Self-fulfilling Volatility and a New Monetary Policy[*]

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Abstract

We demonstrate that macroeconomic models with nominal rigidities feature multiple global solutions supporting alternative equilibria traditionally overlooked in the literature. In these equilibria, conventional Taylor rules give rise to self-fulfilling aggregate volatility, propelling the economy into crises (booms) characterized by elevated (reduced) aggregate risk. This outcome stems from the inability of conventional rules to target the expected growth rate of output, which is determined not only by the policy rate but also by the strength of the precautionary savings channel. We propose a new policy rule that targets both conventional mandates and aggregate volatility, reestablishing equilibrium uniqueness and achieving perfect stabilization.

Keywords: Taylor Rules, Self-fulfilling Volatility, Precautionary Savings
JEL codes: E32, E43, E44, E52

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Note: The ordering of the names of the authors is not indicative of relative contributions; it was chosen arbitrarily (or by mutual agreement) for practical reasons.

1 Introduction

Macroeconomic models with nominal rigidities can exhibit multiple global solutions driven by self-fulfilling aggregate volatility, independent of the central bank's responsiveness to standard business cycle targets such as output gap and inflation. We illustrate this phenomenon within a standard New Keynesian framework.[1] Common approaches in the literature, which rely on linear approximations, often omit higher-order terms associated with aggregate volatility, inadvertently discarding these equilibria. In contrast, our continuous-time formulation preserves these moments in closed-form expressions of the model's equilibrium conditions while remaining tractable. These alternative solutions can generate large and persistent output fluctuations, pushing the economy into prolonged recessions or booms even under active monetary policy.

In the canonical New Keynesian model, aggregate volatility affects consumption demand through the precautionary savings channel. Specifically, increased volatility prompts households to boost precautionary savings, leading to lower aggregate demand and output; output fluctuations then shape aggregate volatility. This mechanism can generate a vicious cycle driven by an aggregate demand externality: if a household expects that others will save more in response to anticipated consumption volatility, it will forecast lower current incomes and higher volatility in its future incomes, prompting it to reduce consumption and further amplify aggregate volatility—eventually rendering the expectation self-fulfilling in certain equilibria.

For instance, consider a scenario in which households in period 0 predict higher aggregate volatility in next period's consumption. They increase precautionary savings and reduce current consumption, triggering a recession in period 0. In period 1, their initial fear about aggregate volatility can be confirmed if, for each consumption realization in period 1, there is a corresponding conditional volatility of period 2 consumption that justifies it. Specifically, higher consumption in period 1 should be paired with lower conditional volatility of period 2 consumption, which reduces precautionary savings. Thus, households' beliefs about current volatility are shaped by past expectations and confirmed by subsequent actions. Note then, that our equilibrium construction relies on nominal rigidities: households' path-dependent consumption strategies determine the stochastic paths of output in a demand-driven economy.

In our first example of such a rational expectations equilibrium, the output gap fol-

[1] See, for example, Galí (2015).

lows a local martingale—meaning that, on average, the economy remains at its current level from one period to the next. We prove that, in this solution, the stabilized path (i.e., the flexible price economy benchmark) acts as an attractor for all sample paths, with the conditional volatility of subsequent consumption declining as the economy approaches it. Consequently, after the emergence of a self-fulfilling volatility shock, the economy is almost surely stabilized in the long run. However, along the equilibrium path—and until the economy is nearly stabilized following the initial shock—it experiences a prolonged recession accompanied by increased aggregate volatility. We demonstrate that a probability-zero event, in which conditional volatility ultimately diverges toward infinity, is key to supporting the appearance of the initial shock, ensuring that the economy follows a local martingale even if it eventually stabilizes. We interpret this property as an endogenously generated rare disaster event arising in a self-fulfilling manner.

Next, we present another class of global solutions. We demonstrate that these solutions generate stable, stationary long-run distributions while exhibiting stochastic fluctuations in both aggregate volatility and the output gap, along with suboptimal steady states characterized by under- or over-production. These findings suggest that economies may sustain equilibria with incessant fluctuations in aggregate volatility and that the welfare costs of business cycles could have been underestimated in the previous literature.

Conventional Taylor rules fail to prevent multiple equilibria and self-fulfilling shocks to aggregate volatility because they do not break the feedback loop linking precautionary savings, endogenous output growth, and its volatility. We show that central banks can address this issue by either (i) establishing explicit output growth mandates while using the policy rate as an intermediate tool, or (ii) incorporating economic volatility directly into their interest rate rules. The latter approach requires that central banks accurately measure and target economic volatility. Even when measurements are imperfect, volatility targeting accelerates economic stabilization under these alternative regimes.

Related literature Our non-linear characterization of the model shares similarities with Caballero and Simsek (2020a,b) in terms of incorporating aggregate volatility, which affects the business cycle fluctuations. However, while their framework focuses on how behavioral biases can generate intriguing crisis dynamics through the feedback loop between asset markets and business cycles,[2] our attention centers on the traditional policy rule un-

[2] Caballero and Simsek (2020b) present a model with optimists and pessimists who hold differing beliefs about the probability of an imminent recession or normal state. During zero lower bound (ZLB) episodes, an

der rational expectations and the existence of alternative equilibria arising from *endogenous* higher-order moments.

While Benhabib et al. (2002) study monetary-fiscal regimes in regards to eliminating indeterminacy issues posed by the ZLB, and Obstfeld and Rogoff (2021) show how a probabilistic (and small) fiscal currency backing can rule out speculative hyperinflation in monetary models, our focus is on the self-fulfilling emergence of aggregate volatility outside the ZLB and the exploration of alternative monetary policy rules.

There is a large macro-finance literature on the self-fulfilling nature of real and financial uncertainty: e.g., Bacchetta et al. (2017), Fajgelbaum et al. (2017),[3] Benhabib et al. (2019, 2024), and Chan (2024). Bacchetta et al. (2017) characterize an endowment economy where current asset prices are affected by a sunspot that shifts the perceived risk of future asset prices. Benhabib et al. (2019) develop a model of "mutual learning" between financial markets and the real economy, leading to strategic complementarity and self-fulfilling uncertainty. Benhabib et al. (2024) study a model of aggregate demand externality,[4] where a positive feedback loop between aggregate output and defaults generates a self-fulfilling default cycle.[5] We instead abstract from defaults and focus on the self-fulfilling appearance of aggregate volatility in a model with nominal rigidities and aggregate demand externalities. Chan (2024) departs from the complete information benchmark in a linearized New Keynesian framework, showing that beliefs about aggregate demand can be self-fulfilling. In contrast, we assume complete information and solve the standard model non-linearly, revealing that this approach can also generate self-fulfilling aggregate volatility.

Our equilibrium multiplicity results resemble those of Acharya and Dogra (2020) and Khorrami and Mendo (2024). While Acharya and Dogra (2020) investigates how determinacy conditions change in the presence of exogenous *idiosyncratic* volatilities that are functions of aggregate output, we explore the existence of self-fulfilling aggregate volatility and examine the monetary policy that restores determinacy. Khorrami and Mendo (2024) study similar equilibrium indeterminacy issues around aggregate volatility and propose fiscal rules as an alternative mechanism to determine equilibrium.

endogenous decline in risky asset valuation, triggered by a reduction in optimists' wealth, leads to a demand recession. We explore related ZLB issues in a separate paper, Dordal i Carreras and Lee (2024).

[3]In Fajgelbaum et al. (2017), higher uncertainty about fundamentals leads to lower investment, slowing down information flows and further discouraging investment. This results in 'uncertainty traps' characterized by self-fulfilling uncertainty and low activity.

[4]For a modern treatment of this issue, see Farhi and Werning (2016).

[5]When aggregate output falls, it raises defaults as firms' revenues and profits decline. As defaults disrupt production, they further decrease aggregate output, ad infinitum.

Layout Section 2 presents the model and derives the key equilibrium conditions. Section 3 provides the main results concerning the existence of multiple equilibria. Section 4 analyzes policy interventions that restore the constrained-efficient equilibrium. Section 5 extends these results to an environment with sticky prices. Section 6 concludes.

2 Standard Non-linear New Keynesian Model

This section describes the main assumptions and presents the exact non-linear optimality conditions of a standard New Keynesian model. To build analytical intuition for our subsequent results on equilibrium multiplicity, we first consider a simplified version of the model with perfectly rigid prices—an assumption relaxed later in Section 5. Appendix II contains detailed derivations and technical aspects underlying the results presented here.

2.1 Households

Consider an economy with a representative household h, whose optimization problem is given by

$$\Gamma_t^h \equiv \max_{\substack{\{C_s^h, L_s^h\}_{s\geq t} \\ \{B_s^h\}_{s>t}}} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \left[\log C_s^h - \frac{\left(L_s^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] ds, \tag{1}$$

subject to the budget constraint

$$\dot{B}_t^h = i_t B_t^h - \bar{p} C_t^h + w_t L_t^h + D_t,$$

where C_t^h and L_t^h denote household consumption and labor supply, respectively. Here, η is the Frisch elasticity of labor supply, ρ is the time discount rate, and B_t^h represents nominal bond holdings, which are in zero aggregate net supply in equilibrium. The household receives lump-sum transfers D_t, which include firms' profits and government transfers. The equilibrium wage is w_t, and the policy rate set by the central bank is i_t. The household's value function is denoted by Γ_t^h.

We simplify the analysis by assuming a perfectly rigid price level: $p_t = \bar{p}$, $\forall t$, implying zero inflation ($\pi_t = 0$ for all t). Although not crucial, this assumption allows us to illustrate clearly the key mechanisms of interest.[6] There is no government spending, and thus

[6]Section 5 relaxes the assumption of rigid prices by incorporating price stickiness à la Rotemberg (1982), showing that the multiplicity of equilibria associated with aggregate volatility persists even in a model with

aggregate consumption fully determines production.

We obtain the intertemporal optimality condition of (1) as

$$-i_t dt = \mathbb{E}_t\left(\frac{d\xi_t^{N,h}}{\xi_t^{N,h}}\right), \quad \text{where } \xi_t^{N,h} = e^{-\rho t}\frac{1}{\bar{p}}\frac{1}{C_t^h}, \tag{2}$$

with $\frac{d\xi_t^{N,h}}{\xi_t^{N,h}}$ representing the instantaneous (nominal) stochastic discount factor, whose expected value equals the (minus) nominal risk-free rate, $-i_t dt$.[7] Due to the rigid price assumption, the real and nominal risk-free rates are equal, $r_t = i_t$, where r_t represents the real interest rate.

We can rewrite equation (2) as

$$\mathbb{E}_t\left(\frac{dC_t^h}{C_t^h}\right) = (i_t - \rho)dt + \underbrace{\mathrm{Var}_t\left(\frac{dC_t^h}{C_t^h}\right)}_{\substack{\text{Endogenous} \\ \text{precautionary savings}}}, \tag{3}$$

where the last term, $\mathrm{Var}_t(\frac{dC_t^h}{C_t^h})$, captures the *endogenous* volatility of consumption growth. Typically, this term is second-order and omitted in log-linearized models. In contrast, our non-linear characterization in equation (3) explicitly accounts for consumption volatility, allowing it to affect the drift. This additional term reflects the standard *precautionary savings channel*: higher business cycle volatility increases households' demand for riskless savings, lowering current consumption and raising expected consumption growth.

Finally, the household must also satisfy the intratemporal optimality condition:

$$\frac{1}{p_t C_t^h} = \frac{\left(L_t^h\right)^{\frac{1}{\eta}}}{w_t}, \tag{4}$$

and the transversality condition imposed on the value function Γ_t^h:

$$\lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t}\Gamma_t^h\right] = 0. \tag{5}$$

inflation. The perfectly rigid price assumption, adopted without loss of generality here, provides intuition by simplifying the New Keynesian Phillips curve ($\pi_t = 0$ for all t) and facilitating analytical derivations of global equilibrium solutions.

[7] Appendix II provides the Hamilton-Jacobi-Bellman (HJB) equation-based derivation for equations (2) and (4), and Online Appendix B derives an analytic form of the equilibrium value function Γ_t^h of households.

2.2 Firms

We assume the usual Dixit-Stiglitz monopolistic competition among firms, where the demand each firm $i \in [0,1]$ faces is given by

$$D_t(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t, \text{ with } p_t = \left(\int_0^1 \left(p_t^i\right)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}},$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. In the assumed rigid price equilibrium, firms never change their prices so $p_t^i = p_t = \bar{p}$ and $D_t(p_t^i, p_t) = D_t(\bar{p}, \bar{p}) = Y_t$ for all $i \in [0,1]$ and $\forall t$, i.e., each firm i produces to meet the aggregate demand Y_t.

An individual firm i produces with the linear production function: $Y_t^i = A_t L_t^i$, taking the aggregate price p_t, wage w_t, and the aggregate output Y_t as given, where L_t^i is firm i's labor hiring, and A_t is the economy's total factor productivity assumed to be exogenous and to follow a geometric Brownian motion with drift:

$$\frac{dA_t}{A_t} = g dt + \sigma dZ_t, \tag{6}$$

where g is its expected growth rate and σ is what we call 'fundamental' volatility, assumed to be constant over time.[8] It follows that firms' profits to be rebated can be written as $D_t = \bar{p} Y_t - w_t L_t$, with $L_t = \int L_t^i di$. We assume that all the aggregate variables are adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ generated by the process in (6) in a given *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$.

Definition 1 (Equilibrium) *An equilibrium is defined by the following sets of variables: household-specific quantities $\{B_t^h, C_t^h, L_t^h\}_h$, firm-specific quantities $\{Y_t^i, L_t^i\}_i$, aggregate quantities $\{B_t, C_t, Y_t, L_t\}$, and prices $\{w_t, \bar{p}\}$. These variables satisfy the household optimality conditions (i.e., equations (2) and (4) together with the transversality condition (5)), the aggregation conditions $B_t^h = B_t = 0$ for all h, $C_t^h = C_t = Y_t$ for all h, $L_t^h = L_t$ for all h, $Y_t^i = Y_t$ and $L_t^i = L_t$ for all i,[9] and a monetary policy rule introduced in Section 2.3.*

Henceforth, we adopt aggregate notation by omitting the superscripts h and i, whenever the distinction between household- or firm-specific and aggregate variables is not essential.

[8] This assumption is made for simplicity and our analysis can be extended to include cases where σ_t is time-varying and adapted to the Brownian motion Z_t.

[9] Due to rigid prices, firm optimization problems can be abstracted away.

Flexible Price Equilibrium We characterize the counterfactual flexible price equilibrium as the equilibrium in which firms freely set prices. The outcomes under this equilibrium are termed 'natural' because central banks, facing price rigidity, target these outcomes using monetary policy instruments. As proven in Appendix II.2.2, the natural output Y_t^n follows:

$$\frac{dY_t^n}{Y_t^n} = \Big(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2 \Big) dt + \underbrace{\sigma}_{\text{Natural volatility}} dZ_t. \tag{7}$$

where $r^n = \rho + g - \sigma^2$ denotes the natural interest rate. From the monetary authority's perspective, the process described in (7) is exogenous and thus beyond the influence of monetary policy. Notice that natural output Y_t^n follows a geometric Brownian motion with volatility σ, matching the volatility of the A_t process in (6).

Rigid Price Equilibrium and the 'Gap' Economy Returning to the rigid-price economy, we introduce σ_t^s as the *excess* volatility of the output growth rate $\{Y_t\}$ relative to the benchmark flexible-price output in (7). By definition, we have:

$$\mathrm{Var}_t\left(\frac{dY_t}{Y_t}\right) = (\sigma + \sigma_t^s)^2 dt, \tag{8}$$

where σ_t^s is an *endogenous* volatility term determined in equilibrium. Substituting equation (8) into the nonlinear Euler equation (3), and using $C_t^h = C_t = Y_t$ for all h, yields:

$$\frac{dY_t}{Y_t} = \left(i_t - \rho + (\sigma + \sigma_t^s)^2\right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{9}$$

Using the standard definition of the output gap $\hat{Y}_t = \ln\left(\frac{Y_t}{Y_t^n}\right)$, we obtain:[10]

$$d\hat{Y}_t = \Bigg(i_t - \Bigg(r^n \overbrace{- \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2}^{\text{New terms}}\Bigg)\Bigg) dt + \sigma_t^s dZ_t. \tag{10}$$

Notice that equation equation (10) includes a feedback effect that is absent in log-linearized models.[11] Given the policy rate i_t, a rise in endogenous volatility σ_t^s increases the drift

[10] In equation (9), we assume output Y_t is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ generated by the technology process in (6). Thus, σ_t^s in (9) can be interpreted as *fundamental* excess volatility.

[11] For comparison, see the linearized IS equation in (14), where endogenous excess volatility σ_t^s does not affect the drift.

term and reduces the output gap $\hat{Y}_t$. The intuition follows directly from the precautionary saving mechanism in equation (3), as higher volatility induces households to save more and consume less, triggering a recession. This feedback loop will become essential for understanding the multiplicity of equilibria discussed in Section 3.

Finally, define the *risk-adjusted* natural rate as:

$$r_t^T = r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2. \tag{11}$$

The rate r_t^T itself is endogenous and negatively related to the aggregate excess volatility σ_t^s. This risk-adjusted natural rate represents the economy's effective "reference" risk-free rate, at which setting $i_t = r_t^T$ completely eliminates the drift of the output gap.

2.3 Taylor Rule

We assume that the central bank sets the risk-free interest rate i_t according to the following Taylor rule:

$$i_t = r^n + \phi_y \hat{Y}_t, \quad \text{with} \quad \phi_y > 0. \tag{12}$$

Condition $\phi_y > 0$, known as the "Taylor principle", ensures a unique equilibrium in the log-linearized version of the model. Substituting equation (12) into equation (10), we derive the following dynamics for $\hat{Y}_t$:

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t - \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^s)^2}{2}\right) dt + \sigma_t^s dZ_t, \tag{13}$$

2.4 Log-Linear Approximation

We first analyze the log-linearized version of the model as a benchmark case. Omitting the volatility terms from the drift of equation (10), we obtain a linear IS equation:

$$d\hat{Y}_t = (i_t - r^n)\, dt + \sigma_t^s dZ_t. \tag{14}$$

Proposition 1 (Benchmark Equilibrium) *The log-linearized model, defined by the Taylor rule (12), the linear IS equation (14), and the transversality condition (5), admits a 'unique' rational expectations equilibrium characterized by perfect stabilization of the output gap $\hat{Y}_t = 0$ and zero excess volatility $\sigma_t^s = 0$ for all t.*

Proof. Substituting (12) into (14), we obtain the standard log-linear approximation of the output gap dynamics:

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t\right) dt + \sigma_t^s dZ_t. \tag{15}$$

With the local dynamics around the flexible-price equilibrium given by (15), Blanchard and Kahn (1980) establish the existence of a *unique* linear rational expectations equilibrium under the Taylor principle $\phi_y > 0$ in (12).[12] The resulting equilibrium with $\hat{Y}_t = \sigma_t^s = 0$ for all t corresponds to a perfectly stabilized economy. ∎

The benchmark equilibrium described in Proposition 1, characterized by perfect stabilization ($\sigma_t^s = \hat{Y}_t = 0$ for all t), remains an equilibrium of the exact non-linear model defined by the IS equation (10) and the Taylor rule (12). However, equilibrium uniqueness is no longer guaranteed in a non-linear framework. The next section analyzes how multiple equilibria emerge when moving beyond the log-linear approximation.

3 Multiple Equilibria

This section illustrates alternative global solutions of the model driven by aggregate volatility and studies the properties of the resulting business cycle dynamics. We proceed in two steps. First, in Section 3.1, we construct a non-stationary equilibrium that allows aggregate volatility to temporarily deviate from the perfectly stabilized path (i.e., benchmark equilibrium) and graphically explore the underlying economic mechanisms. Next, in Section 3.2, we introduce a broader class of stationary equilibria with permanent deviations from the benchmark equilibrium. Appendix II provides a detailed characterization and formal proofs of the results presented here.

3.1 Martingale equilibrium

We begin by presenting a rational expectations equilibrium that supports the emergence of an initial excess volatility $\sigma_0^s > 0$ by explicitly constructing an equilibrium path in which $\hat{Y}_t$

[12] See Buiter (1984) for conditions and results from Blanchard and Kahn (1980) adapted to continuous-time settings. In this log-linearized scenario, the transversality condition (5) becomes

$$\lim_{t\to\infty} \mathbb{E}_0 \left|\hat{Y}_t\right| < \infty, \tag{16}$$

as originally proposed in Blanchard and Kahn (1980).

follows a local martingale. Our martingale equilibrium construction (i) supports an initial jump in excess volatility, $\sigma_0^s > 0$, which arises in a self-fulfilling manner;[13] (ii) satisfies the process defined by the dynamic IS equation (13); and (iii) does not diverge in expectations in the long run, consistent with the transversality condition (5).[14] We also demonstrate that this specific equilibrium is non-stationary by design. It is explicitly constructed through the following steps, with derivation details provided in Appendix I.

Step 1 Assume that $\hat{Y}_t$ is a local martingale consistent with the dynamics in (13). Therefore, the drift of the $\{\hat{Y}_t\}$ process in (13) must be zero, resulting in:

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}. \tag{17}$$

Step 2 Second, we show the existence of a stochastic process for $\{\sigma_t^s\}$ starting from σ_0^s that supports the equilibrium expression for the output gap in (17). Using (13) and (17), we obtain an expression for that process as follows:[15]

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t. \tag{18}$$

Equations (17) and (18) describe the dynamics of our constructed rational expectations equilibrium that supports initial excess volatility $\sigma_0^s > 0$. Online Appendix D shows that the stochastic differential equation (18) admits a strong solution, as required in the filtered probability space assumed in the paper.

Proposition 2 (Martingale Equilibrium) *The model admits a rational expectations equilibrium which supports initial excess volatility $\sigma_0^s > 0$ and is represented by the $\hat{Y}_t$ dynamics in equation* (17)*, and the σ_t^s process in equation* (18)*. We refer to this as the "Martingale" equilibrium, which has the following properties:*

[13] The emergence of the initial excess volatility σ_0^s is not part of the economy's filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. This can be viewed as a "sunspot" shock to the excess volatility σ_t^s, with aggregate variables responding to its appearance.

[14] The transversality condition is proved in Online Appendix B.1.

[15] When $\sigma = 0$, $\forall t$, equation (18) becomes the following Bessel process:

$$d\sigma_t^s = -\frac{(\phi_y)^2}{2\sigma_t^s} dt - \phi_y dZ_t,$$

which stops when σ_t^s reaches zero. For general properties of Bessel processes, see Lawler (2019).

Property 1 *Excess volatility σ_t^s converges to zero almost surely, i.e.,* $\sigma_t^s \xrightarrow{a.s} \sigma_\infty^s = 0$.

Property 2 *Output gap $\hat{Y}_t$ converges to zero almost surely, i.e.,* $\hat{Y}_t \xrightarrow{a.s} \hat{Y}_\infty = 0$.

Property 3 *Non-uniform integrability: aggregate variance $(\sigma + \sigma_t^s)^2$ satisfies*

$$\mathbb{E}_0 \left(\sup_{t \geq 0} (\sigma + \sigma_t^s)^2 \right) = \infty, \quad \textit{and} \quad \lim_{K \to \infty} \sup_{t \geq 0} \left(\mathbb{E}_0 (\sigma + \sigma_t^s)^2 \mathbb{1}_{\left\{(\sigma + \sigma_t^s)^2 \geq K\right\}} \right) > 0.$$

Proof. See Appendix I.2. ■

The results that $\sigma_t^s \xrightarrow{a.s} \sigma_\infty^s = 0$ and $\hat{Y}_t \xrightarrow{a.s} \hat{Y}_\infty = 0$ imply that equilibrium paths originating from an initial excess volatility $\sigma_0^s > 0$ are almost surely stabilized in the long run. Nevertheless, this almost-sure stabilization remains compatible with the self-fulfilling emergence of $\sigma_0^s > 0$. Specifically, by Property 3, we have $\mathbb{E}_0 \left(\sup_{t \geq 0} (\sigma + \sigma_t^s)^2\right) = \infty$, indicating that an initial self-fulfilling shock σ_0^s is sustained by a vanishing probability of infinitely large equilibrium aggregate variance occurring along some future paths.[16]

Intuition Here we explain in detail the intuition for (i) how an initial excess volatility σ_0^s can appear, and (ii) the three Properties in Proposition 2. To this end, we simplify the economic environment and make the following assumptions:

A.1 A shock dZ_t in each period takes one of two possible values: $\{+1, -1\}$, with equal probability.

A.2 Martingale equilibrium: output gap $\hat{Y}_t$ equals the conditional expected value of the next-period output gap, $\hat{Y}_{t+1}$. Thus, if $\hat{Y}_{t+1}$ takes either $\hat{Y}_{t+1}^{(1)}$ or $\hat{Y}_{t+1}^{(2)}$ when $dZ_{t+1} = 1$ or -1, respectively, then

$$\hat{Y}_t = \frac{1}{2} \left(\hat{Y}_{t+1}^{(1)} + \hat{Y}_{t+1}^{(2)} \right).$$

A.3 Aggregate demand (i.e., output gap) $\hat{Y}_t$ falls as the conditional variance of the next period's $\hat{Y}_{t+1}$ rises, due to precautionary savings. Both $\hat{Y}_t$ and σ_t^s are zero on the stabilized path (i.e., flexible-price economy).

Since there are two possible realizations of the shock dZ_t in each period, we can represent this with a tree diagram, as depicted in Figure 1. The thick vertical line represents the stabilized path, with areas to its left and right representing recessions and booms, respectively.

[16]Note that, since every nonnegative local martingale is a supermartingale, σ_t^s is a supermartingale.

