Growth, Heterogeneous Beliefs, and Risk Amplification : Two-Way Interactions*

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Abstract

How do the incomplete information and heterogeneous beliefs about growth prospects affect endogenous volatility and the macroeconomy's transition dynamics? We build a tractable framework that incorporates exogenous technological growth and heterogeneous beliefs, and assess their interaction with the amplification process of the risk during crises. When more productive experts become more optimistic about their growth prospects, trade is facilitated, raising investment, asset prices, and their leverage ratio considerably. However, when the economy transitions into crises, the endogenous volatility and the risk-premium spike in a more amplified way. The economy undergoes a higher number of shorter-lived and more severe crises than under the rational expectations, and spends longer time on average under the crisis regime per year. While more optimistic experts yield more amplified endogenous risks during crises, crisis periods with higher endogenous risks are when the degree of disagreement about asset return prospects across groups is the highest, thereby amplifying the risk during crises again.

Keywords: Heterogeneous Beliefs, Endogenous Risk, Amplification

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1 Introduction

Asset prices before the Global Financial Crisis (GFC) increased considerably and so did capital investments and the leverages of financial market participants. Eventually, all collapsed during the crisis while the market volatility and the risk-premium spiked. This pattern of boom-bust cycles has repeated for a long period of history. The explanation is quite simple: if an investor believes that future asset prices will continue to grow, then she takes more leverage and eventually gets exposed to higher amounts of risk, dragging the economy into a crisis. During crises, asset markets usually feature higher risk-premium levels, thereby facilitating the recapitalization process for many market participants, pushing the economy toward the normal regime.

When markets are turbulent, it is more likely that different market participants have different ideas about the market's direction. During the crisis, there were those who bet against the US housing market, making huge fortunes, for example.[1] Therefore, a natural question arises as to what would happen if different groups of market participants do not share the underlying economic process as common knowledge but instead have their own view about the economy. How do this lack of common knowledge and deviation from the rational expectations affect the market behaviors, i.e., volatility, risk-premium, asset price, and etc? Do heterogeneous beliefs amplify or mitigate crises and recessions? Answering these questions with a tractable introduction of heterogeneous beliefs is our objective in this paper.

We build a continuous-time economy occupied by two groups of agents: experts and households. In a similar way to Brunnermeier and Sannikov (2014), experts have higher productivity in (i) turning capital into outputs (i.e., production); and (ii) turning output into capital (i.e., investment). To introduce heterogeneous beliefs in a tractable way, we first embed exogenous technological growth in our framework: two groups, optimists and pessimists,[2] with imperfect information about the process of their productivity growth, agree to disagree about the expected growth of their productivity measures. We mostly focus on cases where more productive experts believe that the expected growth of their productivity is higher than the other group (i.e., optimistic), but the opposite case is also analyzed.

Our analysis reveals that when experts, who are natural marginal investors for capital due to their higher productivity, become more optimistic about their growth prospects,[3] trade gets facilitated, raising investment, capital asset prices, and the leverage ratio of experts considerably. However, when the economy shows signs of inefficiencies, the endogenous volatility and the risk-premium spike in more amplified ways. Therefore,

[1] For specific such cases, see e.g., Zuckerman (2010).

[2] We model that production technologies of experts and households must have the same growth rate as they are proportional to each other in the true data-generating process. The problem is both experts and households are not aware of it and have their own views about the growth rates of their technologies.

[3] We consider symmetric cases only: when expects become more optimistic, the less productive households have more pessimistic views about their own technological growth.

we have severe crises with higher levels of endogenous risk and risk-premium when experts believe in better growth prospects of his and households believe their expected growth is lower than under the true process.

This amplification of risk under heterogeneous belief can be decomposed into two separate channels: (i) *leverage* effect; and (ii) *market illiquidity* effect. More optimistic experts have higher leverage for funding their purchase of capital than under the benchmark rational expectations cases. Therefore, when negative shocks shift the economy toward inefficiencies, they engage in more intense fire-sale[4] of capital assets, resulting in the higher degree of market turbulence and the amplification of risks.

Another effect comes from the illiquidity of capital markets. Financial crises are when the less productive households hold a portion of capital and the capital market is not liquid. With optimistic experts, a decrease in their 'wealth share' leads to higher decreases in the price of capital[5] than under the rational expectations cases. It creates more market turbulence and the endogenous risk hikes in response. This illiquidity leads to to a higher elasticity of capital price with respect to the wealth share of optimistic experts in equilibrium.

In this case, it turns out the economy undergoes a higher number of shorter-lived and more severe crises than under the rational expectations, and spends more time on average under financial crises. More amplified endogenous volatility and risk-premium levels during the inefficiencies, with the help of their optimism, help experts to recapitalize themselves relatively faster, pushing the economy out of crises and reducing the time the economy lives under crises. On the other hand, more optimistic experts bear too much leverage and risk on their balance sheets, and it makes easier for small negative shocks to push the economy into recessions. On average, this second effect turns out to be stronger than the first effect, and the economy stays for a longer time in the inefficient regime (i.e., crisis) per year.

Therefore, our paper propose a channel through which belief heterogeneity amplifies systemic shocks, contributing to a deeper understanding of what drives *excess volatility* from e.g., Shiller (1980). Furthermore, we have *two-way* interactions between belief heterogeneity about technological growth, and the amplification of endogenous risks: more optimistic experts and pessimistic households yield more amplified endogenous risks during crises, whereas in turn, an increase in the endogenous volatility raises the degree of disagreement about expected asset returns across two groups,[6] which in turn amplifies risks during crises again, ad infinitum. The amplification channel coming from the two-way interaction is novel to the literature.

The elicitation of objectively correct beliefs (i.e., rational expectations) is out of the scope of this paper. All

[4]For fire-sales externalities in the presence of financial frictions, see e.g., Dávila and Korinek (2017).

[5]Since experts are marginal investors at the normal region, their optimism results in a higher price response (i.e., higher elasticity of price) to fluctuations in their wealth share during financial crises.

[6]We fix the degree of disagreement about expected productivity growth rate across two groups (i.e., optimists and pessimists), but a higher level of endogenous risk raises the degree of disagreement between experts and households about the holding return they earn when investing in capital.

agents are assumed to have static beliefs about their expected technological growth rate as in Geanakoplos (2010). This implies that agents' beliefs are not affected by the time variation of sentiments and both experts and households are dogmatic about their own views. In this formulation, disagreement is persistent and two groups do not reach an agreement. This setting is in accordance with the literature,[7] where variation about individual beliefs about expected returns are due to individual fixed effects. In contrast to the models of e.g., Harrison and Kreps (1978), Scheinkman and Xiong (2003), and Geanakoplos (2010), we consider risk-averse, not risk-neutral agents.[8] In terms of the methodological side, we normalize the economy by technology to get our economy stationary, and rely on the Kolmogorov forward equation (KFE) in characterizing the ergodic distribution of our state variable.

Related Literature Our model builds on Brunnermeier and Sannikov (2014) with additional components of exogenous growth prospects and belief heterogeneity in a spirit of Basak (2000).[9] For the effects of uncertainty shocks on macroeconomy, Di Tella (2017) investigates uncertainty shocks that drive balance sheet recessions even in cases when contracting on the macroeconomic state variable is possible. We allow agents to issue risk-free bonds only between themselves, and focus on the general equilibrium interaction between endogenous volatility and macroeconomic variables in the presence of belief heterogeneity about technological growth.

Our paper derives endogenous amplification of aggregate risks in general equilibrium, and see how optimism and pessimism, in addition to the technological differences across two groups, affects the amplification process. Heterogeneous beliefs play a crucial role in economic dynamics as they facilitate trades. We closely follows the literature that focuses on financial frictions, heterogeneous beliefs, and/or other deviations from the rational expectations, e.g., Harrison and Kreps (1978),[10] Detemple and Murthy (1997), Basak and Croitoru (2000), Basak (2000), Croitoru and Basak (2004), Gallmeyer and Hollifield (2008), and Maxted (2022).[11]

Caballero and Simsek (2020) provide a continuous-time *risk-centric* representation of the New-Keynesian model, based on which they analyze interactions among equilibrium asset prices, sentiments (i.e., optimism and pessimism), financial speculation, and macroeconomic outcomes when output is determined by aggre-

[7]See e.g., Giglio et al. (2019)

[8]We consider log-preference for all agents. The literature that uses risk-neutrality usually excludes short-sales for equilibrium to exist. Linearity pushes optimal solutions to boundaries, and it results in extreme behaviors. For example, in Geanakoplos (2010), all agents exposed to the risky asset default in the bad state.

[9]For our continuous-time model, we build on e.g., Basak and Cuoco (1998), He and Krishnamurthy (2011), and He and Krishnamurthy (2013)

[10]Harrison and Kreps (1978) assumes agents agree to disagree about their beliefs, and asset prices can exceed their fundamental value in that case. However, the underlying microfoundation for disagreement about the probability distribution of dividend stream is not provided.

[11]Maxted (2022) incorporates diagnostic expectations into a model with intermediaries based on He and Krishnamurthy (2013) in a tractable way, and finds interactions between diagnostic expectation and financial frictions generates both amplification (in the short-run) and reversal (in the long-run).

gate demand, in the presence of stochastic changes in exogenous volatility.[12] Belief disagreements matter as they induce optimists to speculate during normal times, which exacerbates crashes by reducing their wealth when the economy transitions to recessions. We focus more on the amplification of endogenous risks in the presence of different kinds of belief heterogeneity instead, and introduce two-way interactions between risk amplification and disagreements about asset return prospects.

Outline The remainder of the paper is organised as follows: Section 2 sets out the basic framework. Section 3 characterizes the equilibrium in an analytic way. Then, Section 4 provides simulation results and discusses our model's implications. Finally, Section 5 concludes.

2 The Model

We develop a continuous-time framework with two types of agents, based on which we study how heterogeneous beliefs about technological growth affect leverage choices, asset prices and the endogenous financial volatility, while the endogenous risk affects the degree of belief heterogeneity in asset returns. Our setting is analytical yet tractable, and incorporates exogenous technological growth and heterogeneous beliefs in a general equilibrium sense. Our model is built on e.g., Basak (2000) and Brunnermeier and Sannikov (2014).

2.1 Model Setup

We will begin with the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ which is endowed with a standard Brownian motion Z_t. We assume that $Z_0 = 0$, almost surely. All economic activity will be assumed to take place in the horizon $[0, \infty)$. Let

$$\mathcal{F}^Z(t) \triangleq \sigma\{Z_s; 0 \leq s \leq t\}, \forall t \in [0, T]$$

be the filtration generated by $Z(.)$ and let $\mathcal{N}$ denote the $\mathcal{P}$- null subsets of $\mathcal{F}^Z(T)$. We shall use the augmented filtration as follows:

$$\mathcal{F}(t) \triangleq \sigma\{\mathcal{F}^Z(t) \cup \mathcal{N}\}, \forall t \in [0, T]$$

.

One should interpret the $\sigma-$algebra $\mathcal{F}(t)$ as the information available to agents at time t, in the sense that if $\omega \in \Omega$ is the true state of nature and if $A \in \mathcal{F}(t)$, then all agents will know whether $\omega \in A$.

[12] Caballero and Simsek (2020) defined optimists as those who believe that the probability that the next recession comes is lower during the normal region.

