A Proxy Contract Based Approach to the First-Order Approach in Agency Models

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The agency problem: Holmström (1979)

Principal's canonical problem ($\mathbf{x}$ is the multi-dimensional signal):

$$\max_{a,s(\cdot)} \int (\pi(\mathbf{x}) - s(\mathbf{x}))\, f(\mathbf{x}|a)d\mathbf{x} \quad \text{s.t.}$$

$$(i)\ (\text{PC}) \quad U(s(\cdot), a) \geq \overline{U}$$

$$(ii)\ (\text{IC}) \quad a \in \arg\max_{a'}\ U(s(\cdot), a') = \int u(s(\mathbf{x}))f(\mathbf{x}|a')d\mathbf{x} - a'$$

$$(iii)\ (\text{LL})\ s(\mathbf{x}) \geq \underline{s}$$

Note: the limited-liability (LL) $s(\mathbf{x}) \geq \underline{s}$ for the solution existence (e.g., Mirrlees (1975)): especially when

$$\frac{f_a}{f}(\mathbf{x}|a) \to -\infty, \quad \text{when} \quad x \to \underline{x} \tag{1}$$

First-Order Approach

Principal's canonical problem ($\mathbf{x}$ is the multi-dimensional signal):

$$\max_{a,s(\cdot)} \int (\pi(\mathbf{x}) - s(\mathbf{x}))\, f(\mathbf{x}|a) d\mathbf{x} \quad \text{s.t.}$$

$$(i)\ \text{(PC)} \quad U(s(\cdot), a) \geq \overline{U}$$

$$(ii)'\ \text{(IC)-relaxed} \quad U_a(s(\cdot), a) = \int u(s(\mathbf{x})) f_a(\mathbf{x}|a) d\mathbf{x} - 1 = 0$$

$$(iii)\ \text{(LL)}\ s(\mathbf{x}) \geq \underline{s}$$

Note: the limited-liability (LL) $s(\mathbf{x}) \geq \underline{s}$ for the solution existence (e.g., Mirrlees (1975)): especially when

$$\frac{f_a}{f}(\mathbf{x}|a) \to -\infty, \quad \text{when} \quad x \to \underline{x} \tag{2}$$

Optimal contract $(s^o(x), a^o)$ based on the first-order approach:

$$\frac{1}{u'(s^o(\mathbf{x}))} = \begin{cases} \lambda + \mu \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}, & \text{if } s^o(\mathbf{x}) \geq \underline{s}, \\ \dfrac{1}{u'(\underline{s})}, & \text{otherwise,} \end{cases}$$

with $\lambda \geq 0$ and $\mu > 0$

- Existence and uniqueness: Jewitt, Kadan, and Swinkels (2008)

If the agent's value function $U(s^o(\cdot), a)$,

$$U(s^o(\cdot), a) = \int u(s^o(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - a$$

is 'concave' in a, then the first-order approach is valid (e.g., Mirrlees (1975))

The previous literature since Mirrlees (1975): 'sufficient' conditions for $U(s^o(\cdot), a)$ to be 'concave' in a

The previous literature

Question (Focus of the literature)

How can we make $U(s^o(\cdot), a)$ concave in a?

Strategy 1: put conditions on $f(\mathbf{x}|a)$, the technology, only:

1. One-signal (i.e., $\mathbf{x}$ is scalar): Mirrlees (1975) and Rogerson (1985): MLRP (monotone likelihood ratio property) and CDFC (convexity of the distribution function condition)
2. Multi-signal extension of CDFC: Sinclair-Desgagné (1994, GCDFC: generalized CDFC), Conlon (2009, CISP: concave increasing set property), and Jung and Kim (2015, CD-FCL: convexity of the distribution function condition for the likelihood ratio)
3. Too restricted (e.g., normal, gamma distributions excluded)

Question (Focus of the literature)

How can we make $U(s^o(\cdot), a)$ concave in a?

