Online Appendix (Not for Publication)

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A Suggestive Evidence

Stock market volatility is commonly viewed in the literature as a proxy of financial and economic uncertainty, which Bloom (2009) and later Gilchrist and Zakrajšek (2012), Bachmann et al. (2013), Jurado et al. (2015), Caldara et al. (2016), Baker et al. (2020), **?** further studied as a driving force behind business cycles fluctuations. In this Section, we evaluate these claims and present interesting empirical results. Figure 3 provides the first piece of supportive evidence in that direction. Panel 3a depicts several variables commonly used in the literature to measure financial uncertainty. The correlation between series is remarkably high and they all display positive spikes at the beginning and/or initial months following an NBER-dated recession, which is consistent with the evidence that many of these episodes were financial in nature.[1] Panel 3b plots Ludvigson et al. (2021) (henceforth, LMN) financial and real (i.e. non-financial) uncertainty series. These variables are positively correlated and display a similar propensity to increase around recessions, though a different type of crisis (e.g. financial or not) is correlated with a different type of uncertainty playing the dominant role. For example, the massive spike in real vis-à-vis financial uncertainty following the recent Covid-19 recession, which initially was a health crisis that spilled into the real economy, can be observed in Panel 3b.

The patterns displayed in Figure 3 do not yet constitute a proof of the importance of financial market uncertainty as a driver of the business cycle, as we should worry about the possibility of reverse causation running from unfavorable economic conditions towards uncertainty. We tackle this issue by proposing a simple Vector Autoregression (VAR) with

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[1]See Reinhart and Rogoff (2009) and Romer and Romer (2017) for the classification of the past recessions. Their analysis showed many recessions had roots in financial markets.

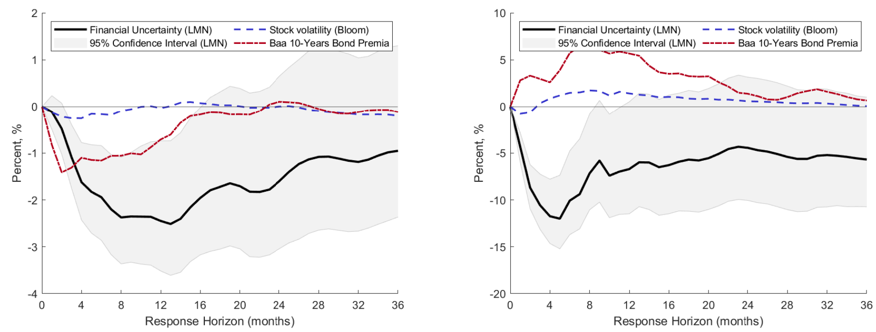
the structural identification strategy based on the timing of macroeconomic shocks similar to Bloom (2009). Equation (1) presents the variables considered and their ordering, with non-financial series first and financial variables last.[2]

$$\text{VAR-11 order:} \quad \begin{bmatrix} \log\left(\text{Industrial Production}\right) \\ \log\left(\text{Employment}\right) \\ \log\left(\text{Real Consumption}\right) \\ \log\left(\text{CPI}\right) \\ \log\left(\text{Wages}\right) \\ \text{Hours} \\ \text{Real Uncertainty (LMN)} \\ \text{Fed Funds Rate} \\ \log\left(\text{M2}\right) \\ \log\left(\text{S\&P-500 Index}\right) \\ \text{Financial Uncertainty (LMN)} \end{bmatrix} \tag{1}$$

Both LMN real and financial uncertainty measures are included to differentiate the effects of financial volatility shocks from the effects from real uncertainty. For similar reasons, we include the S&P-500 index in our VAR to empirically distinguish between shocks affecting the level of financial markets and shocks affecting their volatility. In order to ameliorate possible concerns about the validity of the structural identification strategy, we estimate our VAR using monthly data, where the identification assumptions are more likely to hold. Figure 1 presents the impulse responses to the orthogonalized financial uncertainty shock. Panel 1a plots the response of industrial production, which falls by up to 2.5% and displays moderate persistence following a one standard deviation shock to financial uncertainty. Panel 1b plots the response of the S&P-500 Index, which drops up to 12% within the first four months before gradually recovering. Together, both pictures imply a rise of financial uncertainty depresses both industrial activity and financial markets.

Figure 1 also features alternative estimates using common financial uncertainty proxies such as Bloom (2009) stock market volatility index and 10-years premium on Baa-rated corporate bonds. The responses are generally more muted, and take the opposite sign

[2]The ordering is used by Ludvigson et al. (2021), who, using identification strategy based on event constraints, find that the uncertainty of financial markets tends to be an exogenous source of business cycle fluctuations, while the real uncertainty is more likely an endogenous response to the business cycle fluctuations. We also have implemented alternative specifications and orderings that produced qualitatively similar results (not reported, provided upon request).



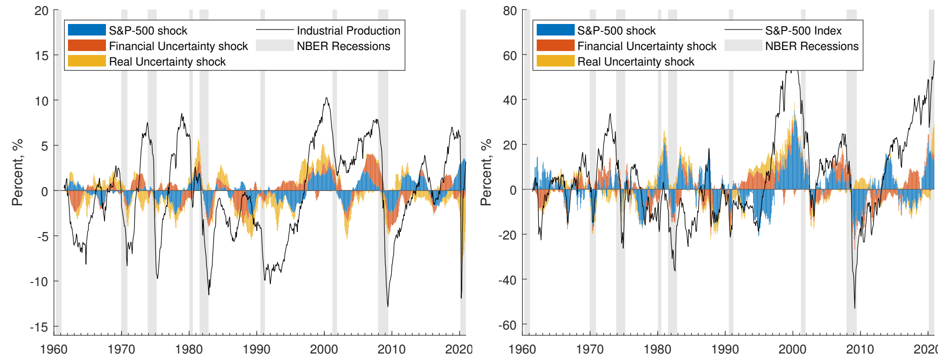
(a) Response: Industrial Production

(b) Response: S&P-500 Index

Figure 1: Impulse Response Functions (IRFs), selected series. Figures 1a and 1b display the response to a one standard deviation financial uncertainty shock of monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with the variable composition and ordering given in (1). Shaded area indicates 95% confidence interval around financial uncertainty measure computed using standard bootstrap techniques.