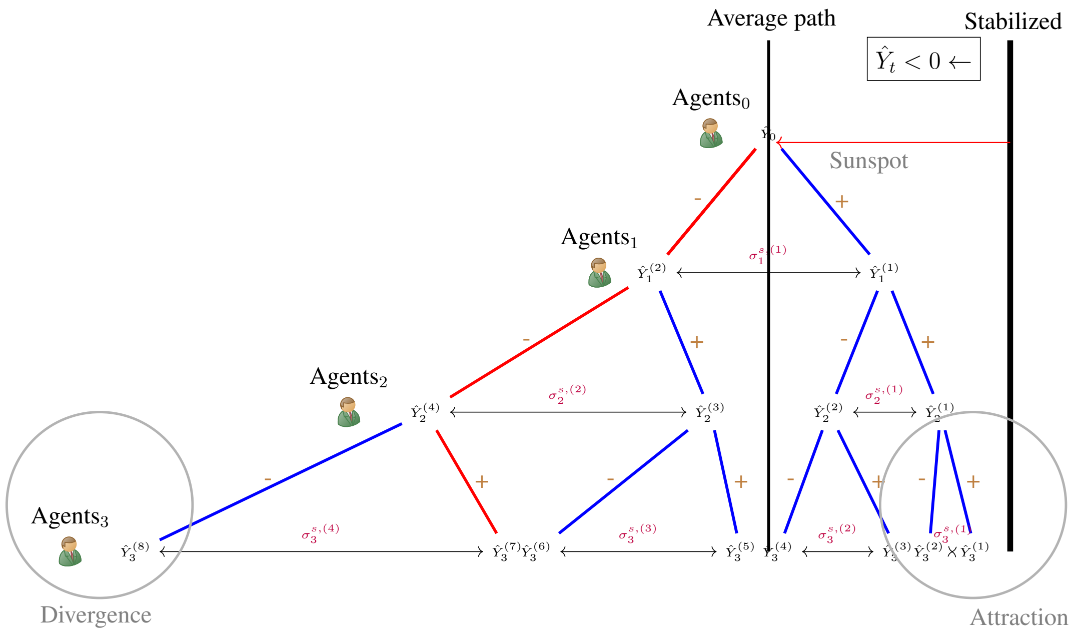


Figure 1: A rise in σ_0^s as a rational expectations equilibrium.

The key to build a rational expectations equilibrium that supports a self-fulfilling jump in excess volatility $\sigma_0^s > 0$ is to construct a path-dependent consumption strategy for agents with time-varying conditional volatilities.

First, let us imagine that the the current period agents (Agents$_0$) suddenly believe that the future agents will choose a path-dependent consumption demand[17] so that the next-period's $\hat{Y}_1$ becomes $\hat{Y}_1^{(1)}$ after $dZ_1 = +1$ is realized and $\hat{Y}_1^{(2)}$ if $dZ_1 = -1$ is realized, with $\hat{Y}_1^{(1)} > \hat{Y}_1^{(2)}$. Then the current output $\hat{Y}_0$ becomes $\hat{Y}_0 = \frac{1}{2}\left(\hat{Y}_1^{(1)} + \hat{Y}_1^{(2)}\right)$ with $\hat{Y}_0$ below the stabilized path, as Agents$_0$ believe that there exists dispersion in next-period outputs, which is given by $\sigma_1^{s,(1)} = \frac{\hat{Y}_1^{(1)} - \hat{Y}_1^{(2)}}{2}$, and which leads to lower consumption through precautionary savings at $t = 0$. Now imagine $dZ_1 = -1$ is realized. For Agents$_0$'s belief in $\hat{Y}_1 = \hat{Y}_1^{(2)}$ to be consistent, Agents$_1$ must believe that future agents will choose their consumption in a way that, at time 2, $\hat{Y}_2$ becomes $\hat{Y}_2^{(3)}$ with $dZ_2 = +1$ and $\hat{Y}_2^{(4)}$ with $dZ_2 = -1$, with conditional volatility $\sigma_2^{s,(2)} = \frac{\hat{Y}_2^{(3)} - \hat{Y}_2^{(4)}}{2}$ higher than $\sigma_1^{s,(1)}$, since $\hat{Y}_1^{(2)}$ is lower than the initial output, $\hat{Y}_0$.

After dZ_2 is realized, Agents$_1$'s belief about $\hat{Y}_2$ can be made consistent through future agents $\{\text{Agents}_{n\geq 2}\}$'s coordination in a forward looking fashion. Observe that all the nodes in Figure 1 satisfy assumptions A.2 and A.3, with distance between adjacent nodes getting

[17] Note that agents' demand determines output in this environment with rigid prices.

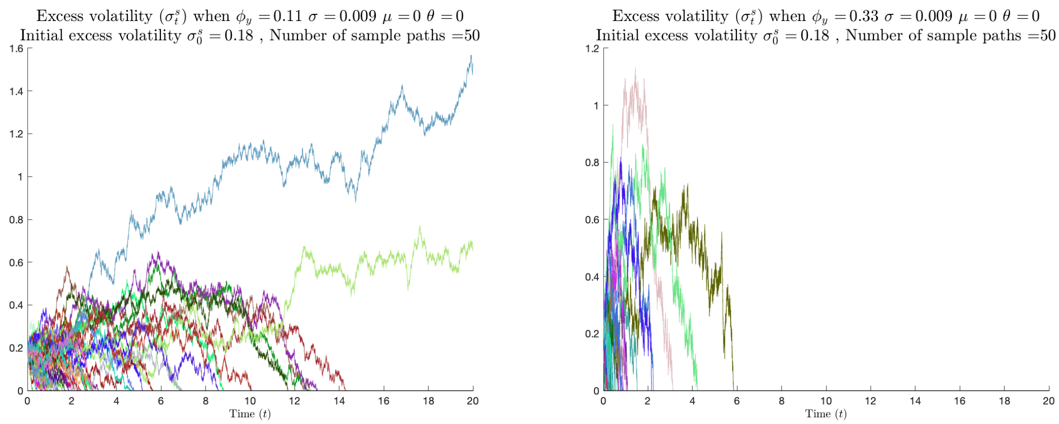
progressively narrower (wider) as output gap gets closer (farther) to the stabilization. This results in divergent and attraction paths offsetting each other, making the output gap $\{\hat{Y}_t\}$ follow a local martingale process in expectation. In sum, Agents$_0$'s initial doubt about volatility in the next-period outcome is validated through coordination among intertemporal agents (i.e., the representative household) at each node.[18]

Note that (i) we obtain an equilibrium with a *stochastic* aggregate volatility: i.e., σ_t^s is dependent on the path of shocks, as output gap $\{\hat{Y}_t\}$ is stochastic and negatively depends on the conditional volatility of its next-period level. Equation (18) specifies the exact stochastic process of $\{\sigma_t^s\}$ starting from $\sigma_0^s > 0$; (ii) Since excess volatility σ_t^s decreases as output gap $\hat{Y}_t$ approaches the stabilized path, this path becomes an attraction point for the set of alternative paths in its neighborhood, justifying the result of Proposition 2 that σ_t^s almost surely converges to zero over time. Nonetheless, as excess volatility σ_t^s rises whenever output $\hat{Y}_t$ deviates farther from the stabilized level, this also aligns with the result of Proposition 2 that maximal σ_t^s diverges, $\mathbb{E}_0(\sup(\sigma+\sigma_t^s)^2) = \infty$. However, this divergent behavior only happens with vanishingly small probability as $\sigma_t^s \xrightarrow{a.s} 0$.

The conclusion in terms of monetary policy is that a conventional Taylor rule almost surely stabilizes the disruption caused by an initial excess volatility shock $\sigma_0^s > 0$ in the long-run, but does not prevent the economy from entering a crisis phase with a positive $\{\sigma_t^s\}$ path starting from σ_0^s.

Simulation Figure 2 illustrates the dynamic paths of $\{\sigma_t^s\}$ under the martingale equilibrium with $\sigma_0^s = 0.18$ and examines the impact of changes in the policy responsiveness to the output gap ϕ_y. Panel 2a employs the default calibration $\phi_y = 0.11$, whereas Panel 2b assumes a more responsive stance with $\phi_y = 0.33$. With $\sigma_0^s > 0$, a higher ϕ_y accelerates convergence toward perfect stabilization, albeit with an increased likelihood of a more severe crisis path over a given period. In other words, for a fixed initial excess volatility to persist under a more responsive monetary policy, higher endogenous volatility (i.e., elevated σ_t^s) and deeper recessions (i.e., lower $\hat{Y}_t$) must occur in the future, albeit with vanishing probability.

[18] Our equilibrium construction is feasible because all future agents have common knowledge of their consumption strategies and there is frictionless communication across intertemporal periods (i.e., perfect recall). For a discussion of how limited recall eliminates indeterminacy, see Angeletos and Lian (2023).



(a) With Taylor coefficient $\phi_y = 0.11$

(b) With Taylor coefficient $\phi_y = 0.33$

Figure 2: Martingale equilibrium with $\phi_y = 0.11$ (Figure 2a), and $\phi_y = 0.33$ (Figure 2b).

Escape Clause If the central bank and/or the government credibly commit to preventing $\hat{Y}_t$ from falling below a predetermined threshold through interventions,[19] then the equilibria generated by the excess volatility σ_0^s (and supported by the paths in Figure 1) cannot be sustained as a rational expectations equilibrium. This clause shows that a credible commitment to intervene when the economy is at risk of a severe recession precludes a crisis phase initiated by a positive volatility shock $\sigma_0^s > 0$.

Negative Volatility Similarly, we can construct a rational expectations equilibrium with an initial negative excess volatility shock, $\sigma_0^s < 0$. This equilibrium is characterized by a boom with strong aggregate demand and low volatility.[20]

In summary, our model's non-linear characterization yields two key predictions: (i) the emergence of boom and crisis phases driven by self-fulfilling volatility shocks; and (ii) the joint evolution of the first (output level) and second (conditional volatility) moments of the model during crises and booms.

[19] For example, governments might commit to incurring significant fiscal deficits during severe recessions. This approach has similar implications for restoring determinate equilibrium as discussed in Benhabib et al. (2002), who examines the role of monetary-fiscal regimes in eliminating the indeterminacy posed by the ZLB. Similarly, Obstfeld and Rogoff (2021) demonstrate how a probabilistic fiscal currency backing can preclude speculative hyperinflation in monetary models.

[20] As seen in equation (8), the process for actual output Y_t has $\sigma + \sigma_t^s$ as its conditional volatility. Therefore, a self-fulfilling negative excess volatility shock $\sigma_0^s < 0$ reduces the volatility of the growth rate of Y_t from σ to $\sigma + \sigma_t^s$.

3.2 Ornstein-Uhlenbeck equilibria

This section introduces a broader class of equilibria with several noteworthy properties: (i) initial *excess* volatility σ_0^s adapted to the economy's filtration (i.e., equilibria that do not require initial sunspot volatility shocks in σ_0^s), (ii) non-degenerate and stationary stochastic processes for the model variables in the long run,[21] and (iii) the potential for alternative deterministic steady states characterized by under- or overproduction.

For that purpose, we conjecture an alternative class of equilibria where the output gap $\{\hat{Y}_t\}$ dynamics follow a process of the form

$$d\hat{Y}_t = \theta \cdot \left[\mu - \hat{Y}_t\right] \mathrm{d}t + \sigma_t^s \mathrm{d}Z_t, \tag{19}$$

where θ and μ are constant parameters. Note that (19) resembles an Ornstein-Uhlenbeck process, with one major difference: the process features an endogenous volatility σ_t^s, which is determined in equilibrium, whereas typical Ornstein-Uhlenbeck processes have constant volatility associated with the diffusion component. When $\theta = 0$, the process (19) becomes the martingale equilibrium studied in Section 3.1. Note that with $\theta > 0$ and $\mu < \frac{\sigma^2}{2\phi_y}$, the $\hat{Y}_t$ process features a mean reversion to μ. To close the model, we equate the drift terms in equation (13) and equation (19) and obtain[22]

$$\hat{Y}_t = \frac{\theta\mu}{\theta + \phi_y} - \frac{1}{2(\theta + \phi_y)} \left[(\sigma + \sigma_t^s)^2 - \sigma^2\right], \tag{20}$$

with

$$\mathrm{d}(\sigma + \sigma_t^s)^2 = -\theta \left[2\mu\phi_y + (\sigma + \sigma_t^s)^2 - \sigma^2\right] \mathrm{d}t - 2(\theta + \phi_y)\sigma_t^s \mathrm{d}Z_t. \tag{21}$$

Proposition 3 (Ornstein-Uhlenbeck Equilibrium) *The model admits a rational expectations equilibrium characterized by the $\hat{Y}_t$ dynamics in equation (20) and the σ_t^s process in equation (21). We refer to this as the "Ornstein-Uhlenbeck" equilibrium, with the following properties:*

Property 1 *For $\theta > 0$, $\mu < \frac{\sigma^2}{2\phi_y}$ and $\mu \neq 0$, the process of σ_t^s defined in (21) is stable and admits a unique stationary distribution. In the limit $\sigma \to 0$, if $(\theta + \phi_y) \neq 0$ and*

[21] In our martingale equilibrium, once σ_t^s reaches zero, it remains at zero thereafter, resulting in a non-stationary solution to the model.

[22] Online Appendix B verifies the transversality condition (5) under the Ornstein-Uhlenbeck equilibrium defined by (20) and (21). Online Appendix D confirms that the stochastic differential equation (21) admits a strong solution, as required in a given *filtered* probability space.

$\mu\phi_y < 0$, this stationary distribution coincides with the generalized gamma distribution $GGD(a,d,p)$, given by[23]

$$a = \sqrt{\frac{2(\theta+\phi_y)^2}{\theta}}, \quad d = -\frac{2\theta\mu\phi_y}{(\theta+\phi_y)^2}, \quad \textit{and} \quad p = 2, \tag{22}$$

where a is the scale parameter, d is the power-law shape parameter, and p is the exponential shape parameter.

Property 2 *For $\theta > 0$ and $\mu = 0$, the process of σ_t^s defined in (21) is non-stationary and its distribution degenerates to the perfectly stabilized equilibrium with $\sigma_t^s = 0$ as $t \to \infty$.*

Property 3 *The long-run expectations of the output gap $\hat{Y}_t$ and excess variance $(\sigma+\sigma_t^s)^2 - \sigma^2$ are given by*

$$\lim_{t\to\infty} \mathbb{E}_0\left[\hat{Y}_t\right] = \mu, \quad \textit{and} \quad \lim_{t\to\infty} \mathbb{E}_0\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right] = -2\mu\phi_y$$

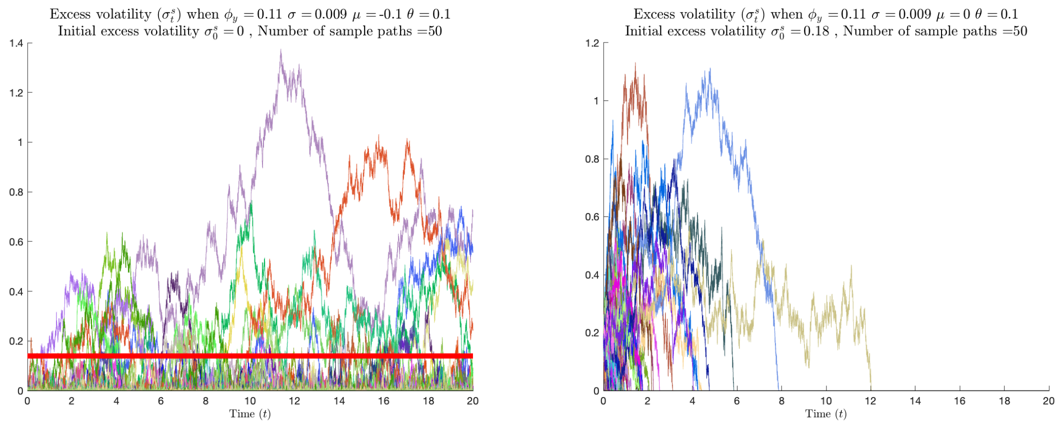
Proof. See Online Appendices B and C. ■

Property 3 of Proposition 3 implies that solutions with $\mu \neq 0$ exhibit a long-run expected excess variance given by $-2\mu\phi_y \neq 0$. Together with Property 1, this result indicates that the volatility process $\{\sigma_t^s\}$ is never permanently fixed at any particular value, including zero. For example, even if the economy initially has $\sigma_0^s = 0$, the aggregate variance $(\sigma+\sigma_t^s)^2$ immediately starts drifting toward $\sigma^2 - 2\mu\phi_y$, since the drift term in (21) is positive when excess volatility is zero. Consequently, the model admits alternative equilibria characterized by an endogenous, stationary stochastic process $\{\sigma_t^s\}$ arising from arbitrary initial conditions.

In contrast, when $\mu = 0$, the equilibrium does not support a stationary stochastic process in the long-run but still allows for temporary deviations from the perfectly stabilized equilibrium, analogous to the Martingale equilibria discussed in Section 3.1. Figure 3 illustrates representative sample paths of the volatility process $\{\sigma_t^s\}$ under both calibrations.

Property 3 also implies that the Ornstein-Uhlenbeck equilibrium features a deterministic steady state for the output gap $\hat{Y}_t$, equal to the parameter μ. Therefore, equilibria with $\theta > 0$ and $\mu < 0$ ($0 < \mu < \frac{\sigma^2}{2\phi_y}$) exhibit steady-state underproduction (overproduction),

[23] See, e.g., Boukai (2022) for an application of the generalized gamma distribution as a benchmark risk-neutral distribution in stochastic volatility models.



(a) $\mu < 0$

(b) $\mu = 0$

Figure 3: Simulated paths for the excess volatility process $\{\sigma_t^s\}$ under the Ornstein-Uhlenbeck solution with $\theta > 0$ and various μ calibrations. In Panel 3a, the red line denotes the long-run expected excess volatility, $\sigma^{s,*}$.

where output is $|100 \times \mu|$ percent below (above) its natural level. The intuition for this result is that a process for $\{\sigma_t^s\}$ with long-run expected excess variance $-2\mu\phi_y > 0$ (< 0) induces a stronger (weaker) precautionary response in household consumption to aggregate volatility relative to the flexible-price benchmark.

Implications This additional class of alternative solutions highlights several implications for the welfare analysis in New Keynesian models. First, by commonly evaluating the economy around the log-linear solution presented in (15), traditional welfare accounting likely overlooks the additional losses stemming from the existence of a non-zero excess volatility process, σ_t^s. Second, and potentially more important, traditional welfare evaluations omit the capacity of monetary policy interventions to generate first-order gains by moving the economy away from steady states featuring suboptimal production levels.

The next section discusses the implementation details of a monetary policy capable of restoring the economy to its constrained efficient equilibrium.

4 A New Monetary Policy

The flexible price equilibrium characterized by equation (7) and Appendix II.2.2 is a constrained-efficient allocation,[24] see, e.g., Woodford (2003) and Galí (2015). Therefore, in light of the previous analysis in Section 2.3, the monetary authority aims to achieve the flexible price allocation as the *unique* equilibrium path, if possible. In this section, we provide a new monetary policy that allows the central bank to accomplish this goal.

4.1 Modified Taylor rule

Instead of the Taylor rule in (12), assume the central bank follows:

$$i_t = r^n + \phi_y \hat{Y}_t - \underbrace{\frac{1}{2}\left((\sigma + \sigma_t^s)^2 - \sigma^2\right)}_{\text{Aggregate volatility targeting}}, \quad \text{with} \;\; \phi_y > 0. \tag{23}$$

This rule targets both the output gap $\hat{Y}_t$ and aggregate excess volatility with a coefficient of $\frac{1}{2}$. Substituting (23) into the IS equation (10) eliminates the volatility feedback terms from the drift. The dynamics then simplify to equation (15), ensuring that the benchmark equilibrium of Proposition 1 (i.e., perfect stabilization, $\hat{Y}_t = \sigma_t^s = 0$ for all t) is the *unique* rational expectations equilibrium whenever the Taylor principle $\phi_y > 0$ holds.

Interpretation The additional volatility target in the policy rule (23) is necessary to offset the feedback loop between the endogenous volatility of the output gap and its drift. To better understand this, we can rewrite equation (23) as $i_t = r_t^T + \phi_y \hat{Y}_t$, where r_t^T is the risk-adjusted natural rate defined in (11). This formulation highlights that monetary policy in a risky environment should target the risk-adjusted, rather than the simple natural, interest rate. Note that r_t^T is time-varying in this setting, as it depends on the endogenous excess volatility σ_t^s.

Following the policy rule in (23) eliminates any excess volatility, ensuring $\sigma_t^s = 0$ for all t. Thus, along the equilibrium path, a central bank adhering to (23) behaves in a manner observationally equivalent to one following the traditional rule (12). The key distinction is that (23) incorporates an *off-equilibrium* threat to target excess volatility should it emerge.

[24] With a proper production subsidy that eliminates the real distortion generated by monopolistic competition, the flexible price equilibrium allocation becomes the first best. For more on this issue, see Woodford (2003).

A direct implication is that, in practice, differences in central banks' perceived credibility in enforcing such threats may explain their varying degrees of success in economic stabilization, even under seemingly similar monetary policy regimes.

Practicality A potential issue with the policy rule (23) is its lack of robustness in practical implementations. Specifically, the coefficient attached to the volatility term, representing the strength of the policymakers' response to deviations in aggregate volatility, must be precisely $\frac{1}{2}$. If the central bank's responsiveness to the volatility term is either too strong or too weak, due to policy mistakes or measurement errors,[25] the rule (23) cannot effectively counteract the precautionary savings feedback loop present in the non-linear IS equation (10).[26]

To examine the consequences of deviating from the $\frac{1}{2}$ volatility target, we consider the following alternative rule within the context of the previously discussed martingale equilibrium of Section 3.1:

$$i_t = r^n + \phi_y \hat{Y}_t - \phi_{\text{vol}} \left((\sigma + \sigma_t^s)^2 - \sigma^2 \right), \tag{24}$$

where ϕ_{vol} is a constant term, which differs from $\frac{1}{2}$.[27] With the policy rule (24), we obtain

$$d\hat{Y}_t = \left[\phi_y \hat{Y}_t + \left(\frac{1}{2} - \phi_{\text{vol}} \right) \left((\sigma + \sigma_t^s)^2 - \sigma^2 \right) \right] dt + \sigma_t^s dZ_t,$$

as the new $\{\hat{Y}_t\}$ dynamics. When $\phi_{\text{vol}} \neq \frac{1}{2}$, the martingale equilibrium with self-fulfilling

[25] In practice, the components of output volatility $\{\sigma,\, \sigma_t^s\}$ and the risk-adjusted natural rate r_t^T may not be directly observable or may be observed with errors. For instance, assume a multiplicative measurement error for the volatility gap $\equiv (\sigma + \sigma_t^s)^2 - \sigma^2$, such that volatility $\text{gap}_t^{\text{obs}} = \varepsilon_t \cdot$ volatility gap_t, where volatility $\text{gap}_t^{\text{obs}}$ represents the measured volatility gap. In these cases, even with the precise targeting strength of $\frac{1}{2}$ on the observed volatility gap, i.e., volatility $\text{gap}_t^{\text{obs}}$, central banks effectively deviate from the $\frac{1}{2}$ response strength on the true volatility gap. Conversely, additive measurement errors result in standard monetary policy shocks.

[26] To clarify the $\frac{1}{2}$ targeting requirement, Online Appendix E extends the model to constant relative risk aversion (CRRA) utility, deriving a risk-adjusted natural rate:

$$r_t^T = r^n - \frac{\gamma^2}{2} \left[(\sigma + \sigma_t^s)^2 - \sigma^2 \right],$$

where γ is the relative risk aversion (RRA) coefficient. The optimal volatility targeting is then $\frac{\gamma^2}{2}$, meaning the central bank must fully offset households' precautionary response to volatility shocks, which scales with their RRA. Under logarithmic utility, RRA equals one.

[27] The policy rule in (24) with $\phi_{vol} \neq \frac{1}{2}$ similarly permits the Ornstein-Uhlenbeck equilibrium to exist. Online Appendix A details the equilibrium conditions for this case.

volatility σ_t^s reappears and is characterized by[28]

$$\hat{Y}_t = -\frac{\left(\sigma + \sigma_t^s\right)^2}{2\phi_{\text{total}}} + \frac{\sigma^2}{2\phi_{\text{total}}}\,, \quad \text{with} \;\; \phi_{\text{total}} \equiv \frac{\phi_y}{1 - 2\phi_{\text{vol}}}, \tag{25}$$

where the $\{\sigma_t^s\}$'s process after an initial volatility shock σ_0^s appears is given by

$$d\sigma_t^s = -\frac{\phi_{\text{total}}^2 \left(\sigma_t^s\right)^2}{2\left(\sigma + \sigma_t^s\right)^3} dt - \phi_{\text{total}} \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t. \tag{26}$$

Note that $\phi_{\text{vol}} \to \frac{1}{2}$, given $\phi_y > 0$, is equivalent to $\phi_y \to \infty$ with $\phi_{\text{vol}} = 0$, both of which lead to $\phi_{\text{total}} \to \infty$ and ensure determinacy. Therefore, there exists an alternative—albeit impractical—stabilization rule that involves an infinitely aggressive *off-equilibrium* threat to output gap deviations.[29] We also find that any combination of parameters (ϕ_y, ϕ_{vol}) that drives the value of ϕ_{total} towards infinity results in faster convergence to perfect stabilization, on average. However, this comes at the cost of an increased likelihood of a more severe crisis path within a given time period, as discussed previously in relation to Figure 2.

Comparison Woodford (2001, 2003) study the Taylor rule in a log-linearized New Keynesian model, summarized by[30,31]

$$\begin{aligned} \mathbb{E}_t(d\hat{Y}_{t+1}) &= \left(i_t^m - r^n\right) dt, \\ i_t &= i_t^* + \phi_y \hat{Y}_t, \quad \text{with} \quad \phi_y > 0, \end{aligned} \tag{27}$$

where i_t^m is the interest rate governing the household's intertemporal consumption smoothing, and i_t^* is the central bank's target for the policy rate, i_t. They uncover that:

B.1 When $i_t^m = i_t$, then $i_t^* = r^n$ guarantees that $\hat{Y}_t = 0$ for all t, as a unique equilibrium. Even if $i_t^* \neq r^n$, there is still a unique equilibrium, but $\hat{Y}_t \neq 0$ on the equilibrium path.