Strategy 2: put conditions on both $u(s)$ and $f(\mathbf{x}|a)$:

1. Theorem 1 in Jewitt (1988):

$$w(z) \equiv u\left(u'^{-1}\left(\frac{1}{z}\right)\right) \text{ is concave in } z > 0 \tag{3}$$

   or Proposition 7 in Jung and Kim (2015):

$$U(s^o(\mathbf{x}), a^o) \equiv r(q) \text{ is concave in } q \equiv \frac{f_a}{f}(\mathbf{x}|a^o) \tag{4}$$

   $\longrightarrow$ (3) and (4) are equivalent

2. Problem: cannot be used when the agent's limited liability $s(\mathbf{x}) \geq \underline{s}$ binds:

$$U(s^o(\mathbf{x}), a^o) \equiv r(q) \text{ becomes convex in } q \equiv \frac{f_a}{f}(\mathbf{x}|a^o)$$

   around $\mathbf{x}$ where $s(\mathbf{x}) \geq \underline{s}$ binds

Our paper: different approach

Big Question (Possibly Non-Concave Expected Monetary Utility of the Agent)

Why should the agent's expected monetary utility $U(s^o(\cdot), a)$ be concave in a?

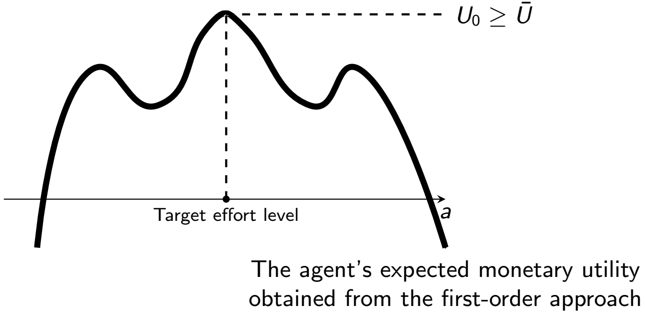
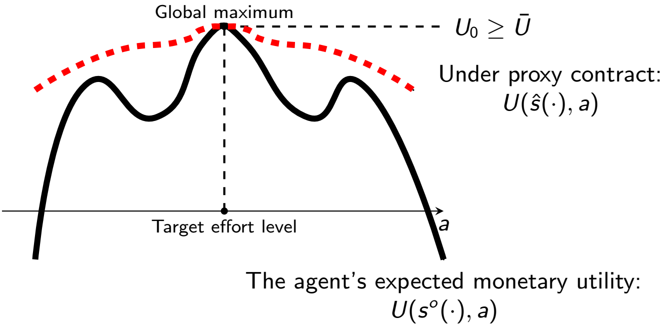


Figure: Possibly Non-Concave Expected Monetary Utility of the Agent



Our approach:

1. Finding a proxy function $\hat{s}(\mathbf{x})$ where the proxy value $U(\hat{s}(\cdot), a)$ is maximized at $a = a^o$, the same target action level

2. Proving $U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a)$, $\forall a$, justifying the first-order approach

Key idea: double-crossing property between $s^o(\cdot)$ and $\hat{s}(\cdot)$ in q-space

Fundamental Lemma

Change of variables to q-space

À la Jung and Kim (2015), define the likelihood ratio

$$\tilde{q} \equiv Q_{a^o}(\tilde{\mathbf{x}}) \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$$

The optimal contract $s^o(x)$ in q-space becomes:

$$s^o(x) \equiv w(q) \equiv (u')^{-1}\left(\frac{1}{\lambda + \mu q}\right)$$

The agent's indirect utility given $s^o(\cdot)$

$$u(s^o(\mathbf{x})) \equiv r(q) = \begin{cases} u(w(q)) \equiv \bar{r}(q), & \text{when } q \geq q_c \\ u(\underline{s}), & \text{when } q < q_c \end{cases}$$

- Threshold q_c solves $u'(\underline{s})^{-1} = \lambda + \mu q_c > 0$:
- $q \leq q_c$: the limited liability binds

Distribution function for q given a (possbly different from a^o)

$$G(q|a) \equiv Pr\left[Q_{a^o}(\tilde{\mathbf{x}}) \leq q|a\right], \quad dG(q|a) = g(q|a)dq$$

Double-crossing: constructing a proxy contract

Define $U^o \geq \overline{U}$ at the optimum:

$$U^o = U(s^o(\mathbf{x}), a^o) = \int u\left(s^o(\mathbf{x})\right) f(\mathbf{x}|a^o) d\mathbf{x} - a^o \tag{5}$$

Lemma (How to construct a proxy contract $\hat{s}(\cdot)$)

(1a) $f(\mathbf{x}|a)$ satisfies that $\dfrac{g(q|a)}{g(q|a^o)}$ is convex in $q = \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a