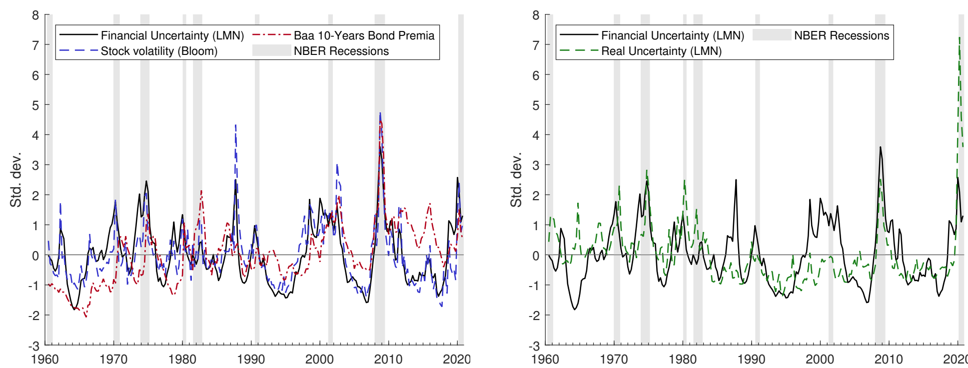
in the case of the S&P Index. These results can be explained by the fact that standard proxies contain information unrelated to financial uncertainty that distorts our estimates (see Jurado et al. (2015) for a discussion), and therefore we choose LMN as our preferred financial uncertainty measure. In Appendix B, we report additional impulse response estimates. Especially, the Figure 5 in Appendix B shows that monetary authorities respond with accommodating interest rate movements to financial uncertainty shocks, while real uncertainty has no statistically significant effect on either interest rates or stock market fluctuations.

Finally, we can further explore the contribution of financial uncertainty to business cycles fluctuations by looking at Table 1 in Appendix B, which reports the Forecast Error Variance Decomposition (FEVD) of Industrial Production and the S&P-500 Index. Financial uncertainty shocks explain close to 5% of the fluctuations in both series, while real uncertainty explains an additional 2-4% of movements in industrial activity in the medium run. Figure 2 provides a more graphical illustration of these results by plotting the historical decomposition of the series. We observe that the contribution of financial uncertainty rivals that of shocks to the level of financial variables captured by the S&P-500 shock, and is especially important in driving industrial production boom-bust patterns during and in the preceding months of recessionary episodes.



(a) Industrial Production (b) S&P-500 Index

Figure 2: Historical Decomposition, selected series. Figures 2a and 2b display the historical decomposition of monthly Industrial Production and S&P-500 Index series, respectively, based on the VAR-11 with variable composition and ordering in (1). Variables are de-trended by subtracting the contribution of initial conditions and constant terms after series decomposition. Columns report a contribution of each shock to the fluctuations around trend of the variable considered.



(a) Financial Uncertainty series (b) Financial vs. Real Uncertainty

Figure 3: Uncertainty series. Figure 3a displays common measures of financial uncertainty. Figure 3b displays Ludvigson et al. (2021) (henceforth, LMN) measures of financial and real economic uncertainty. LMN financial and real economic uncertainty series are constructed as the average volatility of the residuals from predictive regressions on financial and real economic variables, respectively (See Ludvigson et al. (2021)). Bloom (2009)'s stock market volatility is constructed using VXO data from 1987 onward and the monthly volatility of the S&P 500 index normalized to the same mean and variance in the overlapping interval for the 1960-1987 period (See Bloom (2009)). The bond risk-premia series is the Moody's seasoned Baa corporate bond yield relative to the yield on a 10-year treasury bond at constant maturity. The depicted series have a normalized zero mean and one standard deviation.

B Additional Figures and Tables

(i) Industrial Production

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0	0.30	0.21	0.12
h=6	1.27	3.37	2.98	1.36
h=12	4.28	4.38	3.16	1.94
h=36	3.24	1.67	1.98	0.64

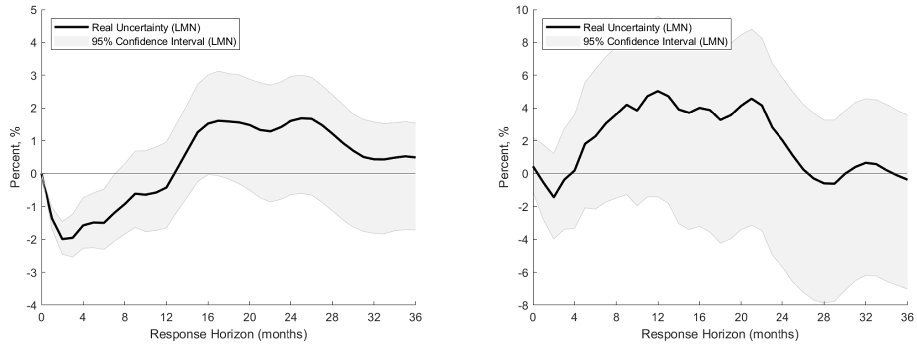
(ii) S&P-500 Index

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.11	0.08	0.39	0.06
h=6	3.30	0.25	3.26	0.62
h=12	4.77	0.54	10.03	2.16
h=36	6.50	0.91	12.16	2.40

