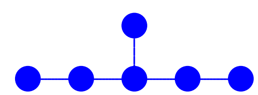

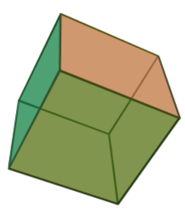

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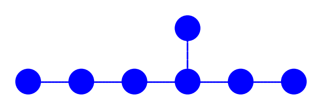

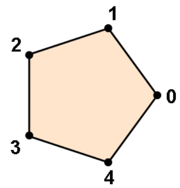

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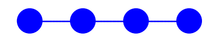

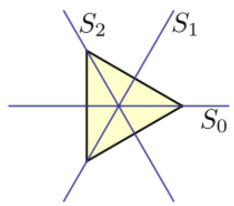

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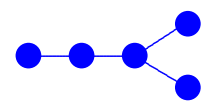

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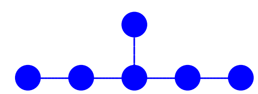

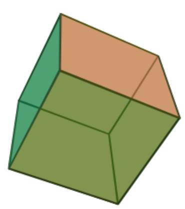

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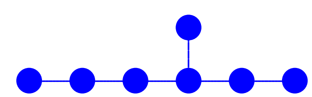

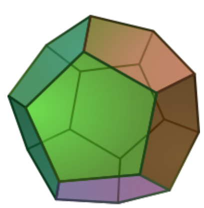

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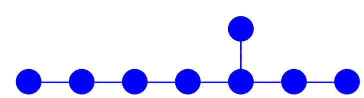
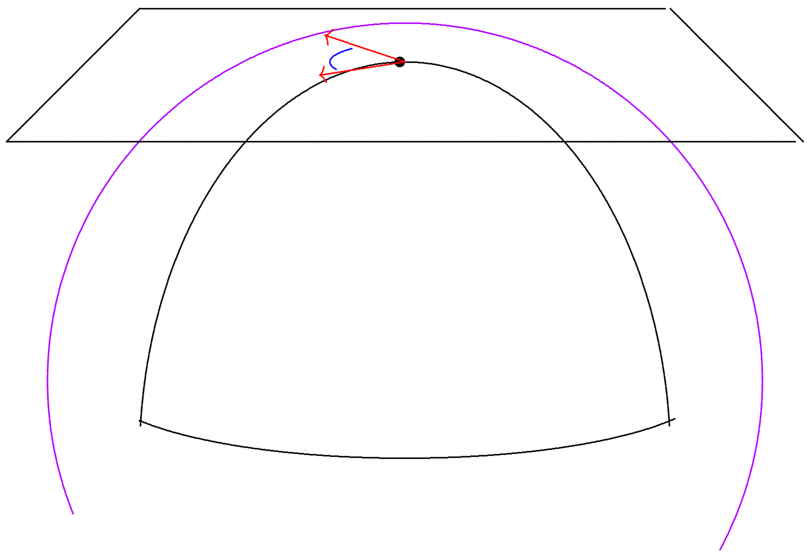


Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

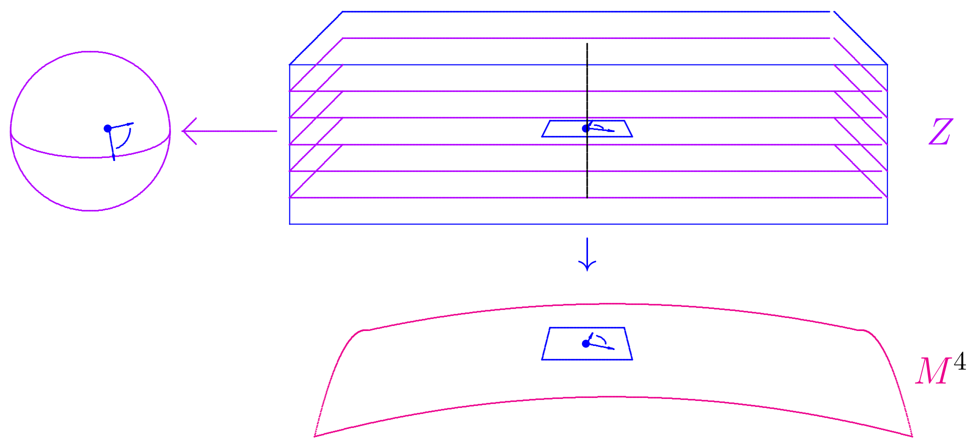
Ricci-flat and Kähler,

for many different complex structures!

All these complex structures can be repackaged as

Penrose Twistor Space (Z, J),

which is a complex 3-manifold.



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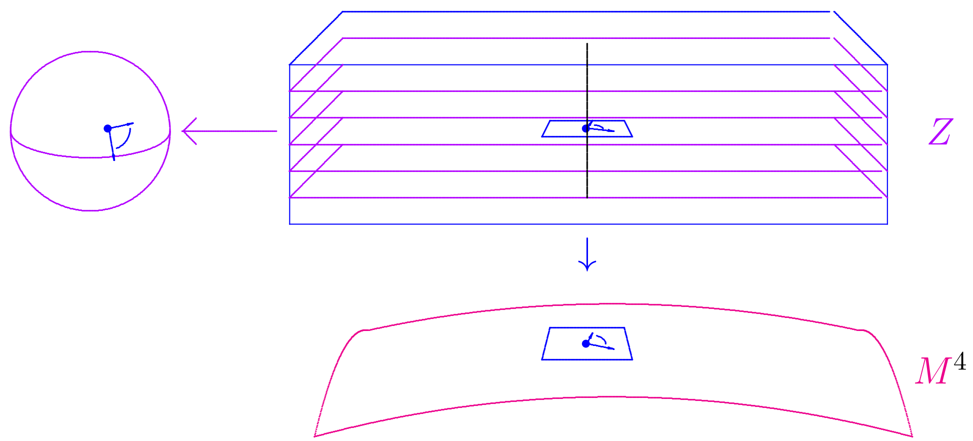

But similar for scalar-flat Kähler surfaces (M^4, g, J)!

Any scalar-flat Kähler surface (M^4, g, J) has a

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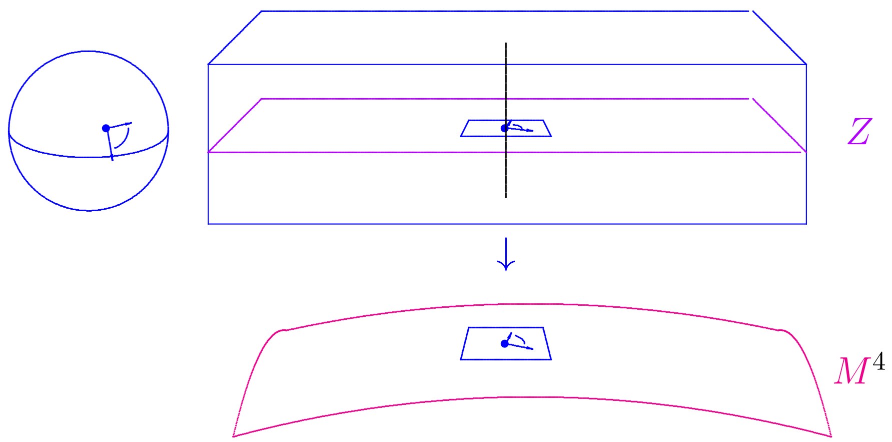

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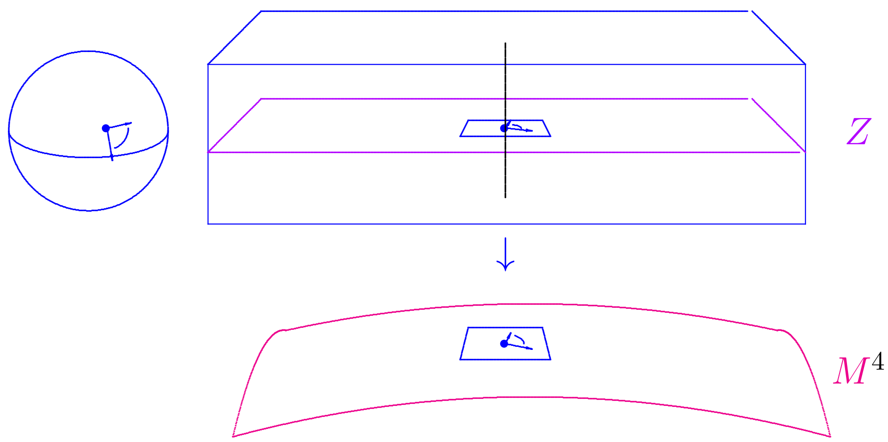

The construction of scalar-flat Kähler surfaces and the study of their twistor spaces was a main focus of my own work during the decade 1985-1994.

