Higher-Order Forward Guidance*

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Abstract

This paper develops a model of business cycles with endogenous volatility at the zero lower bound (ZLB) and central-bank forward guidance. We deliver three main results. First, a credible commitment to future stabilization curbs excess volatility at the ZLB. Second, pledging not to stabilize later can shift the economy onto more favorable equilibrium paths, exposing a trade-off between future stabilization and reduced aggregate volatility at the ZLB. Third, keeping the timing of future stabilization uncertain strictly dominates other forward-guidance strategies.

Keywords: Monetary Policy, Forward Guidance, Aggregate Volatility, Risk Premium
JEL Codes: E32, E43, E44, E52, E62

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1 Introduction

In the aftermath of the Great Recession and the COVID-19 pandemic, prolonged periods of constrained policy rates at the Zero Lower Bound (ZLB) have underscored the need for alternative monetary interventions, notably forward guidance. ZLB episodes are typically associated with heightened economic volatility, a phenomenon compounded by the diminished efficacy of conventional monetary policy tools. In this context, forward guidance is valuable not only for signaling the central bank's economic outlook and policy intentions but also for coordinating market expectations and reducing economy-wide uncertainty. In this paper, we introduce an analytically tractable framework to assess the effects of forward-guidance policies at the ZLB, demonstrating that the central bank can strategically shape the intertemporal evolution of aggregate uncertainty to enhance welfare. For example, by generating controlled uncertainty about future economic conditions, the monetary authority can effectively reduce current market volatility at the ZLB. We refer to such interventions as "higher-order" guidance, distinguishing them from traditional communication strategies that primarily target the levels or expectations of economic variables.

Our analysis starts from a continuous-time New Keynesian model with perfectly rigid prices, chosen for the analytical tractability of second-order moments. We first ask whether aggregate volatility rises when conventional policy is constrained by the ZLB. We show that a credible commitment by the central bank to stabilize the economy *after* the ZLB episode eliminates excess volatility *during* it. This result follows from backward induction: if the monetary authority credibly commits to stabilizing the economy within a finite period, it eliminates the possibility of catastrophic or exuberant outcomes that would otherwise heighten economic volatility. Consequently, such a commitment precludes the emergence of unfavorable coordination equilibrium paths that initially give rise to these scenarios.

We then examine the benefits of various forward guidance strategies. In our framework, traditional forward guidance features an Odyssean component, whereby the central bank credibly commits to maintaining the policy rate at zero for a period longer than the minimum required by economic conditions. After this extended ZLB period, the central bank adopts a policy rule aimed at achieving perfect stabilization once the ZLB constraint is lifted. The outcomes of this policy align with previous research: by committing to a future period of accommodative policy rates, the central bank implicitly accepts a temporary phase of positive excess demand and profits. Anticipating lower rates, forward-looking households reduce desired saving, lifting aggregate demand at the ZLB. This approach effectively distributes the costs of the ZLB over time, yielding welfare benefits. Moreover, the commitment to perfect future stabilization continues to mitigate excess economic volatility at the ZLB, as discussed previously.

Next, we introduce a novel strategy that leverages the inherent coordination problem among households to steer them toward an equilibrium characterized by reduced economic volatility at the ZLB. We refer to this approach as *higher-order forward guidance*. Implementing this policy necessitates that the central bank relinquish its commitment to perfect future stabilization. By pledging not to enforce perfect stabilization at the conclusion of the guidance period, the central bank facilitates the emergence of multiple coordinated equilibria that backward induction would otherwise rule out. This strategy allows the central bank to select equilibrium paths with low volatility and precautionary premia during ZLB episodes, raising expected welfare beyond what traditional forward guidance, a limiting case of our approach, can achieve. The policy entails a trade-off: by foregoing stabilization after the ZLB period, the central bank risks larger future output-gap deviations. Thus, higher-order guidance trades lower ZLB volatility for weaker stabilization in the subsequent economy. Moreover, the strategy remains viable if the bank indicates, even slightly, that perfect stabilization may not be assured at the end of the guidance period.[1]

Finally, we examine the measures available to the monetary authority in enforcing its preferred equilibrium at the ZLB among the multiple solutions permitted by passive future monetary policy. We present an example in which off-equilibrium threats of fiscal intervention—despite involving zero transfers along the equilibrium path—are sufficient to select the optimal higher-order guidance solution as the model's unique equilibrium. While alternative mechanisms for equilibrium selection may exist, this result underscores the importance of coordination between monetary and fiscal authorities at the ZLB and the public's reliance on the credibility of the central bank's communications and commitments. This exercise illustrates a potential benefit of quasi-fiscal interventions (e.g., large-scale asset purchase programs) in steering coordination toward a higher-welfare equilibrium, thereby offering a potential justification for how Mario Draghi's "whatever it takes" speech lowered

[1] Specifically, we prove that if the central bank guarantees a non-zero probability that the business cycle will not be perfectly stabilized at the end of the guidance period, then the higher-order forward guidance strategy becomes feasible. In this regard, our model exhibits a novel discontinuity: if the monetary authority achieves perfect stabilization with certainty after the ZLB period, the system reverts to traditional forward guidance, in which the central bank does not affect excess volatility or risk premiums at the ZLB. Conversely, even a slight but credible probability of not attaining perfect stabilization enables the monetary authority to secure a more favorable equilibrium characterized by lower aggregate volatility and precautionary premiums at the ZLB.

market volatility and peripheral bond yields during the European sovereign debt crisis.

Our framework offers a potential theoretical justification for the Federal Reserve's decision during the COVID-19 crisis to deliberately introduce ambiguity regarding the extent and timing of inflation overshoots following ZLB periods.[2] Under its flexible average inflation targeting (FAIT) framework, the Federal Reserve committed to delaying economic stabilization by allowing inflation to "moderately" overshoot its target after periods of persistent undershooting at the ZLB. Although our analysis abstracts from inflation dynamics by assuming perfectly rigid prices,[3] our higher-order forward guidance experiment illustrates one possible rationale behind the Fed's approach: nudging economic agents toward coordination on a more favorable equilibrium and mitigating the recessionary effects associated with the ZLB.

Featuring a demand-driven economy with perfectly rigid prices, our framework highlights the importance of economic volatility and consumption risk in determining aggregate demand.[4] Consistent with Werning (2012), we assume an exogenous, deterministic transition to the ZLB triggered by a shock that increases an exogenous part of economic volatility and drives the natural rate of interest into negative territory. Our approach diverges from the existing literature by incorporating an endogenous component of economic volatility influenced by both the ZLB and forward guidance. Prior studies (e.g., Bloom (2009), Basu and Bundick (2017), Bloom et al. (2018)) suggest that uncertainty shocks can drive macroeconomic fluctuations. In particular, Basu and Bundick (2017) examine the stabilizing role of monetary policy in the presence of uncertainty shocks, highlighting how the ZLB exacerbates declines in output and its components during periods of heightened uncertainty. While this literature focuses primarily on exogenous uncertainty, our work emphasizes endogenous volatility. Moreover, at the ZLB, we highlight the strategic creation of future uncertainty as a mechanism to reduce present volatility, thereby demonstrating that central

[2] The Fed's 2020 "Statement on Longer-Run Goals and Monetary Policy Strategy" states: "In order to anchor longer-term inflation expectations at this level, the Committee seeks to achieve inflation that averages 2 percent over time, and therefore judges that, following periods when inflation has been running persistently below 2 percent, appropriate monetary policy will likely aim to achieve inflation moderately above 2 percent for some time."

[3] This assumption simplifies the analysis. An extended model with sticky prices à la Calvo (1983) yields qualitatively similar results.

[4] Unlike previous studies (e.g., Akerlof and Yellen (1985), Blanchard and Kiyotaki (1987), Eggertsson and Krugman (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), Schmitt-Grohé and Uribe (2016)) that focus on recessions driven by borrower deleveraging and demand externalities, our model generates ZLB episodes from households' precautionary savings response to a rise in exogenous volatility, as in Caballero and Simsek (2020).

banks can engage in strategic intertemporal uncertainty management through equilibrium selection.

Papers including Eggertsson et al. (2003), Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016), and Caballero and Farhi (2017) explore the implications of forward guidance at the ZLB from both theoretical and empirical perspectives. Our research distinguishes itself by focusing on the impact of forward guidance on higher-order moments, including the endogenous volatility of the aggregate economy.[5] This paper also relates to previous work by Lee and Dordal i Carreras (2025) on determinacy issues and the multiplicity of solutions in the nonlinear New Keynesian model under conventional monetary policy regimes. In contrast, the present study examines whether central bank forward guidance can steer agents toward equilibrium paths—selected from the potential multiplicity of solutions—that are characterized by lower volatility and more rapid economic stabilization at the ZLB, yieldging higher welfare.

Layout The paper is organized as follows. Section 2 introduces the model. Section 3 explains how the ZLB is incorporated into our framework. Section 4 examines the effectiveness of various forward guidance strategies, and Section 5 concludes. Appendix I details the parameter calibration, Appendix II describes standard firm optimization problems, and Appendix III contains the derivations and proofs.

The Online Appendix supplies additional derivations and proofs. Specifically, Online Appendix H examines an alternative Two-Agent New Keynesian (TANK) model with a representative stock-market index and endogenous financial volatility, showing that the equilibrium conditions and results presented here map directly into financial-market volatility and risk-premia dynamics.

2 The Model

We work in continuous time on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. Aggregate technology A_t is the sole exogenous source of variation and generates the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. It follows the geometric Brownian motion

$$\frac{dA_t}{A_t} = g\,dt + \sigma_t\,dZ_t, \tag{1}$$

[5] Our approach, in which central bank communications serve as a device for equilibrium selections, aligns with the concept of 'open-mouth' operations at the ZLB described by Campbell and Weber (2019).

where g is its expected growth rate and σ_t is the economy's fundamental volatility, taken as exogenous. For simplicity, in Section 2 we assume that σ_t is constant and equal to σ. In Section 3, we introduce a deterministic shift in σ_t to examine various scenarios involving the ZLB.

2.1 Households

A representative household chooses paths for consumption C_t, labour supply L_t, and nominal bond holdings B_t to maximise expected lifetime utility

$$\Gamma_0 \equiv \max_{\{B_t, C_t, L_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] dt, \tag{2}$$

subject to the flow budget constraint

$$\dot{B}_t = i_t B_t - \bar{p} C_t + w_t L_t + D_t, \tag{3}$$

where η is the Frisch elasticity of labour supply, D_t denotes lump-sum transfers (firm profits and fiscal transfers), w_t is the wage rate, and i_t is the policy rate set by the central bank. The bond market clears in equilibrium, implying $B_t = 0$ for all t. The subjective discount rate is ρ. Prices are assumed to be perfectly rigid, $p_t = \bar{p}$, which permits closed-form solutions for the key second-order moments.

From the first-order conditions,[6] we have

$$-i_t\, dt = \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right), \qquad \xi_t^N = e^{-\rho t} \frac{1}{\bar{p} C_t}, \tag{5}$$

where $\frac{d\xi_t^N}{\xi_t^N}$ is the instantaneous nominal stochastic discount factor, whose expected value is $-i_t\, dt$, the negative nominal risk-free rate. With perfectly rigid prices, the real and nominal rates coincide, so $r_t = i_t$.

[6] See, e.g., Lee and Dordal i Carreras (2025) for a full derivation of the continuous-time New Keynesian model. An additional first-order condition is $\xi_t^N = \frac{L_t^{1/\eta}}{w_t} e^{-\rho t}$, which, combined with equation (5), yields the standard intratemporal condition:

$$\frac{L_t^{\frac{1}{\eta}}}{w_t} = \frac{1}{\bar{p} C_t} \tag{4}$$

Rearranging the equations in (5) yields the intertemporal condition:

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)\,dt \;+\; \underbrace{\mathrm{Var}_t\left(\frac{dC_t}{C_t}\right)}_{\text{precautionary savings}}\,, \tag{6}$$

where the last term captures consumption volatility and the resulting *precautionary-savings* response of households. Unlike log-linearized models, equation (6) retains this second-order term, allowing volatility to affect the drift of C_t. Define the consumption risk premium $\mathrm{rp}_t \geq 0$, as the instantaneous variance rate, formally $\mathrm{Var}_t\big(\frac{dC_t}{C_t}\big) = \mathrm{rp}_t\,dt$. A higher premium raises desired savings, lowers current consumption, and increases expected consumption growth; hence $\mathbb{E}_t\big(\frac{dC_t}{C_t}\big)$ is increasing in rp_t.

Finally, Appendix II presents the firms' problem following the standard New Keynesian monopolistic-competition framework. With perfectly rigid prices, the aggregate production function is linear in labor and simplifies to $Y_t = A_t L_t$, with market clearing implying $C_t = Y_t$ in equilibrium.

Definition 1 (Equilibrium) *An equilibrium is a collection of processes for household quantities $\{B_t, C_t, L_t\}$, firm quantities $\{Y_t, L_t\}$, and prices $\{w_t, \bar{p}\}$, that satisfy: (i) the household optimality conditions (4) and (6); (ii) the transversality condition for the value function (Γ_t),*

$$\lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t}\Gamma_t\right] = 0;$$

(iii) the aggregation and production conditions $B_t = 0$ and $C_t = Y_t = A_t L_t$,[7] *and (iv) the interest-rate process $\{i_t\}$ defined for each forward-guidance regime considered in Sections 3 and 4. All aggregate quantity and price variables are adapted to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ generated by (1).*

2.2 Efficient Flexible Price Equilibrium

Consistent with the literature, we adopt the efficient flexible-price equilibrium as the benchmark for the monetary authority's policy objectives, denoting variables in this counterfactual equilibrium with the superscript n (for "natural"). In addition to flexible prices, we assume the government implements a production subsidy τ_y, financed by lump-sum taxes,

[7] With rigid prices, firm optimization can be abstracted away; see Appendix II.

to eliminate the real distortions from monopolistic competition (Woodford, 2003). Details are provided in Online Appendix A. Natural output Y_t^n evolves according to

$$\frac{dY_t^n}{Y_t^n} = \left(r^n - \rho + \mathrm{rp}_t^n\right)dt + \underbrace{\sigma}_{\substack{\text{Natural}\\ \text{volatility}}} dZ_t, \tag{7}$$

where $r^n = \rho + g - \sigma^2$ and $\mathrm{rp}_t^n = \sigma^2$ are the natural real rate and risk premium, respectively. From the monetary authority's perspective, the natural output process in (7) is exogenous and beyond its policy control.

2.3 Gap Economy

We define σ_t^s as the *excess* volatility of the output growth rate under rigid prices, relative to the benchmark flexible-price economy described in equation (7). By definition, it follows that risk premium in the the rigid-price economy is given by

$$\mathrm{rp}_t = (\sigma + \sigma_t^s)^2. \tag{8}$$

Note that σ_t^s is an *endogenous* volatility determined in equilibrium.

We define the risk premium gap as $\hat{\mathrm{rp}}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$ and the risk-adjusted natural rate, r_t^T, as:

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{\mathrm{rp}}_t. \tag{9}$$

This adjusted rate accounts for the consumption risk differential between rigid-price and flexible-price economies and serves as a monetary policy anchor in our model. For instance, a positive premium gap ($\hat{\mathrm{rp}}_t > 0$) increases desired savings, depresses current consumption, and may lead to recessionary pressures. Proposition 1 formalizes this effect, demonstrating that a decline in r_t^T relative to the policy rate i_t induces expectations of an expanding output gap.

Proposition 1 (Dynamic IS Equation) *Let $\hat{Y}_t = \log(\frac{Y_t}{Y_t^n})$ denote the output gap. The dynamic IS equation is given by:*

$$d\hat{Y}_t = (i_t - r_t^T)dt + \sigma_t^s dZ_t. \tag{10}$$

Proof. See Online Appendix B.

■

Notably, a log-linearized approximation of the New Keynesian model replaces r_t^T with the natural rate r_t^n in equation (10), thereby removing the direct effect of endogenous risk on the output gap.

Proposition 2 shows that welfare losses from the business cycle fluctuations under rigid prices can be expressed as a function of the expected squared output gap (Woodford, 2003; Galí, 2015). We rely on this welfare loss function to compare various forward guidance paths and compute the optimal higher-order forward guidance equilibrium in Section 4.

Proposition 2 (Welfare Loss Function) *A second-order approximation of households' expected lifetime utility (2) around the efficient flexible-price steady state yields a per-period expected welfare loss proportional to*

$$\mathbb{L}^Y\left(\{\hat{Y}_t\}_{t\geq 0}\right) = \rho\,\mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Y}_t^2\,dt. \tag{11}$$

Proof. See Online Appendix C. ■

3 The Zero Lower Bound

ZLB Recession. Following Werning (2012), we consider a deterministic fall in the natural rate of interest r_t^n that drives the economy to the ZLB. Specifically, we assume that TFP volatility equals $\sigma_t = \bar{\sigma}$ for $0 \leq t < T$ and $\sigma_t = \underline{\sigma}$ for $t \geq T$, with $\underline{\sigma} < \bar{\sigma}$. Hence

$$\underline{r} \equiv r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2 < 0, \qquad \bar{r} \equiv r^n(\underline{\sigma}) = \rho + g - \underline{\sigma}^2 > 0,$$

so the ZLB binds initially because the central bank cannot set $i_t < 0$ to eliminate the output gap in equation (10).[8] Without loss of generality—and as evident from the expression for r_t^n—one may alternatively consider shocks to the growth rate g or the discount rate ρ as drivers of the ZLB spell. Results are unchanged when T is stochastic (Online Appendix F); we therefore focus on the deterministic case.

Recovery Without Guidance. We first consider the benchmark scenario—recovery without forward guidance or macro-prudential intervention. After period T the central bank

[8] Assuming zero endogenous volatility $\sigma_t^s = 0$, $0 \leq t < T$, the bank would need $i_t = \underline{r} < 0$ to ensure $E_t[d\hat{Y}_t] = 0$ in equation (10), which the ZLB constraint $i_t \geq 0$ rules out. Appendix D.1 shows that perfect stabilization is otherwise attainable with an interest-rate rule that closes the gap in the drift of (10).

resumes its active rule via $i_t > 0$, delivering perfect stabilization so that $\hat{Y}_t = \sigma_t^s = 0$ for all $t \geq T$.[9] Backward induction applied to equation (10) then requires $\sigma_t^s = 0$ for every $t < T$: at $T - \Delta$ (with infinitesimal Δ), the only solution consistent with $\hat{Y}_T = 0$ for any draw of $dZ_{T-\Delta}$ is $\sigma_{T-\Delta}^s = 0$; iterating backward yields the same for all earlier dates. Hence, when the central bank can credibly commit to perfect stabilization from T onward, one has $\sigma_t^s = 0$ and $r_t^T = \underline{r} < 0$ for $t < T$. Under these conditions, the dynamics of $\hat{Y}_t$ reduce to

$$d\hat{Y}_t = -\underline{r}\, dt\,, \quad \text{for } t < T\,, \tag{12}$$

with the boundary condition $\hat{Y}_T = 0$ and initial output gap $\hat{Y}_0 = \underline{r}T$. Figure 1 plots the resulting path of $\{\hat{Y}_t\}$.

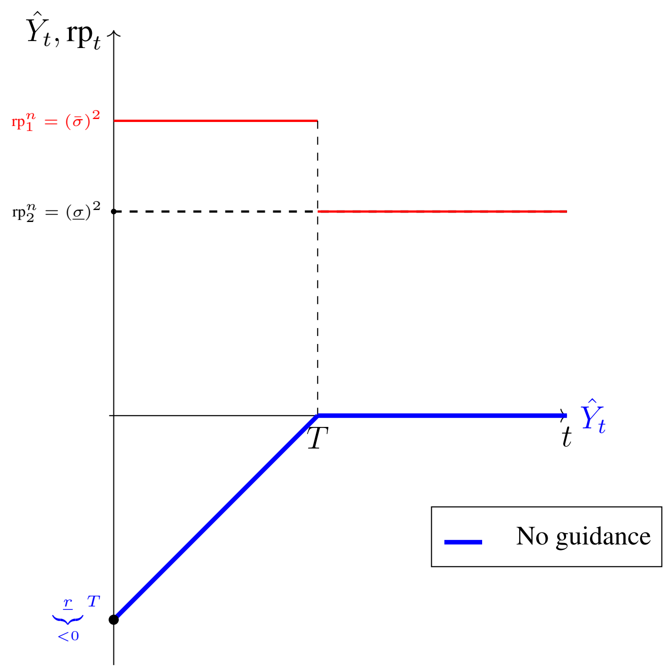

Figure 1: ZLB dynamics, economic recovery without guidance (Benchmark).

The initial rise in σ_t from $\underline{\sigma}$ to $\bar{\sigma}$ increases the natural risk premium from $\text{rp}_2^n = (\underline{\sigma})^2$ to $\text{rp}_1^n = (\bar{\sigma})^2$. As the ZLB prevents the policy rate from turning negative, higher risk premium depresses output gap $\hat{Y}_t$.[10] Note that the dynamics in Figure 1 are consistent with

[9] Appendix D.1 provides an example of how the central bank can achieve perfect stabilization once the economy is away from the ZLB. In general terms, $i_t = \bar{r} > 0$ with $\hat{Y}_t = \sigma_t^s = 0$ for $t \geq T$ can be realized in equilibrium as it satisfies equation (10), and Appendix D.1 explains how the central bank pins it down as a unique equilibrium based on a revised monetary rule.

[10] While Caballero and Farhi (2017) demonstrate that increased demand for safe assets under ZLB con-

findings by the literature (Werning, 2012; Cochrane, 2017), though our model differs by incorporating a distinct IS equation (10), where endogenous volatility σ_t^s directly influences the drift of $\hat{Y}_t$, diverging from the standard New Keynesian framework. In essence, credible commitments to future stabilization for $t \geq T$ suppresses excess endogenous volatility during the ZLB period.

Remarks. Central banks can prevent the emergence of endogenous volatility σ_t^s at the ZLB through a credible commitment to stabilizing the business cycle by a predetermined future date $T < +\infty$. Even when the monetary authority is constrained by the ZLB and unable to actively adjust the policy rate i_t, the costs, stemming from volatility and associated with policy inaction, can be substantially mitigated or even eliminated by pledging stabilization upon exiting the ZLB. Consequently, the impact of the ZLB may differ significantly across countries. Monetary authorities credibly committed to post-ZLB stabilization will likely face only the demand-driven recession described in this section. In contrast, countries lacking either the ability or willingness to commit to future stabilization may incur additional costs due to heightened σ_t^s during the ZLB episode. Further investigation into these differences remains an open area for future research.

4 Forward Guidance

This section analyzes two forward guidance strategies and examines the stabilization tradeoffs inherent in employing these policy tools.

4.1 Traditional Forward Guidance

We define traditional forward guidance as the communication strategy in which the central bank credibly commits to maintain a zero policy rate for a horizon $\hat{T}^{\text{TFG}}$ that exceeds the initial interval T of high fundamental volatility. Once this forward-guidance window closes, the bank reverts to active monetary policy intervention, achieving perfect stabilization of the business cycle for $t \geq \hat{T}^{\text{TFG}}$. By backward induction (see Section 3), the certainty of future stabilization eliminates endogenous economic volatility, implying $\sigma_t^s = 0$ for all $t < \hat{T}^{\text{TFG}}$.

straints might cause a recession, our IS equation (10) emphasizes how aggregate increases in desired savings lower aggregate demand, consistent with Caballero and Simsek (2020) and the Keynesian paradox of thrift.

Under these conditions, the dynamics of the output gap $\hat{Y}_t$ are given by

$$d\hat{Y}_t = \begin{cases} -\underline{r}\,dt, & \text{for } t < T, \\ -\bar{r}\,dt, & \text{for } T \le t < \hat{T}^{\text{TFG}}, \end{cases} \tag{13}$$

with the boundary condition $\hat{Y}_{\hat{T}^{\text{TFG}}} = 0$. This yields an initial output gap of $\hat{Y}_0 = \underline{r}\,T + \bar{r}\,(\hat{T}^{\text{TFG}} - T)$. Figure 2 illustrates the dynamics of $\{\hat{Y}_t\}$ as described by equation (13).

Traditional forward guidance induces an artificial economic boom between T and $\hat{T}^{\text{TFG}}$, thereby mitigating recessionary pressures during $0 \le t < T$. Specifically, by increasing output between T and $\hat{T}^{\text{TFG}}$, this policy raises the initial output gap $\hat{Y}_0$ owing to the forward-looking consumption behavior of households.

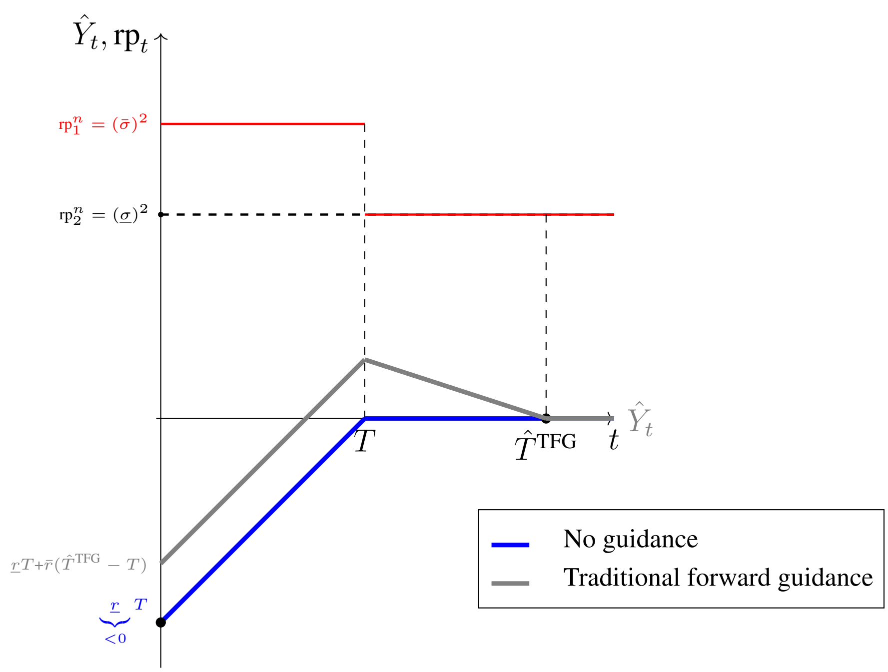

Figure 2: ZLB dynamics under traditional forward guidance.

Optimal Traditional Forward Guidance. To determine the optimal forward guidance duration $\hat{T}^{\text{TFG}}$, we minimize the quadratic welfare loss function in (11) subject to the dynamics specified in equation (13). The first-order condition with respect to $\hat{T}^{\text{TFG}}$ yields

$$\int_0^\infty e^{-\rho t}\hat{Y}_t\,dt = 0\,. \tag{14}$$

Section 4.6 summarizes the key statistics and welfare gains achieved under the optimal traditional forward guidance policy.