(iii) Fed Funds Rate

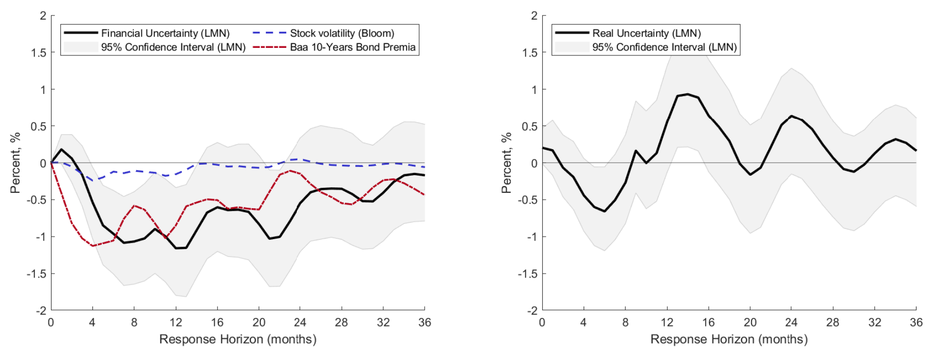
Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.01	0.98	0	0.08
h=6	0.42	0.84	3.11	1.66
h=12	1.47	0.91	4.69	2.30
h=36	2.81	2.05	5.02	3.17

Table 1: Forecast Error Variance Decomposition (FEVD). The table presents the variance contribution (in percentage) of financial and real uncertainty shocks to selected series at different time horizons (in months). The FEVD is constructed using a VAR-11 with equation (1) variable composition and ordering. The first two columns report the contribution of LMN financial and real uncertainty shocks, respectively. The last two columns report alternative VAR specifications where the preferred LMN financial uncertainty measure (column one) is replaced by common proxies employed in the literature, either Bloom (2009) stock market volatility measure or the Baa 10-years corporate bond premia, respectively.



(a) Response: Industrial Production (b) Response: S&P-500 Index

Figure 4: Impulse Response Functions (IRFs), selected series. Figures 4a and 4b display the response to one standard deviation real uncertainty shock by monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (1) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.

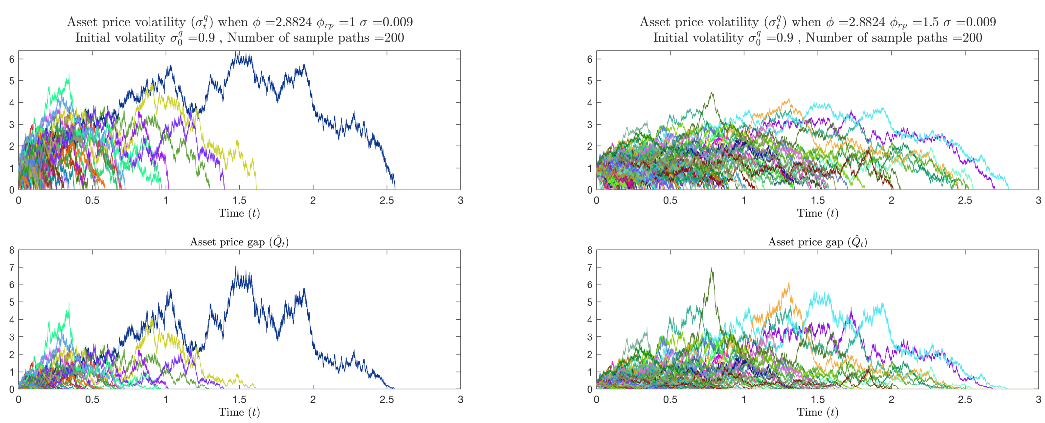


(a) Shock: Financial Uncertainty (b) Shock: Real Uncertainty

Figure 5: Impulse Response Functions (IRFs), Fed Funds Rate. This Figure displays the response to a one standard deviation uncertainty (financial or real) shock by monthly Fed Funds Rate series, using a VAR-11 with equation (1) variable composition and ordering. Panel 5a plots the response to a financial uncertainty shock, and Panel 5b to a real uncertainty shock. Shaded area indicates 95% confidence interval around preferred financial/real uncertainty measure computed using standard bootstrap techniques. Additional lines display alternative impulse responses obtained by substituting preferred LMN financial uncertainty measure with common proxies employed in the literature.

$\phi_{\mathbf{rp}} < 0$ **(Real Bills Doctrine)**	$0 \leq \phi_{\mathbf{rp}} < \frac{1}{2}$
(i) With $\phi_{\mathrm{rp}} \downarrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)	(i) With $\phi_{\mathrm{rp}} \uparrow$, convergence speed$\uparrow$ and more amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)
$\phi_{\mathbf{rp}} = \frac{1}{2}$	$\phi_{\mathbf{rp}} > \frac{1}{2}$
No sunspot (ultra-divine coincidence)	(i) With $\phi_{\mathrm{rp}} \uparrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a boom ($\hat{Q}_t > 0$ and $\pi_t > 0$)
As $\phi \uparrow$, convergence speed$\uparrow$ and $\exists$ more amplified paths	

Table 2: Effects of different parameters $\{\phi_{\mathrm{rp}},\ \phi\}$ on stabilization in Section 4



(a) With $\phi_{\mathrm{rp}} = 1$

(b) With $\phi_{\mathrm{rp}} = 1.5$.

Figure 6: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$, with varying $\phi_{\mathrm{rp}} > \frac{1}{2}$

C Additional Derivations and Proofs

Proof of Lemma 2. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, where we use equation (**??**). Here $\frac{w_t^n}{p_t^n}$ is the real wage level in the flexible price economy. The good market equilibrium condition can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}}. \tag{C.1}$$

We subtract equation (C.1) from the same good market condition in the sticky price economy to obtain

$$A_t \left(\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}} = (C_t - C_t^n) + \left(\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}}, \tag{C.2}$$

where we divide both sides of equation (C.2) by $y_t^n \equiv A_t^{1-\frac{1}{\chi}} (\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$ and obtain

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\widehat{\frac{w_t}{p_t}}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\widehat{\frac{w_t}{p_t}}}, \tag{C.3}$$

which can be written as:

$$\frac{1}{\chi} \frac{\widehat{w_t}}{p_t} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon} \underbrace{\left(1 + \frac{1}{\chi}\right) \frac{\widehat{w_t}}{p_t}}_{=\hat{C^w}(t)}. \tag{C.4}$$

Equation (C.4) with $\hat{C}_t = \hat{Q}_t$ leads to

$$\hat{Q}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \frac{\widehat{w_t}}{p_t} = \underbrace{\frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{C_{W,t}}.$$

We observe that Assumption 1 guarantees that gaps of asset price, consumption of capitalists and workers, employment, and real wage all co-move with positive correlations. Now we can use $\hat{Q}_t$ and $\hat{C}_t$ interchangeably, and if one gap variable becomes 0, then all other gap variables become also stabilized to 0, up to a first order.