We consider an economy with two agents, optimists and pessimists. Both types of agents can own capital, but the former are able to use capital in a more productive way.[13]

2.1.1 Technology

The aggregate amount of capital in the economy is denoted by K_t and capital owned by an individual agent i by k_t^i , where $t \in [0, \infty)$ indicates time. Physical capital k_t^O held by optimists produces output at rate:

$$y_t^O \triangleq \gamma_t^O k_t^O, \quad \forall t \in [0, \infty) \tag{1}$$

per unit of time, where γ_t^O is an exogenous productivity parameter, which evolves according to:[14]

$$\frac{d\gamma_t^O}{\gamma_t^O} \triangleq \alpha dt + \sigma dZ_t, \quad \forall t \in [0, \infty), \tag{2}$$

where dZ_t are exogenous aggregate standard Brownian shocks defined above. In a world without fiat money, output is modeled as a numeraire, and therefore, its price is normalized to one. Capital owned by individual optimists, with state space $\mathscr{F} \subseteq \mathbb{R}$, satisfies the following Ito's process:

$$\frac{dk_t^O}{k_t^O} \triangleq \left(\Lambda^O(\iota_t^O) - \delta^O \right) dt, \quad \forall t \in [0, \infty), \tag{3}$$

where ι_t^O is the portion of the generated output (i.e., $y_t^O = \gamma_t^O k_t^O$) used in creating new capital (i.e., the total amount of investment during the infinitesimal period $(t, t+dt)$ is given by $\iota_t^O \gamma_t^O{}_t k_t^O dt$). Function $\Lambda^O(\cdot)$, which satisfies $\Lambda^O(0) = 0$, $\Lambda^{O\prime}(0) = 1$, $\Lambda^{O\prime}(\cdot) > 0$, and $\Lambda^{O\prime\prime}(\cdot) < 0$, represents a standard investment technology with adjustment costs. In case there is no investment, capital managed by the optimist depreciates at rate δ^O. The concavity of $\Lambda^O(\cdot)$ represents investment formation friction, that can be interpreted as adjustment costs of converting output to new capital and vice versa.

Pessimists are less productive.[15] Capital managed by her, with state space $\mathscr{F} \subseteq \mathbb{R}$, produces the following output:

$$y_t^P \triangleq \gamma_t^P k_t^P, \quad \forall t \in [0, \infty), \tag{4}$$

[13] In Section 4, we consider the opposite case where pessimists use the capital in a more productive way.

[14] Therefore, unlike Brunnermeier and Sannikov (2014), we have stochastic growth which we model as exogenous (i.e., α and σ in (2) are exogenous). Later we normalize our economy by γ_t^O to make it stationary.

[15] Therefore, à la Brunnermeier and Sannikov (2014), optimists are financial experts while pessimists are households. In Section 4, we revisit this assumption and studies the opposite case where pessimists act as experts in our economy.

with γ_t^P is given by $\gamma_t^P = l \cdot \gamma_t^O \leq \gamma^O$, where $l \leq 1$, and evolves according to:

$$\frac{d\gamma_t^P}{\gamma_t^P} = \frac{d\gamma_t^O}{\gamma_t^O}, \quad \forall t \in [0, \infty). \tag{5}$$

In other words, optimists have proportionally higher productivity than pessimists, where the proportionality is given by $l \leq 1$, and productivity of two groups of agents grows at the same rate at every instant. Finally, capital owned by pessimists, which we denote by k_t^P follows:

$$\frac{dk_t^P}{k_t^P} \triangleq \left(\Lambda^P(\iota_t^P) - \delta^P\right) dt, \quad \forall t \in [0, \infty), \tag{6}$$

where in the same way as above, ι_t^P is a capital-owning pessimist's investment rate per unit of output. We assume that $\mathscr{F}$ satisfies the same conditions as for optimists and δ^P is the depreciation rate when capital is managed by pessimists. We assume $\Lambda^P(\iota_t^P) = l \cdot \Lambda^O(\iota_t^P)$ with $l < 1$ for simplicity.[16]

2.1.2 Preferences

Optimists and pessimists have preferences that are generally characterized by the instantaneous utility function $u^i(c_t^i) : \mathbb{R}_+ \to \mathbb{R}$, where $i \in \{O, P\}$, and they also have constant discount factors ρ^i. The consumption space defined above must also be square-integrable:

$$\int_0^\infty \left|c_t^i\right|^2 dt < \infty.$$

With $c^i \equiv \{c_t^i\}_{t=0}^\infty$, agents of type $i \in \{O, P\}$ want to maximize their expected lifetime utility function which is given by:

$$U(c^i) \triangleq \int_0^\infty e^{-\rho^i t} u^i(c_t^i) dt, \quad \forall t \in [0, \infty). \tag{7}$$

The utility function obeys the standard assumption summarized below.

Assumption 1 *The utility function $u^i : \mathbb{R}_+ \to \mathbb{R}$, is concave and continuously diffentiable. Also.*

$$u^{i'}(\cdot) \triangleq \frac{\partial u^i}{\partial c_t^i}(\cdot) > 0 \;\; for \;\; i \in \{O, P\}. \tag{8}$$

[16]Therefore, we effectively assume that both (i) pessimists' productivity in turning capital into output; and (ii) their productivity in turning output into capital are both lower than of optimists with the same proportionality $l \leq 1$. This is for tractability and even if we assume different proportionality for the two productivity measures, most results in this paper do not change qualitatively. Thus, we interchangeably call optimists 'experts' and pessimists 'households' similarly to Brunnermeier and Sannikov (2014).

2.1.3 Market for Capital

Both optimists and pessimists have the opportunity to trade physical capital in a competitive market. We denote the equilibrium market price of capital in terms of output by p_t with state space $\mathscr{F} \subseteq \mathbb{R}$, and assume that it satisfies the following endogenous Ito's process:

$$\frac{dp_t}{p_t} \triangleq \mu_t^p dt + \sigma_t^p dZ_t, \tag{9}$$

where μ_t^p and σ_t^p are drift and volatility of the capital price process (9), respectively. Based on the definition, capital k_t^O costs $p_t k_t^O$ for optimists. Note that, in equilibrium, p_t, μ_t^p and σ_t^p are determined endogenously.

2.1.4 Return to Capital

A straightforward application of Ito's lemma in (3) and (9) reveals that when an optimists hold k_t^O units of capital at price p_t, the total value of this capital (i.e., $p_t k_t^O$) evolves according to:

$$\frac{d(p_t k_t^O)}{p_t k_t^O} = \left(\Lambda^O(\iota_t) - \delta^O + \mu_t^p\right) dt + \sigma_t^p dZ_t. \tag{10}$$

Hence, the total return that experts earn from capital (per unit of wealth invested) is given by:

$$dr_t^{Ok} = \frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} dt + \left(\Lambda^O(\iota_t^O) - \delta^O + \mu_t^p\right) dt + \sigma_t^p dZ_t. \tag{11}$$

Similarly, a pessimist earns the return of

$$dr_t^{Pk} = \frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} dt + \left(\Lambda^P(\iota_t^P) - \delta^P + \mu_t^p\right) dt + \sigma_t^p dZ_t. \tag{12}$$

2.1.5 Beliefs

Agents of the two groups commonly observe their productivity at any instant, but have incomplete information on its exact dynamics. With equation (2) and equation (5), we know that γ_t^O and γ_t^P follow:

$$\frac{d\gamma_t^P}{\gamma_t^P} = \frac{d\gamma_t^O}{\gamma_t^O} = \alpha dt + \sigma dZ_t, \quad \forall t \in [0, \infty) \tag{13}$$

Optimists and pessimists observe the realization of their productivity, γ_t^O and γ_t^P, respectively. However, they do not know their exact processes: agents have incomplete information about the productivity process, represented by filtration $\mathcal{F}^i(t)$, with $i = O, P$, which is possibly different from the actual filtration $\mathcal{F}(t)$. We assume that they can both calculate σ from the quadratic variation of γ_t^O and γ_t^P, but they cannot elicit the exact value of α from the time-series of their productivity. Therefore, although optimists and pessimists have equivalent probability measures $\mathcal{P}^i$, $i = O, P$, respectively, also equivalent to the actual measure $\mathcal{P}$, they disagree about the expected productivity growth. We model a simple type of disagreement here: both types of agents are *dogmatic* and have a fixed bias in their beliefs about the drift of their productivity processes, in a similar manner to Yan (2008). Thus, optimists believe that their productivity γ_t^O follows:

$$\frac{d\gamma_t^O}{\gamma_t^O} = \alpha^O dt + \sigma dZ_t^O, \quad \forall t \in [0, \infty), \tag{14}$$

where α^O is possibly different from the true α, and Z_t^O is a Brownian motion related to their filtration $\mathcal{F}^O(t)$. Similarly, pessimists believe that their productivity γ_t^P evolves according to

$$\frac{d\gamma_t^P}{\gamma_t^P} = \alpha^P dt + \sigma dZ_t^P, \quad \forall t \in [0, \infty). \tag{15}$$

where α^P is possibly different from the true α, and Z_t^P is a Brownian motion related to their filtration $\mathcal{F}^P(t)$. From (13), and (14), we get the following consistency condition for optimists:

$$Z_t^O = Z_t - \frac{\alpha^O - \alpha}{\sigma} t. \tag{16}$$

In other words, optimists regard Z_t^O, not Z_t, as the Brownian motion driving the business cycle, while Z_t^O is in fact not a Brownian motion under the rational expectations. Likewise for pessimists, we get

$$Z_t^P = Z_t - \frac{\alpha^P - \alpha}{\sigma} t. \tag{17}$$

From (16) and (17), we obtain the following consistency condition.

$$Z_t^O = Z_t^P + \frac{\alpha^P - \alpha^O}{\sigma} t. \tag{18}$$

We focus mostly on the case where $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$[17] and call (18) the disagreement process. From (11) and (16), optimists think that the total return that they earn from capital (per unit of wealth invested) would be given by

$$dr_t^{Ok} = \frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} dt + \left(\Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^O - \alpha}{\sigma} \sigma_t^p \right) dt + \sigma_t^p dZ_t^O. \tag{19}$$

Likewise, pessimists believe that the total return that they earn from capital (per unit of wealth invested) is

[17] In Section 4, we also study cases where $\alpha^O < \alpha$ and $\alpha^P > \alpha$: i.e., when experts are pessimists and less productive households are optimists. Our analytical results

given by

$$dr_t^{Pk} = \frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} dt + \left(\Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^p \right) dt + \sigma_t^p dZ_t^P. \tag{20}$$

Two points regarding equation (19) and equation (20) are important to note: (i) with $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$, optimists (pessimists) believe that the expected capital gain they earn when investing in physical capital is higher (lower) than what is implied by rational expectations; (ii) the degree of irrationality and disagreement (i.e., $\frac{\alpha^i - \alpha}{\sigma} \sigma_t^p$ for $i \in \{O, P\}$) between two groups in terms of the expected capital gain becomes proportional to the endogenous capital price risk σ_t^p: thereby, a higher endogenous volatility σ_t^p raises the degree of belief heterogeneity. We believe that it is a reasonable specification: the more turbulent the market becomes, given sample points it is harder to elicit the exact process for the business cycle and financial markets, thereby it becomes more likely that many investors disagree about the expected capital gain that they would receive if invest in capital.[18,19]

2.2 Optimist's Consumption-Portfolio Problem

The non-monetary net worth w_t^O of an optimist who invests fraction x_t of her wealth in capital and consumes with the rate c_t^O evolves according to:

$$dw_t^O = x_t w_t^O dr_t^{Ok} + (1 - x_t) r_t w_t^O dt - c_t^O dt, \tag{21}$$

where r_t is the risk-free interest rate prevailing in the economy. Note that r_t is an equilibrium object to be determined endogenously. x_t represents the share of wealth that optimists invest in capital. It will turn out later that in most cases optimists use greater-than-1 leverage (i.e., $x_t > 1$).