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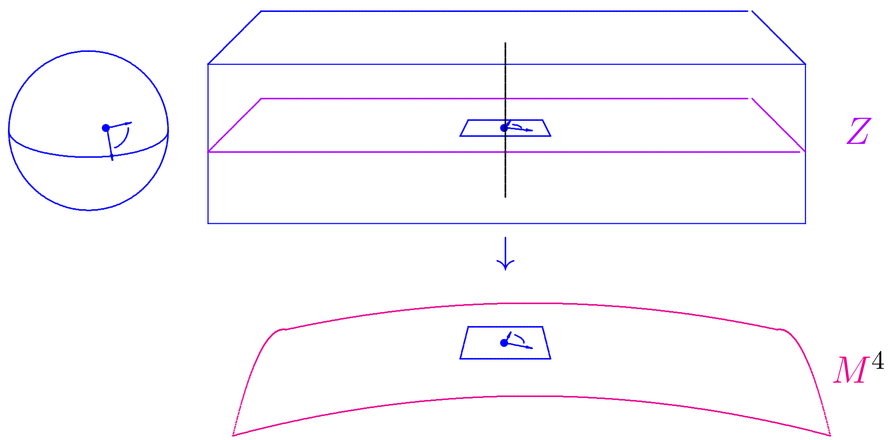

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Constructed ALE examples on line bundles $L \to \mathbb{CP}_1$ with $c_1 < 0$, and on their blow-ups.

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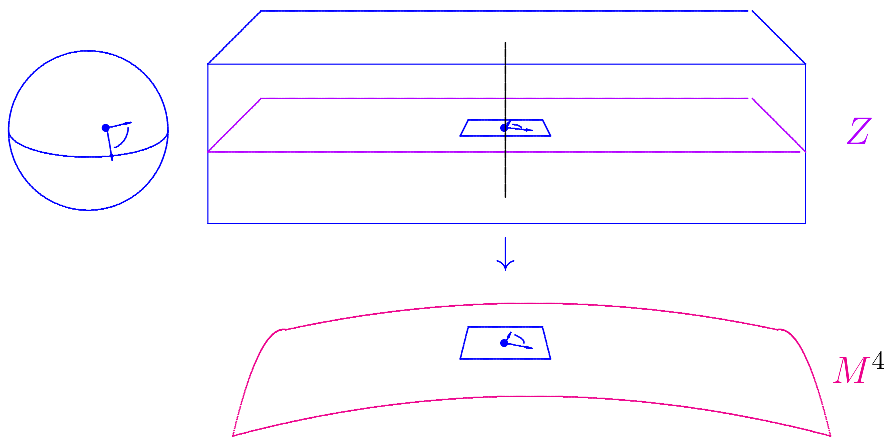

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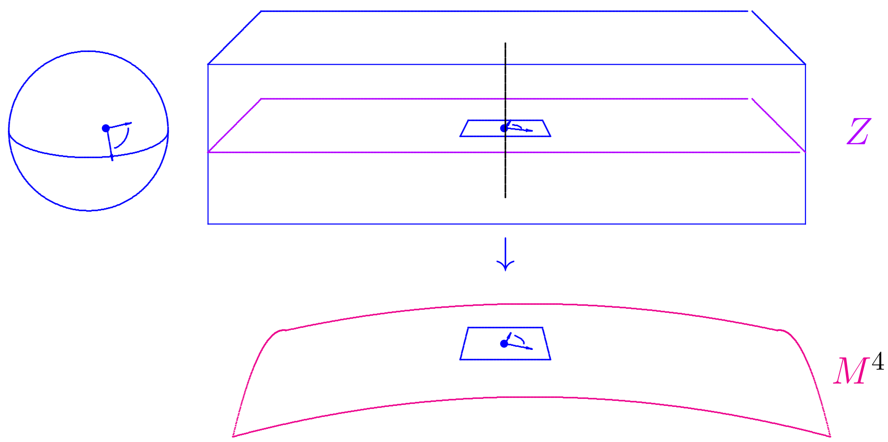

These ALE spaces arise naturally in the study of compact Einstein or Bach-flat 4-manifolds as bubbling modes for sequences of metrics.

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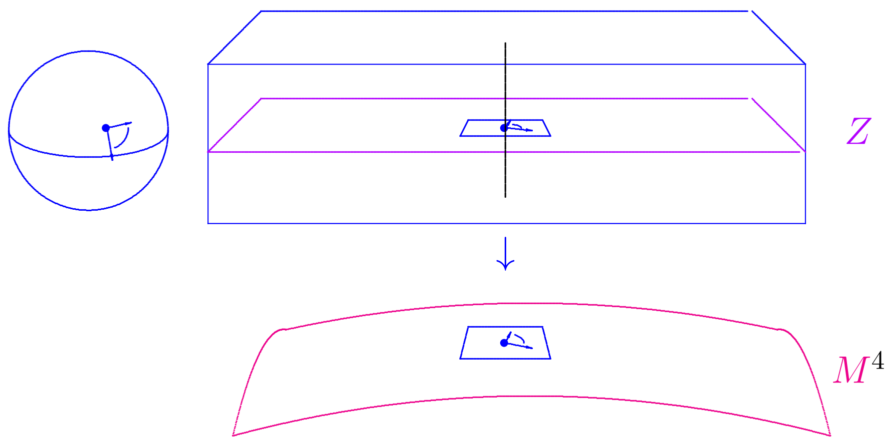

Lots more ALE scalar-flat Kähler surfaces now known:

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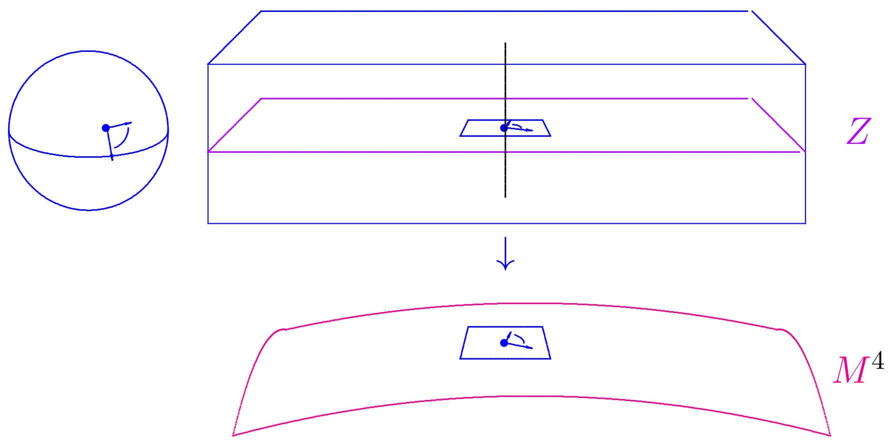

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Joyce, Calderbank-Singer, Lock-Viaclovsky. . .