In the next section, we argue that central banks might deliberately forgo perfect future stabilization to reduce aggregate volatility and risk premium at the ZLB, thereby attaining higher welfare compared to the above traditional forward guidance approach. We refer to this alternative method as *higher-order* forward guidance.

4.2 Higher-Order Forward Guidance

The primary driver of ZLB recessions in our model is an elevated risk premium induced by high fundamental volatility, σ_t. The central bank can counter this by leaning against the positive jump in fundamental volatility, i.e., managing expectations of output-gap volatility, σ_t^s, so that it becomes negative during the ZLB episode, thereby supporting aggregate demand.[11] Under nominal rigidities, aggregate demand endogenously determines labor income and profits, and the household budget constraint links income volatility to the resulting consumption risk. By steering expectations about macroeconomic volatility, the central bank shifts desired consumption and selects an equilibrium with a less volatile output (and therefore income) path. Thus, the endogeneity of household incomes (and consumption) is a key component through which higher-order forward guidance reduces aggregate volatility.

Online Appendix E presents a discrete three-period model illustrating the basic intuition behind the higher-order forward guidance problem. Here, we focus on the results for the general case.

Context. Under traditional forward guidance, the commitment to perfect stabilization at $\hat{T}^{\text{TFG}}$ fixes σ_t^s at its natural value of zero throughout the ZLB period. To sustain alternative equilibria where $\sigma_t^s \neq 0$, the central bank must relinquish its commitment to full stabilization upon exiting the ZLB at $\hat{T}^{\text{TFG}}$.

[11] The risk premium is $\text{rp}_t = (\bar{\sigma} + \sigma_t^s)^2$ for $t < T$ and $\text{rp}_t = (\underline{\sigma} + \sigma_t^s)^2$ for $T \leq t < \hat{T}^{\text{TFG}}$. A negative σ_t^s lowers the premium below its natural level and raises aggregate demand at the ZLB. Note that the central bank can lower aggregate volatility through higher-order forward guidance only if fundamental volatility, σ_t, exists.

Implementation. Let $\hat{T}^{\text{HOFG}}$ denote the horizon over which the policy rate remains at zero under higher-order forward guidance (HOFG). Therefore, the central bank commits to setting $i_t = 0$ for all $t < \hat{T}^{\text{HOFG}}$; afterward, it adopts a *passive* policy rule by fixing $i_t = \bar{r}$, which permits multiple equilibria.

The central bank coordinates expectations of households onto an optimal equilibrium path, subject to two constraints: $\sigma_t^s = 0$ for $t \geq \hat{T}^{\text{HOFG}}$, and $\mathbb{E}_0 \hat{Y}_{\hat{T}^{\text{HOFG}}} = 0$. The first condition ensures that the central bank no longer influences economic volatility after forward guidance expires, while the second guarantees that the economy is fully stabilized in expectation by the end of the guidance period, $\hat{T}^{\text{HOFG}}$. This set of constraints allows HOFG to be directly comparable to traditional forward guidance policies with perfect post-ZLB stabilization considered in Section 4.1. Together with the dynamic IS equation (10), these conditions imply that the expected output gap closes precisely at $\hat{T}^{\text{HOFG}}$. Section 4.3 relaxes these restrictions by allowing the central bank to resume active monetary stabilization after $\hat{T}^{\text{HOFG}}$ with probability less than one.

Formalism. Define the natural risk premia as follows: $\text{rp}_1^n \equiv \bar{\sigma}^2$ for $t < T$ (high fundamental volatility region), $\text{rp}_2^n \equiv \underline{\sigma}^2$ for $T \leq t < \hat{T}^{\text{HOFG}}$ (low fundamental volatility region), and $\text{rp}_3^n \equiv \underline{\sigma}^2$ for $t \geq \hat{T}^{\text{HOFG}}$ (post-guidance, low fundamental volatility region).[12]

For tractability, we assume the central bank maintains constant *excess* volatility within each regime: in specific, we assume $\sigma_t^s = \sigma_1^{s,L}$ for $t < T$, $\sigma_2^{s,L}$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and 0 for $t \geq \hat{T}^{\text{HOFG}}$.[13] The corresponding risk premia are given by

$$\text{rp}_1 = \left(\bar{\sigma} + \sigma_1^{s,L}\right)^2 < \text{rp}_1^n, \quad \text{rp}_2 = \left(\underline{\sigma} + \sigma_2^{s,L}\right)^2 < \text{rp}_2^n, \quad \text{rp}_3 = \underline{\sigma}^2.$$

Proposition 3 establishes that at the optimum, $\sigma_1^{s,L} < 0$ and $\sigma_2^{s,L} < 0$. We impose these sign conventions throughout our analysis throughout this section.

Moreover, the risk-adjusted natural rate defined in equation (9) equals r_1^T for $t < T$ and

[12]The risk premium is $\text{rp}_t = (\sigma_t + \sigma_t^s)^2$. Under flexible prices, $\sigma_t^{s,n} = 0$, implying a natural premium of σ_t^2.

[13]In Online Appendix G, we extend our analysis to the case in which the central bank picks a full path of $\{\sigma_t^s\}_{t \leq \hat{T}^{\text{HOFG}}}$. We prove that all the results in Section 4.2, including Proposition 3, hold under that case. See Proposition G.2.

r_2^T for $T \le t < \hat{T}^{\text{HOFG}}$, with:

$$
\begin{aligned}
r_1^T(\sigma_1^{s,L}) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{\left(\bar{\sigma} + \sigma_1^{s,L}\right)^2}{2} > \underline{r} \equiv r_1^T(0) \quad \text{if } \sigma_1^{s,L} < 0, \\
r_2^T(\sigma_2^{s,L}) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{\left(\underline{\sigma} + \sigma_2^{s,L}\right)^2}{2} > \bar{r} \equiv r_2^T(0) \quad \text{if } \sigma_2^{s,L} < 0.
\end{aligned}
\tag{15}
$$

By reducing the risk premium, a negative $\sigma_j^{s,L}$ increases the risk-adjusted natural rate during the guidance window (up to $\hat{T}^{\text{HOFG}}$). This yields higher output gap values $\{\hat{Y}_t\}$ compared to standard forward guidance of equivalent duration, thereby lowering the expected loss defined in equation (11). However, as indicated by the dynamic IS equation (10), introducing $\sigma_t^s \neq 0$ also generates stochastic fluctuations in the output gap trajectory, potentially incurring future stabilization costs.

Figure 3 illustrates this framework: the green line represents the expected trajectory of $\{\hat{Y}_t\}$ under higher-order forward guidance, while dashed lines depict two possible stochastic realizations.

In summary, higher-order forward guidance with commitment requires the central bank to balance lower risk premia and higher *expected* output before $\hat{T}^{\text{HOFG}}$ against subsequent stabilization costs stemming from stochastic fluctuations in $\hat{Y}_t$. Thus, optimal policy depends crucially on the triple $(\sigma_1^{s,L}, \sigma_2^{s,L}, \hat{T}^{\text{HOFG}})$. Due to the additional stabilization effects from negative $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, the optimal duration of the ZLB interval $\hat{T}^{\text{HOFG}}$ is shorter than $\hat{T}^{\text{TFG}}$, as formally established in Proposition 3.

Optimal Higher-Order Forward Guidance. The initial output gap, $\hat{Y}_0$, is determined by the boundary condition $\mathbb{E}_0[\hat{Y}_{\hat{T}^{\text{HOFG}}}] = 0$ and the dynamic IS equation (10), yielding

$$
\hat{Y}_0 = r_1^T(\sigma_1^{s,L})\, T + r_2^T(\sigma_2^{s,L}) \left(\hat{T}^{\text{HOFG}} - T\right) . \tag{16}
$$

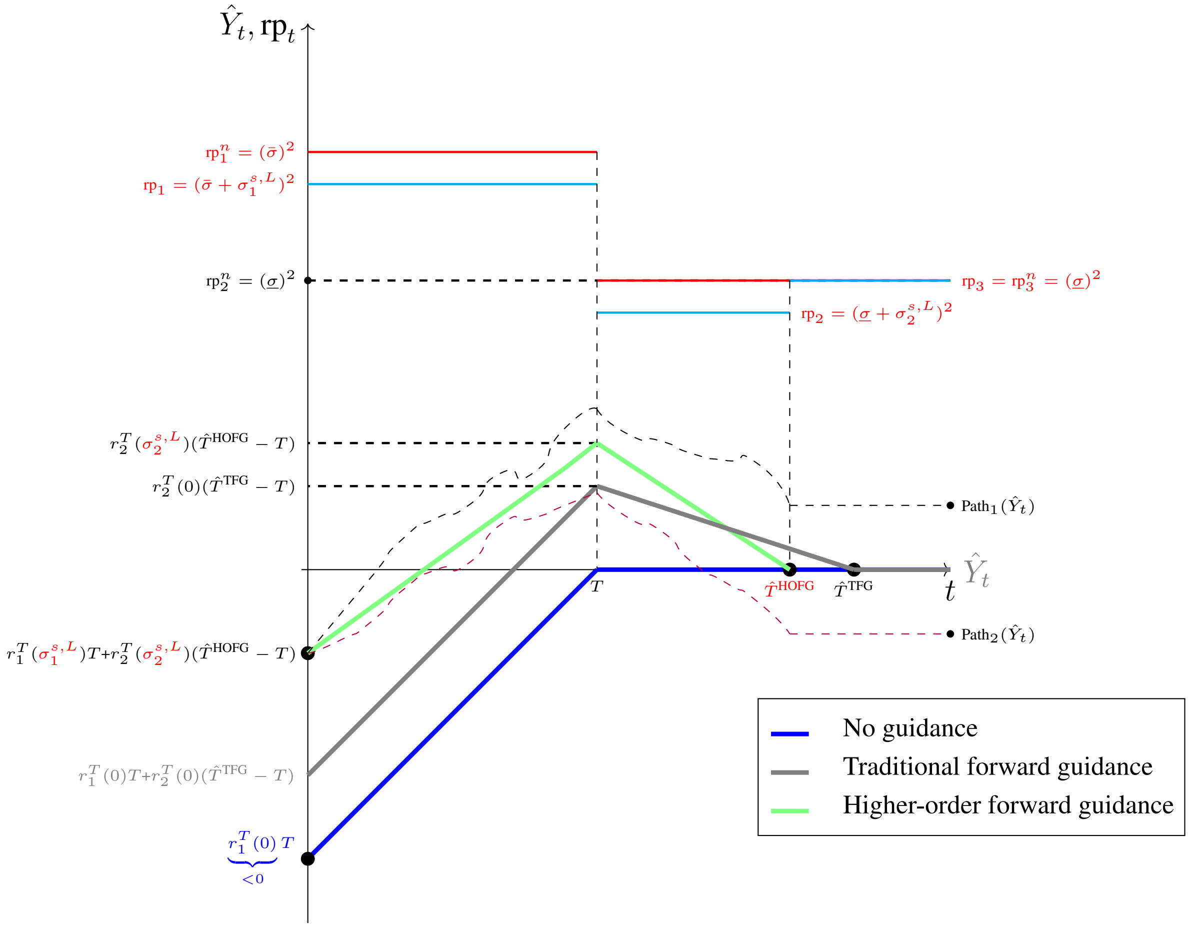

Figure 3: Dynamics of $\{\hat{Y}_t\}$ under intervention with $\sigma_1^{s,L} < 0$, $\sigma_2^{s,L} < 0$, and $\hat{T}^{\mathrm{HOFG}} < \hat{T}^{\mathrm{TFG}}$.

The central bank minimizes the quadratic loss function defined in (11) by optimally choosing $\sigma_1^{s,L}$, $\sigma_2^{s,L}$, and $\hat{T}^{\mathrm{HOFG}}$. The optimization problem is formulated as follows:

$$
\begin{aligned}
&\min_{\sigma_1^{s,L}, \sigma_2^{s,L}, \hat{T}^{\mathrm{HOFG}}} \ \mathbb{E}_0 \int_0^\infty e^{-\rho t}\, \hat{Y}_t^2 \, dt \ , \\
&\text{s.t.} \quad d\hat{Y}_t = \begin{cases} -r_1^T(\sigma_1^{s,L})\, dt + \sigma_1^{s,L}\, dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{s,L})\, dt + \sigma_2^{s,L}\, dZ_t, & \text{for } T \le t < \hat{T}^{\mathrm{HOFG}}, \\ 0, & \text{for } t \ge \hat{T}^{\mathrm{HOFG}}, \end{cases}
\end{aligned} \tag{17}
$$

with $\hat{Y}_0$ given by equation (16).

Proposition 3 (Optimal Commitment Path) *The solution to the central bank's higher-order forward guidance optimization problem in (17) yields an optimal commitment path characterized by $\sigma_1^{s,L} < 0$, $\sigma_2^{s,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. Moreover, the optimal higher-order forward guidance policy results in a lower expected quadratic welfare loss than the traditional forward guidance policy discussed in Section 4.1.*[14]

Proof. See Appendix III. The latter part follows from the observation that when $(\sigma_1^{s,L}, \sigma_2^{s,L}, \hat{T}^{\text{HOFG}}) = (0, 0, \hat{T}^{\text{TFG}})$, the trajectory of the output gap $\{\hat{Y}_t\}$ coincides with that of a traditional forward guidance policy of duration $\hat{T}^{\text{TFG}}$. Therefore, any optimal choice of these parameters yields an equal or lower value of the quadratic loss function in equation (11).

■

4.3 Higher-Order Forward Guidance *with* Stochastic Stabilization

In the previous Section 4.2, we assumed that once the forward guidance regime concludes at $\hat{T}^{\text{HOFG}}$, the monetary authority passively pegs the policy rate i_t to the natural rate $\bar{r}$ and maintains $\sigma_t^s = 0$ indefinitely. Under that arrangement, σ_t^s is permitted to deviate from zero during the ZLB period, as depicted in Figure 3.

In this section, we relax these assumptions while preserving the model's capacity to support multiple equilibria. Specifically, we now assume that after forward guidance ends, the central bank follows the passive rule but simultaneously commits to a stochastic return to perfect stabilization through active monetary policy. This commitment is modeled as a Poisson process with a constant probability event. Consequently, for $t \geq \hat{T}^{\text{HOFG}}$, the output gap evolves according to

$$d\hat{Y}_t = -\hat{Y}_t \, d\Pi_t, \quad \text{with} \quad d\Pi_t = \begin{cases} 1, & \text{with probability } \nu \, dt, \\ 0, & \text{with probability } 1 - \nu \, dt, \end{cases}$$

where $d\Pi_t$ is a Poisson random variable with rate parameter $\nu \geq 0$.[15] Unless $\nu = \infty$, i.e., perfect stabilization upon exiting the ZLB as in Section 4.1, σ_t^s is still permitted to deviate from zero during the ZLB period, allowing the central bank to execute HOFG.

[14]One can easily see that $\sigma_t^s < -\sigma_t$ is not optimal, so still $\sigma_t + \sigma_t^s > 0$ at the ZLB.

[15]Here, ν is treated as an exogenous parameter determined by external factors. If the monetary authority could freely select ν, it would optimally set $\nu \to +\infty$, but $\nu \neq \infty$, as discussed in Online Appendix D.2.

The central bank's optimization problem is then formulated as follows:

$$\min_{\sigma_1^{s,L},\,\sigma_2^{s,L},\,\hat{T}^{\text{HOFG}}} \quad \mathbb{E}_0 \left[\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\, \hat{Y}_t^2 \, dt + \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\, e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)} \hat{Y}_t^2 \, dt \right],$$

$$\text{s.t.} \quad d\hat{Y}_t = \begin{cases} -r_1^T(\sigma_1^{s,L})\, dt + \sigma_1^{s,L}\, dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{s,L})\, dt + \sigma_2^{s,L}\, dZ_t, & \text{for } T \le t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \ge \hat{T}^{\text{HOFG}}, \end{cases} \tag{18}$$

with $\hat{Y}_0$ determined by equation (16).

Proposition 4 (Optimal Commitment Path with Stochastic Stabilization) *The solution to the central bank's optimization problem in (18) yields an optimal commitment path characterized by $\sigma_1^{s,L} < 0$, $\sigma_2^{s,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. Moreover, the optimal higher-order forward guidance policy with a stochastic stabilization probability produces an expected quadratic welfare loss that is lower than that under the traditional forward guidance policy discussed in Section 4.1.*

Furthermore, higher stabilization probability ν leads to smaller optimal values of $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, i.e., more negative $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, thereby further reducing risk premia at the ZLB.

Proof. See Online Appendix D.2. The first part of the proposition extends the results of Proposition 3 to the stochastic stabilization context, while the second part follows from the reduced costs of a more aggressive countercyclical policy when future stabilization is more likely. Specifically, the central bank worries less about $\hat{Y}_{\hat{T}^{\text{HOFG}}}$ deviating from zero, thereby pursuing more negative $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$ up to $\hat{T}^{\text{HOFG}}$.

■

Finally, Corollary 1 demonstrates that introducing even a slight degree of uncertainty regarding the timing of future stabilization is always optimal for the central bank. Such uncertainty enables private agents to coordinate on a stochastic equilibrium where σ_t^s deviates from zero during the ZLB, as illustrated in Figure 3. Thus, higher-order forward guidance generates equilibrium paths that are strictly superior—in terms of quadratic loss—relative to traditional forward guidance outcomes.

Corollary 1 (Discontinuity at the Limit) *When the stabilization parameter ν equals $+\infty$, the problem reduces to the traditional forward guidance case described in Section 4.1. As*

ν approaches $+\infty$ from below, the central bank's expected quadratic loss function exhibits a discontinuity. In particular, the expected quadratic loss is always lower when there is a non-zero probability of not achieving immediate stabilization at the end of the forward guidance period, $\hat{T}$. Formally:

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu = +\infty\right) ,$$

where $\mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu\right)$ denotes the quadratic loss function defined in equation (11), evaluated at its optimum for an economy characterized by a Poisson rate ν.

Proof. See Online Appendix D.2. When $\nu = +\infty$, the probability of immediate stabilization at $\hat{T}^{\text{HOFG}}$ becomes one, which is equivalent to the traditional forward guidance policy of Section 4.1. Moreover, Proposition 4 shows that higher-order guidance always achieves a lower expected quadratic welfare loss compared to traditional guidance, regardless of the value of ν.

■

COVID-19 and the Federal Reserve. Proposition 4 and Corollary 1 theoretically support the Federal Reserve's deliberate strategy during the COVID-19 recession of maintaining ambiguity regarding the extent and timing of inflation stabilization after ZLB periods. Under its flexible average inflation targeting (FAIT) framework, the Federal Reserve committed to allowing inflation to overshoot its target *moderately* following extended episodes of undershooting at the ZLB. Although our model abstracts from inflation explicitly, the corresponding moderate overshoot in the business cycle dynamics is represented by $0 < \nu < +\infty$ in Proposition 4, which is superior to the traditional forward guidance scenario where $\nu = +\infty$.

Our analysis of higher-order forward guidance rationalizes the Fed's policy shift: nudging[16] economic agents toward coordination on a more favorable equilibrium with lower volatility while mitigating the recessionary effects associated with the ZLB.

Equilibrium Selection. We next illustrate how the model's optimal higher-order solution can be practically implemented among multiple potential ZLB equilibria via coordinated reserve-based fiscal policy interventions.

[16]Empirical and experimental evidence on nudging effects in multiple equilibria contexts is documented in, e.g., Barron and Nurminen (2020) and Battiston and Harrison (2024).

4.4 Reserve-Based Fiscal Policy Coordination

Fiscal intervention, together with the forward-guidance policies above, can implement the optimal higher-order forward-guidance solution as the economy's unique equilibrium. While various forms of fiscal coordination are possible, this section presents a reserve-based fiscal policy that supports this outcome.[17]

This policy provides a stylized representation of the effects of fiscal expansions during crises (e.g., under heightened uncertainty) and large-scale asset purchase (LSAP) programs on aggregate volatility. Potential gains or losses on the central bank balance sheet from such policies—and related direct or indirect channels—motivate the literature's emphasis on the quasi-fiscal nature of LSAPs.[18]

The policy acts as an *off-equilibrium threat*: fiscal transfers from reserves are tied to deviations from the optimal path, implying *zero transfers in equilibrium*. Other coordination mechanisms, including non-fiscal ones, may also be feasible. As long as equilibrium selection is achieved via off-equilibrium threats, the properties and welfare gains of higher-order guidance are independent of the specific coordination instrument used by the monetary authority.

Suppose the consolidated government (including the central bank) finances household transfers with reserves. Then transfers D_t in (3) equal firm profits plus an exogenous transfer from government reserves, not financed by taxes:

$$\begin{aligned} \Delta D_0 &= \bar{p}\Delta Y_0 - \Delta(w_0 L_0) + \chi_0, \\ D_t &= \bar{p} Y_t - w_t L_t + \tau_t, \end{aligned} \tag{19}$$

where $\Delta x_0 \equiv x_0 - x_{0-}$ is the initial jump in variable x following the ZLB shock in Section 3, and τ_t and χ_0 are government transfers, with χ_0 the initial jump at $t = 0$.

Proposition 5 specifies a reserve-transfer schedule $\{\tau_t,\, \chi_0\}$ that implements the optimal higher-order forward-guidance solution Y_t^* via an *off-equilibrium threat* scheme. As shown in Corollary 2, transfers are zero in equilibrium.

Proposition 5 (Fiscal Transfers from Reserves and Unique Optimal Equilibrium) *Suppose*

[17] See Online Appendix H for alternative implementations in the context of a two-agent model with explicit financial markets.

[18] See, e.g., Woodford (2016), Chionis et al. (2021), Lee and Dordal i Carreras (2025).

the reserve transfer schedule $\{\tau_t, \chi_0\}$ *is defined by*

$$\begin{aligned} \tau_t &= \bar{p}(Y_t^* - Y_t), \\ \chi_0 &= \bar{p}(Y_0^* - Y_0), \end{aligned} \tag{20}$$

where $\{Y_t^*\}_{t\geq 0}$ *denotes the output process under the optimal higher-order forward guidance equilibrium. Then,* $\{Y_t^*\}_{t\geq 0}$ *path becomes the unique equilibrium at the ZLB: that is,* $\hat{Y}_t = \hat{Y}_t^*$ *and* $\sigma_t^s = \sigma_t^{s,*}$ *for all* $t \geq 0$.

Proof. See Appendix III.

■

Corollary 2 (Zero Equilibrium Transfers) *Under the off-equilibrium threat in Proposition 5, transfers are zero in equilibrium:*

$$\tau_t = \chi_0 = 0, \quad \forall t \geq 0.$$

Proof. Immediate from Proposition 5, since $Y_t = Y_t^*$ and $\sigma_t^s = \sigma_t^{s,*}$ for all $t \geq 0$, together with the fiscal transfer policy in (20).

■

European Sovereign Debt Crisis and *"Whatever It Takes"*. Our analysis provides an economic justification for Mario Draghi's well-known *"whatever it takes"* speech in 2012, delivered during the European sovereign debt crisis when he was president of the European Central Bank (ECB). Given the quasi-fiscal nature of monetary policy interventions, such as large-scale asset purchase (LSAP) programs,[19] Draghi's assertion that *"the ECB is ready to do whatever it takes to preserve the euro"* helped market participants coordinate on an equilibrium characterized by reduced volatility. This coordination notably contributed to a substantial decrease in Italian and Spanish sovereign bond yields (Acharya et al., 2019). Our findings suggest that central banks can effectively steer private agents toward superior equilibria—thus enhancing welfare—by publicly committing to interventions with fiscal characteristics, even when these interventions do not involve immediate expenditures.

[19]In the ECB context, this refers specifically to the Outright Monetary Transactions (OMT) program.

4.5 Commitment

For analytical tractability, and following the literature on forward guidance, we assume the central bank can credibly commit to its policy promises. However, previous research (e.g., Bassetto (2019), Camous and Cooper (2019))[20] shows that "grim-trigger" strategies in repeated monetary games can sustain coordinated equilibria even without full commitment.

We conjecture that a similar mechanism could support the commitment-based equilibria studied here, even when the central bank lacks full commitment capacity.[21] A detailed analysis is left for future work.

4.6 Welfare Comparison

For the quantitative assessment of forward-guidance policies, we simulate the optimal commitment path at the ZLB under three regimes: (i) no forward guidance; (ii) traditional forward guidance; and (iii) higher-order forward guidance with alternative stabilization probabilities. The initial ZLB spell is fixed at $T = 20$ quarters, matching the prolonged periods observed after the global financial crisis. In the higher-order case, we first set the Poisson rate to $\nu = 0$—no chance of returning to active stabilization—and then to $\nu = 1$, implying an expected return one quarter after the guidance period ends. All other parameters follow standard calibrations (see Appendix Table I.1).

We treat the ZLB spell as a one-time event that starts at period zero with no expected recurrence, so the welfare loss $\mathbb{L}^Y$ defined in (11) is the expected conditional loss from a single episode.[22]

[20]Camous and Cooper (2019) studies commitment solutions to state-contingent government policy in the context of debt monetization. Bassetto (2019) shows that, absent private information at the central bank, the set of equilibrium payoffs is independent of the central bank's announcements—i.e., the forward guidance equilibrium can be sustained without additional verbal commitment.

[21]Barthélemy and Mengus (2024) show that, in a large class of static games, the government can guarantee a unique equilibrium for any level of commitment. Our conjecture is that two opposing forces operate in the higher-order forward guidance equilibrium: while the semi-permanent deviation of the output gap $\hat{Y}_t$ after $\hat{T}^{\text{HOFG}}$ can incentivize the central bank to switch to discretion, higher welfare (i.e., lower loss) makes it easier for the monetary authority and households to sustain the equilibrium.

[22]We approximate the welfare loss (11) as

$$\mathbb{L}^Y \equiv \rho \int_0^\infty e^{-\rho t}\, \mathbb{E}_0\big(\hat{Y}_t^2\big)\, \mathrm{d}t \approx \rho \int_0^\infty e^{-\rho t} \frac{1}{s} \sum_{i=1}^{s} \big(\hat{Y}_t^{(i)}\big)^2\, \mathrm{d}t,$$

where $\hat{Y}_t^{(i)}$ is the i-th simulated stochastic sample path of the output gap. We draw $s = 10^4$ Monte-Carlo paths to approximate the loss under higher-order guidance.

Policy	**No guidance**	**Traditional**	**Higher-Order** (no stochastic stabilization)	**Higher-Order** (*with* stoch. stab., $\nu = 1$)
$\sigma_1^{s,L}$	0	0	-1.27%	-4.13%
$\sigma_2^{s,L}$	0	0	-0.24%	-3.79%
$\hat{T}^{\text{HOFG}}$	20	25.27	25.09	24.68
$\mathbb{L}^Y$	1.14%	0.32%	0.29%	0.27%

Table 1: Policy comparisons

Table 1 summarizes the simulation results. The coefficients $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$ are expressed as percentages of the fundamental volatilities $\bar{\sigma}$ and $\underline{\sigma}$, respectively. The first column shows the benchmark welfare loss under a no-guidance regime. The second column evaluates traditional forward guidance: the central bank extends the ZLB by a little more than one year and cuts the total loss by about 0.8 percentage points, consistent with Campbell et al. (2012, 2019), Del Negro et al. (2013), and McKay et al. (2016).[23] The last two columns report the optimal higher-order guidance under the two stabilization regimes discussed above. In line with Propositions 3 and 4, higher-order guidance lowers ZLB costs by 0.03–0.05 percentage points per quarter by reducing aggregate volatility and enables an earlier exit from the ZLB. The gains double when the probability of returning to full stabilization is positive.

