Active Taylor Rules Still Breed Sunspots: Sunspot Volatility, Risk-Premium, and the Business Cycle[*]

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Abstract

This paper demonstrates that in macroeconomic models with nominal rigidities, a global solution exists that supports an alternate equilibrium; in this equilibrium, traditional Taylor rules give rise to self-fulfilling aggregate volatility and excess risk-premium. Within the rational expectations framework, we establish that individually optimal, path-dependent consumption strategies can generate endogenous volatility in a self-fulfilling manner, propelling the entire economy into crises (booms) characterized by elevated (reduced) aggregate risk. This outcome stems from the inability of traditional policy rules to target the expected return on aggregate wealth, which comprises not only the risk-free policy rate but also the market risk-premium; the latter ultimately determining the degree of intertemporal substitution and households' consumption. To address this limitation, we propose a "generalized" Taylor rule that targets both risk-premium and asset prices, thereby restoring the monetary authority's capacity to target aggregate wealth and enforce the socially optimal consumption path. We outline the necessary conditions to reestablish determinacy and attain what we term as the ultra-divine coincidence: the simultaneous stabilization of inflation, output gap, and the risk-premium.

Keywords: Taylor Rules, Sunspot Volatility, Risk-Premium

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1 Introduction

How should monetary policy respond to fluctuations in aggregate market volatility? The prevailing perspective suggests that central banks require two distinct sets of instruments: macroprudential policies and regulations to preserve the stability of financial markets, that is, maintaining a stable level of market volatility, and monetary and fiscal policies to achieve the conventional goal of macroeconomic stabilization.[1] Nevertheless, the debate surrounding this matter remains unresolved for numerous reasons. For instance, the aggregate volatility of both the business cycle and financial markets is endogenous and intricately interconnected. Disentangling their relationship from a theoretical standpoint has proven challenging, as mainstream macroeconomic frameworks typically depend on approximation techniques that either simplify or outrightly eliminate the higher-order terms related to economic volatility and risk, or rely on numerical solution methods which obscure the underlying economic intuition.

In this paper, we demonstrate that within a macroeconomic model featuring nominal rigidities, Taylor rules, irrespective of their responsiveness to typical business cycle mandates (i.e., inflation and output gap), permit aggregate volatility as well as the risk premium to emerge in a self-fulfilling manner. We illustrate this insight within two macroeconomic frameworks: (i) the standard New-Keynesian model,[2] and (ii) a model incorporating stock markets and portfolio decisions. Our continuous-time characterization of the problem allows the models' solutions to remain tractable, yielding closed-form expressions for the time-varying aggregate volatility, risk premium, and business cycle variables, all of which are endogenously determined.

In the standard New-Keynesian model, the economy's time-varying aggregate risk has a first-order impact on aggregate consumption demand through the precautionary savings

[1]For example, at the press conference held on September 16, 2020, the chair of the Federal Reserve, Jerome Powell, stated, "Monetary policy should not be the first line of defense - is not the first line of defense on financial stability. We look to more appropriate tools in the first instance, as a first line of defense. And those would be regulation, supervision, high-capital, high-liquidity stress testing, all of those things, macroprudential tools."

[2]See, for example, Galí (2015).

channel. More specifically, heightened aggregate volatility leads households to increase their precautionary savings, thereby reducing aggregate demand and output, while the aggregate volatility is determined by fluctuations in output. In this setting, agents can generate aggregate volatility through their intertemporal consumption coordination under rational expectations. For instance, consider a scenario where households at time 0 suddenly believe that the economy in the next period will be more volatile. They decrease their current consumption and increase precautionary savings, resulting in a recession at time 0. In period 1, the initial fear at time 0 regarding the volatility of the time 1 economy must be validated. This can be achieved if, for each possible realized consumption at period 1, there exists a corresponding conditional volatility of time 2 consumption. Specifically, a higher realization of time 1 consumption should be accompanied by a lower conditional volatility of time 2 consumption, leading to a decreased degree of precautionary savings. Essentially, households' belief in the current volatility is shaped by their expectations in the previous period and reinforced by their actions in future periods. It is important to note that our equilibrium construction with self-generated volatility is made possible due to nominal rigidities: the path-dependent consumption strategy of households determines the stochastic output paths, as the economy is driven by demand.

In the specific rational expectations equilibrium we refer to as the "martingale" equilibrium, the economy (i.e., output gap) adheres to a martingale, meaning that, on average, the next period economy remains at the current level. As the conditional volatility of the subsequent period's consumption declines when the economy approaches the stabilized path (i.e., the flexible price economy), the stabilized path functions as an attractor for all sample paths. Consequently, after generating a self-fulfilling volatility shock, the economy is almost certainly stabilized in the long run. However, on the equilibrium path, and until the economy is nearly stabilized following the emergence of the sunspot, it experiences a prolonged recession accompanied by increased aggregate volatility. We demonstrate that a *probability-zero event*, in which the self-created conditional volatility ultimately diverges toward infinity, enables the initial appearance of sunspot volatility and ensures that the economy follows a martingale, even if it is almost surely stabilized in the long run. We relate this property to an endogenously generated rare-disaster event that arises in a self-fulfilling manner.

The inability of traditional Taylor rules to prevent the emergence of self-fulfilling volatility stems from their omission and/or incapacity to directly target it. To facilitate a clearer understanding of the problem, we introduce a second macroeconomic model incorporating stock markets and portfolio decisions, wherein aggregate volatility is associated with financial instability and reflected in the financial market risk premium. This model showcases a similar role for aggregate stock price volatility and risk premium in business cycle fluctuations: a more volatile financial market with a higher risk premium reduces aggregate financial wealth through individual investors' portfolio decisions, subsequently diminishing aggregate demand and output. Due to the analogous mathematical structure concerning the influence of aggregate volatility on aggregate demand, we can construct an equilibrium in which aggregate stock price volatility is generated in a self-fulfilling manner and merely reflects the volatility of the underlying firms. The possibility of sunspot volatility in this specific context can also be interpreted as follows: the fear of a financial crisis resulting from an increase in risk premium and stock market volatility renders investors less inclined to invest in the stock market, lowering current asset prices and wealth,[3] thereby producing self-fulfilling increases in the expected stock market return and risk premium.

Our analysis illustrates that although Taylor rules focusing on macroeconomic aggregates (i.e., inflation and output gap) are unable to prevent the emergence of self-fulfilling volatility, adopting a more aggressive stance towards deviations in these targets hastens stabilization following an initial sunspot shock within the constructed rational expectations equilibrium. However, this heightened responsiveness of monetary policy comes with a trade-off: a more aggressive targeting of inflation and output gap intensifies stock price volatility in response to sunspot shocks, leading to more pronounced yet short-lived boom and bust financial cycles driven by sunspot volatility.

The failure of Taylor rules in restoring determinacy stems from their inability to sufficiently target the expected return of financial markets, which influences the intertemporal

[3]A reduction in financial wealth leads to diminished consumption, which in turn decreases firm profits and rationalizes a decline in the stock price level. This occurs because firms are subject to nominal rigidities.

decision-making of agents. Intuitively, households optimally allocate their wealth between risky and risk-free assets, with the return on aggregate wealth —given by the risk-free policy rate plus endogenous market risk-premium— serving as the relevant rate for their intertemporal consumption smoothing decisions. Since conventional Taylor rules operate via the risk-free rate part to enact their macroeconomic objectives, they permit self-fulfilling financial volatility and risk-premium to emerge spontaneously in a rational expectations equilibrium. Consequently, we propose a generalized policy reaction function that precludes the possibility of sunspot volatility in our stochastic environment. Specifically, we contend that optimal policy rules should target the risk-premium of financial markets as a separate factor in addition to their conventional mandates. Essentially, the optimal monetary rule seeks to regulate the expected return on the economy's aggregate wealth in response to the business cycle. Therefore, it must consider the risky component of the portfolio return, which is encapsulated by the risk-premium. Our analysis thus suggests that aggregate wealth should serve as an intermediate target for the central bank in the pursuit of macroeconomic stabilization. This novel policy rule, which specifically targets risk-premium, accomplishes what we term as the "ultra-divine" coincidence: the simultaneous stabilization of inflation, output gap, and risk-premium (equivalently, aggregate stock price volatility) —the latter of which we consider a proxy for financial stability. Consequently, a single monetary policy can stabilize both the business cycle and the risk-premium in stock markets. Implementing this rule presents its own challenges, however, as the central bank must target the risk premium with the appropriate degree of responsiveness. If the policy response is overly dovish or hawkish, monetary policy is once again incapable of preventing the emergence of sunspot volatility. Nevertheless, even when the central bank cannot restore equilibrium determinacy, targeting financial variables remains an optimal strategy, as it facilitates a more rapid convergence to the steady state following a sunspot shock.

Related Literature Our paper connects to an extensive literature on the intersection between macroeconomics and finance. Our model with stock markets builds upon the notion that changes in financial wealth levels influence aggregate outcomes, as documented by Mian et al. (2013), Mian and Sufi (2014),[4] Guerrieri and Iacoviello (2017), Berger et al. (2018), Caballero and Simsek (2020a,b), Di Maggio et al. (2020), Caramp and Silva (2021)[5], and Chodorow-Reich et al. (2021), among others. Consistent with this literature, an endogenous change in stock price shifts aggregate demand in our framework through its effect on aggregate financial wealth. Moreover, our framework emphasizes endogenous risk-premium and financial volatility as crucial factors driving fluctuations in financial markets and the business cycle, aligning with arguments put forth by Gilchrist and Zakrajšek (2012), Brunnermeier and Sannikov (2014), Chodorow-Reich (2014), Stein (2014), Cúrdia and Woodford (2016), Guerrieri and Lorenzoni (2017), Di Tella and Hall (2021), and Basu et al. (2021),[6] among others, who assert that financial (and in particular, credit) disruptions have significant impacts on aggregate demand, especially when monetary policy is constrained. Campbell et al. (2020) note that the New-Keynesian channels —through which higher inflation depresses bond returns while supporting aggregate output, dividends, and stock returns— can explain the correlation reversal between bond and stock returns, which turned negative in recent years. Our framework shares these intuitions and illuminates how stock market fluctuations can be incorporated into conventional New-Keynesian models.

Our finding that monetary policy must systematically address market risk-premium, a measure of financial market stability, is connected to previous literature, including Bernanke and Gertler (2000), Stein (2012), Woodford (2012), Cúrdia and Woodford (2016),[7] Caballero and Simsek (2020a), Cieslak and Vissing-Jorgensen (2021),[8] Kekre and Lenel (2022), and Galí (2021)[9]. In contrast to Bernanke and Gertler (2000)'s conclusion that monetary pol-

[4]In Mian et al. (2013) and Mian and Sufi (2014), consumers with high marginal propensity to consume, who experience substantial declines in their housing values, reduce consumption due to both wealth effects and binding credit constraints —the latter of which is not considered in this paper.

[5]Caramp and Silva (2021) introduce rare-disasters and positive private debt, characterizing time-varying risk-premium and financial wealth in a linearized setting.

[6]Basu et al. (2021) underscore risk-premium as a business cycle driver, demonstrating that the shock explaining fluctuations in risk-premium can account for a large fraction of business cycle movements and comovements. They rely on the third-order perturbation to solve their model.

[7]Woodford (2012) and Cúrdia and Woodford (2016) introduce a friction in intermediation between agents with different marginal propensities to consume and study the optimal monetary policy rule.

[8]Cieslak and Vissing-Jorgensen (2021) demonstrates that stock market performance is a robust predictor of the policy rate, aligning with our specification.

[9]Galí (2021) incorporates rational bubbles into a New-Keynesian model with overlapping generations,

icy should not target stock prices —a finding based on a model with ad-hoc bubbles— our model omits bubble components, and thus only the fundamental stock price serves as the key factor determining aggregate demand. As a result, our specification with the stock price as an aggregate demand shifter leads to the equivalence of targeting stock price levels and more conventional mandates, such as the output gap. Kekre and Lenel (2022), in particular, demonstrate that an accommodative monetary policy shock is redistributive toward those with a higher marginal propensity to take risk, consequently reducing risk-premium levels and amplifying monetary transmission. While their focus is on how monetary policy following the conventional Taylor rule affects the economy through its impacts on economy-wide risk-premia in a heterogeneous agent model, our analytical approach enables us to identify the possibility for self-fulfilling aggregate volatility under Taylor rules,[10] allowing us to propose a more generalized Taylor rule that targets risk-premium as a means of facilitating stabilization and (potentially) restoring model determinacy.

While Giavazzi and Giovannini (2010), Stein (2012), and Caballero and Simsek (2020a) emphasize the preemptive role of monetary policy in averting *future* financial crises, our model features a monetary policy rule targeting the risk-premium of financial markets for current stabilization purposes, in addition to its traditional inflation and output gap targets. Our finding that monetary accommodation supports the business cycle through its effect on stock markets aligns with evidence provided by Rigobon and Sack (2003), Azali et al. (2013), and Kekre and Lenel (2022). Furthermore, we underscore the decline in demand for risky assets as a critical driver behind financial recessions, a channel documented by Caballero and Farhi (2017) and Caballero and Simsek (2020a,b).

Our paper shares similarities with Caballero and Simsek (2020a,b) in terms of incorporating an endogenous asset market interwoven with the fluctuations of the business cycle. However, while their framework focuses on how behavioral biases can generate intriguing

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arguing that a properly specified "leaning against the bubble" policy insulates the economy from aggregate bubble fluctuations.

[10]Additionally, we contribute to the literature by providing an exact stochastic process for each business cycle variable after the emergence of sunspots in the financial market.

crisis dynamics through the feedback loop between asset markets and the business cycle,[11] our attention centers on the traditional policy rule under rational expectations and the existence of sunspot equilibria arising from higher-order moments. Our model's equilibrium determinacy results resemble those of Acharya and Dogra (2020) in terms of how countercyclical risks can lead to indeterminacy. While Acharya and Dogra (2020) investigates how determinacy conditions change in the presence of exogenous idiosyncratic risks that are functions of aggregate output, we explore the existence of sunspots originating from aggregate financial risk, which is countercyclical in nature and influences both financial markets and business cycle fluctuations. Moreover, we examine the monetary policy that restores determinacy and enhances economic and financial stability.

Layout In Section 2, we provide a non-linear treatment of the standard New-Keynesian economy, examining the impact of non-linearity on equilibrium determinacy issues and the appropriate monetary policy rules required for stabilization purposes. In Section 3, we introduce the model with explicit stock markets and delineate the equilibrium conditions. Section 4 concentrates on the suitable monetary policy rules in light of our framework's novel features. Section 5 provides concluding remarks. In Online Appendix A, we present evidence illustrating the significance of financial volatility as a driver of business cycle fluctuations, employing a structural Vector Autoregression (VAR) approach. Appendix B comprises additional figures and tables. Appendix C contains derivations and proofs. Appendix D offers a detailed account of the equilibrium conditions in Section 2.

2 Standard Non-linear New Keynesian Model

In Section 2, we consider a standard New-Keynesian economy[12] where firm profits are transferred in a lump-sum fashion to households. In Section 3, we present a model with stock mar-

[11]Caballero and Simsek (2020b) features optimists and pessimists with different beliefs about the probability of an upcoming recession or boom. During zero lower bound (ZLB) episodes, an endogenous decline in the risky asset valuation, due to a drop in optimists' wealth, generates a demand recession. We study relevant ZLB issues in a separate paper, Lee and Carreras (2022).

[12]See Woodford (2003) for the standard treatment of a textbook New-Keynesian model.

kets where we instead assume that profits are capitalized into dividend-paying stocks traded in financial markets. Our objective in Section 2 is to illustrate that a *non-linear* characterization of the equilibrium enables higher-order moments tied to the aggregate business cycle volatility to have a first-order impact on the business cycle dynamics, even when stock markets are absent. This feature will have important implications for equilibrium determinacy and the proper management of monetary policy needed to stabilize the business cycle. More detailed characterization of optimality conditions for Section 2 is provided in Appendix D.

The representative household owns the firms of this economy and receives the profit stream in a lump-sum fashion. For simplicity, we assume a perfectly rigid price level: $p_t = \bar{p}, \forall t$[13] so there is no inflation in the economy. This assumption is not crucial but allows us to focus on the key mechanism we want to illustrate.

The representative household chooses her usual intertemporal consumption-savings decision by solving the following optimization problem:

$$\max_{\{B_t, C_t, L_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\log C_t - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] dt \;\text{ s.t. }\; \dot{B}_t = i_t B_t - \bar{p} C_t + w_t L_t + D_t, \qquad (1)$$

where C_t and L_t are her consumption and labor supply, respectively, η is the Frisch elasticity of labor supply, B_t is her nominal holding of bonds, and D_t are the entire firms' profits and fiscal transfers from the government. w_t is the equilibrium wage, and i_t is the policy rate set by the central bank. We assume that there is no government spending, and therefore aggregate consumption determines output in this demand-determined environment: $C_t = Y_t$, where Y_t is aggregate output, while the bond market is zero net supplied. Finally, ρ is her time discount rate.

We obtain the following optimality conditions for the representative household's intertemporal consumption-savings decision:

$$-i_t dt = \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right), \;\text{ where }\; \xi_t^N = e^{-\rho t} \frac{1}{\bar{p}} \frac{1}{C_t}, \qquad (2)$$

[13]This assumption can be micro-founded with price stickiness à la Calvo (1983) and a price resetting probability of zero.

where $\frac{d\xi_t^N}{\xi_t^N}$ is the instantaneous (nominal) stochastic discount factor, and its expected value equals the nominal risk-free rate i_t.[14] Due to the rigid price assumption, there is no inflation, i.e., $\pi_t = 0$, $\forall t$, thereby the real and nominal risk-free rates of the economy are equal, i.e., $r_t = i_t$, where r_t is the real interest rate.

We can rewrite equation (2) as

$$\mathbb{E}_t \left(\frac{dC_t}{C_t} \right) = (i_t - \rho)dt + \mathrm{Var}_t \left(\frac{dC_t}{C_t} \right), \tag{3}$$

where the last term $\mathrm{Var}_t(\frac{dC_t}{C_t})$ arises from the endogenous volatility of the aggregate consumption process. Note that this volatility is usually a second-order term and therefore is typically dropped out in log-linearized models. In contrast to those models, our non-linear characterization properly accounts for consumption risk and allows it to affect the drift of the aggregate consumption process, where the volatility as well as the drift is an endogenous object.

This additional term reflects the usual precautionary savings channel, in which a more volatile business cycle leads to an increased demand for riskless savings, which in turn leads to a drop in current consumption and a higher expected growth for the consumption process.

An individual firm i produces with the linear production function: $Y_t^i = A_t L_t^i$ where L_t^i is firm i's labor hiring, and A_t is the economy's total factor productivity assumed to be exogenous and to follow a geometric Brownian motion[15] with drift:

$$\frac{dA_t}{A_t} = g dt + \sigma dZ_t \tag{4}$$

where g is its expected growth rate while σ is what we call 'fundamental' volatility, assumed

[14]In Appendix D, we provide the Hamilton-Jacobi-Bellman (HJB) based derivation for (2).

[15]We assume a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where variables are adapted to the filtration generated by Z_t.

to be constant over time.[16] It follows that firms' profits to be rebated can be written as:

$$D_t = \bar{p}Y_t - w_t L_t.$$

Flexible price equilibrium as benchmark With the usual Dixit-Stiglitz monopolistic competition among firms, we can characterize the flexible price equilibrium where firms can freely choose their prices, in contrast to the fully rigid price, i.e., $p_t = \bar{p}$. The flexible price equilibrium outcomes are called 'natural' as central banks in the presence of price rigidity want to target these outcomes based on monetary tools. The 'natural'[17] (benchmark) economy's output Y_t^n follows the stochastic process:

$$\frac{dY_t^n}{Y_t^n} = \Bigg(\underbrace{r^n}_{\text{Natural rate}} - \rho + (\sigma)^2 \Bigg) dt + \underbrace{\sigma}_{\text{Natural volatility}} dZ_t, \tag{5}$$

where $r^n = \rho + g - \sigma^2$ is defined as the natural interest rate. From monetary authority's perspective, equation (5) is an exogenous process[18] that monetary policy cannot affect nor control. Note that the natural output Y_t^n follows a geometric Brownian motion with the volatility σ, which equals the volatility of A_t process in (4).

Rigid price equilibrium and the 'gap' economy Going back to the 'rigid' price economy, we first introduce σ_t^s as the *excess* volatility the output process $\{Y_t\}$ features, compared with the benchmark flexible price economy output in (5). Then:

$$\text{Var}_t \left(\frac{dY_t}{Y_t} \right) = (\sigma + \sigma_t^s)^2 dt \tag{6}$$

[16] This assumption is made for simplicity and our analysis can be extended to include cases where σ_t is time-varying and adapted to the Brownian motion Z_t.

[17] In Appendix D, we characterize the equilibrium under fully flexible prices, and derive the dynamic equation (5) for Y_t^n.

[18] Equation (5) can be derived from the no-arbitrage condition (i.e., (2)) with $i_t = r_t^n$ and $Y_t = Y_t^n$. Therefore, we can interpret our Brownian motion dZ_t as an aggregate shock that drives the natural output Y_t^n (e.g., a technology shock).

would hold by definition. Note that σ_t^s is the 'endogenous' volatility to be determined later in equilibrium. By plugging equation (6) into the asset pricing equation (2), we obtain

$$\frac{dY_t}{Y_t} = \left(i_t - \rho + (\sigma + \sigma_t^s)^2\right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{7}$$

With the usual definition of output gap $\hat{Y}_t = \ln\left(\frac{Y_t}{Y_t^n}\right)$, we obtain the following dynamic IS equation written in output gap $\hat{Y}_t$:

$$d\hat{Y}_t = \left(i_t - \left(r^n \overbrace{-\frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2}^{\text{New terms}}\right)\right) dt + \sigma_t^s dZ_t, \tag{8}$$

which features an interesting feedback effect that is omitted in log-linearized equations:[19] given the policy rate i_t, a rise in the endogenous volatility σ_t^s pushes up the drift of (8) and lowers output gap $\hat{Y}_t$. The intuition follows from the households' precautionary behavior we see in (3): households respond to a higher economic volatility with increased savings and lower consumption, thereby inducing a recession.

