Justifying the First-Order Approach in Agency Frameworks with the Agent's Possibly Non-Concave Value Function

Jin Yong Jung
Kangnam University

Son Ku Kim
Seoul National University

Seung Joo Lee
Oxford University-Saïd

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Principal's canonical problem (**x** is multi-dimensional):

$$\max_{a,s(\cdot)} \int \left(\underbrace{\pi(\mathbf{x})}_{\text{Principal's value}} - \underbrace{s(\mathbf{x})}_{\text{Contract}} \right) f(\mathbf{x}|a)d\mathbf{x}$$

$$s.t. \quad (i)\ U(s(\cdot), a) \geq \overline{U}$$

$$(ii)\ a \in \arg\max_{a'} \ U(s(\cdot), a') = \int u(s(\mathbf{x}))f(\mathbf{x}|a')d\mathbf{x} - c(a') = 0$$

$$(iii)\ \text{(LL)}\ s(\mathbf{x}) \geq \underline{s}$$

First-Order Approach (FOA): replace (ii) with its first-order condition (ii)'

$$\max_{a,s(\cdot)} \int \left(\underbrace{\pi(\mathbf{x})}_{\text{Principal's value}} - \underbrace{s(\mathbf{x})}_{\text{Contract}} \right) f(\mathbf{x}|a)d\mathbf{x}$$

$$s.t. \quad (i)\ U(s(\cdot), a) \geq \overline{U}$$

$$(ii)'\ U_a(s(\cdot), a) = \int u(s(\mathbf{x}))f_a(\mathbf{x}|a)d\mathbf{x} - c'(a) = 0$$

$$(iii)\ \text{(LL)}\ s(\mathbf{x}) \geq \underline{s}$$

Note: the limited-liability (LL) $s(\mathbf{x}) \geq \underline{s}$ for the solution existence (e.g., Mirrlees (1975))

Optimal contract $(s^o(x), a^o)$ based on the first-order approach:

$$\frac{1}{u'(s^o(\mathbf{x}))} = \begin{cases} \lambda + \mu \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}, & \text{if } s^o(\mathbf{x}) \geq \underline{s}, \\ \dfrac{1}{u'(\underline{s})}, & \text{otherwise}, \end{cases}$$

with $\lambda \geq 0$ and $\mu > 0$

- Existence and uniqueness: Jewitt, Kadan, and Swinkels (2008)

If the agent's value function $U(s^o(\cdot), a)$,

$$U(s^o(\cdot), a) = \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a)$$

is 'concave' in a, then the first-order approach is valid (e.g., Mirrlees (1975))

- The previous literature since Mirrlees (1975): 'sufficient' conditions for $U(s^o(\cdot), a)$ to be 'concave' in a

Question (Focus of the literature)

How can we make $U(s^o(\cdot), a)$ concave in a?

Strategy 1: put conditions on $f(x|a)$, the technology, only:

1. One-signal (i.e., $\mathbf{x}$ is scalar): Mirrlees (1975) and Rogerson (1985): MLRP (monotone likelihood ratio property) and CDFC (convexity of the distribution function condition)
2. Multi-signal extension of CDFC: Sinclair-Desgagné (1994, GCDFC: generalized CDFC), Conlon (2009, CISP: concave increasing set property), and Jung and Kim (2015, CDFCL: convexity of the distribution function condition for the likelihood ratio)
3. Too restricted (e.g., normal, gamma distributions excluded)

Question (Focus of the literature)

How can we make $U(s^o(\cdot), a)$ concave in a?