TANK model with financial markets. The findings in Section 3 and Section 4 extend to a Two-Agent New Keynesian (TANK) model with an explicit financial market. In that environment, higher aggregate volatility depresses risky asset prices by reducing expected firm profits and raising their volatility as well as risk premium, erodes high-income households' wealth, and weakens consumption demand, thereby inducing a recession. The central bank therefore prefers the lower-volatility equilibrium achieved through higher-order guidance during a ZLB episode. A detailed analysis of this framework is provided in Online Appendix H.

5 Conclusion

This paper investigates the potential for increased economic volatility at the ZLB and shows that a credible commitment to future stabilization can prevent excess volatility. We then

[23] This literature also notes that traditional guidance can appear excessively powerful in simple New Keynesian models relative to empirical estimates. We abstract from their proposed quantitative adjustments and focus on the contrast between traditional and higher-order guidance.

examine the effects of traditional forward guidance—defined as the central bank's promise to keep the policy rate at zero for an extended period. Such a commitment creates expectations of higher future output and aggregate demand, thereby raising aggregate consumption at the ZLB, as documented in a large literature on forward guidance.

Our findings indicate that it may not always be optimal for the central bank to commit to perfect future stabilization. By refraining from such a commitment, the central bank allows for alternative equilibrium paths with lower volatility at the ZLB and higher expected welfare. Although this approach improves overall welfare, it involves trade-offs: the absence of commitment, or uncertainty about the timing of future stabilization reduces volatility at the ZLB but can lead to large and costly output gap deviations later.

This paper provides insights for academics and policymakers interested in the relationship between economic uncertainty and unconventional policies at the ZLB, particularly forward guidance. Future research should explore central bank communication strategies in alternative scenarios, such as those involving private information about the state of the economy.

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I Parameter Calibration

	Parameter Description	Value	Source
η	Frisch labor supply elasticity	4	See King and Rebelo (1999).
ρ	Subjective time discount factor	0.020	Target 2.8% natural rate.
g	TFP growth rate	0.0083	Annual growth rate of 3.3%, which corresponds to the US TFP growth rate from 2009 to 2020, as detailed in Table 8 of Comin et al. (2023).
$\underline{\sigma}$	TFP volatility, low volatility regime	0.009	See Dordal i Carreras et al. (2016).
$\bar{\sigma}$	TFP volatility, high volatility regime	0.209	Target -1.5% natural rate (ZLB recession).
T	ZLB duration (quarters)	20	A five-year ZLB duration, consistent with periods such as the Global Financial Crisis and the Great Recession. See Dordal i Carreras et al. (2016).
ν	Stabilization probability parameter	1	Target average duration $1/\nu$ of one quarter before returning to stabilization.
ϵ	Elasticity of substitution intermediate goods	7	Target steady-state mark-up of 16.7%. See Galí (2015).

Table I.1: Parameter calibration used in Section 4.

II Firms' Problem

The economy contains a unit continuum of monopolistically competitive firms indexed by $i \in [0, 1]$. Firm i produces an intermediate good $y_t(i)$. Final output Y_t is a Dixit–Stiglitz aggregate with elasticity of substitution $\epsilon > 0$:

$$Y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} \, di \right)^{\frac{\epsilon}{\epsilon-1}} .$$

Production is linear in labor $y_t(i) = A_t L_t(i)$, where $L_t(i)$ is labor hired by firm i and A_t is aggregate productivity defined in equation (1). The aggregator implies the demand curve

$$y_t(i) = \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon} Y_t, \qquad p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} \, di \right)^{\frac{1}{1-\epsilon}} ,$$

where $p_t(i)$ is the firm-specific price and p_t is the price index. For tractability, prices are perfectly rigid and symmetric: $p_t(i) = p_t = \bar{p}$ for all i and t.[1] Under this restriction, each firm produces the same output, $y_t(i) = Y_t$, determined by aggregate demand.

III Proofs and Derivations

Proof of Proposition 3. In the context outlined in Section 4.2, the central bank solves the following problem:[2]

$$\min_{\sigma_1^{s,L},\sigma_2^{s,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Y}_t^2 dt \ , \quad \text{s.t. } d\hat{Y}_t = \begin{cases} -\underbrace{r_1^T(\sigma_1^{s,L})}_{<0} dt + \sigma_1^{s,L} dZ_t \ , & \text{for } t < T \ , \\ -\underbrace{r_2^T(\sigma_2^{s,L})}_{>0} dt + \sigma_2^{s,L} dZ_t \ , & \text{for } T \le t < \hat{T}^{\text{HOFG}} \ , \\ 0 \ , & \text{for } t \ge \hat{T}^{\text{HOFG}} \ , \end{cases}$$

$$\text{with } \hat{Y}_0 = r_1^T(\sigma_1^{s,L})T + r_2^T(\sigma_2^{s,L})\left(\hat{T}^{\text{HOFG}} - T\right) \ , \tag{III.1}$$

where

$$r_1^T(\sigma_1^{s,L}) \equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma}+\sigma_1^{s,L})^2}{2} < 0 \ , \qquad r_2^T(\sigma_2^{s,L}) \equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma}+\sigma_2^{s,L})^2}{2} > 0 \ .$$

After $\hat{T}^{\text{HOFG}}$, there are no additional fluctuation in $\hat{Y}_t$. Defining r_s^T as $r_1^T(\sigma_1^{s,L})$ for $s < T$ and as $r_2^T(\sigma_2^{s,L})$ for $T \le s \le \hat{T}^{\text{HOFG}}$, the process of $\hat{Y}_t$ can be articulated as follows:

$$\hat{Y}_t = \begin{cases} \underbrace{\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds}_{\equiv \hat{Y}_d(t;\hat{T}^{\text{HOFG}})} + \sigma_1^{s,L} \underbrace{Z_t}_{\sim N(0,t)} \ , & \text{for } t \le T \ , \\ \underbrace{\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds}_{\equiv \hat{Y}_d(t;\hat{T}^{\text{HOFG}})} + \sigma_1^{s,L} Z_T + \sigma_2^{s,L} \underbrace{W_{t-T}}_{\sim N(0,t-T)} \ , & \text{for } T < t \le \hat{T}^{\text{HOFG}} \ , \\ \sigma_1^{s,L} Z_T + \sigma_2^{s,L} \underbrace{W_{\hat{T}^{\text{HOFG}}-T}}_{\sim N(0,\hat{T}^{\text{HOFG}}-T)} \ , & \text{for } \hat{T}^{\text{HOFG}} < t \ . \end{cases} \tag{III.2}$$

[1]Allowing Calvo-style price resetting Calvo (1983) leaves the model's qualitative results unchanged.

[2]For this proof, it is implicitly assumed that $r_1^T(\sigma_1^{s,L}) < 0$ and $r_2^T(\sigma_2^{s,L}) > 0$ hold for the optimal values of $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, ensuring that the ZLB remains effective up to time T.

where it is assumed that after $\hat{T}^{\text{HOFG}}$, the central bank maintains $\sigma_t^s = \sigma_t^{s,n} = 0$. In equation (III.2), Z_t, W_{t-T}, and $U_{\hat{T}-T}$ are independent Brownian motions. If we square each term in equation (III.2) and apply the expectation operator with respect to the information available at $t = 0$, we obtain:

$$\mathbb{E}_0\, \hat{Y}_t^2 = \begin{cases} \hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}})^2 + \left(\sigma_1^{s,L}\right)^2 t\,, & \text{for } t \le T\,, \\ \hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}})^2 + \left(\sigma_1^{s,L}\right)^2 T + \left(\sigma_2^{s,L}\right)^2 (t-T)\,, & \text{for } T < t \le \hat{T}^{\text{HOFG}}\,, \\ \left(\sigma_1^{s,L}\right)^2 T + \left(\sigma_2^{s,L}\right)^2 (\hat{T}^{\text{HOFG}} - T)\,, & \text{for } \hat{T}^{\text{HOFG}} < t\,. \end{cases} \tag{III.3}$$

If we substitute equation (III.3) into the central bank's loss function (11), the central bank's commitment problem can be expressed as follows:

$$\begin{aligned} &\min_{\hat{T}^{\text{HOFG}},\sigma_1^{s,L},\sigma_2^{s,L}} \ \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Y}_t^2\, dt \\ &= \min_{\hat{T}^{\text{HOFG}},\sigma_1^{s,L},\sigma_2^{s,L}} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 dt + \left(\sigma_1^{s,L}\right)^2 \underbrace{\int_0^T t e^{-\rho t} dt}_{=\frac{1}{\rho^2}-\frac{1}{\rho^2}e^{-\rho T}-\cancel{\frac{T}{\rho}e^{-\rho T}}} + \left(\sigma_1^{s,L}\right)^2 T \underbrace{\int_T^\infty e^{-\rho t} dt}_{=\cancel{\frac{1}{\rho}e^{-\rho T}}} \\ &+ \left(\sigma_2^{s,L}\right)^2 \underbrace{\int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}(t-T) dt}_{=\cancel{-\frac{1}{\rho}(\hat{T}^{\text{HOFG}}-T)e^{-\rho\hat{T}^{\text{HOFG}}}}+\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho^2}} + \left(\sigma_2^{s,L}\right)^2 (\hat{T}^{\text{HOFG}}-T) \underbrace{\int_{\hat{T}^{\text{HOFG}}}^\infty e^{-\rho t} dt}_{=\cancel{\frac{1}{\rho}e^{-\rho\hat{T}^{\text{HOFG}}}}} \\ &= \min_{\hat{T},\sigma_1^{s,L},\sigma_2^{s,L}} \underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 dt}_{\text{Deterministic fluctuations}} + \underbrace{\left(\sigma_1^{s,L}\right)^2 \frac{1}{\rho^2}(1-e^{-\rho T}) + \left(\sigma_2^{s,L}\right)^2 \left(\frac{e^{-\rho T} - e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho^2}\right)}_{\text{Stochastic fluctuations}}\,. \end{aligned} \tag{III.4}$$

The central bank now has control over $\sigma_1^{s,L}$, $\sigma_2^{s,L}$, and $\hat{T}^{\text{HOFG}}$, in addition to its conventional monetary policy tool i_t. Initially, we derive the first-order condition for $\hat{T}^{\text{HOFG}}$, which is as follows:

$$2\cdot \underbrace{r_2^T(\sigma_2^{s,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}}) dt + \left(\sigma_2^{s,L}\right)^2 \frac{1}{\rho} e^{-\rho\hat{T}^{\text{HOFG}}} = 0\,, \tag{III.5}$$

from which we obtain

$$\int_0^\infty e^{-\rho t}\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}\|\sigma_1^{s,L}<0,\sigma_2^{s,L}<0)dt < 0\ . \quad \text{(III.6)}$$

The first-order condition for $\hat{T}^{\mathrm{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\mathrm{HOFG}}$ compared to $\hat{T}^{\mathrm{TFG}}$ (traditional forward guidance), as discussed in Section 4.1. This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\mathrm{TFG}}$, the expression above becomes

$$\int_0^{\hat{T}^{\mathrm{TFG}}} e^{-\rho t}\hat{Y}_{\mathrm{d}}(t;\hat{T}\|\sigma_1^{s,L}=\sigma_1^{s,n}=0,\sigma_2^{s,L}=\sigma_2^{s,n}=0)dt = 0\ , \quad \text{(III.7)}$$

which is derived by plugging $\sigma_1^{s,L}=0$ and $\sigma_2^{s,L}=0$ into equation (III.5).

Given that at the optimum, $\sigma_1^{s,L}<0$ and $\sigma_2^{s,L}<0$ (which we will demonstrate),

$$\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}\|\sigma_1^{s,L}=0,\sigma_2^{s,L}=0) < \hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}\|\sigma_1^{s,L}<0,\sigma_2^{s,L}<0)\ .$$

Therefore, we deduce from equation (III.1) that at the optimum, $\hat{T}^{\mathrm{HOFG}} < \hat{T}^{\mathrm{TFG}}$, as evidenced by comparing (III.7) with (III.6).

To characterize the optimal values of $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, a **variational argument** is required. This is because $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$ influence the levels of $r_1^T(\sigma_1^{s,L})$, $r_2^T(\sigma_2^{s,L})$, and $\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$. Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{s,L})}{\partial \sigma_1^{s,L}} = -\left(\bar{\sigma}+\sigma_1^{s,L}\right) < 0,\ \ \frac{\partial r_2^T(\sigma_2^{s,L})}{\partial \sigma_2^{s,L}} = -\left(\underline{\sigma}+\sigma_2^{s,L}\right) < 0\ .$$

Determining $\sigma_1^{s,L}$ An increase in $\sigma_1^{s,L}$ leads to a decrease in $r_1^T(\sigma_1^{s,L})$, which alters the trajectory of $\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$. This change is illustrated in Figure III.1, as depicted by the transition from the thick blue line to the dashed red line.

Differentiating $\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}) = \int_t^{\hat{T}^{\mathrm{HOFG}}} r_s^T ds$ with respect to $\sigma_1^{s,L}$, we obtain:

$$\frac{\partial \hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}{\partial \sigma_1^{s,L}} = \int_t^T -\left(\bar{\sigma}+\sigma_1^{s,L}\right)ds = -\left(\bar{\sigma}+\sigma_1^{s,L}\right)(T-t),\ \ \forall t \leq T\ .$$

To find optimal $\sigma_1^{s,L}$, we differentiate the objective function in (III.4) by $\sigma_1^{s,L}$ and obtain the

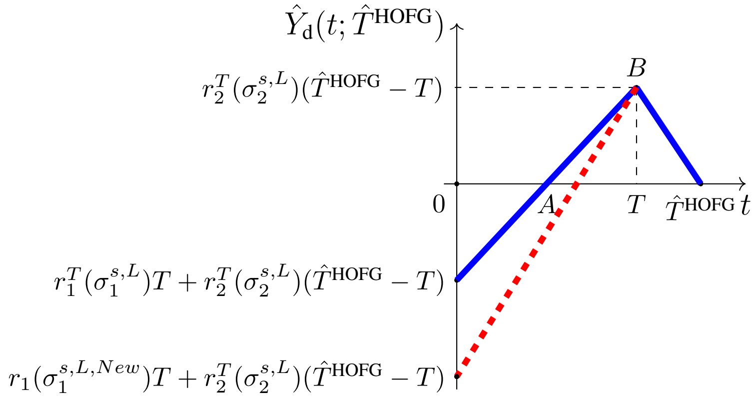

Figure III.1: Variation along $\sigma_1^{s,L}$. Increase to $\sigma_1^{s,L,New} > \sigma_1^{s,L}$.

following condition:

$$\left(\bar{\sigma}+\sigma_1^{s,L}\right)\int_0^T e^{-\rho t}\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})(T-t)dt = \left(\sigma_1^{s,L}\right)\frac{1-e^{-\rho T}}{\rho^2}\ ,$$

from which we can prove that $\sigma_1^{s,L} < 0$ must be satisfied at the optimum, given that

$$\begin{aligned}&\int_0^T e^{-\rho t}\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})(T-t)dt\\ &= \underbrace{\int_0^t e^{-\rho s}\hat{Y}_{\mathrm{d}}(s;\hat{T}^{\text{HOFG}})ds(T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s}\hat{Y}_{\mathrm{d}}(s;\hat{T}^{\text{HOFG}})ds}_{<0}\, dt < 0\ ,\end{aligned}$$

where $\int_0^t e^{-\rho s}\hat{Y}_{\mathrm{d}}(s;\hat{T}^{\text{HOFG}})ds < 0$ for $t \leq T$, as derived in equation (III.6).

Determining $\sigma_2^{s,L}$ An increase in $\sigma_2^{s,L}$ leads to a decrease in $r_2^T(\sigma_2^{s,L})$, which alters the shape of $\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})$. This effect is illustrated in Figure III.2 by the transition from the thick blue line to the dashed red line. To further analyze this, we differentiate $\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})$ with respect to $\sigma_2^{s,L}$ and obtain:

$$\frac{\partial\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})}{\partial\sigma_2^{s,L}} = \begin{cases}\displaystyle\int_T^{\hat{T}^{\text{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{s,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{s,L}\right)(\hat{T}^{\text{HOFG}}-T)\,, & t<T\ ,\\ \displaystyle\int_t^{\hat{T}^{\text{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{s,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{s,L}\right)(\hat{T}^{\text{HOFG}}-t)\,, & T\leq t\leq\hat{T}^{\text{HOFG}}\ .\end{cases}$$

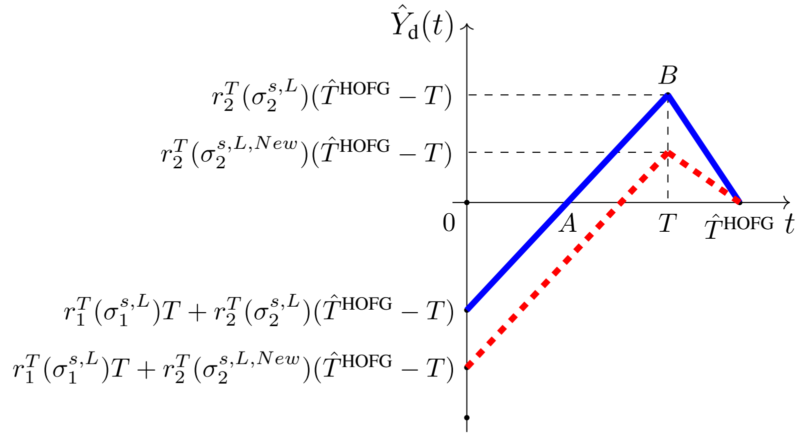

Figure III.2: Variation along $\sigma_2^{s,L}$. Increase to $\sigma_2^{s,L,New} > \sigma_2^{s,L}$.

To find the optimal $\sigma_2^{s,L}$, we differentiate the objective function in (III.4) by $\sigma_2^{s,L}$ and obtain

$$\left(\underline{\sigma}+\sigma_2^{s,L}\right)\left(\int_0^T e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}}-T)dt+\int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}}-t)dt\right)$$
$$=(\sigma_2^{s,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}}}{\rho^2}\ ,$$

from which we can demonstrate that at the optimum, $\sigma_2^{s,L}<0$ must be satisfied, given that

$$\int_0^T e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}}-T)dt+\int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}}-t)dt$$
$$<\int_0^T e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}}-T)dt+\int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}}-T)dt$$
$$=(\hat{T}^{\text{HOFG}}-T)\underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt}_{<0}<0\ ,$$

where the final inequality is derived from equation (III.6). Hence, during periods of high TFP volatility (i.e., $t<T$) and low TFP volatility with forward guidance (i.e., $T\le t\le\hat{T}^{\text{HOFG}}$), a central bank aims to target aggregate volatility levels below those in a flexible price economy: $\sigma_1^{s,L}<\sigma_1^{s,n}=0$ and $\sigma_2^{s,L}<\sigma_2^{s,n}=0$. This intervention reduces the risk premium and increases demand, raising the output gap $\hat{Y}_t$.

■

First-Order Conditions for $\sigma_1^{s,L}$, $\sigma_2^{s,L}$, and $\hat{T}^{\text{HOFG}}$ The deterministic component of the output gap process $\hat{Y}_t$, denoted as $\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$, is defined as follows (with $r_1^T(\sigma_1^{s,L})$ and $r_2^T(\sigma_2^{s,L})$ specified in equation (15)):

$$\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}}) = \int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds = \begin{cases} \underbrace{r_1^T(\sigma_1^{s,L})}_{<0}(T-t) + \underbrace{r_2^T(\sigma_2^{s,L})}_{>0}(\hat{T}^{\text{HOFG}} - T), & \text{for } \forall t \le T\ , \\ r_2^T(\sigma_2^{s,L})(\hat{T}^{\text{HOFG}} - t), & \text{for } T \le \forall t < \hat{T}^{\text{HOFG}}\ , \end{cases}$$

from which we derive the following:

$$\begin{aligned} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt = & \int_0^T e^{-\rho t}\left[r_1^T(\sigma_1^{s,L})(T-t) + r_2^T(\sigma_2^{s,L})(\hat{T}^{\text{HOFG}} - T)\right] dt \\ & + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t} r_2^T(\sigma_2^{s,L})(\hat{T}^{\text{HOFG}} - t)dt\ . \end{aligned} \tag{III.8}$$

The first condition for $\hat{T}^{\text{HOFG}}$ can be written as

$$2 \cdot r_2^T(\sigma_2^{s,L}) \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt + \left(\sigma_2^{s,L}\right)^2 \frac{e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho} = 0\ , \tag{III.9}$$

where

$$\begin{aligned} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt = & r_1^T(\sigma_1^{s,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{s,L})(\hat{T}^{\text{HOFG}} - T)\frac{1-e^{-\rho T}}{\rho} \\ & + r_2^T(\sigma_2^{s,L})\left[\frac{e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho^2} + \frac{\hat{T}^{\text{HOFG}} - T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right]\ , \end{aligned}$$

follows from equation (III.8). Combined with equation (III.9), the first-order condition for $\hat{T}^{\text{HOFG}}$ is expressed as follows:

$$\begin{aligned} & 2 \cdot r_2^T(\sigma_2^{s,L})\Bigg[r_1^T(\sigma_1^{s,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{s,L})(\hat{T}^{\text{HOFG}} - T)\frac{1-e^{-\rho T}}{\rho} \\ & \qquad + r_2^T(\sigma_2^{s,L})\left[\frac{e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho^2} + \frac{\hat{T}^{\text{HOFG}} - T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right]\Bigg] + \left(\sigma_2^{s,L}\right)^2 \frac{e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho} = 0\ . \end{aligned}$$

The first-order condition for $\sigma_1^{s,L}$ is expressed as

$$\left(\bar{\sigma} + \sigma_1^{s,L}\right) \int_0^T e^{-\rho t} \hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}})(T-t)dt = \left(\sigma_1^{s,L}\right) \frac{1-e^{-\rho T}}{\rho^2} , \tag{III.10}$$

where

$$\begin{aligned} \int_0^T e^{-\rho t} \hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}})(T-t)dt =& r_1^T(\sigma_1^{s,L}) \left[-\frac{2}{\rho^3} e^{-\rho T} + \frac{T^2}{\rho} - \frac{2T}{\rho^2} + \frac{2}{\rho^3} \right] \\ &+ r_2^T(\sigma_2^{s,L})(\hat{T}^{\mathrm{HOFG}} - T) \left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] . \end{aligned} \tag{III.11}$$

Substituting equation (III.11) into equation (III.10), we arrive at:

$$\begin{aligned} &\left(\bar{\sigma} + \sigma_1^{s,L}\right) \left[r_1^T(\sigma_1^{s,L}) \left[-\frac{2}{\rho^3} e^{-\rho T} + \frac{T^2}{\rho} - \frac{2T}{\rho^2} + \frac{2}{\rho^3} \right] + r_2^T(\sigma_2^{s,L})(\hat{T}^{\mathrm{HOFG}} - T) \left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] \right] \\ &\qquad = \left(\sigma_1^{s,L}\right) \frac{1-e^{-\rho T}}{\rho^2} , \end{aligned}$$

as the first-order condition for $\sigma_1^{s,L}$. Finally, the first-order condition for $\sigma_2^{s,L}$ is as follows:

$$\begin{aligned} &\left(\underline{\sigma} + \sigma_2^{s,L}\right) \left((\hat{T}^{\mathrm{HOFG}} - T) \int_0^T e^{-\rho t} \hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}}) dt + \int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t} \hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}} - t) dt \right) \\ &\qquad = (\sigma_2^{s,L}) \frac{e^{-\rho T} - e^{-\rho \hat{T}^{\mathrm{HOFG}}}}{\rho^2} , \end{aligned}$$

Therefore, the first-order condition for $\sigma_2^{s,L}$ is expressed as:

$$\begin{aligned} &\left(\underline{\sigma} + \sigma_2^{s,L}\right) \left[\left[r_1^T(\sigma_1^{s,L}) \left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] + r_2^T(\sigma_2^{s,L})(\hat{T}^{\mathrm{HOFG}} - T) \frac{1-e^{-\rho T}}{\rho} \right] (\hat{T}^{\mathrm{HOFG}} - T) \right. \\ &\qquad \left. + r_2^T(\sigma_2^{s,L}) \left[-\frac{2}{\rho^3} e^{-\rho \hat{T}^{\mathrm{HOFG}}} + \frac{(\hat{T}^{\mathrm{HOFG}} - T)^2}{\rho} e^{-\rho T} - \frac{2(\hat{T}^{\mathrm{HOFG}} - T)}{\rho^2} e^{-\rho T} + \frac{2}{\rho^3} e^{-\rho T} \right] \right] \\ &= \left(\sigma_2^{s,L}\right) \frac{e^{-\rho T} - e^{-\rho \hat{T}^{\mathrm{HOFG}}}}{\rho^2} , \end{aligned}$$

where we use the following properties of $\hat{Y}_d\left(t; \hat{T}^{\text{HOFG}}\right)$:

$$\int_0^T e^{-\rho t}\hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}})dt = r_1^T(\sigma_1^{s,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{s,L})(\hat{T}^{\text{HOFG}} - T)\frac{1 - e^{-\rho T}}{\rho},$$

and

$$\begin{aligned}
&\int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - t)dt \\
&= r_2^T(\sigma_2^{s,L})\left[-\frac{2e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho^3} + \frac{(\hat{T}^{\text{HOFG}} - T)^2}{\rho}e^{-\rho T} - \frac{2(\hat{T}^{\text{HOFG}} - T)}{\rho^2}e^{-\rho T} + \frac{2e^{-\rho T}}{\rho^3}\right].
\end{aligned}$$

Proof of Proposition 5. From the household flow budget constraint (3), and noting that bonds are in zero net supply (i.e., $dB_t = 0$ for all t), we have

$$\begin{aligned}
\bar{p}\Delta C_0 &= \Delta(w_0 L_0) + \Delta D_0, \\
\bar{p}C_t &= w_t L_t + D_t,
\end{aligned}$$

which, combined with (19), yields

$$\begin{aligned}
\bar{p}\Delta C_0 &= \bar{p}\Delta Y_0 + \chi_0, \\
\bar{p}C_t &= \bar{p}Y_t + \tau_t.
\end{aligned} \tag{III.12}$$

If the government follows the fiscal transfer policy in Proposition 5, substituting (20) into (III.12) yields $C_t = Y_t = Y_t^*$, leading to $\tau_t = \chi_0 = 0$ for $t \geq 0$ in equilibrium.