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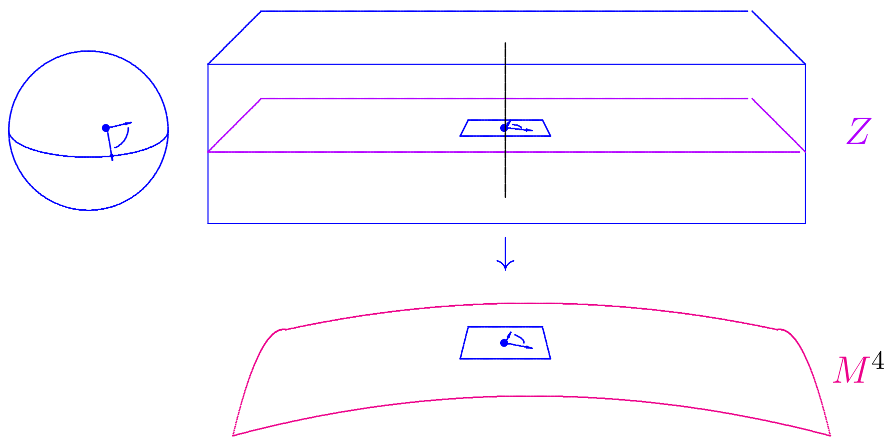
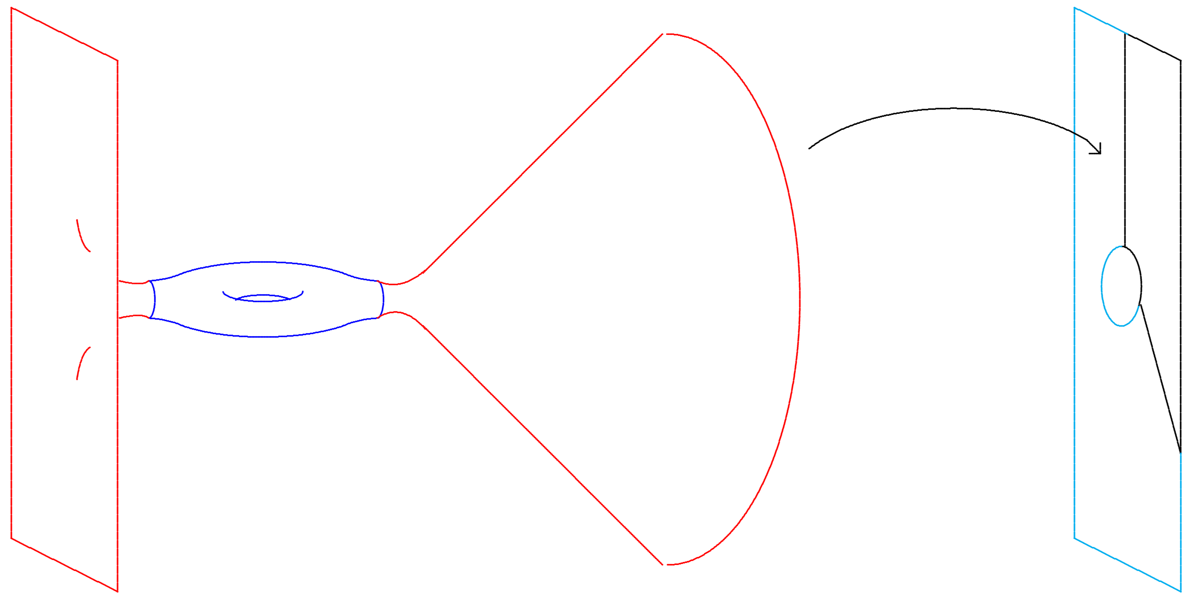


Lots more ALE scalar-flat Kähler surfaces now known:

Joyce, Calderbank-Singer, Lock-Viaclovsky...

But full classification remains an open problem.

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean *(ALE)* *if* $\exists$ *compact set* $K \subset M$ *such that* $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\Gamma_i$, *where* $\Gamma_i \subset \mathbf{O}(\mathbf{n})$, *such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Mass still meaningful in this context. . .

Definition. *The mass (at a given end) of an ALE n-manifold is defined to be*

$$\mathfrak{m}(M, g) := \lim_{\varrho \to \infty} \frac{\Gamma(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{\Sigma(\varrho)} \left[g_{ij,i} - g_{ii,j}\right] \nu^j \alpha_E$$

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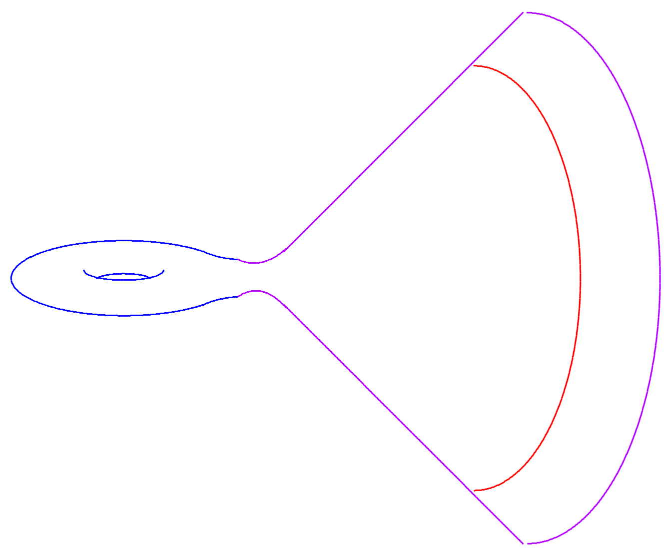
where

- $\Sigma(\varrho) \approx S^{n-1}/\Gamma_i$



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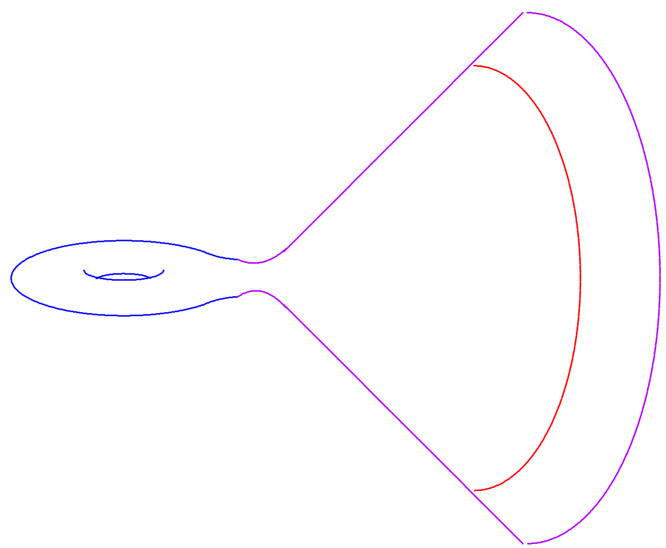

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$$m(M, g) := \lim_{\varrho \to \infty} \frac{\Gamma(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{\Sigma(\varrho)} \left[g_{ij,i} - g_{ii,j} \right] \nu^j \alpha_E$$