Strategy 2: put conditions on both $u(s)$ and $f(x|a)$:

1. Theorem 1 in Jewitt (1988):

$$w(z) \equiv u\left(u'^{-1}\left(\frac{1}{z}\right)\right) \text{ is concave in } z > 0 \tag{1}$$

   or Proposition 7 in Jung and Kim (2015):

$$U(s^o(\mathbf{x}), a^o) \text{ is concave in } q \equiv \frac{f_a}{f}(\mathbf{x}|a^o) \tag{2}$$

   $\longrightarrow$ (1) and (2) are equivalent

2. **Problem**: Cannot be used when the agent's limited liability $s(\mathbf{x}) \geq \underline{s}$ binds:

$$U(s^o(\mathbf{x}), a^o) \text{ becomes convex in } q \equiv \frac{f_a}{f}(\mathbf{x}|a^o)$$

   around $\mathbf{x}$ where $s(\mathbf{x}) \geq \underline{s}$ binds

The first-order approach cannot be justified by the previous literature in:

Example (Exponential distribution: (LL) not binding)

The agent's utility is $u(s) = \frac{1}{r}s^r$, $r \leq \frac{1}{2}$, and cost $c(a)$ is increasing and convex in a. The signal generating function has a multiplicative form, $\tilde{x} = h(a)\tilde{\theta}$, where $h(0) = 0$, $h(a)$ is increasing and convex to a small degree, and $\tilde{\theta}$ is exponentially distributed with mean 1, i.e., the density function of $\tilde{\theta}$ is $p(\theta) = e^{-\theta}$, $\theta \in [0, \infty)$. $\underline{s}$ is low enough. Thereby

$$f(x|a) = \frac{1}{h(a)}e^{-\frac{x}{h(a)}}, \tag{3}$$

- A little convexity of $h(a)$: does not satisfy Jewitt (1988) and Jung and Kim (2015) in **Strategy 2**
- Our Proposition 1 justifies the first-order approach in this case if

$$c(\cdot) \text{ becomes convex in } h(\cdot)$$

The first-order approach cannot be justified by the previous literature in:

Example (Exponential distribution: (LL) not binding)

The agent's utility is $u(s) = \frac{1}{r}s^r$, $1 > r > \frac{1}{2}$ (difference from the above example), and cost $c(a)$ is increasing and convex in a. The signal generating function has a multiplicative form, $\tilde{x} = h(a)\tilde{\theta}$, where $h(0) = 0$, $h(a)$ is increasing and concave,[a] and $\tilde{\theta}$ is exponentially distributed with mean 1, i.e., the density function of $\tilde{\theta}$ is $p(\theta) = e^{-\theta}$, $\theta \in [0, \infty)$. $\underline{s}$ is low enough. Thereby

$$f(x|a) = \frac{1}{h(a)} e^{-\frac{x}{h(a)}}, \tag{4}$$

[a] We assume $h(\cdot)$ is concave enough satisfying a regularity condition.

- Now concave $h(a)$: following Jewitt (1988) and Jung and Kim (2015)
- $1 > r > \frac{1}{2}$: $w(z)$ in Jewitt (1988) is convex in $z > 0$, thereby not satisfying Jewitt (1988) and Jung and Kim (2015)
- Our Proposition 2 justifies the first-order approach in this case

The first-order approach cannot be justified by the previous literature in:

Example (Normal distribution: (LL) binding)

The agent's utility is $u(s) = \frac{1}{r}s^r$, $r \leq 1$, The cost function is $c(a) = D(e^{ka} - 1)$, $D > 0$, $k > 0$, and the signal generating function has an additive form $\tilde{x} = a + \tilde{\theta}$, $\tilde{\theta} \sim N(0, \sigma^2)$ thereby

$$f(x|a) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

- Normal distributions often excluded by the previous literature: ($\leftarrow$ its likelihood ratio unbounded)

- $r < 1$: includes $r > \frac{1}{2}$ with which $w(z)$ in Jewitt (1988) is convex in $z > 0$ not satisfying Jewitt (1988) and Jung and Kim (2015)

- Our Proposition 3 justifies the first-order approach if

$$D \geq \overline{U} - u(\underline{s}) \tag{5}$$

The first-order approach cannot be justified by the previous literature in:

Example (Gamma distribution: (LL) binding)

The agent's utility is $u(s) = \frac{1}{r}s^r$, $r \leq 1$, Cost function is given by $c(a) = ka$, $k > 0$, and $\tilde{x} \in (0, \infty)$ has the gamma distribution with shape parameter a, i.e.,

$$f(x|a) = \frac{x^{a-1}\beta^{-a}}{\Gamma(a)} e^{-\frac{x}{\beta}}. \tag{6}$$

- Gamma distribution often excluded by the previous literature: ($\leftarrow$ its likelihood ratio unbounded)
- $r < 1$: includes $r > \frac{1}{2}$ with which $w(z)$ in Jewitt (1988) is convex in $z > 0$ not satisfying Jewitt (1988) and Jung and Kim (2015)
- Our Proposition 3 justifies the first-order approach (even with linear cost)

Big Question (Possibly Non-Concave Indirect Utility)

Why should the agent's value function $U(s^o(\cdot), a)$ be concave?

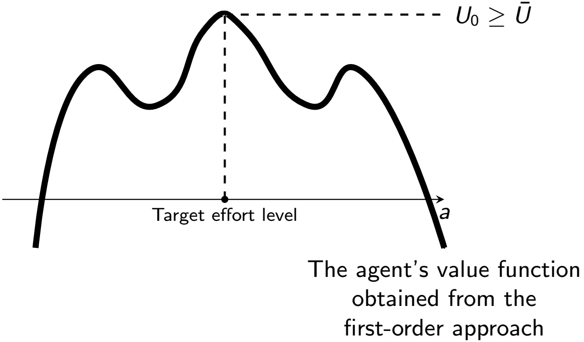
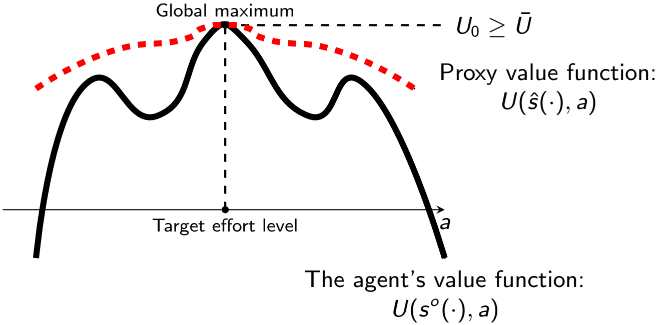


Figure: Possibly Non-Concave Indirect Utility of the Agent



Our approach: justify the first-order approach in all of the above examples

1. Finding a proxy function $\hat{s}(\mathbf{x})$ where the proxy value $U(\hat{s}(\cdot), a)$ is maximized at $a = a^o$, the same target action level
2. Proving $U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a)$, $\forall a$, justifying the first-order approach
3. A proper proxy $\hat{s}(\mathbf{x})$ depends on whether the limited liability binds or not

Note: impose additional conditions on the agent's cost function $c(\cdot)$

Fundamental Lemma

À la Jung and Kim (2015), define the likelihood ratio

$$\tilde{q} \equiv Q_{a^o}(\tilde{\mathbf{x}}) \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$$

The optimal contract $s^o(x)$ in q-space becomes:

$$s^o(x) \equiv w(q) \equiv (u')^{-1}\left(\frac{1}{\lambda + \mu q}\right)$$

The agent's indirect utility (value function) given $s^o(\cdot)$

$$u(s^o(\mathbf{x})) \equiv r(q) = \begin{cases} u(w(q)) \equiv \bar{r}(q), & \text{when } q \geq q_c \\ u(\underline{s}), & \text{when } q < q_c \end{cases}$$

- Threshold q_c solves $u'(\underline{s})^{-1} = \lambda + \mu q_c > 0$: limited liability starts to bind