■

Online Appendix: For Online Publication Only

A Efficient Flexible Price Equilibrium

This section derives the efficient flexible price equilibrium, which serves as the benchmark for economic and welfare analysis. In addition to flexible prices, we assume the government implements a production subsidy τ_Y, financed via lump-sum taxation, to offset the real distortions caused by monopolistic competition. Proposition A.1 summarizes the dynamics of the real wage, output, labour, natural interest rate r_t^n, and household consumption in the efficient flexible price equilibrium.

Proposition A.1 (Efficient Flexible Price Equilibrium) *In the efficient flexible price equilibrium,*[1] *the following results hold:*

1. *Aggregate output, consumption, and real wage are proportional to aggregate technology A_t and are given by $Y_t^n = C_t^n = \frac{w_t^n}{p_t} = A_t$.*

2. *Equilibrium labor supply is constant: $L_t^n = 1$.*

3. *Household consumption evolves according to*

$$\frac{dC_t^n}{C_t^n} = g\, dt + \sigma\, dZ_t = (r^n - \rho + \sigma^2)\, dt + \sigma\, dZ_t,$$

   where $r_t^n = \rho + g - \sigma^2$ is the natural interest rate.

Proof of Proposition A.1. From the firms' problem in Appendix II, the production function of each firm i is linear in labor:

$$y_t(i) = A_t L_t(i),$$

where $y_t(i)$ and $L_t(i)$ denote output and labor for firm i, respectively. Final output Y_t is a Dixit-Stiglitz aggregate of individual varieties, so demand for each firm's output is

$$y_t(i) = \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} Y_t.$$

[1] Variables in the efficient flexible price (natural) equilibrium are denoted by the superscript n.

At each moment t, intermediate firm i sets $p_t(i)$ to maximize profits:

$$\max_{p_t(i)} (1+\tau_Y)p_t(i)\left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} Y_t - \frac{w_t}{A_t}\left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} Y_t, \tag{A.1}$$

taking Y_t as given. Under flexible prices, all firms would set $p_t(i) = p_t$ and hire $L_t(i) = L_t$, so aggregate production is $Y_t = A_t L_t$. We impose $\tau_Y = \frac{1}{\varepsilon - 1}$ following Woodford (2003) to eliminate distortions caused by monopolistic competition. The first-order condition of equation (A.1) yields the real wage:

$$\frac{w_t^n}{p_t} = A_t. \tag{A.2}$$

The household's intratemporal condition, i.e., equation (4), is

$$\frac{L_t^{\frac{1}{\eta}}}{w_t} = \frac{1}{p_t C_t}.$$

Combining this with (A.2), market clearing $C_t = Y_t$, and the aggregate output result, we obtain

$$Y_t^n = C_t^n = A_t. \tag{A.3}$$

From equation (A.3) and aggregate output, equilibrium labor is

$$L_t^n = 1,$$

which is constant over time.

Finally, since C_t^n is proportional to A_t, it follows that

$$\frac{dC_t^n}{C_t^n} = g\,dt + \sigma\,dZ_t = \left(r_t^n - \rho + \sigma^2\right)dt + \sigma\,dZ_t, \tag{A.4}$$

where $r_t^n = \rho + g - \sigma^2$ is the natural real interest rate.

■

B Deriving the IS equation (10)

Proof of Proposition 1. From the expression for the conditional volatility of consumption growth, $Var_t\left(\frac{dC_t}{C_t}\right) = \text{rp}_t\, dt$, and equations (6) and (8), households' consumption C_t follows

$$\frac{dC_t}{C_t} = \left(i_t + (\sigma + \sigma_t^s)^2 - \rho\right) dt + (\sigma_t + \sigma_t^s) dZ_t. \tag{B.1}$$

Define the consumption and output gaps, respectively, as $\hat{C}_t = \log\left(\frac{C_t}{C_t^n}\right)$ and $\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right)$. Using equations (A.4) and (B.1), we obtain

$$\begin{aligned} d\hat{Y}_t = d\hat{C}_t &= \left(i_t - \underbrace{\left(r_t^n - \frac{(\sigma + \sigma_t^s)^2}{2} + \frac{\sigma^2}{2} \right)}_{\equiv r_t^T} \right) dt + \sigma_t^s dZ_t \\ &= \left(i_t - r_t^T\right) dt + \sigma_t^s dZ_t, \end{aligned} \tag{B.2}$$

where the first equality follows by market clearing in equilibrium, $Y_t = C_t$. Since we have risk-premium levels $\text{rp}_t = (\sigma_t + \sigma_t^s)^2$ in the rigid-price economy and $\text{rp}_t^n = \sigma^2$ in the flexible-price economy, we can express our risk-adjusted natural rate r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}\left(\text{rp}_t - \text{rp}_t^n\right) = r_t^n - \frac{1}{2}\hat{\text{rp}}_t. \tag{B.3}$$

■

C Welfare Derivation

Proof of Proposition 2. Here, we derive a quadratic approximation to welfare, i.e., the loss function, for a representative-agent economy with fully rigid prices, following Woodford (2003).

The representative agent's instantaneous utility is

$$U_t = \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}.$$

A second-order approximation around the efficient flexible-price equilibrium yields

$$\begin{aligned} U_t - U_t^n \approx\ & U_{C,t}^n C_t^n \left(\frac{C_t - C_t^n}{C_t^n}\right) + \frac{U_{CC,t}^n}{2} (C_t^n)^2 \left(\frac{C_t - C_t^n}{C_t^n}\right)^2 \\ & + U_{L,t}^n L_t^n \left(\frac{L_t - L_t^n}{L_t^n}\right) + \frac{U_{LL,t}^n}{2} (L_t^n)^2 \left(\frac{L_t - L_t^n}{L_t^n}\right)^2 + \text{h.o.t.}, \end{aligned} \tag{C.1}$$

where h.o.t. denotes higher-order terms. The relevant derivatives are

$$\begin{aligned} U_{C,t}^n &= \frac{1}{C_t^n}, & U_{CC,t}^n &= -\frac{1}{(C_t^n)^2}, \\ U_{L,t}^n &= -(L_t^n)^{\frac{1}{\eta}} = -1, & U_{LL,t}^n &= -\frac{1}{\eta}(L_t^n)^{\frac{1}{\eta}-1} = -\frac{1}{\eta}. \end{aligned} \tag{C.2}$$

where L_t^n and C_t^n are defined in Proposition A.1. Since $Y_t = A_t L_t$ holds in both flexible- and rigid-price economies due to the absence of price dispersion, it follows that

$$\frac{C_t - C_t^n}{C_t^n} = \hat{Y}_t \approx \log\left(\frac{C_t}{C_t^n}\right) = \log\left(\frac{L_t}{L_t^n}\right) \approx \frac{L_t - L_t^n}{L_t^n}. \tag{C.3}$$

Substituting equations (C.2) and (C.3) into equation (C.1) gives

$$U_t - U_t^n \approx -\frac{1}{2}\left(1 + \frac{1}{\eta}\right)\hat{Y}_t^2 + \text{h.o.t.},$$

whose present-discounted value is proportional to the quadratic welfare loss in equation (11):

$$\mathbb{L}^Y\left(\{\hat{Y}_t\}_{t\geq 0}\right) = \rho\,\mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Y}_t^2\,dt,$$

■

D Additional Proofs on Stabilization

D.1 Perfect Stabilization Outside the ZLB

This section shows that the monetary authority can always attain perfect stabilization outside the ZLB—both in the baseline without forward guidance (Section 3) and under traditional forward guidance (Section 4.1). Suppose the central bank adopts the simple rule

$$i_t = \max\left\{ r_t^T + \phi_y \hat{Y}_t,\ 0 \right\}, \tag{D.1}$$

where $\phi_y > 0$ satisfies the Taylor principle when the ZLB is slack.[2] When the ZLB does not bind, combining equation (10) with equation (D.1) yields

$$\mathbb{E}_t\, d\hat{Y}_t = \phi_y \hat{Y}_t,$$

which forces $\hat{Y}_t = 0$ for all t. Hence the output gap is perfectly stabilized, resulting in an unique rational-expectations equilibrium outside the ZLB. See Blanchard and Kahn (1980) and Buiter (1984) for the necessary conditions underlying this result.

D.2 Stochastic Stabilization in Section 4.3

Proof of Proposition 4. Here, we derive the equilibrium when there is a Poisson (with ν as its parameter) probability that the economy returns to full stabilization after $\hat{T}^{\text{HOFG}}$. $\nu \in [0, +\infty)$, where $\nu = 0$ means no return to stabilization (as in Proposition 3). The

[2]Beyond the Taylor principle, Lee and Dordal i Carreras (2025a) show that targeting the risk-adjusted natural rate—or its risk-premium component—is required for equilibrium uniqueness in fully global models with higher-order terms in the dynamic IS equation.

central bank solves:

$$\min_{\sigma_1^{s,L},\sigma_2^{s,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_t^2 dt + \mathbb{E}_0 \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\cdot e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)}\cdot \hat{Y}_t^2 dt,$$
$$\text{s.t.} \begin{cases} d\hat{Y}_t = -(\underbrace{r_1^T(\sigma_1^{s,L})}_{<0})dt + (\sigma_1^{s,L})dZ_t, & \text{for } t < T, \\ d\hat{Y}_t = -(\underbrace{r_2^T(\sigma_2^{s,L})}_{>0})dt + (\sigma_2^{s,L})dZ_t, & \text{for } T \le t < \hat{T}^{\text{HOFG}}, \\ d\hat{Y}_t = 0, & \text{for } t \ge \hat{T}^{\text{HOFG}}, \end{cases} ,$$
$$\text{with } \hat{Y}_0 = r_1^T(\sigma_1^{s,L})T + r_2^T(\sigma_2^{s,L})(\hat{T}^{\text{HOFG}} - T). \tag{D.2}$$

where the discounting becomes $\rho + \nu > \rho$ after $\hat{T}^{\text{HOFG}}$, which is itself endogenous. The loss function in (D.2) can be written then as

$$\begin{aligned} &\min_{\sigma_1^{s,L},\sigma_2^{s,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_t^2 dt + \mathbb{E}_0 \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\cdot e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)}\cdot \hat{Y}_t^2 dt \\ =& \min_{\hat{T},\sigma_1^{s,L},\sigma_2^{s,L}} \underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 dt}_{\text{From deterministic fluctuation}} + \underbrace{\left(\sigma_1^{s,L}\right)^2\left[\frac{1-e^{-\rho T}}{\rho^2} - \frac{Te^{-\rho\hat{T}^{\text{HOFG}}}}{\rho}\left(\frac{\nu}{\rho+\nu}\right)\right]}_{\text{From stochastic fluctuation}} \\ &+ \underbrace{\left(\sigma_2^{s,L}\right)^2\left[\left(\frac{e^{-\rho T} - e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho^2}\right) - \left(\hat{T}^{\text{HOFG}} - T\right)e^{-\rho\hat{T}^{\text{HOFG}}}\frac{\nu}{\rho(\rho+\nu)}\right]}_{\text{From stochastic fluctuation}} \end{aligned} \tag{D.3}$$

where $\hat{Y}_d(t;\hat{T}^{\text{HOFG}})$ is defined in (III.2): we observe new terms appear compared with the baseline case of $\nu = 0$. Notice that if the central bank is allowed to maximize with respect to ν, then we obtain a corner solution with $\nu \to +\infty$. This means that the most efficient would be to immediately return to perfect stabilization, with a very small probability of no adjustment.

The central bank has control over $\sigma_1^{s,L}$, $\sigma_2^{s,L}$, and $\hat{T}^{\text{HOFG}}$, in addition to its conventional monetary policy tool i_t. We derive the first-order condition for $\hat{T}^{\text{HOFG}}$ as follows:

$$2 \cdot \underbrace{r_2^T(\sigma_1^{s,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt + \underbrace{\left(\sigma_1^{s,L}\right)^2 e^{-\rho \hat{T}^{\text{HOFG}}} \left(\frac{\nu}{\rho+\nu}\right) T}_{>0}$$
$$+ \underbrace{\left(\sigma_2^{s,L}\right)^2 \left[\frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho+\nu} + \left(\hat{T}^{\text{HOFG}} - T\right) e^{-\rho \hat{T}^{\text{HOFG}}} \left(\frac{\nu}{\rho+\nu}\right)\right]}_{>0} = 0 \tag{D.4}$$

from which we obtain

$$\int_0^{\infty} e^{-\rho t} \hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt = \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{s,L} < 0, \sigma_2^{s,L} < 0) dt < 0 \,. \tag{D.5}$$

The first-order condition for $\hat{T}^{\text{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\text{HOFG}}$ compared to $\hat{T}^{\text{TFG}}$ (traditional forward guidance). This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$, the expression above becomes

$$\int_0^{\hat{T}^{\text{TFG}}} e^{-\rho t} \hat{Y}_{\text{d}}(t; \hat{T} \| \sigma_1^{s,L} = \sigma_1^{s,n} = 0, \sigma_2^{s,L} = \sigma_2^{s,n} = 0) dt = 0 \,, \tag{D.6}$$

which is derived by plugging $\sigma_1^{s,L} = 0$ and $\sigma_2^{s,L} = 0$ into equation (D.4).

Given that at the optimum, $\sigma_1^{s,L} < 0$ and $\sigma_2^{s,L} < 0$ (which we will demonstrate),

$$\hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{s,L} = 0, \sigma_2^{s,L} = 0) < \hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{s,L} < 0, \sigma_2^{s,L} < 0) \,.$$

Therefore, we deduce from equation (D.2) that at the optimum, $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$, as evidenced by comparing (D.5) with (D.6).

To characterize the optimal values of $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, a **variational argument** is required. This is because $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$ influence the levels of $r_1^T(\sigma_1^{s,L})$, $r_2^T(\sigma_2^{s,L})$, and $\hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}})$.

Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{s,L})}{\partial \sigma_1^{s,L}} = -\left(\bar{\sigma} + \sigma_1^{s,L}\right) < 0, \quad \frac{\partial r_2^T(\sigma_2^{s,L})}{\partial \sigma_2^{s,L}} = -\left(\underline{\sigma} + \sigma_2^{s,L}\right) < 0 \; .$$

Determining $\sigma_1^{s,L}$ An increase in $\sigma_1^{s,L}$ leads to a decrease in $r_1^T(\sigma_1^{s,L})$, which alters the trajectory of $\hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}})$. This variational change is illustrated in Figure D.1, as depicted by the transition from the thick blue line to the dashed red line.

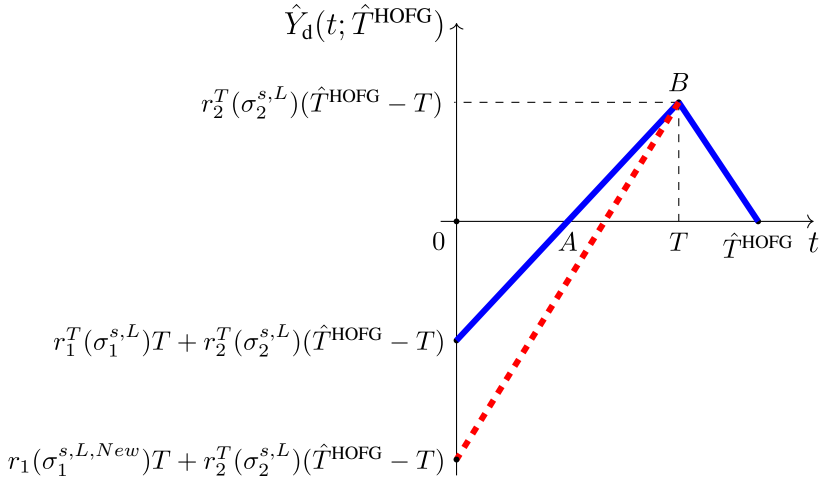

Figure D.1: Variation along $\sigma_1^{s,L}$. Increase to $\sigma_1^{s,L,New} > \sigma_1^{s,L}$.

Differentiating $\hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}}) = \int_t^{\hat{T}^{\mathrm{HOFG}}} r_s^T ds$ with respect to $\sigma_1^{s,L}$, we obtain:

$$\frac{\partial \hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}})}{\partial \sigma_1^{s,L}} = \int_t^T -\left(\bar{\sigma} + \sigma_1^{s,L}\right) ds = -\left(\bar{\sigma} + \sigma_1^{s,L}\right)(T - t), \;\; \forall t \leq T \; .$$

To find optimal $\sigma_1^{s,L}$, we differentiate the objective function in (D.3) by $\sigma_1^{s,L}$ and obtain the following condition:

$$\left(\bar{\sigma} + \sigma_1^{s,L}\right) \int_0^T e^{-\rho t} \hat{Y}_{\mathrm{d}}(t; \hat{T}^{\mathrm{HOFG}})(T - t) dt = \left(\sigma_1^{s,L}\right) \left\{ \frac{1 - e^{-\rho T}}{\rho^2} - \frac{e^{-\rho \hat{T}^{\mathrm{HOFG}}}}{\rho} \left[1 - \frac{\rho}{\rho + \nu}\right] \cdot T \right\} . \tag{D.7}$$

First, we obtain

$$\int_0^T e^{-\rho t}\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = \underbrace{\int_0^t e^{-\rho s}\hat{Y}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds\cdot(T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s}\hat{Y}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds}_{<0}\, dt < 0\ ,$$

where $\int_0^t e^{-\rho s}\hat{Y}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds < 0$ for $t \leq T$, as derived in equation (D.5). Also, as we know

$$\begin{aligned}\frac{1-e^{-\rho T}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]T &\geq \frac{1-e^{-\rho T}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}T \\ &= \underbrace{\int_0^T te^{-\rho t}dt}_{>0} + \underbrace{\frac{T}{\rho}e^{-\rho T} - \frac{T}{\rho}e^{-\rho\hat{T}^{\mathrm{HOFG}}}}_{\geq 0} > 0,\end{aligned} \tag{D.8}$$

from (D.7), we obtain that $\sigma_1^{s,L} < \sigma_1^{s,n} = 0$ at the optimum.[3]

Determining $\sigma_2^{s,L}$ An increase in $\sigma_2^{s,L}$ leads to a decrease in $r_2^T(\sigma_2^{s,L})$, which alters the shape of $\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$. This variational effect is illustrated in Figure D.2 by the transition from the thick blue line to the dashed red line. To further analyze this effect, we differentiate $\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$ with respect to $\sigma_2^{s,L}$ and obtain:

$$\frac{\partial\hat{Y}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}{\partial\sigma_2^{s,L}} = \begin{cases}\displaystyle\int_T^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{s,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{s,L}\right)(\hat{T}^{\mathrm{HOFG}}-T)\,, & t<T\ , \\ \displaystyle\int_t^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{s,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{s,L}\right)(\hat{T}^{\mathrm{HOFG}}-t)\,, & T\leq t\leq\hat{T}^{\mathrm{HOFG}}\ .\end{cases}$$

[3] Note that in (D.7), due to the additional term

$$\frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]\cdot T,$$

$\sigma_1^{s,L}$ becomes more negative at the optimum, taking $\hat{T}^{\mathrm{HOFG}}$ and $\sigma_2^{s,L}$ as given, compared with our benchmark case in which $\nu = 0$.

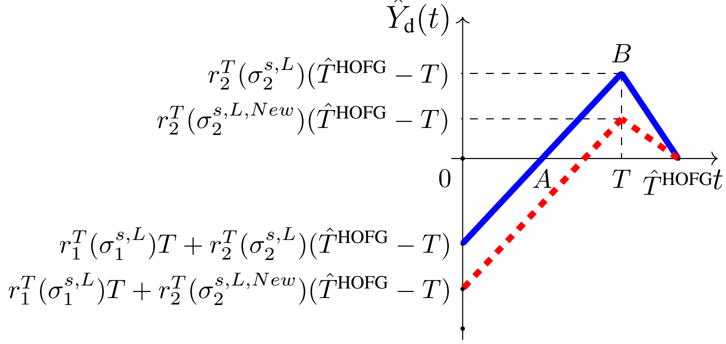

Figure D.2: Variation along $\sigma_2^{s,L}$. Increase to $\sigma_2^{s,L,New} > \sigma_2^{s,L}$.

To find the optimal $\sigma_2^{s,L}$, we differentiate the objective function in (D.3) by $\sigma_2^{s,L}$ and obtain

$$\left(\underline{\sigma} + \sigma_2^{s,L}\right)\left(\int_0^T e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - t)dt\right)$$
$$= (\sigma_2^{s,L})\left\{\frac{e^{-\rho T} - e^{-\rho\hat{T}}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho}\left[1 - \frac{\rho}{\rho+\nu}\right]\left(\hat{T}^{\text{HOFG}} - T\right)\right\}, \tag{D.9}$$

from which we can demonstrate that at the optimum, $\sigma_2^{s,L} < 0$ must be satisfied, given that

$$\int_0^T e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - t)dt$$
$$< \int_0^T e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - T)dt$$
$$= (\hat{T}^{\text{HOFG}} - T)\underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt}_{<0} < 0\,,$$

where the final inequality is derived from equation (D.5), and

$$\frac{e^{-\rho T} - e^{-\rho\hat{T}}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho}\left[1 - \frac{\rho}{\rho+\nu}\right]\left(\hat{T}^{\text{HOFG}} - T\right) \geq \frac{e^{-\rho T} - e^{-\rho\hat{T}}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho}\left(\hat{T}^{\text{HOFG}} - T\right)$$
$$= \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}(t - T)dt > 0. \tag{D.10}$$

Equation (D.9) proves that $\sigma_2^{s,L} < 0$ at the optimum.[4] Thus, we have proven that during periods of high TFP volatility (i.e., $t < T$) and low TFP volatility with forward guidance (i.e., $T \leq t \leq \hat{T}^{\text{HOFG}}$), a central bank aims to target aggregate volatility levels below those in a flexible-price economy: $\sigma_1^{s,L} < \sigma_1^{s,n} = 0$ and $\sigma_2^{s,L} < \sigma_2^{s,n} = 0$. Such intervention reduces the risk premium and raises demand, thereby increasing output gap $\hat{Y}_t$.

■

Proof of Corollary 1. Note that $\nu = +\infty$ implies that full stabilization immediately follows after $\hat{T}^{\text{HOFG}}$ when the zero policy rate regime is over. It corresponds to the traditional forward guidance case of Section 4.1, so when $\nu = +\infty$, the only feasible $\left(\sigma_1^{s,L}, \sigma_2^{s,L}, \hat{T}^{\text{HOFG}}\right)$ would be $(0, 0, \hat{T})$ in this case. Since for every ν, $\left(\sigma_1^{s,L}, \sigma_2^{s,L}, \hat{T}^{\text{HOFG}}\right) = (0, 0, \hat{T}^{\text{TFG}})$ is feasible, we obtain

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu\right) \leq \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu = +\infty\right) .$$

To obtain the strict inequality between the two sides, we compare the first-order conditions for $\hat{T}^{\text{HOFG}}$ when $\nu = +\infty$ and $\nu \to +\infty$. When $\nu = +\infty$, the optimality is given by (14), which can be written as

$$\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}})dt = 0 , \tag{D.11}$$

where $\hat{Y}_{\text{d}}$ is defined in equation (III.2). In contrast, when $\nu \to +\infty$, the first-order condition of $\hat{T}^{\text{HOFG}}$ in (D.4) becomes

$$2 \cdot \underbrace{r_2^T(\sigma_1^{s,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}})dt + \underbrace{\left(\sigma_1^{s,L}\right)^2 e^{-\rho \hat{T}^{\text{HOFG}}} T}_{>0} + \underbrace{\left(\sigma_2^{s,L}\right)^2 \left[\left(\hat{T}^{\text{HOFG}} - T\right) e^{-\rho \hat{T}^{\text{HOFG}}}\right]}_{>0} = 0$$

[4]Note that in (D.9), due to the additional term

$$\frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho}\left[1 - \frac{\rho}{\rho + \nu}\right] \cdot (\hat{T}^{\text{HOFG}} - T),$$

$\sigma_2^{s,L}$ becomes more negative at the optimum, taking $\hat{T}^{\text{HOFG}}$ and $\sigma_1^{s,L}$ as given, compared with our benchmark case in which $\nu = 0$.

which is different from the above (D.11). Therefore, we obtain

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu = +\infty\right) .$$

■

E Discrete Three-Period Model

Here, we consider a discretized, simplified version of the model with equally spaced periods. As in the main text, assume that in period 0 the economy is pushed to the ZLB by a rise in expected fundamental volatility in period 1 to $\sigma_1 = \bar{\sigma}$. The resulting natural rate, $r_1^n \equiv \rho + g - \bar{\sigma}^2 < 0$, makes the ZLB binding.[5] From period 2 onward, volatility declines to $\sigma_t = \underline{\sigma} < \bar{\sigma}$ for all $t \geq 2$, so the natural rate increases to $r_t^n \equiv \rho + g - \underline{\sigma}^2 > 0$, making an escape from the ZLB possible.

For simplicity, assume the forward-guidance horizon is fixed at $\hat{T} = 2$, i.e., $i_1 = i_2 = 0$, and the central bank must choose a policy path that stabilizes the economy in expectation by that date:

$$\mathbb{E}_0[\hat{Y}_2] = 0. \tag{E.1}$$

Define the change in the output gap as $\Delta\hat{Y}_t \equiv \hat{Y}_t - \hat{Y}_{t-1}$. A discretized version of the dynamic IS equation (10) is

$$\Delta\hat{Y}_t = -\left[r_t^n + \frac{1}{2}\left(\sigma_t^2 - (\sigma_t + \sigma_t^s)^2\right)\right] + \sigma_t^s\,\varepsilon_t, \quad \text{for } t = 1,\, 2\,, \tag{E.2}$$

where $\varepsilon_t \sim N(0,1)$ are i.i.d. TFP shocks (which are normalized by fundamental volatility σ_t).

[5] The timing convention here is that r_1^n is the natrual rate of interest in period 0 (i.e., from time 0 to time 1), and σ_1 is the fundamental volatility of technology in period 0 (i.e., volatility of TFP growth from time 0 to time 1).

Expected output gap at period-0 can be written as

$$\mathbb{E}_0[\hat{Y}_t] = \hat{Y}_0 + \sum_{j=1}^{t} \Delta \mathbb{E}_0[\hat{Y}_j].$$

Using equations (E.1) and (E.2) in the previous expression yields the expected output gap for periods $t = 0, 1$,

$$\hat{Y}_0 = \sum_{j=t+1}^{2} r_j^n + \frac{1}{2} \sum_{j=t+1}^{2} \left[\sigma_j^2 - (\sigma_j + \sigma_j^s)^2 \right].$$

and

$$\mathbb{E}_0[\hat{Y}_1] = r_2^n + \frac{1}{2} \left[\sigma_2^2 - (\sigma_2 + \sigma_2^s)^2 \right].$$

The conditional variance from the period-0 perspective is

$$\mathrm{Var}_0(\hat{Y}_t) = \sum_{j=1}^{t} (\sigma_j^s)^2.$$

We compare two regimes: (i) traditional forward guidance (TFG) and (ii) higher-order forward guidance (HOFG) without stochastic stabilization (see Section 4.2). The central bank sets $i_t = r_t^n$ and $\sigma_t^s = 0$ for $t > 2$, so the output gap remains at its period-2 level, thus $\Delta \hat{Y}_t = 0$ for $t > 2$. Under TFG, backward induction gives $\sigma_j^s = 0$ for $j = 1, 2$, as in Section 4.1.