Formally, each optimist solves

$$\max_{i_t \geq 0, x_t \geq 0, c_{t,}^O \geq 0} \mathbb{E}_0^O \left[\int_0^\infty e^{-\rho^O t} u^O \left(c_t^O \right) dt \right], \tag{22}$$

subject to the solvency constraint $w_t^O \geq 0$ and the dynamic budget constraint (21). In optimization (22), the expectation operator $\mathbb{E}^O$ means that optimists believe dZ_t^O, not dZ_t, is the true Brownian motion driving their own filtration $\mathcal{F}^O(t)$. Therefore in characterizing (21), they use (19) for the capital return process dr_t^{Ok} instead of (11) with the true dZ_t. Note that we assume away short-sale for capital, thereby $x_t \geq 0$.

[18]This is a one side of the two-way interaction between belief heterogeneity and amplified volatility. In Section 4, we illustrate that our model generates the other direction: i.e., our heterogeneous belief specification amplifies the endogenous volatility σ_t^p during the crisis regime.

[19]Compared with Brunnermeier and Sannikov (2014) where more productive experts have higher dividend yields given the same investment amount, our framework features an additional channel where more productive experts are more optimistic about the future return through higher capital gain in the case of $\alpha^O \geq \alpha$.

2.3 Pessimist's Consumption-Portfolio Problem

In a similar way to optimists' problem, the non-monetary net worth w_t^P of pessimists who invest fraction $\underline{x}_t$ of their wealth in capital and consume with the rate c_t^P would follow

$$dw_t^P = \underline{x}_t w_t^P dr_t^{Pk} + w_t^P (1 - \underline{x}_t) r_t dt - c_t^P dt \tag{23}$$

Formally, each pessimist solves

$$\max_{i_t \geq 0, \underline{x}_{t,} \geq 0, c^P_{t,} \geq 0} \mathbb{E}_0^P \left[\int_0^\infty e^{-\rho^P t} u\left(c_t^P\right) dt \right], \tag{24}$$

subject to the solvency constraint $w_t^P \geq 0$ and the dynamic budget constraints (23). Likewise, in optimization (24), the expectation operator $\mathbb{E}^P$ means that pessimists believe dZ_t^P, not dZ_t, is the true Brownian motion driving their own filtration $\mathcal{F}^P(t)$. Therefore in characterizing (23), they use (20) for the capital return process dr_t^{Pk} instead of (12) with the true dZ_t. Note that we assume away short-sale for capital, thereby $\underline{x}_t \geq 0$.

2.4 Equilibrium

Intuitively, an equilibrium with full information is characterized by a map from shock histories $\{Z_S, s \in [0,t]\}$, to prices p_t and asset allocations such that, given prices, agents maximize their expected utilities[20]and markets clear.[21] To define an equilibrium more formally, we denote the set of optimists to be an interval $I = [0,1]$ and index individual optimists (i.e., experts) by $i \in I = [0,1]$. Similarly, we denote the set of pessimists by $J = (1,2]$ with index $j \in J = (1,2]$. A similar logic applies to our framework where two groups (i.e., optimists and pessimists) both deviate from the rational expectations with the perceived shocks Z_t^i, $i \in \{O, P\}$. Each group of agents has their own expectation operator $\mathbb{E}^i$, $i \in \{O, P\}$ in this case as already shown in (22) and (24). Note that in equilibrium, every agent in the same group (i.e., optimists or pessimists) chooses the same consumption and portfolio decisions.

We now proceed by stating the market clearing conditions and finally defining the equilibrium.

2.5 Market Clearing

The three markets that must clear in equilibrium at any instant are the capital, commodity and debt markets.

[20]Optimists solve optimization (22) subject to their dynamic budget constraint (21) and the solvency $w_t^O \geq 0$ while pessimists solve optimization (24) subject to their dynamic budget constraint (23) and the solvency $w_t^P \geq 0$.

[21]The physical capital, output, and debt markets must clear in equilibrium.

2.5.1 Capital Market

The total amount of capital demanded by optimists and pessimists should be equal to the aggregate supply of capital in the economy: i.e.,

$$\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj = K_t, \ \ \forall t \in [0, \infty) \tag{25}$$

where K_t is the total supply of capital. The total supply of capital in the model is not fixed as both optimists and pessimists create new capital through their investments. The following (26) describes the evolution of the total supply of capital:

$$dK_t \triangleq \left(\int_0^1 \left(\Lambda^O \left(\iota_t^i \right) - \delta^O \right) k_t^i di + \int_1^2 \left(\Lambda^P \left(\underline{\iota}_t^j \right) - \delta^P \right) \underline{k}_t^j dj \right) dt, \ \ \forall t \in [0, \infty) \tag{26}$$

2.5.2 Good Market

Whatever is produced and not invested, has to be consumed. That is,[22]

$$\int_0^1 k_t^i \left(\gamma_t^O - \iota_t^i \gamma_t^O \right) di + \int_1^2 k_t^j \left(\gamma_t^P - \iota_t^j \gamma_t^P \right) dj = \int_0^1 c_t^i di + \int_1^2 c_t^j dj, \ \ \forall t \in [0, \infty) \tag{27}$$

2.5.3 Debt Market

Debt market clearing condition says that the value of the debt that the optimists receive should equal to the value of loans that the pessimists extend, namely,

$$\int_0^1 \left(w_t^i - p_t k_t^i \right) di + \int_1^2 \left(w_t^j - p_t k_t^j \right) dj = 0 \tag{28}$$

By defining all three markets, we are in a position to define the economy's equilibrium.

Definition 1 *The economy's equilibrium consists of stochastic processes of (i) the price of capital p_t; (ii) interest rate r_t; (iii) investment and consumption, i.e. $\{(k_t^i, \iota_t^i, c_t^i), t \geq 0, i \in [0,1]\}$ for optimists and $\left\{ \left(k_t^j, \iota_t^j, c_t^j \right), t \geq 0, j \in (1,2] \right\}$ for pessimists. These processes should satisfy the following three conditions:*

1. Given their perceived Brownian motion Z_t^O and Z_t^P, respectively, optimists $i \in [0,1]$ solve optimization (22) subject to their dynamic budget constraint (21) and the solvency $w_t^i \geq 0$ while pessimists $j \in (1,2]$ solve optimization (24) subject to their dynamic budget constraint (23) and the solvency $w_t^j \geq 0$.

2. Capital (i.e., (25) and (26)), consumption (i.e., (27)), and debt (i.e., (28)) markets clear.

3. Consistency condition holds. In other words, the disagreement process (18) must hold.

[22] In equation (27), we use the fact that the investment amount is given by $i_t^i \gamma_t^i k_t^i$ for $i \in [0,2]$.

3 Characterization for Equilibrium

In this section, we will discuss how to find the equilibrium price p_t, both optimists and pessimists' consumption decisions, the risk-free interest rates r_t as well as the endogenous drift μ_t^p and volatility σ_t^p of the capital price p_t's process, given the history of perceived shock processes $\{Z_s^1, Z_s^2, 0 \leq s \leq t\}$.

We first start with some definitions.

Definition 2 *The entire wealth of both optimists and pessimists is given by summing up their individual wealth respectively, that is,*[23]

$$W_t = \int_0^1 w_t^i di \qquad \underline{W}_t = \int_1^2 w_t^j dj, \quad \forall t \in [0, \infty)$$

Observe that the capital market clearing condition (25) and the debt market clearing condition (28) become,

$$\int_0^1 x_t^i w_t^i di + \int_1^2 \underline{x}_t^j w_t^j dj = p_t K_t, \quad \forall t \in [0, \infty), \tag{29}$$

where K_t is the supply of aggregate capital that follows the process (26) and

$$W_t + \underline{W}_t = p_t K_t. \tag{30}$$

And the good market equilibrium condition (27) can be written as

$$\int_0^1 \frac{x_t^i w_t^i}{p_t} \left(\gamma_t^O - \iota_t^O \gamma_t^O \right) di + \int_1^2 \frac{\underline{x}_t^j w_t^j}{p_t} \left(\gamma_t^P - \underline{\iota}_t^P \gamma_t^P \right) dj = \int_0^1 c_t^i di + \int_1^2 c_t^j dj \tag{31}$$

We now provide the definition of the state variable of our economy, which is the wealth share optimists possess. By representing each variable in the model in terms of optimists' wealth share which is bounded between 0 and 1, we can fully characterize the equilibrium price and quantity variables in a similar manner to Brunnermeier and Sannikov (2014).

The proportion of wealth that optimists possesses which we denote by η_t is given by,

$$\eta_t = \frac{W_t}{W_t + \underline{W}_t}. \tag{32}$$

We further postulate that the dynamics of η_t, which is endogenous, evolve with the process:

$$\frac{d\eta_t}{\eta_t} = \mu_t^\eta(\eta_t)dt + \sigma_t^\eta(\eta_t)dZ_t. \tag{33}$$

[23] Actually, $w_t^i = w_t^O$ in equation (21) for $i \in [0, 1]$. Likewise, $w_t^j = w_t^P$ in equation (23) for $j \in (1, 2]$.

Based on (30) and (32), we observe that η_t can be written as

$$\eta_t = \frac{W_t}{p_t K_t}. \tag{34}$$

The entire dynamics of the model will be driven by η_t in our Markov equilibrium in a similar way to Brunnermeier and Sannikov (2014). Since every agent of the same group solves the same portfolio problem, let $x_t = x_t^i$ for $i \in [0,1]$ in equilibrium. Evidently, in equilibrium we have,

$$x_t \leq \frac{1}{\eta_t}, \tag{35}$$

which translates to the fact that the maximum leverage that optimists can obtain is bounded above by $\frac{1}{\eta_t}$. In this paper we identify two regions: the first one is when (35) binds and the second one is when leverage is strictly less than $\frac{1}{\eta_t}$, i.e., when (35) does not bind. We call the first the *normal* region and the second one the *crisis* region. In other words, in the *normal* region, all the physical capital will be owned by optimists, while in the *crisis* regime, some of the capital must be purchased by pessimists.

3.1 Internal Investment

Hereafter, we proceed by defining the investment functions $\Lambda^O(\cdot)$ that optimists use as follows:

$$\Lambda^O(i_t^O) = \frac{1}{k}\left(\sqrt{1+2ki} - 1\right), \ \ \forall t \in [0, \infty), \ \ i \in \{O, P\}. \tag{36}$$

which satisfies all standard assumptions discussed in Section 2.1.1. We do not allow for disinvestment, thus $i_t^O \geq 0$ for $i \in [0,1]$. We use the mathematical form in (36) of $\Lambda^O(i_t^i)$ for simplicity and acknowledge that our results do not change qualitatively even if we use different forms for $\Lambda^O(\cdot)$ that satisfy conditions in Section 2.1.1. Similarly as we already explained in Section 2.1.1, we define the internal investment function $\Lambda^P(i_t^P)$ that pessimists use as

$$\Lambda^P(i_t^P) = l \cdot \Lambda^O(i_t^P). \tag{37}$$

From now on, we express our equilibrium with the following normalized asset price:

Definition 3 *The normalized asset price q_t is defined as $q_t \equiv \dfrac{p_t}{\gamma_t^O}$.*

which can be interpreted also as the price-earnings ratio of physical capital. It turns out that when we write our model in terms of q_t instead of p_t, we can characterize the stationary equilibrium, as we have exogenous growth in our economy.