Define the *risk-adjusted* natural rate as

$$r_t^T = r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2. \tag{9}$$

and note that r_t^T is itself endogenous: it negatively depends on the endogenous aggregate (excess) volatility σ_t^s. This risk-adjusted natural rate can be regarded a new reference risk-free rate of the economy at which i_t completely eliminates the drift of the output gap.

We now turn our attention towards monetary policy and study the implications of following a conventional Taylor rule in this environment, as well as possible alternatives. Under Taylor rules that targets output gap $\hat{Y}_t$, it will turn out that a self-fulfilling aggregate volatility

[19] For illustrative purposes, compare equation (8) with the conventional linearized IS equation given by:

$$d\hat{Y}_t = (i_t - r^n)\, dt + \sigma_t^s dZ_t,$$

where the endogenous aggregate volatility σ_t^s has no first-order effect on the drift of output gap.

σ_t^s can always arise out of thin air, through households' intertemporal consumption coordination.

2.1 Taylor rules and Indeterminacy

In this section, we study the conventional Taylor rule and its capacity to guarantee model determinacy and economic stabilization. We assume that the central bank sets the risk-free rate i_t of the economy according to:

$$i_t = r^n + \phi_y \hat{Y}_t, \text{ where } \phi_y > 0. \tag{10}$$

Condition $\phi_y > 0$ is the 'Taylor principle' that prevents the appearance of sunspot equilibria in conventional log-linearized models that omit the first-order effects of aggregate volatility. Here, we ask whether the policy in (10) retains the capacity to prevent sunspot equilibria in our non-linear economy that features the feedback relationship between output gap volatility and its drift explained in (8).

Plugging equation (10) into equation (8), we obtain

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t \underbrace{- \frac{(\sigma_t)^2}{2} + \frac{(\sigma_t + \sigma_t^s)^2}{2}}_{\text{New terms}} \right) dt + \sigma_t^s dZ_t \tag{11}$$

as the output gap dynamics. (11) is our non-linear dynamic IS equation.

Multiple equilibria Omitting the new volatility terms from the drift of equation (11), we obtain the usual log-linearized version of the $\hat{Y}_t$ dynamics as

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t \right) dt + \sigma_t^s dZ_t. \tag{12}$$

With the dynamics described by equation (12), Blanchard and Kahn (1980) proves the existence of a unique rational expectations equilibrium when the Taylor principle $\phi_y > 0$ is satisfied: $\underline{\hat{Y}_t = 0}$, $\forall t$, which corresponds to a fully stabilized economy.

We now claim that this result does not hold in this non-linear version of the $\left\{\hat{Y}_t\right\}$ process. The feedback effect from the endogenous volatility σ_t^s of the output gap to its drift in equation (11) enables the appearance of *self-fulfilling* volatility σ_t^s. We provide a rational expectations equilibrium we call 'martingale equilibrium' that supports a sunspot arise $\sigma_0^s > 0$, by constructing an equilibrium path where the $\{\hat{Y}_t\}$ follows a martingale. The case of negative volatility sunspot (i.e., $\sigma_0^s < 0$) can be similarly constructed. Our martingale equilibrium construction (i) supports an initial sunspot $\sigma_0^s > 0$, i.e., explain why $\sigma_0^s > 0$ arises out of thin air, and (ii) not diverges on expectation in the long-run for it to be a rational expectations equilibrium (see e.g., Blanchard and Kahn (1980)).

Martingale equilibrium We provide the explicit equilibrium in which a sunspot $\sigma_0^s > 0$ appears and $\hat{Y}_t$ follows a martingale process, consistent with the dynamics in (11). First, the $\{\hat{Y}_t\}$ process' drift must be zero in order for it to become martingale, which gives us the following formula for $\hat{Y}_t$:

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{(\sigma)^2}{2\phi_y}. \tag{13}$$

The martingale equilibrium guarantees the rationality of the equilibrium, as on average the path of $\{\hat{Y}_t\}$ stays at the same level (thereby does not diverge in the long run), satisfying $\mathbb{E}_0(\hat{Y}_t) = \hat{Y}_0$. The last step is to show the existence of a stochastic path for $\{\sigma_t^s\}$ starting from σ_0^s that supports this equilibrium. Using (11) and (13), we obtain the stochastic process of σ_t^s starting from σ_0^s as[20]

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma_t + \sigma_t^s} dZ_t. \tag{14}$$

Therefore, equation (13) and equation (14) constitute the dynamics of this particular ra-

[20]When $\sigma = 0$, $\forall t$, equation (14) becomes the following Bessel process:

$$d\sigma_t^s = -\frac{(\phi_y)^2}{2\sigma_t^s} dt - \phi_y dZ_t,$$

which stops when σ_t^s hits zero. For general properties of Bessel processes, see Lawler (2019).

tional expectations equilibrium supporting $\sigma_0^s > 0$. The following Proposition 1 sheds lights on the behavior of $\{\hat{Y}_t, \sigma_t^s\}$ under the martingale equilibrium and finds that: even if the economy can be hit by an arise of self-fulfilling volatility $\sigma_0^s > 0$, the business cycle almost surely converges to the perfectly stabilized path in the long run through the stabilization based on Taylor rules. Nonetheless, a few paths that occur with *tiny* probability do not converge and explode asymptotically, sustaining the initial sunspot $\sigma_0^s > 0$ due to the forward-looking nature of the economy.

Proposition 1 (Taylor Rules and Indeterminacy) *For any value of $\phi_y > 0$:*

1. Indeterminacy*: there is always a rational expectations equilibrium (REE) that supports initial sunspot $\sigma_0^s > 0$ and is represented by $\hat{Y}_t$ dynamics in equation* (13)*, and σ_t^s process in equation* (14)
2. Properties*: the rational expectations equilibrium that supports an initial sunspot $\sigma_0^s > 0$ satisfies:*

$$(i)\ \sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0,\ (ii)\ \hat{Y}_t \overset{a.s}{\to} 0,\ \textit{and}\ (iii)\ \mathbb{E}_0\left(\max_t (\sigma_t^s)^2\right) = \infty$$

The results that $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$ and $\hat{Y}_t \overset{a.s}{\to} 0$ imply that the equilibrium paths starting from an initial sunspot $\sigma_0^s > 0$ are almost surely stabilized in the long run. Still, almost sure stabilization of paths is compatible with a self-fulfilling appearance of $\sigma_0^s > 0$ by the latter result of the Proposition, $\mathbb{E}_0(\max_t (\sigma_t^s)^2) = \infty$, which implies that an initial self-fulfilling arise in σ_0^s is sustained by a *vanishing* probability of an ∞-large equilibrium volatility in some future paths.

Intuition We provide calibrated simulation results in Section 4. Here we explain in a detailed manner the intuition for (i) how an initial sunspot σ_0^s in the aggregate volatility can appear out of thin air, and (ii) the results in Proposition 1. For that purpose, we simplify the economic environment and make the following assumptions:

A.1 A shock dZ_t at each period takes one of two values: $\{+1, -1\}$ with equal probability.

A.2 Martingale equilibrium: the output gap $\hat{Y}_t$ equals the conditional expected value of the next-period gap $\hat{Y}_{t+1}$. Thus, if $\hat{Y}_{t+1}$ takes either $\hat{Y}^{(1)}_{t+1}$ or $\hat{Y}^{(2)}_{t+1}$ when $dZ_{t+1} = 1$ or -1 respectively,

$$\hat{Y}_t = \frac{1}{2}(\hat{Y}^{(1)}_{t+1} + \hat{Y}^{(2)}_{t+1}).$$

A.3 Aggregate demand (i.e., output gap) $\hat{Y}_t$ falls, as the conditional variance of the next-period's $\hat{Y}_{t+1}$ rises, due to precautionary saving. $\{\hat{Y}_t\}$ and $\{\sigma^s_t\}$ are zero on the stabilized path (i.e., flexible-price economy)

Since we have two possible realizations of the shock at each period, we can draw a tree diagram as in Figure 1. In Figure 1, the thick vertical line represents the stabilized path, with

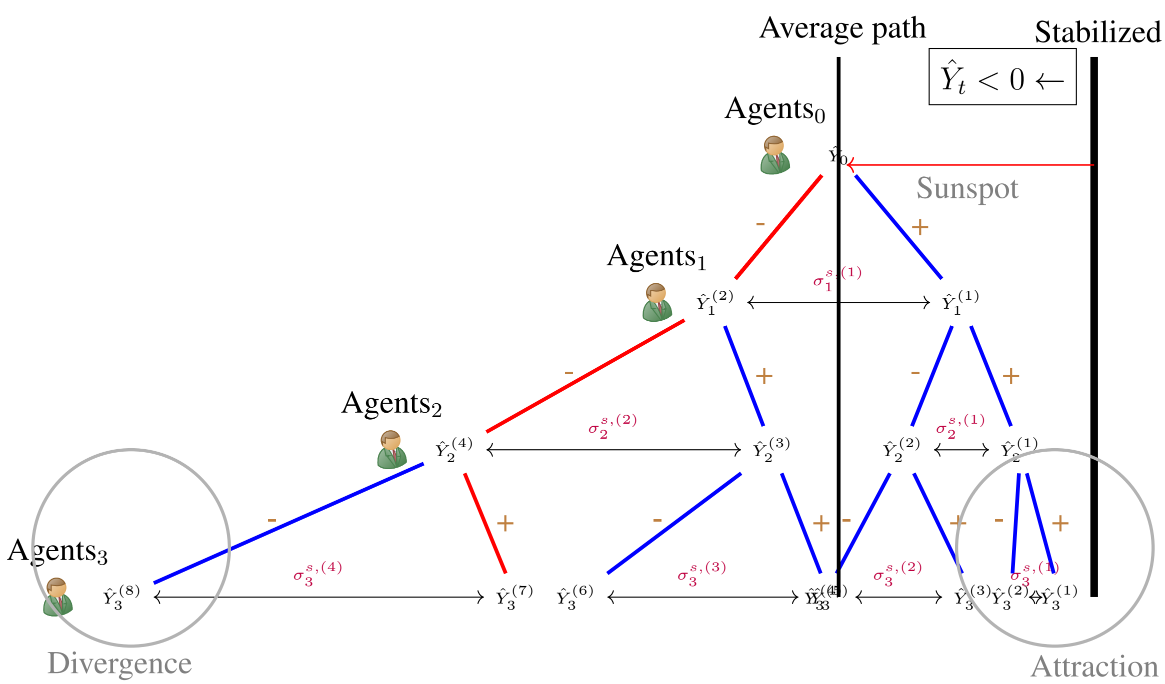


Figure 1: A sunspot in σ^s_0 as a rational expectations equilibrium

areas at its left and right representing recessions and booms, respectively. The key to build a rational expectations equilibrium supporting a sunspot $\sigma^s_0 > 0$ is to construct the agents' path-dependent consumption strategy with time-varying conditional volatilities. First, let us imagine that the the current period agents (Agents$_0$) suddenly believe that the future agents will choose the path-dependent consumption demand[21] so that the next-period's $\hat{Y}_1$ becomes $\hat{Y}_1^{(1)}$ after $dZ_1 = +1$ is realized and $\hat{Y}_1^{(2)}$ if $dZ_1 = -1$ is realized, with $\hat{Y}_1^{(1)} > \hat{Y}_1^{(2)}$. Then the current output $\hat{Y}_0$ becomes $\hat{Y}_0 = \frac{1}{2}(\hat{Y}_1^{(1)} + \hat{Y}_1^{(2)})$ with $\hat{Y}_0$ below the stabilized path, as Agents_0 believe there exists dispersion in next-period outcomes, which is given as $\sigma_1^{s,(1)} = \hat{Y}_1^{(1)} - \hat{Y}_1^{(2)}$, which leads to lower consumption through precautionary savings at $t = 0$. Assume that $dZ_1 = -1$ is realized. For Agents_0's belief $\hat{Y}_1 = \hat{Y}_1^{(2)}$ to be consistent, Agents_1 must believe that future agents will choose their consumption paths in a way that the next period's $\hat{Y}_2$ becomes $\hat{Y}_2^{(3)}$ with $dZ_2 = +1$ and $\hat{Y}_2^{(4)}$ with $dZ_2 = -1$, with conditional volatility $\sigma_2^{s,(2)} = \hat{Y}_2^{(3)} - \hat{Y}_2^{(4)}$ higher than $\sigma_1^{s,(1)}$, since $\hat{Y}_1^{(2)}$ is lower than the initial output $\hat{Y}_0$.

After dZ_2 is realized, Agents_1's belief about $\hat{Y}_2$ can be made consistent through future agents $\{\text{Agents}_{n\geq 2}\}$'s coordination in a forward looking fashion. Observe that all the nodes in Figure 1 satisfy assumptions A.2 and A.3, with distance between adjacent nodes getting progressively narrower (wider) as output gap gets closer (farther) to the stabilized path. This results in divergent and attraction paths balancing each other out, and in expectation, output gap $\{\hat{Y}_t\}$ follows a martingale process.

In sum, Agents_0's initial doubt (sunspot) that the next-period's outcome will be volatile is made consistent by coordination between intertemporal agents (i.e., the representative household) at each node.[22]

Note that (i) we obtain an equilibrium with the *stochastic* aggregate volatility: i.e., σ_t^s is dependent on the path of shocks, as output gap $\{\hat{Y}_t\}$ is stochastic and negatively depends on the conditional volatility of its next-period level. Equation (14) specifies the exact stochastic process of $\{\sigma_t^s\}$ starting from $\sigma_0^s > 0$; (ii) Since volatility σ_t^s decreases as output gap $\hat{Y}_t$ approaches the stabilized path, this path becomes an attraction point for the set of alternative paths in its neighborhood, justifying the result of Proposition 1 that σ_t^s almost surely

[21] Remember, agents' consumption demand determines output in this demand-determined environment with rigid prices.

[22] This equilibrium is feasible since all future agents share the common knowledge of their consumption strategies and there is no behavioral friction blocking communications between agents in intertemporal periods (i.e., perfect recall). Our sunspot equilibrium is closely related to notion of 'self-confirming equilibrium'. See Fudenberg and Levine (1993) for this issue. For how limited recall (friction in memory) removes indeterminacy, see Angeletos and Lian (2022).

converges to zero over time. Nonetheless, as volatility σ_t^s rises whenever output $\hat{Y}_t$ deviates farther from the stabilized level, this also aligns with the result of Proposition 1 that a maximal σ_t^s diverges, $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$. However, this behavior of divergence only happens with vanishing probability as $\sigma_t^s \stackrel{a.s}{\to} 0$.

The conclusion in terms of monetary policy is that a conventional Taylor rule almost surely stabilizes the disruption caused by a $\sigma_0^s > 0$ sunspot in the long-run, but does not prevent the economy from entering a crisis phase with self-created volatility $\{\sigma_t^s\}$ starting from $\sigma_0^s > 0$ and low aggregate demand.

Escape clause If central bank and/or government credibly commit to prevent $\hat{Y}_t$ from going below a predetermined threshold through interventions,[23] these sunspot equilibria arising from the aggregate financial volatility σ_0^s supported by paths in Figure 1 (i.e., martingale equilibrium) are not sustained anymore as a possible rational expectations equilibrium (REE). This escape clause illustrates how the credible commitment of the government entities to intervene whenever the economy probabilistically enters a big recession actually precludes a possibility of the crisis phase initiated by the positive sunspot shock $\sigma_0^s > 0$.

Whether this type of commitment from government and central bank is credible is important, as here we need a 100% credibility to kill our equilibrium construction supporting $\sigma_0^s > 0$.

Negative sunspot We can similarly construct a rational expectations equilibrium that supports an initial negative volatility sunspot $\sigma_0^s < 0$. This equilibrium features a boom phase with buoyant aggregate demand and low business cycle volatility.[24] Therefore, we conclude that our non-linear characterization generates a reasonable prediction of (i) appearance of

[23]For example, government might commit to incur huge fiscal deficits whenever the economy undergoes a severe recession. This prescription entails similar implications about what government can do to restore determinate equilibrium to Benhabib et al. (2002). Benhabib et al. (2002) especially deals with the role of monetary-fiscal regimes in regards to eliminating indeterminacy posed by ZLB. In a similar way, Obstfeld and Rogoff (2021) shows how a probabilistic (and small) fiscal backing to the currency by government rules out speculative hyper-inflations in monetary models.

[24]As seen in equation (6), the actual output Y_t's process features $\sigma + \sigma_t^s$ as its conditional volatility. Thus, a self-created negative excess volatility $\sigma_0^s < 0$ reduces the volatility of the growth rate of Y_t from σ to $\sigma + \sigma_t^s$.

sunspot boom/crisis phases coming from self-created volatilities, and (ii) the joint evolution of the first (output level) and second (conditional volatility) order moments of the model during crises and booms.[25]

Next, we study a monetary policy rule that restores model determinacy.

2.2 A New Monetary Policy

Let's assume, instead, that the central bank follows this alternative policy rule:

$$i_t = r^n + \phi_y \hat{Y}_t - \underbrace{\frac{1}{2}\left(((\sigma + \sigma_t^s)^2 - \sigma^2\right)}_{\text{Aggregate volatility targeting}}, \quad \text{where } \phi_y > 0, \tag{15}$$

which, in addition to output gap $\hat{Y}_t$, targets the aggregate volatility of the output gap with a coefficient $\frac{1}{2}$. By plugging the above monetary policy into the IS equation in (8), the volatility feedback terms in the drift part cancels out and therefore, we obtain an expression equal to (12), which guarantees model determinacy and ensures $\hat{Y}_t = 0, \forall t$ as a unique rational expectations equilibrium when the Taylor principle $\phi_y > 0$ is satisfied. Therefore, we conclude that monetary policy following (15) kills all kinds of potential self-fulfilling aggregate volatilities.

Interpretation The additional volatility target in the policy rule is necessary to offset the feedback channel between the endogenous volatility of the output gap and its drift. To get a more intuitive interpretation of this result, we can rearrange equation (15) as

$$i_t = r_t^T + \phi_y \hat{Y}_t, \quad \text{where } \phi_y > 0, \tag{16}$$

where r_t^T is the risk-adjusted natural rate defined in equation (9). Therefore, an alternative interpretation is that monetary policy in a risky environment should target the risk-adjusted, and not simply the natural, interest rate. Note that r_t^T in our environment is time-varying, as

[25] Our sunspot equilibrium can be interpreted as capturing the occurrence of animal spirit shocks. For the neoclassical treatment of this topic, see Angeletos and La'O (2013).

it depends on the potential excess volatility σ_t^s.

A problem with the new policy rule in equation (16) is that it seems very difficult to implement *in practice*, as neither the output volatility components $\{\sigma,\ \sigma_t^s\}$ nor the risk-adjusted natural rate r_t^T are directly observable. In Section 3, we offer an alternative theoretical model that explicitly incorporates stock markets, and show that the commonly observed measures of *financial volatility* or market *risk-premium* serve as a proxy that can be used to effectively implement the rule.

3 The Model with Stock Markets

In Section 3, we consider a slightly different theoretical framework with explicit stock markets, which enables us to analyze the effects of higher-order moments tied to the *aggregate financial volatility* on aggregate demand, and provides us the practical implications about monetary policy rules.

3.1 Setting

Time is continuous, and a *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ is given as in Section 2. The economy consists of a measure one of capitalists, who we regard as neoclassical agents, and the same measure of hand to mouth workers, who we regard as Keynesian agents. As we describe more in detail later, we assume all of the financial wealth is concentrated in the hands of capitalists, while workers finance their consumption out of labor incomes in a similar manner to Greenwald et al. (2014).[26] There is a single source of exogenous variation in the aggregate production technology A_t, which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and evolves according to a geometric process with volatility σ_t:

$$\frac{dA_t}{A_t} = \underbrace{g}_{\text{Growth}} dt + \underbrace{\sigma}_{\text{Fundamental risk}} dZ_t.$$

[26] Greenwald et al. (2014) focus on redistributive shock that shift the share of income between labor and capital as a systemic risk for cross-sectional asset pricing. We instead introduce nominal rigidities in the framework and analyze monetary policy implications.

We regard the aggregate TFP's volatility σ_t as the economy's *fundamental* risk, which we take as exogenous. We assume both g and σ to be constant.[27]

Finally, there is a standard set of intermediate good producers that face nominal price rigidities, making the economy New-Keynesian in nature. Next, we describe roles of each type of agents (i.e., capitalists and workers) and firms.

3.1.1 Firms and Workers

There are a measure one of monopolistically competitive firms, each producing a differentiated intermediate good $y_t(i)$, $i \in [0, 1]$. There also exists a competitive representative firm which transforms intermediates into a final consumption good y_t according to a Dixit-Stiglitz aggregator with an elasticity of substitution $\epsilon > 0$ in the following way:

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

An intermediate firm i has the same production function $y_t(i) = A_t(N_{W,t})^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the economy's aggregate labor, and $n_t(i)$ is the labor demand of an individual firm i at time t. The reason that we introduce a production externality à la Baxter and King (1991) is that it helps us match empirical regularities on asset price and wage co-movements, and it does not affect other qualitative implications of our model.[28] Firm i faces the downward-sloping demand curve $y_i(p_t(i)|p_t, y_t)$, where $p_t(i)$ is the price of its own intermediate good

[27]This assumption is made for simplicity and our analysis can be extended to include cases where σ_t is time-varying and adapted to the Brownian motion Z_t.