Under HOFG, the central bank commits only to stabilizing the economy in expectation, i.e., (E.1), so any deviation in the output gap by period 2 becomes permanent. This creates multiple equilibrium paths defined by the *excess* volatility pair $\{\sigma_1^s,\, \sigma_2^s\}$, which can deviate from zero and become choice variables in the central bank's optimization. To further simplify the problem, we impose $\sigma_2^s = 0$ under HOFG, ensuring both regimes deliver the same expected gap in period 1: $\hat{Y}_1^{\mathrm{TFG}} = \mathbb{E}_0[\hat{Y}_1^{\mathrm{HOFG}}] = r_2^n > 0$. Policy differences therefore arise solely from the choice of σ_1^s.

The output gap level and variance differences across regimes are:

$$\hat{Y}_0^{\text{HOFG}} - \hat{Y}_0^{\text{TFG}} = \tfrac{1}{2}\left[\overline{\sigma}^2 - (\overline{\sigma} + \sigma_1^s)^2\right] \geq 0, \quad \text{for } \sigma_1^s \in (-\overline{\sigma}, 0],$$
$$\text{Var}_0\left[\hat{Y}_t^{\text{HOFG}} - \hat{Y}_t^{\text{TFG}}\right] = (\sigma_1^s)^2 \geq 0, \quad \text{for } t \geq 1.$$

Thus, the welfare loss function (11) highlights the key trade-off faced by the central bank: narrowing the initial output gap by choosing $\sigma_1^s < 0$ increases future output gap volatility, which is also costly. However, since HOFG can always match the TFG outcome by setting $\sigma_1^s = 0$, it weakly dominates TFG in welfare terms. Figure E.3 shows the equilibrium paths in this three-period example.[6]

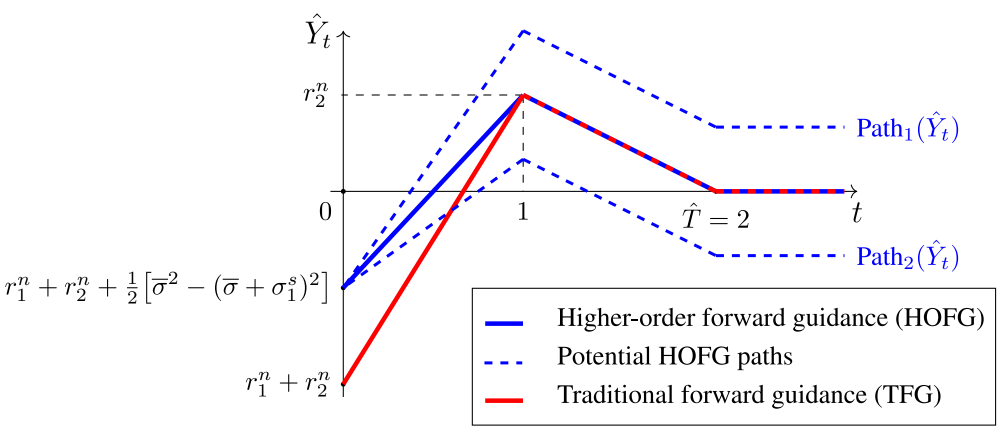

Figure E.3: Intervention dynamics of $\{\hat{Y}_t\}$ with $\sigma_1^s < 0$.

F Stochastic T in Section 3 and Section 4.1

Here we prove the results of Sections 3 and 4.1, showing that $\sigma_t^s = \sigma_t^{s,n} \equiv 0$ holds even when the mandatory ZLB duration T is stochastic. We first present the proof under the no-guidance regime; the case with traditional forward guidance follows directly.

For illustration purposes, we assume that T follows a discrete distribution: T_1, T_2, and T_3 with probabilities p_1, p_2, and p_3 with $p_1 + p_2 + p_3 = 1$. The same logic can be applied

[6] Once can easily see that choosing $\sigma_1^s > 0$ is suboptimal.

to more general cases where T has a continuous distribution. We keep assuming that after T is realized, i.e., the ZLB ends, the monetary authority enforces perfect stabilization of the economy as describe in the baseline case of Section 3. We similarly rely on backward induction. First, we know certainly that after T_3, the economy is fully stabilized, implying $\sigma_t^s = 0$ for $t \geq T_3$. For $t \in [T_2, T_3)$,

1. If the ZLB already ended at T_1 or T_2, then $\sigma_t^s = 0$.

2. The ZLB has not ended: then it is certain that $T = T_3$ and $\hat{Y}_t = 0$ for $t \geq T_3$, which means that $\sigma_t^s = 0$ for $t \in (T_2, T_3)$. In that case, $\hat{Y}_{T_2} = \underline{r}(T_3 - T_2) < 0$ is determined.

For $t \in [T_1, T_2)$, we know that

1. If the ZLB already ended at T_1, then $\sigma_t^s = 0$.

2. The ZLB has not ended: then it is for sure that $T = T_2$ or $T = T_3$. At $t = T_2 - dt$ for small $dt > 0$, $\hat{Y}_{T_2 - dt}$ is determined by a conditional probability-weighted linear combination of 0 (when $T = T_2$) and $\underline{r}(T_3 - T_2)$ (when $T = T_3$), so that

$$\hat{Y}_{T_2 - dt} = \underline{r}dt + \frac{p_2}{1 - p_1} \cdot 0 + \frac{p_3}{1 - p_1} \cdot \underline{r}(T_3 - T_2).$$

   Since $\hat{Y}_{T_2 - dt}$ is determined, $\sigma_t^s = 0$ for $t \in [T_1, T_2)$.

For $t < T_1$, we know that

1. $T = T_1$ or T_2 or T_3. At $t = T_1 - dt$ for small $dt > 0$, $\hat{Y}_{T_1 - dt}$ is determined by a conditional probability-weighted linear combination of 0 (when $T = T_1$), $\underline{r}(T_2 - T_1)$ (when $T = T_2$) and $\underline{r}(T_3 - T_1)$ (when $T = T_3$), so that

$$\hat{Y}_{T_1 - dt} = \underline{r}dt + p_1 \cdot 0 + p_2 \cdot \underline{r}(T_2 - T_1) + p_3 \cdot \underline{r}(T_3 - T_1).$$

   Since $\hat{Y}_{T_1 - dt}$ is determined, $\sigma_t^s = 0$ for $t < T_1$.

Therefore, $\sigma_t^s = \sigma_t^{s,n} = 0$ for all t even if ZLB duration T is stochastic.

Traditional forward guidance When T is stochastic, the zero rate duration under traditional forward guidance, i.e., $\hat{T}$ in Section 4.1, becomes stochastic as well and dependent on T. The above logic can be applied in this case, and we can similarly prove that if the monetary authority commits to perfectly stabilizing the economy after any realized $\hat{T}$, then $\sigma_t^s = \sigma_t^{s,n} = 0$ for $t \leq \hat{T}$.

G Time-Varying Endogenous Volatility

In Section 4.2, we assumed that the central bank maintains *constant* excess volatility within each regime: in specific, we assumed $\sigma_t^s = \sigma_1^{s,L}$ for $t < T$, $\sigma_2^{s,L}$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and 0 for $t \geq \hat{T}^{\text{HOFG}}$. In this section, we extend our analysis and assume that the central bank can choose a full path of $\{\sigma_t^s\}_{t\geq 0}$ to minimize the loss function (11), under the assumption that $\sigma_t^s = 0$ for $t \geq \hat{T}^{\text{HOFG}}$ and $\mathbb{E}_0 \hat{Y}_{\hat{T}^{\text{HOFG}}} = 0$. In this case, the central bank solves the following optimization:

$$\min_{\{\sigma_t^s\}_{t\leq \hat{T}^{\text{HOFG}}},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\,\hat{Y}_t^2\,dt\ ,$$

$$\text{s.t.}\quad d\hat{Y}_t = \begin{cases} -r_1^T(\sigma_t^s)\,dt + \sigma_t^s\,dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_t^s)\,dt + \sigma_t^s\,dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} \tag{G.1}$$

with $\hat{Y}_0$ given by

$$\hat{Y}_0 = \int_0^T r_1^T(\sigma_s^s)ds + \int_T^{\hat{T}^{\text{HOFG}}} r_2^T(\sigma_s^s)ds,$$

where $r_1^T(\cdot)$ and $r_2^T(\cdot)$ are defined in (15).

As in Proposition 3, we can prove that at optimum, $\sigma_t^s < 0$ for $t \leq \hat{T}^{\text{HOFG}}$ and $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$ for the above optimization.

Proposition G.2 (Optimal Commitment Path) *The solution to the central bank's higher-*

order forward guidance optimization problem in (G.1) yields an optimal commitment path characterized by $\sigma_t^s < 0$ for $t \leq \hat{T}^{HOFG}$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. Moreover, the optimal higher-order forward guidance policy results in a lower expected quadratic welfare loss than the traditional forward guidance policy discussed in Section 4.1.

Proof. After $\hat{T}^{\text{HOFG}}$, there are no additional fluctuation in $\hat{Y}_t$. Defining r_t^T as $r_1^T(\sigma_t^s)$ for $t < T$ and as $r_2^T(\sigma_t^s)$ for $T \leq t \leq \hat{T}^{\text{HOFG}}$, the process of $\hat{Y}_t$ can be articulated as follows:

$$\hat{Y}_t = \begin{cases} \underbrace{\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds}_{\equiv \hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})} + \int_0^t \sigma_s^s dZ_s, & \text{for } t \leq \hat{T}^{\text{HOFG}}\,, \\ \int_0^{\hat{T}^{\text{HOFG}}} \sigma_s^s dZ_s, & \text{for } \hat{T}^{\text{HOFG}} < t\,. \end{cases} \tag{G.2}$$

where for $t \leq T$,

$$\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds = \int_t^T r_1^T(\sigma_s^s) ds + \int_T^{\hat{T}^{\text{HOFG}}} r_2^T(\sigma_s^s) ds$$

and for $T \leq t \leq \hat{T}^{\text{HOFG}}$,

$$\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds = \int_t^{\hat{T}^{\text{HOFG}}} r_2^T(\sigma_s^s) ds\,.$$

From (G.2), we obtain:

$$\mathbb{E}_0\, \hat{Y}_t^2 = \begin{cases} \hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 + \int_0^t (\sigma_s^s)^2 ds\,, & \text{for } t \leq \hat{T}^{\text{HOFG}}\,, \\ \int_0^{\hat{T}^{\text{HOFG}}} (\sigma_s^s)^2 ds\,, & \text{for } \hat{T}^{\text{HOFG}} < t\,. \end{cases} \tag{G.3}$$

If we substitute equation (G.3) into the central bank's loss function (11), the central bank's commitment problem can be expressed as follows:

$$
\begin{aligned}
&\min_{\{\sigma_t^s\}_{t\leq\hat{T}^{\text{HOFG}}},\hat{T}^{\text{HOFG}}} \ \mathbb{E}_0 \int_0^\infty e^{-\rho t}\,\hat{Y}_t^2\,dt \\
=&\min_{\{\sigma_t^s\}_{t\leq\hat{T}^{\text{HOFG}}},\hat{T}^{\text{HOFG}}} \ \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 dt + \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\left(\int_0^t (\sigma_s^s)^2 ds\right) dt \\
&\qquad + \int_{\hat{T}^{\text{HOFG}}}^\infty e^{-\rho t} dt \cdot \int_0^{\hat{T}^{\text{HOFG}}} (\sigma_s^s)^2 ds \\
=&\min_{\{\sigma_t^s\}_{t\leq\hat{T}^{\text{HOFG}}},\hat{T}^{\text{HOFG}}} \underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 dt}_{\text{Deterministic fluctuations}} + \underbrace{\frac{1}{\rho}\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho s}(\sigma_s^s)^2 ds}_{\text{Stochastic fluctuations}} \ ,
\end{aligned}
\tag{G.4}
$$

where we used

$$
\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\left(\int_0^t (\sigma_s^s)^2 ds\right) dt = \frac{1}{\rho}\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho s}(\sigma_s^s)^2 ds - \frac{1}{\rho}e^{-\rho\hat{T}^{\text{HOFG}}}\int_0^{\hat{T}^{\text{HOFG}}} (\sigma_s^s)^2 ds \, .
$$

The central bank now has control over $\{\sigma_t^s\}_{t\leq\hat{T}^{\text{HOFG}}}$, and $\hat{T}^{\text{HOFG}}$, in addition to its conventional monetary policy tool i_t. Initially, we derive the first-order condition for $\hat{T}^{\text{HOFG}}$, which is as follows:

$$
2\cdot\underbrace{r_2^T\left(\sigma_{\hat{T}^{\text{HOFG}}}^s\right)}_{>0}\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt + \left(\sigma_{\hat{T}^{\text{HOFG}}}^s\right)^2\frac{1}{\rho}e^{-\rho\hat{T}^{\text{HOFG}}} = 0 \ , \tag{G.5}
$$

from which we obtain

$$
\int_0^\infty e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt = \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}}\|\sigma_t^s<0,\forall t\leq\hat{T}^{\text{HOFG}})dt < 0 \ . \tag{G.6}
$$

The first-order condition for $\hat{T}^{\text{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\text{HOFG}}$ compared to $\hat{T}^{\text{TFG}}$ in the same way as in Section 4.2. This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$, the expression above becomes

$$\int_0^{\hat{T}^{\text{TFG}}} e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}\|\sigma_t^s = 0, \forall t \leq \hat{T}^{\text{HOFG}})dt = 0\ . \tag{G.7}$$

Given that at the optimum, $\sigma_t^s < 0$ for $t \leq \hat{T}^{\text{HOFG}}$ (which we will demonstrate),

$$\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}}\|\sigma_t^s = 0, \forall t \leq \hat{T}^{\text{HOFG}}) < \hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}}\|\sigma_t^s < 0, \forall t \leq \hat{T}^{\text{HOFG}})\ .$$

Therefore, from equations (G.6) and (G.7) we deduce that at the optimum, $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$.

To characterize optimal $\{\sigma_t^s\}_{t\leq\hat{T}^{\text{HOFG}}}$, a similar **variational argument** is required. This is because σ_t^s for a particular t influences the levels of $r_1^T(\sigma_t^s)$ or $r_2^T(\sigma_t^s)$ depending on whether $t \leq T$ or $T < t \leq \hat{T}^{\text{HOFG}}$, and thus, $\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})$. Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_t^s)}{\partial \sigma_t^s} = -\left(\bar{\sigma} + \sigma_t^s\right) < 0,\ \ \frac{\partial r_2^T(\sigma_t^s)}{\partial \sigma_t^s} = -\left(\underline{\sigma} + \sigma_t^s\right) < 0\ .$$

Determining $\{\sigma_t^s\}_{t\leq\hat{T}^{\text{HOFG}}}$ For $t \leq T$, an increase in σ_t^s lowers $r_1^T(\sigma_t^s)$, altering the trajectory of $\hat{Y}_{\text{d}}(s;\hat{T}^{\text{HOFG}})$ for $s \leq t$. Differentiating $\hat{Y}_{\text{d}}(s;\hat{T}^{\text{HOFG}}) = \int_s^{\hat{T}^{\text{HOFG}}} r_h^T dh$ with respect to σ_t^s, we obtain:

$$\frac{\partial \hat{Y}_{\text{d}}(s;\hat{T}^{\text{HOFG}})}{\partial \sigma_t^s} = -\left(\bar{\sigma} + \sigma_t^s\right),\ \ \forall s \leq t \leq T\ .$$

To find optimal σ_t^s for $t \leq T$, we differentiate the objective function in (G.4) by σ_t^s and obtain the following condition:

$$\left(\bar{\sigma} + \sigma_t^s\right)\underbrace{\int_0^t e^{-\rho t}\hat{Y}_{\text{d}}(t;\hat{T}^{\text{HOFG}})dt}_{<0 \text{ from (G.6)}} = \frac{1}{\rho}e^{-\rho t}\cdot\sigma_t^s\ ,$$

from which we prove that $\sigma_t^s < 0$ for $t \leq T$. For $T \leq t \leq \hat{T}^{\text{HOFG}}$, the first-order condition becomes

$$(\underline{\sigma} + \sigma_t^s) \underbrace{\int_0^t e^{-\rho t} \hat{Y}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt}_{<0 \text{ from (G.6)}} = \frac{1}{\rho} e^{-\rho t} \cdot \sigma_t^s \ ,$$

from which we can also prove that $\sigma_t^s < 0$ for $T < t \leq \hat{T}^{\text{HOFG}}$.

■

H Two-Agent New Keynesian (TANK) Model

This section proposes an alternative Two-Agent New Keynesian (TANK) model that captures the links between financial volatility, risk premia, aggregate wealth, and aggregate demand, while remaining tractable for analyzing macroprudential policies. We show that the equilibrium conditions and results presented in the main text can be mapped directly into financial-market volatility and risk-premia dynamics.

H.1 Setting

As in Section 2, we work in continuous time on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. The economy now comprises two equally sized groups: capitalists, modeled as neoclassical agents, and hand-to-mouth workers, modeled as Keynesian agents. Following Greenwald et al. (2014) and Caballero et al. (2024), all financial wealth is held by capitalists, while workers consume exclusively out of labor income. Aggregate technology, denoted A_t, is the sole source of exogenous variation and generates the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. The process for A_t is defined as in Section 2 of the main text.

H.1.1 Firms

The economy comprises a unit measure of monopolistically competitive firms, each producing an intermediate good $y_t(i)$ for $i \in [0,1]$. These intermediate goods are aggregated into the final output Y_t via a Dixit-Stiglitz aggregation function characterized by a substitution elasticity $\epsilon > 0$, as follows:

$$Y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

Each intermediate firm i employs a production function $y_t(i) = A_t \left(N_{W,t}\right)^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ denotes the total labor in the economy and $n_t(i)$ represents the labor demand of firm i at time t. Incorporating a production externality à la Baxter and King (1991) allows our model to be aligned with observed asset price and wage co-movements, without altering its other qualitative outcomes.[7]

Intermediate firms face a downward-sloping demand curve $y_i(p_t(i) \mid p_t, Y_t)$, where $p_t(i)$ is the price of an individual firm's good and p_t is the aggregate price index. Specifically, demand is given by

$$y_i(p_t(i) \mid p_t, Y_t) = \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon} Y_t,$$

with the aggregate price index defined as $p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} \, di \right)^{\frac{1}{1-\epsilon}}$. For tractability, we assume perfect price rigidity and symmetry, so that $p_t(i) = p_t = \bar{p}$ for all i and t.[8] As a result, each firm produces an identical level of output $y_t(i) = Y_t$, determined by aggregate demand.

[7]Without the Baxter and King (1991) externality, rising asset prices are typically associated with lower wages, which contradicts empirical evidence (Chodorow-Reich et al., 2021) on the effects of stock price increases on aggregate demand, employment, and wages. Incorporating the Baxter and King (1991) externality allows our calibration to capture these empirical trends by linking higher asset prices and aggregate demand with increased labor demand and wages.

[8]An alternative assumption of sticky price resetting à la Calvo (1983) does not significantly alter the model's dynamics or its qualitative results.

H.1.2 Workers

A representative hand-to-mouth worker supplies labor to intermediate firms, earning wage income $w_t N_{W,t}$ that is entirely devoted to final good consumption. The worker maximizes

$$\max_{C_{W,t},N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0}\ , \quad \text{s.t.} \quad \bar{p}C_{W,t} = w_t N_{W,t}\ , \tag{H.1}$$

where $C_{W,t}$ denotes consumption, $N_{W,t}$ denotes labor supply, w_t represents the wage, and χ_0 is the inverse Frisch elasticity of labor supply. Following Mertens and Ravn (2011), we normalize consumption by technology A_t to allow for a linearly additive utility specification consistent with a balanced growth path.[9]

Under the assumption of perfectly rigid prices, equilibrium labor demand by each firm i aggregates directly into total labor $N_{W,t}$, implying that $n_t(i) = N_{W,t}$ for all i. Substituting this result into the production function yields equilibrium output as a linear function of total labor, $Y_t = A_t N_{W,t}$.

H.1.3 Financial Market and Capitalists

Unlike a conventional New Keynesian model (e.g., Section 2) in which a representative household owns firms and receives lump-sum rebated profits every period, we assume that firm profits are capitalized in the stock market via a representative index fund. Capitalists face an optimal portfolio allocation problem at each moment t, choosing between investment in a risk-free bond and the stock index.

The aggregate nominal value of the stock index fund is given by $\bar{p}A_t Q_t$, where Q_t is the normalized real index price. The price Q_t is determined endogenously and evolves with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ as follows:

$$\frac{dQ_t}{Q_t} = \mu_t^q\, dt + \sigma_t^q\, dZ_t,$$

[9] This normalization simplifies the analysis without altering the qualitative results of our model.

where μ_t^q and σ_t^q represent the endogenous drift and volatility of the process, respectively. We interpret σ_t^q as a measure of endogenous financial uncertainty. Consequently, aggregate financial wealth $\bar{p}A_tQ_t$ evolves according to a geometric Brownian motion with a combined volatility of $\sigma + \sigma_t^q$. Notably, since σ_t^q is determined in equilibrium and may be either positive or negative, the volatility of the aggregate real stock market value A_tQ_t may exceed or fall below the volatility of the technology process $\{A_t\}$. In particular, when σ_t^q is negative, the overall financial wealth volatility $\sigma + \sigma_t^q$ is lower than the fundamental volatility σ.[10] This endogenous financial volatility σ_t^q will take up a role of excess volatility σ_t^s of Section 2.3.

In addition to the stock market, we introduce a risk-free bond with a nominal interest rate i_t set by the central bank, assuming that bonds are in zero net supply in equilibrium. A unit measure of identical capitalists allocates their wealth between risk-free bonds and the risky stock index. By holding the stock index, capitalists earn profits from the intermediate goods sector—distributed as dividends—and benefit from stock price revaluations driven by changes in A_t and Q_t. Given the competitive nature of financial markets, each capitalist takes the nominal risk-free rate i_t, the expected stochastic stock market return i_t^m, and the total risk level $\sigma + \sigma_t^q$ as given when making portfolio decisions.[11] If a capitalist allocates a fraction θ_t of their nominal wealth a_t to the stock market, the risk borne over the interval $[t, t+dt]$ is $\theta_t a_t(\sigma + \sigma_t^q)$; thus, the portfolio's riskiness is directly proportional to the investment share θ_t. Being risk-averse, capitalists require a risk premium $i_t^m - i_t$ for investing in the risky asset, which is determined in equilibrium.

[10]When $\sigma_t^q < 0$, we have $\mathrm{Cov}_t\left(dA_t, dQ_t\right) = \sigma\sigma_t^q A_t Q_t\, dt < 0$, implying a negative covariance between total factor productivity and asset prices.

[11]The competitive market assumption is essential in our model to account for inefficiencies stemming from the aggregate demand externality imposed by each capitalist's investment decision. For further details, see Farhi and Werning (2016).

A representative capitalist solves the following problem:

$$\begin{aligned} \max_{C_t,\theta_t} \quad & \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dt, \\ \text{s.t.} \quad & da_t = \Big(a_t \big(i_t + \theta_t (i_t^m - i_t) \big) - \bar{p} C_t \Big) dt + \theta_t a_t (\sigma + \sigma_t^q) \, dZ_t, \end{aligned} \tag{H.2}$$

where ρ denotes the subjective discount rate and C_t represents the final good consumption of capitalists. At each moment, the capitalist receives returns from both risk-free bonds and risky stock investments and allocates these returns to consumption.

H.2 Equilibrium and Asset Pricing

The nominal state price density of capitalists, denoted by ξ_t^N, is given by

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{\bar{p}} \ , \ \text{ with } \ \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right) = -i_t dt \ . \tag{H.3}$$

The stochastic discount factor between time t and a future time s is ξ_s^N / ξ_t^N. Under the assumption that capitalists are the equilibrium marginal investors, the aggregate stock market wealth, $\bar{p} A_t Q_t$, is determined as the sum of discounted profit streams from the intermediate goods sector, using ξ_t^N for discounting.

At time t, the total profit of the intermediate goods sector, D_t, is defined by

$$D_t \equiv \bar{p} Y_t - w_t N_{W,t} = \bar{p}(Y_t - C_{W,t}) = \bar{p} C_t \ , \tag{H.4}$$

where $w_t N_{W,t}$ (wage income) equals the consumption expenditure of hand-to-mouth workers, $\bar{p} C_{W,t}$. Hence, total dividends equal the capitalists' aggregate consumption expenditure. Substituting this result into the asset pricing equation yields

$$\bar{p} A_t Q_t = \mathbb{E}_t \left[\frac{1}{\xi_t^N} \int_t^\infty \xi_s^N D_s \, ds \right] = \frac{\bar{p} C_t}{\rho}, \tag{H.5}$$

which implies $C_t = \rho A_t Q_t$. Thus, in equilibrium, capitalists' consumption is a fixed proportion ρ of their aggregate financial wealth.

Agents of the same type—whether workers or capitalists—are identical and make symmetric decisions in equilibrium. Because bonds are in zero net supply, capitalists invest their entire wealth in the stock market, implying that the wealth share θ_t is equal to one at all times. This condition determines the equilibrium risk premium demanded by capitalists. By combining equations (H.2), (H.3), and (H.5), the risk premium is given by

$$\mathrm{rp}_t \equiv i_t^m - i_t = (\sigma + \sigma_t^q)^2, \tag{H.6}$$

indicating that rp_t increases with the total volatility $\sigma + \sigma_t^q$ of aggregate financial wealth $\bar{p}A_tQ_t$. Notably, a capitalist's wealth gain or loss corresponds to the nominal revaluation of the stock market index. These equilibrium conditions, as specified in equations (H.5) and (H.6), are consistent with Merton (1971).