3.2 Solving an Optimist's Consumption-Portfolio Problem

Each optimist $i \in [0,1]$ solve optimization (22) subject to her dynamic budget constraint (21) and the solvency $w_t^i \geq 0$ We now fully characterize her optimal consumption c_t^{O*}, her optimal investment i_t^{O*}, and the equilibrium risk free interest rate r_t^*. We also assume that all the optimists have the same logarithmic utility function, i.e., $u^O(c_t^O) = \log c_t^O$ for mathematical tractability.

Proposition 1 *Assume that all optimists have the same logarithmic utility, i.e.,* $u^O(c_t^O) = \log c_t^O$*. Then:*
(i) The optimal consumption c_t^{O*} *is given by:*

$$c_t^{O*} = \rho^O w_t^O.$$

(ii) The equilibrium interest rate r_t^* *is given by:*

$$r_t^* = \left(\frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma} \sigma_t^p \right) - x_t \left(\sigma_t^p\right)^2,$$

where x_t *is her optimal portfolio choice (i.e., leverage) as defined in (21).*
(iii) The optimal investment rate ι_t^{O*} *is given by:*

$$i_t^{O*}(q_t) = \frac{q_t^2 - 1}{2k}.$$

Proof. Since optimists believe that dZ_t^O, not dZ_t, is the true Brownian motion driving the business cycle, they believe the wealth share η_t follows:

$$\frac{d\eta_t}{\eta_t} = \left(\mu_t^\eta(\eta_t) + \frac{\alpha^O - \alpha}{\sigma} \sigma_t^\eta(\eta_t) \right) dt + \sigma_t^\eta(\eta_t) dZ_t^O, \tag{38}$$

which is consistent with the true η_t process in (33). Based on Merton (1971), we conjecture her value function $V(\cdot)$ will depend on her own wealth w_t^O and the aggregate wealth share of optimists η_t with the following form:

$$V\left(w_t^O, \eta_t\right) = \frac{\log w_t^O}{\rho^O} + f\left(\eta_t\right). \tag{39}$$

Based on (21) and (38), the Hamiltonian-Jacobi-Bellman (HJB) equation for an individual optimist's problem can be written as[24]

[24] We use the following relation from equation (19):

$$\mathbb{E}_t^O\left(dr^{Ok}\right) = \left(\frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma} \sigma_t^p = \frac{1 - \iota_t^O}{q_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma} \sigma_t^p \right) dt.$$

$$\rho^O V(\cdot) = \max_{x_t, c_t^O, i_t^O} \log c_t^O + \left[w_t^O \left(x_t \left(\frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma} \sigma_t^p \right) + (1 - x_t) r_t \right) - c_t^O \right] \frac{dV_t}{dw_t^O}(\cdot)$$
$$+ \frac{\left(x_t w_t^O \sigma_t^p\right)^2}{2} \frac{d^2 V_t}{d\left(w_t^O\right)^2}(\cdot) + \left(\eta_t \left(\mu_t^\eta(\eta_t) + \frac{\alpha^O - \alpha}{\sigma} \sigma_t^\eta(\eta_t) \right) \right) \frac{dV}{d\eta_t}(\cdot) + \frac{\left(\eta_t \sigma_t^\eta(\eta_t)\right)^2}{2} \frac{d^2 V}{d\eta_t^2}(\cdot).$$

The first order condition with respect to c_t^O are given by:

$$\frac{1}{c_t^{O*}} = \frac{dV_t}{dw_t^O}\left(w_t^O, \eta_t\right) = \frac{1}{\rho^O w_t^O}, \tag{40}$$

which gives $c_t^{O*} = \rho^O w_t^O$ at optimum. As every aggregate variable including q_t, i_t^O, μ_t^p, σ_t^p, x_t, and r_t will be expressed as functions of the aggregate wealth share η_t, Merton (1971) justifies our choice of value function in (39). The first-order condition with respect to x_t gives the optimal portfolio choice given in (ii).

To prove (iii), we observe that investment appears only in a intra-temporal manner. Thus the corresponding optimization problem is static and satisfies the first-order condition: $\Lambda^{O\prime}(i_t) = q_t^{-1}$ (i.e., marginal Tobin's q). With the form of $\Lambda^O(i_t)$ from Section 3.1, we finally obtain:

$$i_t^{O*}(q_t) = \frac{q_t^2 - 1}{2k}. \tag{41}$$

■

Note from (ii) that given i_t^O, μ_t^p, σ_t^p, and x_t, the equilibrium interest rate r_t^* increases due to the distorted belief of optimists when $\alpha^O > \alpha$. If optimists believe that the expected capital gain they earn when investing in capital will be higher, they will try to get more loans from pessimists, raising the equilibrium interest rate r_t^*.

3.3 Solving a Pessimist's Consumption-Portfolio Problem

Each pessimist $j \in (1, 2]$ optimizes her lifetime utility (24) subject to his wealth evolution (23) and the solvency $w_t^j \geq 0$. The next proposition solves her optimal choices for consumption c_t^{P*}, leverage $\underline{x}_t$, and investment ι_t^{P*}.

Proposition 2 *Assume that all pessimists have the same logarithmic utility, i.e., $u^P(c_t^P) = \log c_t^P$. Then:*
(i) The optimal consumption c_t^{P*} is given by:

$$c_t^{P*} = \rho^P w_t^P.$$

(ii) The equilibrium interest rate r_t^* is given by:

$$r_t^* = \left(\frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^p \right) - \underline{x}_t \left(\sigma_t^p\right)^2,$$

where $\underline{x}_t$ is his optimal portfolio choice (i.e., leverage) as defined in (23).

(iii) The optimal investment rate ι_t^{P*} is given by:

$$i_t^{P*}(q_t) = \frac{q_t^2 - 1}{2k},$$

which is the same as i_t^{O*}. Therefore, optimists and pessimists have the same rate of investment, i.e., $i_t^{O*}(\cdot) = i_t^{P*}(\cdot)$.

Proof. Since pessimists believe that dZ_t^P, not dZ_t, is the true Brownian motion driving the business cycle, they believe the wealth share η_t follows:

$$\frac{d\eta_t}{\eta_t} = \left(\mu_t^\eta(\eta_t) + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^\eta(\eta_t) \right) dt + \sigma_t^\eta(\eta_t) dZ_t^P, \tag{42}$$

which is consistent with the true η_t process in (33). Based on Merton (1971), we conjecture her value function $\underline{V}(\cdot)$ will depend on her own wealth w_t^P and the aggregate wealth share of optimists η_t with the following form:

$$\underline{V}\left(w_t^P, \eta_t\right) = \frac{\log w_t^P}{\rho^P} + \underline{f}\left(\eta_t\right). \tag{43}$$

Based on (23) and (42), the Hamiltonian-Jacobi-Bellman (HJB) equation for an individual optimist's problem can be written as[25]

$$\begin{aligned} \rho^P \underline{V}(\cdot) = \max_{\underline{x}_t, c_t^P, i_t^P} \log c_t^P + \left[w_t^P \left(\underline{x}_t \left(\frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^p \right) + (1 - \underline{x}_t) r_t \right) - c_t^P \right] \frac{d\underline{V}_t}{dw_t^P}(\cdot) \\ + \frac{\left(\underline{x}_t w_t^P \sigma_t^p\right)^2}{2} \frac{d^2 \underline{V}_t}{d\left(w_t^P\right)^2}(\cdot) + \left(\eta_t \left(\mu_t^\eta(\eta_t) + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^\eta(\eta_t) \right) \right) \frac{d\underline{V}}{d\eta_t}(\cdot) + \frac{\left(\eta_t \sigma_t^\eta(\eta_t)\right)^2}{2} \frac{d^2 \underline{V}}{d\eta_t^2}(\cdot). \end{aligned}$$

The first order condition with respect to c_t^P are given by:

$$\frac{1}{c_t^{P*}} = \frac{d\underline{V}_t}{dw_t^P}\left(w_t^P, \eta_t\right) = \frac{1}{\rho^P w_t^P}, \tag{44}$$

which gives $c_t^{P*} = \rho^P w_t^P$ at optimum. As every aggregate variable including q_t, i_t^P, μ_t^p, σ_t^p, $\underline{x}_t$, and r_t will be

[25] We use the following relation from equation (20):

$$\mathbb{E}_t^P\left(dr^{Pk}\right) = \left(\frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^p = l \cdot \frac{1 - \iota_t^P}{q_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^p \right) dt.$$

expressed as functions of the aggregate wealth share η_t, Merton (1971) justifies our choice of value function in (43). The first-order condition with respect to x_t gives the optimal portfolio choice given in (ii).

To prove (iii), we observe that investment appears only in a intra-temporal manner. Thus the corresponding optimization problem is static and satisfies the first-order condition: $\Lambda^{P\prime}(i_t) = l \cdot q_t^{-1}$. With equation (37) from Section 3.1 that $\Lambda^P(\cdot) = l \cdot \Lambda^O(\cdot)$, we finally obtain:

$$i_t^{P*}(q_t) = \frac{q_t^2 - 1}{2k}, \tag{45}$$

which is the same as $i_t^{O*}(q_t)$ in (41).

■

Finally, in order to conclude with the characterization of equilibrium, we need to derive the evolution of the state variable η_t. First, we define the fraction of physical capital held by optimists and pessimists by

$$\psi_t \equiv \frac{\int_0^1 k_t^i di}{K_t} = \frac{k_t^O}{K_t}, \;\; 1 - \psi_t = \frac{\int_1^2 k_t^j dj}{K_t} = \frac{k_t^P}{K_t}, \tag{46}$$

where the aggregate capital K_t follows the process (26). Then leverages of two groups of agents, i.e., x_t and $\underline{x}_t$, respectively, can be also characterized by the following proposition.

Proposition 3 *In equilibrium, leverages of two groups of agents, i.e., x_t and $\underline{x}_t$ are given by*

$$x_t = \frac{\psi_t}{\eta_t} \qquad \underline{x}_t = \frac{1 - \psi_t}{1 - \eta_t} \tag{47}$$

Proof. It follows immediately from the definitions of wealth share and capital share, i.e., (34) and (46).

■

Thus the evolution of the proportion of wealth held by optimists is established in the following Proposition 4.