■

Proof of Proposition 4. Firms change their prices with instantaneous probability δdt à la Calvo (1983). If there is price dispersion Δ_t, as defined in (20), across intermediate goods firms, then labor market equilibrium condition can be written as

$$N_{W,t} = \int_0^1 n_t(i)di = \left(\frac{y_t}{A_t\left(N_{W,t}\right)^{\alpha}}\right)^{\frac{1}{1-\alpha}} \underbrace{\int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} di,}_{\equiv \Delta_t^{\frac{1}{1-\alpha}}} \tag{C.5}$$

where

$$y(t) = \frac{A_t N_{W,t}}{\Delta_t} = C_t + C_{W,t}. \tag{C.6}$$

We know that the good market equilibrium condition in (26) can be written as

$$\rho A_t Q_t + A_t \left(\frac{w_t}{p_t A_t}\right)^{1+\frac{1}{\chi}} = A_t \left(\frac{w_t}{p_t A_t}\right)^{\frac{1}{\chi}} \frac{1}{\Delta_t}. \tag{C.7}$$

Since a price process (i.e., (21)) does not affect the resource allocation in the flexible price economy, we can regard $\hat{x}_t$ to be the $\log$-deviation of x_t from the flexible price economy *where the price is constant*. From price aggregator in (17), we obtain

$$\hat{p}_t = \int_0^1 \widehat{p_t(i)} di. \tag{C.8}$$

To study price dispersion Δ_t up to a first-order, we illustrate Woodford (2003)'s treatment of Δ_t up to a second-order. From

$$\begin{aligned} \frac{1}{1-\alpha}\hat{\Delta}_t &= \ln \int_0^1 \left(1 - \frac{\epsilon}{1-\alpha}\left(\cancel{\widehat{p_t(i)} - \hat{p}_t}\right) + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \left(\widehat{p_t(i)} - \hat{p}_t\right)^2\right) di + \text{h.o.t.} \\ &= \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 Var_i\left(\widehat{p_t(i)}\right) + \text{h.o.t,} \end{aligned} \tag{C.9}$$

where h.o.t stands for higher-order terms, we observe that $\Delta_t \simeq 1$ up to a first-order because Δ_t is in nature the second order as (C.9) suggests. Pricing à la Calvo (1983) is standard, except that our model is in continuous time. For dt period from t to $t + dt$, individual firm i change the price with δdt probability. From time 0 perspective, a probability that firm resets its price for the first time at time t is

$$\delta e^{-\delta t} dt = \underbrace{\delta dt}_{\text{Change now}} \cdot \underbrace{e^{-\delta t}}_{\text{No change until t}}. \tag{C.10}$$

At time t, a price-changing firm i chooses $p_t(i)$ to solve

$$\begin{aligned}
&\max_{p_t(i)} \frac{1}{\xi_t^N p_t} \mathbb{E}_t \int_t^\infty e^{-\delta(s-t)} \xi_s^N p_s \left(\frac{p_t(i)}{p_s} y_{s|t}(i) - \frac{1}{p_s} C(y_{s|t}(i)) \right) ds, \text{ with } y_{s|t}(i) = \left(\frac{p_t(i)}{p_s} \right)^{-\epsilon} y_s \\
&= \frac{1}{\xi_t^N p_t} \mathbb{E}_t \int_t^\infty e^{-\delta(s-t)} \xi_s^N p_s \left(\left(\frac{p_t(i)}{p_s} \right)^{1-\epsilon} y_s - \frac{1}{p_s} C\left(\left(\frac{p_t(i)}{p_s} \right)^{-\epsilon} y_s \right) \right) ds,
\end{aligned} \tag{C.11}$$

where $C(\cdot)$ is defined as an individual firm's nominal production cost as a function of its output produced, which is to be written explicitly. Let $MC_{s|t}$ and $\varphi_{s|t}$ be the nominal and real marginal cost at time s conditional on price resetting at prior time t. Using the nominal pricing kernel ξ_s^N formula in (23), we obtain

$$\frac{\xi_s^N p_s}{\xi_t^N p_t} = e^{-\rho(s-t)} \frac{C_t}{C_s}. \tag{C.12}$$

By plugging (C.12) into (C.11) and solving (C.11), the optimal adjusted price p_t^*[3] is given as

$$p_t^* = \frac{\mathbb{E}_t \displaystyle\int_t^\infty e^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} \frac{\varphi_{s|t}}{\bar{\varphi}} p_s^\epsilon ds}{\mathbb{E}_t \displaystyle\int_t^\infty e^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} p_s^{\epsilon-1} ds}, \tag{C.13}$$

where $\varphi_{s|t}$, the real marginal cost of firms at time s given the price resetting at previous time t, appears, and $\bar{\varphi}$ is its level in the flexible-price equilibrium, which is $\frac{\epsilon-1}{\epsilon}$. If we log-linearize (C.13) around the flexible price equilibrium with constant price as in (C.8), we can log-linearize $\widehat{p_t^*}$ expressed as

$$\widehat{p_t^*} = (\delta+\rho) \mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)(s-t)} \left(\hat{\varphi}_{s|t} + \hat{p}_s \right) ds. \tag{C.14}$$

We know that the conditional real production cost and the conditional real marginal cost

[3] We use the property that every price-setting firm at any time t chooses the same price, so we drop the firm index i in $p_t^*(i)$ and use p_t^*.