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Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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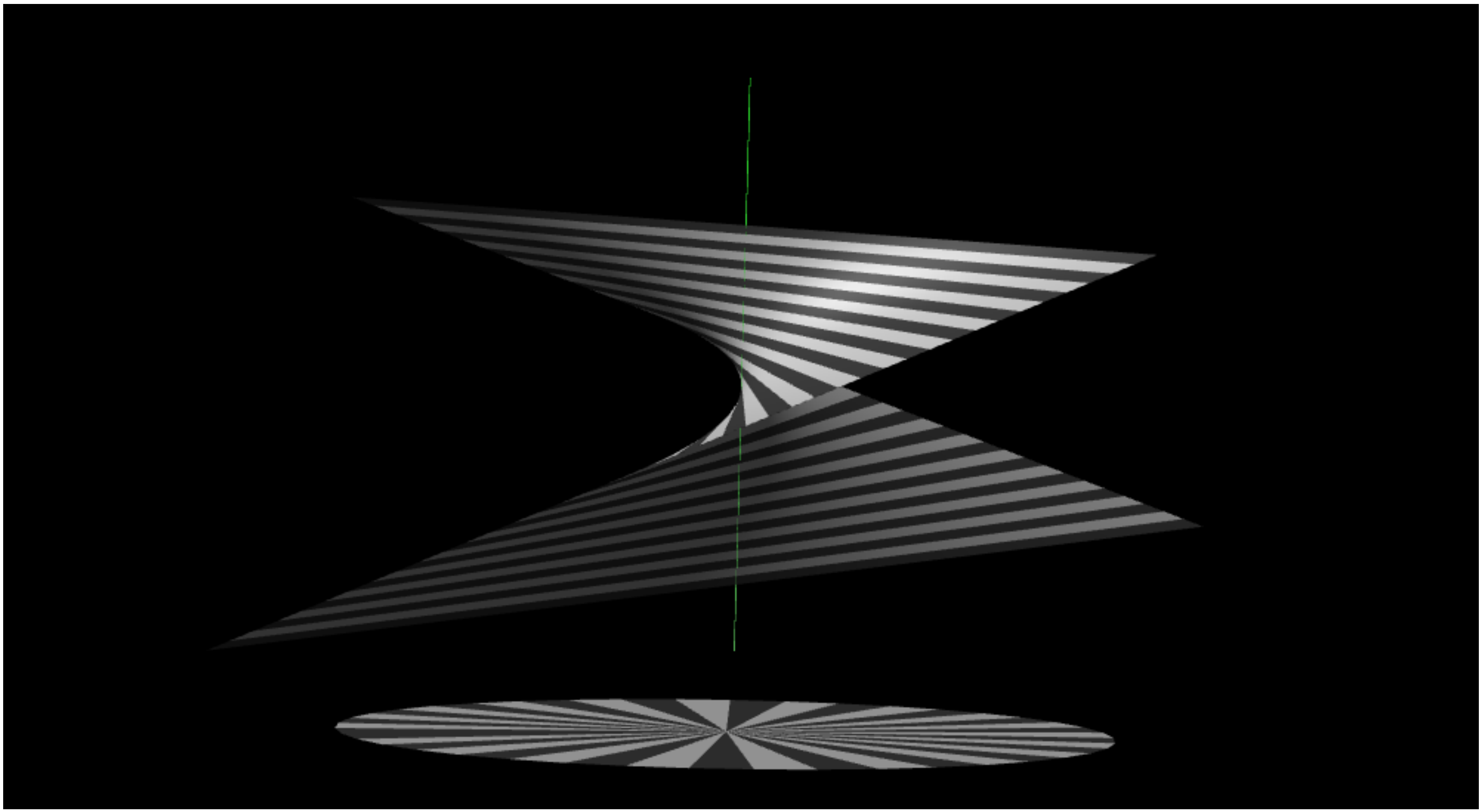

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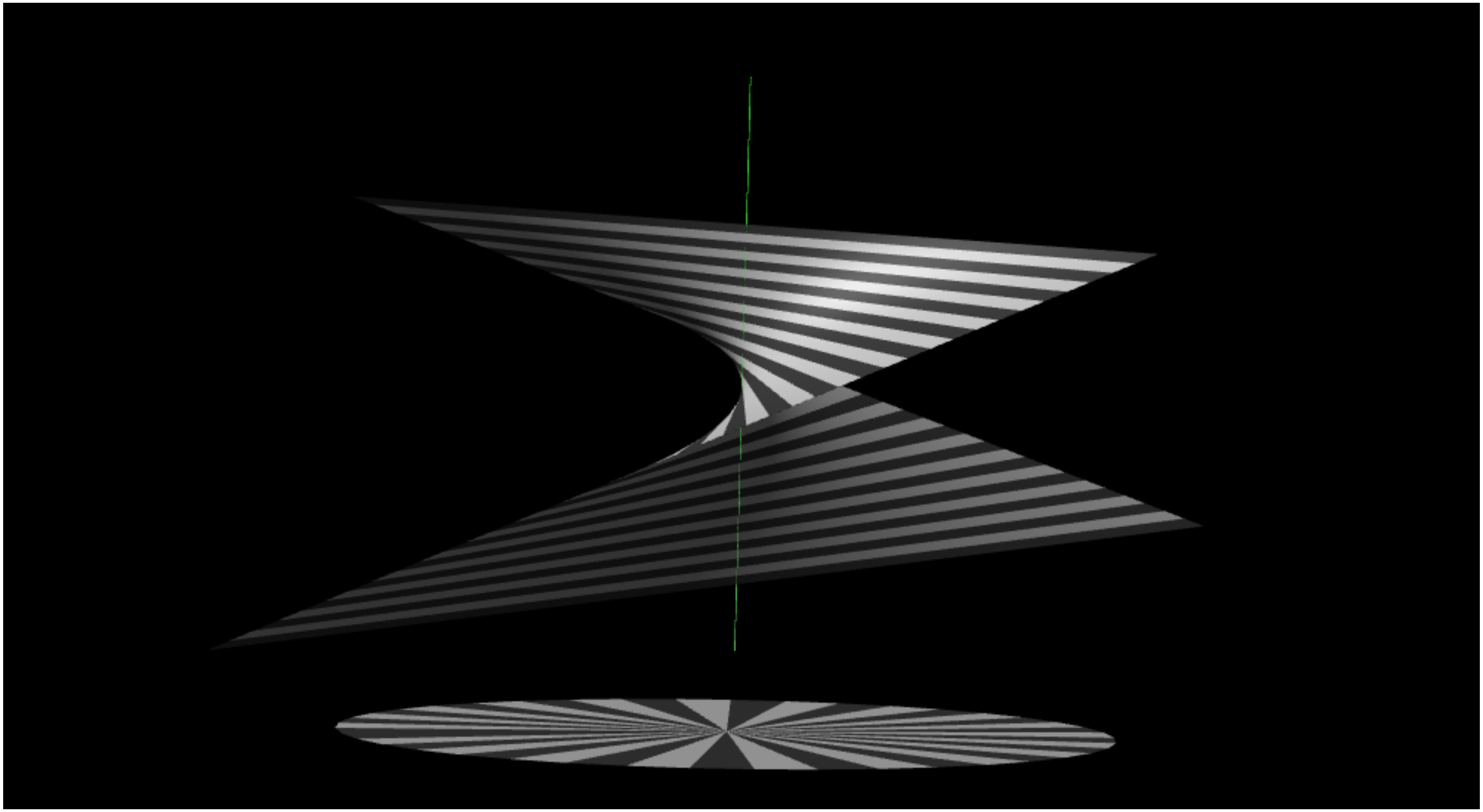