Distribution function for q given a (possbly different from a^o)

$$G(q|a) \equiv Pr\left[Q_{a^o}(\tilde{\mathbf{x}}) \leq q|a\right], \quad dG(q|a) = g(q|a)dq$$

Properties of a proxy contract

Define $U^o \geq \overline{U}$ at the optimum:

$$U^o = U(s^o(\mathbf{x}), a^o) = \int u\left(s^o(\mathbf{x})\right) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) \qquad (7)$$

Lemma (How to construct a proxy contract $\hat{s}(\cdot)$)

(1a) $f(\mathbf{x}|a)$ satisfies that $\dfrac{g(q|a)}{g(q|a^o)}$ is convex in $q = \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a

(2a) (DOUBLE-CROSSING PROPERTY) $\exists$a contract $\hat{s}(\mathbf{x})$ satisfying

$$(i) \quad \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) = U^o \qquad (8)$$

$$(ii) \quad \int u(\hat{s}(\mathbf{x})) f_a(\mathbf{x}|a^o) d\mathbf{x} - c'(a^o) = 0 \qquad (9)$$

such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ crosses $r(q) \equiv u(s^o(\mathbf{x}))$ twice starting from above

(3a) $E[\hat{r}(q)|a]$ is concave in $c(a)$

then using the first-order approach is justified

(1a) and (2a) jointly imply:

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int (r(q) - \hat{r}(q))\, g(q|a) dq \leq 0, \quad \forall a$$

Why?: we know that $U(s^o(\cdot), a^o) = U(\hat{s}(\cdot), a^o)$ when $a = a^o$

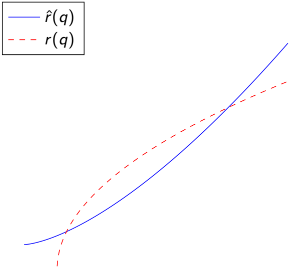


Figure: $r(q)$ and $\hat{r}(q)$: double-crossing

As $a\uparrow$ from a^o: $g(q|a)$ moves toward higher q, where $r(q) - \hat{r}(q)$ is more likely to be negative. When $a\downarrow$ from a^o, the same

- (1a) condition operationalizes this intuition

(1a) and (2a) jointly imply:

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int (r(q) - \hat{r}(q))\, g(q|a) dq \leq 0, \quad \forall a$$

But: It might be the following case

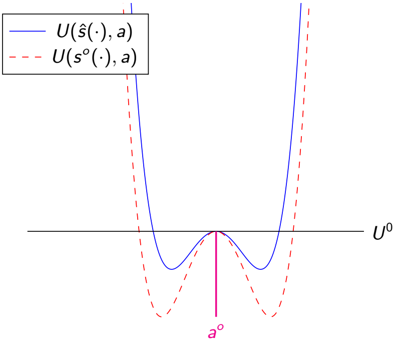


Figure: First-order approach not justified?

(3a) makes sure that $U(\hat{s}(\cdot), a)$ is maximized at $a = a^o$, therefore:

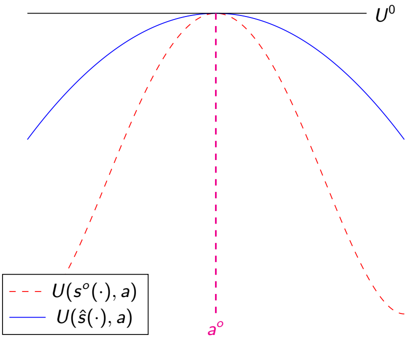


Figure: First-Order Approach Justified

So $U(s^o(\cdot), a)$ must be maximized at $a = a^o$

- The first-order approach (FOA) justified

When the Limited Liability (LL) Not Binds

Proposition (When (LL) does not bind)

Given that the likelihood ratio, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below,[a] given a^o,

(1a) $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a

(2c) $c(a)$ is convex in $m(a) \equiv \int qg(q|a)dq$, and

(3c) $r(q) = \overline{r}(q)$ is concave in q

then the first-order approach is justified

[a] We assume $\underline{s}$ is small enough, so (LL) does not bind at optimum.