B.2 When $i_t^m \neq i_t$, setting $i_t^* = r^n + (i_t - i_t^m)$ achieves $\hat{Y}_t = 0$ for all t, as a unique

[28] Equations (25) and (26) are easily derived similarly to Proposition 2.

[29] See Cochrane (2007) for a comprehensive discussion on this topic in traditional New Keynesian frameworks.

[30] For comparison, inflation is abstracted away in equation (27).

[31] We thank an anonymous referee for suggesting this comparison.

equilibrium. If $i_t - i_t^m$ is an exogenous process, then even when $i_t^* \neq r^n + (i_t - i_t^m)$, there is still a unique equilibrium, but $\hat{Y}_t \neq 0$ on the equilibrium path.

What we do corresponds to neither case: in our model, $i_t - i_t^m$ depends on the endogenous volatility of the $\{\hat{Y}_t\}$ process, with $r_t^T \equiv r^n + (i_t - i_t^m)$ in equation (11). We show that

C.1 If $i_t^* = r_t^T$, we achieve $\hat{Y}_t = 0$ for all t, as a unique equilibrium. In this case, the policy rule corresponds to the new rule proposed in (23).

C.2 In contrast to Woodford (2001, 2003), where $i_t - i_t^m$ is exogenous, if $i_t^* \neq r_t^T$, we cannot guarantee a unique equilibrium, and the martingale equilibrium of Section 3.1 with a self-fulfilling initial volatility σ_0^s or the Ornstein-Uhlenbeck equilibrium of Section 3.2 may potentially appear.

C.3 $i_t - i_t^m$ depends only on the volatility gap, i.e., $(\sigma + \sigma_t^s)^2 - \sigma^2$. Thus, in a knife-edge case where $i_t^* - (i_t - i_t^m)$ does not contain any multiple of the volatility gap (or more generally, is not a function of the *excess* volatility σ_t^s), even if $i_t^* - (i_t - i_t^m) \neq r^n$, we have a unique equilibrium, but $\hat{Y}_t \neq 0$ along the equilibrium path.

4.2 Policy reformulation and growth mandates

We can rewrite the policy rule in (23) as

$$\underbrace{\frac{\mathbb{E}_t\left(d\log Y_t\right)}{dt}}_{\text{Growth rate}} = \underbrace{\frac{\mathbb{E}_t\left(d\log Y_t^n\right)}{dt}}_{\substack{\text{Benchmark}\\ \text{growth rate}}} + \underbrace{\phi_y \hat{Y}_t}_{\substack{\text{Business cycle}\\ \text{targeting}}} = \left(g - \frac{1}{2}\sigma^2\right) + \phi_y \hat{Y}_t.$$

Thus, an output growth rule centered around the natural growth rate can restore model determinacy and stabilize the economy. From a practical perspective, such a policy reformulation has several advantages, as it does not require the monetary authority to measure or target deviations in the aggregate volatility with precise strength. Forecast errors in the output growth rate or its natural counterpart are actually more forgiving in this implementation, resulting in traditional monetary policy shocks instead of multiple equilibria.

To understand the intuition behind this result, recall that the source of equilibrium multiplicity lies in the feedback loop between the endogenous components of the economy's (expected) growth rate and its conditional volatility, generated by the intertemporal consumption decisions of agents and captured by the drift and volatility components of equation (9). To break this loop, the monetary authority must establish a (direct or indirect) tight grip over at least one of these components. Examining the definition of the expected growth rate and its first-order linear approximation, i.e., $\frac{\mathbb{E}_t(d\log Y_t)}{dt} = i_t - \rho + \frac{1}{2}(\sigma + \sigma_t^s)^2 \approx i_t - \rho$, we observe that a traditional Taylor rule in a log-linearized framework can exert the same degree of control over the endogenous components of economic growth as a direct growth mandate. However, this statement is no longer true when considering the global solution of the model, which properly accounts for economic risk, $(\sigma + \sigma_t^s)^2$.

Therefore, to avoid sunspot shocks and equilibrium multiplicity, the monetary authority faces the dilemma of either: (a) establishing clear economic growth mandates,[32] using the policy rate i_t as an intermediate tool toward attaining these objectives, or (b) precisely targeting deviations in the aggregate volatility of the economy when following an interest rate rule.

Thus far, we have abstracted from inflation dynamics by assuming perfectly rigid prices, highlighting that the multiplicity of global solutions originates from the dynamic IS equation—driven by households' intertemporal decisions—and the central bank's policy rule. The next section demonstrates that these results extend to standard sticky-price models commonly studied in the literature.

5 Model with Sticky Prices

We briefly summarize the key assumptions underlying the derivation of the New Keynesian Phillips curve with sticky prices following Rotemberg (1982), and present the main results regarding equilibrium multiplicity in this framework. Detailed derivations are provided in Online Appendix F. For robustness, Online Appendix G replicates these findings under the alternative assumption of sticky price adjustments à la Calvo (1983).

Rotemberg (1982) assumes a continuum of identical firms indexed by the interval $[0, 1]$, operating under monopolistic competition. The price process for firm i is given by

$$dp_t^i = \pi_t^i p_t^i \, dt,$$

[32] That is, in addition to any inflation mandates dictated by traditional considerations.

where each firm can adjust its price p_t^i by choosing an inflation rate π_t^i. These adjustments incur convex adjustment costs $\Theta(\pi_t^i)$, specified by

$$\Theta(\pi_t^i) = \frac{\tau}{2}(\pi_t^i)^2 p_t Y_t,$$

with $\tau \geq 0$ determining the penalty on the speed of adjustment. In equilibrium, the model yields symmetric inflation rates across firms, resulting in a nonlinear version of the New Keynesian Phillips curve given by:

$$d\pi_t = \left[[2(\rho + \pi_t) - i_t - (\sigma + \sigma_t^s)(\sigma + \sigma_t^s + \sigma_t^\pi)]\, \pi_t - \left(\frac{\epsilon - 1}{\tau}\right) \left(e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t} - 1\right) \right] dt + \sigma_t^\pi\, \pi_t\, dZ_t, \tag{28}$$

where σ_t^π is the endogenous volatility of inflation growth. The IS equation then becomes:

$$d\hat{Y}_t = \left[i_t - \pi_t - r_t^T\right]\, dt + \sigma_t^s\, dZ_t, \tag{29}$$

which modifies the rigid-price benchmark equation (10) by subtracting the inflation rate π_t from its drift.

Proposition 4 (Model with Sticky Prices) *The model with sticky prices à la Rotemberg (1982), represented by the New Keynesian Phillips curve (28), IS equation (29), Taylor rule (12), and the transversality condition (5), admits an alternative solution to the benchmark equilibrium given by:*

$$\begin{aligned} d\hat{Y}_t &= \theta \left[\mu - \hat{Y}_t\right]\, dt + \sigma_t^s\, dZ_t, \\ \pi_t &= f(\sigma_t^s), \end{aligned} \tag{30}$$

where $f(\cdot)$ is a smooth function of excess volatility σ_t^s. This alternative equilibrium solution exists for any positive degree of price stickiness, as captured by the adjustment rate parameter $\tau > 0$.

Proof. See Online Appendix F. ■

Proposition 4 extends the Ornstein-Uhlenbeck equilibria of Section 3.2 to the sticky-price framework and remains valid under alternative Phillips curve specifications. In particular, the results hold when the Phillips curve is generated via Calvo pricing or expressed in a linear form, such as:

$$d\pi_t = \left(\rho\, \pi_t - \kappa\, \hat{Y}_t\right) dt + \sigma_t^\pi\, \pi_t\, dZ_t,$$

for some $\kappa > 0$. This linear specification typically arises from linearizing the Phillips curve in a wide variety of sticky-price models. Under this linearized form, it is straightforward to verify that the IS equation (29) together with an extension of the Taylor rule in (23) that incorporates additional inflation targeting,

$$i_t = r^n + \phi_y \hat{Y}_t + \phi_\pi \pi_t - \underbrace{\tfrac{1}{2}\Big((\sigma + \sigma_t^s)^2 - \sigma^2\Big)}_{\text{Aggregate volatility targeting}}, \quad \text{with} \quad \underbrace{\phi_y + \frac{\kappa(\phi_\pi - 1)}{\rho}}_{\equiv \phi} > 0,$$

delivers perfect stabilization as the unique rational expectations equilibrium, with $\phi > 0$ satisfying the Taylor principle.

Finally, note that any positive degree of price stickiness ($\tau > 0$) permits equilibria of the form (30), whereas perfectly flexible prices ($\tau = 0$) yield a unique equilibrium described by (7), with $\sigma_t^s = 0$ and $\hat{Y}_t = 0$ for all t. Even minimal stickiness prevents aggregate supply from fully offsetting coordinated household consumption, thereby generating and sustaining self-fulfilling aggregate volatility along alternative equilibrium paths, as illustrated in Figure 1.

6 Conclusion

This paper establishes that standard New Keynesian models exhibit multiple global solutions previously unknown to the literature. In these solutions, self-fulfilling volatility arises under conventional Taylor rules because such rules do not target the economy's aggregate volatility—a key driver of precautionary savings. When households perceive higher future volatility, they consume less today, triggering the very fluctuations they fear.

To address this issue, we propose a modified interest rate rule that incorporates a response to aggregate volatility. By fully neutralizing the intertemporal feedback loop between volatility and precautionary savings, this new policy stabilizes the economy along a unique equilibrium path. An alternative way to implement the same idea is to assign mandates for output growth, allowing the monetary authority to guide expectations directly.

Our results indicate that central banks risk fueling boom-bust cycles along alternative equilibrium paths if they concentrate solely on conventional targets such as output or inflation and overlook second-moment shocks. Future work might investigate: (i) the quantitative relevance of these alternative equilibria and which ones align most closely with empirical data, (ii) the interplay between volatility-targeting policy and financial stability mandates (e.g., macroprudential policies), and (iii) how forward guidance and fiscal tools can reinforce volatility-targeting frameworks to mitigate boom-bust cycles arising in these equilibria.

I Proofs and Derivations

I.1 Derivations in Section 2

Derivation of Equation (3) From the definition of (nominal) state-price density $\xi_t^N = e^{-\rho t}\frac{1}{C_t}\frac{1}{p_t}$, we obtain

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} - \frac{dp_t}{p_t} + \frac{1}{2}\left(\frac{dC_t}{C_t}\right)^2 + \frac{1}{2}\left(\frac{dp_t}{p_t}\right)^2 + \frac{dC_t}{C_t}\frac{dp_t}{p_t}. \tag{I.1}$$

Since we have a perfectly rigid price (i.e., $p_t = \bar{p}$ for all t), the above (I.1) becomes

$$\begin{aligned} \frac{d\xi_t^N}{\xi_t^N} &= -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \\ &= -\rho dt - \frac{dC_t}{C_t} + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right). \end{aligned} \tag{I.2}$$

Plugging equation (I.2) into equation (2), we obtain

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)\,dt + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right).$$

Derivation of Equation (10) From equation (9), we obtain

$$d\ln Y_t = \left(i_t - \rho + \frac{1}{2}(\sigma + \sigma_t^s)^2\right)dt + (\sigma + \sigma_t^s)dZ_t. \tag{I.3}$$

From (7), we obtain

$$d\ln Y_t^n = \left(r^n - \rho + \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t. \tag{I.4}$$

Therefore, by subtracting equation (I.4) from equation (I.3), we obtain equation (10):

$$d\hat{Y}_t = \left(i_t - \left(r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2\right)\right)dt + \sigma_t^s dZ_t.$$

I.2 Proofs of Section 3.1

The Construction of the Martingale Equilibrium in Section 3.1 and the Proof of Proposition 2. Setting the drift of the $\hat{Y}_t$ process in (13) to zero, i.e.,

$$d\hat{Y}_t = \left(\underbrace{\phi_y \hat{Y}_t - \frac{\sigma^2}{2} + \frac{(\sigma+\sigma_t^s)^2}{2}}_{=0}\right) dt + \sigma_t^s dZ_t = \sigma_t^s dZ_t, \tag{I.5}$$

leads to

$$\hat{Y}_t = -\frac{(\sigma+\sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}, \tag{I.6}$$

proving equation (17). From equations (I.5) and (I.6), we obtain

$$d\hat{Y}_t = -\frac{1}{\phi_y}(\sigma+\sigma_t^s)d\sigma_t^s - \frac{1}{2\phi_y}\left(d\sigma_t^s\right)^2 = \sigma_t^s dZ_t,$$

which leads to

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma+\sigma_t^s} dZ_t,$$

proving equation (18).

From equations (I.5) and (I.6), it is evident that $\mathcal{E}_t \equiv (\sigma+\sigma_t^s)^2 - \sigma^2$ has no drift, thereby qualifying as a local martingale. This can also be demonstrated as follows:

$$\begin{aligned}
d\mathcal{E}_t &= 2\left(\sigma+\sigma_t^s\right) d\sigma_t^s + \left(d\sigma_t^s\right)^2 \\
&= 2(\sigma+\sigma_t^s)\left(\cancel{-\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}dt} - \phi_y \frac{\sigma_t^s}{\sigma+\sigma_t^s} dZ_t\right) + \cancel{(\phi_y)^2 \frac{(\sigma_t^s)^2}{(\sigma+\sigma_t^s)^2}dt} \\
&= -2\phi_y(\sigma_t^s)dZ_t = 2\phi_y\left(\sigma - \sqrt{\sigma^2+\mathcal{E}_t}\right) dZ_t.
\end{aligned} \tag{I.7}$$

Our logic proceeds as follows:

Step 1 Starting from $\mathcal{E}_0 > 0$, $\mathcal{E}_t > 0$ for all t almost surely, i.e., 0 is a natural boundary of the $\mathcal{E}_t$ process.

Proof. We use the methodology of Linetsky (2007): by defining a function $m(\mathcal{E})$ that measures the speed of convergence in the process (I.7) as follows:

$$m\left(\mathcal{E}\right) \equiv \frac{1}{\left(\sqrt{\sigma^2+\mathcal{E}} - \sigma\right)^2},$$

which determines the behavior of $\mathcal{E}_t$ process.

For small $\Delta > 0$, we calculate the following two integrals:

$$\begin{aligned} I_0 &\equiv \int_0^{\Delta} \mathcal{E} \cdot m(\mathcal{E})\, d\mathcal{E} = \int_0^{\Delta} \mathcal{E} \cdot \frac{1}{\left(\sqrt{\sigma^2 + \mathcal{E}} - \sigma\right)^2} d\mathcal{E} \\ &= \int_0^{\sqrt{\Delta+\sigma^2}-\sigma} \left(\frac{t+2\sigma}{t}\right) \cdot 2(t+\sigma)dt \to \infty, \end{aligned} \tag{I.8}$$

with $t \equiv \sqrt{\mathcal{E} + \sigma^2} - \sigma$ as a change of variable. Similarly,

$$\begin{aligned} J_0 &\equiv \int_0^{\Delta} (\Delta - \mathcal{E}) \cdot m(\mathcal{E})\, d\mathcal{E} = \int_0^{\Delta} (\Delta - \mathcal{E}) \cdot \frac{1}{\left(\sqrt{\sigma^2 + \mathcal{E}} - \sigma\right)^2} d\mathcal{E} \\ &= \int_0^{\sqrt{\Delta+\sigma^2}-\sigma} \left[\frac{(\Delta + \sigma^2) - (t+\sigma)^2}{t^2}\right] \cdot 2(t+\sigma)dt \to \infty. \end{aligned} \tag{I.9}$$

With $I_0 = \infty$ and $J_0 = \infty$, process $\mathcal{E}_t$ has zero as a natural boundary, i.e., $\mathcal{E}_t$ never reaches the boundary at zero if it starts in the interior of the state space. In other words, if $\mathcal{E}_0 > 0$, then $\mathcal{E}_t \geq 0$ almost surely. ■

Step 2 As $\mathcal{E}_t$ is a local martingale that is non-negative due to Step 1, it becomes a supermartingale: see e.g., Le Gall (2016) about how to use Fatou's lemma in proving this statement.[1,2]

Step 3 Since $\mathcal{E}_t$ is a supermartingale that is non-negative (or more generally, bounded from below), we can apply the famous martingale convergence theorem (see e.g., Williams (1991) and Le Gall (2016)), that implies:

$$\mathcal{E}_t \xrightarrow{a.s} \mathcal{E}_\infty,$$

point-wise, where $\mathcal{E}_\infty$ exists and is finite almost surely.

Step 4 Now, we define a function that is globally concave:[3]

$$\Phi(x) \equiv 4\left[\sigma \log\left(\sqrt{\sigma^2 + x} - \sigma\right) + \sqrt{\sigma^2 + x}\right],$$

[1] $\mathcal{E}_t$ might not be a true martingale as $\sigma - \sqrt{\sigma^2 + \mathcal{E}_t}$ in equation (I.7) is not bounded.

[2] As a result, equation (16), the transversality condition under log-linear approximation (Blanchard and Kahn, 1980), is proved. Online Appendix B.1 proves that the actual transversality condition (5) is satisfied under the martingale equilibrium.

[3] We appreciate Victor Kleptsyn at CNRS à Institut de Recherche Mathématique de Rennes for suggesting function $\Phi(x)$.

which yields

$$\Phi'(x) = \frac{1}{\sqrt{\sigma^2 + x} - \sigma},$$

and $\Phi(x) \to -\infty$ as $x \to 0$. Additionally, we can obtain

$$\begin{aligned} d\left(\Phi\left(\mathcal{E}_t\right)\right) &= \Phi'(\mathcal{E}_t) d\mathcal{E}_t + \frac{1}{2}\Phi''(\mathcal{E}_t)(d\mathcal{E}_t)^2 \\ &= -\phi_y^2 \frac{1}{\sqrt{\sigma^2 + \mathcal{E}_t}} dt - 2\phi_y dZ_t. \end{aligned} \tag{I.10}$$

From Step 3, we know that $\mathcal{E}_\infty$ is finite with probability one, which implies that the drift $\frac{1}{\sqrt{\sigma^2+\mathcal{E}_\infty}}$ of (I.10) is finite and positive as well. Then, the only way to satisfy (I.10) in the long run is for $\Phi(\mathcal{E}_t) \to -\infty$, implying $\mathcal{E}_\infty = 0$.

Step 5 Finally, $\mathcal{E}_t \to 0$ implies that $\sigma_t^s \to 0$ almost surely. It can be easily shown that this satisfies our stochastic process (18) as follows:

$$\underbrace{d\sigma_t^s}_{\xrightarrow{a.s} 0} = -\underbrace{\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}}_{\xrightarrow{a.s} 0} dt - \phi_y \underbrace{\frac{\sigma_t^s}{\sigma+\sigma_t^s}}_{\xrightarrow{a.s} 0} dZ_t. \tag{I.11}$$

Step 6 Finally from Le Gall (2016), we know that if

$$\mathbb{E}_0\left(\sup_{t\geq 0}|\mathcal{E}_t|\right) < \infty, \quad \text{or} \quad \lim_{K\to\infty}\sup_{t\geq 0}\left(\mathbb{E}_0\left(|\mathcal{E}_t|\,\mathbb{1}_{\{|\mathcal{E}_t|\geq K\}}\right)\right) > 0,$$

then $\mathcal{E}_t$, which is a local martingale, becomes an uniformly integrable martingale. But then, if $\mathcal{E}_t$ is an uniformly integrable martingale,

$$0 < \mathcal{E}_0 = \lim_{t\to\infty}\mathbb{E}_0\mathcal{E}_t = \mathbb{E}_0\underbrace{\mathcal{E}_\infty}_{=0} = 0,$$

which is a contradiction. Therefore, Property 3 of Proposition 2 is proved.

Special case With $\sigma = 0$, the stochastic process (I.11) becomes:

$$d\sigma_t^s = -\frac{(\phi_y)^2}{2\sigma_t^s}dt - \phi_y dZ_t, \tag{I.12}$$

which is known as a Bessel process and widely studied in the literature. The process stops when σ_t^s reaches zero. In this case, we can observe that equation (I.8) becomes[4]

$$I_0 \equiv \int_0^{\Delta} \mathcal{E} \cdot \frac{1}{\left(\sqrt{0^2 + \mathcal{E}} - 0\right)^2} d\mathcal{E} = \Delta < \infty,$$

with $J_0 = \infty$. This implies that the $\mathcal{E}_t$ process has zero as an exit boundary, meaning the $\mathcal{E}_t$ process is instantaneously terminated the first time this boundary is reached. The behavior of the Bessel process (I.12) hitting time (i.e., the first time it reaches zero) is well known. For example, its hitting time τ has a well-defined distribution, as derived in Lawler (2019).

■

I.3 Derivations of Section 3.2

The equilibrium output gap $\hat{Y}_t$ follows:

$$\mathrm{d}\hat{Y}_t = \left[\phi_y \hat{Y}_t - \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^s)^2}{2}\right] \mathrm{d}t + \sigma_t^s \mathrm{d}Z_t. \tag{I.13}$$

Now we guess that the solution of the model represented by equation (I.13) has the following form:

$$\mathrm{d}\hat{Y}_t = \theta \cdot \left[\mu - \hat{Y}_t\right] \mathrm{d}t + \sigma_t^s \mathrm{d}Z_t, \tag{I.14}$$

where θ and μ are constant parameters. The process I.14 is similar to the Ornstein-Uhlenbeck process, except for the fact that it has an endogenous volatility σ_t^s which is to be determined in equilibrium.

Note that when $\theta = 0$, the process becomes the martingale conjectured in Section 3.1. For this new conjectured solution to be valid, the drift term in equation (I.13) and equation (I.14) should be equal, implying that the output gap under this conjecture is:

$$\hat{Y}_t = \frac{\theta \mu}{\theta + \phi_y} - \frac{1}{2(\theta + \phi_y)} \left[(\sigma + \sigma_t^s)^2 - \sigma^2\right]. \tag{I.15}$$

[4] We appreciate an anonymous referee for pointing out this discontinuity.

We know that σ_t^s follows a process of the form:[5]

$$\mathrm{d}\sigma_t^s = \mu_t^\sigma \mathrm{d}t + \tilde{\sigma}_t \mathrm{d}Z_t,$$

where μ_t^σ and $\tilde{\sigma}_t$ are unknown variables. Applying Ito's Lemma to I.15 we obtain:

$$\mathrm{d}\hat{Y}_t = -\left(\frac{1}{\theta+\phi_y}\right)\left[(\sigma+\sigma_t^s)\cdot\mu^\sigma + \frac{\tilde{\sigma}_t^2}{2}\right]\mathrm{d}t - \tilde{\sigma}_t \cdot \left(\frac{\sigma+\sigma_t^s}{\theta+\phi_y}\right)\mathrm{d}Z_t. \tag{I.16}$$

By equating the drift and volatility terms of equations I.16 and I.15, and based on equation (20), we obtain the unknown variables μ_t^σ and $\tilde{\sigma}_t$ consistent with our guessed solution as follows:

$$\begin{aligned}
\tilde{\sigma}_t &= -(\theta+\phi_y)\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right), \\
\mu_t^\sigma &= -\left(\frac{\theta}{\sigma+\sigma_t^s}\right)\left[\mu\phi_y + \frac{1}{2}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right] - (\theta+\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}.
\end{aligned}$$

Therefore, the σ_t^s process consistent with our guessed solution in (19) can be written as:

$$\begin{aligned}
\mathrm{d}\sigma_t^s = &-\left[\left(\frac{\theta}{\sigma+\sigma_t^s}\right)\left[\mu\phi_y + \frac{1}{2}\left[(\sigma+\sigma_t^s)^2-\sigma^2\right]\right] + (\theta+\phi_y)^2\frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}\right]\mathrm{d}t \\
&-(\theta+\phi_y)\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)\mathrm{d}Z_t,
\end{aligned} \tag{I.17}$$

leading to

$$\mathrm{d}(\sigma+\sigma_t^s)^2 = -\theta\left[2\mu\phi_y + (\sigma+\sigma_t^s)^2 - \sigma^2\right]\mathrm{d}t - 2(\theta+\phi_y)\sigma_t^s\mathrm{d}Z_t.$$

Notice the two following important observations:

Observation 1 When $\theta = 0$, the solution (I.17) becomes the martingale equilibrium solution of Section 3.1.

Observation 2 We obtain solutions with different values of θ and μ parameters, so there is a chance that different parametrization of this pair of parameters can be consistent with a rational expectations equilibrium solution considered in Section 3.2.

[5]As in Section 2, we assume that all the aggregate variables are adapted to the given filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Brownian motion dZ_t.

II Detailed Derivations in Section 2

II.1 Model Setup

A representative household h solves the following intertemporal optimization consumption-savings decision problem:

$$\max_{\substack{\{C_s^h, L_s^h\}_{s\geq t} \\ \{B_s^h\}_{s>t}}} \mathbb{E}_t \int_t^{\infty} e^{-\rho(s-t)} \left[\log C_s^h - \frac{\left(L_s^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] \mathrm{d}s, \quad \text{s.t.} \quad dB_t^h = \left[i_t B_t^h - p_t C_t^h + w_t L_t^h + D_t \right] dt,$$

where C_t^h is consumption, L_t^h aggregate labor, w_t is the equilibrium wage level, B_t^h are risk-free bonds held by the household h at the beginning of t (hence, B_t^h at t is taken as given for each household), i_t is the nominal interest rate, D_t is a lump-sum transfer of any firm profits/losses towards the household, p_t the nominal price of consumption goods and ρ is the subjective discount rate of the household.[6]

An individual firm i produces in this economy with the following production function:

$$Y_t^i = A_t L_t^i, \text{ with } \frac{\mathrm{d}A_t}{A_t} = g\mathrm{d}t + \underbrace{\sigma}_{\text{Fundamental risk}} \mathrm{d}Z_t,$$

where A_t is the economy's total factor productivity, assumed to be exogenous and to follow a geometric Brownian motion with drift, where g is the expected growth rate of A_t, σ is its volatility, which we assume to be constant over time and which we define as the *fundamental* volatility, and Z_t is a standard Brownian motion process. It follows that firms' profits are defined as:

$$D_t = p_t Y_t - w_t L_t.$$

Finally, we assume bonds are in zero net supply in equilibrium (i.e., $B_t = 0, \ \forall t$), and that there is no government spending, so market clearing in this economy results in $C_t = Y_t$.