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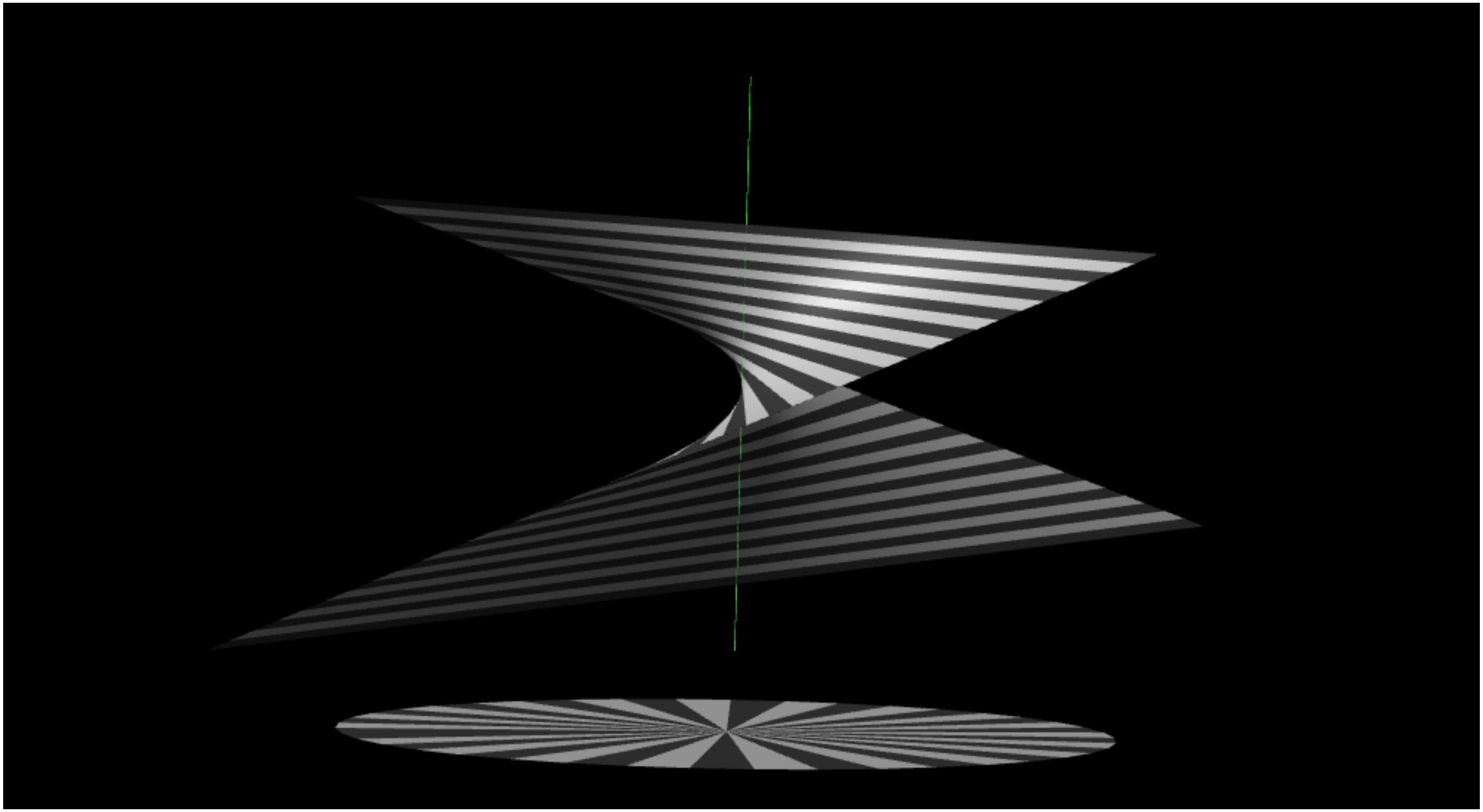

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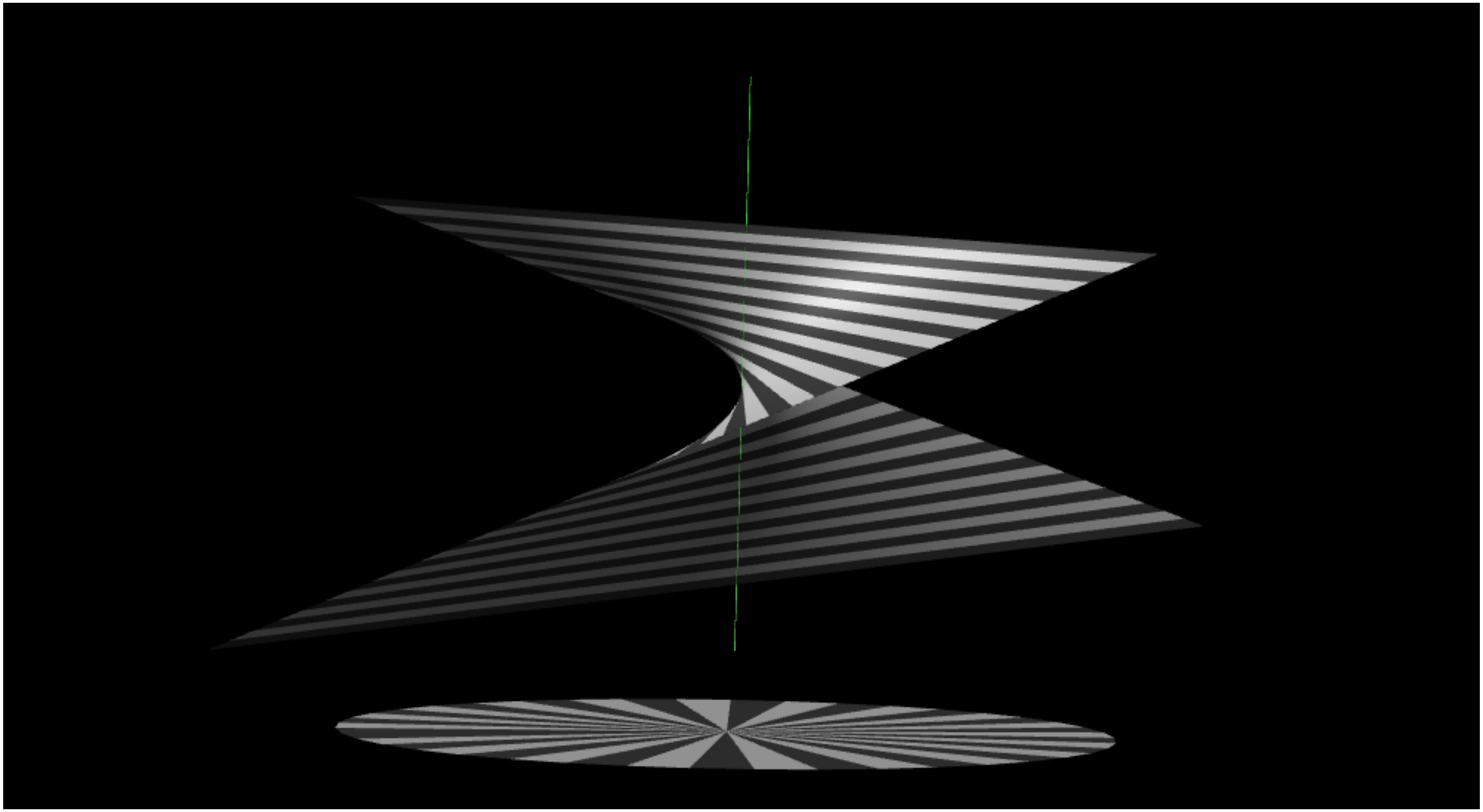