[28]In our model, rising asset prices tend to be correlated with the decreasing wage compensation to workers since firm value (stock price) usually rises if firms can pay less to workers. It violates empirical regularities documented by Chodorow-Reich et al. (2021) in which an increase in stock price tends to push up local aggregate demand variables such as employment and wage. Our production function with externality à la Baxter and King (1991) provides us a reasonable calibration that matches these empirical regularities because higher asset prices and aggregate demand raise the firms' marginal product of labor, thus increasing labor demand and wages. Basically, our externality plays similar roles to the capital in the production function, with a higher degree of tractability.

and p_t, y_t are the aggregate price index and output, respectively:

$$y_i(p_t(i)|p_t, y_t) = y_t \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon}.$$

The set of prices charged by intermediate good firms, $\{p_t(i)\}$, is aggregated into the price index p_t as

$$p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}. \tag{17}$$

We also impose a nominal price rigidity à la Calvo (1983), and firms can change prices of their own intermediate goods with δdt probability in a given time interval dt. In the cross-section, this implies that a total δdt portion of firms reset their prices during a given dt time interval.

A representative hand-to-mouth worker supplies labor to intermediate good producers, receives the equilibrium wage income, and spends every dollar he earns on final good consumption. Each worker solves

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad \text{s.t.} \quad p_t C_{W,t} = w_t N_{W,t}, \tag{18}$$

at every moment t, where $C_{W,t}$, $N_{W,t}$ and w_t are his consumption, labor supply, and equilibrium wage at time t, respectively, and χ_0 is the inverse Frisch elasticity of labor supply. Note that we normalize consumption $C_{W,t}$ by technology A_t, which governs the economy's size.[29] As wage w_t is homogeneous across firms, labor demanded by each firm i, $\{n_t(i)\}$, are simply combined into aggregate labor $N_{W,t}$ in a linear manner as in

$$N_{W,t} = \int_0^1 n_t(i) di. \tag{19}$$

[29]We introduce the consumption normalization by the aggregate TFP due to the economy's growth. The qualitative results of the model are not affected by this consumption normalization, which we introduce to simplify the analytic expressions of the model.

Final good output y_t can be written as

$$y_t = \frac{A_t N_{W,t}}{\Delta_t}, \quad \text{where} \quad \Delta_t \equiv \left(\int_0^1 \left(\frac{p_t(i)}{p_t} \right)^{-\frac{\epsilon}{1-\alpha}} di \right)^{1-\alpha}. \tag{20}$$

where Δ_t is defined as the price dispersion measure. Due to the externality à la Baxter and King (1991), the aggregate production function becomes linear in $N_{W,t}$.

3.1.2 Financial Market and Capitalists

Unlike conventional New-Keynesian models where a representative household owns the intermediate goods sector and receives rebated profits in a lump sum way,[30] we assume firm profits are capitalized in financial markets as a representative index fund. Capitalist then faces an optimal portfolio decision problem involving the allocation of their wealth between a risk-free bond and the risky stock at every instant t.

The nominal aggregate financial wealth of the economy is $p_t A_t Q_t$, where Q_t is the normalized (or TFP detrended) real asset price. Q_t is an endogenous variable adapted to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and assumed to evolve according to

$$\frac{dQ_t}{Q_t} = \mu_t^q dt + \underbrace{\sigma_t^q}_{\text{Financial volatility}} dZ_t,$$

with endogenous drift μ_t^q and volatility σ_t^q. In particular, we interpret σ_t^q as a measure of financial uncertainty or disruption, as spikes in asset price volatility σ_t^q is empirically observed during a financial crisis. Like Q_t, we assume that the price aggregator p_t follows such a general stochastic process as

$$\frac{dp_t}{p_t} = \pi_t dt + \underbrace{\sigma_t^p}_{\text{Inflation risk}} dZ_t, \tag{21}$$

in which drift π_t and volatility σ_t^p are endogenous. Therefore, it follows that the total financial

[30] We already studied a non-linear standard New-Keynesian model in Section 2.

market wealth $p_tA_tQ_t$ evolves with a geometric Brownian motion with total volatility $\sigma + \sigma_t^q + \sigma_t^p$. Intuitively, if a capitalist invests in the stock market, they have to bear all three risks: inflation risk, technology (fundamental) risk, and (detrended) real asset price risk.

Here, σ_t^q is determined in equilibrium and can be either positive or negative, i.e., $\sigma_t^q < 0$ corresponds to the case where total real wealth A_tQ_t is less volatile than the TFP process $\{A_t\}$. The nominal price process has inflation rate π_t as its drift, and in general has a volatility part σ_t^p, which we call an inflation risk. In most cases other than the flexible price benchmark, we show that $\sigma_t^p = 0$ holds and do not need to concern ourselves with this term.

In addition to the stock market, we assume that there is a risk-free bond with an associated nominal rate i_t that is controlled by the central bank. Bonds are in zero net supply in equilibrium since all capitalists are equal. A measure one of identical capitalists chooses the portfolio allocation between a risk-free bond and a risky index stock, where in the latter case, they earn the profits of the intermediate goods sector as dividends, as well as the nominal price revaluation of the index due to changes in p_t, A_t and Q_t. Financial markets are competitive, thus each capitalist takes the nominal risk-free rate i_t, expected stochastic stock market return i_t^m, and the risk level $\sigma + \sigma_t^q + \sigma_t^p$ as given when choosing her portfolio decision.[31] If a capitalist invests a share θ_t of her wealth a_t in the stock market, she bears a total risk $\theta_t a_t(\sigma + \sigma_t^q + \sigma_t^p)$ between t and $t + dt$. Therefore, the riskiness of her portfolio increases proportionally to the investment share θ_t in the index. Capitalists are risk-averse, and ask for a risk-premium compensation $i_t^m - i_t$ when they invest in the risky index market, which is to be determined in equilibrium.

Each capitalist with nominal wealth a_t has log-utility and solves

$$\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \;\text{ s.t. }\; da_t = \left(a_t\left(i_t + \theta_t(i_t^m - i_t)\right) - p_tC_t\right)dt + \theta_t a_t\left(\sigma + \sigma_t^q + \sigma_t^p\right)dZ_t, \tag{22}$$

where ρ, C_t are her discount rate and final good consumption, respectively. At every instant,

[31] This competitive market assumption is related to the reason we assume a measure one of identical capitalists. This assumption turns out to be an important aspect of our model for explaining inefficiencies caused by aggregate demand externality that individual capitalist's financial investment decision imposes on the aggregate economy. For this issue, see Farhi and Werning (2016).

she earns returns out of both the risk-free bond and the risky stock investments, and spends on final good consumption.

3.2 Equilibrium and Asset Pricing

Due to the log-utility of capitalists, their nominal state price density ξ_t^N[32] is given by

$$\xi_t^N = e^{-\rho t}\frac{1}{C_t}\frac{1}{p_t}, \text{ where } \mathbb{E}_t\left(\frac{d\xi_t^N}{\xi_t^N}\right) = -i_t dt \tag{23}$$

where the stochastic discount factor between time t (now) and s (future) is by definition given as $\frac{\xi_s^N}{\xi_t^N}$. Aggregate stock market wealth, $p_t A_t Q_t$, is by definition the sum of discounted profit streams from the intermediate goods sector, priced by the above ξ_t^N, as capitalists are marginal stock market investors in equilibrium. We know that at time t, the entire profit of the intermediate goods sector is given by

$$D_t \equiv \int (p_t(i)y_t(i) - w_t n_t(i))di = \underbrace{\int p_t(i)y_t(i)di}_{=p_t y_t} - \underbrace{w_t N_{W,t}}_{=p_t C_{W,t}} = p_t(y_t - C_{W,t}) = p_t C_t,$$

where we use the Dixit-Stiglitz aggregator properties that the total expenditure equals a sum of expenditures on intermediate goods and the linear aggregation of labor, i.e., equation (19). Regardless of the price dispersion across firms, the aggregate dividend D_t is equal to the consumption expenditure of capitalists, who are the natural stock market investors as hand-to-mouth workers spend all of their incomes on consumption.

Plugging the above expressions into the fundamental asset pricing equation yields the following condition:

$$p_t A_t Q_t = \mathbb{E}_t \frac{1}{\xi_t^N}\int_t^\infty \xi_s^N \left(\underbrace{D_s}_{=p_s C_s \text{ from (26)}}\right) ds = \frac{p_t C_t}{\rho}, \tag{24}$$

which becomes $p_t C_t = \rho\left(p_t A_t Q_t\right)$, which is equal to ρa_t in equilibrium with $a_t = p_t A_t Q_t$,

[32] A superscript N means a nominal state-price density, where a superscript r implies a real one.

i.e., in equilibrium, capitalists hold a wealth amount that equals the total financial market wealth.

Every agent with the same type (i.e., worker or capitalist) is identical and chooses the same decisions in equilibrium. Because bonds are in zero net supply, each capitalist's wealth share θ_t in the stock market must satisfy $\theta_t = 1$, which pins down the equilibrium risk-premium demanded by capitalists. Using (22), (23), and (24), risk-premium is given by[33]

$$\mathrm{rp}_t \equiv i_t^m - i_t = \underbrace{(\sigma + \sigma_t^q + \sigma_t^p)^2}_{\text{Risk-premium}}, \tag{25}$$

where the risk-premium rp_t demanded by capitalists increases with either of the three volatilities $\{\sigma, \sigma_t^q, \sigma_t^p\}$. As the financial volatility σ_t^q is endogenous, the risk-premium rp_t term is endogenous as well and needs to be determined in equilibrium. Note that the wealth gain/loss of the capitalist is equal to the nominal revaluation of the stock. Also note that our equilibrium conditions in (24) and (25) align with Merton (1971).

We characterize the good's market equilibrium and the equilibrium asset pricing condition of the expected stock return i_t^m as follows: Since capitalists spends ρ portion of their wealth a_t on consumption expenditure and they hold the entire wealth, $C_t = \rho A_t Q_t$ holds in equilibrium. Therefore, we can write the equilibrium condition for the final good market as[34]

$$\rho A_t Q_t + \frac{w_t}{p_t} N_{W,t} = \frac{A_t N_{W,t}}{\Delta_t} = y_t. \tag{26}$$

The nominal expected return on the stock market i_t^m consists of the dividend yield from the intermediate goods sector profits and the nominal stock price re-valuation (i.e., capital gain) due to fluctuations in $\{p_t, A_t, Q_t\}$. Within our specifications, the dividend yield always is equal to ρ, the discount rate of capitalists. Therefore, when i_t^m changes, only nominal stock prices can adjust endogenously, as the dividend yield is fixed.

If we define $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process with $\mathbb{E}_t\left(dI_t^m\right) = i_t^m dt$, the following (27) shows the decomposition of i_t^m into dividend yield and stock revaluation

[33] In Appendix C, we derive equation (25) more in detail.

[34] Here $N_{W,t}$ is the solution of the worker's optimization problem in (18).

in equilibrium:

$$d\mathbf{I}_t^m = \frac{\overbrace{\cancel{p_t}\left(\underbrace{y_t - \frac{w_t}{p_t}N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\cancel{p_t}A_tQ_t}_{\text{Total capital market wealth}}}dt + \underbrace{\frac{d\left(p_tA_tQ_t\right)}{p_tA_tQ_t}}_{\text{Capital gain}} = \rho \cdot dt + \frac{d\left(p_tA_tQ_t\right)}{p_tA_tQ_t} \tag{27}$$

$$= \underbrace{\left(\rho + \underbrace{\pi_t}_{\text{Inflation}} + g + \mu_t^q + \sigma_t^q\sigma_t^p + \sigma(\sigma_t^p + \sigma_t^q)\right)}_{=i_t^m} dt + \underbrace{\left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{\text{Risk term}} dZ_t.$$

The equilibrium conditions we have obtained consist of the worker's optimization (i.e., solution of (18)), labor aggregation (i.e., (19)), output formula (i.e., (20)), capitalist's optimization (i.e., (24) and (25)), the good market equilibrium (i.e., (26)), and determination of the risky stock return (i.e., (27)). To close the model, we also have to derive the supply block of the economy (i.e., pricing decisions of intermediate good firms à la Calvo (1983)) and define the monetary policy rule, which is the most important topic of our interest.

Before we characterize the benchmark case without nominal rigidity, the following Lemma 1 re-derives the Fisher equation when there is a correlation between the (aggregate) price process and the wealth process. It shows that the inflation premium should be added to the original Fisher relation.

Lemma 1 (Inflation Premium) *Real interest rate is given by the following variant of the Fisher identity.*

$$r_t = i_t - \pi_t + \underbrace{\sigma_t^p \underbrace{\left(\sigma + \sigma_t^p + \sigma_t^q\right)}_{\textit{Wealth volatility}}}_{\textit{Inflation premium}} \tag{28}$$

Lemma 1 is useful when we characterize the flexible price equilibrium of the model where the nominal price process is arbitrary and does not affect the real economy.

3.3 Flexible Price Equilibrium

As a benchmark case, we study the flexible price equilibrium. When firms can freely reset their prices (i.e., $\delta \to \infty$ case), the real wage $\frac{w_t}{p_t}$ becomes proportional to aggregate technology A_t. The following proposition summarizes real wage, asset price, natural rate of interest r_t^n (i.e., the real risk-free rate that prevails in the benchmark economy), and consumption process of the capitalist in the flexible price equilibrium. Before we proceed, we define the following parameter, which is the effective labor supply elasticity of workers taking their optimal consumption decision into account.

Definition 1 *Effective labor supply elasticity of workers:* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0 + \varphi}$

Proposition 2 (Flexible Price Equilibrium) *In the flexible price equilibrium,*[35] *we obtain the analytic characterization of real wage* $\frac{w_t^n}{p_t^n}$*, asset price* Q_t^n*, natural rate of interest* r_t^n*, and consumption of capitalists* C_t^n *as given below:*

(i) Every firm charges the same price leading to $\Delta_t = 1, \forall t$*, and the real wage is proportional to aggregate technology* A_t*, and given by*

$$\frac{w_t^n}{p_t^n} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon} A_t$$

(ii) The equilibrium detrended asset price Q_t^n *is constant and given by*

$$Q_t^n = \frac{1}{\rho}\left(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{\frac{1}{\chi}}\left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \quad \text{and} \quad \mu_t^{q,n} = \sigma_t^{q,n} = 0 \tag{29}$$

(iii) The natural interest rate r_t^n *is constant, and depends on parameters* ρ, g, σ *in the following way:*

$$r_t^n \equiv r^n = \rho + g - \sigma^2 \tag{30}$$

[35] We assign superscript n to denote variables in the flexible price (i.e., natural) equilibrium of the economy.

(iv) Consumption of capitalists evolves with

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \left(\underbrace{r^n - \rho + \sigma^2}_{\equiv \mu_t^{c,n}}\right) dt + \underbrace{\sigma}_{\equiv \sigma_t^{c,n}} dZ_t, \tag{31}$$

whose drift and volatility $\mu_t^{c,n}$ and $\sigma_t^{c,n}$ are constant and depend on parameters r^n, ρ, σ. Equation (31) has the same form with equation (5) in Section 2.

In flexible price equilibrium, proposition 2 shows that we can characterize closed-form expressions of the real wage $\frac{w_t^n}{p_t^n}$, detrended stock price Q_t^n, and the natural rate r_t^n. A few points are worth mentioning. In the flexible price economy, $\sigma_t^{q,n} = 0$ holds, which implies that there is no additional financial risk running in the economy, in addition to the TFP risk, σ. This feature arises because our economy features no explicit frictions (other than nominal rigidity, which is absent for now) and thus every variable other than the labor supply $N_{W,t}^n$ becomes proportional to A_t. This means that real wealth $A_t Q_t^n$ has the exact same volatility as A_t itself, and the financial market imposes no additional risk on the economy.

A higher ϵ increases competition among firms, raising real wage $\frac{w_t^n}{p_t^n}$. It also has two competing effects on asset price Q_t^n. A higher real wage pushes down the profit of the intermediate sector and reduces the stock price Q_t^n. On the other hand, a higher wage induces workers to supply more labor to firms, raising output and stock price Q_t^n. The effective labor supply elasticity χ^{-1} matters in this second effect, thus equation (29) features χ^{-1} exponent on the term that increases with ϵ. As Q_t^n is constant, its drift $\mu_t^{q,n}$ also satisfies $\mu_t^{q,n} = 0$ for all t.

The natural real interest rate r_t^n consists of two parts with countervailing forces. A higher growth rate g raises the expected value of future cash flows generated by firms, driving up the wealth of capitalists and adding upward pressures on aggregate demand, thereby requiring a higher natural interest rate to balance demand and supply in the flexible price economy. On the other hand, a higher σ pushes down the natural rate r_t^n through the risk-premium channel: a higher σ raises the stock market's equilibrium risk-premium level, inducing capitalists to reduce their portfolio demand for the index, forcing r_t^n to go down in order to prevent a fall in their financial wealth. This channel is a little bit different from the well-documented precautionary savings channel in Section 2's standard New-Keynesian framework even though two models share the similar mathematical structure (e.g., equation (5) is of the same form as (31)): in a standard New-Keynesian model, a rise in σ induces capitalists to save more in a precautionary way, bringing down the natural rate r_t^n.[36] The risk-premium channel is present in our framework as we explicitly model portfolio decisions of capitalists, which collectively pin down the equilibrium wealth and thus aggregate demand. However, as we clearly show in Section 2, our purpose here is to illustrate that not only the exogenous volatility σ, but also the endogenous volatility σ_t^q plays a key role in generating business cycles, and isomorphic mathematical structure between two models will be clearer when we explain those parts.

With the flexible price equilibrium as a benchmark, we move on to the sticky price equilibrium and show how our framework features a role of endogenous financial volatility σ_t^q.

3.4 Sticky Price Equilibrium

When price resetting is sticky à la Calvo (1983), we obtain the Phillips curve that describes inflation dynamics. Since a fixed portion δdt of firms can change their prices on a given infinitesimal time interval dt, we have no stochastic fluctuation in the price process in (21) up to a first order, thus $\sigma_t^p = 0$ holds.[37] Now, we just need a monetary policy rule to close the model. Before analyzing the proper monetary rule in this framework, we first describe the 'gap' economy, which is defined as the economy where every variable is a $\log$-deviation from the corresponding level in the flexible price economy. That is, we define any business cycle variable x_t's gap, $\hat{x}_t$, to be the $\log$-deviation of x_t from its natural level x_t^n, which is the level of the variable in the flexible price equilibrium:

$$\hat{x}_t \equiv \ln \frac{x_t}{x_t^n}.$$

[36]For example, see Acharya and Dogra (2020) for the recent treatment of precautionary saving in the New-Keynesian environment in the presence of idiosyncratic risks.

[37]Following Section 2, we solve the model's demand block globally, accounting for the non-linear effects of higher-order terms. To simplify the analysis, we linearize the supply block, following Woodford (2003).

Because the asset price acts as an endogenous aggregate demand shifter, we first write every other variable's gap in terms of the asset price gap. The following Assumption 1 is the first step.[38]

Assumption 1 (Labor Supply Elasticity) $\chi^{-1} > \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Assumption 1 guarantees the positive co-movement between asset price and other business cycle variables (e.g., real wage and consumptions of capitalists and workers) observed in data. With large ϵ, firms' mark-ups decrease as competition between them intensifies, and real wage rises as a result. This has a negative impact on the stock price as firm profits decrease, making it harder to satisfy a positive co-movement between gaps in asset price and real wage.[39] An increase in α amplifies the effect of the Baxter and King (1991) externality, raising labor demand: so that a rise in asset price can result in higher labor demand and real wage. Without Assumption 1, a positive gap in the asset price depresses wages, labor, and consumption of workers, which can be regarded a redistributive shock from labor to capital, or in the longer-run, might explain a portion of the observed trend towards increased wealth inequality and income stagnation.[40]

The following Lemma 2 argues that given Assumption 1, gaps in consumptions of capitalists and workers, asset price, employment, and real wage all co-move with one another up to a first-order. For stabilization purposes, therefore, the central bank only needs to deal with the asset price gap $\hat{Q}_t$. From $C_t = \rho A_t Q_t$, we infer that $\hat{Q}_t = \hat{C}_t$ holds. Thus from now on we can interchangeably use $\hat{Q}_t$ or $\hat{C}_t$ to denote gaps of asset price Q_t and consumption of capitalists C_t.

[38] Assumption 1 ensures that our model matches empirical regularities observed in the data, and holds under a standard calibration of the model: see Table 3 in Appendix. Even without Assumption 1, the main qualitative features of our model remain unchanged.

[39] When ϵ increases, firms' profits per revenue decrease with more elastic demand. Drops in profits lead to decreases in the asset price as well as consumption of capitalists, while hand-to-mouth workers consume more with higher wage incomes. A higher χ^{-1} leads to a higher aggregate TFP elasticity of output, which tends to generate a positive correlation between capitalists' and workers' consumption.

[40] For instance, see Saez and Zucman (2020) for historical trends on rising wealth and income inequality in the US. Also, see Autor et al. (2020) for evidence on a decreasing labor share and effects from the rise of market concentration. Especially, growth in pre-tax income for bottom 50% has been only 0.2% on average per year since 1980s, while S&P-500 index has risen almost by 8% per year. For shorter-run interpretation, Greenwald et al. (2014) interpret redistributive shocks that shift the share of income between labor and capital as a systemic risk to explain various asset pricing phenomena.