can be written as

$$\frac{1}{p_s}C(y_{s|t}) = \frac{w_s}{p_s}\left(\frac{y_{s|t}}{A_s(N_{W,s})^\alpha}\right)^{\frac{1}{1-\alpha}}, \tag{C.15}$$

and

$$\varphi_{s|t} \equiv \frac{1}{p_s}C'(y_{s|t}) = \frac{w_s}{p_s}\left(\frac{y_{s|t}}{A_s(N_{W,s})^\alpha}\right)^{\frac{\alpha}{1-\alpha}}\frac{1}{A_s(N_{W,s})^\alpha}. \tag{C.16}$$

From equation (C.16)), we obtain the conditional real marginal cost gap at time s conditional on price resetting at time t, which is given by

$$\hat{\varphi}_{s|t} = \underbrace{\frac{\hat{w}_s}{p_s}}_{\equiv\hat{\varphi}_s} - \frac{\alpha\epsilon}{1-\alpha}\left(\widehat{p_t^*} - \hat{p}_s\right) = \hat{\varphi}_s - \frac{\alpha\epsilon}{1-\alpha}\left(\widehat{p_t^*} - \hat{p}_s\right). \tag{C.17}$$

where $\hat{\varphi}_s$ is defined as the aggregate marginal cost index: as production is linear in aggregate level, $\hat{\varphi}_s$ becomes equal to the real wage gap. Using (C.8), we then characterize the change in aggregate price gap $\hat{p}_t$ as

$$\begin{aligned} d\hat{p}_t &= \delta dt\left(\widehat{p_t^*} - \hat{p}_t\right) \\ &= \delta dt(\delta+\rho)\mathbb{E}_t\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)}\left(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t\right)ds, \quad \text{where } \Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \end{aligned} \tag{C.18}$$

Since we log-linearize our economy around the flexible price equilibrium with constant price (i.e., $\pi_t = \sigma_t^p = 0$ in (21)), $\hat{p}_t$ changes with a rate of current π_t,[4] we have

$$\pi_t = \frac{d\hat{p}_t}{dt} = \delta(\delta+\rho)\mathbb{E}_t\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)}\left(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t\right)ds. \tag{C.19}$$

Now that we have (C.19) for the instantaneous inflation π_t. we manipulate (C.19) as:

$$\begin{aligned} \pi_t + \delta\hat{p}_t &= \delta(\delta+\rho)\mathbb{E}_t\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s + \hat{p}_s)ds \\ &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t\int_t^\infty \mathrm{e}^{-(\delta+\rho)s}\left(\Theta\hat{\varphi}_s + \hat{p}_s\right)ds \\ &= \delta(\delta+\rho)(\Theta\hat{\varphi}_t + \hat{p}_t)dt + \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t\int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}\left(\Theta\hat{\varphi}_s + \hat{p}_s\right)ds, \end{aligned} \tag{C.20}$$

[4]In the case of positive inflation targets, see e.g., Coibion et al. (2012).

where we can rewrite the first line of equation (C.20) at time $t+dt$ instead of t as

$$\begin{aligned}\pi_{t+dt}+\delta\hat{p}_{t+dt} &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)(t+dt)}\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}\left(\Theta\hat{\varphi}_s+\hat{p}_s\right)ds\\ &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\left(1+(\delta+\rho)dt\right)\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}\left(\Theta\hat{\varphi}_s+\hat{p}_s\right)ds.\end{aligned} \tag{C.21}$$

Due to the *martingale representation theorem* (see e.g., Oksendal (1995)), there exists a measurable H_t such that

$$\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds=\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds+H_t dZ_t, \tag{C.22}$$

holds. We plug (C.22) into equation (C.21) to obtain[5]

$$\begin{aligned}\pi_{t+dt}+\delta\hat{p}_{t+dt}=\delta(\delta+\rho)\Bigg(&\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds+\mathrm{e}^{(\delta+\rho)t}H_t dZ_t\\ &+\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds\Bigg).\end{aligned} \tag{C.23}$$

We subtract (C.20) from (C.23) to obtain

$$\begin{aligned}&d\pi_t+\cancel{\delta}\pi_t dt\\ &=\delta(\delta+\rho)\left(\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds+\mathrm{e}^{(\delta+\rho)t}H_t dZ_t-(\Theta\hat{\varphi}_t+\hat{p}_t)dt\right)\\ &=\underbrace{\delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}H_t}_{\equiv\sigma_{\pi,t}}dZ_t-\delta(\delta+\rho)\Theta\hat{\varphi}_t dt\\ &\quad+\underbrace{\delta(\delta+\rho)\left((\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds\right)}_{=(\cancel{\delta}+\rho)\pi_t dt},\end{aligned} \tag{C.24}$$

where we use

$$(\delta+\rho)dt\cdot\mathbb{E}_t\int_{\color{red}{t+dt}}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds=(\delta+\rho)dt\cdot\mathbb{E}_t\int_{\color{red}{t}}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds, \tag{C.25}$$

which holds from the property $(dt)^2=0$. Note that in (C.24), we define $\sigma_{\pi,t}$ as an instanta-

[5] We use the property that $dt\cdot dZ_t=0$.

neous volatility of the inflation process. Finally from equation (C.24) we get the continuous time version of New Keynesian Phillips curve (NKPC), written as[6]

$$d\pi_t = \rho\pi_t dt - \delta(\delta+\rho)\Theta\hat{\varphi}_t dt + \sigma_{\pi,t} dZ_t. \tag{C.26}$$

Due to the linear aggregate production function up to a first-order, we obtain:[7]

$$\hat{\varphi}_t = \frac{\widehat{w_t}}{p_t} = \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1}\hat{Q}_t \equiv \frac{\kappa}{\delta(\delta+\rho)\Theta}\hat{Q}_t. \tag{C.27}$$