II.2 Flexible Price Economy

We first solve the flexible price economy as our benchmark economy. For that purpose, we assume the usual Dixit-Stiglitz monopolistic competition among firms, where the demand

[6] Later, we will impose the equilibrium condition: $C_t^h = C_t, \forall h$, $L_t^h = L_t, \forall h$, $B_t^h = B_t = 0, \forall h$, where C_t, L_t, and B_t are aggregate variables.

each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. Each firm i takes the aggregate price p_t, wage w_t, and the aggregate output Y_t as given.

II.2.1 Household problem

In the flexible price economy, each household takes the $\{A_t, p_t, i_t\}$ processes as given:

$$\frac{dp_t}{p_t} = \pi_t dt + \sigma_t^p dZ_t,$$

and

$$di_t = \mu_t^i dt + \sigma_t^i dZ_t,$$

where π_t, σ_t^p, μ_t^i, and σ_t^i are all endogenous, so the state variables for each household would become $\{B_t^h, A_t, p_t, i_t\}$.[7]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function of household h as:

$$\Gamma^h \equiv \Gamma^h\left(B_t^h, A_t, p_t, i_t, t\right) = \max_{\substack{\{C_s^h, L_s^h\}_{s\geq t} \\ \{B_s^h\}_{s>t}}} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \left[\log C_s^h - \frac{\left(L_s^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s.$$

subject to $dB_t^h = \left[i_t B_t^h - p_t C_t^h + w_t L_t^h + D_t\right] dt$. The HJB equation is given by

$$\rho \cdot \Gamma^h = \max_{C_t^h, L_t^h} \left\{\log C_t^h - \frac{\left(L_t^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma^h\right]}{\mathrm{d}t}\right\}. \tag{II.1}$$

[7]This is a conjectural but correct statement due to the classical dichotomy between real and nominal sectors: output, consumption, and labor in equilibrium turn out to depend on A_t (only), and it turns out that p_t and i_t do not matter for the real economy and the welfare of the households.

Using Ito's Lemma, we compute:

$$d\Gamma^h = \mu_t^{\Gamma,h} dt + \sigma_t^{\Gamma,h} dZ_t, \tag{II.2}$$

where

$$\begin{aligned}
\mu_t^{\Gamma,h} =& \Gamma_t^h + \Gamma_B^h \cdot \left(i_t B_t^h - p_t C_t^h + w_t L_t^h + D_t\right) + \Gamma_A^h \cdot A_t g + \Gamma_p^h \cdot p_t \pi_t + \Gamma_i^h \cdot \mu_t^i \\
&+ \frac{1}{2}\Gamma_{AA}^h \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{pp}^h \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{ii}^h \cdot (\sigma_t^i)^2 \\
&+ \Gamma_{Ap}^h \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{Ai}^h \cdot (\sigma A_t)\sigma_t^i + \Gamma_{pi}^h \cdot (p_t \sigma_t^p)\sigma_t^i,
\end{aligned}$$

and $\sigma_t^{\Gamma,h} = \Gamma_A^h(\sigma A_t) + \Gamma_p^h(p_t \sigma_t^p) + \Gamma_i^h(\sigma_t^i)$. In the same way, we compute $d\Gamma_B^h = \mu_t^{\Gamma_B,h} dt + \sigma_t^{\Gamma_B,h} dZ_t$, where

$$\begin{aligned}
\mu_t^{\Gamma_B,h} =& \Gamma_{Bt}^h + \Gamma_{BB}^h \cdot \left(i_t B_t^h - p_t C_t^h + w_t L_t^h + D_t\right) + \Gamma_{BA}^h \cdot A_t g + \Gamma_{Bp}^h \cdot p_t \pi_t + \Gamma_{Bi}^h \cdot \mu_t^i \\
&+ \frac{1}{2}\Gamma_{BAA}^h \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{Bpp}^h \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{Bii}^h \cdot (\sigma_t^i)^2 \\
&+ \Gamma_{BAp}^h \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{BAi}^h \cdot (\sigma A_t)\sigma_t^i + \Gamma_{Bpi}^h \cdot (p_t \sigma_t^p)\sigma_t^i,
\end{aligned} \tag{II.3}$$

and $\sigma_t^{\Gamma_B,h} = \Gamma_{BA}^h(\sigma A_t) + \Gamma_{Bp}^h(p_t \sigma_t^p) + \Gamma_{Bi}^h(\sigma_t^i)$. Note that $\Gamma_\Delta^h = \frac{\partial \Gamma^h}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B^h, A, p, i\}$. Now plug equation (II.2) into equation (II.1) to obtain:

$$\rho \cdot \Gamma^h = \max_{C_t^h, L_t^h} \left\{ \log C_t^h - \frac{\left(L_t^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,h} \right\}. \tag{II.4}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t^h and L_t^h from equation (II.4), we obtain:

$$\Gamma_B^h = \frac{1}{p_t C_t^h}, \tag{II.5}$$

$$\Gamma_B^h = \frac{\left(L_t^h\right)^{\frac{1}{\eta}}}{w_t}. \tag{II.6}$$

Finally, merging (II.5) with (II.6) gives us the intratemporal optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^{N,h} = e^{-\rho t}\frac{1}{p_t C_t^h}$, and $dQ_t^h = \frac{\mathrm{d}\zeta_t^{N,h}}{\zeta_t^{N,h}}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is $C_t^{h,*}$ and $L_t^{h,*}$. Then, we can express equation (II.4) as:

$$\rho \cdot \Gamma^h = \log C_t^{h,*} - \frac{\left(L_t^{h,*}\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,h,*}. \tag{II.7}$$

Taking the derivative of both sides of equation (II.7) with respect to B_t, using the envelope theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B^h = \mu_t^{\Gamma_B,h,*}, \tag{II.8}$$

where $\mu_t^{\Gamma_B,h,*}$ is from equation (II.3) and it is evaluated at the optimum. Plugging (II.8) into the process for Γ_B^h, we obtain a simplified expression:

$$\mathrm{d}\Gamma_B^h = (\rho - i_t) \cdot \Gamma_B^h \mathrm{d}t + \underbrace{\left(\Gamma_{BA}^h (A_t \sigma) + \Gamma_{Bp}^h (p_t \sigma_t^p) + \Gamma_{Bi}^h \left(\sigma_t^i\right)\right)}_{\equiv \sigma_t^{\Gamma_B,h}} \mathrm{d}Z_t. \tag{II.9}$$

Note that $\zeta_t^{N,h} = e^{-\rho t}\Gamma_B^h$, then, using equation (II.9) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^{N,h} = -\, \zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B,h}}{\Gamma_B^h}\right] \mathrm{d}Z_t.$$

From the definition of $\mathrm{d}Q_t$, we obtain:

$$dQ_t^h \equiv \frac{\mathrm{d}\zeta_t^{N,h}}{\zeta_t^{N,h}} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B,h}}{\Gamma_B^h}\right] \mathrm{d}Z_t, \tag{II.10}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t^h\right] = -i_t \mathrm{d}t$ follows by taking expectations, which proves (2) in the flexible price equilibrium.

Equilibrium is defined as in Definition 1. From now, we interchangeably use variables with and without h superscript.

Nominal and real interest rates Prices and consumption should be adapted to the filtration generated by the Brownian motion Z_t process. Let us express the processes for consumption and price as:

$$
\begin{aligned}
\mathrm{d}p_t &= \pi_t p_t \mathrm{d}t + \sigma_t^p p_t \mathrm{d}Z_t, \\
\mathrm{d}C_t &= g_t^C C_t \mathrm{d}t + \sigma_t^C C_t \mathrm{d}Z_t,
\end{aligned} \tag{II.11}
$$

where π_t, g_t^C, σ_t^p and σ_t^C are variables to be determined in equilibrium, and which can be interpreted as the inflation rate, the expected consumption growth, and the volatilities of the price and consumption processes, respectively. As the real state density is defined as $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$, the real interest rate r_t is defined by the relation $\mathbb{E}_t\left[\frac{d\zeta_t^r}{\zeta_t^r}\right] = -r_t dt$, similarly to (2).

With (II.11), applying Ito's Lemma to the real state density $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$ results in

$$
\frac{d\zeta_t^r}{\zeta_t^r} = -\underbrace{\left[\rho + g_t^C - \left(\sigma_t^C\right)^2\right]}_{\equiv r_t} \mathrm{d}t - \sigma_t^C \mathrm{d}Z_t, \tag{II.12}
$$

which determines the real interest rate $r_t = \rho + g_t^C - (\sigma_t^C)^2$. We also apply Ito's Lemma to $\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$ and use the above processes for p_t and C_t to obtain:

$$
dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\left[\rho + g_t^C + \pi_t - (\sigma_t^p)^2 - \left(\sigma_t^C\right)^2 - \sigma_t^p\sigma_t^C\right] \mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right] \mathrm{d}Z_t,
$$

which can be rearranged as:

$$
dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\underbrace{\left[r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right)\right]}_{=i_t} \mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right] \mathrm{d}Z_t. \tag{II.13}
$$

Comparing equation (II.10) and equation (II.13), we obtain

$$
i_t = r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right), \text{ where: } r_t = \rho + g_t^C - \left(\sigma_t^C\right)^2.
$$

II.2.2 Firm problem and equilibrium

Firm optimization The demand faced by each firm i is given by

$$
D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,
$$

where p_t^i is an individual firm's price, p_t is the price aggregator, and Y_t is the aggregate output. Each firm i solves the following problem:

$$\max_{p_t^i} \quad p_t^i \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t - \frac{w_t}{A_t}\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

which results in the following first-order condition for the firm:[8]

$$p_t = \left(\frac{\varepsilon}{\varepsilon - 1}\right)\frac{w_t}{A_t}, \tag{II.14}$$

which is intuitive as it tells us that in equilibrium, price is equal to the marginal cost of production multiplied by the constant mark-up, due to the constant elasticity of demand $\varepsilon > 1$. Using equation (II.14) and the equilibrium condition $C_t = Y_t = A_t L_t$ in the first-order condition of the household in (II.5) and (II.6), we obtain $L_t^n = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$,[9] which is a constant. This implies that in the flexible price equilibrium, we have $C_t^n = Y_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$. It follows that the stochastic process for Y_t^n is the same as that for A_t, as follows:

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \frac{\mathrm{d}C_t^n}{C_t^n} = g\mathrm{d}t + \sigma \mathrm{d}Z_t. \tag{II.15}$$

Equation (II.15) implies that the growth rate of consumption and its volatility are $g_t^C = g$ and $\sigma_t^C = \sigma$, so the real interest rate in the flexible price economy, i.e., the natural rate of interest, can be expressed as $r_t^n \equiv r^n = \rho + g - \sigma^2$ from (II.12), which finally gives

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \Bigg(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2\Bigg)\mathrm{d}t + \sigma\mathrm{d}Z_t,$$

which proves equation (7).

II.3 Rigid Price Economy

We now solve the equilibrium of the rigid price economy with $p_t = \bar{p}$ for all t. The rigid price economy's consumption volatility, which we define as σ_t^C, is given by $\sigma_t^C = \sigma + \sigma_t^s$

[8]In equilibrium, $p_t^i = p_t$ as every firm chooses the same price level.

[9]We impose the superscript n (i.e., natural) in variables to denote that those are the equilibrium values in the flexible price economy.

(i.e. volatility of the flexible price equilibrium in (II.15), plus excess volatility of rigid price equilibrium). Therefore, the consumption process can be written as:

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + (\sigma + \sigma_t^s) C_t \mathrm{d}Z_t. \tag{II.16}$$

Let us conjecture that this endogenous 'excess' volatility σ_t^s, which is one of the state variables in the rigid price economy, follows the process $d\sigma_t^s = \mu_t^\sigma dt + \sigma_t^\sigma dZ_t$. With price rigidity (i.e., $p_t = \bar{p}$ for all t), the agent takes the $\{A_t, \sigma_t^s\}$ processes as given, so the state variables for each household become $\{B_t^h, A_t, \sigma_t^s\}$.[10]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function of household h as:

$$\Gamma^h \equiv \Gamma^h\left(B_t^h, A_t, \sigma_t^s, t\right) = \max_{\substack{\{C_s^h, L_s^h\}_{s\geq t} \\ \{B_s^h\}_{s>t}}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s^h - \frac{\left(L_s^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s.$$

subject to $dB_t^h = \left[i_t B_t^h - p_t C_t^h + w_t L_t^h + D_t\right] dt$. The HJB equation can be written as:

$$\rho \cdot \Gamma^h = \max_{C_t^h, L_t^h} \left\{ \log C_t^h - \frac{\left(L_t^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma^h\right]}{\mathrm{d}t} \right\}, \tag{II.17}$$

Using Ito's Lemma, we compute:

$$d\Gamma^h = \mu_t^{\Gamma,h} dt + \sigma_t^{\Gamma,h} dZ_t, \tag{II.18}$$

where

$$\begin{aligned} \mu_t^{\Gamma,h} =& \Gamma_t^h + \Gamma_B^h \cdot \left(i_t B_t^h - \bar{p} \cdot C_t^h + w_t L_t^h + D_t\right) + \Gamma_A^h \cdot A_t g + \Gamma_\sigma^h \cdot \mu_t^\sigma \\ &+ \frac{1}{2}\Gamma_{AA}^h \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{\sigma\sigma}^h \cdot (\sigma_t^\sigma)^2 + \Gamma_{A\sigma}^h \cdot (A_t\sigma)(\sigma_t^\sigma), \end{aligned}$$

[10]This is a conjectural (but correct) statement as the actual output (thereby, consumption and other variables including inflation, nominal interest rate (that follows the Taylor rule), etc) would turn out to only depend on A_t and σ_t^s under our equilibrium construction.

and $\sigma_t^{\Gamma,h} = \Gamma_A^h(\sigma A_t) + \Gamma_\sigma^h(\sigma_t^\sigma)$. Applying Ito's Lemma to Γ_B^h, we compute $d\Gamma_B^h = \mu_t^{\Gamma_B,h}dt + \sigma_t^{\Gamma_B,h}dZ_t$, where

$$\begin{aligned}\mu_t^{\Gamma_B,h} =& \Gamma_{Bt}^h + \Gamma_{BB}^h \cdot \left(i_t B_t^h - \bar{p} \cdot C_t^h + w_t L_t^h + D_t\right) + \Gamma_{BA}^h \cdot A_t g + \Gamma_{B\sigma}^h \cdot \mu_t^\sigma \\ &+ \frac{1}{2}\Gamma_{BAA}^h \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{B\sigma\sigma}^h \cdot (\sigma_t^\sigma)^2 + \Gamma_{BA\sigma}^h \cdot (A_t\sigma)(\sigma_t^\sigma),\end{aligned} \tag{II.19}$$

and $\sigma_t^{\Gamma_B,h} = \Gamma_{BA}^h \cdot (\sigma A_t) + \Gamma_{B\sigma}^h \cdot \sigma_t^\sigma$. Note $\Gamma_\Delta^h = \frac{\partial \Gamma^h}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B^h, A, \sigma^s\}$. Now plug equation (II.18) into equation (II.17) to obtain:

$$\rho \cdot \Gamma^h = \max_{C_t^h, L_t^h} \left\{ \log C_t^h - \frac{\left(L_t^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,h} \right\}. \tag{II.20}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t^h and L_t^h from equation (II.20), we obtain:

$$\Gamma_B^h = \frac{1}{\bar{p}C_t^h}, \tag{II.21}$$

$$\Gamma_B^h = \frac{\left(L_t^h\right)^{\frac{1}{\eta}}}{w_t}. \tag{II.22}$$

Finally, merging (II.21) with (II.22) gives us the intratemporal condition of the problem.

State price density and pricing kernel We know that the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^{N,h} = e^{-\rho t}\frac{1}{\bar{p}C_t^h}$, and $dQ_t^h = \frac{d\zeta_t^{N,h}}{\zeta_t^{N,h}}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is $C_t^{h,*}$ and $L_t^{h,*}$. Then, we can express equation (II.20) as:

$$\rho \cdot \Gamma^h = \log C_t^{h,*} - \frac{\left(L_t^{h,*}\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,h,*}. \tag{II.23}$$

Taking the derivative of both sides of equation (II.23) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B^h = \mu_t^{\Gamma_B,h,*}, \tag{II.24}$$

where $\mu_t^{\Gamma_B,h,*}$ follows from equation (II.19) evaluated at the optimum. Plugging equation (II.24) into the process for Γ_B^h, we obtain a simplified expression at the optimum:

$$\mathrm{d}\Gamma_B^h = (\rho - i_t) \cdot \Gamma_B^h \mathrm{d}t + \underbrace{\left(\Gamma_{BA}^h \cdot (A_t \sigma) + \Gamma_{B\sigma}^h \cdot (\sigma_t^\sigma)\right)}_{\equiv \sigma_t^{\Gamma_B,h}} \mathrm{d}Z_t. \tag{II.25}$$

Notice that $\zeta_t^{N,h} = e^{-\rho t}\Gamma_B^h$, then using equation (II.25) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^{N,h} = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^{N,h} \cdot \left[\frac{\sigma_t^{\Gamma_B,h}}{\Gamma_B^h}\right] \mathrm{d}Z_t.$$

From the previous equation, we obtain:

$$dQ_t^h \equiv \frac{\mathrm{d}\zeta_t^{N,h}}{\zeta_t^{N,h}} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B,h}}{\Gamma_B^h}\right] \mathrm{d}Z_t, \tag{II.26}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t^h\right] = -i_t \mathrm{d}t$ also follows in the rigid price economy by taking conditional expectations.

Again, the equilibrium is defined by Definition 1. From now, we interchangeably use variables with and without h superscript.

Verification of Equilibria Now let us verify that our Ornstein-Uhlenbeck equilibrium,[11] characterized by equations (I.15) and (I.17), satisfies the equilibrium conditions derived above. From (I.15) and (I.17),

$$\hat{Y}_t = \frac{\theta\mu}{\theta + \phi_y} - \frac{1}{2(\theta + \phi_y)}\left[(\sigma + \sigma_t^s)^2 - \sigma^2\right], \tag{II.27}$$

$$\begin{aligned}\mathrm{d}\sigma_t^s = &-\left[\left(\frac{\theta}{\sigma + \sigma_t^s}\right)\left[\mu\phi_y + \frac{1}{2}\left[(\sigma + \sigma_t^s)^2 - \sigma^2\right]\right] + (\theta + \phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3}\right]\mathrm{d}t \\ &- (\theta + \phi_y)\left(\frac{\sigma_t^s}{\sigma + \sigma_t^s}\right)\mathrm{d}Z_t,\end{aligned} \tag{II.28}$$

These equations will be a solution to the model, as long as there is no contradiction with the equilibrium conditions. In order to check if (II.27) and (II.28) satisfy the equilibrium

[11] With $\theta = 0$, it becomes the martingale equilibrium of Section 3.1.

conditions, first, the output gap is defined as:

$$\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right) = \log\left(\frac{C_t}{C_t^n}\right) = \log\left(\frac{C_t}{A_t}\right) - \frac{\eta}{\eta+1}\log\left(\frac{\varepsilon-1}{\varepsilon}\right), \tag{II.29}$$

where the last equality follows from $C_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$, as shown above for the flexible price equilibrium. Combining (II.27) and (II.29), we obtain:

$$C_t = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}} \cdot \exp\left\{\frac{\theta\mu}{\theta+\phi_y} - \frac{1}{2(\theta+\phi_y)}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right\},$$

which is a function of A_t and σ_t^s. We can now compute the derivative of equation (II.21) with respect to A_t and σ_t^s as:

$$\Gamma_{BA} = -\frac{\Gamma_B}{C_t}\cdot\frac{\partial C_t}{\partial A_t}, \tag{II.30}$$

$$\Gamma_{B\sigma} = -\frac{\Gamma_B}{C_t}\cdot\frac{\partial C_t}{\partial \sigma_t^s}. \tag{II.31}$$

Plugging equations (II.30) and (II.31) into equation (II.25), we obtain:

$$\mathrm{d}\Gamma_B = (\rho - i_t)\cdot\Gamma_B\mathrm{d}t - \Gamma_B\left[\frac{A_t}{C_t}\cdot\frac{\partial C_t}{\partial A_t}\cdot\sigma + \frac{1}{C_t}\cdot\frac{\partial C_t}{\partial\sigma_t^s}\cdot\sigma_t^\sigma\right]\mathrm{d}Z_t. \tag{II.32}$$

Using Ito's Lemma in equation (II.21) together with equation (II.16), we obtain

$$\mathrm{d}\Gamma_B = -\Gamma_B\left(g_t^C - (\sigma_t^C)^2\right)\mathrm{d}t - \Gamma_B(\sigma+\sigma_t^s)\mathrm{d}Z_t. \tag{II.33}$$

Comparing the volatility terms in (II.32) and (II.33) (i.e., the terms multiplying $\mathrm{d}Z_t$), it must follow that:

$$\sigma + \sigma_t^s = \frac{A_t}{C_t}\cdot\frac{\partial C_t}{\partial A_t}\cdot\sigma + \frac{1}{C_t}\cdot\frac{\partial C_t}{\partial\sigma_t^s}\cdot\sigma_t^\sigma. \tag{II.34}$$

We can now compute the derivative of C_t with respect to A_t and σ_t^s as:

$$\frac{\partial C_t}{\partial A_t} = \frac{C_t}{A_t},$$

and

$$\frac{\partial C_t}{\partial \sigma_t^s} = C_t \cdot \left(\frac{-(\sigma + \sigma_t^s)}{\theta + \phi_y} \right) = C_t \cdot (\sigma_t^\sigma)^{-1} \cdot \sigma_t^s,$$

which satisfies (II.34). Therefore, the conjectured martingale solution is verified as a valid equilibrium of the model.

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A Equilibrium Dynamics: Summary

As in the main text, we use the superscript h to denote household-specific variables (e.g., C_t^h); variables without the superscript refer to aggregates. We assume a representative-agent economy with a unit measure of households, where each household solves an identical optimization problem and selects the same consumption, labor, and bond investment in equilibrium. This results in an equilibrium given by: $C_t^h = C_t = Y_t$ for all h, $L_t^h = L_t$ for all h, $D_t^h = D_t$ for all h, $Y_t = A_t L_t$ and $B_t = 0$.

In Online Appendix, we consider the most general version of the model, with a monetary policy rule targeting aggregate volatility with responsiveness ϕ_{vol}, i.e.,

$$i_t = r^n + \phi_y \hat{Y}_t - \phi_{vol}\left[(\sigma + \sigma_t^s)^2 - \sigma^2\right]. \tag{A.1}$$

With this rule, individual optimality conditions, after imposing equilibrium conditions $C_t^h = C_t = Y_t$ for all h, $L_t^h = L_t$ for all h, $D_t^h = D_t$ for all h, $Y_t = A_t L_t$ and $B_t = 0$, are characterized by

$$\begin{aligned}
\dot{B}_t &= i_t B_t - \bar{p} C_t + w_t L_t + D_t, \\
B_t &= \dot{B}_t = 0, \\
D_t &= \bar{p} Y_t - w_t L_t, \\
\frac{dC_t}{C_t} &= \left(i_t - \rho + (\sigma + \sigma_t^s)^2\right) dt + (\sigma + \sigma_t^s)\, dZ_t, \\
L_t &= \left(\frac{w_t}{\bar{p}} \frac{1}{C_t^h}\right)^{\eta}.
\end{aligned}$$

Ornstein-Uhlehbeck equilibrium Under the Ornstein-Uhlenbeck equilibrium described in Section 3.2, output gap $\hat{Y}_t$ follows $d\hat{Y}_t = \theta\left(\mu - \hat{Y}_t\right) dt + \sigma_t^s dZ_t$. With an adjusted policy rule following (A.1), the equilibrium is characterized by the following conditions:

$$\begin{aligned}
Y_t &= A_t L_t, \\
A_t &= \underbrace{A_0}_{=1} e^{\left(g - \frac{1}{2}\sigma^2\right)\cdot t + \sigma Z_t}, \\
Y_t^n &= A_t \left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}} = \left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}} e^{\left(g - \frac{1}{2}\sigma^2\right)\cdot t + \sigma Z_t}.
\end{aligned} \tag{A.2}$$

with

$$\begin{aligned} r^n &= \rho + g - \sigma^2, \\ i_t &= r^n + \phi_y \hat{Y}_t - \phi_{vol}\left[(\sigma + \sigma_t^s)^2 - \sigma^2\right], \\ \hat{Y}_t &= \mu - \left(\frac{1 - 2\phi_{vol}}{2(\theta + \phi_y)}\right)\left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right], \\ &= \frac{\mu\theta}{\theta + \phi_y} - \left(\frac{1 - 2\phi_{vol}}{2(\theta + \phi_y)}\right)\left[(\sigma + \sigma_t^s)^2 - \sigma^2\right]. \end{aligned} \tag{A.3}$$

where the endogenous excess volatility σ_t^s follows a stochastic differential equation given by

$$\begin{aligned} d\sigma_t^s = & \overbrace{-\frac{1}{2}\left(\frac{1}{\sigma + \sigma_t^s}\right)\left[\theta\left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right] + \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)^2\left(\frac{\sigma_t^s}{\sigma + \sigma_t^s}\right)^2\right]}^{\mu_t^\sigma \equiv \tilde{\mu}(\sigma_t^s)} dt \\ & \underbrace{-\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\left(\frac{\sigma_t^s}{\sigma + \sigma_t^s}\right)}_{\sigma_t^\sigma \equiv \tilde{\sigma}(\sigma_t^s)} dZ_t. \end{aligned} \tag{A.4}$$

which leads to

$$d\left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right] = -\theta\left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right] dt - 2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\sigma_t^s dZ_t. \tag{A.5}$$

Note that with $\phi_{vol} = 0$, equation (A.4) becomes (21) of Section 3.2. With $\theta = 0$, we return to the martingale equilibrium of Section 3.1 and equation (A.4) becomes equation (18).

B Equilibrium Value Function for the Representative Household and the Transversality Condition

In this section, we characterize the equilibrium value function of households under the general equilibrium described in Online Appendix A, and prove the transversality condition for individual households.