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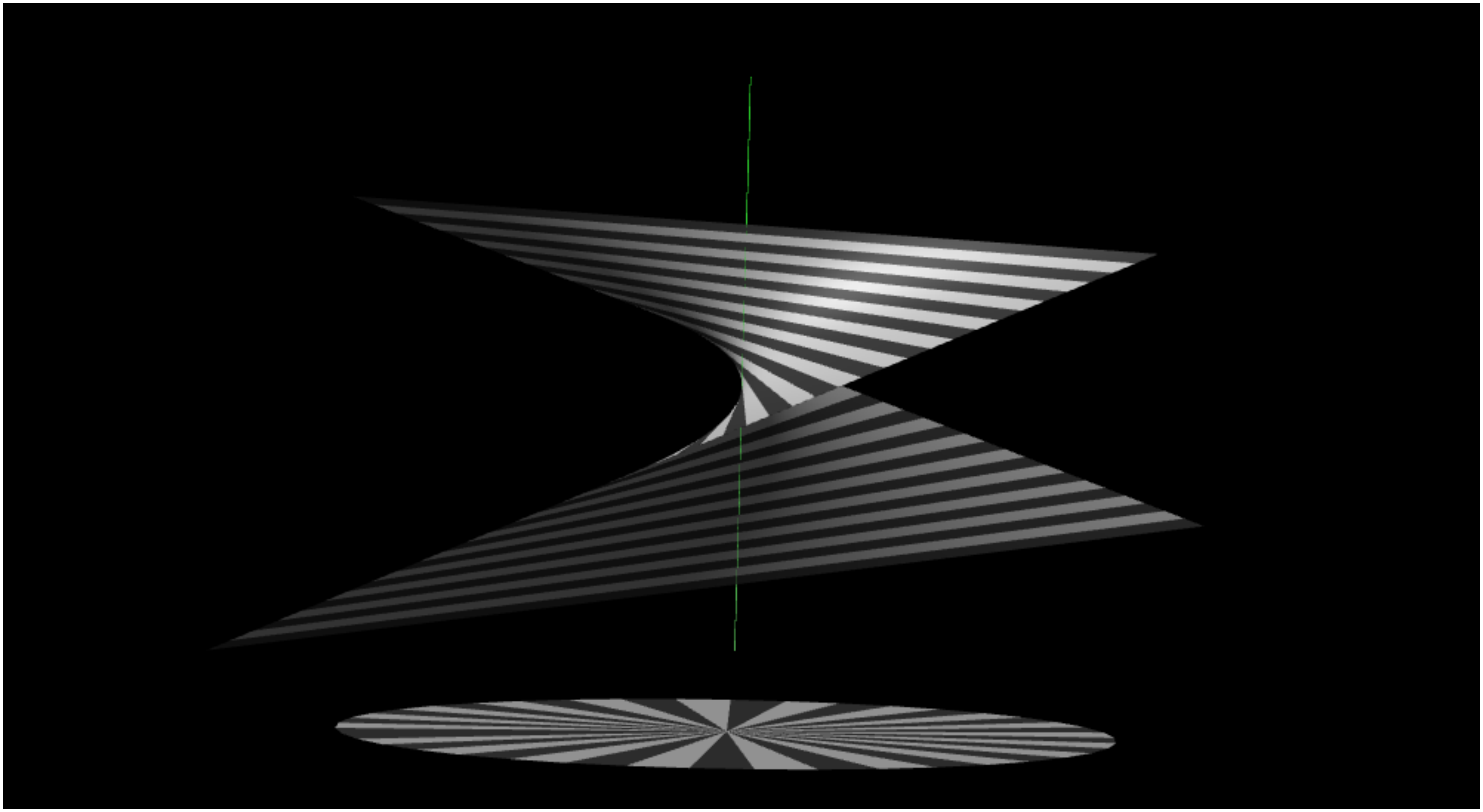

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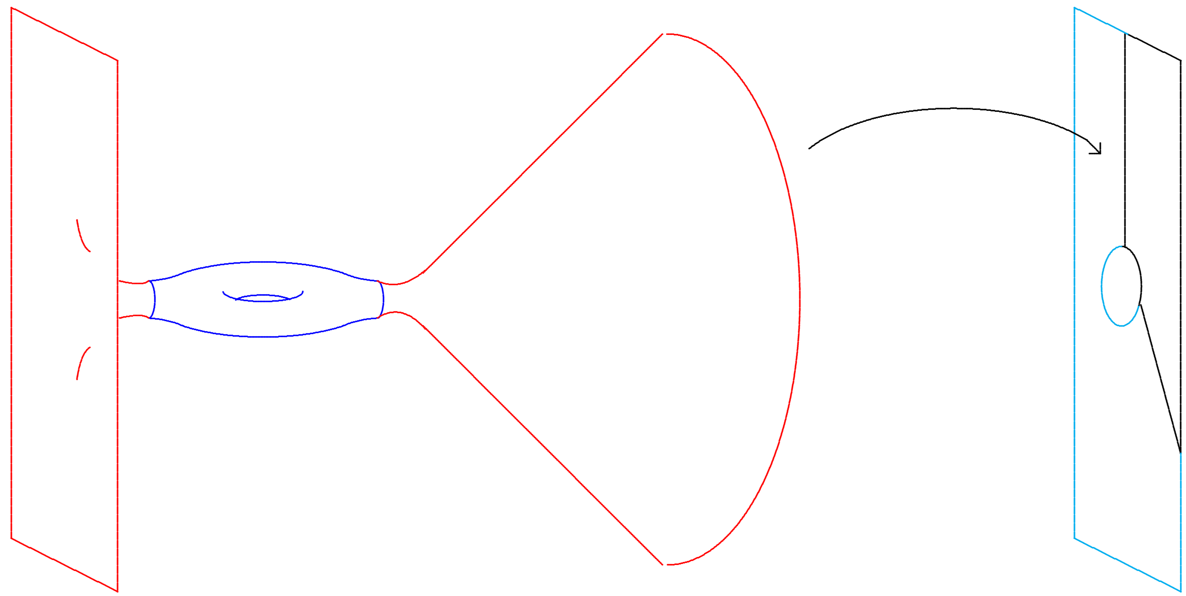
But $m \neq 0$ for some other scalar-flat Kähler ALEs:

Scalar-flat-Kähler Burns metric on $\widetilde{\mathbb{C}^2} \subset \mathbb{C}^2 \times \mathbb{CP}_1$:

$$\omega = \frac{i}{2}\partial\bar{\partial}\left[u + 3m \log u\right], \qquad u = |z_1|^2 + |z_2|^2$$

has mass $m > 0$.

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean *(ALE) if* $\exists$ *compact set* $K \subset M$ *such that* $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\Gamma_i$*, where* $\Gamma_i \subset \mathbf{O(n)}$*, such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Positive Mass Conjecture:

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Any AE manifold with $s \geq 0$ has $\boldsymbol{m} \geq 0$.

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on line bundles $L \to \mathbb{CP}_1$ of Chern-class ≤ -3.