Finally plugging equation (C.27) into equation (C.26), we represent New-Keynesian Phillips curve in terms of asset price gap $\hat{Q}_t$ in the following way:

$$d\pi_t = \left(\rho\pi_t - \kappa\hat{Q}_t\right)dt + \sigma_{\pi,t}dZ_t, \text{ and } \mathbb{E}_t d\pi_t = \left(\rho\pi_t - \kappa\hat{Q}_t\right)dt, \tag{C.28}$$

which proves the proposition 4.[8] We know $\kappa > 0$ due to Assumption 1. ∎

D Detailed Derivations in Section 2

D.0. Model Setup

A representative household solves the following intertemporal optimization consumption-savings decision problem:

$$\max_{\{C_s,L_s\}_{s\geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)}\left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s \quad \text{s.t.} \quad \mathrm{d}B_t = \left[i_t B_t - p_t C_t + w_t L_t + D_t\right]\mathrm{d}t$$

where C_t is consumption, L_t aggregate labor, w_t is the equilibrium wage level, B_t are risk-free bonds held by the household at the beginning of t (hence, B_t at t is taken as given for each household), i_t is the nominal interest rate, D_t is a lump-sum transfer of any firm profits/losses towards the household, p_t the nominal price of consumption goods and ρ is the subjective discount rate of the household.

[6]Our continuous-time version of the Phillips curve in (C.24) is of the same form as in Werning (2012) and Cochrane (2017) after taking expectation on both sides.

[7]We use Lemma 2's log-linearization result to represent the real aggregate marginal cost gap $\frac{\hat{w}_t}{p_t}$ as a function of capitalists' consumption gap $\hat{C}_t = \hat{Q}_t$.

[8]Since $\hat{y}_t = \zeta\hat{Q}_t$, Phillips curve can be represented in terms of output gap $\hat{y}_t$ as in Proposition 4.

An individual firm i produces in this economy with the following production function:

$$Y_t^i = A_t L_t^i, \text{ where}$$
$$\frac{\mathrm{d}A_t}{A_t} = g\mathrm{d}t + \underbrace{\sigma}_{\text{Fundamental risk}} \mathrm{d}Z_t$$

where A_t is the economy's total factor productivity, assumed to be exogenous and to follow a geometric Brownian motion with drift, where g is the expected growth rate of A_t, σ is its volatility, which we assume to be constant over time and call *fundamental* volatility, and Z_t is a standard Brownian motion process. It follows that firms' profits are defined as:

$$D_t = p_t Y_t - w_t L_t$$

Finally, we assume that in equilibrium, bonds are in zero net supply (i.e. $B_t = 0, \ \forall t$) and that there is no government spending, so market clearing in this economy results in $C_t = Y_t$.

D.1. Flexible Price Economy

We first solve the flexible price economy as our benchmark economy. In that purpose, we assume the usual Dixit Stiglitz monopolistic competition among firms, where the demand each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. Each firm i takes the aggregate price p_t, wage w_t, and the aggregate output Y_t as given.

D.1.1. Household problem

In the flexible price economy, each household takes $\{A_t, p_t, i_t\}$ process as given:

$$\frac{dp_t}{p_t} = \pi_t dt + \sigma_t^p dZ_t \tag{D.1}$$

and

$$di_t = \mu_t^i dt + \sigma_t^i dZ_t \tag{D.2}$$

where π_t, σ_t^p, μ_t^i, and σ_t^i are all endogenous, so the state variable for each household would become $\{B_t, A_t, p_t, i_t\}$.[9]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, p_t, i_t, t\right) = \max_{\{C_s, L_s\}_{s\geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s.$$

The formula for the stochastic HJB equation is given as:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t} \right\}. \tag{D.3}$$

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^\Gamma dt + \sigma_t^\Gamma dZ_t \tag{D.4}$$

where

$$\begin{aligned} \mu_t^\Gamma = & \Gamma_t + \Gamma_B \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_p \cdot p_t \pi_t + \Gamma_i \cdot \mu_t^i \\ & + \frac{1}{2}\Gamma_{AA} \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{pp} \cdot (p_t\sigma_t^p)^2 + \frac{1}{2}\Gamma_{ii} \cdot (\sigma_t^i)^2 \\ & + \Gamma_{Ap} \cdot (\sigma A_t)(p_t\sigma_t^p) + \Gamma_{Ai} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{pi} \cdot (p_t\sigma_t^p)\sigma_t^i \end{aligned} \tag{D.5}$$

and $\sigma_t^\Gamma = \Gamma_A(\sigma A_t) + \Gamma_p(p_t\sigma_t^p) + \Gamma_i(\sigma_t^i)$. In the same way, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$ where

$$\begin{aligned} \mu_t^{\Gamma_B} = & \Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{Bp} \cdot p_t \pi_t + \Gamma_{Bi} \cdot \mu_t^i \\ & + \frac{1}{2}\Gamma_{BAA} \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{Bpp} \cdot (p_t\sigma_t^p)^2 + \frac{1}{2}\Gamma_{Bii} \cdot (\sigma_t^i)^2 \\ & + \Gamma_{BAp} \cdot (\sigma A_t)(p_t\sigma_t^p) + \Gamma_{BAi} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{Bpi} \cdot (p_t\sigma_t^p)\sigma_t^i \end{aligned} \tag{D.6}$$

[9] This is a conjectural but correct statement due to the classical dichotomy between real and nominal sectors: output, consumption, and labor in equilibrium turn out to depend on A_t only and it turns out that p_t and i_t do not matter for the real economy and the welfare of the households.

and $\sigma_t^{\Gamma_B} = \Gamma_{BA}(\sigma A_t) + \Gamma_{Bp}(p_t\sigma_t^p) + \Gamma_{Bi}(\sigma_t^i)$. Note $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, p, i\}$. Now plug equation (D.4) into equation (D.3) to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^\Gamma \right\}. \tag{D.7}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (D.7), we obtain:

$$\Gamma_B = \frac{1}{p_t C_t} \tag{D.8}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t} \tag{D.9}$$

Finally, merging (D.8) with (D.9) gives us the optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$, and $dQ_t = \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (D.7) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*} \tag{D.10}$$

Taking the derivative of both sides of equation (D.10) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*} \tag{D.11}$$

where $\mu_t^{\Gamma_B,*}$ is from equation (D.6) and it is evaluated at the optimum. Plugging (D.11) into the process for Γ_B, we obtain a simplified expression:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA}(A_t\sigma) + \Gamma_{Bp}(p_t\sigma_t^p) + \Gamma_{Bi}\left(\sigma_t^i\right)\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t \tag{D.12}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then, using equation (D.12) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t$$

From the definition of $\mathrm{d}Q_t$, we obtain:

$$\mathrm{d}Q_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t \tag{D.13}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ follows by taking expectations, which proves (2) in the flexible price equilibrium.