Lemma 2 (Co-movement) *Given Assumption 1, gaps in consumption of capitalists C_t and workers $C_{W,t}$, employment $N_{W,t}$, and real wage $\frac{w_t}{p_t}$ co-move with a positive correlation. Up to a first-order, the following approximation holds:*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{\frac{w_t}{p_t}} = \frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right) \widehat{C_{W,t}}.$$

Using Lemma 2, we can actually get the following relation between $\hat{Q}_t$ and $\hat{y}_t$.

$$\hat{y}_t = \zeta \hat{Q}_t, \quad \text{where} \ \zeta \equiv \chi^{-1} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1} > 0, \tag{32}$$

Demand block Now we formulate one of the key building blocks of this paper, a $\{\hat{Q}_t\}$ process. This $\{\hat{Q}_t\}$ process serves as the demand block of the model, while the Phillips curve will serve as a supply block.

The dynamic IS equation of $\{\hat{Q}_t\}$ in our model features some important modifications from the canonical New-Keynesian framework. Before we characterize it, we define the risk-premium level $\mathrm{rp}_t \equiv (\sigma + \sigma_t^q)^2$ and its natural level in the flexible price economy $\mathrm{rp}_t^n \equiv \sigma^2$ with $\sigma_t^{q,n} = 0$, as we characterized in equation (29). By subtracting rp_t^n from the current risk-premium level rp_t, we define risk-premium gap $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$. Basically, as the risk-premium gap becomes positive in the absence of monetary policy responses, capitalists ask for a higher compensation in bearing financial market risks, causing asset prices to fall below its natural level. We also define the risk-adjusted natural rate r_t^T in the similar way to (9) in Section 2:

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t. \tag{33}$$

r_t^T serves as a real rate anchor for monetary policy. A positive risk-premium gap (i.e., $\hat{rp}_t > 0$), for example, lowers the portfolio demand of capitalists for the stock market compared with the benchmark economy, and thus decreases the risk-free rate r_t^T that monetary policy must target for stabilization purposes.

In the following proposition, we characterize an asset price gap $\hat{Q}_t$'s stochastic process. As in equation (8) of Section 2's standard non-linear New-Keynesian framework, the natural rate r_t^n is replaced with the risk-adjusted natural rate r_t^T.

Proposition 3 (Asset Price Gap Process: IS Equation) *With inflation $\{\pi_t\}$, we obtain*

$$d\hat{Q}_t = (i_t - \pi_t - r_t^T)dt + \sigma_t^q dZ_t, \tag{34}$$

where r_t^T takes the role of the natural rate r_t^n. Thus, the aggregate and endogenous financial volatility σ_t^q directly affects the drift of the $\{\hat{Q}_t\}$ process, governing how all other gap variables fluctuate over time.

With $\sigma_t^p = 0$, capitalists bear $(\sigma + \sigma_t^q)$ amount of risk when investing in the stock market. Due to their $\log$ preference, the risk-premium level is determined to be $(\sigma + \sigma_t^q)^2$. In the flexible price equilibrium, the natural rate is given by $r_t^n = r^n = \rho + g - \sigma^2$ and σ_t^q equals $\sigma_t^{q,n} = 0$. Thus, the level of expected (instantaneous) real return on the stock market becomes $r_t^n + \sigma^2 - \frac{1}{2}\sigma^2$, where the factor $\frac{1}{2}\sigma^2$ is from the quadratic variation factor that arises from the second-order Taylor expansion. In our sticky price equilibrium with endogenous asset price volatility σ_t^q, risk premium changes from σ^2 to $(\sigma + \sigma_t^q)^2$. Therefore, with monetary policy rate i_t and inflation π_t, the (real) expected stock market return becomes $i_t - \pi_t + \frac{1}{2}(\sigma + \sigma_t^q)^2$. If this value differs from $r_t^n + \frac{1}{2}\sigma^2$, then asset price gap $\hat{Q}_t$ endogenously adjusts, and this adjustment creates a real distortion from its effect on aggregate demand.

Equation (34) has the same mathematical structure as equation (8) in the standard New-Keynesian model. In Section 2, the endogenous business cycle volatility has a first-order impact on aggregate demand through precautionary savings channel, whereas in the current framework with stock markets, the aggregate financial volatility affects risk-premium and financial wealth, thereby determining stock prices and aggregate demand. Due to this isomorphic structure between two frameworks, we will show that our novel findings in Section 2 continue to hold here, with important implications about monetary policy.

Thus, we get the lesson that the monetary policy i_t should take deviation in risk-premium from its natural level into account as well as the natural rate of interest r_t^n, since otherwise asset price Q_t will deviate from its natural level and generate business cycle fluctuation. r_t^T can be interpreted as the real risk-free rate that ensures that the real return on stock market investment is equal to its level in the benchmark economy, as shown by

$$r_t^n + \frac{1}{2}\underbrace{(\sigma_t)^2}_{=\mathrm{rp}_t^n} = r_t^T + \frac{1}{2}\underbrace{(\sigma_t + \sigma_t^q)^2}_{=\mathrm{rp}_t}.$$

When $\sigma_t^q = \sigma_t^{q,n} = 0$ holds, the risk-adjusted rate r_t^T equals the natural rate r_t^n and equation (34) becomes the canonical New-Keynesian IS equation:

$$d\hat{C}_t = (i_t - \pi_t - r_t^n)dt. \tag{35}$$

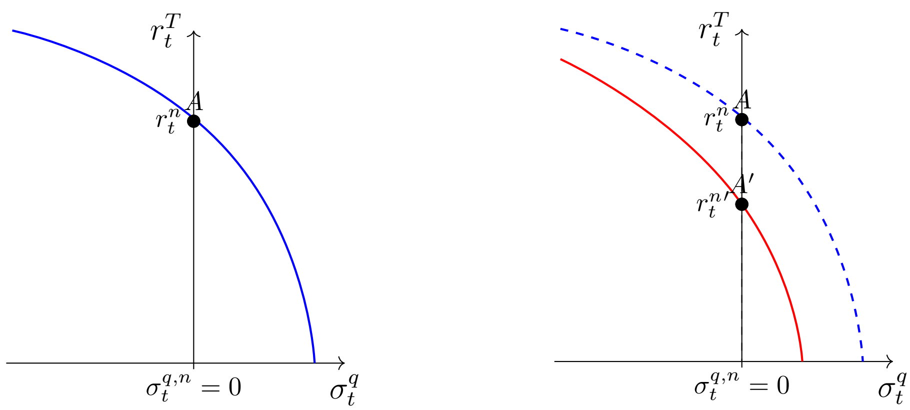
The crux of the problem is that σ_t^q is itself an endogenous variable to be determined in equilibrium, with no guarantee that it will equate its natural level $\sigma_t^{q,n} = 0$.

The endogenous financial volatility σ_t^q can be interpreted a measure of financial disruption, as its rise, given monetary policy rate i_t, reduces stock prices and thus aggregate demand, dragging the economy into recessions. This financial channel has been pointed out by many authors including Gilchrist and Zakrajšek (2012), Stein (2014), Chodorow-Reich (2014), Guerrieri and Lorenzoni (2017), Di Tella and Hall (2021) among others, with different aspects of financial disruption affecting economic activity. Woodford (2012) and Cúrdia and Woodford (2016) especially introduced a friction in credit intermediation between borrowers and savers in the New-Keynesian model and derived similar dynamics for the business cycle, but their friction is exogenous and not related to the aggregate stock market volatility.

The existence of this new stock market volatility channel invites us to re-think the traditional monetary policy framework, to which we devote Section 4. Before we jump on to the next topic, if we plug equation (30) into equation (33), we get the following expression for r_t^T.

$$r_t^T = \rho + g - \frac{\sigma^2}{2} - \frac{(\sigma + \sigma_t^q)^2}{2}.$$

Figure 2a represents r_t^T as a function of σ_t^q given σ level. Intuitively, when σ_t^q jumps up, a rise in risk-premium rp_t ensues and the rate r_t^T falls. We see r_t^T aligns with the natural rate r_t^n when σ_t^q equals $\sigma_t^{q,n} = 0$. Figure 2b illustrates the effect of a spike in σ. When σ rises, the curve in Figure 2a uniformly shifts down. The formula $\sigma_t^{q,n} = 0$ in equation (29) implies that $\sigma_t^{q.n}$ remains unchanged, but the natural rate of interest r_t^n still falls in this case due to equation (30).



(a) r^T as a function of σ_t^q given σ_t

(b) σ rises to $\sigma^{'} > \sigma$

Figure 2: r^T as a function of σ_t^q and σ

Supply block We follow the standard literature on pricing à la Calvo (1983) to determine inflation dynamics. The above Lemma 2 allows us to express the firms' aggregate marginal cost gap in terms of the asset price gap up to a first order, as asset price determines aggregate demand, which in turn determines such variables as the aggregate marginal cost.

The following Phillips curve in Proposition 4 describes π_t dynamics, and is of the same form as in many New-Keynesian models.

Proposition 4 (Phillips Curve) *Inflation π_t evolves according to*

$$\mathbb{E}_t d\pi_t = (\rho\pi_t - \frac{\kappa}{\zeta}\hat{y}_t)dt \;\; \textit{where} \;\; \kappa \equiv \delta(\delta+\rho)\Theta\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1}, \;\; \Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}, \tag{36}$$

where $\hat{Q}_t$ enters in the position of output gap in conventional New-Keynesian models.[41]

Plugging equation (32) into the Phillips curve, we get $\mathbb{E}_t d\pi_t = (\rho\pi_t - \kappa\hat{Q}_t)dt$, which is expressed in terms of $\hat{Q}_t$. Under Assumption 1, a higher asset price gap $\hat{Q}_t$ means the economy is over-heated, and thus inflation rates would jump up. Note that when price resetting probability increases (i.e., $\delta \to \infty$), we have $\kappa \to \infty$ and $\hat{Q}_t = 0$ in equilibrium. Basically, we achieve the flexible price equilibrium when $\delta \to \infty$.

Now that we characterize the model's demand block (i.e., equation (34)) and the supply block (i.e., equation (36)), we need to specify the policy reaction function i_t to close the model. Before we move on to the analysis of policy rules, we briefly discuss the traditional approach to the problem of financial stability and macroeconomic stabilization in the literature.

Macroprudential policies and regulations There are in general two goals in short (and/or medium)-run macroeconomics: *macro-stabilization* and *financial stability*. Many policymakers including central bankers and academic economists believe that financial stability should be dealt with by regulations and macroprudential policies imposed on banks and financial institutions, with business cycle stabilization being the sole focus of monetary policy. Because our model is parsimonious and does not include any complex financial market participants, those macroprudential regulations that tackle potential financial instabilities can be regarded as a policy avenue to prevent σ_t^q from deviating from $\sigma_t^{q,n} = 0$. If $\sigma_t^q = \sigma_t^{q,n} = 0$, then as in equation (35), our model features exactly the same dynamics as conventional New Keynesian models. Therefore, in that case a conventional monetary policy rule can solely focus on business cycle stabilization.

One interesting aspect built in our model is that financial stability issues (i.e., volatility and risk-premium) are intertwined with macro-stabilization. The more volatile financial markets features higher risk-premium levels, driving down both aggregate financial wealth and

[41] We observe that the coefficient $\chi\delta(\delta+\rho)\Theta$ is attached to the output gap $\hat{y}_t$ in (36). In standard New-Keynesian models with a representative agent whose utility is of the same form as our worker's, the coefficient becomes $(\chi_0+\varphi)\delta(\delta+\rho)\Theta$, which is different from $\chi\delta(\delta+\rho)\Theta$ when $\chi \neq \chi_0+\varphi$.

aggregate demand. Our view is that even without perfect macroprudential policies to guarantee $\sigma_t^q = \sigma_t^{q,n} = 0$, monetary policy might be able to tackle both concerns simultaneously, as stabilization in one dimension might help stabilize the other.[42]

Now we move onto the analysis of distinct monetary policy rules and revisit the classical question on the role of monetary policy in regards to financial stability.

4 Monetary Policy

In Section 4, we study the monetary policy's roles of macroeconomic stabilization in the context of our model. First, we analyze conventional Taylor rules with inflation and output gap as policy targets. After showing limitations of such policies and how a self-fulfilling financial volatility can arise, we propose a generalized version of the Taylor rule for stochastic environments that successfully achieve our twin objectives of financial stability and macroeconomic stability. Note that the natural rate of interest and the natural risk-premium are given by $r_t^n = r^n = \rho + g - \sigma^2 > 0$[43] and $\mathrm{rp}_t^n = \mathrm{rp}^n = \sigma^2$.

4.1 Old Monetary Rule

4.1.1 Conventional Taylor rule and Bernanke and Gertler (2000) rule

We start with a conventional Taylor rule with a constant intercept equal to the natural rate r^n, and standard inflation and output gap targets:

$$i_t = r^n + \phi_\pi \pi_t + \phi_y \hat{y}_t,$$

where $\hat{y}_t$ and π_t are the output gap and inflation, respectively. Note that we implicitly assume a zero trend inflation target. As output gap $\hat{y}_t$ is positively correlated with the asset price gap

[42] To compare, our non-linear analysis of a conventional New-Keynesian model in Section 2 shows that the policy reaction function in (15) perfectly stabilizes the business cycle in the presence of first-order roles of aggregate volatility, as it eliminates the possibility that the sunspot volatility is self-created out of thin air.

[43] So that we have no zero lower bound (ZLB) problem throughout Section 4.

$\hat{Q}_t$ from equation (32), we can rewrite the above equation as

$$\begin{aligned} i_t &= r^n + \phi_\pi \pi_t + \underbrace{\phi_y \zeta}_{\equiv \phi_q > 0} \hat{Q}_t \\ &= r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t, \end{aligned} \tag{37}$$

which is the policy reaction function that targets asset price $\hat{Q}_t$ as well as inflation π_t. Bernanke and Gertler (2000), by incorporating stochastic ad-hoc bubble components to asset prices in a model based on financial frictions à la Bernanke et al. (1999), study whether the monetary policy rule that directly targets asset price as in (37) is an effective business cycle stabilizer. Their conclusion is such rules are undesirable as they deter real economic activities when bubbles appear and burst.[44] In contrast, our framework features no *irrational* asset price bubble: fluctuations in $\hat{Q}_t$ reflect the *rational expectations* about business cycle fluctuations, and thus from monetary authority's perspective, targeting the stock price gap $\hat{Q}_t$ becomes equivalent to targeting the output gap $\hat{y}_t$, as the two gaps are perfectly correlated up to a first-order. Therefore in our framework, a conventional monetary policy rule is equivalent to the rule of Bernanke and Gertler (2000).

Now we study whether equation (37) achieves divine coincidence as in textbook New-Keynesian models. Our objective now is to show that (i) this rule cannot guarantee equilibrium determinacy even if it satisfies the so-called Taylor principle; (ii) the aggregate financial volatility σ_t^q can be created in a self-fulfilling way as in Section 2. We define a coefficient $\phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0$, which is the total responsiveness of monetary policy to inflation and asset price gap. $\phi > 0$ corresponds to the conventional Taylor principle that excludes the possibility of sunspot in inflation in log-linearized models. Thus, i_t follows

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t, \text{ where } \phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0. \tag{38}$$

[44]More recently, Galí (2021) introduces rational bubbles in a New-Keynesian model with overlapping generations. He argues that 'leaning against the bubble' monetary policy, if properly specified, can insulate the economy from the aggregate fluctuations due to rational bubbles.

Plugging equation (38) into equation (34), we get the following $\hat{Q}_t$ dynamics:

$$d\hat{Q}_t = \left((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t \underbrace{- \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^q)^2}{2}}_{\text{New terms}} \right) dt + \sigma_t^q dZ_t. \tag{39}$$

Multiple equilibria Omitting the new volatility terms in the above (39), we obtain the usual log-linearized version of the $\hat{Q}_t$ dynamics as

$$d\hat{Q}_t = \left((\phi_\pi - 1)\, \pi_t + \phi_q \hat{Q}_t \right) dt + \sigma_t^q dZ_t,$$

with which the Taylor principle $\phi > 0$ ensures that we achieve the famous *divine coincidence*: $\hat{Q}_t = \pi_t = 0$ for $\forall t$ is the unique possible rational expectations equilibrium from Blanchard and Kahn (1980). In contrast, now that the financial volatility σ_t^q affects the drift of equation (39), we have multiple equilibria and sunspots in σ_t^q might possibly appear out of thin air. The reason is similar to why we have sunspots in endogenous volatility (i.e., σ_t^s in Section 2) in Section 2.[45] Here, the dynamic IS equation in (39) features the countercyclical financial volatility σ_t^q: an increase in σ_t^q raises the risk-premium. It in turn brings down the financial wealth and aggregate demand,[46] thus raising the drift of (39).

Here is an intuitive way to think about the core reason why the financial volatility σ_t^q is created in a self-fulfilling way. Imagine that capitalists in our model suddenly fear of a potential financial crisis that features higher levels of risk-premium and financial volatility: they respond by reducing their portfolio demand for the stock market, which leads to the collapse of the asset price, and self-justifies a higher expected return in the stock market and a rise in risk-premium. This result is related to Acharya and Dogra (2020)'s findings about equilibrium determinacy in models with countercyclical income risks, even though their paper focuses on *idiosyncratic* risks and the effects from precautionary savings, while ours centers on the sunspot equilibria stemming from *aggregate* endogenous risk.

[45] Due to the isomorphic mathematical structure between the dynamics in (34) and equation (8), we easily predict that sunspots in σ_t^q can arise similarly to the ways sunspots in σ_t^s arise in Section 2.

[46] The policy reaction function in (37) responds when its mandates (i.e., $\hat{Q}_t$ and π_t) are affected by the financial volatility σ_t^q, but does not directly target σ_t^q itself.

We now formalize the multiple equilibrium intuition presented above by constructing a rational expectations equilibrium that supports an initial sunspot σ_0^q. For simplicity, we focus on the case in which σ_0^q jumps off from $\sigma_0^{q,n} = 0$ (i.e., $\sigma_0^q > 0$), and study how the sunspot σ_0^q can be rationally sustained in equilibrium. For that purpose, a rational expectations equilibrium must: (i) support an initial hike $\sigma_0^q > 0$, and (ii) not diverge (on expectation) in the long-run, following Blanchard and Kahn (1980).

Martingale equilibrium[47] As in Section 2, we study one particular form of rational expectations equilibrium that supports an initial sunspot σ_0^q: the equilibrium in which asset price gap $\hat{Q}_t$ follows a martingale after the initial sunspot σ_0^q happens. As $\hat{Q}_t$ is martingale, we obtain

$$\pi_t = \kappa \int_t^\infty e^{\rho(s-t)} \underbrace{\mathbb{E}_t(\hat{Q}_s)}_{=\hat{Q}_t} ds = \frac{\kappa}{\rho} \hat{Q}_t, \tag{40}$$

for π_t by iterating equation (36) over time, which implies that inflation π_t closely follows the trajectory of $\hat{Q}_t$. Plugging (40) into equation (39) and imposing a martingale condition, we obtain

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi} + \frac{\sigma^2}{2\phi}, \text{[48]} \tag{41}$$

and

$$\pi_t = \frac{\kappa}{\rho} \left(-\frac{(\sigma + \sigma_t^q)^2}{2\phi} + \frac{\sigma^2}{2\phi} \right).$$

Our martingale equilibrium trajectory does not diverge on expectation in the long-run, as $\{\hat{Q}_t, \pi_t\}$ paths stay, on expectation, at their initial values, thus satisfying $\mathbb{E}_0(\pi_t) = \pi_0$ and $\mathbb{E}_0\left(\hat{Q}_t\right) = \hat{Q}_0$, $\forall t \geq 0$. The last step is to show that there exists a stochastic path of $\{\sigma_t^q\}$ starting from σ_0^q that supports this equilibrium. This equilibrium then both (i) supports an initial sunspot $\sigma_0^q > 0$ and (ii) does not diverge in the long-run. Using equations (39) and

[47] A martingale process for $\hat{Q}_t$ is consistent with the previous findings of the literature on the 'Efficient Market Hypothesis (EMH)' (e.g., see Fama (1970)).

[48] Hence in this particular equilibrium, $\sigma_t^q > \sigma^{q,n} = 0$ causes $\hat{Q}_t$ and $\hat{Y}_t$ to drop below zero, causing a recession.

(41),[49] we obtain the stochastic process of σ_t^q as given by[50]

$$d\sigma_t^q = -\frac{\phi^2 \left(\sigma_t^q\right)^2}{2\left(\sigma + \sigma_t^q\right)^3} dt - \phi \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t. \tag{42}$$

Both (41) and (42) constitute the dynamics of $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ in this particular rational expectations equilibrium supporting $\sigma_0^q > 0$. What does this equilibrium look like? The next Proposition 5 sheds light on the behavior of $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ paths and argues that similarly to Section 2, $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ almost surely converge to a perfectly stabilized path (i.e., $\hat{Q}_t = \pi_t = \sigma_t^q = 0$) in the long run. Few paths that do not converge blow up asymptotically with vanishing probability and together with the forward-looking nature of the economy, help sustain the initial crisis.

Proposition 5 (Bernanke and Gertler (2000) Rule and Indeterminacy) *For any value of Taylor responsiveness $\phi > 0$:*

1. Indeterminacy*: there is always a rational expectations equilibrium (REE) that supports initial sunspot $\sigma_0^q > 0$ and is represented by $\hat{Q}_t$ and π_t dynamics in (41), and σ_t^q process in (42)*

2. Properties*: the rational expectations equilibrium that supports an initial sunspot $\sigma_0^q > 0$ satisfies:*

$$(i)\ \sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0,\ \ (ii)\ \hat{Q}_t \overset{a.s}{\to} 0 \text{ and } \pi_t \overset{a.s}{\to} 0,\ \text{ and }\ (iii)\ \mathbb{E}_0\left(\max_t(\sigma_t^q)^2\right) = \infty$$

Proposition 5 is similar to Proposition 1 due to the isomorphic equilibrium structure between Sections 2 and 4.[51] The conditions $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0$, $\hat{Q}_t \overset{a.s}{\to} 0$, and $\pi_t \overset{a.s}{\to} 0$

[49] Since $\hat{Q}_t$ process is a martingale, the drift part in equation (39) must be 0 almost surely.