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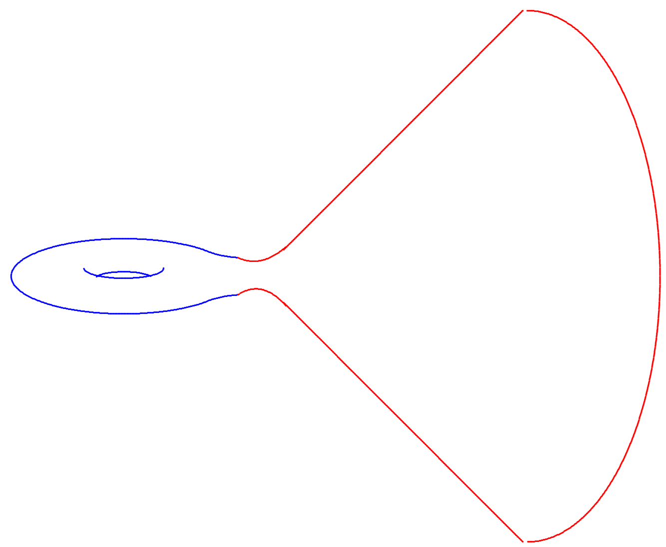

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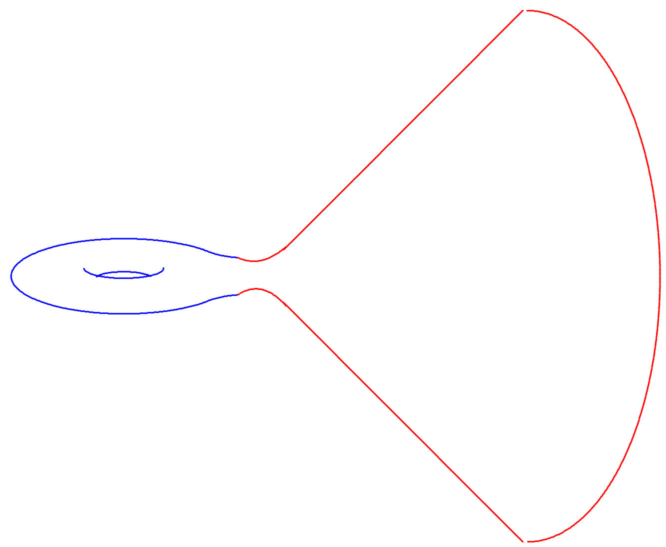

$$\boxed{n = 2m \geq 4}$$

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Does not depend on the choice of an end!

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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Theorem B. *Let (M^4, g, J) be an ALE scalar-flat Kähler surface, and suppose that (M, J) is the minimal resolution of a surface singularity.*

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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$$\frac{4\pi^m(2m-1)}{(m-1)!}\mathfrak{m}(M,g) = -\frac{4\pi}{(m-1)!}\langle \clubsuit(c_1), [\omega]^{m-1}\rangle + \int_M s_g d\mu_g$$

For a compact Kähler manifold (M^{2m}, g, J),

$$\int_M s_g d\mu_g = \frac{4\pi}{(m-1)!}\langle c_1, [\omega]^{m-1}\rangle$$

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M,g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}}.$$

Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has* mass *given by*

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So **Theorem A** is an immediate consequence!

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Since g is Kähler, the complex coordinates

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Special Case: Suppose

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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However, since $s = 0$,

$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4}\omega^2 = 0.$$

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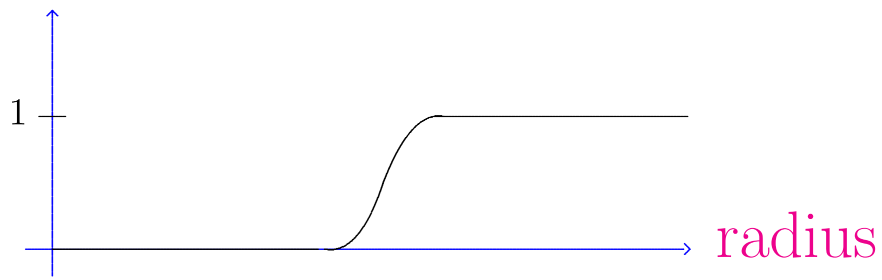

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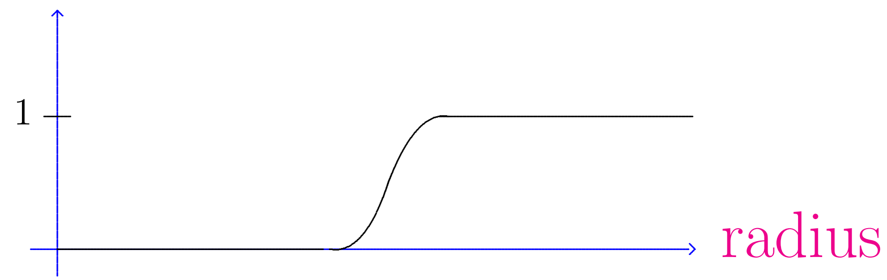

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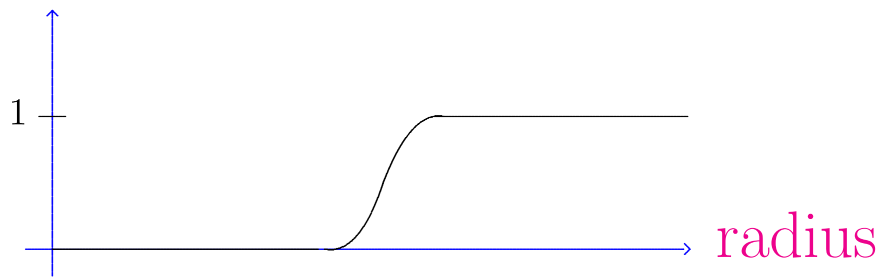

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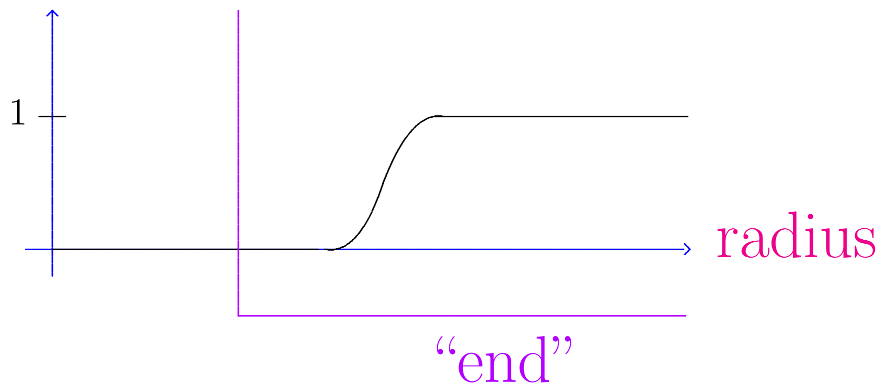

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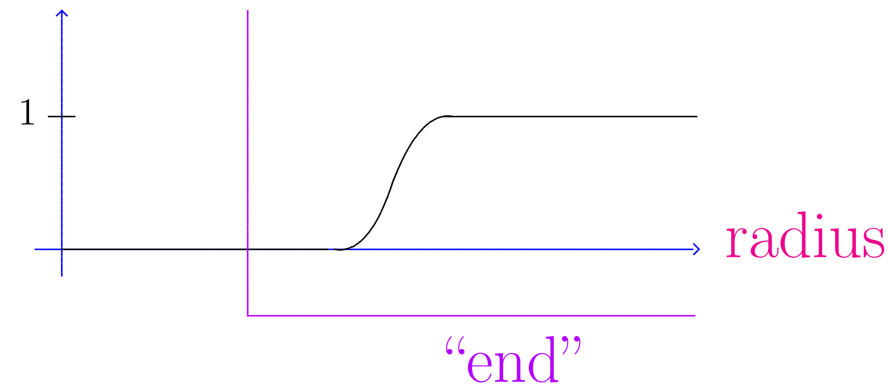