Nominal and real interest rates Prices and consumption would be adapted to the filtration generated by our Brownian motion Z_t process. Let us express the processes for consumption and price as:

$$\mathrm{d}p_t = \pi_t p_t \mathrm{d}t + \sigma_t^p p_t \mathrm{d}Z_t \tag{D.14}$$

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + \sigma_t^C C_t \mathrm{d}Z_t \tag{D.15}$$

where π_t, g_t^C, σ_t^p and σ_t^C are variables to be determined in equilibrium, which can be interpreted as inflation rate, expected consumption growth, and volatilities of prices and consumption processes, respectively. As the real state density is defined as $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$, the real interest rate r_t is defined by the relation $\mathbb{E}_t\left[\frac{d\zeta_t^r}{\zeta_t^r}\right] = -r_t dt$, similarly to (2).

With (D.15), applying Ito's Lemma to the real state density $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$ results in

$$\frac{d\zeta_t^r}{\zeta_t^r} = -\underbrace{\left[\rho + g_t^C - \left(\sigma_t^C\right)^2\right]}_{\equiv r_t} \mathrm{d}t - \sigma_t^C \mathrm{d}Z_t. \tag{D.16}$$

which determines the real interest rate $r_t = \rho + g_t^C - (\sigma_t^C)^2$. We also apply Ito's Lemma to $\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$ and use the above processes for p_t and C_t to obtain:

$$\mathrm{d}Q_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\left[\rho + g_t^C + \pi_t - (\sigma_t^p)^2 - \left(\sigma_t^C\right)^2 - \sigma_t^p \sigma_t^C\right] \mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right] \mathrm{d}Z_t$$

which can be rearranged as:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\underbrace{\left[r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right)\right]}_{=i_t}\mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right]\mathrm{d}Z_t \tag{D.17}$$

Comparing equation (D.13) and equation (D.17), we obtain

$$\begin{aligned} i_t &= r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right), \\ \text{where:} \quad r_t &= \rho + g_t^C - \left(\sigma_t^C\right)^2. \end{aligned}$$

D.1.2. Firm problem and equilibrium

Firm optimization As the demand each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t$$

as usual where p_t^i is an individual firm's price, p_t is the price aggregator, and Y_t is the aggregate output, each firm i solves the following problem:

$$\max_{p_t^i} \quad p_t^i\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t - \frac{w_t}{A_t}\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t, \tag{D.18}$$

which results in the following first-order condition for the firm:[10]

$$p_t = \left(\frac{\varepsilon}{\varepsilon - 1}\right)\frac{w_t}{A_t}, \tag{D.19}$$

which is intuitive as it tells us that in equilibrium, price is equal to the marginal cost of production multiplied by the constant mark-up, due to the constant elasticity of demand $\varepsilon > 1$. Using equation (D.19) and the equilibrium condition $C_t = Y_t = A_t L_t$ in the first-order condition of the household in (D.8) and (D.9), we obtain $L_t^n = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$,[11] which is a constant. This implies: in the flexible price equilibrium, we have $C_t^n = Y_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$.

[10] In equilibrium $p_t^i = p_t$ as every firm chooses the same price level.

[11] We impose the superscript n (i.e., natural) in variables to denote that those are the equilibrium values in the flexible price economy.

It follows that the stochastic process for Y_t^n is the same as that for A_t as follows:

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \frac{\mathrm{d}C_t^n}{C_t^n} = g\mathrm{d}t + \sigma\mathrm{d}Z_t. \tag{D.20}$$

(D.20) implies that the growth rate of consumption and its volatility are $g_t^C = g$ and $\sigma_t^C = \sigma$, so the real interest rate in the flexible price economy, i.e., the natural rate of interest, can be expressed as $r_t^n \equiv r^n = \rho + g - \sigma^2$ from (D.16), which finally gives

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \left(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2 \right) \mathrm{d}t + \sigma\mathrm{d}Z_t$$

that proves equation (5).

D.2. Rigid Price Economy

We then solve our rigid price economy with $p_t = \bar{p}$ for $\forall t$. First, let us say the rigid price economy's consumption volatility, which we call σ_t^C is given as $\sigma_t^C = \sigma + \sigma_t^s$ (i.e. volatility of flexible price equilibrium in (D.20), plus excess volatility of rigid price equilibrium). Therefore, the consumption process can be written as:

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + (\sigma + \sigma_t^s) C_t \mathrm{d}Z_t. \tag{D.21}$$

And let us conjecture that this endogenous 'excess' volatility σ_t^s follows $d\sigma_t^s = \mu_t^\sigma dt + \sigma_t^\sigma dZ_t$, which turns out to be one of state variables in the rigid price economy. With price rigidity (i.e., $p_t = \bar{p}$ for $\forall t$), the agent takes $\{A_t, \sigma_t^s\}$ process as given, so the state variable for each household would become $\{B_t, A_t, \sigma_t^s\}$.[12]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, \sigma_t^s, t\right) = \max_{\{C_s, L_s\}_{s \geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] \mathrm{d}s$$

[12]This is a conjectural (but correct) statement as the actual output (thereby, consumption and other variables including inflation, nominal interest rate (that follows the Taylor rule), etc) would turn out to only depend on A_t and σ_t^s under our equilibrium construction.