The equilibrium in the goods market and the expected stock return i_t^m are characterized as follows. Given that in equilibrium capitalists' consumption satisfies $C_t = \rho A_t Q_t$, the final goods market equilibrium condition can be written as

$$\rho A_t Q_t + \frac{w_t}{\bar{p}} N_{W,t} = Y_t = A_t N_{W,t}\,. \tag{H.7}$$

The nominal expected return on stocks, i_t^m, consists of two components: the dividend yield from firm profits and the nominal revaluation of stock prices due to fluctuations in A_t and Q_t. In equilibrium, changes in i_t^m affect only nominal stock prices because the dividend yield remains fixed at ρ. Define $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process, such that $\mathbb{E}_t[d\mathbf{I}_t^m] = i_t^m\,dt$. Equation equation (H.8) decomposes $\mathbf{I}_t^m$ into its dividend yield and stock revaluation components:

$$d\mathbf{I}_t^m = \frac{\overbrace{\not{p}\left(\underbrace{Y_t - \frac{w_t}{\bar{p}} N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\not{p} A_t Q_t}_{\text{Total stock market wealth}}} dt + \underbrace{\frac{d\left(\not{p} A_t Q_t\right)}{\not{p} A_t Q_t}}_{\text{Stock revaluation}} \tag{H.8}$$

$$= \underbrace{\left(\rho + g + \mu_t^q + \sigma\sigma_t^q\right)}_{=i_t^m} dt + \underbrace{\left(\sigma + \sigma_t^q\right)}_{\text{Risk term}} dZ_t \ .$$

The real stock price Q_t plays a critical role in driving the business cycle in equilibrium. An increase in Q_t raises capitalists' consumption, which in turn leads to higher wages and greater labor demand by firms, ultimately boosting aggregate consumption.

Flexible Price Equilibrium. Consistent with the literature, we adopt the flexible price equilibrium as the benchmark guiding the monetary authority's policy objectives. Details of this equilibrium are provided in Online Appendix I.1. Additionally, Online Appendix I.2 specifies the conditions necessary for positive co-movement among the gaps in asset prices, wages, labor supply, and consumption for both capitalists and workers. As in Section 2, here 'gaps' refer to the log-deviations from the flexible price equilibrium. As demonstrated in Online Appendix I.2, these gaps are proportional to one another; henceforth, we express equilibrium conditions in terms of the asset price gap $\hat{Q}_t$, defined as $\hat{Q}_t \equiv \log\left(\frac{Q_t}{Q_t^n}\right)$.

In the flexible price equilibrium, denoted by the superscript n (for 'natural'), we have $\mu_t^{q,n} = 0$ and $\sigma_t^{q,n} = 0$, implying a constant natural stock price Q_t^n. The natural interest rate, $r^n \equiv \rho + g - \sigma^2$, represents the real risk-free rate in the flexible price economy and remains constant in equilibrium. Note that r^n here is the same as in equation (7) of Section 2.

Gap Economy We define the risk-premium gap as $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$, where rp_t^n denotes the risk premium in the natural (flexible-price) equilibrium. We then introduce the risk-adjusted natural rate, r_t^T, defined by

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t \ . \tag{H.9}$$

This rate adjusts the natural rate of return to account for the risk differential between rigid-price and flexible-price economies, serving as an anchor for monetary policy in our model. For example, a positive risk-premium gap ($\hat{rp}_t > 0$) reduces the demand for the stock market portfolio by capitalists relative to the benchmark economy, potentially triggering a recession. This effect is formalized in Proposition H.3, which shows that a decline in r_t^T relative to the risk-free policy rate i_t fosters expectations of future asset price revaluations. These expectations translate into lower current asset prices and a widening output gap.

Proposition H.3 (Dynamic IS Equation) *The dynamic IS equation of the model, expressed in terms of the asset price gap, is given by:*[12]

$$d\hat{Q}_t = (i_t - r_t^T)dt + \sigma_t^q dZ_t. \tag{H.10}$$

Proof of Proposition H.3. With (H.2) with $\theta_t = 1$ and (H.6), capitalists' consumption C_t follows

$$\frac{dC_t}{C_t} = \left(i_t + (\sigma + \sigma_t^q)^2 - \rho\right) dt + (\sigma_t + \sigma_t^q)dZ_t. \tag{H.11}$$

where we use $i_t^m = i_t + (\sigma + \sigma_t^q)^2$. Thus, with equation (I.7), we obtain

$$\begin{aligned} d\hat{Q}_t = d\hat{C}_t &= \left(i_t - \underbrace{\left(r_t^n - \frac{(\sigma + \sigma_t^q)^2}{2} + \frac{\sigma^2}{2} \right)}_{\equiv r_t^T} \right) dt + \sigma_t^q dZ_t \\ &= \left(i_t - r_t^T\right) dt + \sigma_t^q dZ_t. \end{aligned} \tag{H.12}$$

Since we have risk-premium levels $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$ in the rigid-price economy and $\text{rp}_t^n =$

[12] A conventional formulation using the output gap yields a similar expression, as both variables are proportional in equilibrium.

σ^2 in the flexible-price economy, we can express our risk-adjusted natural rate r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}\left(\mathrm{rp}_t - \mathrm{rp}_t^n\right) = r_t^n - \frac{1}{2}\hat{rp}_t, \tag{H.13}$$

where $r_t^n \equiv \rho + g - \sigma^2$ represents the real risk-free rate in the flexible-price economy.

■

Notably, in a linearized approximation of the New Keynesian model, the natural rate r_t^n replaces r_t^T in equation (H.10), muting the impact of risk on asset prices.

H.3 The Zero Lower Bound

ZLB Recession. Following Werning (2012), we consider a deterministic fall in the natural rate of interest r_t^n that drives the economy to the ZLB. Specifically, we assume that TFP volatility equals $\sigma_t = \bar{\sigma}$ for $0 \le t < T$ and $\sigma_t = \underline{\sigma}$ for $t \ge T$, with $\underline{\sigma} < \bar{\sigma}$. Hence

$$\underline{r} \equiv r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2 < 0, \qquad \bar{r} \equiv r^n(\underline{\sigma}) = \rho + g - \underline{\sigma}^2 > 0,$$

so the ZLB binds initially because the central bank cannot set $i_t < 0$ to eliminate the asset-price gap in equation (H.10).[13] Without loss of generality—and as evident from the expression for r_t^n—one may alternatively consider shocks to the growth rate g or the discount rate ρ as drivers of the ZLB spell. Results are unchanged when T is stochastic;[14] we therefore focus on the deterministic case.

Recovery Without Guidance. We first consider the benchmark scenario—recovery without forward guidance or macro-prudential intervention. After period T the central bank resumes its active intervention via $i_t > 0$, delivering perfect stabilization so that $\hat{Q}_t = 0$

[13] Assuming zero financial volatility $\sigma_t^q = 0$, $0 \le t < T$, the bank would need $i_t = \underline{r} < 0$ to ensure $E_t[d\hat{Q}_t] = 0$ in equation (H.10), which the ZLB constraint $i_t \ge 0$ rules out. Appendix D.1 shows that perfect stabilization is otherwise attainable with an interest-rate rule that closes the gap in the drift of (H.10).

[14] See Online Appendix F for a proof in the representative agent model of the main text. Since the asset price gap and output gap are linearly related up to a first-order approximation, $\hat{Y}_t = \zeta\hat{Q}_t$, the result extends directly to the TANK setting.

for all $t \geq T$.[15] Backward induction applied to equation (H.10) then requires $\sigma_t^q = 0$ for every $t < T$: at $T - \Delta$ (with infinitesimal Δ), the only solution consistent with $\hat{Q}_T = 0$ for any draw of $dZ_{T-\Delta}$ is $\sigma_{T-\Delta}^q = 0$; iterating backward yields the same for all earlier dates. Hence, when the authority can credibly commit to perfect stabilization from T onward, one has $\sigma_t^q = 0$ and $r_t^T = \underline{r} < 0$ for $t < T$. Under these conditions, the dynamics of $\hat{Q}_t$ reduce to

$$d\hat{Q}_t = -\underline{r}\, dt \,, \qquad \text{for } t < T \,, \tag{H.14}$$

with the boundary condition $\hat{Q}_T = 0$ and initial asset price gap $\hat{Q}_0 = \underline{r}T$. Figure H.4 plots the resulting path of $\{\hat{Q}_t\}$.

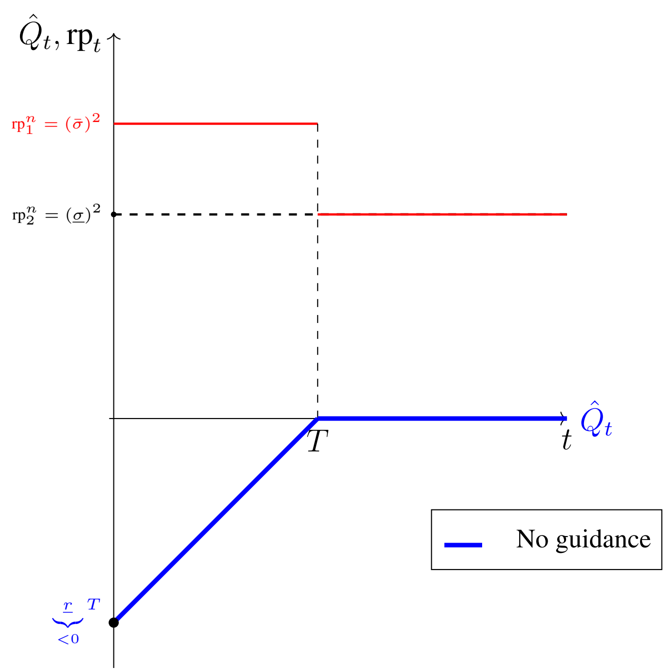

Figure H.4: ZLB dynamics, economic recovery without guidance (Benchmark).

The initial increase in σ_t from $\underline{\sigma}$ to $\bar{\sigma}$ raises the natural risk premium from $\text{rp}_2^n = (\underline{\sigma})^2$ to $\text{rp}_1^n = (\bar{\sigma})^2$. Since the ZLB prevents the risk-free rate from declining into negative

[15] Online Appendix D.1 gives an example of how the central bank can achieve perfect stabilization once the economy is away from the ZLB.

territory, the increased risk premium induces a decline in asset prices, $\hat{Q}_t$. This reduction in asset prices diminishes capitalists' appetite for stock market investments, leading to lower aggregate financial wealth and reduced consumption demand.[16] These dynamics align with the findings of Werning (2012) and Cochrane (2017), even though our model incorporates a distinct IS equation (H.10) with endogenous volatility σ_t^q affecting the drift of $\hat{Q}_t$—a departure from traditional New Keynesian models. In essence, the credible commitment to future stabilization for $t \geq T$ eliminates excess endogenous volatility during the ZLB episode.

Remarks. Central banks can avert the emergence of endogenous volatility, σ_t^q, at the ZLB by making a credible commitment to stabilize the business cycle by a predetermined future date $T < +\infty$. Even if the monetary authority is constrained by the ZLB and unable to actively adjust the policy rate i_t, the additional financial stability costs associated with policy inaction can be effectively managed, or even entirely eliminated, by pledging to stabilize the economy upon exiting the ZLB. One implication of this result is that the impact of the ZLB may vary considerably across countries. Monetary authorities committed to post-ZLB stabilization are likely to experience only the demand-driven recession described in this Section, whereas countries that lack either the capacity or the willingness to stabilize in the future might incur additional costs due to increases in σ_t^q during a ZLB episode. Further exploration of these scenarios is left for future research.

H.4 Higher-Order Forward Guidance

This section analyzes the two forward guidance regimes from the main text—traditional and higher-order guidance—within the financial TANK model.

[16] While Caballero and Farhi (2017) show that an increased demand for safe assets under ZLB constraints can drive a recession, our analysis suggests that investors withdraw from the stock market, thereby reducing both stock market value and aggregate demand, consistent with Caballero and Simsek (2020).

H.4.1 Traditional Forward Guidance

We define traditional forward guidance as the communication strategy in which the central bank credibly commits[17] to maintain a zero policy rate for a horizon $\hat{T}^{\text{TFG}}$ that exceeds the initial interval T of high fundamental volatility. Once this forward-guidance window closes, the bank reverts to active monetary policy intervention, achieving perfect stabilization of the business cycle and financial markets for $t \geq \hat{T}^{\text{TFG}}$. By backward induction (see Section 3), the certainty of future stabilization eliminates endogenous financial volatility, implying $\sigma_t^q = 0$ for all $t < \hat{T}^{\text{TFG}}$.

Under these conditions, the dynamics of the asset price gap $\hat{Q}_t$ are given by

$$d\hat{Q}_t = \begin{cases} -\underline{r}\, dt, & \text{for } t < T, \\ -\bar{r}\, dt, & \text{for } T \leq t < \hat{T}^{\text{TFG}}, \end{cases} \tag{H.15}$$

with the boundary condition $\hat{Q}_{\hat{T}^{\text{TFG}}} = 0$. This yields an initial asset price gap of $\hat{Q}_0 = \underline{r}\, T + \bar{r}\, (\hat{T}^{\text{TFG}} - T)$. Figure H.5 illustrates the dynamics of $\{\hat{Q}_t\}$ as described by equation (H.15).

Traditional forward guidance induces an artificial economic boom between T and $\hat{T}^{\text{TFG}}$, thereby mitigating recessionary pressures during $0 \leq t < T$. Specifically, by increasing asset prices between T and $\hat{T}^{\text{TFG}}$, this policy reduces the initial asset price gap $\hat{Q}_0$ owing to the forward-looking behavior of stock markets.

Optimal Traditional Forward Guidance. To determine the optimal forward guidance duration $\hat{T}^{\text{TFG}}$, we minimize the quadratic welfare loss function

$$\mathbb{L}^Q \left(\{\hat{Q}_t\}_{t \geq 0} \right) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2 \, dt \,, \tag{H.16}$$

[17] For analytical tractability, we assume the central bank can credibly commit to its policy promises. However, Camous and Cooper (2019) (in the context of debt monetization) shows that "grim-trigger" strategies in repeated monetary games can sustain coordinated equilibria even without such commitment. We conjecture that a similar mechanism could support the commitment-based equilibria studied here when the central bank lacks full commitment capacity; a detailed analysis is left for future work.

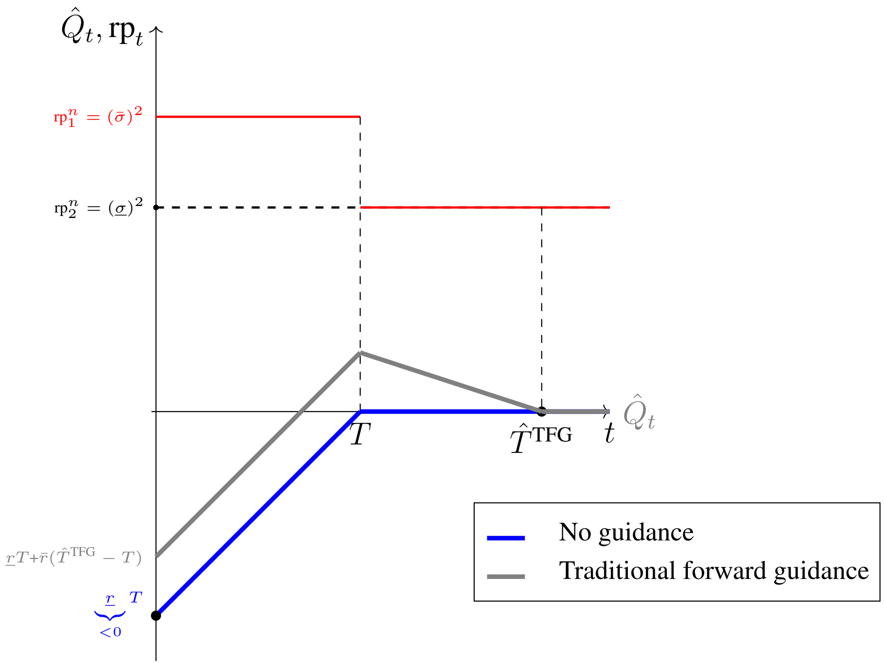

Figure H.5: ZLB dynamics under traditional forward guidance.

subject to the dynamics specified in equation (H.15).[18] The first-order condition with respect to $\hat{T}^{\mathrm{TFG}}$ yields

$$\int_0^\infty e^{-\rho t} \hat{Q}_t \, dt = 0 \; . \tag{H.17}$$

Section H.6 summarizes the key statistics and welfare gains achieved under the optimal traditional forward guidance policy.

In the next section, we argue that central banks might deliberately forgo perfect future stabilization to reduce financial volatility at the ZLB, thereby attaining higher welfare compared to the traditional forward guidance approach. We refer to this alternative method as *higher-order* forward guidance.

[18]The derivation of the quadratic welfare loss function in equation (H.16) is provided in Online Appendix I.5.

H.4.2 Higher-Order Forward Guidance

The primary driver of ZLB recessions in our model is an excessively elevated risk premium generated by higher fundamental volatility σ_t. A central bank can counter this by steering expectations about asset-price volatility $\{\sigma_t^q\}$ during the ZLB episode, thereby supporting asset prices and consumption demand.[19]

Context. Under traditional forward guidance, the commitment to perfect stabilization at $\hat{T}^{\text{TFG}}$ pins σ_t^q at its natural value of zero throughout the ZLB. To sustain alternative equilibria where $\sigma_t^q \neq 0$, the central bank must forgo that commitment to perfect stabilization when the economy exits the ZLB at $\hat{T}^{\text{TFG}}$.

Implementation. Similar to Section 4.2, let $\hat{T}^{\text{HOFG}}$ denote the horizon over which the policy rate is kept at zero under higher-order forward guidance (HOFG). The central bank commits to $i_t = 0$ for $t < \hat{T}^{\text{HOFG}}$; afterwards it follows a *passive* rule, fixing $i_t = \bar{r}$ and thus admitting multiple equilibria. It coordinates agents onto an optimal equilibrium path subject to the constraints $\sigma_t^q = 0$ for $t \geq \hat{T}^{\text{HOFG}}$ and $\mathbb{E}_0 \hat{Q}_{\text{HOFG}} = 0$. The first condition removes the bank's influence on financial-market volatility once guidance ends, while the second ensures that the economy is, in expectation, fully stabilized by $\hat{T}^{\text{HOFG}}$, making HOFG comparable to the traditional guidance policies with perfect post-ZLB stabilization examined in this paper. Together with the dynamic IS equation (H.10), they imply that the expected asset-price gap closes by $\hat{T}^{\text{HOFG}}$, i.e. $\mathbb{E}_0 \hat{Q}_{\hat{T}^{\text{HOFG}}} = 0$. Section H.4.3 relaxes these restrictions by allowing the bank to revert to the active monetary stabilization after $\hat{T}^{\text{HOFG}}$ with probability less than one.

Formalism. Define the natural risk premia as $\text{rp}_1^n \equiv \bar{\sigma}^2$ for $t < T$ (high fundamental volatility region), $\text{rp}_2^n \equiv \underline{\sigma}^2$ for $T \leq t < \hat{T}^{\text{HOFG}}$ (low fundamental volatility region), and

[19] The risk premium is $\text{rp}_t = (\bar{\sigma} + \sigma_t^q)^2$ for $t < T$ and $\text{rp}_t = (\underline{\sigma} + \sigma_t^q)^2$ for $T \leq t < \hat{T}^{\text{TFG}}$. A negative σ_t^q lowers the premium below its natural level, boosting asset prices and aggregate demand at the ZLB.

$\mathrm{rp}_3^n \equiv \underline{\sigma}^2$ for $t \geq \hat{T}^{\mathrm{HOFG}}$ (post-guidance low fundamental volatility region).[20]

For tractability, assume the central bank maintains constant levels of financial volatility within each regime: $\sigma_t^q = \sigma_1^{q,L}$ for $t < T$, $\sigma_2^{q,L}$ for $T \leq t < \hat{T}^{\mathrm{HOFG}}$, and 0 for $t \geq \hat{T}^{\mathrm{HOFG}}$. The corresponding risk premia are

$$\mathrm{rp}_1 = (\bar{\sigma} + \sigma_1^{q,L})^2 < \mathrm{rp}_1^n, \quad \mathrm{rp}_2 = (\underline{\sigma} + \sigma_2^{q,L})^2 < \mathrm{rp}_2^n, \quad \mathrm{rp}_3 = \underline{\sigma}^2.$$

Proposition H.4 shows that $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ at the optimum; we impose these signs throughout the discussion.

Moreover, the risk-adjusted natural rate in equation (H.9) equals r_1^T for $t < T$ and r_2^T for $T \leq t < \hat{T}^{\mathrm{HOFG}}$, with

$$\begin{aligned} r_1^T\left(\sigma_1^{q,L}\right) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{\left(\bar{\sigma} + \sigma_1^{q,L}\right)^2}{2} > \underline{r} \equiv r_1^T(0) \quad \text{if } \sigma_1^{q,L} < 0, \\ r_2^T\left(\sigma_2^{q,L}\right) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{\left(\underline{\sigma} + \sigma_2^{q,L}\right)^2}{2} > \bar{r} \equiv r_2^T(0) \quad \text{if } \sigma_2^{q,L} < 0. \end{aligned} \tag{H.18}$$

By lowering the risk premium, a negative $\sigma_j^{q,L}$ raises the risk-adjusted natural rate during the guidance window (i.e., up to $\hat{T}^{\mathrm{HOFG}}$), leads to higher values of the asset price gap $\{\hat{Q}_t\}$ relative to standard forward guidance of the same duration, and reduces the expected loss in (H.16). However, the dynamic IS equation (H.10) indicates that $\sigma_t^q \neq 0$ also introduces stochastic fluctuations in the trajectory of $\hat{Q}_t$, which may entail future stabilization costs.

Figure H.6 illustrates this framework: the green line traces the expected path of $\{\hat{Q}_t\}$ under higher-order forward guidance, while the dashed lines show two possible stochastic path realizations.

In summary, higher-order forward guidance with commitment forces the central bank to trade lower risk premia and higher asset prices before $\hat{T}^{\mathrm{HOFG}}$ against higher post-guidance stabilization costs. Optimal policy therefore depends on the triple $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\mathrm{HOFG}})$. Owing to the additional stabilization effects induced by negative $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, the optimal

[20] The risk premium is $\mathrm{rp}_t = (\sigma_t + \sigma_t^q)^2$. Under flexible prices, $\sigma_t^{q,n} = 0$, so the natural level is σ_t^2.

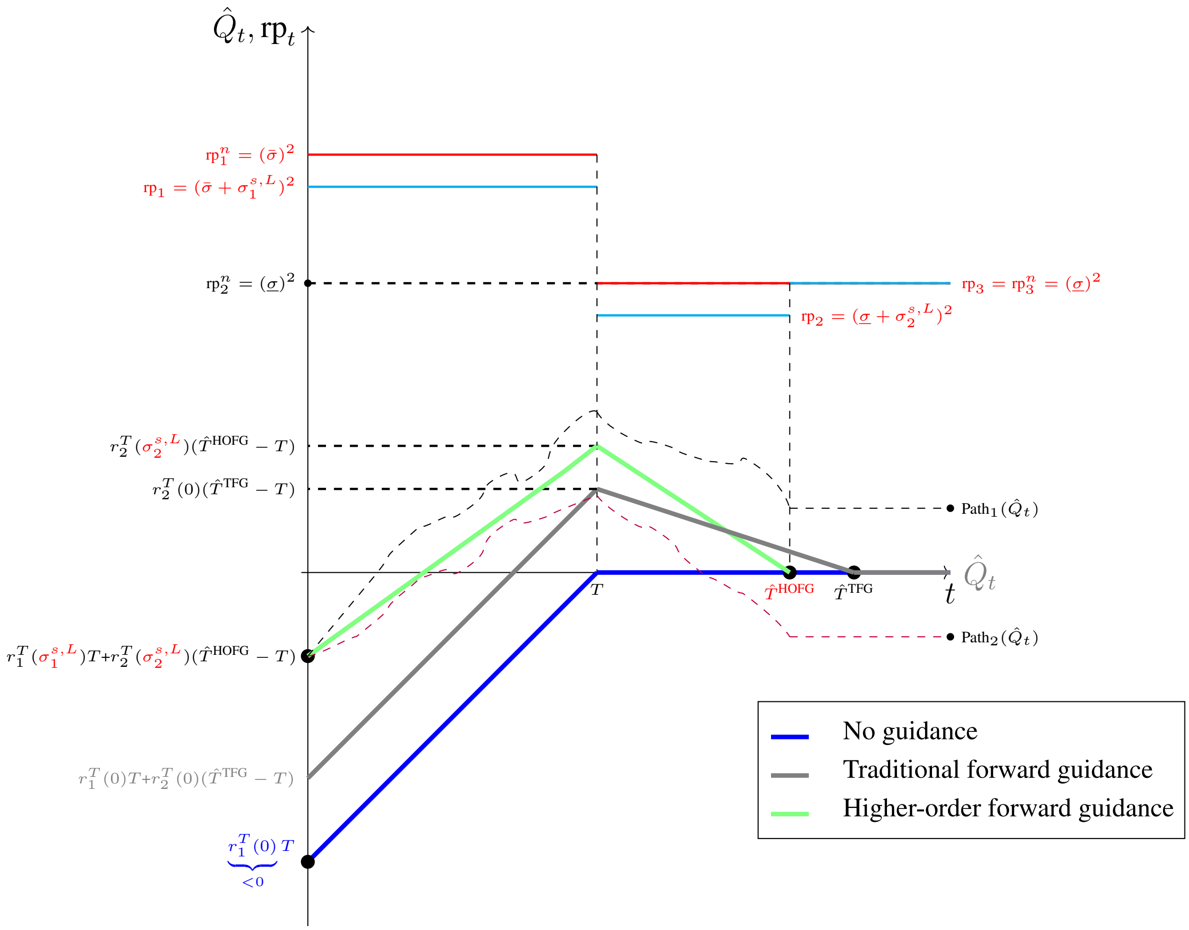

Figure H.6: Intervention dynamics of $\{\hat{Q}_t\}$ with $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{\mathrm{HOFG}} < \hat{T}^{\mathrm{TFG}}$.

duration of the zero-policy-rate period, $\hat{T}^{\mathrm{HOFG}}$, is shorter than $\hat{T}^{\mathrm{TFG}}$, as proven in Proposition H.4.

Optimal Higher-Order Forward Guidance. The initial asset price gap, $\hat{Q}_0$, is determined by the boundary condition $\mathbb{E}_0[\hat{Q}_{\hat{T}^{\mathrm{HOFG}}}] = 0$ and the dynamic IS equation in (H.10), yielding

$$\hat{Q}_0 = r_1^T(\sigma_1^{q,L})\,T + r_2^T(\sigma_2^{q,L})\left(\hat{T}^{\mathrm{HOFG}} - T\right). \tag{H.19}$$

The central bank minimizes the quadratic loss function in (H.16) by optimally choosing $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$. The optimization problem is formulated as follows:

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\,\hat{Q}_t^2\,dt\ ,$$

$$\text{s.t.} \quad d\hat{Q}_t = \begin{cases} -r_1^T(\sigma_1^{q,L})\,dt + \sigma_1^{q,L}\,dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L})\,dt + \sigma_2^{q,L}\,dZ_t, & \text{for } T \le t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \ge \hat{T}^{\text{HOFG}}, \end{cases} \tag{H.20}$$

with $\hat{Q}_0$ given by equation (H.19).

Proposition H.4 (Optimal Commitment Path) *The solution to the central bank's higher-order forward guidance optimization problem in (H.20) yields an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. Moreover, the optimal higher-order forward guidance policy results in an equal or lower expected quadratic loss than the traditional forward guidance policy discussed in Section H.4.1.*

Proof. Identical to Proposition 3 in Section 4.2.