The Hamilton-Jacobi-Bellman (HJB) equation of household h is given by:

$$\rho \cdot \Gamma^h = \max_{C_t^h, L_t^h} \left\{ \log C_t^h - \frac{\left(L_t^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,h} \right\}. \tag{B.1}$$

where we assume B_t^h as the only endogenous state variable,[1] and $\{A_t, \sigma_t^s\}$ as exogenous state variables from individual household's perspective, as σ_t^s is an aggregate variable that affects aggregate price and quantity variables. We calculate $\mu_t^{\Gamma,h}$ as follows:

$$\begin{aligned}\mu_t^{\Gamma,h} =& \Gamma_t^h + \Gamma_B^h \cdot \left(i_t B_t^h - \bar{p} \cdot C_t^h + w_t L_t^h + D_t\right) + \Gamma_A^h \cdot A_t g + \Gamma_\sigma^h \cdot \mu_t^\sigma \\ &+ \frac{1}{2}\Gamma_{AA}^h \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{\sigma\sigma}^h \cdot (\sigma_t^\sigma)^2 + \Gamma_{A\sigma}^h \cdot (A_t\sigma)(\sigma_t^\sigma),\end{aligned}$$

where subscripts denote differentiation with respect to the corresponding variable, yielding the following first-order conditions:

$$\Gamma_B^h = \frac{1}{\bar{p}C_t} = \frac{L_t^{\frac{1}{\eta}}}{w_t},$$

where we impose equilibrium conditions $C_t^h = C_t$ and $L_t^h = L_t$.

Under the general Ornstein-Uhlenbeck equilibrium described in Online Appendix A, we can characterize the exact functional form of $\Gamma^h(B_t^h, A_t, \sigma_t^s, t)$. From (A.4), together with[2]

$$i_t = \rho + g - \sigma^2 + \frac{\theta\phi_y\mu}{\theta+\phi_y} - \frac{\phi_y + 2\theta\phi_{vol}}{2(\theta+\phi_y)} \cdot \left[\left(\sigma + \sigma_j^s\right)^2 - \sigma^2\right], \tag{B.2}$$

[1] Eventually, we will impose the bond market equilibrium condition, i.e., $B_t = 0$.

[2] In equilibrium, $B_t^h = 0$. We characterize the individual value Γ^h as a function of B_t^h, which is specific to an individual household, and (A_t, σ_t^s), which are exogenous to an individual household, under the equilibrium dynamics of Online Appendix A.

and using the fact that $w_t L_t - \bar{p} \cdot C_t + D_t = 0$, together with $\log(C_t) = \left(\frac{\eta}{\eta+1}\right) \log\left(\frac{\varepsilon-1}{\varepsilon}\right) + \log(A_t) + \hat{Y}_t$ and equation (A.2) to substitute for labor L_t, we can express the value function Γ^h of the household evaluated at the optimum as:[3]

$$\begin{aligned}
\Gamma^h =& \frac{1}{\rho} \cdot \left(\frac{\eta}{\eta+1}\right) \log\left(\frac{\varepsilon-1}{\varepsilon}\right) + \frac{1}{\rho} \cdot \Gamma_B^h \cdot i_t B_t^h + \frac{1}{\rho} \cdot \log A_t + \frac{1}{\rho} \cdot \Gamma_A^h \cdot A_t g + \frac{1}{2\rho} \Gamma_{AA}^h \cdot (A_t \sigma)^2 \\
& - \frac{1}{\rho} \Gamma_{A\sigma}^h \cdot (A_t \sigma) \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right) \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s}\right) + \frac{1}{\rho} \cdot \Gamma_t^h \\
& + \frac{1}{\rho} \cdot \left[\mu - \left(\frac{1 - 2\phi_{vol}}{2(\theta + \phi_y)}\right) \left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right]\right] \\
& - \frac{1}{\rho} \cdot \left(\frac{\varepsilon - 1}{\varepsilon}\right) \frac{e^{\left(\frac{\eta+1}{\eta}\right)\left[\mu - \left(\frac{1-2\phi_{vol}}{2(\theta+\phi_y)}\right)\left[(\sigma+\sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1-2\phi_{vol}}\right]\right]}}{1 + \frac{1}{\eta}} \\
& - \frac{1}{2\rho} \Gamma_\sigma^h \cdot \left(\frac{1}{\sigma + \sigma_t^s}\right) \left[\theta \left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right] + \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)^2 \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s}\right)^2\right] \\
& + \frac{1}{2\rho} \Gamma_{\sigma\sigma}^h \cdot \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)^2 \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s}\right)^2 .
\end{aligned} \tag{B.3}$$

To further simplify the problem, we adopt a guess-and-verify approach by assuming that the value function Γ^h takes the following form:

$$\Gamma^{guess} = \frac{1}{\rho} \cdot \log(A_t) + h(\sigma_t^s) \cdot \frac{B_t^h}{A_t} + \tilde{\Gamma}(\sigma_t^s).$$

Therefore, we assume that the value function depends on $\log(A_t)$; on a term that is a multiple of a function of σ_t^s and the ratio $\frac{B_t^h}{A_t}$; and on a function solely of σ_t^s (plus constants). Differentiating yields:

$$\begin{aligned}
\Gamma_B^{guess} &= \frac{h(\sigma_t^s)}{A_t}, \quad \Gamma_A^{guess} = \frac{1}{\rho A_t} - h(\sigma_t^s) \cdot \frac{B_t^h}{A_t^2}, \\
\Gamma_{AA}^{guess} &= -\frac{1}{\rho (A_t)^2} + 2h(\sigma_t^s) \cdot \frac{B_t^h}{A_t^3}, \quad \Gamma_{A\sigma}^{guess} = -h'(\sigma_t^s) \cdot \frac{B_t^h}{A_t^2}, \\
\Gamma_\sigma^{guess} &= h'(\sigma_t^s) \cdot \frac{B_t^h}{A_t} + \tilde{\Gamma}'(\sigma_t^s), \quad \Gamma_{\sigma\sigma}^{guess} = h''(\sigma_t^s) \cdot \frac{B_t^h}{A_t} + \tilde{\Gamma}''(\sigma_t^s).
\end{aligned}$$

[3] Note that i_t following our modified Taylor rule will be a sole function of σ_t^s in equilibrium, as shown in (B.2).

Finding $h(\sigma_t^s)$ First, collecting all the terms that have $\frac{B_t^h}{A_t}$, we see that $h(\sigma_t^s)$ satisfies

$$
\begin{aligned}
h(\sigma_t^s) =& \frac{1}{\rho} i_t h(\sigma_t^s) - \frac{1}{\rho} g h(\sigma_t^s) + \frac{1}{\cancel{2}\rho}\sigma^2 \cdot \cancel{2} h(\sigma_t^s) \\
& - \frac{1}{\rho} h'(\sigma_t^s)\sigma \left[- \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}} \right) \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s} \right) \right] \\
& - \frac{1}{2\rho} h'(\sigma_t^s) \left(\frac{1}{\sigma + \sigma_t^s} \right) \left[\theta \left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}} \right] + \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}} \right)^2 \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s} \right)^2 \right] \\
& + \frac{1}{2\rho} h''(\sigma_t^s) \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}} \right)^2 \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s} \right)^2 .
\end{aligned}
\tag{B.4}
$$

We conjecture

$$
h(\sigma_t^s) \propto \exp \left[\left(\frac{1 - 2\phi_{vol}}{2\left(\theta + \phi_y\right)} \right) \left[(\sigma + \sigma_t^s)^2 \right] \right]. \tag{B.5}
$$

If the conjecture (B.5) is true, we obtain

$$
\begin{aligned}
h'(\sigma_t^s) &= h(\sigma_t^s) \cdot \frac{1 - 2\phi_{vol}}{(\theta + \phi_y)} (\sigma + \sigma_t^s), \\
h''(\sigma_t^s) &= h(\sigma_t^s) \cdot \left[\frac{1 - 2\phi_{vol}}{(\theta + \phi_y)} + \left(\frac{1 - 2\phi_{vol}}{(\theta + \phi_y)} \right)^2 (\sigma + \sigma_t^s)^2 \right],
\end{aligned}
$$

which can be plugged into (B.4) and lead to

$$
\begin{aligned}
\rho =& (i_t - g + \sigma^2) + \sigma\sigma_t^s \\
& - \frac{1}{2} \frac{1 - 2\phi_{vol}}{(\theta + \phi_y)} \cancel{(\sigma + \sigma_t^s)} \left(\frac{1}{\cancel{\sigma + \sigma_t^s}} \right) \left[\theta \left[(\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}} \right] + \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}} \right)^2 \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s} \right)^2 \right] \\
& + \frac{1}{2} \left[\frac{1 - 2\phi_{vol}}{(\theta + \phi_y)} + \left(\frac{1 - 2\phi_{vol}}{(\theta + \phi_y)} \right)^2 (\sigma + \sigma_t^s)^2 \right] \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}} \right)^2 \left(\frac{\sigma_t^s}{\sigma + \sigma_t^s} \right)^2 ,
\end{aligned}
$$

leading to the equilibrium interest rate (B.2), thereby confirming our conjectural functional form (B.5). Finally, from the optimality condition

$$
\Gamma_B^h = \Gamma_B = h(\sigma_t^s) \frac{1}{A_t} = \frac{1}{\bar{p} C_t} = \frac{1}{\bar{p} Y_t},
$$

and with the help of (A.3), we obtain

$$h\left(\sigma_t^s\right)=\frac{1}{\bar{p}\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\left(\frac{\eta}{\eta+1}\right)}}\cdot e^{-\left[\mu-\left(\frac{1-2\phi_{vol}}{2(\theta+\phi_y)}\right)\left[(\sigma+\sigma_t^s)^2-\sigma^2+\frac{2\mu\phi_y}{1-2\phi_{vol}}\right]\right]}.$$

Finding $\tilde{\Gamma}(\sigma_t^s)$ The final term $\tilde{\Gamma}(\sigma_t^s)$ should satisfy the following ordinary differential equation (ODE):

$$\begin{aligned}\tilde{\Gamma}\left(\sigma_t^s\right)=&\frac{1}{\rho}\cdot\left(\frac{\eta}{\eta+1}\right)\log\left(\frac{\varepsilon-1}{\varepsilon}\right)+\frac{1}{\rho^2}\left(g-\frac{1}{2}\sigma^2\right)+\frac{1}{\rho}\cdot\left[\mu-\left(\frac{1-2\phi_{vol}}{2\left(\theta+\phi_y\right)}\right)\left[\left(\sigma+\sigma_t^s\right)^2-\sigma^2+\frac{2\mu\phi_y}{1-2\phi_{vol}}\right]\right]\\&-\frac{1}{\rho}\left(\frac{\eta}{\eta+1}\right)\left(\frac{\varepsilon-1}{\varepsilon}\right)\cdot e^{\left(\frac{\eta+1}{\eta}\right)\left[\mu-\left(\frac{1-2\phi_{vol}}{2(\theta+\phi_y)}\right)\left[(\sigma+\sigma_t^s)^2-\sigma^2+\frac{2\mu\phi_y}{1-2\phi_{vol}}\right]\right]}\\&-\frac{1}{2\rho}\tilde{\Gamma}'\left(\sigma_t^s\right)\cdot\left(\frac{1}{\sigma+\sigma_t^s}\right)\left[\theta\left[\left(\sigma+\sigma_t^s\right)^2-\sigma^2+\frac{2\mu\phi_y}{1-2\phi_{vol}}\right]+\left(\frac{\theta+\phi_y}{1-2\phi_{vol}}\right)^2\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)^2\right]\\&+\frac{1}{2\rho}\tilde{\Gamma}''\left(\sigma_t^s\right)\cdot\left(\frac{\theta+\phi_y}{1-2\phi_{vol}}\right)^2\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)^2.\end{aligned}$$

We can rearrange the previous equation as

$$\tilde{\Gamma}\left(\sigma_t^s\right)+\mathcal{A}\left(\sigma_t^s\right)\tilde{\Gamma}'\left(\sigma_t^s\right)+\mathcal{B}\left(\sigma_t^s\right)\tilde{\Gamma}''\left(\sigma_t^s\right)=\mathcal{R}\left(\sigma_t^s\right) \tag{B.6}$$

where:

$$\begin{aligned}\mathcal{A}\left(\sigma_t^s\right)=&\frac{1}{2\rho}\left(\frac{1}{\sigma+\sigma_t^s}\right)\left[\theta\left[\left(\sigma+\sigma_t^s\right)^2-\sigma^2+\frac{2\mu\phi_y}{1-2\phi_{vol}}\right]+\left(\frac{\theta+\phi_y}{1-2\phi_{vol}}\right)^2\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)^2\right]\\\mathcal{B}\left(\sigma_t^s\right)=&-\frac{1}{2\rho}\left(\frac{\theta+\phi_y}{1-2\phi_{vol}}\right)^2\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)^2\\\mathcal{R}\left(\sigma_t^s\right)=&\frac{1}{\rho}\cdot\left(\frac{\eta}{\eta+1}\right)\log\left(\frac{\varepsilon-1}{\varepsilon}\right)+\frac{1}{\rho^2}\left(g-\frac{1}{2}\sigma^2\right)+\frac{1}{\rho}\cdot\left[\mu-\left(\frac{1-2\phi_{vol}}{2\left(\theta+\phi_y\right)}\right)\left[\left(\sigma+\sigma_t^s\right)^2-\sigma^2+\frac{2\mu\phi_y}{1-2\phi_{vol}}\right]\right]\\&-\frac{1}{\rho}\left(\frac{\eta}{\eta+1}\right)\left(\frac{\varepsilon-1}{\varepsilon}\right)\cdot e^{\left(\frac{\eta+1}{\eta}\right)\left[\mu-\left(\frac{1-2\phi_{vol}}{2(\theta+\phi_y)}\right)\left[(\sigma+\sigma_t^s)^2-\sigma^2+\frac{2\mu\phi_y}{1-2\phi_{vol}}\right]\right]}.\end{aligned}$$

Then, by Peano's Theorem (Walter, 1973), we know that a function $\tilde{\Gamma}\left(\sigma_t^s\right)$ solving the ODE in equation (B.6) exists. Several points are worth noting. First, the initial assumption that Γ depends on $\{B_t^h, A_t, \sigma_t^s, t\}$, i.e. $\Gamma=\Gamma(B_t^h, A_t, \sigma_t^s, t)$, can be simplified to $\Gamma(B_t, A_t, \sigma_t^s)$ by dropping the explicit time dependence (and hence eliminating Γ_t from the derivations).

Second, with aggregate bonds in zero net supply, $B_t = 0$ for all t, the representative household's value function simplifies to:

$$\Gamma\left(A_t, \sigma_t^s\right) = \frac{1}{\rho} \cdot \log(A_t) + \tilde{\Gamma}\left(\sigma_t^s\right). \tag{B.7}$$

B.1 Transversality condition

Using equation (B.7), the transversality condition of the representative household is given by:

$$\lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t} \cdot \Gamma\left(A_t, \sigma_t^s\right)\right] = \lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t}\frac{1}{\rho}\log(A_t)\right] + \lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t}\tilde{\Gamma}\left(\sigma_t^s\right)\right] = 0.$$

The first limit tends to zero, as shown by

$$\lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t}\frac{1}{\rho}\log(A_t)\right] = \lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t}\frac{1}{\rho}\left[\left(g - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\right]\right] = \frac{1}{\rho}\lim_{t\to\infty} e^{-\rho t}\left(g - \frac{1}{2}\sigma^2\right)t = 0.$$

The second limit requires

$$\lim_{t\to\infty} \mathbb{E}_0\left[e^{-\rho t}\tilde{\Gamma}\left(\sigma_t^s\right)\right] = 0,$$

which is trivially satisfied by Oksendal (1995) as (i) $\tilde{\Gamma}(\sigma_t^s)$ is well-defined by the ODE in (B.6), (ii) the distribution of σ_t^s is stationary in the long run, as proven in Online Appendix C.3; (iii) As shown in Online Appendix C.2, the process (A.4) is irreducible and stable, thus converges to the stationary distribution.[4]

Sufficiency According to Bertsekas (2005) (Proposition 3.2.1) and Liberzon (2012) (Section 5.1.4), the solution to the Hamilton-Jacobi-Bellman equation (B.1) is both necessary and sufficient, as it satisfies the transversality condition and the the utility function is concave in consumption and labor, with the budget constraint linear in bond holdings.[5] Therefore, our conjectured class of solutions, which follows an Ornstein-Uhlenbeck process, satisfies the optimality conditions for the individual household.

[4] Global solutions converging to degenerate distributions with zero excess volatility ($\sigma_t^s = 0$ for all t)—i.e., perfectly stabilized equilibrium, Martingale equilibrium, or Ornstein-Uhlenbeck process with $\mu = 0$—trivially satisfy these conditions as well.

[5] It leads to a weakly concave stochastic Hamiltonian function. See Liberzon (2012) for details.

C Characterizing Stability and Stationary Properties of the Distribution: the Ornstein-Uhlenbeck Process

As explained in Online Appendix A, we examine the most general version of the model, which incorporates the policy rule in (A.1) and defines the output gap as in equation (A.3).

C.1 Limit Behavior

Define

$$u_t \equiv (\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}},$$

and $m_t \equiv \mathbb{E}_0\left(u_t\right)$, then due to (A.5), u_t follows

$$du_t = -\theta u_t dt - 2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\sigma_t^s \mathrm{d}Z_t,$$

leading to

$$u_t = -\theta \int_0^t u_h dh - 2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\int_0^t \sigma_h^s dZ_h + p_0.$$

Imposing the expectations operator $\mathbb{E}_0$, we obtain

$$\frac{dm_t}{dt} + \theta m_t = 0.$$

Solving the equation above:

$$m_T \equiv \mathbb{E}_0\left[(\sigma + \sigma_T^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right] = e^{-\theta T} m_0, \tag{C.1}$$

which can be rewritten as

$$\mathbb{E}_0\left[(\sigma + \sigma_T^s)^2\right] = \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}} + e^{-\theta T}\mathbb{E}_0\left[(\sigma + \sigma_0^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right].$$

Taking the limit as $T \to \infty$, assuming that $\theta > 0$, and using equation (C.1), we obtain:

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}_0\left[(\sigma + \sigma_T^s)^2\right] &= \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}} + \lim_{T\to\infty} e^{-\theta T}\mathbb{E}_0\left[(\sigma + \sigma_0^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1 - 2\phi_{vol}}\right] \\ &= \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}.\end{aligned}$$

Similarly, taking the limit of equation (A.3), assuming that $\theta > 0$, and using equation (C.1), we obtain:

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}_0\left[\hat{Y}_T\right] &= \mu - \frac{1-2\phi_{vol}}{2(\theta+\phi_y)} \lim_{T\to\infty} e^{-\theta T}\mathbb{E}_0\left[(\sigma+\sigma_0^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1-2\phi_{vol}}\right] \\ &= \mu.\end{aligned}$$

Therefore, parameter μ in our Ornstein-Uhlenbeck process can be regarded as the long-run expectation of $\hat{Y}_t$.

C.2 Process Stability: Foster-Lyapunov Drift Condition

We now demonstrate that the process $\{\sigma_t^s\}$ defined in (A.4) is stable and converges to the stationary distribution described in Online Appendix C. Following the techniques of Meyn and Tweedie (1993, 1998, 2012), we select the Lyapunov function $\Phi(\sigma_t^s) = (\sigma_t^s)^2$, which is positive definite and radially unbounded. The Lyapunov generator is given by:

$$\mathcal{L}\Phi(\sigma_t^s) = \Phi'(\sigma_t^s)\tilde{\mu}(\sigma_t^s) + \frac{1}{2}(\tilde{\sigma}(\sigma_t^s))^2\Phi''(\sigma_t^s).$$

The Foster-Lyapunov drift condition requires for there to exist some constants $c > 0$ and $d \leq 0$ such that

$$\mathcal{L}\Phi(\sigma_t^s) \leq -c\Phi(\sigma_t^s) + d, \text{ for all } |\sigma_t^s| \geq R, \tag{C.2}$$

for some $R > 0$. This condition implies that outside a compact set $|\sigma_t^s| \leq R$, the process exhibits a negative drift relative to Φ. This negative drift is critical for establishing both the tightness and stability of the process, ensuring convergence to its invariant measure (i.e., the stationary distribution).

Substituting the drift and diffusion terms into the Lyapunov generator yields:

$$\mathcal{L}\Phi(\sigma_t^s) = -\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)\theta\left[(\sigma+\sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1-2\phi_{vol}}\right] + \sigma\left(\frac{\theta+\phi_y}{1-2\phi_{vol}}\right)^2\frac{(\sigma_t^s)^2}{(\sigma+\sigma_t^s)^3}. \tag{C.3}$$

It is easy to notice that as $\sigma_t^s \to \infty$, we obtain:[6]

$$\lim_{\sigma_t^s \to \infty} \mathcal{L}\Phi(\sigma_t^s) \simeq -\theta(\sigma_t^s)^2.$$

Thus, for some constants $0 < c \leq \theta$ (with $\theta > 0$) and $d = 0$, there exists an R sufficiently large to satisfy the Foster-Lyapunov condition in equation (C.2). Since the process is smooth and irreducible, the results in Meyn and Tweedie (1993, 1998, 2012) ensure its long-run stability.

C.3 Stationary Distribution

We define $n(\sigma_t^s, t)$ as the probability density function of σ_t^s, which satisfies the following Kolmogorov Forward Equation:

$$\frac{\partial n\left(\sigma^s, t\right)}{\partial t} = -\frac{d}{d\sigma^s}\left[\tilde{\mu}\left(\sigma^s\right) n\left(\sigma^s, t\right)\right] + \frac{d^2}{d(\sigma^s)^2}\left[\frac{1}{2}\tilde{\sigma}(\sigma^s)^2 n\left(\sigma^s, t\right)\right], \tag{C.4}$$

where $\tilde{\mu}(\sigma^s)$ and $\tilde{\sigma}(\sigma^s)$ are defined in process (A.4).

Setting $\frac{\partial n(\sigma^s,t)}{\partial t} = 0$ to obtain the stationary distribution, denoted $n(\sigma^s)$, and integrating (C.4) once:

$$\tilde{\mu}\left(\sigma^s\right) n\left(\sigma^s\right) = \frac{d}{d\sigma^s}\left[\frac{1}{2}\tilde{\sigma}(\sigma^s)^2 n\left(\sigma^s\right)\right] + \mathcal{C}_1,$$

where $\mathcal{C}_1$ is the integration constant. Imposing the no-flux condition, $\mathcal{C}_1 = 0$, and defining $D(\sigma^s) \equiv \tilde{\sigma}(\sigma^s)^2 n(\sigma^s)$, we have

$$\frac{dD(\sigma^s)}{d\sigma^s} = \frac{2\tilde{\mu}(\sigma^s)}{\tilde{\sigma}(\sigma^s)^2} D(\sigma^s),$$

which leads to

$$D(\sigma^s) \propto \exp\left(\int \frac{2\tilde{\mu}(\sigma^s)}{\tilde{\sigma}(\sigma^s)^2} d\sigma^s\right),$$

and results in a probability density function for the stationary distribution proportional to

$$n(\sigma^s) \propto \frac{1}{\tilde{\sigma}(\sigma^s)^2} \exp\left(\int \frac{2\tilde{\mu}(\sigma^s)}{\tilde{\sigma}(\sigma^s)^2} d\sigma^s\right). \tag{C.5}$$

[6] More specifically, we obtain from (C.3) that $\lim_{\sigma_t^s \to \infty} \frac{\mathcal{L}\Phi(\sigma_t^s)}{(\sigma_t^s)^2} = -\theta$.

Substituting the expressions for $\tilde{\mu}(\sigma^s)$ and $\tilde{\sigma}(\sigma^s)$ defined in (A.4) into equation (C.5) and evaluating the integral yields:

$$n\left(\sigma^s\right) = \frac{\mathcal{C}_2}{\left(\frac{\theta+\phi_y}{1-2\phi_{vol}}\right)^2 \left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)^2} \cdot \exp\left\{ - \left[\begin{array}{l} \theta\left(\frac{1-2\phi_{vol}}{\theta+\phi_y}\right)^2 \left[\frac{(\sigma^s)^2}{2} + 3\sigma^2 \log|\sigma^s| + 3\sigma\sigma^s - \frac{\sigma^3}{\sigma^s}\right] \\ +\,\theta\left(\frac{1-2\phi_{vol}}{\theta+\phi_y}\right)^2 \left(\frac{2\mu\phi_y}{1-2\phi_{vol}} - \sigma\right) \left[\log|\sigma^s| - \frac{\sigma}{\sigma^s}\right] \\ +\log|\sigma+\sigma^s| \end{array} \right] \right\}, \tag{C.6}$$

where $\mathcal{C}_2$ is the integration constant chosen so that the density integrates to one over the domain of σ^s. Equation (C.6) defines the stationary distribution of σ_t^s, demonstrating that the Ornstein–Uhlenbeck solution attains a stationary stochastic equilibrium. Note that this distribution does not belong to any standard family.

Note that for $\mu = 0$ and $\theta > 0$ (see Property 2 of Proposition 3), we obtain $\mathcal{C}_2 \to 0$ to ensure that $\eta(\sigma_t^s)$ is a proper probability density function. This is consistent with the result that the distribution degenerates at $\sigma_\infty^s = 0$.

C.3.1 Limiting Case: Zero Fundamental Volatility

As fundamental volatility tends to zero ($\sigma \to 0$), the long-run stationary distribution of $\{\sigma_t^s\}$ becomes

$$n\left(\sigma^s\right) = \tilde{\mathcal{C}}_2 \cdot \left(\sigma^s\right)^{-\left(1+\frac{2\theta\mu\phi_y}{(\theta+\phi_y)^2}(1-2\phi_{vol})\right)} e^{-\left(\frac{\theta(1-2\phi_{vol})^2(\sigma^s)^2}{2(\theta+\phi_y)^2}\right)}. \tag{C.7}$$

Moreover, if (i) the Ornstein-Uhlenbeck parameters satisfy $\theta > 0$ and $\mu < 0$, and (ii) the Taylor rule coefficients satisfy $\phi_y > 0$ and $\phi_{vol} < \frac{1}{2}$, this distribution corresponds to the generalized gamma distribution.