(2a) (Double-crossing) $\exists$a contract $\hat{s}(\mathbf{x})$ satisfying

$$(i) \quad \text{Same (PC)} \quad U(\hat{s}(\cdot), a^o) = \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - a^o = U^o \tag{6}$$

$$(ii) \quad \text{Same (IC)} \quad a^o \in \arg\max_{a'} \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a') d\mathbf{x} - a' \tag{7}$$

such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ double-crosses $r(q) \equiv u(s^o(\mathbf{x}))$ from above in q-space

then using the first-order approach is justified

Intuition

(1a) and **(2a)** jointly imply:

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int \left(r(q) - \hat{r}(q)\right) g(q|a) dq \leq 0, \quad \forall a$$

Why? We know that $U(s^o(\cdot), a^o) = U(\hat{s}(\cdot), a^o)$ when $a = a^o$

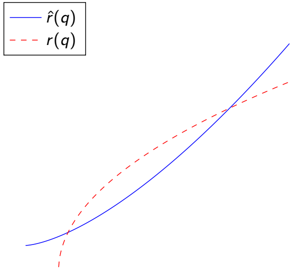


Figure: $r(q)$ and $\hat{r}(q)$: double-crossing

For example, when $a\uparrow$ from a^o, $G(q|a)$ shifts toward higher q, where $r(q) - \hat{r}(q)$ becomes more negative

- **(1a)** condition operationalizes this intuition

(1a) and **(2a)** jointly imply:

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int (r(q) - \hat{r}(q))\, g(q|a) dq \leq 0, \quad \forall a$$

But, it might be the following case

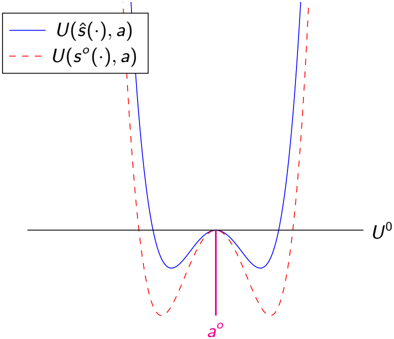


Figure: First-order approach not justified?

(2a) makes sure that $U(\hat{s}(\cdot), a)$ is maximized at $a = a^o$, therefore:

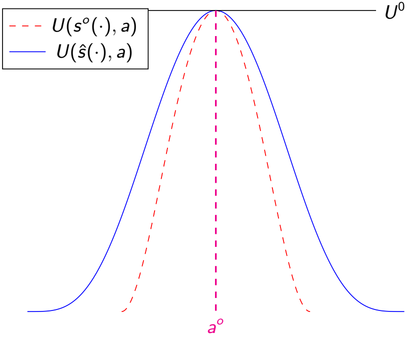


Figure: First-Order Approach Justified

So $U(s^o(\cdot), a)$ must be maximized at $a = a^o$

- The first-order approach (FOA) justified

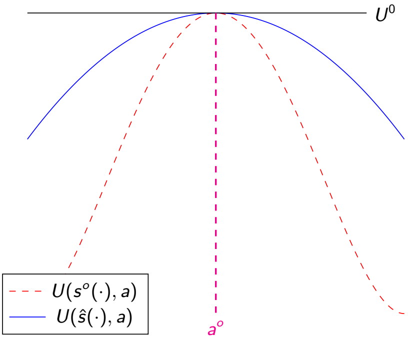


Figure: First-Order Approach Justified

So $U(s^o(\cdot), a)$ must be maximized at $a = a^o$

- The first-order approach (FOA) justified

When the Limited Liability (LL) Not Binds

Simplest case: linear proxy contract in q-space

Proposition (Proposition 1)

Given that the likelihood ratio, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below,[a] given a^o,

(1a) $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ for all a

(2b) $m(a) \equiv \int qg(q|a)dq$ is concave in a

(3b) $r(q)$ is concave in q

then the first-order approach is justified

[a] We assume $\underline{s}$ is small enough, so (LL) does not bind at optimum

Note: Now $\overline{r}(q) = r(q)$ due to the nonbinding (LL)

- **(2b)** and **(3b)** are from Jewitt (1988) and Jung and Kim (2015)
- Find $\hat{s}(\mathbf{x})$ such that $u(\hat{s}(\mathbf{x})) \equiv \hat{r}(q)$ becomes linear in q

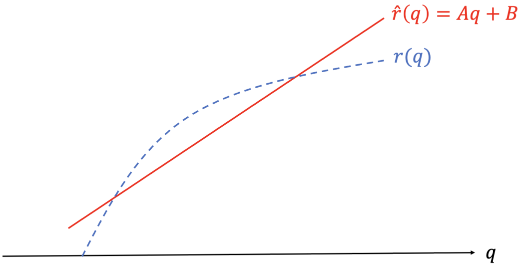


Figure: When the Agent's Limited Liability Constraint Does Not Bind

Simplest case: our proxy contract $\hat{r}(q)$ is linear in q

- **(2b)** makes sure under $\hat{r}(q)$, the agent will choose $a = a^o$
- With **(1a)** and **(3b)**, we apply the lemma above (double-crossing)

Violating **(3b)**: what if $\overline{r}(q)$ becomes convex in q?