[50] When $\sigma = 0$, this process becomes the following Bessel process:

$$d\sigma_t^q = -\frac{\phi^2}{2\sigma_t^q} dt - \phi dZ_t.$$

which stops when σ_t^q reaches $\sigma^{q,n} = 0$. For general properties of Bessel processes, see Lawler (2019).

[51] Even with the presence of nontrivial inflation π_t, Figure 1 illustrates the construction of the martingale equilibrium in Proposition 5.

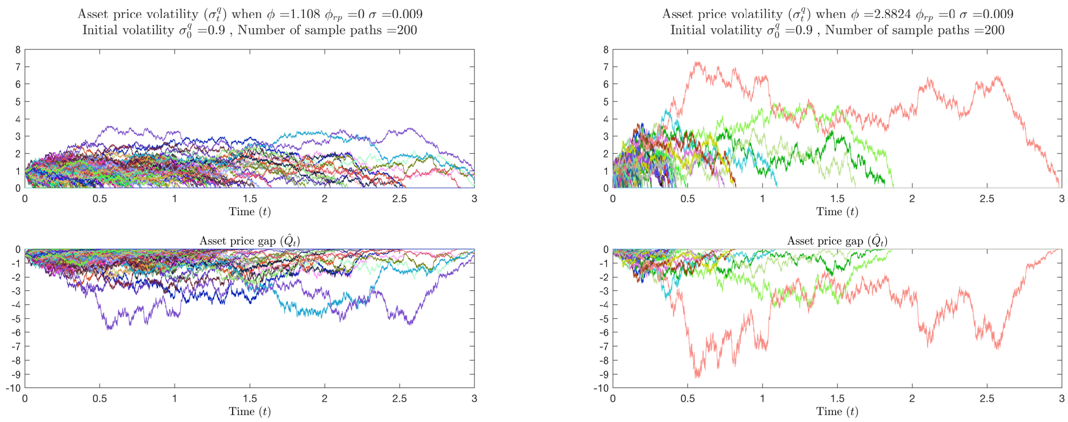
imply that equilibrium paths supporting an initial sunspot $\sigma_0^q > 0$ are almost surely stabilized in the long run. Then, how is it possible for a sunspot $\sigma_0^q > 0$ to appear at first? The finding $\mathbb{E}_0(\max_t(\sigma_t^q)^2) = \infty$ implies that an initial self-fulfilling creation of σ_0^q and the ensuing crisis can be sustained by the vanishing probability of an ∞-severe financial disruption in the long future. This result has similar implications to Martin (2012) in a sense that our framework does not assume the existence of specific disasters but disaster risk is always present even if monetary authority satisfies the Taylor principle and actively stabilizes the business cycle. Martin (2012) applied the similar logic to pure asset pricing contexts and showed that the pricing of a broad class of long-dated assets is driven by the possibility of extraordinarily bad news in the future.

Calibration and Simulation For the rest of the paper, we calibrate our model parameters based on values commonly found in the previous literature: see Table 3 in Appendix B for details. A few points are worth mentioning. For worker's risk-aversion parameter φ, we use $\varphi = 0.2$ following Gandelman and Hernández-Murillo (2014).[52] For an individual firm's labor share in production, we use $1 - \alpha = 0.6$ following Alvarez-Cuadrado et al. (2018), as we regard the aggregate labor in the production function as a proxy for the capital in conventional macroeconomic models. With this calibration, our co-movement condition (i.e., Assumption 1) is satisfied.

Figure 3 illustrates the martingale equilibrium's dynamic paths of $\{\sigma_t^q, \hat{Q}_t\}$ supporting $\sigma_0^q = 0.9 > \sigma^{q,n} = 0$. Normalization shows that as σ_0^q jumps off by σ, stock price falls by $2 - 10\%$, which is consistent with our empirical findings in Figure 5b of Appendix A.

Figure 3 also explores the effects on the martingale equilibrium paths of a change in policy responsiveness to inflation ϕ_π. The right panel 3b uses the default calibration value $\phi_\pi = 2.5$, while the left panel 3a assumes a more accomodating stance $\phi_\pi = 1.5$. As we raise ϕ_π, we observe that sample paths are likely to converge faster towards full stabilization at

[52] Gandelman and Hernández-Murillo (2014)'s estimates of φ range between 0.2 and 10. In our environment, a higher risk aversion of workers makes their labor supply (and therefore, output) less responsive to business cycle fluctuations. In that scenario, a higher asset price tends to translate into less wage income distributed to workers, making it harder to satisfy the co-movement condition (i.e., Assumption 1). Thus, we pick a value on the lower end of the acceptable range and set $\varphi = 0.2$.



(a) With $\phi_\pi = 1.5$

(b) With $\phi_\pi = 2.5$.

Figure 3: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$

the expense of an increased likelihood of a more severe crisis path in a given period of time. The intuition is simple: for the *given* level of initial sunspot $\sigma_0^q > 0$ to be sustained under a more responsive policy rate with higher ϕ_π, it must be true that more amplified endogenous volatility (i.e., high σ_t^q) and severe recession (i.e., low $\hat{Q}_t$) arise with vanishing probability in the future. We obtain similar results when looking at changes in policy responsiveness to the asset price gap ϕ_q (alternatively, output gap ϕ_y), and find that a change in $\phi \equiv \phi_q + (\phi_\pi - 1)\frac{\kappa}{\rho}$, the measure of combined responsiveness of monetary policy i_t, brought by any combination $\{\phi_\pi, \phi_q\}$ follows the same patterns depicted in Figure 3.

Booms In an analogous way, we can construct a rational expectations equilibrium that supports an initial downward sunspot $\sigma_0^q < \sigma_t^{q,n} \equiv 0$. The equilibrium paths feature a boom phase with buoyant production and consumption with lower levels of financial volatility and risk-premium. A higher ϕ value speeds up the stabilization process, but increases the likelihood of an equilibrium path with an overheated economy.[53]

[53]We have two singular points in the $\{\sigma_t^q\}$ process in (42): as σ_t^q hits $-\sigma$, both drift and volatility of the process diverge, and $\{\sigma_t^q\}$ process would feature a jump. When σ_t^q hits 0, it stays there forever so $\sigma_t^q = 0$ thereafter. Thus, when σ_0^q is below $-\sigma$, we might end up in paths where we have a jump in σ_t^q to a positive value, which eventually converges to 0.

4.2 Modified Monetary Rule

A modified monetary policy rule includes risk-premium as a separate factor in the following way:

$$i_t = \underbrace{r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Bernanke and Gertler (2000)}} - \underbrace{\frac{1}{2}\hat{rp}_t}_{\text{Risk-premium targeting}}, \quad \text{where } \hat{rp}_t \equiv \text{rp}_t - \text{rp}^n. \tag{43}$$

The above monetary policy rule in (43) contains a 'risk-premium gap term' as a factor in addition to inflation and asset price gap. It also can be written in terms of the risk-adjusted natural rate r_t^T as

$$i_t = r_t^T + \phi_\pi \pi_t + \phi_q \hat{Q}_t,$$

where a higher $\hat{rp}_t$ brings down r_t^T, forcing i_t to fall. The following Proposition 6 establishes that a monetary policy rule following (43) and that satisfies the Taylor principle, i.e., $\phi > 0$ restores equilibrium determinacy and fully stabilizes the economy.

Proposition 6 (Risk-Premium Targeting and Ultra-Divine Coincidence) *The monetary policy rule*

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \frac{1}{2}\hat{rp}_t, \quad \textit{where } \phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0, \tag{44}$$

achieves $\hat{Q}_t = \pi_t = \hat{rp}_t = 0$ as unique rational expectations equilibrium. Therefore, the monetary policy rule in (44) attains (i) output (asset price) stabilization, (ii) price level (inflation) stabilization, and (iii) financial market (financial volatility and risk-premium) stabilization. We call it the ultra-divine coincidence.

This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984). The reason central banks must target risk-premium as a separate factor is that this term directly appears in the drift part of our dynamic IS equation (i.e., (34)). According to the policy rule in (44), central bank lowers the policy rate i_t when $\text{rp}_t > \text{rp}^n$ to boost $\hat{Q}_t$ and $\hat{C}_t$[54], since a higher risk-premium drags down asset price and business cycle levels. If monetary policy kills an initial excess volatility (or excess risk-premium) with this additional target

[54]Even with Bernanke and Gertler (2000) rule, monetary policy responds to a rise in risk-premium since it negatively affects the asset price gap $\hat{Q}_t$ and inflation π_t. Our point here is that the policy rate must systematically respond to deviations of rp_t from its natural level rp^n given $\hat{Q}_t$ and π_t levels.

in its rule, it precludes the possibility of sunspots in financial volatility that we discussed. Since the Taylor principle (i.e., $\phi > 0$) guarantees there is no sunspot inflation, the policy rule in equation (44) restores equilibrium determinacy and achieves both macro stability (with $\hat{Q}_t = \pi_t = 0$) and financial stability (with $\hat{rp}_t = 0$, which implies $\mathrm{rp}_t = \mathrm{rp}^n$ and $\sigma_t^q = \sigma_t^{q,n} = 0$). The interest rate on the equilibrium path then becomes $i_t = r^n$, which is the same level as in the equilibrium path of a canonical New-Keynesian model. Therefore, the ultra-divine coincidence result implies: one policy tool (i_t rule) achieves an additional objective (financial stability) in addition to the two usual mandates (output gap and inflation stability). This is possible in our framework because financial markets and the business cycle are tightly interwoven and real and financial instabilities are equivalent to each other.

A common view in the literature holds that monetary policy must respond to financial market disruptions only when they affect (or to the degree that they affect) the original mandates (i.e., inflation stability and full employment). This premise is at odds with our results: the failure to target the risk-premium of financial markets subjects the economy to the apparition of sunspot shocks in financial volatility and risk-premium, and the corresponding recessions and overheating episodes that ensue. Only by targeting risk-premium in the particular way characterized in (43), the monetary authority can re-establish equilibrium determinacy and achieve the ultra-divine coincidence outlined in the previous paragraphs.

Interpretation We can rewrite our modified Taylor rule in (44) as

$$\underbrace{i_t + \mathrm{rp}_t}_{=i_t^m} - \underbrace{\frac{1}{2}\mathrm{rp}_t}_{\text{Ito term}} = \underbrace{r^n + \mathrm{rp}^n}_{=i_t^{m,n}} - \underbrace{\frac{1}{2}\mathrm{rp}^n}_{\text{Ito term}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}},$$

or equivalently as

$$\underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\\text{yield}}} + \underbrace{\frac{\mathbb{E}_t\left(d \log a_t\right)}{dt}}_{\substack{\text{Internal rate of return}\\\text{of aggregate wealth}}}}_{\text{Cum-dividend stock return}} = \underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\\text{yield}}} + \frac{\mathbb{E}_t\left(d \log a_t^n\right)}{dt}}_{\text{Benchmark cum-dividend stock return}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}}, \tag{45}$$

where a_t is the economy's aggregate financial wealth,[55] i.e., $p_t A_t Q_t$, and a_t^n is the aggregate wealth of the natural (i.e., flexible price) economy. Our modified monetary policy that targets a risk-premium as prescribed in equation (44) thus can be interpreted as the rule on the rate of change of $\log$-aggregate wealth as a function of traditional inflation and output gap (asset price) targets. Basically, the rate that determines the households' intertemporal substitution should be the expected return on stock markets, instead of just the risk-free policy rate, and therefore in order to achieve determinacy as well as stabilization in our model, the expected return on stock markets must target business cycles.

We interpret the rule in (45) as the *generalized Taylor rule*. With this rule, the central bank uses the aggregate wealth and its rate of return as *intermediate* targets towards achieving business cycle stabilization.

Practicality Some issues still remain about the feasibility to implement this new policy rule in (44). First, the risk premium gap $\hat{\mathrm{rp}}_t$ in (43) depends on the natural level of risk-premium, rp^n, which is a counterfactual variable by definition, and therefore its observation is subject to some form of measurement error. Second, there are multiple kinds of risk-premia in financial markets that can be possibly targeted through monetary policy, and the construction of the aggregate risk-premium index as featured in our model might be subject to error as well.[56,57]

More importantly, and related to the previous two points, the coefficient attached to the risk-premium in (43) is exactly $\frac{1}{2}$. Given the potential for measurement error in $r\hat{p}_t$, it might be impossible for the central bank to target the risk-premium with the exact strength de-

[55](45) is derived from equations (24) and (25).

[56]Our framework features only an 'index' of the stock market as a feasible vehicle to invest in, but there are multiple risk-premia (including term-premia) covering stocks and bonds in the real world.

[57]There have been long-standing debates about whether monetary authority should adjust policy rates in response to fluctuations in risk-premia of financial markets. For example, Doh et al. (2015) argued "adjusting short-term interest rates in response to various estimated risk premium levels could be appropriate, especially if the risk premiums are low for a sustained period. In contrast, if policymakers are predominantly concerned about the most likely macroeconomic outcome, monitoring the estimated risk premium and adjusting the monetary policy stance accordingly may be of little benefit.". This argument is based on the fact that information about possible tail risks is summarized by the risk-premia levels in financial markets.

manded by (43).[58] To understand the consequences of deviating from the $\frac{1}{2}$ risk-premium target, we consider the following alternative rule:

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \phi_{\text{rp}} \hat{rp}_t, \tag{46}$$

where ϕ_{rp} is a constant term potentially different from $\frac{1}{2}$. With the policy rule in (46), we obtain

$$d\hat{Q}_t = \left((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t + \left(\frac{1}{2} - \phi_{\text{rp}} \right) \hat{rp}_t \right) dt + \sigma_t^q dZ_t. \tag{47}$$

as $\{\hat{Q}_t\}$ dynamics. With $\phi_{\text{rp}} = \frac{1}{2}$, we return to determinacy (i.e., Proposition 6). With $\phi_{\text{rp}} \neq \frac{1}{2}$, the martingale equilibrium with self-created volatility σ_t^q reappears and is characterized by[59]

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\phi_{\text{rp}}}} + \frac{\sigma^2}{2\phi_{\phi_{\text{rp}}}} \text{ with } \phi_{\phi_{\text{rp}}} \equiv \frac{\phi}{1 - 2\phi_{\text{rp}}}, \tag{48}$$

and

$$\pi_t = \frac{\kappa}{\rho} \left(-\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\phi_{\text{rp}}}} + \frac{\sigma^2}{2\phi_{\phi_{\text{rp}}}} \right),$$

where $\{\sigma_t^q\}$'s stochastic process after an initial sunspot σ_0^q appears is given as

$$d\sigma_t^q = -\frac{\phi_{\phi_{\text{rp}}}^2 (\sigma_t^q)^2}{2(\sigma + \sigma_t^q)^3} dt - \phi_{\phi_{\text{rp}}} \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t. \tag{49}$$

Table 1 summarizes how the magnitude of ϕ_{rp} affects the convergence process after the sunspot volatility σ_0^q arises. When $\phi_{\text{rp}} < \frac{1}{2}$, including the case of $\phi_{\text{rp}} = 0$ of Proposition 5, an increase in ϕ_{rp} leads to an increase in $\phi_{\phi_{\text{rp}}}$ from (48). From (49), we observe that a higher $\phi_{\phi_{\text{rp}}}$ accelerates the convergence of sample paths while creating more amplified ones given initial sunspot σ_0^q. As far as $\phi_{\text{rp}} < \frac{1}{2}$, a higher ϕ_{rp} means monetary policy responds more strongly to fluctuations in $\hat{rp}_t$, which allows the faster stabilization. As ϕ_{rp} goes up from 0

[58] As an example, consider a multiplicative measurement error ε_t such that $\hat{rp}_t^{obs} = \varepsilon_t \cdot \hat{rp}_t$, where $\hat{rp}_t^{obs}$ is the measured premium. It is easy to see that the central bank following the policy rule in (43) will target the 'true' risk-premium with a coefficient $\neq \frac{1}{2}$.

[59] Equations (46) and (48) are easily derived in a similar way to Proposition 5.

$\phi_{\mathbf{rp}} < 0$ **(Real Bills Doctrine)**	$0 \leq \phi_{\mathbf{rp}} < \frac{1}{2}$
(i) With $\phi_{\text{rp}} \downarrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)	(i) With $\phi_{\text{rp}} \uparrow$, convergence speed$\uparrow$ and more amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)
$\phi_{\mathbf{rp}} = \frac{1}{2}$	$\phi_{\mathbf{rp}} > \frac{1}{2}$
No sunspot (ultra-divine coincidence)	(i) With $\phi_{\text{rp}} \uparrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a boom ($\hat{Q}_t > 0$ and $\pi_t > 0$)
As $\phi \uparrow$, convergence speed$\uparrow$ and $\exists$ more amplified paths	

Table 1: Effects of different parameters $\{\phi_{\text{rp}},\ \phi\}$ on stabilization

to $\frac{1}{2}$, fluctuations in $\hat{rp}_t$ have the weaker direct effect on the drift of (47). Thus, the volatility of $\{\sigma_t^q\}$ process in (49) must rise to ensure that the initial sunspot σ_0^q is supported, as on average the economy is better stabilized with a higher ϕ_{rp}. $\{\hat{Q}_t\}$ eventually is stabilized, which results, on average, on shorter but more amplified sample paths.

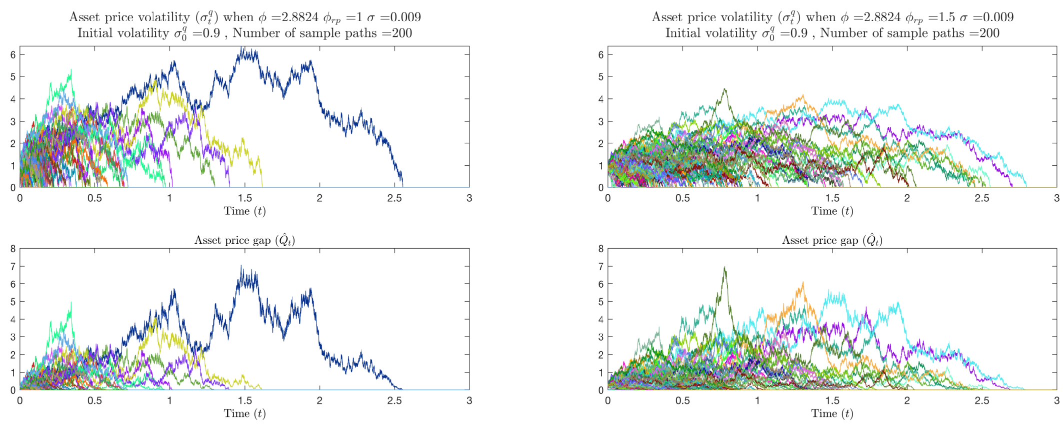
$\phi_{\text{rp}} < 0$ case is interesting since it implies central bank raises the policy rate when risk-premia rise in financial markets. It is consistent with the *Real Bills Doctrine* which was a popular idea during the first half of the 20th century. Basically, the doctrine advocated for the Fed discount rate to track the average interest rate of the financial markets, as a means of stabilization.[60] In our framework, $\phi_{\text{rp}} < 0$ pushes down $\phi_{\phi_{\text{rp}}}$ from ϕ, thereby effectively slowing down the pace of stabilization after sunspots hit the stock market. So we confirm the *Real Bills Doctrine* with $\phi_{\text{rp}} < 0$ is not suitable for stabilization purposes, as empirically documented by Richardson and Troost (2009).

With $\phi_{\text{rp}} > \frac{1}{2}$, monetary policy responds too strongly to fluctuations in risk-premium, thus with an initial positive sunspot $\sigma_0^q > 0$, the policy rate drops too excessively and creates an artificial boom instead of crisis.[61] A higher ϕ_{rp} reduces $|\phi_{\phi_{\text{rp}}}|$ and slows down stabilization

[60] Richardson and Troost (2009) studied the effects of such policies during the Great Depression era, exploiting the fact that the state of Mississippi is divided by the Federal Reserve act between the 6th (Atlanta) and 8th (St. Louis) districts which had different approaches to the economy-wide banking panics and depressions.

[61] With $\phi_{\text{rp}} > \frac{1}{2}$, we have $\phi_{\phi_{\text{rp}}} < 0$ from (48), thus $\sigma_t^q > 0$ is equivalent to the boom phase with $\pi_t > 0$ and $\hat{Q}_t > 0$.

since a higher ϕ_{rp} means monetary policy deviates more from determinacy (the case of $\phi_{\text{rp}} = \frac{1}{2}$), and therefore becomes less efficient at stabilization. Figure 4 illustrates that with $\phi_{\text{rp}} > \frac{1}{2}$, a spike in financial volatility, $\sigma_t^q > 0$, actually acts as a boon to the economy, as we have $\hat{Q}_t > 0$ and $\pi_t > 0$ along sample paths. Moreover, with $\phi_\pi = 2.5$ fixed, as we raise ϕ_{rp} from 1 to 1.5, stabilization slows down[62] as we further deviate from the determinacy case, $\phi_{\text{rp}} = \frac{1}{2}$.



(a) With $\phi_{\text{rp}} = 1$

(b) With $\phi_{\text{rp}} = 1.5$.

Figure 4: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$, with varying $\phi_{\text{rp}} > \frac{1}{2}$

Zero lower bound (ZLB)? In Figure 3, we observe that an initial sunspot $\sigma_0^q > 0$ can be amplified endogenously through monetary policy's responses to the business cycle fluctuation, which might drag the economy into the zero lower bound (ZLB) episodes when the $\{\sigma_t^q\}$ path hits some threshold from below. When monetary policy is constrained at those episodes, both asset market and business cycle would collapse, which we observed in the 2007-2009 Global Financial Crisis (GFC). We study these issues in a follow-up paper, Lee and Carreras (2022).