Generalized Gamma Distribution The generalized gamma distribution, $GGD(a, d, p)$, is defined by the probability density function

$$\tilde{n}(x) = \frac{p/a^d}{\Gamma(d/p)}\, x^{d-1} \exp\left[-\left(\frac{x}{a}\right)^p\right].$$

where $a > 0$ is the scale parameter, $d > 0$ is the power-law shape parameter, $p > 0$ is the exponential shape parameter, and $\Gamma(\cdot)$ denotes the gamma function. It is straightforward to verify that equation (C.7) conforms to this definition with the parametrization

$$
\begin{aligned}
a &= \sqrt{\frac{2(\theta+\phi_y)^2}{\theta\left(1-2\phi_{vol}\right)^2}}, \\
d &= -\frac{2\theta\mu\phi_y}{(\theta+\phi_y)^2}\left(1-2\phi_{vol}\right), \\
p &= 2.
\end{aligned}
$$

Thus, in the long run, σ_t^s follows a generalized gamma distribution when $\sigma \to 0$ and the additional parameter restrictions hold.

Furthermore, by the properties of the generalized gamma distribution, the stationary distribution of $\left(\sigma_\infty^s\right)^2$ is given by $GGD(a^2,\, d/2,\, p/2)$. Since $p/2 = 1$ in this case, the distribution reduces to a standard Gamma distribution, $\left(\sigma_\infty^s\right)^2 \sim Gamma\left(a', d'\right)$, where $a' = a^2$ is the scale parameter and $d' = d/2$ is the shape parameter.

D Strong Solution

Based on Chapter 5.3, *Weak and Strong Solutions* of Øksendal (2003), proving that our solution is strong requires verifying that the proposed solution for σ_t^s (or a transformation such as $(\sigma + \sigma_t^s)^2$) satisfies both the *Lipschitz continuity* and the *linear growth conditions*. Here, we focus on the case where $\mu < 0$ and $\theta > 0$;[7] proofs for other cases are analogous.

To verify these conditions, define $y_t = (\sigma + \sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1-2\phi_{vol}}$ and rearrange equation (A.5) as follows:

$$dy_t = \underbrace{-\theta y_t}_{\equiv v(y_t)} dt + \underbrace{\left[2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\sigma - 2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\sqrt{y_t + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}}\right]}_{\equiv g(y_t)} dZ_t,$$

where we defined $v(y_t)$ as the drift, and $g(y_t)$ as the volatility of the $\{y_t\}$ process.

D.1 Lipschitz Continuity

Lipschitz Continuity requires that for any y_1, y_2:

$$|v(y_1) - v(y_2)| + |g(y_1) - g(y_2)| \leq \mathcal{L} \cdot |y_1 - y_2|,$$

which we prove by showing that the following conditions separately hold:

$$|v(y_1) - v(y_2)| \leq \mathcal{L}_v \cdot |y_1 - y_2|,$$
$$|g(y_1) - g(y_2)| \leq \mathcal{L}_g \cdot |y_1 - y_2|,$$

which then allows us to trivially prove the original Lipschitz Continuity with $\mathcal{L} = \mathcal{L}_v + \mathcal{L}_g$.

Drift term Since

$$|v(y_1) - v(y_2)| = |\theta| \cdot |y_1 - y_2|,$$

which leads to

$$|v(y_1) - v(y_2)| \leq \mathcal{L}_v \cdot |y_1 - y_2|,$$

[7] In this case, $\sigma_t^s \geq 0$ almost surely for all t, assuming $\sigma_0^s = 0$.

with $\mathcal{L}_v = |\theta|$, the drift function $v(y)$ satisfies Lipschitz continuity.

Diffusion term From

$$|g(y_1) - g(y_2)| = \left| -2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right) \cdot \left(\sqrt{y_1 + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}} - \sqrt{y_2 + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}}\right)\right|,$$

we apply the mean value theorem: there is a constant ξ for any pair $\{y_1,\, y_2\}$ such that

$$\sqrt{y_1 + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}} - \sqrt{y_2 + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}} = \frac{1}{2\sqrt{\xi + \sigma^2 - \frac{2\mu\phi_y}{1-2\phi_{vol}}}} \cdot (y_1 - y_2).$$

Therefore, we can express the previous expression as:

$$|g(y_1) - g(y_2)| = \left| \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right) \frac{1}{\sqrt{\xi + \sigma^2 - \frac{2\mu\phi_y}{1-2\phi_{vol}}}} \right| \cdot |y_1 - y_2|, \tag{D.1}$$

The constant ξ defined above is specific to each pair; therefore, we require a constant that uniformly satisfies the Lipschitz condition for any pair. Differentiating with respect to ξ, we find that:

$$\frac{d\left(\frac{1}{\sqrt{\xi + \sigma^2 - \frac{2\mu\phi_y}{1-2\phi_{vol}}}}\right)}{d\xi} < 0.$$

Note from (A.4) that with $\theta > 0$ and $\mu < 0$, we have $\tilde{\sigma}(0) = 0$ and $\tilde{\mu}(0) > 0$. This implies that $\sigma_t^s = 0$ is a natural boundary for the process $\{\sigma_t^s\}$. Consequently, the minimum of y_t occurs at $\sigma_t^s = 0$, yielding $y^{min} = \frac{2\mu\phi_y}{1-2\phi_{vol}}$. Substituting this into the expression for ξ in (D.1) gives:

$$|g(y_1) - g(y_2)| \leq \mathcal{L}_g \cdot |y_1 - y_2|\ ,$$

where

$$\mathcal{L}_g = \left|\frac{1}{\sigma} \cdot \left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\right|,$$

satisfying Lipschitz continuity for the volatility function $g(y)$.

D.2 Linear Growth Condition

The Linear Growth Condition requires:

$$|v(y)|^2 + |g(y)|^2 \leq \mathcal{K} \cdot \left(1 + |y|^2\right).$$

We prove the above equation by showing that the following conditions separately hold:

$$|v(y)|^2 \leq \mathcal{K}_v \cdot \left(1 + |y|^2\right),$$
$$|g(y)|^2 \leq \mathcal{K}_g \cdot \left(1 + |y|^2\right),$$

which then allow us to prove the Linear Growth Condition with $\mathcal{K} = \mathcal{K}_v + \mathcal{K}_g$.

Drift term It is trivial to see that $|v(y)|^2 = |\theta|^2 |y|^2 \leq \mathcal{K}_v \cdot (1 + |y|^2)$ with $\mathcal{K}_v = \theta^2$.

Diffusion term We see that

$$\begin{aligned} |g(y)|^2 &= \left| 2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\sigma - 2\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)\sqrt{y + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}}\right|^2 \\ &= 4\left(\frac{\theta + \phi_y}{1 - 2\phi_{vol}}\right)^2 \left[\sigma - \sqrt{y + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}}\right]^2. \end{aligned}$$

Since[8]

$$y + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}} \geq \sigma^2,$$

we obtain

$$\begin{aligned} \left[\sigma - \sqrt{y + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}}\right]^2 &< y + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}} \\ &< 1 + |y|^2 + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}} \\ &< \max\left\{1 + \sigma^2 - \frac{2\mu\phi_y}{1 - 2\phi_{vol}}, 1\right\}\left(1 + |y|^2\right), \end{aligned}$$

[8]Given that $\sigma_t^s \geq 0$ for all t almost surely, when $\sigma_t^s = 0$, $y_t = y^{min} = \frac{2\mu\phi_y}{1-2\phi_{vol}}$

leading to

$$|g(y)|^2 < \underbrace{4\left(\frac{\theta+\phi_y}{1-2\phi_{vol}}\right)^2 \max\left\{1+\sigma^2-\frac{2\mu\phi_y}{1-2\phi_{vol}},1\right\}}_{\equiv\mathcal{K}_g}\left(1+|y|^2\right),$$

satisfying the Linear Growth Condition for the volatility function $g(y)$.

Note that we defined $y_t = (\sigma+\sigma_t^s)^2 - \sigma^2 + \frac{2\mu\phi_y}{1-2\phi_{vol}}$ and proved that it permits a strong solution. Thus, it is trivial to see that the conditions will also hold for $(\sigma+\sigma_t^s)^2$ and for σ_t^s.

E Constant Relative Risk Aversion (CRRA) Utility

We modify the baseline model by assuming a CRRA utility function with parameter γ. The representative household solves:[9]

$$\max_{\substack{\{C_s^h, L_s^h\}_{s\geq t} \\ \{B_s^h\}_{s>t}}} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \left[\frac{\left(C_s^h\right)^{1-\gamma} - 1}{1-\gamma} - \frac{\left(L_s^h\right)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] \mathrm{d}s, \quad \text{s.t.} \quad \mathrm{d}B_t^h = \left[i_t B_t^h - p_t C_t^h + w_t L_t^h + D_t \right] \mathrm{d}t.$$

Since most equilibrium expressions are similar to those in the main text with log-utility, we summarize only the key equations. First, the real and nominal state price densities and stochastic discount factors are given by:

$$\frac{d\xi_t^r}{\xi_t^r} = -\underbrace{\left[\rho + \gamma g_t^C - \frac{\gamma(\gamma+1)}{2} \left(\sigma_t^C\right)^2 \right]}_{\equiv r_t} dt - \gamma \sigma_t^C dZ_t,$$

and

$$dQ_t \equiv \frac{d\xi_t^N}{\xi_t^N} = -\underbrace{\left[r_t + \pi_t - \sigma_t^P (\gamma \sigma_t^C + \sigma_t^P) \right]}_{\equiv i_t} dt - \left[\gamma \sigma_t^C + \sigma_t^P \right] dZ_t.$$

From the Euler equation, the expected consumption growth is given by

$$E_t \left[\frac{dC_t}{C_t} \right] = \frac{1}{\gamma} (i_t - \rho) dt + \frac{(\gamma+1)}{2} Var_t \left(\frac{dC_t}{C_t} \right).$$

Let σ_t^s denote the excess volatility of output growth, as defined earlier. Then, output Y_t follows the process:

$$\frac{dY_t}{Y_t} = \frac{1}{\gamma} \left[i_t - \rho + \frac{\gamma(\gamma+1)}{2} (\sigma + \sigma_t^s)^2 \right] dt + (\sigma + \sigma_t^s) dZ_t, \tag{E.1}$$

leading to

$$d \ln Y_t = \frac{1}{\gamma} \left[i_t - \rho + \frac{1}{2} \gamma^2 (\sigma + \sigma_t^s)^2 \right] dt + (\sigma + \sigma_t^s) dZ_t. \tag{E.2}$$

[9] As in the main text, in equilibrium, we impose $C_t^h = C_t$, $L_t^h = L_t$, $B_t^h = B_t = 0$, $\forall h$, and $C_t = Y_t$, $D_t = \bar{p} Y_t - w_t L_t$.

In a similar way to (E.1), natural output Y_t^n follow

$$\frac{dY_t^n}{Y_t^n} = \frac{1}{\gamma}\left[r^n - \rho + \frac{\gamma(\gamma+1)}{2}\sigma^2\right]dt + \sigma dZ_t, \tag{E.3}$$

where the natural rate of interest r^n is given by

$$r^n = \rho + \gamma g - \frac{\gamma(\gamma+1)}{2}\sigma^2.$$

Equation (E.3) can be written as

$$d\ln Y_t^n = \frac{1}{\gamma}\left[r^n - \rho + \frac{1}{2}\gamma^2\sigma^2\right]dt + \sigma dZ_t. \tag{E.4}$$

From equations (E.2) and (E.4), output gap $\hat{Y}_t$ follows[10]

$$\begin{aligned} d\hat{Y}_t &= \frac{1}{\gamma}\left[i_t - r^n + \gamma^2\frac{1}{2}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right]dt + \sigma_t^s dZ_t \\ &= \frac{1}{\gamma}\left[i_t - r_t^T\right]dt + \sigma_t^s dZ_t, \end{aligned} \tag{E.5}$$

where the risk-adjusted natural rate r_t^T is similarly defined:

$$r_t^T = r^n - \frac{1}{2}\gamma^2\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right].$$

Precautionary premium Defining the precautionary premium as $pp_t \equiv \gamma^2(\sigma+\sigma_t^s)^2$, we can express r_t^T as[11]

$$r_t^T = r^n - \frac{1}{2}\left(pp_t - pp_t^n\right), \tag{E.6}$$

which has the same form as the log-utility case.

[10]Note that $\frac{1}{\gamma}$ is multiplied to the gap between i_t and r_t^T as it acts as an elasticity of intertemporal substitution.

[11]This definition of pp_t is equivalent to the risk premium in a canonical consumption-based asset pricing model.

Martingale equilibrium We can construct the martingale equilibrium similarly in a similar manner. Plugging policy rule (A.1) into equation (E.5):

$$d\hat{Y}_t = \underbrace{\frac{1}{\gamma}\left[\phi_y \hat{Y}_t + \left[\gamma^2 - 2\phi_{vol}\right]\frac{1}{2}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right]}_{=0} dt + \sigma_t^s dZ_t.$$

Under the assumption that $\hat{Y}_t$ is a local martingale, we obtain

$$\hat{Y}_t = -\frac{1}{2\phi_y}\left[\gamma^2 - 2\phi_{vol}\right]\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right].$$

Defining $\phi_{total} \equiv \frac{\phi_y}{\gamma^2 - 2\phi_{vol}}$, the previous expression becomes:

$$\hat{Y}_t = -\frac{1}{2\phi_{total}}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]. \tag{E.7}$$

Then, from equation (E.7), we obtain

$$d(\sigma+\sigma_t^s)^2 = -2\phi_{total}\sigma_t^s dZ_t,$$

and

$$d\sigma_t^s = -\phi_{total}^2 \frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3} dt - \phi_{total}\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right) dZ_t. \tag{E.8}$$

When $\gamma = 1$ (i.e., under log-utility), equation (E.8) reduces to equation (26) in Section 4.

Ornstein-Uhlenbeck equilibrium If we assume that $\hat{Y}_t$ follows an Ornstein-Uhlenbeck process:

$$d\hat{Y}_t = \theta\left[\mu - \hat{Y}_t\right] dt + \sigma_t^s dZ_t, \tag{E.9}$$

then from

$$d\hat{Y}_t = \underbrace{\frac{1}{\gamma}\left[\phi_y \hat{Y}_t + \left[\gamma^2 - 2\phi_{vol}\right]\frac{1}{2}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right]}_{=\theta\left(\mu - \hat{Y}_t\right)} dt + \sigma_t^s dZ_t,$$

we obtain

$$\hat{Y}_t = \left(\frac{\gamma\theta}{\gamma\theta+\phi_y}\right)\mu - \frac{1}{2}\left(\frac{1}{\gamma\theta+\phi_y}\right)\left[\gamma^2 - 2\phi_{vol}\right]\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]. \tag{E.10}$$

If we define $\tilde{\phi}_{total} \equiv \frac{\gamma\theta+\phi_y}{\gamma^2-2\phi_{vol}}$,[12] we obtain

$$\hat{Y}_t = \left(\frac{\gamma\theta}{\gamma\theta+\phi_y}\right)\mu - \frac{1}{2\tilde{\phi}_{total}}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right],$$

from which with the help of equations (E.9) and (E.10), we obtain

$$d(\sigma+\sigma_t^s)^2 = -2\theta\tilde{\phi}_{total}\left[\left(\frac{\phi_y}{\gamma\theta+\phi_y}\right)\mu + \frac{1}{2\tilde{\phi}_{total}}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right]dt - 2\tilde{\phi}_{total}\sigma_t^s dZ_t,$$

and

$$\begin{aligned} d\sigma_t^s =& -\theta\tilde{\phi}_{total}\left(\frac{1}{\sigma+\sigma_t^s}\right)\left[\left(\frac{\phi_y}{\gamma\theta+\phi_y}\right)\mu + \frac{1}{2\tilde{\phi}_{total}}\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right] - \tilde{\phi}_{total}^2\frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}\right]dt \\ & - \tilde{\phi}_{total}\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)dZ_t. \end{aligned}$$

Perfect stabilization and growth targeting The policy rule (A.1) with $\phi_{vol} = \gamma^2$ achieves full stabilization and yields $\hat{Y}_t = 0$ for all t as the unique stable equilibrium. Or, with precautionary premium as defined in (E.6), we can write:

$$i_t = \underbrace{r^n - \frac{1}{2}\left(pp_t - pp_t^n\right)}_{=r_t^T} + \phi_y\hat{Y}_t, \tag{E.11}$$

which is of the same form as in the log-utility case of Section 4.

Finally, we can rewrite (E.11) as a growth mandate rule:

$$\frac{E_t\left(d\log Y_t\right)}{dt} = \frac{E_t\left(d\log Y_t^n\right)}{dt} + \frac{\phi_y}{\gamma}\hat{Y}_t. \tag{E.12}$$

[12] Note that it is just a natural extension of ϕ_{total} constant in (E.7), as $\theta = 0$ corresponds to the martingale equilibrium case.

The expected growth rate of natural output is given by:

$$\frac{E_t\left(d\log Y_t^n\right)}{dt} = g - \frac{1}{2\gamma}\sigma^2,$$

so equation (E.12) can be rewritten as:

$$\frac{E_t\left(d\log Y_t\right)}{dt} = \left(g - \frac{1}{2\gamma}\sigma^2\right) + \frac{\phi_y}{\gamma}\hat{Y}_t.$$

A possible interpretation is that the central bank in practice follows the rule

$$\frac{E_t\left(d\log Y_t\right)}{dt} = \left(g - \frac{1}{2\gamma}\sigma^2\right) + \tilde{\phi}_y\hat{Y}_t,$$

which results in an implicit output gap target in the Taylor rule for interest rates, with $\phi_y = \gamma\tilde{\phi}_y$.

F Model with Sticky Prices à la Rotemberg (1982)

In this section, we derive a New Keynesian Phillips curve based on nominal rigidities à la Rotemberg (1982). First, we present the equilibrium conditions when firms are monopolistically competitive as in a canonical New Keynesian model (Woodford, 2003).

F.1 Equilibrium Conditions

Firm i setting its price p_t^i when p_t is the price aggregator faces the demand given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\epsilon} Y_t. \tag{F.1}$$

Other equilibrium conditions do not change. The optimality conditions for households are given by

$$\frac{1}{p_t C_t} = \frac{L_t^{1/\eta}}{w_t}, \tag{F.2}$$

and

$$\frac{dY_t}{Y_t} = \underbrace{\left(i_t + (\sigma + \sigma_t^s)^2 - \pi_t - \rho\right)}_{\equiv \mu_t^Y} dt + (\sigma + \sigma_t^s) dZ_t. \tag{F.3}$$

In equilibrium, $Y_t = C_t = A_t L_t$ hold. Natural output Y_t^n is given by

$$Y_t^n = A_t \left(\frac{\epsilon - 1}{\epsilon}\right)^{\frac{\eta}{\eta+1}}, \tag{F.4}$$

and output gap $\hat{Y}_t$ is similarly defined as follows:

$$\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right). \tag{F.5}$$

Combining equations (F.2) to (F.5), we obtain:

$$\frac{w_t}{p_t A_t} = \left(\frac{\epsilon - 1}{\epsilon}\right) e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t}. \tag{F.6}$$

F.2 Firm Price Setting

The price p_t^i of an individual firm i evolves according to:

$$dp_t^i = \pi_t^i p_t^i dt. \tag{F.7}$$

From firm i's perspective, the current price p_t^i is taken as given and acts as a state variable when solving for π_t^i, which the firm controls at each point t.

We assume that firms face price nominal rigidities à la Rotemberg (1982), i.e., they need to pay convex adjustment costs whenever $\pi_t^i \neq 0$, given by:

$$\Theta(\pi_t^i) = \frac{\tau}{2}(\pi_t^i)^2 p_t Y_t.$$

For simplicity, we assume that this costs are rebated to the representative household in a lump-sum fashion, so no output ends up "missing" in the final equilibrium.

Therefore, nominal firm profits Ψ_t^i at time t are given by:

$$\begin{aligned}\Psi_t^i &= \left[p_t^i - \frac{w_t}{A_t}\right] D(p_t^i, p_t) - \Theta(\pi_t^i) \\ &= p_t Y_t \left[\left(\frac{p_t^i}{p_t}\right)^{1-\epsilon} - \frac{w_t}{p_t A_t}\left(\frac{p_t^i}{p_t}\right)^{-\epsilon} - \frac{\tau}{2}(\pi_t^i)^2\right].\end{aligned}$$

Let S_t^i be a vector containing the state variables of the firm's dynamic optimization problem. Therefore, it contains the current price of the firm, $p_t^i \in S_t^i$. We can express this vector of states as following an Ito process of the form:

$$dS_t^i = \mu_t^{S,i} dt + G_t^{S,i} dZ_t,$$

where $\mu_t^{S,i}$ and $G_t^{S,i}$ are vectors containing the drift and stochastic components of the states. For example, $\pi_t^i p_t^i$ is a component of vector $\mu_t^{S,i}$.

We define the value function of firm i, denoted $V(S_t^i)$, as the present discounted sum of the real profits of the firm on the optimal path, formally:

$$\begin{aligned} V(S_t^i) &= \max_{\{\pi_s^i\}_{s\geq t}} \mathbb{E}_t \int_t^\infty \frac{\xi_s^r}{\xi_t^r}\frac{\Psi_s^i}{p_s}ds \\ &= \max_{\{\pi_s^i\}_{s\geq t}} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)}\frac{C_t}{C_s}Y_s\left[\left(\frac{p_s^i}{p_s}\right)^{1-\epsilon} - \frac{w_s}{p_sA_s}\left(\frac{p_s^i}{p_s}\right)^{-\epsilon} - \frac{\tau}{2}(\pi_s^i)^2\right]ds, \end{aligned}$$

where $\xi_t^r = e^{-\rho t}\frac{1}{C_t}$ is the (real) state price density as defined in the main text. We then can write the Hamilton-Jacobi-Bellman (HJB) equation of the firm's problem as:

$$\rho V(S_t^i) = \max_{\{\pi_s^i\}_{s\geq t}} \left\{ Y_t\left[\left(\frac{p_t^i}{p_t}\right)^{1-\epsilon} - \frac{w_t}{p_tA_t}\left(\frac{p_t^i}{p_t}\right)^{-\epsilon} - \frac{\tau}{2}(\pi_t^i)^2\right] + \frac{E_t\left[dV(S_t^i)\right]}{dt}\right\}. \tag{F.8}$$

Based on Ito's lemma, we can express the process for $V(S_t^i)$ as:

$$dV(S_t^i) = \left[(\nabla_S V)^T\mu_t^{S,i} + \frac{1}{2}Tr\left[\left(G_t^{S,i}\right)^T(H_SV)\,G_t^{S,i}\right]\right]dt + (\nabla_S V)^T G_t^{S,i}dZ_t,$$

where $\nabla_S V$ and $H_S V$ stand for the gradient and Hessian of the value function V with respect to the vector of states S_t^i, respectively.

This allows to alternatively write the value function as:

$$\rho V(S_t^i) = \max_{\{\pi_s^i\}_{s\geq t}} \left\{ Y_t\left[\left(\frac{p_t^i}{p_t}\right)^{1-\epsilon} - \frac{w_t}{p_tA_t}\left(\frac{p_t^i}{p_t}\right)^{-\epsilon} - \frac{\tau}{2}(\pi_t^i)^2\right] + (\nabla_S V)^T\mu_t^{S,i} + \frac{1}{2}Tr\left[\left(G_t^{S,i}\right)^T(H_SV)\,G_t^{S,i}\right]\right\}. \tag{F.9}$$

Computing the first-order condition of equation (F.9) with respect to π_s^i, we obtain:

$$\frac{\partial V(S_t^i)}{\partial p_t^i} = \tau Y_t\frac{\pi_t^i}{p_t^i}. \tag{F.10}$$

Taking derivative of the HJB equation in (F.8) evaluated at the optimum with respect to the price of the individual firm, we obtain:

$$\rho\frac{\partial V_t^i}{\partial p_t^i} = Y_t\left[\epsilon\frac{w_t}{p_tA_t}\left(\frac{1}{p_t}\right)\left(\frac{p_t^i}{p_t}\right)^{-(\epsilon+1)} - (\epsilon-1)\left(\frac{1}{p_t}\right)\left(\frac{p_t^i}{p_t}\right)^{-\epsilon}\right] + \frac{E_t\left[d\left(\frac{\partial V_t^i}{\partial p_t^i}\right)\right]}{dt}. \tag{F.11}$$

Plugging equation (F.10) into (F.11), we obtain:

$$\rho\left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right) = Y_t\left[\epsilon \frac{w_t}{p_t A_t}\left(\frac{1}{p_t}\right)\left(\frac{p_t^i}{p_t}\right)^{-(\epsilon+1)} - (\epsilon-1)\left(\frac{1}{p_t}\right)\left(\frac{p_t^i}{p_t}\right)^{-\epsilon}\right] + \frac{E_t\left[d\left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right)\right]}{dt}. \tag{F.12}$$

Let us assume that individual firm inflation π_t^i follows a geometric process of the form:

$$d\pi_t^i = \mu_t^{\pi,i}\pi_t^i dt + \sigma_t^{\pi,i}\pi_t^i dZ_t. \tag{F.13}$$

Based on Ito's lemma, together with equations (F.3), (F.7) and (F.13), we obtain:

$$d\left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right) = \left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right)\left[\mu_t^Y + \mu_t^{\pi,i} - \pi_t^i + \sigma_t^{\pi,i}(\sigma + \sigma_t^s)\right]dt + \left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right)(\sigma + \sigma_t^s + \sigma_t^{\pi,i})dZ_t.$$

Taking expectations and using the fact that $\mu_t^{\pi,i} = \frac{\mathbb{E}_t\left[d\pi_t^i\right]}{\pi_t^i dt}$, we obtain:

$$\frac{\mathbb{E}_t\left[d\left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right)\right]}{dt} = \left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right)\left[\frac{\mathbb{E}_t\left[d\pi_t^i\right]}{\pi_t^i dt} + \mu_t^Y - \pi_t^i + \sigma_t^{\pi,i}(\sigma + \sigma_t^s)\right]. \tag{F.14}$$

Plugging equation (F.14) into equation (F.12) and rearranging, we obtain:

$$\begin{aligned}\rho\left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right) &= \left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right)\left(\frac{1}{\pi_t^i}\right)\left(\frac{\epsilon-1}{\tau}\right)\left[\left(\frac{\epsilon}{\epsilon-1}\right)\frac{w_t}{p_t A_t}\left(\frac{p_t^i}{p_t}\right)^{-1} - 1\right]\left(\frac{p_t^i}{p_t}\right)^{1-\epsilon} \\ &\quad + \left(\tau Y_t \frac{\pi_t^i}{p_t^i}\right)\left[\frac{\mathbb{E}_t\left[d\pi_t^i\right]}{\pi_t^i dt} + \mu_t^Y - \pi_t^i + \sigma_t^{\pi,i}(\sigma + \sigma_t^s)\right],\end{aligned}$$

which becomes

$$\mathbb{E}_t\left[d\pi_t^i\right] = \left[(\rho + \pi_t^i - \mu_t^Y - \sigma_t^{\pi,i}(\sigma + \sigma_t^s))\pi_t^i - \left(\frac{\epsilon-1}{\tau}\right)\left[\left(\frac{\epsilon}{\epsilon-1}\right)\frac{w_t}{p_t A_t}\left(\frac{p_t^i}{p_t}\right)^{-1} - 1\right]\left(\frac{p_t^i}{p_t}\right)^{1-\epsilon}\right]dt. \tag{F.15}$$

Under the symmetric equilibrium, i.e., $p_t^i = p_t, \forall i$, $\pi_t^i = \pi_t, \forall i$, and $\sigma_t^{\pi,i} = \sigma_t^{\pi}, \forall i$, equation

(F.15) becomes:

$$\mathbb{E}_t\left[d\pi_t\right]=\left[(\rho+\pi_t-\mu_t^Y-\sigma_t^\pi(\sigma+\sigma_t^s))\pi_t-\left(\frac{\epsilon-1}{\tau}\right)\left[\left(\frac{\epsilon}{\epsilon-1}\right)\frac{w_t}{p_tA_t}-1\right]\right]dt. \tag{F.16}$$

Finally, we plug equation (F.6) into (F.16) and obtain

$$\mathbb{E}_t\left[d\pi_t\right]=\left[(\rho+\pi_t-\mu_t^Y-\sigma_t^\pi(\sigma+\sigma_t^s))\pi_t-\left(\frac{\epsilon-1}{\tau}\right)\left[e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t}-1\right]\right]dt. \tag{F.17}$$

which is the New Keynesian Phillips curve in our environment.