■

H.4.3 Higher-Order Forward Guidance *with* Stochastic Stabilization

In the previous section, we assumed that once the forward guidance regime ends at $\hat{T}^{\text{HOFG}}$, the monetary authority passively pegs the policy rate i_t to the natural rate $\bar{r}$ and sets $\sigma_t^q = 0$ indefinitely. This arrangement permits σ_t^q to deviate from zero during the ZLB period, as illustrated in Figure H.6. In this section, we relax those assumptions while preserving the framework's support for multiple equilibria. Specifically, we now assume that after forward guidance ends, the central bank follows the passive rule but also commits to a stochastic return to perfect stabilization via active monetary policy. This commitment is modeled as a constant probability event governed by a Poisson process. Consequently, for $t \ge \hat{T}^{\text{HOFG}}$, the asset price gap evolves according to

$$d\hat{Q}_t = -\hat{Q}_t \, d\Pi_t, \quad \text{with} \quad d\Pi_t = \begin{cases} 1, & \text{with probability } \nu \, dt, \\ 0, & \text{with probability } 1 - \nu \, dt, \end{cases}$$

where $d\Pi_t$ is a Poisson random variable with rate parameter $\nu \geq 0$.[21]

The central bank's optimization problem is then formulated as follows:

$$\min_{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}} \quad \mathbb{E}_0 \left[\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \, \hat{Q}_t^2 \, dt + \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t} \, e^{-\nu\left(t - \hat{T}^{\text{HOFG}}\right)} \hat{Q}_t^2 \, dt \right],$$

$$\text{s.t.} \quad d\hat{Q}_t = \begin{cases} -r_1^T(\sigma_1^{q,L}) \, dt + \sigma_1^{q,L} \, dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L}) \, dt + \sigma_2^{q,L} \, dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} \tag{H.21}$$

with $\hat{Q}_0$ determined by equation (H.19).

Proposition H.5 (Optimal Commitment Path with Stochastic Stabilization) *The solution to the central bank's optimization problem in (H.21) yields an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. Moreover, the optimal higher-order forward guidance policy with a stochastic stabilization probability produces an expected quadratic loss that is equal to or lower than that under the traditional forward guidance policy discussed in Section H.4.1.*

Furthermore, an increased stabilization probability (i.e., higher ν) leads to smaller optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, thereby reducing risk premia at the ZLB.

Proof. Identical to Proposition 4 in Section 4.3. ■

Finally, Corollary H.1 shows that introducing even a minimal degree of uncertainty about the timing of future stabilization is always optimal for the central bank. This uncer-

[21] Here, ν is treated as an exogenous parameter determined by external factors. If the central bank could choose ν, it would set $\nu \to +\infty$, but $\nu \neq \infty$, as shown in Online Appendix D.2.

tainty enables private agents to coordinate on a stochastic equilibrium in which σ_t^q deviates from zero during the ZLB, as depicted in Figure H.6. Consequently, higher-order forward guidance yields equilibrium paths that are strictly superior—in terms of the quadratic loss—compared to those under traditional forward guidance.

Corollary H.1 (Discontinuity at the Limit) *When the stabilization parameter ν equals $+\infty$, the problem reduces to the traditional forward guidance case described in Section H.4.1. As ν approaches $+\infty$ from below, the central bank's expected quadratic loss function exhibits a discontinuity. In particular, the expected quadratic loss is always lower when there is a non-zero probability of not achieving immediate stabilization at the end of the forward guidance period, $\hat{T}$. Formally:*

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu = +\infty\right) ,$$

where $\mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right)$ denotes the quadratic loss function defined in equation (H.16), evaluated at its optimum for an economy characterized by a Poisson rate ν.

Proof. See Online Appendix D.2. The intuition is that when $\nu = +\infty$, the probability of immediate stabilization at $\hat{T}^{\text{HOFG}}$ becomes one, which is equivalent to the traditional forward guidance policy discussed in Section H.4.1. Moreover, Proposition H.5 shows that higher-order guidance always achieves an equal or lower expected quadratic loss compared to traditional guidance, regardless of the value of ν.

■

Equilibrium Selection. We next present an example demonstrating how the model's optimal higher-order solution can be implemented in practice among the multiple existing ZLB equilibria through coordinated fiscal policy measures. This part is different from Section 4.4, as we now provide fiscal policies based on wealth transfer to capitalists.

H.4.4 Fiscal Policy Coordination

Fiscal intervention, alongside the forward guidance policies discussed earlier, can enforce the optimal higher-order equilibrium as the unique solution of the model. Although various forms of fiscal coordination are feasible, we focus on a fiscal subsidy (or tax) that adjusts stock returns relative to the risk-free rate.[22] Alternatively, these transfers can be interpreted as a stylized representation of the impact of large-scale asset purchase (LSAP) programs on the relative returns of risky assets. The potential for balance sheet losses (and gains) from such policies—as well as other direct or indirect effects—has led many authors to highlight the quasi-fiscal nature of these interventions (see, e.g., Woodford (2016), Chionis et al. (2021), and Lee and Dordal i Carreras (2025b)).

We consider a subsidy scheme financed by withdrawals from the fiscal authority's monetary reserves, denoted F_t. The fiscal authority provides funds in response to unexpected shocks in stock returns, and for simplicity, we assume that the reserves are affected solely by these transfers. The process evolves as follows:

$$\begin{aligned} dF_t &= -\theta_t \, a_t \, \tau_t \, dZ_t \ , \\ F_0 &= F_{0-} - \chi \, \theta_{0-} \, a_{0-} \ , \end{aligned} \tag{H.22}$$

where τ_t and χ are parameters determined by the fiscal authority. The subscript $0-$ denotes the values immediately prior to the ZLB shock, allowing for an initial jump in the subsidy process of magnitude $\chi \, \theta_{0-} \, a_{0-}$. The flow budget constraint for capitalists becomes:

$$\begin{aligned} da_t &= \left(a_t \left(i_t + \theta_t \left(i_t^m - i_t\right)\right) - \bar{p} \, C_t\right) dt + \theta_t \, a_t \left[\left(\sigma_t + \sigma_t^q\right) + \tau_t\right] dZ_t \ , \\ \Delta a_0 &= \chi \, \theta_{0-} \, a_{0-} + \bar{p} \, A_{0-} \, \Delta Q_0 \ , \end{aligned} \tag{H.23}$$

where $\Delta x_t \equiv x_t - x_{t-}$ represents the initial jump in the corresponding variable following the ZLB shock. The second equation in (H.23) indicates that the change in initial capitalist

[22]In our model, a subsidy (or tax) on stock investments functions similarly to a tax break or hike on capital income, a policy commonly employed by governments. We choose the subsidy formulation for notational simplicity.

wealth equals the change in the value of stocks (with full allocation in equilibrium at time $0-$) plus the transfer from the fiscal authority.

Proposition H.6 describes the implementation of the subsidy scheme that enforces the optimal equilibrium paths detailed in Sections H.4.2 and H.4.3 as the unique equilibrium outcome.

Proposition H.6 (Fiscal Coordination and Unique Optimal Equilibrium) *In the ZLB environment described in Section 3, and under the forward guidance policies of Sections H.4.2 and H.4.3, if the subsidy scheme is governed by the variables $\{\tau_t, \chi\}$ defined below, the optimal higher-order solution is the unique equilibrium of the model:*

$$\tau_t = (\sigma_t^{q,*} - \sigma_t^q) \;\; , \quad \textit{and} \quad \chi = \bar{p} A_{0-} \frac{Q_0^* - Q_0}{\theta_{0-} a_{0-}} \; , \tag{H.24}$$

where the star superscript denotes variables along the optimal higher-order forward guidance path. Thus, $\hat{Q}_t = \hat{Q}_t^*$ and $\sigma_t^q = \sigma_t^{q,*}$ for all $t \geq 0$ in equilibrium.

Proof. See Online Appendix I.3. The intuition is that the transfer schedule in (H.24) targets two sources of deviation from the optimal path: (i) the initial stock price response Q_0 to the shock that leads to the ZLB, and (ii) the sensitivity σ_t^q of agents' responses to subsequent stochastic shocks.

■

Under our proposed policy, fiscal intervention acts as an off-equilibrium threat by linking subsidies to deviations from the optimal path, resulting in zero transfers in equilibrium (see Corollary H.2). We conjecture that alternative coordination mechanisms may exist—potentially involving elements beyond fiscal intervention—but as long as equilibrium selection is achieved through off-equilibrium threats, the properties and welfare benefits of higher-order guidance can be assessed independently of the specific coordination mechanism employed by the monetary authority.

Corollary H.2 (Zero Equilibrium Transfers) *Under the scheme described in Proposition H.6, subsidies and tax transfers are zero in equilibrium:*

$$\begin{aligned} \tau_t &= \chi = 0 \, , \\ F_t &= F_{0-} \, , \end{aligned} \qquad \forall t \geq 0 \, .$$

Proof. This follows directly from Proposition H.6 since $Q_t = Q_t^*$ and $\sigma_t^q = \sigma_t^{q,*}$ for all $t \geq 0$, together with the definitions in equation (H.24).

■

Note that these results do not preclude the fiscal authority from implementing conventional fiscal transfer schemes to address output gap deviations caused by the ZLB. The effects of such policies and related implementation details are discussed next.

H.5 Macroprudential Policies

This section examines two types of macroprudential policies designed to stimulate the economy at the ZLB. Firstly, we consider a fiscal subsidy aimed at encouraging capitalists to undertake higher levels of risk, thereby boosting asset prices and other real economic activities. Secondly, we explore the impact of direct fiscal redistribution from capitalists to hand-to-mouth workers, who typically exhibit a higher marginal propensity to consume. This policy is shown to increase overall stock market dividends, and consequently, asset prices $\hat{Q}_t$ and consumption. To assess the impact of macroprudential policies on the business cycle, forward guidance is excluded from our analysis in this section. We maintain the same scenario as outlined in Section 3, and assume that the monetary authority perfectly stabilizes the economy for $t \geq T$.

H.5.1 Fiscal Subsidy on Stock Market Investment

In the period up to T, where $r_t^n = \underline{r} < 0$ and monetary policy is constrained by the ZLB, the risk-premium level $\mathrm{rp}_1^n = \bar{\sigma}^2$ required by capitalists leads to a reduction in asset prices,

$\hat{Q}_t$. To counteract this, we propose a subsidy policy aimed at incentivizing capitalists' holdings of the stock market index. This intervention is expected to increase $\hat{Q}_t$, addressing the aggregate demand externalities responsible for dragging the economy into a ZLB recession.[23]

We begin by examining a government subsidy for the purchase of (risky) stock market index shares.[24] Specifically, instead of the usual expected return i_t^m, a capitalist earns an expected return of $(1+\tau)i_t^m$ for every dollar invested in the stock market, where $\tau \geq 0$ is the stock subsidy. To fund this intervention, the government imposes a 'lump-sum' tax L_t on capitalists. Consequently, a capitalist solves the optimization problem with a modified flow budget constraint given by:

$$\begin{aligned} &\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dt \\ &\text{s.t. } da_t = \left(a_t \left(i_t + \theta_t((1+\tau)i_t^m - i_t)\right) - \bar{p}C_t - L_t\right) dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \,. \end{aligned} \tag{H.25}$$

In equilibrium, capitalists finance the stock market subsidy by paying taxes L_t equal to $\tau \bar{p} A_t Q_t i_t^m$. Setting $\theta_t = 1$ in equilibrium, we express the stock market's expected return as follows:

$$i_t^m = \frac{i_t + \left(\bar{\sigma} + \sigma_t^q\right)^2}{1+\tau} = \underbrace{\rho}_{\substack{\text{Dividend} \\ \text{yield}}} + \underbrace{g + \mu_t^q + \sigma_t \sigma_t^q}_{\text{Capital gain}} \,. \tag{H.26}$$

As detailed in Section 3, given that σ_t^q and i_t equal zero for $t \leq T$, equation (H.26) simplifies to

$$i_t^m = \frac{\bar{\sigma}^2}{1+\tau} \,,$$

which is lower than $\bar{\sigma}^2$ and inversely proportional to τ. Thus, a positive subsidy rate $\tau > 0$ increases $\hat{Q}_t$ along the path up to time T, when the economy achieves perfect stabilization

[23]Numerous studies have examined the link between externalities (e.g., pecuniary or aggregate demand) and macroprudential policies. Notable references include Caballero and Krishnamurthy (2001), Lorenzoni (2008), Farhi et al. (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010), Stein (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), Dàvila and Korinek (2018), among others.

[24]A subsidy for stock investments functions similarly to a tax break on capital income, a policy commonly implemented *in practice* by governments. We opt for the subsidy model for simplicity in notation.

with $\hat{Q}_T = 0$. Proposition H.7 summarizes this result.

Proposition H.7 (Fiscal Subsidy on Stock Market Expected Returns) *Under the ZLB environment of Section 3, where a fiscal subsidy $\tau \geq 0$ is applied to the expected return of stock markets, the dynamics of $\hat{Q}_t$ during the period $t < T$ are given by:*

$$d\hat{Q}_t = -\left(\underbrace{\underline{r}}_{\equiv r^n(\bar{\sigma})<0} + \underbrace{\frac{\tau}{1+\tau}\bar{\sigma}^2}_{>0}\right) dt \;, \tag{H.27}$$

for $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 < 0$ and $\hat{Q}_T = 0$. When $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 > 0$, the subsidy $\tau > 0$ lifts the economy out of the ZLB and immediate stabilization becomes possible via active monetary policy with $i_t > 0$.

Proof. See Online Appendix I.4.

■

In equation (H.27), a positive subsidy $\tau > 0$ increases the effective natural rate from $\underline{r}$ to $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2$. This rise narrows the gap between the ZLB and the 'effective' natural rate, consequently raising $\hat{Q}_t$ relative to the scenario described in Section 3. It is important to note that as τ approaches infinity, the expression $\underline{r}+\frac{\tau}{1+\tau}\bar{\sigma}^2$ converges to $\underline{r}+\bar{\sigma}^2 = \rho+g > 0$. In this situation, the economy moves away from the ZLB and the monetary authority can achieve perfect stabilization via active monetary policy with $i_t > 0$.

Tax on whom? We consider an alternative funding scheme for the stock market subsidy τ by imposing a lump-sum tax L_t on hand-to-mouth workers. Under this policy, the budget constraint of the workers (H.1) becomes

$$\frac{w_t}{\bar{p}} N_{W,t} = C_{W,t} + \frac{L_t}{\bar{p}} \,. \tag{H.28}$$

Hand-to-mouth workers, characterized by a marginal propensity to consume of one, experience a proportional reduction in their consumption due to taxation. This fall in workers' consumption adversely impacts stock dividends and prices, $\hat{Q}_t$. In this context, the formula for the stock market's expected return i_t^m is as follows:

$$i_t^m = \underbrace{\frac{Y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t}}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right] = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right], \quad \text{(H.29)}$$

where we used an equilibrium tax equal to $\tau i_t^m \bar{p} A_t Q_t$ to obtain the last equality. Proposition H.8 summarizes our findings, highlighting the crucial role of tax scheme design in determining the effectiveness of the macroprudential policy.

Proposition H.8 (Fiscal Subsidy and Tax on Workers) *The positive impact of a subsidy τ on asset prices is precisely offset by the reduced consumption of hand-to-mouth workers due to taxation L_t. Consequently, this results in no net effect on the dynamics of $\{\hat{Q}_t\}$ during a ZLB episode, apart from a redistribution of wealth from workers to capitalists. The trajectory of asset prices under this taxation scheme corresponds with the benchmark scenario, which lacks forward guidance and macroprudential interventions, as depicted in Figure H.4.*

Proof. See Online Appendix I.4.

■

H.5.2 Fiscal Redistribution

Lastly, we consider a redistribution policy in the form of a fiscal transfer $L_t > 0$ from capitalists to hand-to-mouth workers during a ZLB episode.[25] This policy increases aggregate demand due to the high marginal propensity to consume of workers and, in turn, the total dividends paid by the stock market index. The expected return on the stock market i_t^m then

[25] A policy subsidizing firms' payroll, financed through a lump-sum tax L_t on capitalists, produces identical results. When firms incur net payroll costs of $w_t N_{W,t} - L_t$, the consequent rise in employment effectively creates a transfer of income equivalent to L_t to the workers. We opt for the direct transfer formulation for simplicity in notation.

becomes:

$$i_t^m = \frac{Y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t} + \mathbb{E}_t\left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt}\right] = \rho + \underbrace{\frac{L_t}{\bar{p} A_t Q_t}}_{>0} + \mathbb{E}_t\left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt}\right] .$$

Assuming capitalists finance this transfer L_t by paying a portion φ of their wealth a_t, the dividend yield increases to $\rho + \varphi$ from a baseline yield (before transfers) of ρ. This adjustment raises the effective natural rate of interest from $\underline{r}$ to $\underline{r} + \varphi$, resulting in an increase in asset prices $\hat{Q}_t$ and a narrower output gap during a ZLB episode. Proposition H.9 summarizes this result.

Proposition H.9 (Fiscal Redistribution) *In the ZLB environment presented in Section 3, and under a redistribution scheme where a $\varphi \geq 0$ portion of capitalists' wealth is transferred to hand-to-mouth workers, the dynamic IS equation for $\hat{Q}_t$ becomes:*

$$d\hat{Q}_t = -(\underbrace{\underline{r}}_{<0} + \varphi)\, dt \ , \tag{H.30}$$

for $\underline{r} + \varphi < 0$. After time T, the central bank perfectly stabilizes the economy and eliminates the volatility in asset prices, $\sigma_t^q = 0$, for all $t \geq T$. When $\underline{r} + \varphi > 0$, fiscal transfers lift the economy out of the ZLB and immediate stabilization is possible via active monetary policy, with $\underline{r} + \varphi$ as the effective natural rate.

Proof. See Online Appendix I.4.

■

From the perspective of capitalists, this policy effectively reduces their expected growth in wealth by φ, taking the expected stock market return i_t^m as given. At the ZLB, i_t^m does not react to fiscal transfers as the policy rate i_t is fixed at zero.[26] As a result, the equilibrium growth rates of capitalists' wealth and the stock price index fall by φ, due to a less

[26]Note from the capitalists' optimization that risk premium rp_t is given by $\bar{\sigma}^2$ during the ZLB, and $i_t^m = i_t + \mathrm{rp}_t$.

significant initial decline in asset prices $\hat{Q}_0$ at the start of the ZLB episode. Therefore, fiscal transfers to workers with a high marginal propensity to consume not only enhance aggregate demand but also create additional wealth effects which manifest through increases in dividend yields and asset prices, $\hat{Q}_t$.

H.6 Welfare Comparison

For the quantitative evaluation of the forward guidance policies discussed in this paper, we simulate optimal commitment paths at the ZLB under three scenarios: (i) no forward guidance; (ii) traditional forward guidance; and (iii) higher-order forward guidance with varying stabilization probabilities. The initial ZLB duration, T, is set to 20 quarters, reflecting the extended ZLB periods following the global financial crisis. The Poisson rate parameter, ν, in the higher-order forward guidance policy is calibrated first to zero—indicating a zero probability of reverting to an active policy rule—and then to one, indicating an expectation of resuming an active policy rule one quarter after the guidance period. All remaining model parameters are set to values commonly used in the literature and summarized in Table H.1.

We define the loss function $\mathbb{L}$ as the per-period quadratic output loss and approximate it by

$$\mathbb{L}^{Y}_{\text{Per-period}} \equiv \rho \int_0^\infty e^{-\rho t}\mathbb{E}_0\left(\hat{Y}_t^2\right) dt \approx \zeta^2 \cdot \rho \int_0^\infty e^{-\rho t}\frac{1}{s}\sum_{i=1}^{s}\left(\hat{Q}_t^{(i)}\right)^2 dt\ ,$$

where the constant $\zeta > 0$ is defined by the relationship $\hat{Y}_t = \zeta \hat{Q}_t$.[27] Here, $\hat{Q}_t^{(i)}$ denotes the i^{th} simulated stochastic sample path of the asset price gap.[28] We consider a scenario in which a one-time ZLB recession begins at period zero with no expectation of recurrence; thus, $\mathbb{L}$ is interpreted as the expected conditional loss from a single ZLB episode.

Table H.2 presents our simulation results. In this table, $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ are reported as

[27] For this, see equation (I.8) in Online Appendix I.2.

[28] We simulate $s = 10^4$ sample paths to approximate the quadratic loss under higher-order forward guidance.

	Parameter Description	Value	Source
φ	Relative Risk Aversion	0.2	Within the admissible calibration ranges specified by Gandelman and Hernández-Murillo (2014).
χ_0	Inverse Frisch labor supply elasticity	0.25	See King and Rebelo (1999).
ρ	Subjective time discount factor	0.020	Target 2.8% natural rate.
g	TFP growth rate	0.0083	Annual growth rate of 3.3%, which corresponds to the US TFP growth rate from 2009 to 2020, as detailed in Table 8 of Comin et al. (2023).
$\underline{\sigma}$	TFP volatility, low volatility regime	0.009	See Dordal i Carreras et al. (2016).
$\bar{\sigma}$	TFP volatility, high volatility regime	0.209	Target -1.5% natural rate (ZLB recession).
T	ZLB duration (quarters)	20	A five-year ZLB duration, consistent with periods such as the Global Financial Crisis and the Great Recession. See Dordal i Carreras et al. (2016).
ν	Stabilization probability parameter	1	Target average duration $1/\nu$ of one quarter before returning to stabilization.
α	$1-$ Labor income share	0.4	See Alvarez-Cuadrado et al. (2018).
ϵ	Elasticity of substitution intermediate goods	7	Target steady-state mark-up of 16.7%. See Galí (2015).

Table H.1: Parameter calibration TANK model.

Policy	No guidance	Traditional	**Higher-Order** (no stochastic stabilization)	**Higher-Order** (*with* stoch. stab., $\nu = 1$)
$\sigma_1^{q,L}$	0	0	-1.27%	-4.13%
$\sigma_2^{q,L}$	0	0	-0.24%	-3.79%
$\hat{T}^{\text{HOFG}}$	20	25.27	25.09	24.68
$\mathbb{L}^{Y}_{\text{Per-period}}$	7%	1.93%	1.81%	1.69%

Table H.2: Policy comparisons.

percentages of the fundamental volatilities $\bar{\sigma}$ and $\underline{\sigma}$, respectively. The initial columns assess the effectiveness of traditional forward guidance, showing that the central bank extends the ZLB period by just over one year and reduces total loss by approximately five percentage points. These findings are consistent with the existing literature (e.g., Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016)).[29] The final two columns provide summary statistics on the optimal implementation of higher-order guidance under the two stabilization regimes discussed above. Consistent with Propositions H.4 and H.5, higher-order guidance further reduces ZLB costs by 0.12–0.24 percentage points per quarter via lower financial market volatility during the guidance period, and permits an earlier exit from the ZLB. Moreover, gains from higher-order guidance double when there is a positive probability of returning to full stabilization in the future.

I Derivations for Online Appendix H

I.1 Flexible Price Equilibrium in the Two-Agent New Keynesian (TANK) model

This section derives the flexible price equilibrium of the model, establishing it as the benchmark for economic and welfare analysis. We begin by revisiting the Fisherian identity, incorporating an inflation premium linked to wealth volatility into the relation. Lemma I.3

[29]These studies also note that traditional forward guidance can be overly potent in plain-vanilla New Keynesian models relative to empirical estimates. This paper does not incorporate the quantitative adjustments proposed to address that discrepancy, focusing instead on the differences between traditional and higher-order forward guidance.

summarizes the modified identity.

Lemma I.3 (Inflation Premium) *The real interest rate of the economy is given by:*

$$r_t = i_t - \pi_t + \overbrace{\sigma_t^p \underbrace{\left(\sigma + \sigma_t^p + \sigma_t^q\right)}_{\text{Wealth volatility}}}^{\text{Inflation Premium}} . \tag{I.1}$$

Proof of Lemma I.3. The wealth of capitalists is equal to the value of the stock market index, $a_t = p_t A_t Q_t$, since bonds are zero net supplied. The nominal state-price density of capitalists, ξ_t^N, satisfies the following condition:

$$\frac{d\xi_t^N}{\xi_t^N} = -i_t dt - (\sigma + \sigma_t^q) dZ_t \ , \tag{I.2}$$

and the real state price density ξ_t^r, which is given by

$$\xi_t^r = e^{-\rho t}\frac{1}{C_t} = p_t \xi_t^N \ . \tag{I.3}$$

Utilizing equations (I.2) and (H.2), and considering that $\theta_t = 1$ in equilibrium, the application of Ito's Lemma to equation (I.3) yields the following expression:

$$\frac{d\xi_t^r}{\xi_t^r} = \left(\underbrace{\pi_t - i_t - \sigma_t^p \left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{=-r_t} \right) dt - (\sigma + \sigma_t^q) dZ_t \ ,$$

resulting in the modified Fisherian identity detailed in equation (I.1).

■

Definition I.1 *Let* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0 + \varphi}$ *represent the effective labor supply elasticity of workers, conditional on their optimal consumption decision.*

Proposition I.10 summarizes the dynamics of the real wage, asset price, natural interest rate r_t^n, and the consumption process of capitalists within the flexible price equilibrium.