The formula for the stochastic HJB equation is:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t} \right\} \tag{D.22}$$

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^{\Gamma} dt + \sigma_t^{\Gamma} dZ_t \tag{D.23}$$

where

$$\begin{aligned} \mu_t^{\Gamma} =& \Gamma_t + \Gamma_B \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_\sigma \cdot \mu_t^{\sigma} \\ &+ \frac{1}{2}\Gamma_{AA} \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{\sigma\sigma} \cdot (\sigma_t^{\sigma})^2 + \Gamma_{A\sigma} \cdot (A_t\sigma)(\sigma_t^{\sigma}) \end{aligned} \tag{D.24}$$

and $\sigma_t^{\Gamma} = \Gamma_A(\sigma A_t) + \Gamma_\sigma(\sigma_t^{\sigma})$. Applying Ito's Lemma to Γ_B, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$ where

$$\begin{aligned} \mu_t^{\Gamma_B} =& \Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{B\sigma} \cdot \mu_t^{\sigma} \\ &+ \frac{1}{2}\Gamma_{BAA} \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{B\sigma\sigma} \cdot (\sigma_t^{\sigma})^2 + \Gamma_{BA\sigma} \cdot (A_t\sigma)(\sigma_t^{\sigma}) \end{aligned} \tag{D.25}$$

and $\sigma_t^{\Gamma_B} = \Gamma_{BA} \cdot (\sigma A_t) + \Gamma_{B\sigma} \cdot \sigma_t^{\sigma}$. Note $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, \sigma_t^s\}$. Now plug equation (D.23) into equation (D.22) to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma} \right\} \tag{D.26}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (D.26), we obtain:

$$\Gamma_B = \frac{1}{\bar{p} C_t} \tag{D.27}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t} \tag{D.28}$$

Finally, merging (D.27) with (D.28) gives us the optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t} \frac{1}{\bar{p}C_t}$, and $dQ_t = \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (D.26) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*} \tag{D.29}$$

Taking the derivative of both sides of equation (D.29) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*} \tag{D.30}$$

where $\mu_t^{\Gamma_B,*}$ is from equation (D.25) and it is evaluated at the optimum. Plugging equation (D.30) into the process for Γ_B, we obtain a simplified expression at the optimum:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA} \cdot (A_t \sigma) + \Gamma_{B\sigma} \cdot (\sigma_t^\sigma)\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t \tag{D.31}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then using equation (D.31) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t$$

From the previous equation, we obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t \tag{D.32}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ also follows in the rigid price economy by taking conditional expectations.

Verification of the Martingale Equilibrium Now let us verify that our martingale equilibrium, characterized by equations (13) and (14), satisfies our equilibrium conditions derived above. From (13) and (14),

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}, \tag{D.33}$$

$$\mathrm{d}\sigma_t^s = \underbrace{-(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3}}_{=\mu_t^\sigma} \mathrm{d}t \underbrace{-\phi_y \left(\frac{\sigma_t^s}{\sigma_t + \sigma_t^s}\right)}_{=\sigma_t^\sigma} \mathrm{d}Z_t. \tag{D.34}$$

These equations will be a solution to the model, as long as there is no contradiction with the equilibrium conditions. In order to check if (D.33) and (D.34) satisfy the equilibrium conditions, first, the output gap is defined as:

$$\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right) = \log\left(\frac{C_t}{C_t^n}\right) = \log\left(\frac{C_t}{A_t}\right) - \frac{\eta}{\eta+1}\log\left(\frac{\varepsilon-1}{\varepsilon}\right) \tag{D.35}$$

where the last equality follows from $C_t^n = A_t \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$, as shown above for the flexible price equilibrium. Combining (D.33) and (D.35), we obtain:

$$C_t = A_t \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}} \cdot \exp\left\{-\frac{(\sigma+\sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}\right\}, \tag{D.36}$$

which is a function of A_t and σ_t^s. Under fully sticky prices (i.e. $p_t = \bar{p}$, for), From equation (D.27) we knows

$$\Gamma_B = \frac{1}{\bar{p}C_t}. \tag{D.37}$$

We can now compute the derivative of equation (D.37) with respect to A_t and σ_t^s as:

$$\Gamma_{BA} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial A_t}, \tag{D.38}$$

$$\Gamma_{B\sigma} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s}. \tag{D.39}$$

Plugging equations (D.38) and (D.39) into equation (D.31), we obtain:

$$\mathrm{d}\Gamma_B = (\rho - i_t)\cdot\Gamma_B \mathrm{d}t - \Gamma_B\left[\frac{A_t}{C_t}\cdot\frac{\partial C_t}{\partial A_t}\cdot\sigma + \frac{1}{C_t}\cdot\frac{\partial C_t}{\partial \sigma_t^s}\cdot\sigma_t^\sigma\right]\mathrm{d}Z_t. \tag{D.40}$$

Using Ito's Lemma in equation (D.37) together with equation (D.21), we obtain

$$d\Gamma_B = -\Gamma_B \left(g_t^C - (\sigma_t^C)^2\right) dt - \Gamma_B(\sigma + \sigma_t^s) dZ_t. \tag{D.41}$$

Comparing the volatility terms in (D.40) and (D.41) (i.e., terms multiplied to dZ_t), it must follow that:

$$\sigma + \sigma_t^s = \frac{A_t}{C_t} \cdot \frac{\partial C_t}{\partial A_t} \cdot \sigma + \frac{1}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s} \cdot \sigma_t^\sigma. \tag{D.42}$$

We can now compute the derivative of C_t with respect to A_t and σ_t^s as:

$$\frac{\partial C_t}{\partial A_t} = \frac{C_t}{A_t}, \tag{D.43}$$

and

$$\frac{\partial C_t}{\partial \sigma_t^s} = C_t \cdot \left(\frac{-(\sigma + \sigma_t^s)}{\phi_y}\right) = C_t \cdot (\sigma_t^\sigma)^{-1} \cdot \sigma_t^s, \tag{D.44}$$

which satisfies (D.42). Therefore, our martingale equilibrium is verified as an equilibrium.

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