Define the moment generating function (MGF) of $g(q|a)$:

$$M(a;t) \equiv \int e^{tq} g(q|a)dq$$

Proposition (Proposition 2)

Given that $u(s) > 0$ for all s and $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, given a^o,

(1a) $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ for all a

(2b′) $\phi\left(a;t,\overline{U}\right) \equiv \frac{M_a(a;t)}{M(a;t)} \times [\overline{U}+a]$ is decreasing in a for any given $t > 0$

(3b′) $\ln r(q)$ is concave in q

then the first-order approach is justified

(3b'): $\ln \overline{r}(q)$, not $\overline{r}(q)$, is concave so $\overline{r}(q)$ can be convex ($\rightarrow$ weaker)

- We prove **(2b')** is a bit stronger than (i.e., implies) **(2b)** instead
- In this case, our proxy contract $\hat{r}(q)$ is exponential in q

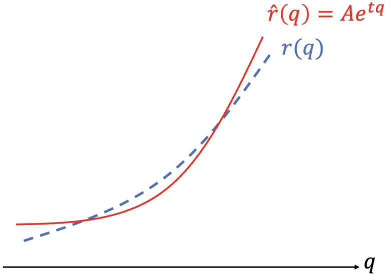


Figure: Double-Crossing: $\hat{r}(q)$ and $\bar{r}(q)$

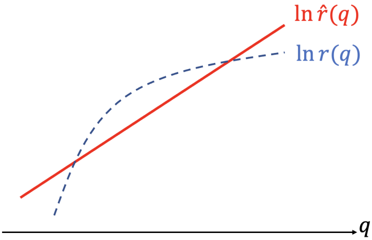


Figure: $\ln \hat{r}(q)$ and $\ln \bar{r}(q)$

Double-crossing: proxy contract $\hat{r}(q)$ is exponential in q so $\ln \hat{r}(q)$ is linear

- **(2b')** makes sure under $\hat{r}(q)$, the agent will choose $a = a^o$
- **(1a)** and **(3b')** allow us to apply the lemma above (double-crossing)

Example

Violating **(2b)**: what if $m(a)$ becomes convex in a?

Proposition (Proposition 3)

Given that $u(s) < 0$ for all s and $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below,[a] given a^o,

(1a) $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ for all a

(2b$''$) $\phi\left(a; t, \overline{U}\right) \equiv \frac{M_a(a;t)}{M(a;t)} \times [\overline{U} + a]$ is decreasing in a for any given $t < 0$

(3b$''$) $-\ln[-r(q)]$ is concave in q

then the first-order approach is justified

[a] We assume $\underline{s}$ is small enough, so (LL) does not bind at optimum

(3b"): $-\ln[-\overline{r}(q)]$, not $\overline{r}(q)$, is concave so $\overline{r}(q)$ is *more* concave ($\rightarrow$ stronger)

- We prove **(2b")** is a bit weaker than (i.e., implied by) **(2b)** instead
- In this case, our proxy contract $\hat{r}(q)$ is negative exponential (concave) in q

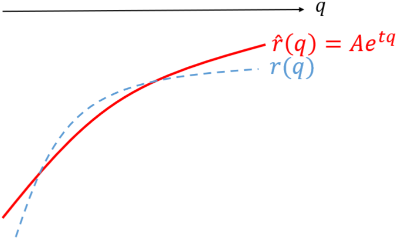


Figure: Double-Crossing: $\hat{r}(q)$ and $\overline{r}(q)$

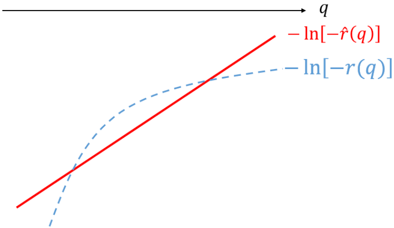


Figure: $\ln \hat{r}(q)$ and $\ln \overline{r}(q)$

Double-crossing: proxy contract $\hat{r}(q)$ is less concave in q so $-\ln[-\hat{r}(q)]$ is linear