Note: Now $\overline{r}(q) = r(q)$ due to the nonbinding (LL)

- In this case, finding a proxy contract $\hat{s}(\mathbf{x})$ is easier (no need to respect the limited liability (LL))
- Find $\hat{s}(\mathbf{x})$ such that $u(\hat{s}(\mathbf{x})) \equiv \hat{r}(q)$ becomes linear in q

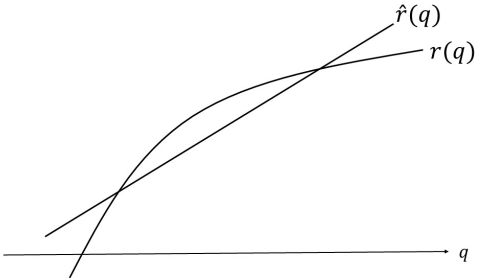


Figure: When the Limited Liability Constraint Does Not Bind

Simplest case: our proxy contract $\hat{r}(q)$ is linear in q

- (2c) makes sure under $\hat{r}(q)$, the agent will choose $a = a^o$
- With (1c) and (3c), we apply the lemma above (double-crossing) while (3c) is from Jewitt (1988) and Jung and Kim (2015)
- This case justifies Example 1 (the exponential distribution case with $r < \frac{1}{2}$)

What happens if $\overline{r}(q)$ becomes convex in q?

Proposition (When (LL) does not bind)

Given that the likelihood ratio, $\tilde{q} \equiv \dfrac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below,[a] given a^o,

(1a) $\dfrac{g(q|a)}{g(q|a^o)}$ is convex in $q = \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a

(2c') (i) there exists $t > 0$ such that

$$\frac{c'(a^o)}{M'(a^o;t)} M(a^o;t) - c(a^o) = \overline{U}$$

and (ii) $c(a)$ is convex in $M(a;t)$ for such $t > 0$, and

(3c') $\ln \overline{r}(q)$ is concave in q

then the first-order approach is justified

[a] We assume $\underline{s}$ is small enough, so (LL) does not bind at optimum.

Note: Now $\ln \overline{r}(q)$, not $\overline{r}(q)$, is concave so $\overline{r}(q)$ can be convex ($\rightarrow$ weaker)

- (2c') is a bit stronger than (2c) instead
- In this case, our proxy contract $\hat{r}(q)$ is exponential in q

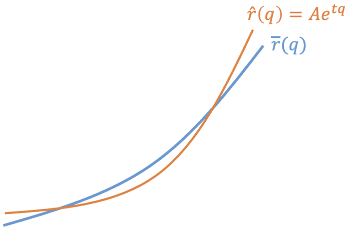


Figure: $\hat{r}(q)$ and $\overline{r}(q)$

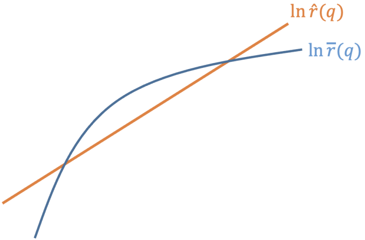


Figure: $\ln \hat{r}(q)$ and $\ln \overline{r}(q)$

Simplest case: our proxy contract $\hat{r}(q)$ is exponential in q so $\ln \hat{r}(q)$ is linear

- (2c') makes sure under $\ln \hat{r}(q)$, the agent will choose $a = a^o$
- (1c) and (3c') allow us to apply the lemma above (double-crossing)
- This case justifies Example 2 (the exponential distribution case with $r > \frac{1}{2}$ and concave $h(a)$)

When the Limited Liability (LL) Binds

Define the moment generating function (MGF) of $g(q|a)$:

$$M(a;t) \equiv \int e^{tq} g(q|a)dq.$$

Proposition (When (LL) binds for $q \leq q_c$)