We can further replace the expression for μ_t^Y in equation (F.3) into (F.17), and use equation (F.13) to obtain:

$$d\pi_t=\left[\left[2(\rho+\pi_t)-i_t-(\sigma+\sigma_t^s)(\sigma+\sigma_t^s+\sigma_t^\pi)\right]\pi_t-\left(\frac{\epsilon-1}{\tau}\right)\left[e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t}-1\right]\right]dt+\sigma_t^\pi\pi_t dZ_t. \tag{F.18}$$

Interpretation Equation (F.17) can be interpreted as follows. An increase in μ_t^Y prompts households to borrow against the future and raise current consumption, leading to higher aggregate demand and higher current inflation while reducing $d\pi_t$ on average. Similarly, a rise in current income, $\hat{Y}_t$, increases inflation now and lowers $d\pi_t$ on average.

Moreover, a higher $\sigma_t^\pi(\sigma+\sigma_t^s)$ implies that inflation is typically higher when aggregate output is higher too. It raises the expected marginal price-adjustment cost (see equation (F.10)) in the future (after dt period), inducing firms to raise inflation now and leading to higher $d\pi_t$ on average.

Linearization By approximating $(\pi_t-\mu_t^Y-\sigma_t^\pi(\sigma+\sigma_t^s))\pi_t\simeq 0$ and using

$$e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t}-1\simeq\left(\frac{\eta+1}{\eta}\right)\hat{Y},$$

equation (F.17) becomes

$$\mathbb{E}_t(d\pi_t)=\left(\rho\pi_t-\kappa\hat{Y}_t\right)dt,\quad\text{where}\quad\kappa\equiv\frac{(\epsilon-1)(\eta+1)}{\tau\eta}>0,$$

leading to a standard continuous-time New Keynesian Phillips curve, e.g., Werning (2012).

F.3 Equilibrium with Inflation and Endogenous Volatility

In this section, we prove that our main result—namely, that an equilibrium with endogenous $\sigma_t^s \neq 0$ exists unless monetary policy is adjusted to follow the rule in Section 4—holds even in the presence of inflation. We base our proof on the derivation of the Phillips curve in equation (F.18).

We work with the following three-equations system of the non-linear New Keynesian model with pricing à la Rotemberg (1982), given by:

$$d\pi_t = \left[\left[2(\rho + \pi_t) - i_t - (\sigma + \sigma_t^s)(\sigma + \sigma_t^s + \sigma_t^\pi)\right] \pi_t - \left(\frac{\epsilon - 1}{\tau}\right) \left[e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t} - 1\right] \right] dt + \sigma_t^\pi \pi_t dZ_t, \tag{F.19}$$

$$d\hat{Y}_t = \left[i_t - \pi_t - \left(r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2 \right) \right] dt + \sigma_t^s dZ_t, \tag{F.20}$$

with monetary policy given by

$$i_t = r^n + \phi_y \hat{Y}_t + \phi_\pi \pi_t - \phi_{vol} \left[(\sigma + \sigma_t^s)^2 - \sigma^2 \right], \tag{F.21}$$

where the natural rate of interest is given by $r^n = \rho + g - \sigma^2$.

Plugging monetary policy (F.21) into (F.19) and (F.20), we obtain:

$$\begin{aligned} d\pi_t = \Bigg[& \left[2\rho - r^n - \phi_{vol}\sigma^2 + (2 - \phi_\pi)\pi_t - \phi_y \hat{Y}_t - (\sigma + \sigma_t^s) \left[(1 - \phi_{vol})(\sigma + \sigma_t^s) + \sigma_t^\pi \right] \right] \pi_t \\ & - \left(\frac{\epsilon - 1}{\tau}\right) \left[e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t} - 1 \right] \Bigg] dt + \sigma_t^\pi \pi_t dZ_t, \end{aligned} \tag{F.22}$$

and

$$d\hat{Y}_t = \left[\phi_y \hat{Y}_t + (\phi_\pi - 1)\pi_t + \left(\frac{1}{2} - \phi_{vol} \right) \left[(\sigma + \sigma_t^s)^2 - \sigma^2 \right] \right] dt + \sigma_t^s dZ_t. \tag{F.23}$$

In a manner similar to the baseline case with fully rigid prices, we now conjecture a solution to the system given by equations (F.22) and (F.23) and verify that the conjectured equilibrium indeed holds. Because the system now includes inflation π_t, we introduce an additional assumption involving π_t. Formally, we conjecture:

$$d\hat{Y}_t = \theta\big[\mu - \hat{Y}_t\big]dt + \sigma_t^s dZ_t, \tag{F.24}$$

$$\pi_t = f(\sigma_t^s), \tag{F.25}$$

where (F.24) is the usual Ornstein-Uhlenbeck conjecture for the output gap process. Equation (F.25) conjectures that inflation is a smooth function of excess volatility, σ_t^s. Our goal is to prove that there exists a smooth function $f(\cdot)$ such that (F.24) and (F.25) jointly constitute an equilibrium.

Combining equations (F.23), (F.24) and (F.25), we obtain an expression for output gap as:

$$\hat{Y}_t = \left(\frac{\theta}{\theta+\phi_y}\right)\mu - \left(\frac{1}{\theta+\phi_y}\right)\left[(\phi_\pi - 1)f(\sigma_t^s) + \left(\frac{1}{2} - \phi_{vol}\right)\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right]. \tag{F.26}$$

Plugging equations (F.26) and (F.25) into equations (F.22) and (F.24), we obtain:

$$d\pi_t = \left[\left[\psi_1 + \psi_2 f(\sigma_t^s) - (\sigma+\sigma_t^s)\left[\psi_3(\sigma+\sigma_t^s) + \sigma_t^\pi\right]\right]f(\sigma_t^s) - \psi_4\left[e^{\psi_5 - \psi_6 f(\sigma_t^s) - \psi_7(\sigma+\sigma_t^s)^2} - 1\right]\right]dt$$
$$+ \sigma_t^\pi f(\sigma_t^s)dZ_t, \tag{F.27}$$

and

$$d\hat{Y}_t = \left[\psi_8 + \psi_9 f(\sigma_t^s) + \psi_{10}(\sigma+\sigma_t^s)^2\right]dt + \sigma_t^s dZ_t, \tag{F.28}$$

where the constants are defined as:

$$\psi_1 = 2\rho - r^n - \left[\left(\frac{\theta}{\theta+\phi_y}\right)\phi_{vol} + \frac{1}{2}\left(\frac{\phi_y}{\theta+\phi_y}\right)\right]\sigma^2 - \left(\frac{\theta\phi_y}{\theta+\phi_y}\right)\mu,$$
$$\psi_2 = 1 + \left(\frac{\theta}{\theta+\phi_y}\right)(1-\phi_\pi), \quad \psi_3 = \left(\frac{\theta}{\theta+\phi_y}\right)(1-\phi_{vol}) + \left(\frac{\phi_y}{\theta+\phi_y}\right)\frac{1}{2},$$
$$\psi_4 = \left(\frac{\epsilon-1}{\tau}\right), \quad \psi_5 = \left(\frac{\eta+1}{\eta}\right)\left(\frac{\theta}{\theta+\phi_y}\right)\mu + \left(\frac{\eta+1}{\eta}\right)\left(\frac{1}{\theta+\phi_y}\right)\left(\frac{1}{2}-\phi_{vol}\right)\sigma^2,$$
$$\psi_6 = \left(\frac{\eta+1}{\eta}\right)\left(\frac{\phi_\pi-1}{\theta+\phi_y}\right), \quad \psi_7 = \left(\frac{\eta+1}{\eta}\right)\left(\frac{1}{\theta+\phi_y}\right)\left(\frac{1}{2}-\phi_{vol}\right),$$
$$\psi_8 = \left(\frac{\theta\phi_y}{\theta+\phi_y}\right)\mu - \left(\frac{\theta}{\theta+\phi_y}\right)\left(\frac{1}{2}-\phi_{vol}\right)\sigma^2, \quad \psi_9 = \left(\frac{\theta}{\theta+\phi_y}\right)(\phi_\pi - 1),$$
$$\psi_{10} = \left(\frac{\theta}{\theta+\phi_y}\right)\left(\frac{1}{2}-\phi_{vol}\right).$$

Equation (F.26) can be rewritten as $N(\hat{Y}_t, \sigma_t^s) = 0$, where:

$$N(\hat{Y}_t, \sigma_t^s) \equiv \hat{Y}_t - \left(\frac{\theta}{\theta+\phi_y}\right)\mu + \left(\frac{1}{\theta+\phi_y}\right)\left[(\phi_\pi - 1)f(\sigma_t^s) + \left(\frac{1}{2} - \phi_{vol}\right)\left[(\sigma+\sigma_t^s)^2 - \sigma^2\right]\right],$$

which implicitly determines σ_t^s as a function of $\hat{Y}_t$. To compute the implicit derivatives, we compute the following, with $N_{\hat{Y}} = 1, N_{\hat{Y}\hat{Y}} = 0, N_{\hat{Y}\sigma^s} = 0$:

$$N_{\sigma^s} = \left(\frac{1}{\theta+\phi_y}\right)\left[(\phi_\pi - 1)f'(\sigma_t^s) + (1-2\phi_{vol})(\sigma+\sigma_t^s)\right],$$
$$N_{\sigma^s\sigma^s} = \left(\frac{1}{\theta+\phi_y}\right)\left[(\phi_\pi - 1)f''(\sigma_t^s) + (1-2\phi_{vol})\right],$$

from which we can compute the implicit derivative of σ_t^s with respect to $\hat{Y}_t$ as:

$$\frac{\partial\sigma_t^s}{\partial\hat{Y}_t} = -\frac{N_{\hat{Y}}}{N_{\sigma^s}} = -\left[\frac{\theta+\phi_y}{(\phi_\pi - 1)f'(\sigma_t^s) + (1-2\phi_{vol})(\sigma+\sigma_t^s)}\right], \tag{F.29}$$

and the second-order implicit derivative as:

$$\frac{\partial^2\sigma_t^s}{\partial^2\hat{Y}_t} = -(\theta+\phi_y)^2\left[\frac{(\phi_\pi - 1)f''(\sigma_t^s) + (1-2\phi_{vol})}{\left[(\phi_\pi - 1)f'(\sigma_t^s) + (1-2\phi_{vol})(\sigma+\sigma_t^s)\right]^3}\right]. \tag{F.30}$$

Using Ito's lemma, together with equations (F.28), (F.29), and (F.30), we obtain the following expression for the process of σ_t^s:

$$\begin{aligned} d\sigma_t^s = &-\left[\psi_{11}\frac{\psi_8 + \psi_9 f(\sigma_t^s) + \psi_{10}(\sigma+\sigma_t^s)^2}{\psi_{12}f'(\sigma_t^s) + \psi_{13}(\sigma+\sigma_t^s)} + \psi_{14}\frac{\psi_{12}f''(\sigma_t^s) + \psi_{13}}{\left[\psi_{12}f'(\sigma_t^s) + \psi_{13}(\sigma+\sigma_t^s)\right]^3}(\sigma_t^s)^2\right]dt \\ &- \left[\frac{\psi_{11}}{\psi_{12}f'(\sigma_t^s) + \psi_{13}(\sigma+\sigma_t^s)}\right]\sigma_t^s dZ_t, \end{aligned} \tag{F.31}$$

where the constants are defined as:

$$\psi_{11} = \theta + \psi_y,$$
$$\psi_{12} = \phi_\pi - 1,$$
$$\psi_{13} = 1 - 2\phi_{vol},$$
$$\psi_{14} = \frac{(\theta+\psi_y)^2}{2}.$$

Next, applying Ito's lemma to equation (F.25) and using equation (F.31), we obtain:

$$
\begin{aligned}
d\pi_t = &- \left[\psi_{11} f'(\sigma_t^s) \frac{\psi_8 + \psi_9 f(\sigma_t^s) + \psi_{10}(\sigma + \sigma_t^s)^2}{\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)} + \psi_{14} \frac{\psi_{13}\,(\sigma_t^s)^2}{\left[\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)\right]^3} \left(f'(\sigma_t^s) - f''(\sigma_t^s)(\sigma + \sigma_t^s)\right)\right] dt \\
&- \left[\frac{\psi_{11} f'(\sigma_t^s)}{\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)}\right] \sigma_t^s dZ_t,
\end{aligned}
\tag{F.32}
$$

which expresses the process of inflation π_t as a sole function of σ_t^s. Now, we can compare (F.32) with the process described by the Phillips curve in equation (F.27). By comparing the stochastic components of equations (F.27) and (F.32), we find that the volatility of inflation must be equal to:

$$
\sigma_t^\pi = -\left[\frac{\psi_{11} f'(\sigma_t^s)}{\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)}\right] \left(\frac{\sigma_t^s}{f(\sigma_t^s)}\right). \tag{F.33}
$$

Finally, comparing the trend components of equations (F.27) and (F.32), and substituting equation (F.33) where necessary, we obtain that the following condition must be satisfied:

$$
\begin{aligned}
&- \left[\psi_{11} f'(\sigma_t^s) \frac{\psi_8 + \psi_9 f(\sigma_t^s) + \psi_{10}(\sigma + \sigma_t^s)^2}{\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)} + \psi_{14} \frac{\psi_{13}\,(\sigma_t^s)^2}{\left[\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)\right]^3} \left(f'(\sigma_t^s) - f''(\sigma_t^s)(\sigma + \sigma_t^s)\right)\right] \\
&= \left[\psi_1 + \psi_2 f(\sigma_t^s) - (\sigma + \sigma_t^s)\left[\psi_3(\sigma + \sigma_t^s) - \left[\frac{\psi_{11} f'(\sigma_t^s)}{\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)}\right]\left(\frac{\sigma_t^s}{f(\sigma_t^s)}\right)\right]\right] f(\sigma_t^s) \\
&\quad - \psi_4 \left[e^{\psi_5 - \psi_6 f(\sigma_t^s) - \psi_7(\sigma + \sigma_t^s)^2} - 1\right],
\end{aligned}
$$

which can be rearranged as

$$
f''(\sigma_t^s) = \underbrace{\frac{f'(\sigma_t^s)}{\sigma + \sigma_t^s} + \frac{\left[\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)\right]^3}{\psi_{14}\psi_{13}(\sigma_t^s)^2(\sigma + \sigma_t^s)} \left[\begin{aligned} &\psi_{11} f'(\sigma_t^s) \frac{\psi_8 + \psi_9 f(\sigma_t^s) + \psi_{10}(\sigma + \sigma_t^s)^2}{\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)} \\ &+ \left[\psi_1 + \psi_2 f(\sigma_t^s) \right. \\ &\quad \left. - (\sigma + \sigma_t^s)\left[\psi_3(\sigma + \sigma_t^s) - \left[\frac{\psi_{11} f'(\sigma_t^s)}{\psi_{12} f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)}\right] \frac{\sigma_t^s}{f(\sigma_t^s)}\right]\right] f(\sigma_t^s) \\ &- \psi_4 \left[e^{\psi_5 - \psi_6 f(\sigma_t^s) - \psi_7(\sigma + \sigma_t^s)^2} - 1\right] \end{aligned}\right]}_{\equiv K(\sigma_t^s, f(\sigma_t^s), f'(\sigma_t^s))} \tag{F.34}
$$

Define the following vector of variables $x(\sigma_t^s) = [x_1(\sigma_t^s), x_2(\sigma_t^s)]'$, with elements defined as:

$$x_1(\sigma_t^s) = f(\sigma_t^s),$$
$$x_2(\sigma_t^s) = f'(\sigma_t^s).$$

Then the ordinary differential equation (ODE) in (F.34) can be rewritten as the following first-order system:

$$\begin{cases} x_1'(\sigma_t^s) & = x_2(\sigma_t^s), \\ x_2'(\sigma_t^s) & = K\left(\sigma_t^s, x_1(\sigma_t^s), x_2(\sigma_t^s)\right). \end{cases}$$

In vectorized form, we can define $\tilde{K}\left(\sigma_t^s, x(\sigma_t^s)\right) = [x_2(\sigma_t^s), K\left(\sigma_t^s, x_1(\sigma_t^s), x_2(\sigma_t^s)\right)]'$, so that:

$$x'(\sigma_t^s) = \tilde{K}\left(\sigma_t^s, x(\sigma_t^s)\right).$$

As $\tilde{K}(\sigma_t^s, x_1, x_2)$ is continuous in a closed "box" (or "rectangle") around any $(\sigma_0^s,\ x_1(\sigma_0^s),\ x_2(\sigma_0^s))$, we can apply Peano's theorem (Walter, 1973), which guarantees that there exists at least one local solution $\big(x_1(\sigma_t^s), x_2(\sigma_t^s)\big)$ that satisfies the ordinary differential equation (F.34). Therefore, $f(\sigma_t^s) = x_1(\sigma_t^s)$ is a local solution to the original second-order ordinary differential equation (F.34). Since this proof holds over the entire domain of σ_t^s, it guarantees the existence of a function $f(\cdot)$ such that the output gap follows the process in (F.24) for any value of σ_t^s.

This solution differs from the perfectly stabilized path, defined by $\sigma_t^s = \pi_t = \hat{Y}_t = 0$ for all t. In particular, setting $\pi_t = \sigma_t^s = 0$ contradicts equation (F.31), which implies that $d\sigma_t^q$ generally exhibits a nonzero drift when evaluated at zero excess volatility. This contradicts our assumption that $\sigma_t^s = 0$ for all t. Hence, the conjectured equilibrium in (F.24) and (F.25) is distinct from the perfectly stabilized path.

G Model with Sticky Prices à la Calvo (1983)

In this section, we derive a New Keynesian Phillips curve based on nominal rigidities à la Calvo (1983). First, we present the equilibrium conditions when firms are monopolistically competitive as in a canonical New Keynesian model (Woodford, 2003).

G.1 Equilibrium Conditions

Firm i setting its price p_t^i when p_t is the price aggregator faces the demand given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\epsilon} Y_t.$$

and the following production function:

$$Y_t^i = A_t L_t^i, \tag{G.1}$$

with demand equal to supply in equilibrium, $D(p_t^i, p_t) = Y_t^i$. The price aggregatior is given by

$$p_t = \left(\int_0^1 (p_t^i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}. \tag{G.2}$$

Other equilibrium conditions remain the same. The optimality conditions for households are given by

$$\frac{1}{p_t C_t} = \frac{L_t^{1/\eta}}{w_t}, \tag{G.3}$$

and

$$\frac{dY_t}{Y_t} = \underbrace{\left(i_t + (\sigma + \sigma_t^s)^2 - \pi_t - \rho\right)}_{\equiv \mu_t^Y} dt + (\sigma + \sigma_t^s) dZ_t. \tag{G.4}$$

In equilibrium, $Y_t = C_t$ holds. Natural output Y_t^n is given by

$$Y_t^n = A_t \left(\frac{\epsilon - 1}{\epsilon}\right)^{\frac{\eta}{\eta+1}},$$

and output gap $\hat{Y}_t$ is similarly defined as follows:

$$\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right). \tag{G.5}$$

Combining equations (G.3) to (G.5), we obtain:

$$d\hat{Y}_t = \left[i_t - \pi_t - \left(r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2\right)\right] dt + \sigma_t^s dZ_t, \tag{G.6}$$
$$\frac{w_t}{p_t A_t} = \left(\frac{\epsilon - 1}{\epsilon}\right) e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t}.$$

We define price dispersion as:

$$\Delta_t = \int_0^1 \left(\frac{p_t^i}{p_t}\right)^{-\epsilon} di, \tag{G.7}$$

and (G.1) and the linear aggregation of labor, i.e.,$L_t = \int_0^1 L_t^i di$, we obtain:

$$Y_t = \frac{A_t L_t}{\Delta_t}.$$

Finally, we conjecture that the aggregate price follows a diffusion process of the form:

$$dp_t = \pi_t p_t dt + \sigma_t^p p_t dZ_t \tag{G.8}$$

where π_t stands for inflation, and σ_t^p is a potentially endogenous and unknown price volatility.

G.2 Firms Problem

Firms set prices following Calvo (1983), with the framework adapted to continuous time. Over an interval of length dt, from t to $t + dt$, an individual firm i adjusts its price with probability $\delta\, dt$. From the perspective of time 0, the probability that a firm resets its price for the first time at time t is:

$$\delta e^{-\delta t} dt = \underbrace{\delta dt}_{\text{change now}} \cdot \underbrace{e^{-\delta t}}_{\text{No change until } t}.$$

Formally, we can describe the evolution of individual firm prices as jump processes:

$$dp_t^i = \left(p_t^{i,*} - p_{t-}^i\right) d\Lambda_t^i, \quad \text{where: } d\Lambda_t^i = \begin{cases} 1, & \text{with probability } \delta dt \\ 0, & \text{with probability } 1 - \delta dt \end{cases}, \tag{G.9}$$

where $d\Lambda_t^i$ is an i.i.d Poisson random variable, with rate parameter $\delta \geq 0$, p_{t-}^i stands for the individual price of the firm just before t, and $p_t^{i,*}$ stands for the optimal reset price for firm i at time t.

Nominal firm profits Ψ_t^i at time t are given by:

$$\begin{aligned}\Psi_t^i &= \left[p_t^i - \frac{w_t}{A_t}\right] D(p_t^i, p_t) \\ &= p_t Y_t \left[\left(\frac{p_t^i}{p_t}\right)^{1-\epsilon} - \frac{w_t}{p_t A_t}\left(\frac{p_t^i}{p_t}\right)^{-\epsilon}\right].\end{aligned}$$

Define $\Psi_{s|t}^i$ as the nominal profits at time $s \geq t$ of an individual firm i that last reset its prices in time t, formally:

$$\Psi_{s|t}^i = p_s Y_s \left[\left(\frac{p_t^i}{p_s}\right)^{1-\epsilon} - \frac{w_s}{p_s A_s}\left(\frac{p_t^i}{p_s}\right)^{-\epsilon}\right].$$

At time t, a price-changing firm i chooses $p_t^{i,*}$ to solve

$$\max_{p_t^{i,*}} \mathbb{E}_t \int_t^\infty e^{-\delta(s-t)} \frac{\xi_s^r}{\xi_t^r} \frac{\Psi_{s|t}^i}{p_s} ds = \mathbb{E}_t \int_t^\infty e^{-(\rho+\delta)(s-t)} \frac{C_t}{C_s} Y_s \left[\left(\frac{p_t^{i,*}}{p_s}\right)^{1-\epsilon} - \frac{w_s}{p_s A_s}\left(\frac{p_t^{i,*}}{p_s}\right)^{-\epsilon}\right] ds,$$

where $\xi_t^r = e^{-\rho t}\frac{1}{C_t}$ is the (real) state price density as defined in the main text.

Computing the first-order condition with respect to $p_t^{i,*}$ and rearranging, we obtain:

$$\frac{p_t^*}{p_t} \equiv \frac{p_t^{i,*}}{p_t} = \frac{\overbrace{\mathbb{E}_t \int_t^\infty e^{-(\rho+\delta)(s-t)} \left(\frac{p_s}{p_t}\right)^{\epsilon} e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_s} ds}^{\equiv F_t}}{\underbrace{\mathbb{E}_t \int_t^\infty e^{-(\rho+\delta)(s-t)} \left(\frac{p_s}{p_t}\right)^{\epsilon-1} ds}_{\equiv G_t}} = \frac{F_t}{G_t}, \tag{G.10}$$

where it follows that the optimal reset price is the same for all firms, $p_t^{i,*} = p_t^*$ for all i.