- **(2b")** makes sure under $\hat{r}(q)$, the agent will choose $a = a^o$
- **(1a)** and **(3b")** allow us to apply the lemma above (double-crossing)

Example

When the Limited Liability (LL) Binds

Finding a proxy contract when (LL) binds for $q \leq q_c$

Define the moment generating function (MGF) of $g(q|a)$:

$$M(a;t) \equiv \int e^{tq} g(q|a) dq$$

Proposition (Proposition 4)

Given that the likelihood ratio, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is unbounded below, given a^o,

(1a) $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a

(2c) $\phi\left(a; t, \overline{U} - u(\underline{s})\right) \equiv \frac{M_a(a;t)}{M(a;t)} \times [\overline{U} - u(\underline{s}) + a]$ is decreasing in a for any given $t > 0$

(3c) $\ln\left[\overline{r}(q) - u(\underline{s})\right]$ is concave in q for all $q > q_c$, where q_c solves $\overline{r}(q_c) = u(\underline{s})$

then the first-order approach is justified

(3c): $\ln[\overline{r}(q) - u(\underline{s})]$, not $\overline{r}(q)$, is concave so $\overline{r}(q)$ can be convex ($\rightarrow$ weaker)

- **(2c)**: **(2b')** with $\bar{U} - u(\underline{s})$

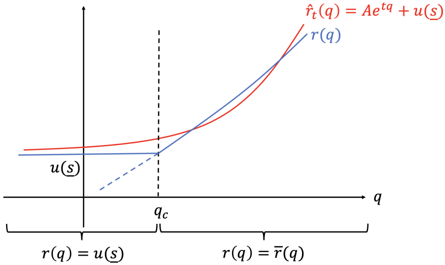


Figure: Double-Crossing: $\hat{r}(q)$ and $\overline{r}(q)$

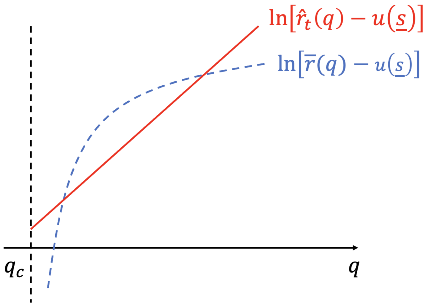


Figure: $\ln \hat{r}(q)$ and $\ln \overline{r}(q)$

Double-crossing: proxy contract $\hat{r}(q)$ is affine-exponential in q

- **(2c)** makes sure under $\hat{r}(q)$, the agent will choose $a = a^o$
- **(1a)** and **(3c)** allow us to apply the lemma above (double-crossing)

Example

Comparison with the earlier literature

To compare with Jung and Kim (2015)'s conditions **(1J-1)** and **(1J-2)**:

- We introduce the total positivity of degree 3 **(TP_3)** (Karlin (1968))
- Our **(1a)** condition is equivalent to ($\mathbf{TP}_3$) under MLRP
- Thus, Proposition 1 implies Jung and Kim (2015)

$$((\mathbf{TP}_3), (\mathbf{1J-2})) \underset{\text{Given MLRP}}{\overset{\text{Lemma 5}}{\longleftrightarrow}} ((\mathbf{1a}), (\mathbf{2b})) \xrightarrow{\text{Implies}} ((\mathbf{1J-1}), (\mathbf{1J-2}))$$

Figure: Relation Diagram between Conditions

Still, Our Propositions 2, 3, 4 extend the first-order approach's applicability

Thank you very much!
(Appendix)

Example 2

Example (Poisson distribution: (LL) not binding)

1. The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r > \frac{1}{2}$
2. The single-dimensional signal x, which is non-negative integer, follows

$$f(x|a) = \frac{[h(a)]^x}{\Gamma(x+1)} e^{-h(a)}. \tag{8}$$

which is the Poisson distribution with mean $h(a)$ that is increasing in a

Issue with $r > \frac{1}{2}$:

- $U(s^o(\mathbf{x}), a^o) \equiv r(q)$ becomes convex in q, not satisfying Jewitt (1988) and Jung and Kim (2015)
- Our Proposition 2 justifies the first-order approach in this case if

  $h(a)$ becomes concave 'enough'
- Jewitt (1988) and Jung and Kim (2015) imposes $h(a)$ is concave