Given that the likelihood ratio, $\tilde{q} \equiv \dfrac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is unbounded below, given a^o,

(1a) $\dfrac{g(q|a)}{g(q|a^o)}$ is convex in $q = \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a

(2b) (i) there exists $t > 0$ such that

$$\frac{c'(a^o)}{M'(a^o;t)} M(a^o;t) - c(a^o) \leq \overline{U} - u(\underline{s})$$

and (ii) $c(a)$ is convex in $M(a;t)$ for such t, and

(3b) $\ln(\overline{r}(q) - u(\underline{s}))$ is concave in q

then the first-order approach is justified

Note: Concave $\overline{r}(q) \longrightarrow$ concave $\ln(\overline{r}(q) - u(\underline{s}))$ ($\rightarrow$ weaker)

Intuition: a proxy contract $\hat{s}(\mathbf{x})$ must respect the limited liability constraint (LL)

$$u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = Ae^{tq} + B$$

which has a good property: $\hat{r}_t(q) \longrightarrow \underbrace{B \geq u(\underline{s})}_{\text{by (i) of (2b)}}$ as $q \longrightarrow -\infty$

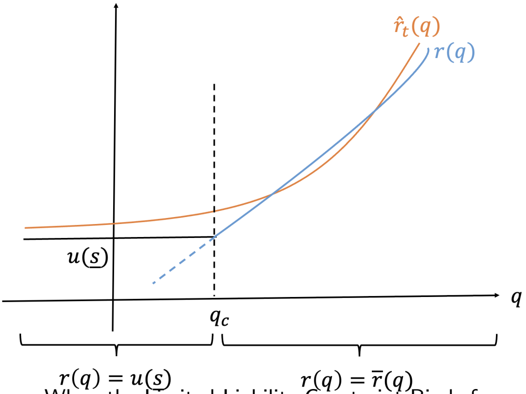


Figure: When the Limited Liability Constraint Binds for $q \leq q_c$

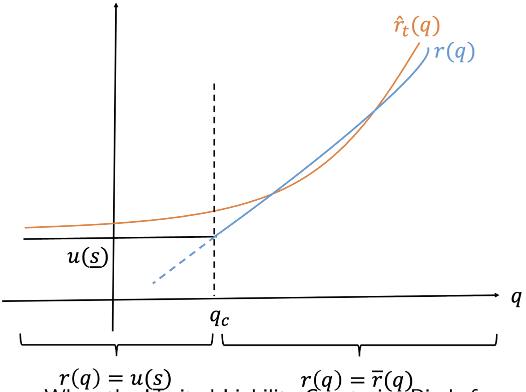


Figure: When the Limited Liability Constraint Binds for $q \leq q_c$

Note: Examples 3 and 4 (Normal and Gamma distributions) can be justified of their use of the first-order approach

- $\bar{r}(q)$ for $q \geq q_c$ can be convex
- Both distribution features unbounded likelihood ratio (thus we need (LL)): Jewitt (1988) and Jung and Kim (2015) assume away (LL) in contrast

Comparison with the earlier literature

To compare with Jung and Kim (2015)'s conditions (1J-1) and (1J-2):

- We introduce the total positivity of degree 3 (TP_3) (Karlin (1968))
- Our (1a) condition is related to this (TP_3) condition

$$
\begin{tikzcd}
((TP_3),(1J-2)) \arrow[r, leftrightarrow, "\text{Lemma 5}", "\text{Given MLRP}"'] \arrow[dr] & ((1a),(1J-2)) \arrow[r] & ((1a),(2c)) \arrow[dl, leftrightarrow, "\text{No inclusion}"'] \\
& ((1J-1),(1J-2)) &
\end{tikzcd}
$$

Figure: Relation Diagram between Conditions

Conclusion: no direct inclusion between our paper and Jung and Kim (2015)

- Jung and Kim (2015): applicable only when (LL) does not bind