[62] Also, a higher ϕ_{rp} causes less amplification from the initial sunspot $\sigma_0^q > 0$.

5 Conclusion

In this paper, we illustrate the following main features of the macroeconomic models with nominal rigidity: (i) properly accounting for higher-order moments related to the business cycle and stock markets changes the business cycle dynamics of the model economy. In specific, we incorporate the feature that a rise in aggregate volatility depresses aggregate output (as well as stock price) in a tractable manner, where both volatility and output are endogenous. (ii) Under our usual Taylor-rule based monetary policy, the aggregate volatility can be generated in a self-fulfilling manner and we provide how to construct the rational expectations equilibrium that supports a self-created volatility; To that purpose, we provide two macroeconomic frameworks that illustrate the above findings: the standard non-linear New-Keynesian framework and the model with the endogenous risk-premium and portfolio decisions in regards to stock markets.

Therefore, our analysis reveals that conventional Taylor rules, even with the aggressive targeting of traditional macroeconomic measures, cannot guarantee determinacy. This failure of conventional rules in ensuring determinacy lies in their inability to adequately target the *expected return of risky financial markets*, the relevant rate for the households' intertemporal substitution in a stochastic environment. We then propose a generalized Taylor rule that restores determinacy, under which central banks target not only conventional mandates (i.e., inflation and output gap), but also the risk-premium in a specific way, thus effectively managing the expected rate of return on the aggregate financial wealth. This new policy rule achieves what we describe as the *ultra-divine coincidence*: the joint stabilization of inflation, output gap and risk-premium.

Our framework opens new avenues for the future research on understanding the effects of the higher order variables of business cycles and financial markets. For example, we largely abstract from the wealth inequality and potentially heterogenous sensitivities of economic players to aggregate volatility.[63] We view future works incorporating these realistic features

[63]For example, Cioffi (2021) incorporates the households' heterogeneous sensitivities to aggregate risks according to their position in the wealth distribution, as households choose different portfolios depending on their position in the distribution.

as a particularly fruitful direction to pursue.

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Online Appendices

A Suggestive Evidence

Stock market volatility is commonly viewed in the literature as a proxy of financial and economic uncertainty, which Bloom (2009) and later Gilchrist and Zakrajšek (2012), Bachmann et al. (2013), Jurado et al. (2015), Caldara et al. (2016), Baker et al. (2020), Coibion et al. (2021) further studied as a driving force behind business cycles fluctuations. In this Section, we evaluate these claims and present interesting empirical results. Figure 7 in Appendix B provides the first piece of supportive evidence in that direction. Panel 7a depicts several variables commonly used in the literature to measure financial uncertainty. The correlation between series is remarkably high and they all display positive spikes at the beginning and/or initial months following an NBER-dated recession, which is consistent with the evidence that many of these episodes were financial in nature.[64] Panel 7b plots Ludvigson et al. (2021) (henceforth, LMN) financial and real (i.e. non-financial) uncertainty series. These variables are positively correlated and display a similar propensity to increase around recessions, though a different type of crisis (e.g. financial or not) is correlated with a different type of uncertainty playing the dominant role. For example, the massive spike in real vis-à-vis financial uncertainty following the recent Covid-19 recession, which initially was a health crisis that spilled into the real economy, can be observed in Panel 7b.

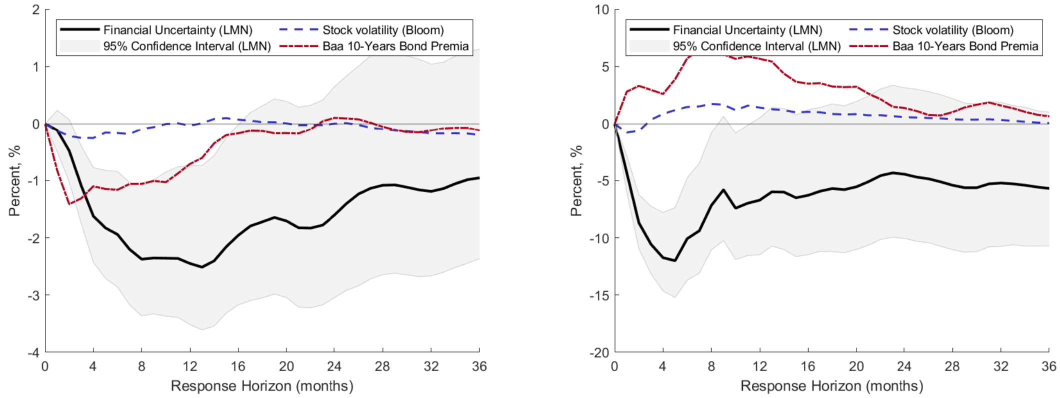
The patterns displayed in Figure 7 do not yet constitute a proof of the importance of financial market uncertainty as a driver of the business cycle, as we should worry about the possibility of reverse causation running from unfavorable economic conditions towards uncertainty. We tackle this issue by proposing a simple Vector Autoregression (VAR) with the structural identification strategy based on the timing of macroeconomic shocks similar to Bloom (2009). Equation (50) presents the variables considered and their ordering, with non-financial series first and financial variables last.[65]

[64] See Reinhart and Rogoff (2009) and Romer and Romer (2017) for the classification of the past recessions. Their analysis showed many recessions had roots in financial markets.

[65] The ordering is used by Ludvigson et al. (2021), who, using identification strategy based on event constraints, find that the uncertainty of financial markets tends to be an exogenous source of business cycle fluctuations, while the real uncertainty is more likely an endogenous response to the business cycle fluctuations. We also have implemented alternative specifications and orderings that produced qualitatively similar results (not reported, provided upon request).

$$
\text{VAR-11 order:} \quad \begin{bmatrix} \log(\text{Industrial Production}) \\ \log(\text{Employment}) \\ \log(\text{Real Consumption}) \\ \log(\text{CPI}) \\ \log(\text{Wages}) \\ \text{Hours} \\ \text{Real Uncertainty (LMN)} \\ \text{Fed Funds Rate} \\ \log(\text{M2}) \\ \log(\text{S\&P-500 Index}) \\ \text{Financial Uncertainty (LMN)} \end{bmatrix} \tag{50}
$$

Both LMN real and financial uncertainty measures are included to differentiate the effects of financial volatility shocks from the effects from real uncertainty. For similar reasons, we include the S&P-500 index in our VAR to empirically distinguish between shocks affecting the level of financial markets and shocks affecting their volatility. In order to ameliorate possible concerns about the validity of the structural identification strategy, we estimate our VAR using monthly data, where the identification assumptions are more likely to hold. Figure 5 presents the impulse responses to the orthogonalized financial uncertainty shock. Panel 5a plots the response of industrial production, which falls by up to 2.5% and displays moderate persistence following a one standard deviation shock to financial uncertainty. Panel 5b plots the response of the S&P-500 Index, which drops up to 12% within the first four months before gradually recovering. Together, both pictures imply that an increase of financial uncertainty tends to depress both industrial activity and financial markets.



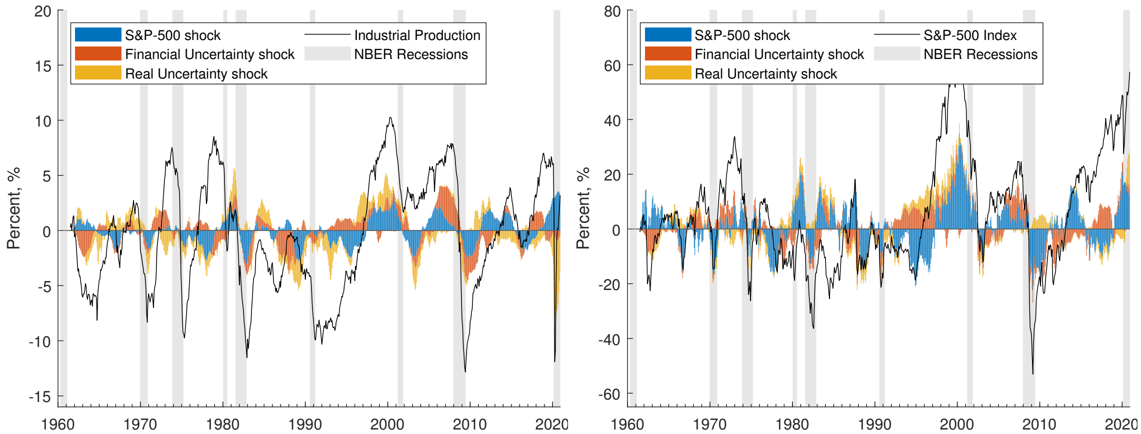
(a) Response: Industrial Production

(b) Response: S&P-500 Index

Figure 5: Impulse Response Functions (IRFs), selected series. Figures 5a and 5b display the response to a one standard deviation financial uncertainty shock of monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with the variable composition and ordering given in (50). Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.

Figure 5 also features alternative estimates using common financial uncertainty proxies such as Bloom (2009) stock market volatility index and 10-years premium on Baa-rated corporate bonds. The responses are generally more muted, and take the opposite sign in the case of the S&P Index. These results can be explained by the fact that standard proxies contain information unrelated to financial uncertainty that distorts our estimates (see Jurado et al. (2015) for a discussion), and therefore we choose LMN as our preferred financial uncertainty measure. In Appendix B, we report additional impulse response estimates. Especially, the Figure 9 in Appendix B shows that monetary authorities respond with accommodating interest rate movements to financial uncertainty shocks, while real uncertainty has no statistically significant effect on either interest rates or stock market fluctuations. We further discuss optimal monetary policy response to financial volatility shocks in Section 4.

Finally, we can further explore the contribution of financial uncertainty to business cycles fluctuations by looking at Table 2 in Appendix B, which reports the Forecast Error Variance Decomposition (FEVD) of Industrial Production and the S&P-500 Index. Financial uncertainty shocks explain close to 5% of the fluctuations in both series, while real uncertainty explains an additional 2-4% of movements in industrial activity in the medium run. Figure 6 provides a more graphical illustration of these results by plotting the historical decomposition of the series. We observe that the contribution of financial uncertainty rivals that of shocks to the level of financial variables captured by the S&P-500 shock, and is especially important in driving industrial production boom-bust patterns during and in the preceding months of recessionary episodes, as it can be seen during the global financial crisis (2007).

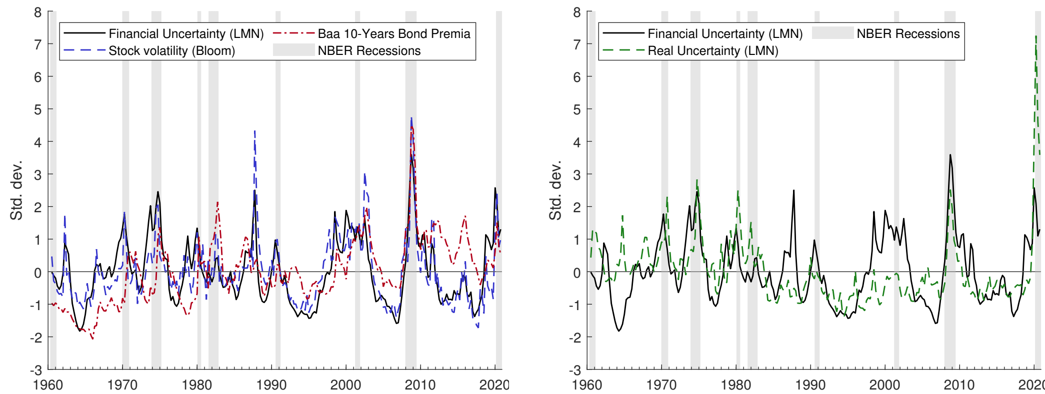


(a) Industrial Production (b) S&P-500 Index

Figure 6: Historical Decomposition, selected series. Figures 6a and 6b display the historical decomposition of monthly Industrial Production and S&P-500 Index series, respectively, based on the VAR-11 with variable composition and ordering in (50). Shaded areas indicate NBER dated recessions (peak trough the through). Variables of interest are de-trended by subtracting the contribution of initial conditions and constant terms after series decomposition. Columns report a contribution of each shock to the fluctuations around trend of the variable considered.

In this Appendix A, we have revisited the empirical evidence on financial market volatility and shown that it acts as a major driving force of the business cycle.

B Additional Figures and Tables



(a) Financial Uncertainty series

(b) Financial vs. Real Uncertainty

Figure 7: Uncertainty series. Figure 7a displays common measures of financial uncertainty. Figure 7b displays Ludvigson et al. (2021) (henceforth, LMN) measures of financial and real economic uncertainty. Shaded areas indicate NBER dated recessions (peak trough the through). LMN financial and real economic uncertainty series are constructed as the average volatility of the residuals from predictive regressions on financial and real economic variables, respectively (See Ludvigson et al. (2021) for the series construction). Bloom (2009)'s stock market volatility variable is constructed using VXO data from 1987 onward and the monthly volatility of the S&P 500 index normalized to the same mean and variance in the overlapping interval for the 1960-1987 period (See Bloom (2009) for the series construction). The bond risk-premia series is the Moody's seasoned Baa corporate bond yield relative to the yield on a 10-year treasury bond at constant maturity. For graphical comparison purposes, the depicted series have a normalized zero mean and one standard deviation.

(i) Industrial Production

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0	0.30	0.21	0.12
h=6	1.27	3.37	2.98	1.36
h=12	4.28	4.38	3.16	1.94
h=36	3.24	1.67	1.98	0.64

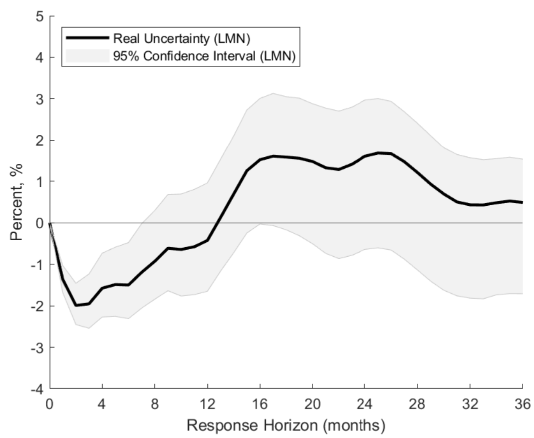
(ii) S&P-500 Index

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.11	0.08	0.39	0.06
h=6	3.30	0.25	3.26	0.62
h=12	4.77	0.54	10.03	2.16
h=36	6.50	0.91	12.16	2.40

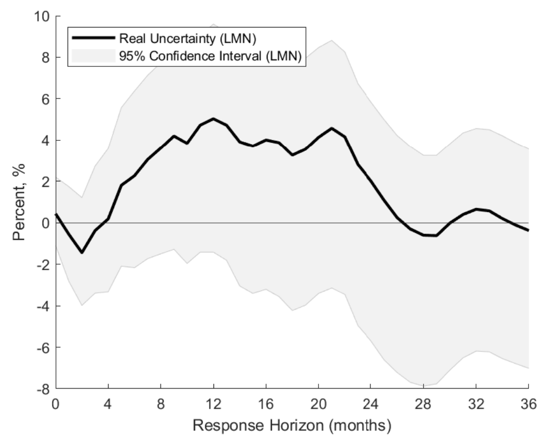
(iii) Fed Funds Rate

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.01	0.98	0	0.08
h=6	0.42	0.84	3.11	1.66
h=12	1.47	0.91	4.69	2.30
h=36	2.81	2.05	5.02	3.17

Table 2: Forecast Error Variance Decomposition (FEVD). The table presents the variance contribution (in percentage) of financial and real uncertainty shocks to selected series at different time horizons (in months). The FEVD is constructed using a VAR-11 with equation (50) variable composition and ordering. The first two columns report the contribution of LMN financial and real uncertainty shocks, respectively. The last two columns report alternative VAR specifications where the preferred LMN financial uncertainty measure (column one) is replaced by common proxies employed in the literature, either Bloom (2009) stock market volatility measure or the Baa 10-years corporate bond premia, respectively.

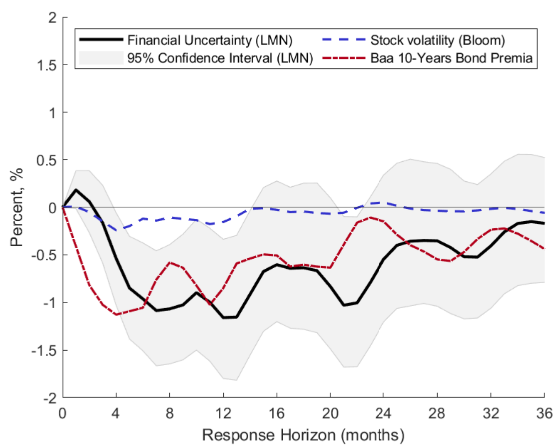


(a) Response: Industrial Production

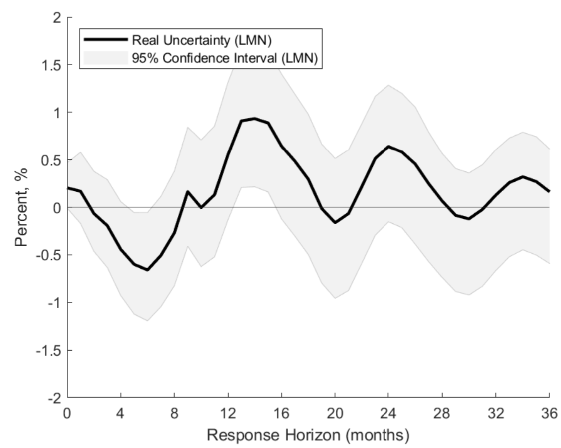


(b) Response: S&P-500 Index

Figure 8: Impulse Response Functions (IRFs), selected series. Figures 8a and 8b display the response to one standard deviation real uncertainty shock by monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (50) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.



(a) Shock: Financial Uncertainty



(b) Shock: Real Uncertainty

Figure 9: Impulse Response Functions (IRFs), Fed Funds Rate. This Figure displays the response to a one standard deviation uncertainty (financial or real) shock by monthly Fed Funds Rate series, using a VAR-11 with equation (50) variable composition and ordering. Panel 9a plots the response to a financial uncertainty shock, and Panel 9b to a real uncertainty shock. Shaded area indicates 95% confidence interval around preferred financial/real uncertainty measure computed using standard bootstrap techniques. Additional lines display alternative impulse responses obtained by substituting preferred LMN financial uncertainty measure with common proxies employed in the literature.

Parameter	Value	Description
φ	0.2	Relative Risk Aversion
χ_0	0.25	Inverse Frisch labor supply elasticity
ρ	0.020	Subjective time discount factor
σ	0.0090	TFP volatility
g	0.0083	TFP growth rate
α	0.4	$1-$ Labor income share
ϵ	7	Elasticity of substitution intermediate goods
δ	0.45	Calvo price resetting probability
ϕ_π	2.50	Policy rule inflation response
ϕ_y	0.11	Policy rule output gap response
ϕ_{rp}	0	Policy rule risk premium response
$\bar{\pi}$	0	Steady state trend inflation target

Table 3: The table presents the baseline parameter calibration used in Sections 4 of the paper.

C Derivations and Proofs for Sections 2, 3, and 4

C.0. Section 2

Derivation of equation (3) From the definition of (nominal) state-price density $\xi_t^N = e^{-\rho t}\frac{1}{C_t}\frac{1}{p_t}$, we obtain

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} - \frac{dp_t}{p_t} + \left(\frac{dC_t}{C_t}\right)^2 + \left(\frac{dp_t}{p_t}\right)^2 + \frac{dC_t}{C_t}\frac{dp_t}{p_t}. \tag{C.1}$$

Since we have a perfectly rigid price (i.e., $p_t = \bar{p}$ for $\forall t$), the above (C.1) becomes

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \tag{C.2}$$

$$= -\rho dt - \frac{dC_t}{C_t} + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right). \tag{C.3}$$

Plugging equation (C.2) into equation (2), we obtain

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)\, dt + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right). \tag{C.4}$$

Derivation of equation (8) From equation (7), we obtain

$$d\ln Y_t = \left(i_t - \rho + \frac{1}{2}\left(\sigma + \sigma_t^s\right)^2\right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{C.5}$$

From (5), we obtain

$$d\ln Y_t^n = \left(r^n - \rho + \frac{1}{2}\sigma^2\right) dt + \sigma dZ_t. \tag{C.6}$$

Therefore, by subtracting equation (C.6) from equation (C.5), we obtain

$$d\hat{Y}_t = \left(i_t - \left(r^n - \frac{1}{2}\left(\sigma + \sigma_t^s\right)^2 + \frac{1}{2}\sigma^2\right)\right) dt + \sigma_t^s dZ_t, \tag{C.7}$$

which derives equation (8).

Proof of Proposition 1. From equation (14), $\{\sigma_t^s\}$ process can be written as

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t. \tag{C.8}$$

Using Ito's lemma, we get the process for $(\sigma + \sigma_t^s)^2$ which is a martingale, as given by

$$
\begin{aligned}
d(\sigma+\sigma_t^s)^2 &= 2(\sigma+\sigma_t^s)d\sigma_t^s + (d\sigma_t^s)^2 \\
&= 2(\sigma+\sigma_t^s)\left(-\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}dt - \phi_y\frac{\sigma_t^s}{\sigma+\sigma_t^s}dZ_t\right) + (\phi_y)^2\frac{(\sigma_t^s)^2}{(\sigma+\sigma_t^s)^2}dt \\
&= -2\phi_y(\sigma_t^s)dZ_t.
\end{aligned}
\tag{C.9}
$$

Therefore, we have $\mathbb{E}_0((\sigma+\sigma_t^s)^2) = (\sigma+\sigma_0^s)^2$. By applying Doob's martingale convergence theorem as $(\sigma+\sigma_t^s)^2 \geq 0, \forall t$, we know $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$ since:

$$
\underbrace{d\sigma_t^s}_{\overset{a.s}{\to}0} = -\underbrace{\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}}_{\overset{a.s}{\to}0}dt - \phi_y\underbrace{\frac{\sigma_t^s}{\sigma+\sigma_t^s}}_{\overset{a.s}{\to}0}dZ_t. \tag{C.10}
$$

Thus equation (C.10) proves $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$. From equation (13) $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^q = 0$ leads to $\hat{Y}_t \overset{a.s}{\to} 0$. Finally, we must have $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$, since otherwise the uniform integrability says $\mathbb{E}_0((\sigma+\sigma_\infty^s)^2) = (\sigma+\sigma_0^s)^2$, which is a contradiction to our earlier result $\sigma_t^s \overset{a.s}{\to} 0$ since $\sigma_\infty^s = 0$ and $\sigma_0^s > 0$ by assumption in Proposition 1.