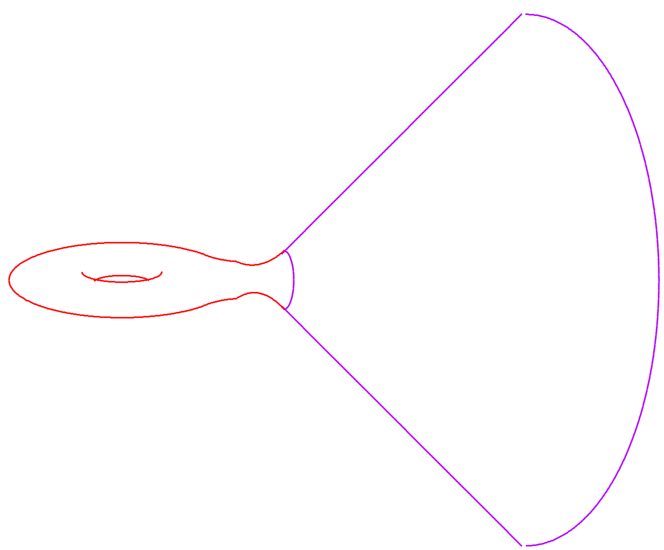

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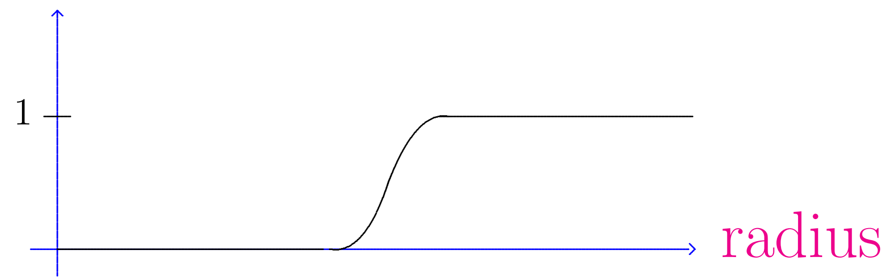

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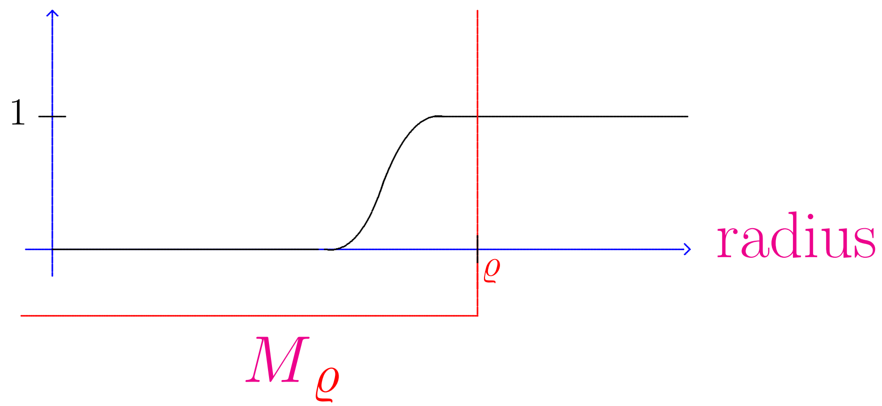

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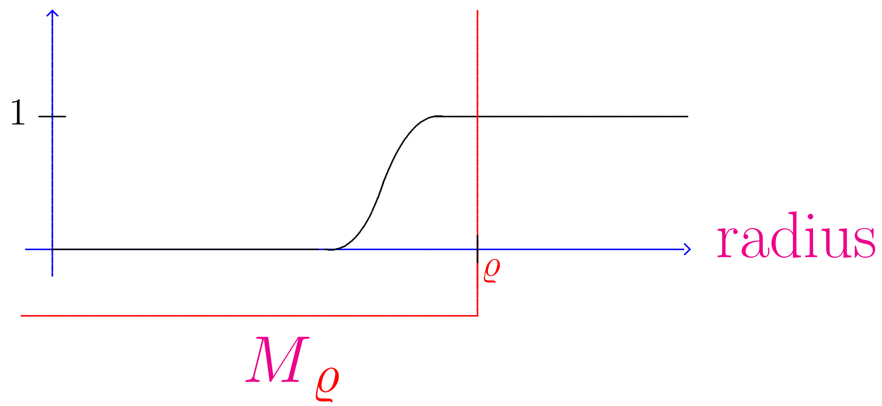

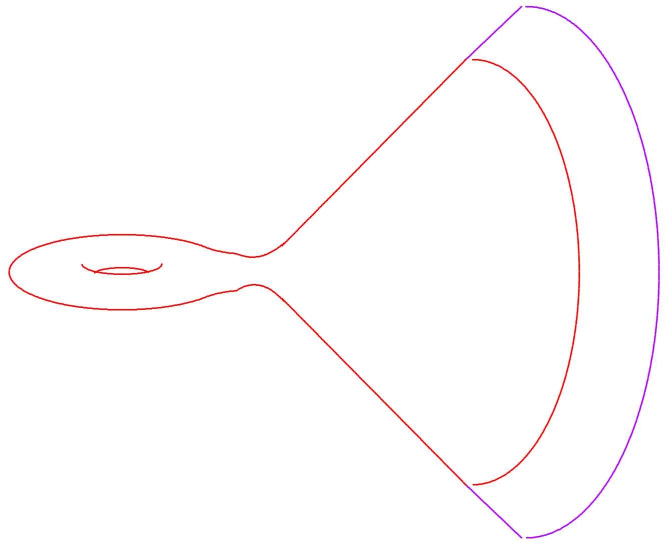

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Compactly supported, because $d\theta = \rho$ near infinity.

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The last point is serious.

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Seen in "gravitational instantons"

and other explicit examples.

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$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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Complete analytic family encodes info about J.

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This has some interesting consequences...

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Proof actually shows something stronger!

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so the mass formula implies the claim.

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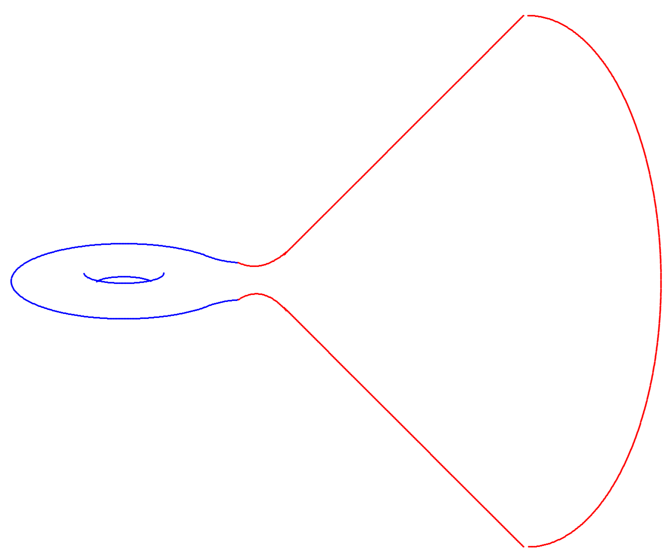

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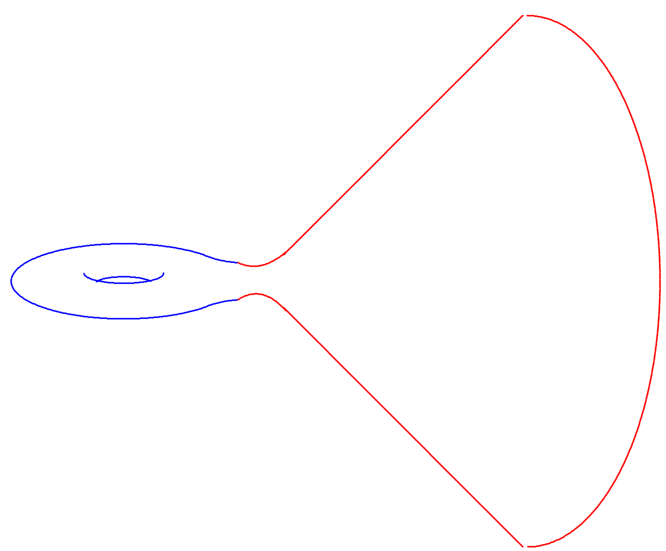

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