Proposition 4 *The evolution of optimists' wealth relative to the entire economy is given by*

$$\frac{d\eta_t}{\eta_t} = \mu^\eta(\eta_t)dt + \sigma^\eta(\eta_t)dZ_t \tag{48}$$

where

$$\mu^\eta(\eta_t) = \left(\frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p\right)^2 - \frac{\psi_t - \eta_t}{\eta_t}\frac{\alpha^O - \alpha}{\sigma}\sigma_t^p + \frac{1 - \iota_t^O}{q_t} + (1 - \psi_t)\left(\delta^P - \delta^O\right) + (1 - l)\,(1 - \psi_t)\,\Lambda^O(i_t^O) - \rho^O,$$

and

$$\sigma^{\eta}(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t} \sigma_t^p.$$

Proof. Optimists' aggregate wealth W_t defined in Definition 2 would evolve with the process

$$dW_t = r_t W_t dt + \psi_t p_t K_t \left(dr_t^{Ok} - r_t dt \right) - c_t^O dt \tag{49}$$

where we use $c_t^i = c_t^O$ for $i \in [0,1]$ for all optimists.[26] By applying Ito's quotient rule on (34),[27], we have

$$\frac{d\eta_t}{\eta_t} = \frac{dW_t}{W_t} - \frac{d\left(p_t K_t\right)}{p_t K_t} + \left(\frac{d\left(p_t K_t\right)}{p_t K_t} \right)^2 - \frac{dW_t}{W_t} \frac{d\left(p_t K_t\right)}{p_t K_t}. \tag{50}$$

In addition, from the process (26) of the aggregate capital K_t, we obtain

$$\begin{aligned} \frac{1}{K_t}\frac{dK_t}{dt} &= \left(\Lambda^O \left(i_t^O \right) - \delta^O \right) \psi_t + \left(\Lambda^P \left(i_t^P \right) - \delta^P \right) (1 - \psi_t) \\ &= \left(\Lambda^O \left(i_t^O \right) - \delta^O \right) - \left(\delta^P - \delta^O \right) (1 - \psi_t) - (1 - \psi_t)(1 - l) \Lambda^O \left(i_t^O \right), \end{aligned} \tag{51}$$

where we used the property that $i_t^O = i_t^P$ in equilibrium as seen in (41) and (45). Applying Ito's product rule to the price process (9) and the capital process (51), and comparing with (19), we obtain

$$\frac{d\left(p_t K_t\right)}{p_t K_t} = dr_t^{Ok} - \frac{1 - \iota_t^O}{q_t} dt - \left(\delta^P - \delta^O \right) (1 - \psi_t)\, dt - (1 - \psi_t)(1 - l) \Lambda^O \left(i_t^O \right) dt. \tag{52}$$

Since $\mathbb{E}_t^O \left(dr_t^{Ok} \right) - r_t = x_t \left(\sigma_t^p \right)^2$ from the optimists' optimal portfolio decision à la Merton (1971), and from the fact that

$$\mathbb{E}_t^O \left(dr_t^{Ok} \right) = \mathbb{E}_t \left(dr_t^{Ok} \right) + \frac{\alpha^O - \alpha}{\sigma} \sigma_t^p dt, \tag{53}$$

where $\mathbb{E}_t$ is the expectation operator corresponding to the rational expectations, we can finally plug in (49), (52), and (53) into (50) and obtain

$$\begin{aligned} \frac{d\eta_t}{\eta_t} &= \left(\frac{\psi_t - \eta_t}{\eta_t} \sigma_t^p \right)^2 dt - \frac{\psi_t - \eta_t}{\eta_t} \frac{\alpha^o - \alpha}{\sigma} \sigma_t^p dt + \frac{1 - \iota_t^O}{q_t} dt + (1 - \psi_t) \left(\underline{\delta}^P - \delta^O \right) dt \\ &\quad + (1 - l)(1 - \psi_t) \Lambda^O \left(i_t^O \right) dt - \rho^O dt + \frac{\psi_t - \eta_t}{\eta_t} \sigma_t^p dZ_t, \end{aligned} \tag{54}$$

[26] We know $c_t^O = \rho^O W_t$ holds at optimum from equation (40).

[27] Ito's quotient rule states that:

$$\frac{d\left(XY^{-1}\right)}{XY^{-1}} = \frac{dX}{X} - \frac{dY}{Y} + \left(\frac{dY}{Y} \right)^2 - \frac{dX}{X} \frac{dY}{Y}.$$

where we used $c_t^O = \rho^O W_t$ in the equilibrium. From (54), we finally obtain

$$\mu^\eta(\eta_t) = \left(\frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p\right)^2 - \frac{\psi_t - \eta_t}{\eta_t}\frac{\alpha^O - \alpha}{\sigma}\sigma_t^p + \frac{1-\iota_t^O}{q_t} + (1-\psi_t)\left(\delta^P - \delta^O\right) + (1-l)(1-\psi_t)\Lambda^O(i_t^O) - \rho^O, \tag{55}$$

and

$$\sigma^\eta(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p. \tag{56}$$

■

Now by knowing the drift μ_t^η and the volatility σ_t^η of our state variable η, we calculate the the drift μ_t^p and the volatility σ_t^p of the price of capital p_t.

Proposition 5 *In the Markov equilibrium where q_t is a function of our state variable η_t with $q_t = q(\eta_t)$, the drift μ_t^p of the price of capital p_t is given by*

$$\mu_t^p = \alpha + \frac{q'(\eta_t)}{q_t}\mu^\eta(\eta_t)\eta_t + \frac{1}{2}\sigma^\eta(\eta_t)\eta_t\frac{q''(\eta_t)}{q(\eta_t)} + \sigma\sigma^\eta(\eta_t)\eta_t\frac{q'(\eta_t)}{q(\eta_t)}, \tag{57}$$

where $\sigma_t^\eta(\eta_t)$ is given by equation (56) and the volatility of the price of capital, σ_t^p, is given by

$$\sigma_t^p = \frac{\sigma}{1 - (x_t\eta_t - \eta_t)\dfrac{q'(\eta_t)}{q(\eta_t)}}. \tag{58}$$

Proof. Applying Ito's lemma to the Markov relation $q_t = q(\eta_t)$, we obtain

$$dq_t = q'(\eta_t)\,d\eta_t + \frac{1}{2}q''(\eta_t)(d\eta_t)^2. \tag{59}$$

Plugging in equation (48) to (59) and using $(d\eta_t)^2 = \eta_t^2\sigma^\eta(\eta_t)^2 dt$, we have

$$\frac{dq_t}{q_t} = \left(\eta_t\mu^\eta(\eta_t)\frac{q'(\eta_t)}{q(\eta_t)} + \frac{\eta_t^2\sigma^\eta(\eta_t)^2}{2}\frac{q''(\eta_t)}{q(\eta_t)}\right)dt + \eta_t\sigma^\eta(\eta_t)\frac{q'(\eta_t)}{q(\eta_t)}dZ_t. \tag{60}$$

Now, with the definition of normalized asset price (i.e., price-earnings ratio) $q_t = \frac{p_t}{\gamma_t^O}$, we also have:

$$\frac{dq_t}{q_t} = \left(\mu_t^p - \alpha + \sigma^2 - \sigma_t^p\sigma\right)dt + \left(\sigma_t^p - \sigma\right)dZ_t. \tag{61}$$

Comparing (60) and (61) yields:

$$\sigma_t^p = \sigma + \eta_t\sigma^\eta(\eta_t)\frac{q'(\eta_t)}{q(\eta_t)}, \tag{62}$$

and

$$\mu_t^p = \alpha + \eta_t \mu^\eta(\eta_t) \frac{q'(\eta_t)}{q(\eta_t)} + \frac{\eta_t^2 \sigma^\eta(\eta_t)^2}{2} \frac{q''(\eta_t)}{q(\eta_t)} + \sigma \sigma_t^\eta \eta_t \frac{q'(\eta_t)}{q(\eta_t)}. \tag{63}$$

From (56) in Proposition 4, we have that

$$\sigma^\eta(\eta_t) = \frac{x_t \eta_t - \eta_t}{\eta_t} \sigma_t^p, \tag{64}$$

where we used $x_t \eta_t = \psi_t$ from (47). Therefore, by (62) and (64) we have

$$\sigma_t^p = \frac{\sigma}{1 - (x_t \eta_t - \eta_t) \dfrac{q'(\eta_t)}{q(\eta_t)}}. \tag{65}$$

■

3.4 Price of Capital and Optimal Leverages

We derive a closed formed solution for leverage and a first order differential equation for the price of capital. The following theorems summarise the main results of the paper.

Proposition 6 *The equilibrium domain consists of sub-intervals $[0, \eta^\psi)$, where $\psi(\eta) < 1$, and $[\eta^\psi, 1]$ where $\psi(\eta) = 1$. The capital price function $q(\eta)$ in our Markov equilibrium satisfies:*

$$q(0) = l \cdot \frac{1 - i(q(0))}{\rho^P} \tag{66}$$

and

$$q(\eta) = \frac{1 - i(q(\eta))}{\rho^P(1-\eta) + \rho^O \eta} \quad on \quad [\eta^\psi, 1] \tag{67}$$

The following procedures can be used to compute $\psi(\eta)$ and $q'(\eta)$ from $(\eta, q(\eta))$ on $(\eta^\psi, 0)$:

(i) Find ψ that satisfies

$$\left(\rho^P(1-\eta) + \rho^O \eta\right) q(\eta) = \psi + (1-\psi) l - i(q)\left(\psi + (1-\psi) l\right) \tag{68}$$

(ii) Compute $q(\eta)$ where $q(\eta)$ is given by the solution of the equation:

$$\frac{(1-l)(1-i(q(\eta)))}{q(\eta)} + (1-l)\Lambda^O(i(q(\eta))) + \delta^P - \delta^O + \frac{\alpha^O - \alpha^P}{\sigma} \sigma^p(\eta) = \left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right) \sigma^p(\eta)^2, \tag{69}$$

where $\sigma^p(\eta)$ is given by (65). From (69), $\sigma^p(\eta)$ can also be expressed as:

$$\sigma^p(\eta) = \frac{\frac{\alpha^O - \alpha^P}{\sigma} + \sqrt{\left(\frac{\alpha^O - \alpha^P}{\sigma}\right)^2 + 4\left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right)\left(\frac{(1-l)(1-i(q))}{q(\eta)} + (1-l)\Lambda^O(i(q)) + \delta^P - \delta^O\right)}}{2\left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right)}, \tag{70}$$

where $q = q(\eta)$.

Proof. From the good market equilibrium condition (27), we have

$$k_t^O\left(\gamma_t^O - \iota_t^O\gamma_t^O\right) + k_t^P\left(\gamma^P - \iota_t^P\gamma^P\right) = c_t^O + c_t^P, \quad \forall t \in [0, \infty). \tag{71}$$

Observing that $c_t^O = \rho^O w_t^O$ and $c_t^P = \rho^P w_t^P$ in equilibrium, we can divide (71) by $p_t K_t$ and obtain

$$\left(\rho^P(1-\eta) + \rho^O\eta\right) q = \psi + (1-\psi)l - i(q)\left(\psi + (1-\psi)l\right). \tag{72}$$

Since $\mathbb{E}_t^O\left(dr_t^{Ok}\right) - r_t = x_t\left(\sigma_t^p\right)^2$ from the optimists' optimal portfolio decision, we obtain

$$\frac{1-i(q(\eta))}{q(\eta)} + \Lambda^O(i(q(\eta))) - \delta^O + \frac{\alpha^O - \alpha}{\sigma}\sigma^p(\eta) - r_t = \left(\frac{\psi}{\eta}\right)\sigma^p(\eta)^2. \tag{73}$$

For pessimists, it must be the case where

$$l \cdot \frac{1-i(q(\eta))}{q(\eta)} + l \cdot \Lambda^O(i(q(\eta))) - \delta^P + \frac{\alpha^P - \alpha}{\sigma}\sigma^p(\eta) - r_t \le \left(\frac{1-\psi}{1-\eta}\right)\sigma^p(\eta)^2. \tag{74}$$

with equality when $\psi < 1$. Finally, by subtracting (74) from (73), we get

$$\frac{(1-l)(1-i(q(\eta)))}{q(\eta)} + (1-l)\Lambda^O\left(i(q(\eta))\right) + \delta^P - \delta^O + \frac{\alpha^O - \alpha^P}{\sigma}\sigma^p(\eta) = \left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right)\sigma^p(\eta)^2, \tag{75}$$

when $\psi_t < 1$, i.e., pessimists hold some physical capital.