Proposition I.10 (Flexible Price Equilibrium) *In the flexible price equilibrium,*[30] *the following results are obtained:*

1. *The real wage is proportional to aggregate technology A_t, and given by*

$$\frac{w_t^n}{p_t} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon} A_t \ .$$

2. *The equilibrium asset price Q_t^n is constant and given by*

$$Q_t^n = \frac{1}{\rho}\left(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{\frac{1}{\chi}}\left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \ , \quad \textit{and} \quad \mu_t^{q,n} = \sigma_t^{q,n} = 0 \ .$$

3. *The natural interest rate r_t^n is constant and defined as $r_t^n \equiv r^n = \rho + g - \sigma^2$. The consumption of capitalists evolves according to the following equation:*

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \underbrace{(r^n - \rho + \sigma^2)}_{\equiv \mu_t^{c,n}} dt + \underbrace{\sigma}_{\equiv \sigma_t^{c.n}} dZ_t \ .$$

Proof of Proposition I.10. Starting with the optimization problem of intermediate firms, the presence of an externality à la Baxter and King (1991) imposes extra steps on the aggregation process of individual decisions across firms. Utilizing the production function, the employed labor of firm i can be expressed as

$$n_t(i) = \left(\frac{y_t(i)}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \ ,$$

where we defined $E_t \equiv (N_{W,t})^\alpha$. At any given time t, each intermediate firm i determines the optimal price $p_t(i)$ to maximize its profits,

$$\max_{p_t(i)} p_t(i)\left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} Y_t - w_t \left(\frac{Y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} \ , \tag{I.4}$$

[30] Variables in the flexible price (i.e., natural) equilibrium are denoted with the superscript n.

taking the aggregate demand of the economy Y_t as given. In the flexible price equilibrium, all firms charge the same price, $p_t(i) = p_t$ for all i, and hire the same amount of labor, $n_t(i) = N_{w,t}$ for all i. From the first-order condition (I.4), we obtain the real wage as

$$\frac{w_t^n}{p_t} = \frac{\epsilon - 1}{\epsilon}(1-\alpha) Y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} = \frac{\epsilon - 1}{\epsilon}(1-\alpha) Y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} \left(\frac{w_t^n}{p_t}\right)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}} \ ,$$

which can be further simplified to the following expression:

$$\frac{w_t^n}{p_t} = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} Y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}} \ .$$

Aggregate production in the flexible price equilibrium is linear, $Y_t = A_t N_{W,t}$. We obtain:

$$Y_t = A_t \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} Y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}} \ .$$

The previous expression allows us to write the natural level of output Y_t^n and the natural real wage $\frac{w_t^n}{p_t}$ as

$$Y_t^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \ \text{ and } \ \frac{w_t^n}{p_t} = \frac{\epsilon - 1}{\epsilon}(1-\alpha) A_t \ ,$$

from which we obtain

$$N_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \ \text{ and } \ C_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t \ . \tag{I.5}$$

In equilibrium, the combined consumption of capitalists and workers equates to the total final output. Following from equation (I.5), we obtain:

$$\rho A_t Q_t^n + \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \ .$$

where we defined Q_t^n to be the natural stock price. Therefore, we obtain an expression for

Q_t^n as

$$Q_t^n = \frac{1}{\rho}\left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) ,$$

and $C_t^n = \rho A_t Q_t^n$. Since Q_t^n is constant in equilibrium, its process in a flexible price economy exhibits neither drift nor volatility, which implies $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To determine the natural interest rate r_t^n, we start from the capital gain component outlined in equation (H.8). The application of Ito's lemma yields:

$$\mathbb{E}_t \frac{d\left(p_t A_t Q_t\right)}{p_t A_t Q_t}\frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0}\sigma_t^p + \sigma\left(\sigma_t^p + \underbrace{\sigma_t^q}_{=0}\right) .$$

Given a constant dividend yield equal to ρ, applying expectations to both sides of equation (H.8) and combining this expression with the equilibrium condition presented in equation (H.6) results in:

$$i_t^m = \rho + \pi_t + g + \sigma\sigma_t^p = i_t + (\sigma + \sigma_t^p)^2 .$$

Inserting the previous expression into the Fisherian identity in equation (I.1), we express the natural rate of interest r_t^n as

$$r_t^n = i_t - \pi_t + \sigma_t^p\left(\sigma + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p\right) = \rho + g - \sigma^2 , \tag{I.6}$$

which is a function of structural parameters, including σ, thereby proving the final point of Proposition I.10. As the consumption of capitalists C_t^n is directly proportional to the level of technology A_t, it follows that:

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \left(r_t^n - \rho + \sigma^2\right)dt + \sigma dZ_t , \tag{I.7}$$

where the last equality is derived using equation equation (I.6).

■

I.2 Co-movements between gap variables

The following Lemma I.4 demonstrates that Assumption I.1 serves as a sufficient condition for the model to exhibit the empirical regularities of positive co-movements between asset prices and various business cycle variables, such as real wage and consumption (of capitalists and workers), as observed in data.[31]

Assumption I.1 (Labor Supply Elasticity) *The effective labor supply elasticity of workers satisfies:* $\chi^{-1} > \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Lemma I.4 (Positive comovement) *Under Assumption I.1, the consumption gaps of capitalists C_t and workers $C_{W,t}$, employment $N_{W,t}$, and the real wage $\frac{w_t}{p_t}$ exhibit joint positive comovement. This relationship is approximated up to a first-order as follows:*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{\frac{w_t}{p_t}} = \frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right) \widehat{C_{W,t}} \ ,$$

and is related to the output gap of the economy by:

$$\hat{Y}_t = \zeta \hat{Q}_t \ , \quad \text{where} \ \ \zeta \equiv \chi^{-1} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1} > 0 \ . \tag{I.8}$$

Proof of Lemma I.4. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, which can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} \ , \tag{I.9}$$

where $\frac{w_t^n}{p_t^n}$ is the real wage in the flexible price economy. We subtract equation (I.9) from the analogous good market condition in the sticky price economy, and divide by $Y_t^n \equiv$

[31] See Table I.1 in the Appendix for a plausible calibration of the model parameters.

$A_t^{1-\frac{1}{\chi}}(\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$, which yields the following result:

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\frac{\widehat{w_t}}{p_t}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}}\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\frac{\widehat{w_t}}{p_t}} ,$$

which can be written as

$$\frac{1}{\chi}\frac{\widehat{w_t}}{p_t} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\underbrace{\left(1+\frac{1}{\chi}\right)\frac{\widehat{w_t}}{p_t}}_{=\hat{C}_{W,t}} ,$$

which, together with $\hat{C}_t = \hat{Q}_t$, leads to

$$\begin{aligned} \hat{Q}_t &= \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\frac{\widehat{w_t}}{p_t} \\ &= \underbrace{\frac{1}{1+\chi^{-1}}\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\widehat{C_{W,t}} . \end{aligned}$$

Finally, equation (I.8) follows by combining the previous expression with the market clearing condition $Y_t = C_t + C_{W,t}$, from which we obtain

$$\hat{Y}_t = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{Q}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\hat{C}_{W,t} = \zeta\hat{Q}_t .$$

■

I.3 Fiscal Policy Coordination Derivation

Integrating equation (H.22), we obtain an expression for the monetary reserves as:

$$F_t = \underbrace{F_{0-} - \chi\theta_{0-}a_{0-}}_{\equiv F_0} - \int_0^t \theta_s a_s \left(\sigma_s^{q,*} - \sigma_s^s\right) dZ_s \ . \tag{I.10}$$

From the first order condition of capitalists, and setting $\theta_t = 1$ in equilibrium, we can express the stock market's expected returns for $t \geq 0$ as:

$$i_t^m = i_t + \left[(\sigma_t + \sigma_t^q) + \tau_t\right]^2 \ . \tag{I.11}$$

Using equations (H.5), (H.24), (I.11) and the equilibrium condition $\theta_t = 1$ into the equations in (H.23), we obtain:

$$\frac{da_t}{a_t} = \left(i_t + (\sigma_t + \sigma_t^{q,*})^2 - \rho\right) dt + (\sigma_t + \sigma_t^{q,*})\, dZ_t \ , \tag{I.12}$$

$$a_0 = a_{0-} + \bar{p}A_{0-}\Delta Q_0^* = \underbrace{\bar{p}A_0 Q_0^*}_{=a_0^*} \ , \tag{I.13}$$

where the last equality in equation (I.13) follows from the fact that wealth is fully allocated into stocks in equilibrium, $a_{0-} = \bar{p}A_{0-}Q_{0-}$, and that TFP does not experience any jump at the limit, $A_0 = A_{0-}$. From equation (I.12) we obtain an expression for the ouput gap process as:

$$d\hat{Q}_t = \left(i_t - r_t^{T,*}\right) dt + \sigma_t^{q,*} dZ_t \ , \tag{I.14}$$

which aligns with the process of the output gap under the optimal forward guidance solution. We previously defined the volatility of the asset price gap as $\sigma_t^q = Var_t\left(\frac{d\hat{Q}_t}{\hat{Q}_t}\right)$, so equation (I.14) demonstrates that $\sigma_t^q = \sigma_t^{q,*}$ in equilibrium under the subsidy scheme. Similarly, equation (I.13) implies that $Q_0 = Q_0^*$, and therefore $\hat{Q}_0 = \hat{Q}_0^*$ in equilibrium. Finally, substituting these findings into equations (H.24) and (I.10), we establish the statements in

Corollary H.2.

I.4 Macroprudential Policy Derivation

Proof of Proposition H.7. We begin by solving the capitalist's problem presented in equation (H.25), considering a subsidy rate τ on stock market investments for $t \leq T$:

$$\begin{aligned} &\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \\ \text{s.t. } \; & da_t = (a_t(i_t + \theta_t((1+\tau)\, i_t^m - i_t)) - \bar{p}C_t - L_t)dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \; . \end{aligned} \tag{I.15}$$

Since the subsidy τ is financed through a lump-sum tax on capitalists, the dividend process in equation (H.4) and the stock market valuation equation (H.5) remain unchanged. As a result, $\bar{p}C_t = \rho a_t$ and $C_t = \rho A_t Q_t$. Equilibrium taxes L_t equal to $\tau i_t^m a_t$, and the budget constraint in equation (I.15) becomes

$$\begin{aligned} \frac{dC_t}{C_t} = \frac{da_t}{a_t} &= ((1+\cancel{\tau})i_t^m - \rho - \cancel{\tau i_t^m})dt + (\bar{\sigma} + \sigma_t^q)dZ_t \\ &= (i_t^m - \rho)dt + (\bar{\sigma} + \sigma_t^q)dZ_t \; , \end{aligned} \tag{I.16}$$

where we used equilibrium condition $\theta_t = 1$. Since $\xi_t^N = e^{-\rho t}\frac{1}{\bar{p}C_t}$, we obtain:

$$\begin{aligned} \frac{d\xi_t^N}{\xi_t^N}(i_t^m, \sigma_t^q) &= -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \\ &= -\cancel{\rho} dt - [(i_t^m - \cancel{\rho})dt + (\bar{\sigma} + \sigma_t^q)dZ_t] + (\bar{\sigma} + \sigma_t^q)^2 dt \\ &= -\left[i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] dt - (\bar{\sigma} + \sigma_t^q)dZ_t \; . \end{aligned} \tag{I.17}$$

The subsidy τ on the expected return i_t^m alters the original Euler equation $\mathbb{E}_t \frac{d\xi_t^N}{\xi_t^N} = -i_t dt$. Consequently, the revised expression with a subsidy τ must be :

$$\mathbb{E}_t\left[\frac{d\xi_t^N}{\xi_t^N}((1+\tau)i_t^m, \sigma_t^q)\right] = -\left[(1+\tau)i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] = -i_t dt \; ,$$

from which we obtain equation (H.26):

$$i_t^m = \frac{i_t + (\bar{\sigma} + \sigma_t^q)^2}{1+\tau} = \frac{\bar{\sigma}^2}{1+\tau} \ ,$$

where the final equality results from substituting $i_t = 0$ and $\sigma_t^q = 0$ into the equation. From equation (I.16),it follows that:

$$\frac{dC_t}{C_t} = (i_t^m - \rho)dt + \bar{\sigma}dZ_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho\right) dt + \bar{\sigma}dZ_t \ , \tag{I.18}$$

with which we obtain

$$d\ln C_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho - \frac{\bar{\sigma}^2}{2}\right) dt + \bar{\sigma}dZ_t \ .$$

Finally, by using equation (I.7) from Online Appendix I.1, we derive the natural counterpart to the above expression:

$$d\ln C_t^n = \left(\underbrace{\bar{r}}_{<0} - \rho + \frac{\bar{\sigma}^2}{2}\right) + \bar{\sigma}dZ_t \ . \tag{I.19}$$

Combining both expressions, we obtain the dynamic IS equation in (H.27).

■

Proof of Proposition H.8. By equation (H.29), the condition that characterizes the equilibrium stock market return i_t^m is given by:

$$i_t^m = \frac{Y_t - \overbrace{\frac{w_t}{\bar{p}}}^{=C_{W,t}+\frac{L_t}{\bar{p}}} N_{W,t}}{A_t Q_t} + \frac{d(\bar{p}A_t Q_t)}{\bar{p}A_t Q_t}\frac{1}{dt} = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \frac{d(\bar{p}A_t Q_t)}{\bar{p}A_t Q_t}\frac{1}{dt} \ ,$$

from which we obtain $(1+\tau)i_t^m = \rho + g + \mu_t^q$ using $\sigma_t^q = 0$. Since $(1+\tau)i_t^m = \bar{\sigma}^2$ by equation (H.26), we infer that μ_t^q remains constant in comparison to the scenario without subsidy, conditional on $i_t = 0$ and $\sigma_t^q = 0$. Therefore, the subsidy policy does not alter the $\{\hat{Q}_t\}$ process. To align this intuition with the mathematical representation, we begin by examining the process for C_t, which is different from that in equation (I.18), as capitalists are now exempt from paying taxes L_t:

$$\begin{aligned}\frac{dC_t}{C_t} &= ((1+\tau)i_t^m - \rho)dt + \bar{\sigma}dZ_t \\ &= (\bar{\sigma}^2 - \rho)dt + \bar{\sigma}dZ_t \ .\end{aligned}$$

Given that the previous expression remains unchanged in the presence of subsidy τ, it can be inferred that a policy subsidizing the expected return of the stock market and financed by a lump-sum tax on workers does not impact the $\{\hat{Q}_t\}$ process. Consequently, the dynamics of $\{\hat{Q}_t\}$ are identical to those in an economy without this policy.

■

Proof of Proposition H.9. A fiscal transfer $L_t > 0$ from capitalists to hand-to-mouth workers increases the aggregate dividends in the financial market. This results in a reduced need for expected future capital gains, which translates into higher asset prices $\hat{Q}_t$ at the ZLB. The expected stock market return i_t^m under these circumstances is given by:

$$\begin{aligned}i_t^m &= \frac{A_t N_{W,t} - \overbrace{\frac{w_t}{\bar{p}} N_{W,t}}^{=C_{W,t} - \frac{L_t}{\bar{p}}}}{A_t Q_t} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt} = \rho + \underbrace{\frac{L_t}{\bar{p} A_t Q_t}}_{>0} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt} \\ &= \rho + \varphi + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t}\frac{1}{dt} \ ,\end{aligned}$$

where the last equality follows from L_t being equal to $\varphi \bar{p} A_t Q_t$ in equilibrium.

To derive equation (H.30), we start from the capitalists' optimization problem:

$$\begin{aligned}&\max_{C_t, \theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \\ &\text{s.t. } da_t = (a_t(i_t + \theta_t(i_t^m - i_t)) - \bar{p} C_t - L_t)dt + \theta_t a_t(\bar{\sigma} + \sigma_t^q)dZ_t \ ,\end{aligned}$$

which features equilibrium conditions for C_t and θ_t identical to those described in equations (H.5) and (H.6), together with $\sigma_t^q = 0$. As a result, $C_t = \rho\bar{p}A_tQ_t$ and $i_t^m = i_t + (\bar{\sigma} + \sigma_t^q)^2$ follows. In an equilibrium where $\sigma_t^q = 0$ and i_t is constrained by the ZLB, the wealth process for capitalists is given by:

$$\frac{dC_t}{C_t} = \frac{da_t}{a_t} = (i_t^m - \rho - \varphi)\, dt + \bar{\sigma}_t dZ_t = (\bar{\sigma}^2 - \varphi - \rho)dt + \bar{\sigma}_t dZ_t \ ,$$

from which we derive

$$d\ln C_t = \left(\frac{\bar{\sigma}^2}{2} - \varphi - \rho\right) dt + \bar{\sigma}_t dZ_t \ .$$

Subtracting the process for C_t^n in equation (I.19) yields the dynamic IS equation in (H.30).

■

I.5 Welfare Derivation

In this section, we derive the quadratic welfare function in equation (H.16), in a similar way to Woodford (2003) with a key difference: as there are two types of agents in the economy, we need to consider some welfare weights attached to each type.

I.5.1 First-Best Allocation

A first-best allocation must be the solution of the following optimization problem.

$$\max_{C_t, N_{W,t}, C_{W,t}} \omega_1 \log \frac{C_t}{A_t} + \omega_2 \left(\frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \right) \quad \text{s.t. } C_t + C_{W,t} = A_t N_{W,t}, \tag{I.20}$$

where $\omega_1 > 0$ and $\omega_2 > 0$ are welfare weights attached to capitalists and workers, respectively. For expositional purposes, we define $x_t \equiv N_{W,t}$ and $Y_t \equiv \frac{C_{W,t}}{A_t}$: then the first-order conditions for (I.20) can be written as

$$Y_t^{-\varphi} = x_t^{\chi_0}, \quad \frac{\omega_1}{\omega_2} = x_t^{\chi_0}(x_t - Y_t). \tag{I.21}$$

I.5.2 Optimization for workers and firms

Following Woodford (2003), we introduce a production subsidy $\tau > 0$ offered to the firms, financed through a lump-sum tax on workers. The production subsidy ensures that our flexible price equilibrium (or steady-state) allocation $\left(N^n_{W,t}, \frac{C^n_{W,t}}{A_t}, \frac{C^n_t}{A_t}\right)$ is efficient and satisfies equation (I.21). With the subsidy τ, workers solve

$$\max_{C_{W,t},N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad \text{s.t. } p_t C_{W,t} = w_t N_{W,t} - p_t T_t, \tag{I.22}$$

where $T_t = \tau Y_t$ is the (real) lump-sum tax imposed on workers. Equation (I.22)'s first-order condition is written as:

$$(N_{W,t})^{\chi_0+\varphi}\left(\frac{w_t}{p_t A_t} - \tau\right)^{\varphi} = \frac{w_t}{p_t A_t}. \tag{I.23}$$

We can express $N_{W,t}$ that satisfies equation (I.23) as a function of the normalized real wage $\frac{w_t}{p_t A_t}$, i.e., $N_{W,t} \equiv f_N(\frac{w_t}{p_t A_t})$. Under the flexible price equilibrium, each firm's optimization is changed from (I.4) with the introduction of τ as follows, with $E_t = (N_{W,t})^{\alpha}$:

$$\max_{p_t(i)} (1+\tau) p_t(i) \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} Y_t - w_t \left(\frac{Y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t}\right)^{\frac{-\epsilon}{1-\alpha}}, \tag{I.24}$$

which at the optimum leads to

$$\frac{w^n_t}{p^n_t A_t} = \frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}. \tag{I.25}$$

Based on equation (I.23) and equation (I.25), we can obtain

$$\begin{aligned} N^n_{W,t} &= f_N\left(\frac{w^n_t}{p^n_t A_t}\right) = f_N\left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\right), \\ \frac{C^n_{W,t}}{A_t} &= \left[\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau\right] f_N\left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\right). \end{aligned} \tag{I.26}$$

Since our goal is to align the allocation implied by equation (I.26) with the first-best allocation implied by equation (I.21), $N^n_{W,t}$ and $\frac{C^n_{W,t}}{A_t}$ in equation (I.26) must satisfy (I.21):

$$\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau = f_N\left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{-\frac{\chi_0+\varphi}{\varphi}}. \tag{I.27}$$

Plugging equation (I.25) into equation (I.23), we obtain

$$\left(N^n_{W,t}\right)^{\chi_0+\varphi}\left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau\right)^{\varphi} = \frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}. \tag{I.28}$$

Solving jointly equation (I.27) and equation (I.28), we conclude the optimal τ^* must satisfies

$$\frac{(1+\tau^*)(\epsilon-1)(1-\alpha)}{\epsilon} = 1.^{32} \tag{I.29}$$

This optimal τ^* in equation (I.29) eliminates the mark-up of firms and restores efficiency. With $\tau = \tau^*$, the normalized real wage $\frac{w^n_t}{p^n_t A_t}$ in (I.25) becomes 1 and we obtain the following benchmark efficient allocation from equation (I.26):

$$N^n_{W,t} \equiv \bar{x} = (1-\tau^*)^{-\frac{\varphi}{\chi_0+\varphi}}, \quad \frac{C^n_{W,t}}{A_t} \equiv \bar{y} = (1-\tau^*)^{\frac{\chi_0}{\chi_0+\varphi}}, \tag{I.30}$$

and

$$\frac{C^n_t}{A_t} = \bar{x} - \bar{y} = (1-\tau^*)^{-\frac{\varphi}{\chi_0+\varphi}} \cdot \tau^*. \tag{I.31}$$

[32] As in Woodford (2003), τ^* is a function of primitive parameters ϵ and α.

The last step is to ensure that the welfare weights ω_1 and ω_2 in (I.20) satisfy equation (I.21).[33] By plugging equation (I.30) into the second condition of equation (I.21), we obtain

$$\frac{\omega_1}{\omega_2} = (N^n_{W,t})^{\chi_0}\left(N^n_{W,t} - \frac{C^n_{W,t}}{A_t}\right) = (1-\tau^*)^{-\frac{(\chi_0+1)\varphi}{\chi_0+\varphi}} \cdot \tau^*. \tag{I.32}$$

Thus, with $\omega_1 > 0$ and $\omega_2 > 0$ satisfying equation (I.32), our allocation with $\tau = \tau^*$ is efficient. We now approximate a joint welfare in equation (I.20) with ω_1 and ω_2 satisfying (I.32) up to a second-order.

I.5.3 Derivation of a quadratic loss function

The steady-state values $\bar{x}$ and $\bar{y}$ (or the flexible price equilibrium values) of x_t and Y_t are provided in equation (I.30). From the economy-wide resource constraint and given our assumption of perfectly rigid prices, we express

$$\frac{C_t}{A_t} = N_{W,t} - \frac{C_{W,t}}{A_t} = x_t - Y_t, \tag{I.33}$$

With (I.33), we express our social welfare in (I.20) with ω_1 and ω_2 satisfying equation (I.32) as

$$U(x_t, Y_t, \Delta_t) \equiv \omega_1 \log(x_t - Y_t) + \omega_2\left(\frac{Y_t^{1-\varphi}}{1-\varphi} - \frac{x_t^{1+\chi_0}}{1+\chi_0}\right), \tag{I.34}$$

which achieves its maximum value $\bar{U}$ when $x_t = \bar{x}$, and $Y_t = \bar{y}$.[34] A second-order approximation of equation (I.34) around the efficient benchmark allocation $(\bar{x}, \bar{y})$ in equation (I.30) results in:

$$U_t - \bar{U} = \frac{1}{2}U_{xx} \cdot \bar{x}^2 \cdot (\hat{x}_t)^2 + \frac{1}{2}U_{yy} \cdot \bar{y}^2 \cdot (\hat{y}_t)^2 + U_{xy} \cdot \bar{x} \cdot \bar{y} \cdot \hat{x}_t \cdot \hat{y}_t + h.o.t, \tag{I.35}$$

where all the second-order partial derivatives (U_{xx}, U_{yy}, U_{xy}) are evaluated at the bench-

[33] Since ω_1 and ω_2 are chosen arbitrarily, we make sure that our allocation with a production subsidy can be on the efficient frontier, which is generated by a varying set of $\{\omega_1, \omega_2\}$.

[34] We have $U_x = U_y = 0$ at $x_t = \bar{x}$ and $Y_t = \bar{y}$ where U_x and U_y are the partial derivatives with respect to x_t and Y_t, respectively and $\bar{x}$ and $\bar{y}$ are defined in (I.30).

mark point $(\bar{x}, \bar{y})$ and given by

$$\begin{aligned} U_{xx} &= -\omega_2(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \left(\frac{1}{\tau^*} + \chi_0\right), \\ U_{yy} &= -\omega_2(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \left(\frac{1}{\tau^*} + \frac{\varphi}{1-\tau^*}\right), \\ U_{xy} &= \omega_2(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \frac{1}{\tau^*}, \end{aligned} \tag{I.36}$$

where we use the relation between ω_1 and ω_2 in equation (I.32) in the process of derivation. Since ω_2 can be regarded a free parameter, we set $\omega_2 \equiv 1$ from now on.

Log-linearization Log-linearizing the worker's optimization condition (I.23), with τ^* given by (I.29), results in

$$\widehat{N_{W,t}} = \frac{1 - \frac{\varphi}{1-\tau^*}}{\chi_0 + \varphi} \widehat{\left(\frac{w_t}{p_t}\right)}. \tag{I.37}$$

Log-linearizing the budget constraint of workers in (I.20) results in

$$\widehat{C_{W,t}} = \frac{1 + \frac{\chi_0}{1-\tau^*}}{\chi_0 + \varphi} \widehat{\left(\frac{w_t}{p_t}\right)}. \tag{I.38}$$

Linearizing the economy-wide resource constraint (I.33) with $\hat{Q}_t = \hat{C}_t$ and solving jointly with equations (I.37) and (I.38), we can obtain

$$\begin{aligned} \widehat{\left(\frac{w_t}{p_t}\right)} &= \frac{\tau^*(\chi_0+\varphi)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)} \hat{Q}_t \\ \hat{x}_t \equiv \widehat{N_{W,t}} &= \frac{\tau^*\left(1 - \frac{\varphi}{1-\tau^*}\right)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)} \hat{Q}_t, \\ \hat{y}_t \equiv \widehat{C_{W,t}} &= \frac{\tau^*\left(1 + \frac{\chi_0}{1-\tau^*}\right)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)} \hat{Q}_t. \end{aligned} \tag{I.39}$$

Plugging equation (I.36) into the second-order approximation to the welfare function, i.e., equation (I.35), we obtain

$$\begin{aligned}
U_t - \bar{U} = &-\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)(1-\tau^*)^{\frac{-2\varphi}{\chi_0+\varphi}}\left(\hat{x}_t\right)^2 \\
&-\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\frac{\varphi}{1-\tau^*}\right)(1-\tau^*)^{\frac{2\chi_0}{\chi_0+\varphi}}\left(\hat{y}_t\right)^2+(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\frac{1}{\tau^*}(1-\tau^*)^{\frac{\chi_0-\varphi}{\chi_0+\varphi}}\hat{x}_t\hat{y}_t \qquad \text{(I.40)} \\
= &-\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)\left(\hat{x}_t\right)^2-\frac{1}{2}(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\left(\frac{1-\tau^*}{\tau^*}+\varphi\right)\left(\hat{y}_t\right)^2+(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\frac{1}{\tau^*}\hat{x}_t\hat{y}_t.
\end{aligned}$$

Finally by plugging equation (I.39) into equation (I.40), we obtain

$$U_t - \bar{U} = \tilde{\gamma}_q\left(\hat{Q}_t\right)^2 + h.o.t, \qquad \text{(I.41)}$$

with

$$\begin{aligned}
\tilde{\gamma}_q = &-\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)\left(\frac{\tau^*\left(1-\frac{\varphi}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)^2 \\
&-\frac{1}{2}(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\left(\frac{1-\tau^*}{\tau^*}+\varphi\right)\left(\frac{\tau^*\left(1+\frac{\chi_0}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)^2 \qquad \text{(I.42)} \\
&+(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\frac{1}{\tau^*}\left(\frac{\tau^*\left(1-\frac{\varphi}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)\left(\frac{\tau^*\left(1+\frac{\chi_0}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)<0.
\end{aligned}$$

Conditional loss function Equations (I.41) and (I.42) lead to our dynamic loss function in (H.16):

$$\mathbb{L}^Q\left(\left\{\hat{Q}_t\right\}_{t\geq 0}\right) = \mathbb{E}_0\int_0^\infty e^{-\rho t}\left(\hat{Q}_t\right)^2 dt. \qquad \text{(I.43)}$$

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