Based on the Hamilton-Jacobi-Bellman (HJB) method, we can find a recursive expression for F_t and G_t as:

$$
\begin{aligned}
(\rho+\delta)F_t &= e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_s} + \frac{E_t\left[dF_t\right]}{dt}, \\
(\rho+\delta)G_t &= 1 + \frac{E_t\left[dG_t\right]}{dt}.
\end{aligned}
$$

This can be rewritten as:

$$
dF_t = \underbrace{\left[(\rho+\delta) - \frac{1}{F_t}e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_s}\right]}_{\equiv\mu_t^F} F_t dt + \sigma_t^F F_t dZ_t, \tag{G.11}
$$

$$
dG_t = \underbrace{\left[(\rho+\delta) - \frac{1}{G_t}\right]}_{\equiv\mu_t^G} G_t dt + \sigma_t^G G_t dZ_t, \tag{G.12}
$$

where σ_t^F and σ_t^G are endogenous unknown variables.

G.3 Price Process

Using equation (G.2), we can find an expression for the following derivatives:

$$
\begin{aligned}
\frac{dp_t}{d\left(\int_0^1 (p_t^i)^{1-\epsilon}di\right)} &= -\left(\frac{1}{\epsilon-1}\right)p_t^{\epsilon}, \\
\frac{d^2 p_t}{d^2\left(\int_0^1 (p_t^i)^{1-\epsilon}di\right)} &= \frac{\epsilon}{(\epsilon-1)^2}p_t^{2\epsilon-1},
\end{aligned}
$$

which we can use to obtain an expression for the price process as:

$$
dp_t = -\left(\frac{1}{\epsilon-1}\right)p_t^{\epsilon} d\left(\int_0^1 (p_t^i)^{1-\epsilon}di\right) + \frac{1}{2}\frac{\epsilon}{(\epsilon-1)^2}p_t^{2\epsilon-1}\left[d\left(\int_0^1 (p_t^i)^{1-\epsilon}di\right)\right]^2. \tag{G.13}
$$

Now notice that by the individual price process in (G.9) and $p_t^{i,*} = p^*$, we have that:

$$
d(p_t^i)^{1-\epsilon} = \left[(p_t^*)^{1-\epsilon} - (p_{t-}^i)^{1-\epsilon}\right] d\Lambda_t^i
$$

Then, we can compute:

$$\begin{aligned} d\int_0^1 (p_t^i)^{1-\epsilon} di &= \int_0^1 d(p_t^i)^{1-\epsilon} di = \mathbb{E}_{i,t}\left[d(p_t^i)^{1-\epsilon}\right] \\ &= \delta\left[(p_t^*)^{1-\epsilon} - (p_t)^{1-\epsilon}\right] dt = \delta (p_t)^{1-\epsilon}\left[\left(\frac{p_t^*}{p_t}\right)^{1-\epsilon} - 1\right] dt. \end{aligned} \tag{G.14}$$

Plugging equations (G.10) and (G.14) into equation (G.13) and eliminating all terms of order higher than dt, we obtain

$$dp_t = \left(\frac{\delta}{\epsilon - 1}\right)\left[1 - \left(\frac{F_t}{G_t}\right)^{1-\epsilon}\right] p_t dt.$$

which validates the conjecture in equation (G.8) with $\sigma_t^p = 0$ and inflation given by:

$$\pi_t = \left(\frac{\delta}{\epsilon - 1}\right)\left[1 - \left(\frac{F_t}{G_t}\right)^{1-\epsilon}\right]. \tag{G.15}$$

For later use, we can rearrange the previous expression as:

$$\frac{F_t}{G_t} = \left[1 - \left(\frac{\epsilon - 1}{\delta}\right)\pi_t\right]^{\frac{1}{1-\epsilon}}. \tag{G.16}$$

G.4 Price Dispersion

From equation (G.7), we observe that price dispersion can be alternatively interpreted as a cross-sectional expectation on price dispersion

$$\Delta_t = \mathbb{E}_{i,t}\left[\left(\frac{p_t^i}{p_t}\right)^{-\epsilon}\right],$$

where $\mathbb{E}_{i,t}$ stands for the expectations operator over the cross section i. As reset prices $p_t^{i,*}$ are the same for firms resetting on the same instant, i.e., $p_t^{i,*} = p_t^*$,[13] we obtain

$$\Delta_t = \int_{-\infty}^t \delta e^{-\delta(t-s)}\left(\frac{p_s}{p_t}\right)^{-\epsilon} ds. \tag{G.17}$$

[13] See Woodford (2003) for the derivation of (G.17).

We can now differentiate equation (G.17) with respect to time to obtain:

$$
\begin{aligned}
\frac{d\Delta}{dt} &= \underbrace{\delta e^{-\delta(t-t)}\left(\frac{p_t}{p_t}\right)^{-\epsilon}}_{\text{boundary term}} + \int_{-\infty}^{t} \frac{d}{dt}\left\{\delta e^{-\delta(t-s)}\left(\frac{p_s}{p_t}\right)^{-\epsilon}\right\} ds \\
&= \delta + \left[\epsilon\left(\frac{1}{p_t}\right)\frac{dp_t}{dt} - \delta\right]\int_{-\infty}^{t}\delta e^{-\delta(t-s)}\left(\frac{p_s}{p_t}\right)^{-\epsilon} ds \\
&= \delta + \left[\epsilon\pi_t - \delta\right]\Delta_t,
\end{aligned} \tag{G.18}
$$

where the last line follows from equations (G.17) and (G.8). We can rearrange (G.18) further as:

$$
d\Delta_t = \left[\delta(1-\Delta_t) + \epsilon\pi_t\Delta_t\right] dt.
$$

G.5 Inflation Process

We now compute the following derivatives from (G.15) and (G.16):

$$
\begin{aligned}
\frac{\partial\pi_t}{\partial F_t} &= \delta\left(\frac{F_t}{G_t}\right)^{1-\epsilon}\left(\frac{1}{F_t}\right) = [\delta-(\epsilon-1)\pi_t]\left(\frac{1}{F_t}\right), \\
\frac{\partial\pi_t}{\partial G_t} &= -\delta\left(\frac{F_t}{G_t}\right)^{1-\epsilon}\left(\frac{1}{G_t}\right) = -\left[\delta-(\epsilon-1)\pi_t\right]\left(\frac{1}{G_t}\right), \\
\frac{\partial^2\pi_t}{\partial F_t^2} &= -\epsilon\delta\left(\frac{F_t}{G_t}\right)^{1-\epsilon}\left(\frac{1}{F_t}\right)^2 = -\epsilon\left[\delta-(\epsilon-1)\pi_t\right]\left(\frac{1}{F_t}\right)^2, \\
\frac{\partial^2\pi_t}{\partial G_t^2} &= -(\epsilon-2)\delta\left(\frac{F_t}{G_t}\right)^{1-\epsilon}\left(\frac{1}{G_t}\right)^2 = -(\epsilon-2)\left[\delta-(\epsilon-1)\pi_t\right]\left(\frac{1}{G_t}\right)^2, \\
\frac{\partial^2\pi_t}{\partial F_t\partial G_t} &= (\epsilon-1)\delta\left(\frac{F_t}{G_t}\right)^{1-\epsilon}\left(\frac{1}{F_tG_t}\right) = (\epsilon-1)\left[\delta-(\epsilon-1)\pi_t\right]\left(\frac{1}{F_tG_t}\right).
\end{aligned}
$$

Using equations (G.11), (G.12), (G.15) and the derivatives above together with Ito's Lemma to find an expression for the inflation process:

$$
\begin{aligned}
d\pi_t = &\left[\delta-(\epsilon-1)\pi_t\right]\left[(\mu_t^F-\mu_t^G) - \frac{1}{2}\left[\epsilon(\sigma^F)^2 + (\epsilon-2)(\sigma_t^G)^2\right] + (\epsilon-1)\sigma_t^F\sigma_t^G\right] dt \\
&+ \left[\delta-(\epsilon-1)\pi_t\right](\sigma_t^F-\sigma_t^G)dZ_t,
\end{aligned}
$$

which, after substituting the expressions for μ_t^F and μ_t^G in (G.11) and (G.12), becomes

$$d\pi_t = [\delta - (\epsilon - 1)\pi_t] \left[\frac{1}{G_t} \left[1 - \frac{1}{\delta^{\frac{1}{\epsilon-1}}} [\delta - (\epsilon-1)\pi_t]^{\frac{1}{\epsilon-1}} e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t} \right] - \frac{1}{2} \left[(\epsilon-1)(\sigma_t^F - \sigma_t^G)^2 + (\sigma_t^F + \sigma_t^G)(\sigma_t^F - \sigma_t^G) \right] \right] dt$$
$$+ [\delta - (\epsilon-1)\pi_t] (\sigma_t^F - \sigma_t^G) dZ_t,$$

which corresponds to the non-linear Phillips curve under Calvo (1983) pricing.

Note that the volatility term $(\sigma_t^F - \sigma_t^G)$ appears in the drift as in the dynamic IS equation (G.4).

G.6 Solving the Model

Formally, we conjecture:

$$d\hat{Y}_t = \theta^y \left[\mu^y - \hat{Y}_t \right] dt + \sigma_t^s dZ_t, \tag{G.19}$$

$$d\pi_t = \theta^\pi \left[\mu^\pi - \pi_t \right] dt + \sigma_t^\pi dZ_t, \tag{G.20}$$

$$\pi_t = f(\sigma_t^s), \tag{G.21}$$

$$\mathbb{E}_t \left[dG_t \right] = 0. \tag{G.22}$$

Therefore, our conjecture is that output gap and inflation follow an Ornstein-Uhlenbeck processes, while the process G_t is a martingale. We conjecture that inflation depends only on excess volatility. Our objective is to prove the existence of a smooth function $f(\cdot)$ such that equations (G.19), (G.20), (G.21), and (G.22) jointly characterize an equilibrium.

Comparing the drift terms of (G.12) and (G.22), we obtain:

$$G_t = \frac{1}{\rho + \delta},$$

which implies $dG_t = 0$ and $\sigma_t^G = 0$. From equation (G.16), we obtain

$$F_t = \left(\frac{1}{\delta^{\frac{1}{1-\epsilon}} (\rho + \delta)} \right) [\delta - (\epsilon - 1)\pi_t]^{\frac{1}{1-\epsilon}}. \tag{G.23}$$

Computing the derivatives of F_t with respect to π_t:

$$\frac{\partial F_t}{\partial \pi_t} = \left(\frac{1}{\delta^{\frac{1}{1-\epsilon}}(\rho+\delta)}\right)[\delta-(\epsilon-1)\pi_t]^{\frac{\epsilon}{1-\epsilon}},$$
$$\frac{\partial^2 F_t}{\partial^2 \pi_t} = \left(\frac{\epsilon}{\delta^{\frac{1}{1-\epsilon}}(\rho+\delta)}\right)[\delta-(\epsilon-1)\pi_t]^{\frac{2\epsilon-1}{1-\epsilon}}.$$

Now we apply Ito's Lemma using equations (G.20), (G.23) and the derivatives above to obtain:

$$\begin{aligned} dF_t = &\left(\frac{1}{\delta^{\frac{1}{1-\epsilon}}(\rho+\delta)}\right)[\delta-(\epsilon-1)\pi_t]^{\frac{\epsilon}{1-\epsilon}}\left[\theta^\pi[\mu^\pi-\pi_t]+\frac{\epsilon}{2}\frac{(\sigma_t^\pi)^2}{\delta-(\epsilon-1)\pi_t}\right]dt \\ &+\left(\frac{1}{\delta^{\frac{1}{1-\epsilon}}(\rho+\delta)}\right)[\delta-(\epsilon-1)\pi_t]^{\frac{\epsilon}{1-\epsilon}}\sigma_t^\pi dZ_t. \end{aligned} \tag{G.24}$$

Equating the diffusion terms in (G.11) and (G.24), and using (G.23), we obtain:

$$\sigma_t^F = \frac{\sigma_t^\pi}{\delta-(\epsilon-1)\pi_t}.$$

Equating the drift terms in (G.11) and (G.24), we obtain:

$$F_t = \left(\frac{1}{\rho+\delta}\right)\left[\left(\frac{1}{\delta^{\frac{1}{1-\epsilon}}(\rho+\delta)}\right)[\delta-(\epsilon-1)\pi_t]^{\frac{\epsilon}{1-\epsilon}}\left[\theta^\pi[\mu^\pi-\pi_t]+\frac{\epsilon}{2}\frac{(\sigma_t^\pi)^2}{\delta-(\epsilon-1)\pi_t}\right]+e^{\left(\frac{\eta+1}{\eta}\right)\hat{Y}_t}\right]. \tag{G.25}$$

Combining equations (G.6), (G.19) and (G.21), we obtain an expression for output gap as:

$$\hat{Y}_t = \left(\frac{\theta^y}{\theta^y+\phi_y}\right)\mu^y - \left(\frac{1}{\theta^y+\phi_y}\right)\left[(\phi_\pi-1)f(\sigma_t^s)+\left(\frac{1}{2}-\phi_{vol}\right)\left[(\sigma+\sigma_t^s)^2-\sigma^2\right]\right]. \tag{G.26}$$

Plugging equations (G.21), (G.23), and (G.26) into (G.25), we obtain:

$$\varphi_1[\delta-\varphi_2 f(\sigma_t^s)] = \theta^\pi[\mu^\pi-f(\sigma_t^s)]+\frac{\epsilon}{2}\left[\frac{(\sigma_t^\pi)^2}{\delta-\varphi_2 f(\sigma_t^s)}\right]+\varphi_3[\delta-\varphi_2 f(\sigma_t^s)]^{\varphi_4}e^{\varphi_5-\varphi_6 f(\sigma_t^s)-\varphi_7(\sigma+\sigma_t^s)^2}, \tag{G.27}$$

where the constants are defined as:

$$
\begin{aligned}
\varphi_1 &= \rho + \delta, \\
\varphi_2 &= \epsilon - 1, \\
\varphi_3 &= \delta^{\frac{1}{1-\epsilon}}(\rho + \delta), \\
\varphi_4 &= \frac{\epsilon}{\epsilon - 1}, \\
\varphi_5 &= \left(\frac{\eta+1}{\eta}\right)\left(\frac{\theta^y}{\theta^y + \phi_y}\right)\mu^y + \left(\frac{\eta+1}{\eta}\right)\left(\frac{1}{\theta^y + \phi_y}\right)\left(\frac{1}{2} - \phi_{vol}\right)\sigma^2, \\
\varphi_6 &= \left(\frac{\eta+1}{\eta}\right)\left(\frac{\phi_\pi - 1}{\theta^y + \phi_y}\right), \\
\varphi_7 &= \left(\frac{\eta+1}{\eta}\right)\left(\frac{1}{\theta^y + \phi_y}\right)\left(\frac{1}{2} - \phi_{vol}\right).
\end{aligned}
$$

Plugging equations (G.26) and (G.21) into (G.19), we obtain:

$$
d\hat{Y}_t = \left[\varphi_8 + \varphi_9 f(\sigma_t^s) + \varphi_{10}(\sigma + \sigma_t^s)^2\right] dt + \sigma_t^s dZ_t, \tag{G.28}
$$

where the constants are defined as:

$$
\begin{aligned}
\varphi_8 &= \left(\frac{\theta^y \phi_y}{\theta^y + \phi_y}\right)\mu^y - \left(\frac{\theta^y}{\theta^y + \phi_y}\right)\left(\frac{1}{2} - \phi_{vol}\right)\sigma^2, \\
\varphi_9 &= \left(\frac{\theta^y}{\theta^y + \phi_y}\right)(\phi_\pi - 1), \\
\varphi_{10} &= \left(\frac{\theta^y}{\theta^y + \phi_y}\right)\left(\frac{1}{2} - \phi_{vol}\right).
\end{aligned}
$$

Equation (G.26) can be rewritten as $M\left(\hat{Y}_t, \sigma_t^s\right) = 0$, where:

$$
M\left(\hat{Y}_t, \sigma_t^s\right) \equiv \hat{Y}_t - \left(\frac{\theta^y}{\theta^y + \phi_y}\right)\mu^y + \left(\frac{1}{\theta^y + \phi_y}\right)\left[(\phi_\pi - 1)f(\sigma_t^s) + \left(\frac{1}{2} - \phi_{vol}\right)\left[(\sigma + \sigma_t^s)^2 - \sigma^2\right]\right],
$$

which implicitly determines σ_t^s as a function of $\hat{Y}_t$. To compute the implicit derivatives, we compute the following, with $M_{\hat{Y}} = 1, M_{\hat{Y}\hat{Y}} = 0, M_{\hat{Y}\sigma^s} = 0$:

$$M_{\sigma^s} = \left(\frac{1}{\theta^y + \phi_y}\right)\left[(\phi_\pi - 1)f'(\sigma_t^s) + (1 - 2\phi_{vol})(\sigma + \sigma_t^s)\right],$$

$$M_{\sigma^s\sigma^s} = \left(\frac{1}{\theta^y + \phi_y}\right)\left[(\phi_\pi - 1)f''(\sigma_t^s) + (1 - 2\phi_{vol})\right],$$

from which we can compute the implicit derivative of σ_t^s with respect to $\hat{Y}_t$ as:

$$\frac{\partial \sigma_t^s}{\partial \hat{Y}_t} = -\frac{M_{\hat{Y}}}{M_{\sigma^s}} = -\left[\frac{\theta^y + \phi_y}{(\phi_\pi - 1)f'(\sigma_t^s) + (1 - 2\phi_{vol})(\sigma + \sigma_t^s)}\right], \tag{G.29}$$

and the second-order implicit derivative as:

$$\frac{\partial^2 \sigma_t^s}{\partial^2 \hat{Y}_t} = -(\theta^y + \phi_y)^2\left[\frac{(\phi_\pi - 1)f''(\sigma_t^s) + (1 - 2\phi_{vol})}{\left[(\phi_\pi - 1)f'(\sigma_t^s) + (1 - 2\phi_{vol})(\sigma + \sigma_t^s)\right]^3}\right]. \tag{G.30}$$

Using Ito's lemma, together with equations (G.28), (G.29), and (G.30), we obtain the following expression for the process of σ_t^s:

$$\begin{aligned} d\sigma_t^s = &-\left[\varphi_{11}\frac{\varphi_8 + \varphi_9 f(\sigma_t^s) + \varphi_{10}(\sigma + \sigma_t^s)^2}{\varphi_{12}f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)} + \varphi_{14}\frac{\varphi_{12}f''(\sigma_t^s) + \varphi_{13}}{\left[\varphi_{12}f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)\right]^3}(\sigma_t^s)^2\right]dt \\ &-\left[\frac{\varphi_{11}}{\varphi_{12}f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)}\right]\sigma_t^s dZ_t, \end{aligned} \tag{G.31}$$

where the constants are defined as:

$$\varphi_{11} = \theta^y + \psi_y, \;\; \varphi_{12} = \phi_\pi - 1,$$
$$\varphi_{13} = 1 - 2\phi_{vol}, \;\; \varphi_{14} = \frac{(\theta^y + \psi_y)^2}{2}.$$

Next, applying Ito's lemma to (G.21) and using (G.31), we obtain:

$$\begin{aligned} d\pi_t = &-\left[\varphi_{11}f'(\sigma_t^s)\frac{\varphi_8 + \varphi_9 f(\sigma_t^s) + \varphi_{10}(\sigma + \sigma_t^s)^2}{\varphi_{12}f'(\sigma_t^s) + \psi_{13}(\sigma + \sigma_t^s)} + \varphi_{14}\frac{\varphi_{13}\,(\sigma_t^s)^2}{\left[\varphi_{12}f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)\right]^3}\left(f'(\sigma_t^s) - f''(\sigma_t^s)(\sigma + \sigma_t^s)\right)\right]dt \\ &-\left[\frac{\varphi_{11}f'(\sigma_t^s)}{\varphi_{12}f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)}\right]\sigma_t^s dZ_t, \end{aligned} \tag{G.32}$$

which expresses the process of inflation π_t as a sole function of σ_t^s.

Comparing the diffusion terms of (G.20) and (G.32), we find that:

$$\sigma_t^\pi = -\left[\frac{\varphi_{11} f'(\sigma_t^s)}{\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)}\right] \sigma_t^s. \tag{G.33}$$

Comparing the drift terms of (G.20) and (G.32), and using (G.21), we obtain:

$$\theta^\pi\left[\mu^\pi - f(\sigma_t^s)\right] = -\left[\varphi_{11} f'(\sigma_t^s)\frac{\varphi_8 + \varphi_9 f(\sigma_t^s) + \varphi_{10}(\sigma + \sigma_t^s)^2}{\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)} + \varphi_{14}\frac{\varphi_{13}\left(\sigma_t^s\right)^2}{\left[\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)\right]^3}\left(f'(\sigma_t^s) - f''(\sigma_t^s)(\sigma + \sigma_t^s)\right)\right]. \tag{G.34}$$

We must identify a function $f(\cdot)$ that satisfies the differential equation in (G.34) while also meeting the condition in (G.27). Note that (G.27) contains the term $\theta^\pi\left[\mu^\pi - f(\sigma_t^s)\right]$. By substituting (G.33) and (G.34) into (G.27), we obtain an ordinary differential equation (ODE) that fulfills all requirements simultaneously:

$$\begin{aligned}\varphi_1\left[\delta - \varphi_2 f(\sigma_t^s)\right] = &-\left[\varphi_{11} f'(\sigma_t^s)\frac{\varphi_8 + \varphi_9 f(\sigma_t^s) + \varphi_{10}(\sigma + \sigma_t^s)^2}{\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)} + \varphi_{14}\frac{\varphi_{13}\left(\sigma_t^s\right)^2}{\left[\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)\right]^3}\left(f'(\sigma_t^s) - f''(\sigma_t^s)(\sigma + \sigma_t^s)\right)\right]\\
&+\frac{\epsilon}{2}\left(\frac{1}{\delta - \varphi_2 f(\sigma_t^s)}\right)\left[\frac{\varphi_{11} f'(\sigma_t^s)}{\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)}\right]^2 (\sigma_t^s)^2\\
&+\varphi_3\left[\delta - \varphi_2 f(\sigma_t^s)\right]^{\varphi_4} e^{\varphi_5 - \varphi_6 f(\sigma_t^s) - \varphi_7(\sigma + \sigma_t^s)^2},\end{aligned}$$

which can be rearranged as

$$f''(\sigma_t^s) = \underbrace{\left(\frac{f'(\sigma_t^s)}{\sigma + \sigma_t^s}\right) + \frac{\left[\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)\right]^3}{\varphi_{14}\varphi_{13}(\sigma_t^s)^2(\sigma + \sigma_t^s)}\cdot\left[\begin{aligned}&\varphi_1\left[\delta - \varphi_2 f(\sigma_t^s)\right] + \varphi_{11} f'(\sigma_t^s)\frac{\varphi_8 + \varphi_9 f(\sigma_t^s) + \varphi_{10}(\sigma + \sigma_t^s)^2}{\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)}\\ &-\frac{\epsilon}{2}\left(\frac{1}{\delta - \varphi_2 f(\sigma_t^s)}\right)\left[\frac{\varphi_{11} f'(\sigma_t^s)}{\varphi_{12} f'(\sigma_t^s) + \varphi_{13}(\sigma + \sigma_t^s)}\right]^2 (\sigma_t^s)^2\\ &-\varphi_3\left[\delta - \varphi_2 f(\sigma_t^s)\right]^{\varphi_4} e^{\varphi_5 - \varphi_6 f(\sigma_t^s) - \varphi_7(\sigma + \sigma_t^s)^2}\end{aligned}\right]}_{\equiv J(\sigma_t^s, f(\sigma_t^s), f'(\sigma_t^s))}. \tag{G.35}$$

Define the following vector of variables $x(\sigma_t^s) = [x_1(\sigma_t^s), x_2(\sigma_t^s)]'$, with elements defined as:

$$\begin{aligned}x_1(\sigma_t^s) &= f(\sigma_t^s),\\ x_2(\sigma_t^s) &= f'(\sigma_t^s).\end{aligned}$$

Then the ordinary differential equation (ODE) in (G.35) can be rewritten as the following first-order system:

$$\begin{cases} x_1'(\sigma_t^s) & = x_2(\sigma_t^s) \\ x_2'(\sigma_t^s) & = J\left(\sigma_t^s, x_1(\sigma_t^s), x_2(\sigma_t^s)\right) \end{cases}.$$

In vectorized form, we can define $\tilde{J}\left(\sigma_t^s, x(\sigma_t^s)\right) = \left[x_2(\sigma_t^s), J\left(\sigma_t^s, x_1(\sigma_t^s), x_2(\sigma_t^s)\right)\right]'$, so that:

$$x'(\sigma_t^s) = \tilde{J}\left(\sigma_t^s, x(\sigma_t^s)\right).$$

As $\tilde{J}(\sigma_t^s, x_1, x_2)$ is continuous in a closed "box" (or "rectangle") around any $(\sigma_0^s,\ x_1(\sigma_0^s),\ x_2(\sigma_0^s))$, we can apply Peano's theorem (Walter, 1973), which guarantees that there exists at least one local solution $\left(x_1(\sigma_t^s), x_2(\sigma_t^s)\right)$ that satisfies the ordinary differential equation (G.35). Therefore, $f(\sigma_t^s) = x_1(\sigma_t^s)$ is a local solution to the original second-order ordinary differential equation (G.35). Since this proof holds over the entire domain of σ_t^s, it guarantees the existence of a function $f(\cdot)$ such that the output gap and inflation follow the processes in (G.19) and (G.20) for any value of σ_t^s.

This solution differs from the perfectly stabilized path, defined by $\sigma_t^s = \pi_t = \hat{Y}_t = 0$ for all t. In particular, setting $\pi_t = \sigma_t^s = 0$ contradicts equation (G.31), which implies that $d\sigma_t^q$ generally exhibits a nonzero drift when evaluated at zero excess volatility. This contradicts our assumption that $\sigma_t^s = 0$ for all t. Hence, the conjectured equilibrium in (G.19) and (G.21) is distinct from the perfectly stabilized path.

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