∎

4 Analysis

We solve our model numerically in a similar way to Brunnermeier and Sannikov (2014). For our numerical solutions, we study two cases: (i) when more productive experts are optimists and less productive households are pessimists, i.e., $\alpha^O \ge \alpha$ and $\alpha^P \le \alpha$; (ii) the opposite case where experts are pessimists and households are optimists, i.e., $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$.[28,29] We use the following parametrization:

	l	δ^O	δ^P	ρ^O	ρ^P	χ	σ	k	α
Values	0.6	0	0	0.09	0.05	1	0.1	18	0.05

Table 1: Parameterization

4.1 Results

Figure 1 shows the normalized capital price (i.e., price-earnings ratio) q_t as a function of our state variable η_t. Figure 1a corresponds to the original case where more productive experts are optimists (i.e., $\alpha^O \geq \alpha$), while figure 1b represents the opposite case where less productive households are optimists (i.e., $\alpha^P \geq \alpha$). As we already discussed in Section 3.4, two regions are identified. The first region, is when more productive experts manage the entire aggregate capital. We call this region (i.e., when $\psi_t = 1$) the *efficient* region. When experts' wealth share η_t holds reach a level η^ψ defined in Proposition 6, so that the economy is at the efficient region, the price of capital q_t reaches its peak regardless of (α^O, α^P). After then, it starts declining slowly as η_t rises: wealth share of households goes down, thereby the leverage ratio experts can attain from households falls as seen in Figure 2, lowering their demand for physical capital. When η_t is lower than the threshold η^ψ, the economy is in the *inefficient* region. In this region, the less productive households pessimists participate in the capital market as marginal investors and will hold a proportion of capital, lowering the equilibrium capital price as η_t falls.

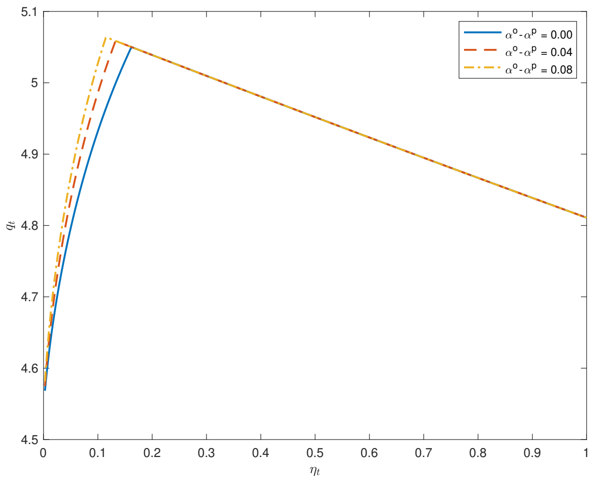
As we observe, belief heterogeneity does not affect the price of capital when the economy is at the efficient region. This is expected since the entire capital is held by experts (i.e., $\psi_t = 1$).[30] In this region, the return that the households obtain from holding capital is less than the prevailing risk-free rate and thus households prefer to invest in risk-free bonds issued by experts.

However, in the inefficient region where less productive households hold a portion of capital, the degree and direction of belief heterogeneity start to matter. As experts become more optimistic, i.e., α^O gets higher from α and households become more pessimistic, i.e., α^P decreases from α, the threshold for the efficient region η^ψ falls: as experts, who are marginal investors, become more optimistic about productivity growth, they still demand capital and do not engage in fire-sale even when their wealth share η_t is not high enough.

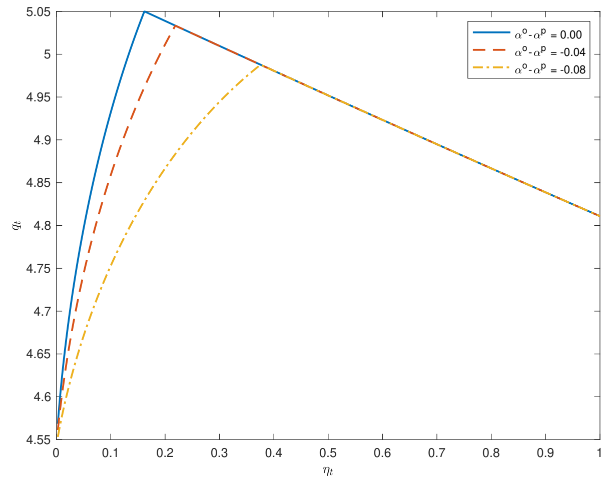
[28] Until now, we have called those who have higher productivity in both turning capital into output and output into new investment 'optimists'. This nomenclature is partially true as they are optimistic only when $\alpha^O \geq \alpha$. In Section 4, we call them 'experts' as we also consider the other case where $\alpha^O < \alpha$.

[29] When we vary α^O or α^P, we always consider symmetric cases where $|\alpha^O - \alpha| = |\alpha - \alpha^P|$ in both cases.

[30] This can be seen in Figure A2 in Appendix.



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$



(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 1: Price-earnings ratio q_t as a function of η_t

It can be seen in Figure 2a where the threshold where experts start deleveraging[31] falls as they become more optimistic. When more productive experts become pessimistic and less productive households are optimistic in contrast, i.e., $\alpha^O < \alpha$ and $\alpha^P > \alpha$, the threshold η^ψ increases in the degree of belief heterogeneity instead. Also comparing Figure 1a and 1b, we observe the effect of belief heterogeneity in this case turns out to be stronger than the former case. It is because experts, who are already marginal investors, become pessimistic and thereby have lower demands for capital now in general, allowing inefficient households to participate in the market as marginal investors and raising the crisis threshold η^ψ strongly.[32]

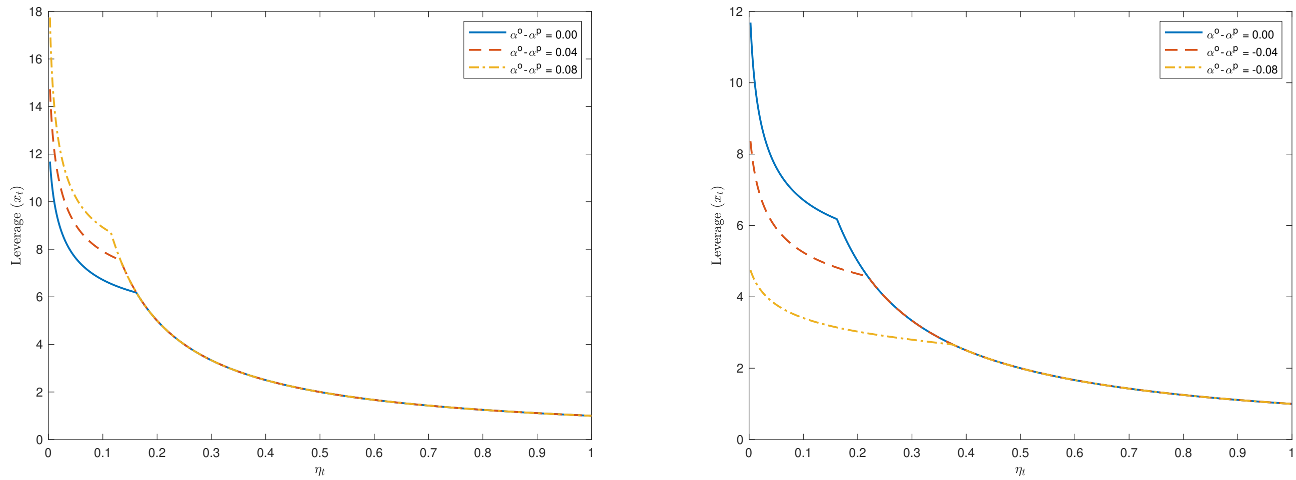
Figure 3 represents the endogenous volatility σ_t^p as a function of the experts' wealth share η_t. We observe that as more productive experts become more optimistic, i.e., α^O rises from α, and less productive households are more pessimistic, i.e., α^P falls from α, we have more amplification of the endogenous risk in the inefficient (crisis) region.[33] It can be understood with regard to equation (58): we can rewrite (58) as

$$\sigma_t^p \left(1 - (x_t - 1) \frac{\dfrac{dq(\eta_t)}{q(\eta_t)}}{\dfrac{d\eta_t}{\eta_t}} \right) \equiv \sigma_t^p \left(1 - (x_t - 1)\, \varepsilon_{q,\eta} \right) = \sigma, \tag{76}$$

[31] In Figure 2, the points where the slope of the graphs becomes discontinuous are the thresholds where experts start to deleverage. Even as experts fire-sell their capital assets and try to lower their leverage, we observe the leverage x_t rises at the inefficient (crisis) region in equilibrium. It is because the capital price q_t drops too much as experts engage in fire-sale of their capital assets. For the issue of procyclicality of leverage, see e.g., Adrian and Shin (2014).

[32] As investment rate ι_t is an increasing function the capital price q_t from (41), it is expected to have the same shape and properties as Figure 1. The full dynamics of investment per unit of capital can be seen in Figure A1.

[33] As we already discussed in Figure 1 and 2, we still observe the threshold η^ψ decreases in this case. More optimistic experts still demand capital when their wealth share is low enough.



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

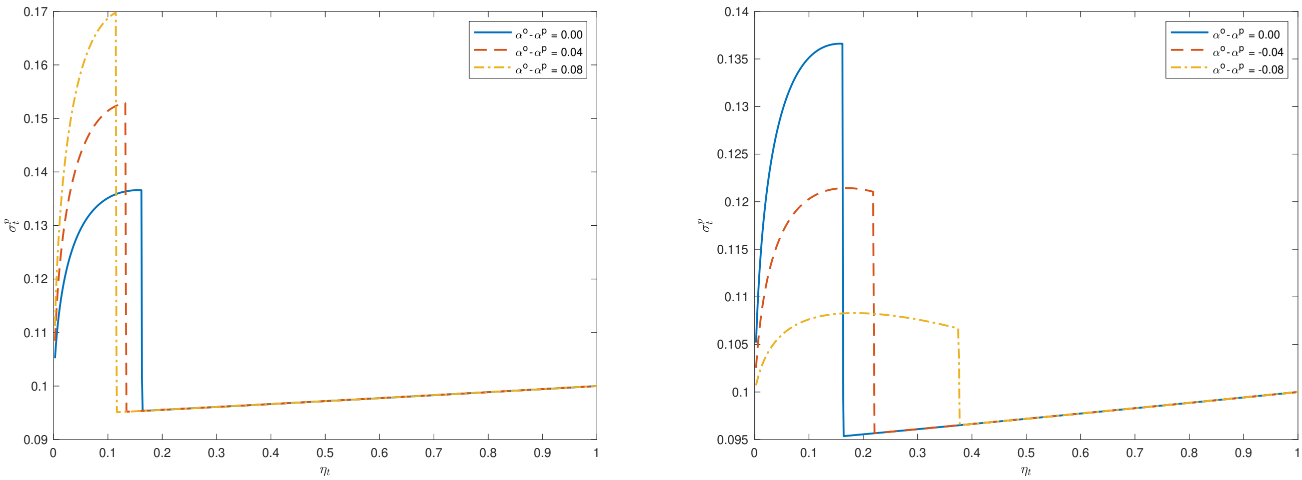
(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 2: Leverage x_t as a function of η_t

where $\varepsilon_{q,\eta}$ is defined as the elasticity of the price-earnings ratio (i.e., normalized capital price) with respect to the experts' wealth share η_t. When experts become more optimistic, their leverage ratio x_t becomes higher around the new threshold η^ψ compared with the benchmark rational expectations case as seen in Figure 2a, as the threshold point η^ψ where they start fire-selling their capital assets falls due to higher demands coming from their optimism. With experts' higher leverage ratio, we have more intense fire-sale of their capital assets when negative shocks are realized, yielding amplification of the endogenous risk at the inefficient region. Therefore, this *leverage* effect raises the equilibrium endogenous risk σ_t^p from (76).