Go back

Example (Poisson distribution: (LL) not binding)

1. The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r > \frac{1}{2}$
2. The single-dimensional signal x, which is non-negative integer, follows

$$f(x|a) = \frac{[h(a)]^x}{\Gamma(x+1)} e^{-h(a)}. \tag{9}$$

which is the Poisson distribution with mean $h(a)$ that is increasing in a

Even with $r > \frac{1}{2}$:

- Our Proposition 2 justifies the first-order approach in this case if

$$\frac{h''(a)}{h'(a)} \leq -\frac{1}{\overline{U} + a} < 0. \tag{10}$$

- Jewitt (1988) and Jung and Kim (2015) imposes $h(a)$ is concave

Trade-off: $h(a)$ should be more strictly concave, but $r > \frac{1}{2}$ allowed

Go back

Example 3

Example (Exponential distribution: (LL) not binding)

1. The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r < 0$
2. The single-dimensional signal x, which is non-negative, follows
$$f(x|a) = \frac{1}{h(a)} e^{-\frac{x}{h(a)}}, \tag{11}$$
which is the exponential distribution with mean $h(a)$ that is increasing in a
3. $h(a)$ is convex to some degree (not too much) in a

Issue with $h(a)$:

- Jewitt (1988) and Jung and Kim (2015) imposes $h(\cdot)$ is concave
- Our Proposition 3 justifies the first-order approach in this case if $U(s^o(\mathbf{x}), a^o) \equiv r(q)$ becomes concave 'enough'

Trade-off: $r(q)$ should be more strictly concave, but convex $h(a)$ allowed

Go back

Example (Exponential distribution: (LL) not binding)

1. The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r < 0$
2. The single-dimensional signal x, which is non-negative, follows
$$f(x|a) = \frac{1}{h(a)} e^{-\frac{x}{h(a)}}, \tag{12}$$
which is the exponential distribution with mean $h(a)$ that is increasing in a
3. $h(a)$ is convex to some degree (not too much) in a

- Our Proposition 3 justifies the first-order approach in this case if
$$\frac{h''(a)}{h'(a)} \leq \underbrace{-\frac{1}{\overline{U} + a}}_{>0}, \quad \forall a \in (0, \bar{a}], \tag{13}$$
if $h(a) + h'(a)[\overline{U} + a] < 0$ for $\forall a \in (0, \bar{a}]$

Trade-off: $r(q)$ should be more strictly concave, but convex $h(a)$ allowed

Go back

Example 4

Example (Normal distribution: (LL) binding)

1. The agent's utility is $u(s) = \frac{1}{r} s^r$, with $r < 1$
2. The single-dimensional signal $x \sim N(h(a), \sigma^2)$, i.e.,

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-h(a))^2}{2\sigma^2}}$$

   which is a normal distribution with mean $h(a)$ that is increasing in a

3. In this case, (LL) $s(x) \geq \underline{s}$ must be imposed

Issue with normal distribution:

- The previous literature cannot justify the first-order approach in this simple example
- $U(s^o(\mathbf{x}), a^o) \equiv r(q)$ becomes convex at points where (LL) binds, not satisfying Jewitt (1988) and Jung and Kim (2015): even if $r < \frac{1}{2}$

Go back

Example (Normal distribution: (LL) binding)

1. The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r < 1$

2. The single-dimensional signal $x \sim N(h(a), \sigma^2)$, i.e.,

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-h(a))^2}{2\sigma^2}}$$

   which is a normal distribution with mean $h(a)$ that is increasing in a

3. In this case, (LL) $s(x) \geq \underline{s}$ must be imposed

Issue with normal distribution:

- Our Proposition 4 justifies the first-order approach in this case if $h(a)$ becomes concave 'enough', regardless of $r < 1$
- Jewitt (1988) and Jung and Kim (2015) imposes $h(a)$ is concave

Trade-off: $h(a)$ should be more strictly concave, but (LL) allowed

Go back

Example (Normal distribution: (LL) binding)

1. The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r < 1$

2. The single-dimensional signal $x \sim N(h(a), \sigma^2)$, i.e.,

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   which is a normal distribution with mean $h(a)$ that is increasing in a

3. In this case, (LL) $s(x) \geq \underline{s}$ must be imposed

- Our Proposition 4 justifies the first-order approach if

$$\frac{h''(a)}{h'(a)} \leq -\frac{1}{\overline{U} - u(\underline{s}) + a} < 0, \quad \forall a, \tag{14}$$

Go back