■

C.1. Section 3

C.1.1. Section 3.1

Here we solve the optimization problems of workers (i.e., equation (18)) and capitalists (i.e., equation (22)).

Worker's optimization At each time t, each hand-to-mouth worker solves

$$
\max_{C_{W,t},N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad \text{s.t.} \quad p_tC_{W,t} = w_tN_{W,t}. \tag{C.11}
$$

If we let $\lambda_t A_t^{\varphi-1}$ be the Lagrange multiplier on the budget constraint, the first-order conditions are given by

$$
C_{W,t}^{-\varphi} = \lambda_t p_t, \quad A_t^{1-\varphi}(N_{W,t})^{\chi_0} = \lambda_t w_t = \frac{w_t}{p_t}C_{W,t}^{-\varphi} = \left(\frac{w_t}{p_t}\right)^{1-\varphi}N_{W,t}^{-\varphi}, \tag{C.12}
$$

which leads to

$$
N_{W,t} = \left(\frac{w_t}{p_t}\right)^{\frac{1-\varphi}{\chi_0+\varphi}}\frac{1}{A_t^{\frac{1-\varphi}{\chi_0+\varphi}}} = \left(\frac{w_t}{p_tA_t}\right)^{\frac{1}{\chi}}, \quad C_{W,t} = \frac{w_t}{p_t}N_{W,t} = \left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}}A_t^{-\frac{1}{\chi}}, \tag{C.13}
$$

where we use $\chi \equiv \dfrac{\chi_0+\varphi}{1-\varphi}$ in Definition 1.

Capitalist's optimization In equilibrium, each capitalist chooses $\theta_t = 1$ as the bond market is zero net supplied. Plugging $\rho a_t = p_t C_t$ from equation (24), the budget flow constraint of capitalists in (22) becomes:

$$\frac{da_t}{a_t} = (i_t^m - \rho)\, dt + (\sigma + \sigma_t^q + \sigma_t^p)\, dZ_t. \tag{C.14}$$

The capitalist's state price density in equilibrium is thereby given by

$$\xi_t^N = e^{-\rho t}\frac{1}{p_t C_t} = e^{-\rho t}\frac{1}{\rho a_t}, \tag{C.15}$$

on which we can apply Ito's Lemma and obtain

$$\begin{aligned} -\frac{d\xi_t^N}{\xi_t^N} &= \frac{da_t}{a_t} - \left(\frac{da_t}{a_t}\right)^2 + \rho dt \\ &= \underbrace{\left(i_t^m - (\sigma + \sigma_t^q + \sigma_t^p)^2\right)}_{=i_t} dt + (\sigma + \sigma_t^q + \sigma_t^p)\, dZ_t = i_t dt + (\sigma + \sigma_t^q + \sigma_t^p) dZ_t \end{aligned} \tag{C.16}$$

with which we obtain $i_t + (\sigma + \sigma_t^q + \sigma_t^q)^2 = i_t^m$ (i.e., equation (25)) from $\mathbb{E}_t\left(-\frac{d\xi_t^N}{\xi_t^N}\right) = i_t dt$. Note that (24) and (C.16) are the same conditions as in Merton (1971).

C.1.2. Section 3.2

We know that in equilibrium, each capitalist holds the financial wealth $a_t = p_t A_t Q_t$ since all of them are identical both ex-ante and ex-post. Now we prove Lemma 1.

Proof of Lemma 1. First, we start by stating capitalist's nominal state-price density ξ_t^N and real state-price density ξ_t^r. The nominal state-price density is relevant to the nominal interest rate, while the real state-price density matters when we calculate the real interest rate. The real state price density ξ_t^r is given by

$$\xi_t^r = e^{-\rho t}\frac{1}{C_t} = p_t \xi_t^N. \tag{C.17}$$

Using (C.16), we can apply Ito's Lemma to (C.17) and obtain

$$\frac{d\xi_t^r}{\xi_t^r} = \left(\underbrace{\pi_t - i_t - \sigma_t^p\left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{=-r_t}\right) dt - (\sigma + \sigma_t^q) dZ_t, \tag{C.18}$$

from which we obtain the following Fisher identity with the inflation premium in equation (28):

$$r_t = i_t - \pi_t + \sigma_t^p\left(\sigma + \sigma_t^q + \sigma_t^p\right). \tag{C.19}$$

■

C.1.3. Section 3.3

Here we prove the Proposition 2 based on the results above.

Proof of Proposition 2. We start from the intermediate firms' optimization problem. As we have the externality à la Baxter and King (1991), we need to go through additional steps in aggregeing individual decisions across firms. Let firm i take its demand function as given and choose the optimal price $p_t(i)$ at any t. With $E_t \equiv (N_{W,t})^\alpha$, from the production function, we have

$$n_t(i) = \left(\frac{y_t(i)}{A_t E_t}\right)^{\frac{1}{1-\alpha}}. \tag{C.20}$$

Then each firm i chooses p_i that maximizes its profit, solving

$$\max_{p_t(i)} p_t(i)\left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} y_t - w_t \left(\frac{y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}}. \tag{C.21}$$

In the flexible price economy, all firms charge the same price (i.e., $p_t(i) = p_t \ \forall i$) and hire the same amount of labor (i.e., $n_t(i) = N_{w,t} \ \forall i$). The solution of (C.21) combined with these conditions yields

$$\begin{aligned}\frac{w_t^n}{p_t^n} &= \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t E_t)^{\frac{1}{1-\alpha}} \\ &= \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} = \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}},\end{aligned} \tag{C.22}$$

from which we obtain the following equilibrium real wage:

$$\frac{w_t^n}{p_t^n} = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}}. \tag{C.23}$$

In flexible price equilibrium, we know the aggregate production is linear, i.e., $y_t = A_t N_{W,t}$. Therefore, we obtain

$$\begin{aligned} y_t = A_t N_{W,t} &= A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} (A_t)^{-\frac{1}{\chi}} \\ &= A_t \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}}. \end{aligned} \tag{C.24}$$

Solving (C.24), we can write the natural level of output y_t^n and the natural level of real wage $\frac{w_t^n}{p_t^n}$ as

$$y_t^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t, \tag{C.25}$$

and

$$\frac{w_t^n}{p_t^n} = \frac{\epsilon - 1}{\epsilon}(1-\alpha)A_t, \tag{C.26}$$

from which in equilibrium, we obtain

$$N_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}}, \tag{C.27}$$

and

$$C_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t. \tag{C.28}$$

In equilibrium, consumption of capitalists and workers add up to the final output produced (i.e., equation (26)). Based on (C.27) and (C.28), we obtain

$$\rho A_t Q_t^n + \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t. \tag{C.29}$$

where we define Q_t^n to be the natural level of detrended stock price. Therefore we obtain Q_t^n and C_t^n, given by

$$Q_t^n = \frac{1}{\rho}\left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}}\left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right), \tag{C.30}$$

and

$$C_t^n = \rho A_t Q_t^n = A_t\left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}}\left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right). \tag{C.31}$$

Since Q_t^n is constant, there is no drift and volatility for its process in the flexible price economy, thus we have $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To calculate the natural interest rate r_t^n, we start from the capital gain component in equation (27). By applying Ito's lemma, we obtain

$$\mathbb{E}\frac{d\left(p_t A_t Q_t\right)}{p_t A_t Q_t}\frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0}\sigma_t^p + \sigma\left(\sigma_t^p + \underbrace{\sigma_t^q}_{=0}\right). \tag{C.32}$$

As the dividend yield is always ρ, imposing expectation on both sides of (27) and combining with the equilibrium condition in equation (25) yields

$$i_t^m = \rho + \pi_t + g + \sigma\sigma_t^p = i_t + \left(\sigma + \sigma_t^p\right)^2. \tag{C.33}$$

Plugging (C.33) to the real interest rate formula in Lemma 1, we express the natural rate of interest r_t^n as

$$r_t^n = i_t - \pi_t + \sigma_t^p\left(\sigma + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p\right) = \rho + g - \sigma^2, \tag{C.34}$$

which is a function of structural parameters and σ_t, proving (iii) of Proposition 2. For the consumption process of capitalists in the flexible price case, since their consumption C_t^n is directly proportional to TFP A_t, we know

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \left(r_t^n - \rho + \sigma^2\right) dt + \sigma dZ_t, \tag{C.35}$$

where we use $r_t^n - \rho + \sigma^2 = g$ from equation (C.34). ■

C.1.4. Section 3.4

Proof of Lemma 2. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, where we use equation (C.13). Here $\frac{w_t^n}{p_t^n}$ is the real wage level in the flexible price economy. The good market equilibrium condition can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}}. \tag{C.36}$$

We subtract equation (C.36) from the same good market condition in the sticky price economy to obtain

$$A_t \left(\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}} = (C_t - C_t^n) + \left(\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}}, \tag{C.37}$$

where we divide both sides of equation (C.37) by $y_t^n \equiv A_t^{1-\frac{1}{\chi}} (\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$ and obtain

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\widehat{\frac{w_t}{p_t}}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\widehat{\frac{w_t}{p_t}}}, \tag{C.38}$$

which can be written as:

$$\frac{1}{\chi}\frac{\widehat{w_t}}{p_t} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon} \underbrace{\left(1 + \frac{1}{\chi}\right) \frac{\widehat{w_t}}{p_t}}_{=\hat{C^w}(t)}. \tag{C.39}$$

Equation (C.39) leads to

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \frac{\widehat{w_t}}{p_t} = \underbrace{\frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{C_{W,t}}. \tag{C.40}$$

We observe that Assumption 1 guarantees that gaps of asset price, consumption of capitalists and workers, employment, and real wage all co-move with positive correlations. Now we can use $\hat{Q}_t$ and $\hat{C}_t$ interchangeably, and if one gap variable becomes 0, then all other gap variables become also stabilized to 0, up to a first order.

■

Proof of Proposition 3. In the sticky price equilibrium, we would have $\sigma_t^p \equiv 0$, since over the small time period dt, a δdt portion of firms get to change their prices and there is no stochastic change in aggregate price level p_t up to a first-order. With the equilibrium wealth dynamics represented by (C.14) and the optimal consumption in (24), the capitalist's consumption C_t follows

$$\begin{aligned}\frac{dC_t}{C_t} &= (i_t^m - \pi_t - \rho)\,dt + (\sigma + \sigma_t^q)\,dZ_t \\ &= \Big(i_t + (\sigma + \sigma_t^q)^2 - \pi_t - \rho\Big)\,dt + (\sigma_t + \sigma_t^q)dZ_t.\end{aligned} \tag{C.41}$$

where we use the equilibrium condition in (25): $i_t^m = i_t + (\sigma + \sigma_t^q)^2$. Thus, the processes for $\ln C_t$ and $\ln C_t^n$ can be written as

$$d\ln C_t = \left(i_t - \pi_t + \frac{(\sigma_t + \sigma_t^q)^2}{2} - \rho\right)dt + (\sigma + \sigma_t^q)\,dZ_t, \tag{C.42}$$

and

$$d\ln C_t^n = \left(r_t^n - \rho + \frac{\sigma^2}{2}\right)dt + \sigma dZ_t, \tag{C.43}$$

of which the latter is from equation (C.35). From (C.42), we obtain

$$\begin{aligned} d\hat{Q}_t = d\hat{C}_t &= \left(i_t - \pi_t - \underbrace{\left(r_t^n - \frac{(\sigma + \sigma_t^q)^2}{2} + \frac{\sigma^2}{2}\right)}_{\equiv r_t^T}\right)dt + \sigma_t^q dZ_t \\ &= \left(i_t - \pi_t - r_t^T\right)dt + \sigma_t^q dZ_t. \end{aligned} \tag{C.44}$$

Since we have risk-premium levels $\mathrm{rp}_t = (\sigma_t + \sigma_t^q)^2$ in the sticky price economy and $\mathrm{rp}_t^n = \sigma^2$ in the flexible price economy, we can express our risk-adjusted natural rate r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}\left(\mathrm{rp}_t - \mathrm{rp}_t^n\right) = r_t^n - \frac{1}{2}\hat{rp}_t, \tag{C.45}$$

from which we know that when $\sigma_t^q = 0 (= \sigma_t^{q,n})$ holds, then we have $\hat{rp}_t = 0$ and $r_t^T = r_t^n$.

■

Proof of Proposition 4. Firms change their prices with instantaneous probability δdt à la Calvo (1983). If there is price dispersion Δ_t, as defined in (20), across intermediate goods firms, then labor market equilibrium condition can be written as

$$N_{W,t} = \int_0^1 n_t(i)di = \left(\frac{y_t}{A_t\left(N_{W,t}\right)^{\alpha}}\right)^{\frac{1}{1-\alpha}} \underbrace{\int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} di}_{\equiv \Delta_t^{\frac{1}{1-\alpha}}}, \tag{C.46}$$

where

$$y(t) = \frac{A_t N_{W,t}}{\Delta_t} = C_t + C_{W,t}. \tag{C.47}$$

We know that the good market equilibrium condition in (26) can be written as

$$\rho A_t Q_t + A_t \left(\frac{w_t}{p_t A_t}\right)^{1+\frac{1}{\chi}} = A_t \left(\frac{w_t}{p_t A_t}\right)^{\frac{1}{\chi}} \frac{1}{\Delta_t}. \tag{C.48}$$

Since a price process (i.e., (21)) does not affect the resource allocation in the flexible price economy, we can regard $\hat{x}_t$ to be the log-deviation of x_t from the flexible price economy *where the price is constant*. From price aggregator in (17), we obtain

$$\hat{p}_t = \int_0^1 \widehat{p_t(i)} di. \tag{C.49}$$

To study price dispersion Δ_t up to a first-order, we illustrate Woodford (2003)'s treatment of Δ_t up to a second-order. From

$$\begin{aligned}\frac{1}{1-\alpha}\hat{\Delta}_t &= \ln \int_0^1 \left(1 - \frac{\epsilon}{1-\alpha}\left(\cancel{\widehat{p_t(i)} - \hat{p}_t}\right) + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \left(\widehat{p_t(i)} - \hat{p}_t\right)^2\right) di + \text{h.o.t.} \\ &= \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 Var_i\left(\widehat{p_t(i)}\right) + \text{h.o.t},\end{aligned} \tag{C.50}$$

where h.o.t stands for higher-order terms, we observe that $\Delta_t \simeq 1$ up to a first-order because Δ_t is in nature the second order as (C.50) suggests. Pricing à la Calvo (1983) is standard, except that our model is in continuous time. For dt period from t to $t + dt$, individual firm i change the price with δdt probability. From time 0 perspective, a probability that firm resets its price for the first time at time t is

$$\delta \mathrm{e}^{-\delta t} dt = \underbrace{\delta dt}_{\text{Change now}} \cdot \underbrace{\mathrm{e}^{-\delta t}}_{\text{No change until t}}. \tag{C.51}$$

At time t, a price-changing firm i chooses $p_t(i)$ to solve

$$\begin{aligned}&\max_{p_t(i)} \frac{1}{\xi_t^N p_t} \mathbb{E}_t \int_t^{\infty} \mathrm{e}^{-\delta(s-t)} \xi_s^N p_s \left(\frac{p_t(i)}{p_s} y_{s|t}(i) - \frac{1}{p_s} C(y_{s|t}(i))\right) ds, \ \text{where } y_{s|t}(i) = \left(\frac{p_t(i)}{p_s}\right)^{-\epsilon} y_s \\ &= \frac{1}{\xi_t^N p_t} \mathbb{E}_t \int_t^{\infty} \mathrm{e}^{-\delta(s-t)} \xi_s^N p_s \left(\left(\frac{p_t(i)}{p_s}\right)^{1-\epsilon} y_s - \frac{1}{p_s} C\left(\left(\frac{p_t(i)}{p_s}\right)^{-\epsilon} y_s\right)\right) ds,\end{aligned} \tag{C.52}$$

where $C(\cdot)$ is defined as an individual firm's nominal production cost as a function of its output produced, which is to be written explicitly. Let $MC_{s|t}$ and $\varphi_{s|t}$ be the nominal and real marginal cost at time s conditional on price resetting at prior time t. Using the nominal pricing kernel ξ_s^N formula in (23), we obtain

$$\frac{\xi_s^N p_s}{\xi_t^N p_t} = \mathrm{e}^{-\rho(s-t)} \frac{C_t}{C_s}. \tag{C.53}$$

By plugging (C.53) into (C.52) and solving (C.52), the optimal adjusted price p_t^*[66] is given as

$$p_t^* = \frac{\mathbb{E}_t \displaystyle\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} \frac{\varphi_{s|t}}{\bar{\varphi}} p_s^\epsilon ds}{\mathbb{E}_t \displaystyle\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} p_s^{\epsilon-1} ds}, \tag{C.54}$$

where $\varphi_{s|t}$, the real marginal cost of firms at time s given the price resetting at previous time t, appears, and $\bar{\varphi}$ is its level in the flexible-price equilibrium, which is $\frac{\epsilon-1}{\epsilon}$. If we log-linearize (C.54) around the flexible price equilibrium with constant price as in (C.49), we can log-linearize $\widehat{p_t^*}$ expressed as

$$\widehat{p_t^*} = (\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\hat{\varphi}_{s|t} + \hat{p}_s\right) ds. \tag{C.55}$$

We know that the conditional real production cost and the conditional real marginal cost can be written as

$$\frac{1}{p_s} C(y_{s|t}) = \frac{w_s}{p_s} \left(\frac{y_{s|t}}{A_s (N_{W,s})^\alpha}\right)^{\frac{1}{1-\alpha}}, \tag{C.56}$$

and

$$\varphi_{s|t} \equiv \frac{1}{p_s} C'(y_{s|t}) = \frac{w_s}{p_s} \left(\frac{y_{s|t}}{A_s (N_{W,s})^\alpha}\right)^{\frac{\alpha}{1-\alpha}} \frac{1}{A_s (N_{W,s})^\alpha}. \tag{C.57}$$

From equation (C.57)), we obtain the conditional real marginal cost gap at time s conditional on price resetting at time t, which is given by

$$\hat{\varphi}_{s|t} = \underbrace{\frac{\hat{w}_s}{p_s}}_{\equiv \hat{\varphi}_s} - \frac{\alpha\epsilon}{1-\alpha} \left(\widehat{p_t^*} - \hat{p}_s\right) = \hat{\varphi}_s - \frac{\alpha\epsilon}{1-\alpha} \left(\widehat{p_t^*} - \hat{p}_s\right). \tag{C.58}$$

where $\hat{\varphi}_s$ is defined as the aggregate marginal cost index: since production is linear in aggregate level, $\hat{\varphi}_s$ becomes equal to the real wage gap. Using (C.49), we then characterize the change in aggregate price gap $\hat{p}_t$ as

$$\begin{aligned} d\hat{p}_t &= \delta dt \left(\widehat{p_t^*} - \hat{p}_t\right) \\ &= \delta dt(\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t\right) ds, \quad \text{where } \Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \end{aligned} \tag{C.59}$$

Since we log-linearize our economy around the flexible price equilibrium with constant price (i.e.,

[66] We use the property that every price-setting firm at any time t chooses the same price, so we drop the firm index i in $p_t^*(i)$ and use p_t^*.