Another effect comes from the elasticity term $\varepsilon_{q,\eta}$. This elasticity of the price of capital with respect to the wealth share of marginal investors (i.e., experts) can be interpreted as a measure of market illiquidity (or the inverse of market depth). As experts become more optimistic, a % increase in their wealth share obviously leads to higher % increases in the price of capital in the inefficient region as seen in Figure 1a: the inefficient (crisis) region is where the households hold a portion of capital and when the market risk-premium is the highest. With a higher degree of optimism, a rise in the wealth share of experts affects the capital price more strongly, raising the elasticity $\varepsilon_{q,\eta}$. This *market illiquidity* effect raises the equilibrium endogenous risk σ_t^p from (76) too. Adding *leverage* effect and *market illuqidity* effect yields more amplified endogenous risks during crises.

Figure 3b represents the case where experts become pessimistic and households are optimistic in contrast, i.e., $\alpha^O < \alpha$ and $\alpha^P > \alpha$. We have less amplification of endogenous risks and the higher thresholds η^ψ. Also comparing Figure 3a and 3b, we observe the effect of belief heterogeneity in this case turns out to be stronger



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 3: Endogenous Volatility σ_t^p as a function of η_t

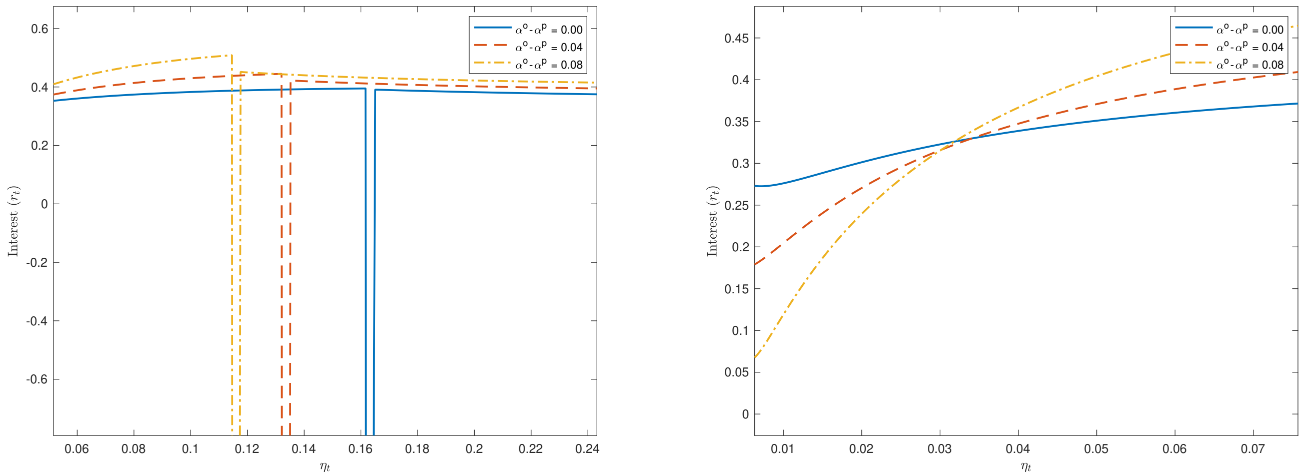
than the former case as before.

Two-Way Interactions Therefore, we have *two-way* interactions between belief heterogeneity about growth, (α^O, α^P), and the amplification of endogenous risk, σ_t^p: (i) more optimistic experts and pessimistic households yield more amplified endogenous risks during crises, i.e., As α^O increases and α^P decreases from the true α, we have higher σ_t^p at the inefficient region as seen in Figure 3a; (ii) In turn, a higher endogenous volatility σ_t^p raises the degree of disagreement about the expected capital gain earned when investing in physical capital, as seen in (19) and (20) given (α^O, α^P), which in turn amplifies the risk during crises again.

Interest Rates To study the behavior of the equilibrium risk-free interest rate r_t, we focus on the case where experts become optimists, i.e., $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$. Figure 4a and Figure 4b look at different regions of η_t. In Figure 4a, we observe that (i) the risk-free rate drops a lot only around the crisis threshold η^ψ; (ii) the interest rate becomes higher as experts become more optimistic. As experts become more optimistic, their demands for leverage get higher, leading to on average higher interest rates.[34,35] And this effect gets bigger when the market risk-premium is higher (i.e., crises).

[34] Of course, as less productive households become more pessimistic, this leads to their higher demands for risk-free loans issued by experts, pushing down the risk-free rate. In our framework, as experts are more productive and participate in the capital market as *natural* marginal investors, this effect is weaker than the effect the higher demands of experts for leverage has on the interest rate.

[35] When experts (optimists in this case) start deleveraging and fire-sell their capital assets, the interest rate drops a lot in an almost discontinuous manner. It is expected: as experts suddenly fire-sell their capital and reduce their demands for leverage, the interest rate must drop to clear the bond market. After the economy enters the inefficient region, i.e., η_t is slightly lower than η^ψ, the interest rate level becomes restored.



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

Figure 4: Interest Rate r_t as a function of η_t: $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

Figure 4b illustrates the region around very low levels of η_t: when η_t is extremely low and close to 0, the interest rate actually falls as experts become more optimistic. When optimistic experts have almost zero net worth, the amounts of loans they issue must be very small in absolute terms,[36] compared with the demands of households for risk-free loans, thereby yielding drops in the interest rate.

Other relevant figures of our numerical simulation are provided in Appendix.

4.2 Ergodic Distribution

Now, we consider the stationary (ergodic) distribution of η_t in the long run: the objective here is to analyze how the optimism and pessimism of experts and households affect the *average* time the economy lives under the inefficient (i.e., crisis) regime.

Starting form equation (48), the dynamics of η_t, we can employ the Kolmogorov forward equation (KFE) in a similar way to Brunnermeier and Sannikov (2014). If we let $d(\eta)$ as the stationary density of η_t, it should satisfy

$$0 = -\frac{\partial}{\partial \eta}\left(\mu^\eta(\eta)\eta \cdot d(\eta)\right) + \frac{1}{2}\frac{\partial^2}{\partial \eta^2}\left(\left(\sigma^\eta(\eta)\eta\right)^2 d(\eta)\right). \tag{77}$$

With the transformation $D(\eta) \equiv \left(\sigma^\eta(\eta)\eta\right)^2 \cdot d(\eta)$, equation (77) can be written as

$$\frac{D'(\eta)}{D(\eta)} = 2\frac{\mu^\eta(\eta)\eta}{\left(\sigma^\eta(\eta)\eta\right)^2}, \tag{78}$$

[36] Due to the solvency constraint of experts, they cannot issue many loans for the purpose of funding their capital purchase when their net wealth share is very low.

which can be solved easily by integrating both sides of (78).

Figure 5 draws the stationary distribution of our state variable η_t in the presence of belief heterogeneity: we observe that the time the economy spends the most is around its stochastic steady states.[37] When negative shocks shift the economy toward the inefficient region however, experts' higher degree of optimism (thereby households' higher degree of pessimism) on average makes the economy more vulnerable to financial crises: the economy spends more time in its inefficient region.

It can be understood as follows: as we see in Figure 3, experts' more optimism (i.e., higher α^O) amplifies the endogenous volatility σ_t^p during crises. In that case, experts receive higher risk-premium[38] when investing in physical capital during the crises, getting recapitalized faster and moving the economy out of the inefficient region. This feature is checked in Figure A3 where the drift of the wealth share process $\mu^\eta(\eta)\eta$ is higher in the inefficient region when experts become more optimistic. This channel reduces the time the economy spends in crises.

However, when experts become more optimistic, they bear too much risk due to higher leverage ratio x_t around the new threshold η^ψ, which itself falls. It raises the probability that the economy is pushed from the efficient to inefficient region when negative shocks are realized. This feature is checked in Figure A3 where the drift of the wealth share process $\mu^\eta(\eta)\eta$ becomes more negative in the efficient region as experts become more optimistic. This channel increases the time the economy spends in the inefficient region (i.e., crises). It turns out that the second channel (i.e., from efficient to inefficient) is stronger than the first channel (i.e., from inefficient to efficient): so our economy undergoes a higher number of shorter-lived and severe (with higher levels of volatility) financial crises than under the rational expectations as experts who are natural marginal investors become more optimistic.

Figure 5b represents the case where experts become pessimistic and households are optimistic in contrast, i.e., $\alpha^O < \alpha$ and $\alpha^P > \alpha$. The economy undergoes a lower number of long-lived and mild financial crises, as it takes more time for experts to get recapitalized during the crises.[39]

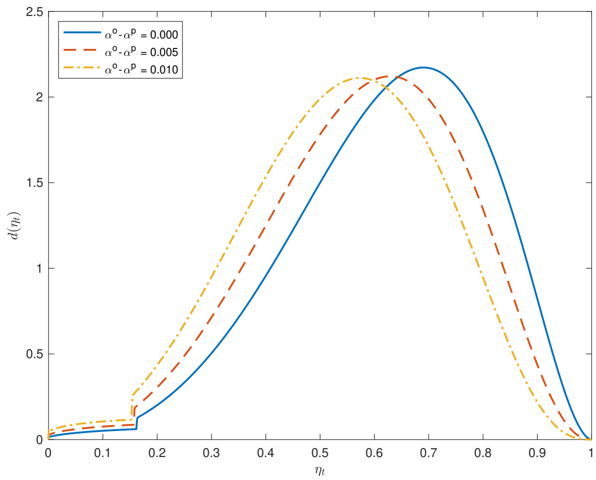
5 Conclusion

In an economy with two groups of agents who have different productivity in (i) turning capital into outputs; (ii) turning output into capital (i.e., investment), we introduce heterogeneous beliefs in a tractable way: two

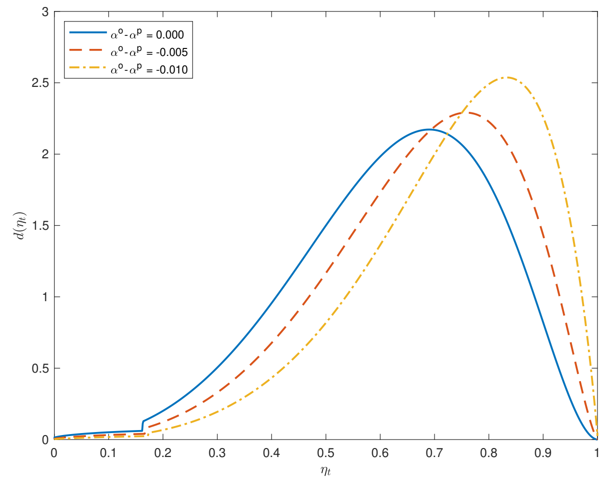
[37] There are two stochastic steady states where $\mu^\eta(\eta) = 0$, one of which is $\eta = 1$, definitely

[38] From Proposition 1, we know that the risk-premium is given by $(\sigma_t^p)^2$ in our log-utility case.