$\pi_t = \sigma_t^p = 0$ in (21)), $\hat{p}_t$ changes with a rate of current π_t,[67] we have

$$\pi_t = \frac{d\hat{p}_t}{dt} = \delta(\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t\right) ds. \tag{C.60}$$

Now that we have (C.60) for the instantaneous inflation π_t. we manipulate (C.60) as:

$$\begin{aligned}\pi_t + \delta\hat{p}_t &= \delta(\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s + \hat{p}_s)ds = \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds \\ &= \delta(\delta+\rho)(\Theta\hat{\varphi}_t + \hat{p}_t)dt + \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds,\end{aligned} \tag{C.61}$$

where we can rewrite the first line of equation (C.61) at time $t + dt$ instead of t as

$$\begin{aligned}\pi_{t+dt} + \delta\hat{p}_{t+dt} &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)(t+dt)}\mathbb{E}_{t+dt} \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds \\ &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t} \left(1 + (\delta+\rho)dt\right) \mathbb{E}_{t+dt} \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds.\end{aligned} \tag{C.62}$$

Due to the *martingale representation theorem* (see e.g., Oksendal (1995)), there exists a measurable H_t such that

$$\mathbb{E}_{t+dt} \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds = \mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds + H_t dZ_t, \tag{C.63}$$

holds. We plug (C.63) into equation (C.62) to obtain[68]

$$\begin{aligned}\pi_{t+dt} + \delta\hat{p}_{t+dt} = \delta(\delta+\rho)\Bigg(&\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds + \mathrm{e}^{(\delta+\rho)t} H_t dZ_t \\ &+ \mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt \cdot \mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds\Bigg).\end{aligned} \tag{C.64}$$

We subtract (C.61) from (C.64) to obtain

$$\begin{aligned}d\pi_t + \cancel{\delta}\pi_t dt &= \delta(\delta+\rho)\left(\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt \cdot \mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds + \mathrm{e}^{(\delta+\rho)t} H_t dZ_t - (\Theta\hat{\varphi}_t + \hat{p}_t)dt\right) \\ &= \underbrace{\delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t} H_t}_{\equiv \sigma_{\pi,t}} dZ_t - \delta(\delta+\rho)\Theta\hat{\varphi}_t dt \\ &\quad + \underbrace{\delta(\delta+\rho)\left((\delta+\rho)dt \cdot \mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t)ds\right)}_{=(\cancel{\delta}+\rho)\pi_t dt},\end{aligned} \tag{C.65}$$

[67] In the case of positive inflation targets, see e.g., Coibion et al. (2012).
[68] We use the property that $dt \cdot dZ_t = 0$.

where we use

$$(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds=(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds, \tag{C.66}$$

which holds from $(dt)^2=0$. Note that in (C.65), we define $\sigma_{\pi,t}$ as an instantaneous volatility of the inflation process. Finally from equation (C.65) we get the continuous time version of New Keynesian Phillips curve (NKPC), written as[69]

$$d\pi_t=\rho\pi_t dt-\delta(\delta+\rho)\Theta\hat{\varphi}_t dt+\sigma_{\pi,t}dZ_t. \tag{C.67}$$

Due to the linear aggregate production function up to a first-order, we obtain:[70]

$$\hat{\varphi}_t=\frac{\widehat{w_t}}{p_t}=\left(\chi^{-1}-\frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1}\hat{Q}_t\equiv\frac{\kappa}{\delta(\delta+\rho)\Theta}\hat{Q}_t. \tag{C.68}$$

Finally plugging equation (C.68) into equation (C.67), we represent New-Keynesian Phillips curve in terms of asset price gap $\hat{Q}_t$ in the following way:

$$d\pi_t=\left(\rho\pi_t-\kappa\hat{Q}_t\right)dt+\sigma_{\pi,t}dZ_t,\ \text{ and }\ \mathbb{E}_t d\pi_t=\left(\rho\pi_t-\kappa\hat{Q}_t\right)dt, \tag{C.69}$$

which proves the proposition 4.[71] We know $\kappa>0$ due to Assumption 1.

■

C.2. Section 4

C.2.1. Section 4.2

Proof of Proposition 6. This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984). ■

B.2.2. Section 4.1

Proof of Proposition 5. The proof strategy is similar to Proposition 1. From (42), $\{\sigma_t^q\}$ process can be written as

$$d\sigma_t^q=-\frac{\phi^2(\sigma_t^q)^2}{2(\sigma+\sigma_t^q)^3}dt-\phi\frac{\sigma_t^q}{\sigma+\sigma_t^q}dZ_t. \tag{C.70}$$

Using Ito's lemma on (C.70), we write the process for $(\sigma+\sigma_t^q)^2$, which is a martingale itself, as

[69] Our continuous-time version of the Phillips curve in (C.65) is of the same form as in Werning (2012) and Cochrane (2017) after taking expectation on both sides.

[70] We use Lemma 2's log-linearization result to represent the real aggregate marginal cost gap $\frac{\hat{w}_t}{p_t}$ as a function of capitalists' consumption gap $\hat{C}_t=\hat{Q}_t$.

[71] Since $\hat{y}_t=\zeta\hat{Q}_t$, Phillips curve can be represented in terms of output gap $\hat{y}_t$ as in Proposition 4.

$$
\begin{aligned}
d\left(\sigma+\sigma_t^q\right)^2 &= 2\left(\sigma+\sigma_t^q\right) d\sigma_t^q + \left(d\sigma_t^q\right)^2 \\
&= 2\left(\sigma+\sigma_t^q\right)\left(-\frac{\phi^2(\sigma_t^q)^2}{2\left(\sigma+\sigma_t^q\right)^3}dt - \phi\frac{\sigma_t^q}{\sigma+\sigma_t^q}dZ_t\right) + \phi^2\frac{(\sigma_t^q)^2}{(\sigma+\sigma_t^q)^2}dt \\
&= -2\phi\left(\sigma_t^q\right)dZ_t.
\end{aligned}
\tag{C.71}
$$

Therefore, we would have $\mathbb{E}_0((\sigma+\sigma_t^q)^2) = (\sigma+\sigma_0^q)^2$ where $\mathbb{E}_0$ is an expectation operator with respect to the $t=0$ filtration. By Doob's martingale convergence theorem (as $(\sigma+\sigma_t^q)^2 \geq 0, \forall t$), we know $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0$ since:

$$
\underbrace{d\sigma_t^q}_{\overset{a.s}{\to}0} = -\underbrace{\frac{\phi^2\left(\sigma_t^q\right)^2}{2\left(\sigma+\sigma_t^q\right)^3}}_{\overset{a.s}{\to}0}dt - \phi\underbrace{\frac{\sigma_t^q}{\sigma+\sigma_t^q}}_{\overset{a.s}{\to}0}dZ_t.
\tag{C.72}
$$

Thus, (C.72) proves $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = 0$. From (41) $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = 0$ leads to $\hat{Q}_t \overset{a.s}{\to} 0$ and $\pi_t \overset{a.s}{\to} 0$. Finally, we must have $\mathbb{E}(\max_t(\sigma_t^q)^2) = \infty$, since otherwise, the uniform integrability implies $\mathbb{E}((\sigma+\sigma_\infty^q)^2) = (\sigma+\sigma_0^q)^2$, which is a contradiction to our earlier result $\sigma_t^q \overset{a.s}{\to} \sigma^{q,n} = 0$ since $\sigma_\infty^q = 0$ and $\sigma_0^q > \sigma^{q,n} = 0$ by assumption in Proposition 5.

■

D Detailed Derivations in Section 2

D.0. Model Setup

A representative household solves the following intertemporal optimization consumption-savings decision problem:

$$
\max_{\{C_s,L_s\}_{s\geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)}\left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right]\mathrm{d}s \quad \text{s.t.} \quad \mathrm{d}B_t = \left[i_tB_t - p_tC_t + w_tL_t + D_t\right]\mathrm{d}t
$$

where C_t is consumption, L_t aggregate labor, w_t is the equilibrium wage level, B_t are risk-free bonds held by the household at the beginning of t (hence, B_t at t is taken as given for each household), i_t is the nominal interest rate, D_t is a lump-sum transfer of any firm profits/losses towards the household, p_t the nominal price of consumption goods and ρ is the subjective discount rate of the household.

An individual firm i produces in this economy with the following production function:

$$
\begin{aligned}
Y_t^i &= A_tL_t^i, \quad \text{where} \\
\frac{\mathrm{d}A_t}{A_t} &= g\mathrm{d}t + \underbrace{\sigma}_{\text{Fundamental risk}}\mathrm{d}Z_t
\end{aligned}
$$

where A_t is the economy's total factor productivity, assumed to be exogenous and to follow a geometric Brownian motion with drift, where g is the expected growth rate of A_t, σ is its volatility, which we assume to be constant over time and call *fundamental* volatility, and Z_t is a standard Brownian motion process. It follows that firms' profits are defined as:

$$D_t = p_t Y_t - w_t L_t$$

Finally, we assume that in equilibrium, bonds are in zero net supply (i.e. $B_t = 0,\ \forall t$) and that there is no government spending, so market clearing in this economy results in $C_t = Y_t$.

D.1. Flexible Price Economy

We first solve the flexible price economy as our benchmark economy. In that purpose, we assume the usual Dixit Stiglitz monopolistic competition among firms, where the demand each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. Each firm i takes the aggregate price p_t, wage w_t, and the aggregate output Y_t as given.

D.1.1. Household problem

In the flexible price economy, each household takes $\{A_t, p_t, i_t\}$ process as given:

$$\frac{dp_t}{p_t} = \pi_t dt + \sigma_t^p dZ_t \tag{D.1}$$

and

$$di_t = \mu_t^i dt + \sigma_t^i dZ_t \tag{D.2}$$

where π_t, σ_t^p, μ_t^i, and σ_t^i are all endogenous, so the state variable for each household would become $\{B_t, A_t, p_t, i_t\}$.[72]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, p_t, i_t, t\right) = \max_{\{C_s, L_s\}_{s \geq t}} \mathbb{E}_t \int_s^{\infty} e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s.$$

The formula for the stochastic HJB equation is given as:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{\log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t}\right\}. \tag{D.3}$$

[72] This is a conjectural but correct statement due to the classical dichotomy between real and nominal sectors: output, consumption, and labor in equilibrium turn out to depend on A_t only and it turns out that p_t and i_t do not matter for the real economy and the welfare of the households.

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^{\Gamma} dt + \sigma_t^{\Gamma} dZ_t \tag{D.4}$$

where

$$\begin{aligned}
\mu_t^{\Gamma} =& \Gamma_t + \Gamma_B \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_p \cdot p_t \pi_t + \Gamma_i \cdot \mu_t^i \\
&+ \frac{1}{2}\Gamma_{AA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{pp} \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{ii} \cdot (\sigma_t^i)^2 \\
&+ \Gamma_{Ap} \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{Ai} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{pi} \cdot (p_t \sigma_t^p)\sigma_t^i
\end{aligned} \tag{D.5}$$

and $\sigma_t^{\Gamma} = \Gamma_A(\sigma A_t) + \Gamma_p(p_t \sigma_t^p) + \Gamma_i(\sigma_t^i)$. In the same way, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$ where

$$\begin{aligned}
\mu_t^{\Gamma_B} =& \Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{Bp} \cdot p_t \pi_t + \Gamma_{Bi} \cdot \mu_t^i \\
&+ \frac{1}{2}\Gamma_{BAA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{Bpp} \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{Bii} \cdot (\sigma_t^i)^2 \\
&+ \Gamma_{BAp} \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{BAi} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{Bpi} \cdot (p_t \sigma_t^p)\sigma_t^i
\end{aligned} \tag{D.6}$$

and $\sigma_t^{\Gamma_B} = \Gamma_{BA}(\sigma A_t) + \Gamma_{Bp}(p_t \sigma_t^p) + \Gamma_{Bi}(\sigma_t^i)$. Note $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, p, i\}$. Now plug equation (D.4) into equation (D.3) to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma} \right\}. \tag{D.7}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (D.7), we obtain:

$$\Gamma_B = \frac{1}{p_t C_t} \tag{D.8}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t} \tag{D.9}$$

Finally, merging (D.8) with (D.9) gives us the optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$, and $dQ_t = \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (D.7) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*} \tag{D.10}$$

Taking the derivative of both sides of equation (D.10) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*} \tag{D.11}$$

where $\mu_t^{\Gamma_B,*}$ is from equation (D.6) and it is evaluated at the optimum. Plugging (D.11) into the process for Γ_B, we obtain a simplified expression:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA}(A_t\sigma) + \Gamma_{Bp}(p_t\sigma_t^p) + \Gamma_{Bi}\left(\sigma_t^i\right)\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t \tag{D.12}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then, using equation (D.12) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t$$

From the definition of $\mathrm{d}Q_t$, we obtain:

$$\mathrm{d}Q_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t \tag{D.13}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ follows by taking expectations, which proves (2) in the flexible price equilibrium.

Nominal and real interest rates Prices and consumption would be adapted to the filtration generated by our Brownian motion Z_t process. Let us express the processes for consumption and price as:

$$\mathrm{d}p_t = \pi_t p_t \mathrm{d}t + \sigma_t^p p_t \mathrm{d}Z_t \tag{D.14}$$

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + \sigma_t^C C_t \mathrm{d}Z_t \tag{D.15}$$

where π_t, g_t^C, σ_t^p and σ_t^C are variables to be determined in equilibrium, which can be interpreted as inflation rate, expected consumption growth, and volatilities of prices and consumption processes, respectively. As the real state density is defined as $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$, the real interest rate r_t is defined by the relation $\mathbb{E}_t\left[\frac{d\zeta_t^r}{\zeta_t^r}\right] = -r_t dt$, similarly to (2).

With (D.15), applying Ito's Lemma to the real state density $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$ results in

$$\frac{d\zeta_t^r}{\zeta_t^r} = -\underbrace{\left[\rho + g_t^C - \left(\sigma_t^C\right)^2\right]}_{\equiv r_t} \mathrm{d}t - \sigma_t^C \mathrm{d}Z_t. \tag{D.16}$$

which determines the real interest rate $r_t = \rho + g_t^C - (\sigma_t^C)^2$. We also apply Ito's Lemma to

$\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$ and use the above processes for p_t and C_t to obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\left[\rho + g_t^C + \pi_t - (\sigma_t^p)^2 - \left(\sigma_t^C\right)^2 - \sigma_t^p\sigma_t^C\right]\mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right]\mathrm{d}Z_t$$

which can be rearranged as:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\underbrace{\left[r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right)\right]}_{=i_t}\mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right]\mathrm{d}Z_t \tag{D.17}$$

Comparing equation (D.13) and equation (D.17), we obtain

$$\begin{aligned} & i_t = r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right), \\ \text{where:} \quad & r_t = \rho + g_t^C - \left(\sigma_t^C\right)^2. \end{aligned}$$

D.1.2. Firm problem and equilibrium

Firm optimization As the demand each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t$$

as usual where p_t^i is an individual firm's price, p_t is the price aggregator, and Y_t is the aggregate output, each firm i solves the following problem:

$$\max_{p_t^i} \quad p_t^i\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t - \frac{w_t}{A_t}\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t, \tag{D.18}$$

which results in the following first-order condition for the firm:[73]

$$p_t = \left(\frac{\varepsilon}{\varepsilon - 1}\right)\frac{w_t}{A_t}, \tag{D.19}$$

which is intuitive as it tells us that in equilibrium, price is equal to the marginal cost of production multiplied by the constant mark-up, due to the constant elasticity of demand $\varepsilon > 1$. Using equation (D.19) and the equilibrium condition $C_t = Y_t = A_t L_t$ in the first-order condition of the household in (D.8) and (D.9), we obtain $L_t^n = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$,[74] which is a constant. This implies: in the flexible price equilibrium, we have $C_t^n = Y_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$. It follows that the stochastic process for Y_t^n is the same as that for A_t as follows:

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \frac{\mathrm{d}C_t^n}{C_t^n} = g\mathrm{d}t + \sigma\mathrm{d}Z_t. \tag{D.20}$$

[73] In equilibrium $p_t^i = p_t$ as every firm chooses the same price level.

[74] We impose the superscript n (i.e., natural) in variables to denote that those are the equilibrium values in the flexible price economy.

(D.20) implies that the growth rate of consumption and its volatility are $g_t^C = g$ and $\sigma_t^C = \sigma$, so the real interest rate in the flexible price economy, i.e., the natural rate of interest, can be expressed as $r_t^n \equiv r^n = \rho + g - \sigma^2$ from (D.16), which finally gives

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \Bigg(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2 \Bigg) \mathrm{d}t + \sigma \mathrm{d}Z_t$$

that proves equation (5).

D.2. Rigid Price Economy

We then solve our rigid price economy with $p_t = \bar{p}$ for $\forall t$. First, let us say the rigid price economy's consumption volatility, which we call σ_t^C is given as $\sigma_t^C = \sigma + \sigma_t^s$ (i.e. volatility of flexible price equilibrium in (D.20), plus excess volatility of rigid price equilibrium). Therefore, the consumption process can be written as:

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + (\sigma + \sigma_t^s) C_t \mathrm{d}Z_t. \tag{D.21}$$

And let us conjecture that this endogenous 'excess' volatility σ_t^s follows

$$d\sigma_t^s = \mu_t^\sigma dt + \sigma_t^\sigma dZ_t,$$

which turns out to be one of state variables in the rigid price economy. With price rigidity (i.e., $p_t = \bar{p}$ for $\forall t$), the agent takes $\{A_t, \sigma_t^s\}$ process as given, so the state variable for each household would become $\{B_t, A_t, \sigma_t^s\}$.[75]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, \sigma_t^s, t\right) = \max_{\{C_s, L_s\}_{s \geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] \mathrm{d}s$$

The formula for the stochastic HJB equation is:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t} \right\} \tag{D.22}$$

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^\Gamma dt + \sigma_t^\Gamma dZ_t \tag{D.23}$$

[75] This is a conjectural (but correct) statement as the actual output (thereby, consumption and other variables including inflation, nominal interest rate (that follows the Taylor rule), etc) would turn out to only depend on A_t and σ_t^s under our equilibrium construction.

where

$$\mu_t^\Gamma = \Gamma_t + \Gamma_B \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_\sigma \cdot \mu_t^\sigma$$
$$+ \frac{1}{2}\Gamma_{AA} \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{\sigma\sigma} \cdot (\sigma_t^\sigma)^2 + \Gamma_{A\sigma} \cdot (A_t\sigma)(\sigma_t^\sigma) \tag{D.24}$$

and $\sigma_t^\Gamma = \Gamma_A(\sigma A_t) + \Gamma_\sigma(\sigma_t^\sigma)$. Applying Ito's Lemma to Γ_B, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$ where

$$\mu_t^{\Gamma_B} = \Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{B\sigma} \cdot \mu_t^\sigma$$
$$+ \frac{1}{2}\Gamma_{BAA} \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{B\sigma\sigma} \cdot (\sigma_t^\sigma)^2 + \Gamma_{BA\sigma} \cdot (A_t\sigma)(\sigma_t^\sigma) \tag{D.25}$$

and $\sigma_t^{\Gamma_B} = \Gamma_{BA} \cdot (\sigma A_t) + \Gamma_{B\sigma} \cdot \sigma_t^\sigma$. Note $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, \sigma_t^s\}$. Now plug equation (D.23) into equation (D.22) to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^\Gamma \right\} \tag{D.26}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (D.26), we obtain:

$$\Gamma_B = \frac{1}{\bar{p} C_t} \tag{D.27}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t} \tag{D.28}$$

Finally, merging (D.27) with (D.28) gives us the optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t} \frac{1}{\bar{p} C_t}$, and $dQ_t = \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (D.26) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*} \tag{D.29}$$

Taking the derivative of both sides of equation (D.29) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*} \tag{D.30}$$

where $\mu_t^{\Gamma_B,*}$ is from equation (D.25) and it is evaluated at the optimum. Plugging equation (D.30) into the process for Γ_B, we obtain a simplified expression at the optimum:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA} \cdot (A_t \sigma) + \Gamma_{B\sigma} \cdot (\sigma_t^{\sigma})\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t \tag{D.31}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then using equation (D.31) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t$$

From the previous equation, we obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t \tag{D.32}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ also follows in the rigid price economy by taking conditional expectations.

D.3. Verification of the Martingale Equilibrium

Now let us verify that our martingale equilibrium, characterized by equations (13) and (14), satisfies our equilibrium conditions derived above. From (13) and (14),

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}, \tag{D.33}$$

$$\mathrm{d}\sigma_t^s = \underbrace{-(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3}}_{=\mu_t^{\sigma}} \mathrm{d}t \underbrace{-\phi_y \left(\frac{\sigma_t^s}{\sigma_t + \sigma_t^s}\right)}_{=\sigma_t^{\sigma}} \mathrm{d}Z_t. \tag{D.34}$$

These equations will be a solution to the model, as long as there is no contradiction with the equilibrium conditions. In order to check if (D.33) and (D.34) satisfy the equilibrium conditions, first, the output gap is defined as:

$$\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right) = \log\left(\frac{C_t}{C_t^n}\right) = \log\left(\frac{C_t}{A_t}\right) - \frac{\eta}{\eta+1}\log\left(\frac{\varepsilon-1}{\varepsilon}\right) \tag{D.35}$$

where the last equality follows from $C_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$, as shown above for the flexible price equilibrium. Combining (D.33) and (D.35), we obtain:

$$C_t = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}} \cdot \exp\left\{-\frac{(\sigma+\sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}\right\}, \tag{D.36}$$

which is a function of A_t and σ_t^s. Under fully sticky prices (i.e. $p_t = \bar{p},\ \forall t$), From equation (D.27) we knows

$$\Gamma_B = \frac{1}{\bar{p}C_t}. \tag{D.37}$$

We can now compute the derivative of equation (D.37) with respect to A_t and σ_t^s as:

$$\Gamma_{BA} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial A_t}, \tag{D.38}$$

and

$$\Gamma_{B\sigma} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s}. \tag{D.39}$$

Plugging equations (D.38) and (D.39) into equation (D.31), we obtain:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t - \Gamma_B \left[\frac{A_t}{C_t} \cdot \frac{\partial C_t}{\partial A_t} \cdot \sigma + \frac{1}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s} \cdot \sigma_t^\sigma \right] \mathrm{d}Z_t. \tag{D.40}$$

Using Ito's Lemma in equation (D.37) together with equation (D.21), we obtain

$$\mathrm{d}\Gamma_B = -\Gamma_B \left(g_t^C - (\sigma_t^C)^2 \right) \mathrm{d}t - \Gamma_B(\sigma + \sigma_t^s)\mathrm{d}Z_t. \tag{D.41}$$

Comparing the volatility terms in (D.40) and (D.41) (i.e., terms multiplied to $\mathrm{d}Z_t$), it must follow that:

$$\sigma + \sigma_t^s = \frac{A_t}{C_t} \cdot \frac{\partial C_t}{\partial A_t} \cdot \sigma + \frac{1}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s} \cdot \sigma_t^\sigma. \tag{D.42}$$

We can now compute the derivative of C_t with respect to A_t and σ_t^s as:

$$\frac{\partial C_t}{\partial A_t} = \frac{C_t}{A_t}, \tag{D.43}$$

and

$$\frac{\partial C_t}{\partial \sigma_t^s} = C_t \cdot \left(\frac{-(\sigma + \sigma_t^s)}{\phi_y} \right) = C_t \cdot (\sigma_t^\sigma)^{-1} \cdot \sigma_t^s, \tag{D.44}$$

which satisfies (D.42). Therefore, our martingale equilibrium is verified as an equilibrium.