[39] From Figure 3b, we know that more pessimistic experts (i.e., when $\alpha^O \leq \alpha$) reduce the amplification of endogenous risk σ_t^p in the inefficient region, thereby bringing down the risk-premium level: therefore experts need more time in getting recapitalized and pushing the economy toward the efficient region. This is checked in Figure A3b where the drift of the wealth share process $\mu^\eta(\eta)\eta$ falls in the inefficient region as experts become more pessimistic.



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$



(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 5: Ergodic Distribution of η_t

groups, optimists and pessimists, with imperfect information about the process of their productivity growth, agree to disagree about the expected growth of their productivity measures. We mostly focus on cases where more productive experts believe that the expected growth of their productivity is higher than the other group (i.e., optimistic), but the opposite case is also analyzed.

The setting is close to Brunnermeier and Sannikov (2014) with additional components of growth prospects and belief heterogeneity. The main objective is to study how imperfect information and heterogeneous beliefs in productivity growth affect aggregate investment, leverage choices, capital asset price, and the endogenous volatility in general equilibrium. When more productive experts become more optimistic about their growth prospects, trade is facilitated, raising investment, asset prices, and their leverage ratio considerably. However, when the economy shows signs of inefficiencies, the endogenous volatility and the risk-premium spike in a more amplified way. The economy undergoes a higher number of shorter-lived and more severe crises than under the rational expectations, and spends more time on average in the inefficient region (i.e., crises) per given amount of time. On the other hand, when experts become more pessimistic, the economy would have a low number of longer-lived and mild crises, spending less time per year in the inefficient region.

We believe that our joint characterization of (i) amplified risk from the heterogeneous beliefs about growth prospects; (ii) 'the more turbulent, the more disagreeing about financial market returns' is novel and useful in understanding many financial market behaviors in general equilibrium sense.

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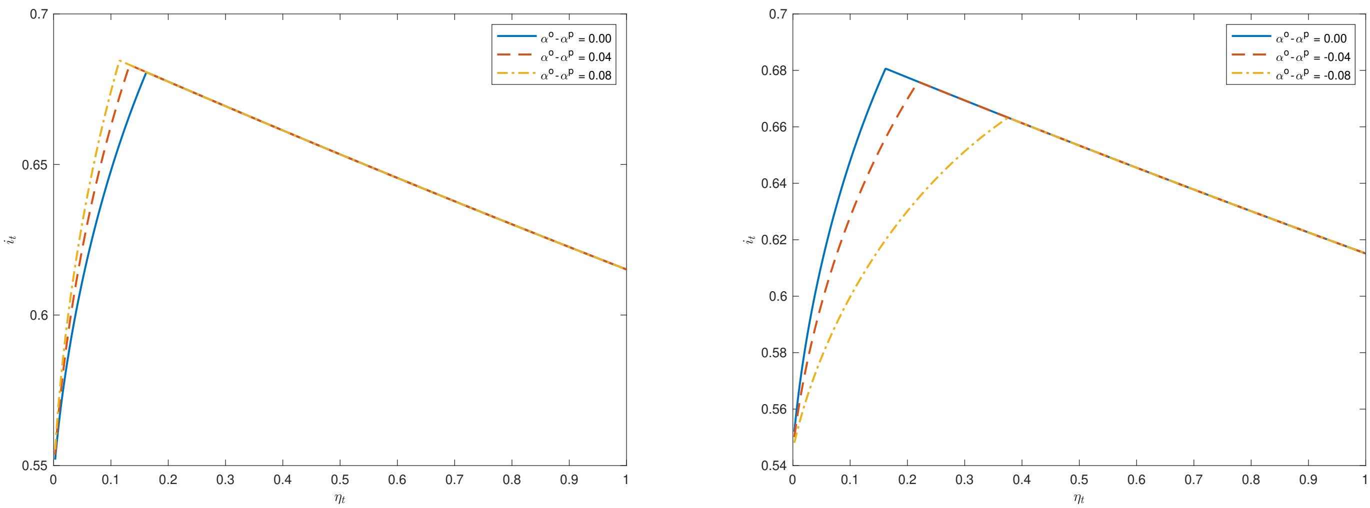
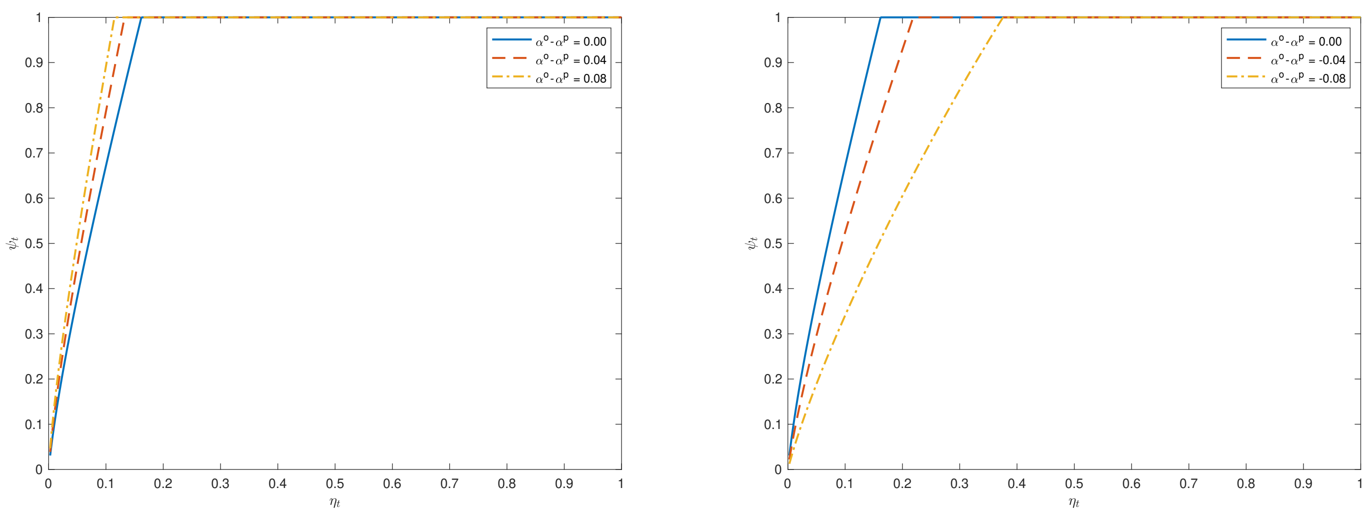
Appendix. Additional Figures



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

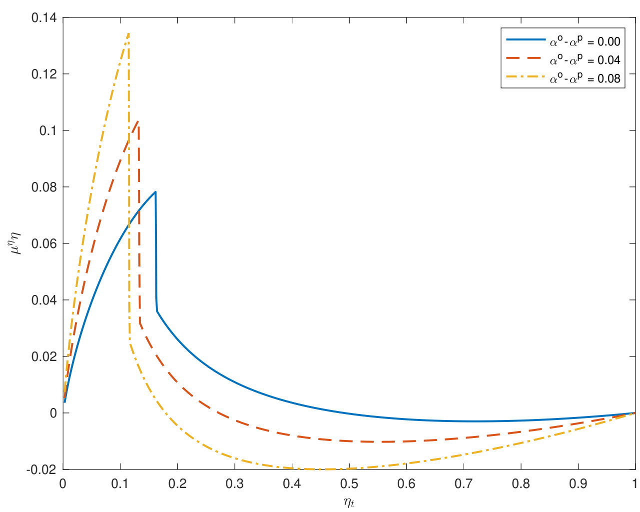
Figure A1: Investment Rate ι_t as a Function of η_t



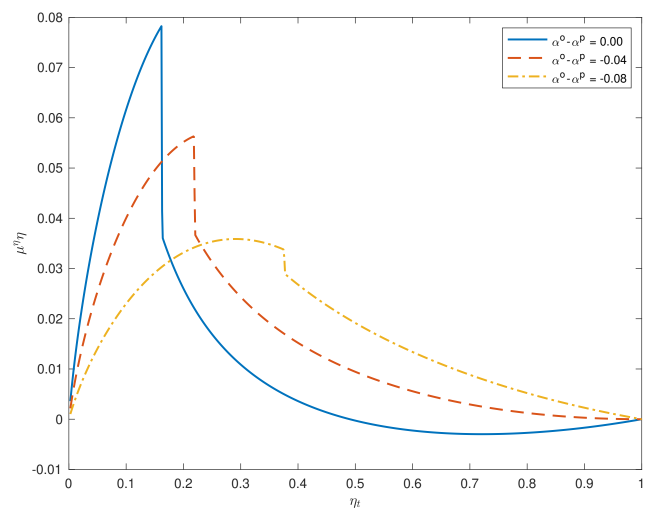
(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure A2: Capital Share ψ_t as a Function of η_t

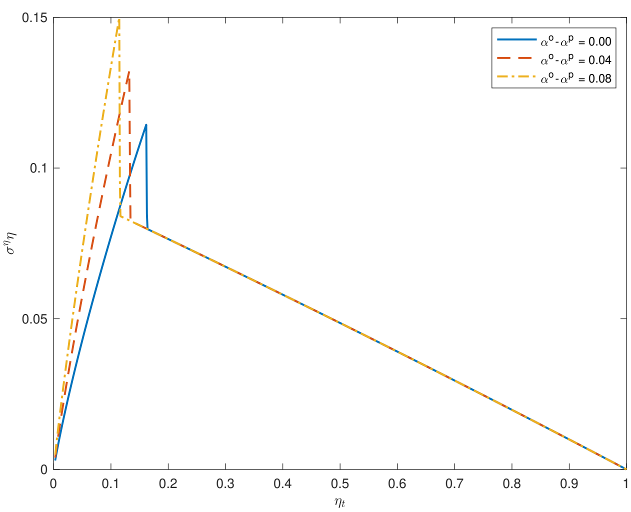


(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

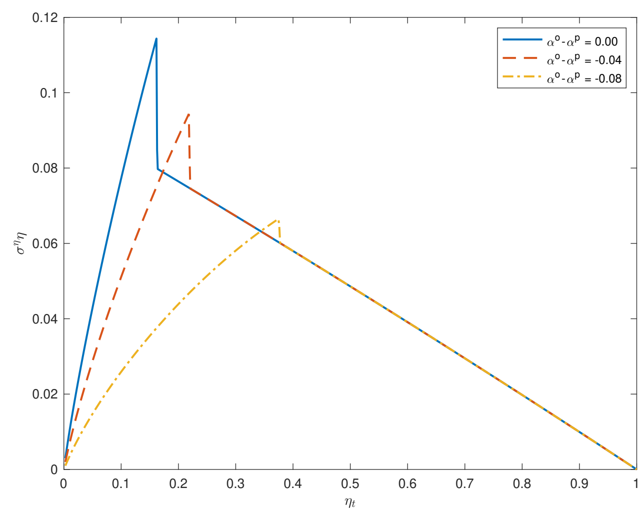


(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure A3: Wealth Share Drift $\mu_\eta(\eta_t) \cdot \eta_t$ as a Function of η_t



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$



(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure A4: Wealth Share Volatility $\sigma^\eta(\eta_t) \cdot \eta_t$ as a Function of η_t