# Justifying the First-Order Approach in Agency Frameworks with the Agent's Possibly Non-Concave Value Function

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Principal's canonical problem:

$$\max_{a,s(\cdot)} \int \left( \underbrace{\pi(\mathbf{x})}_{\text{Principal's value}} - \underbrace{s(\mathbf{x})}_{\text{Contract}} \right) f(\mathbf{x}|a)d\mathbf{x}$$

$$s.t. \quad (i)\ U(s(\cdot), a) \geq \overline{U}$$

$$(ii)\ a \in \arg\max_{a'}\ U(s(\cdot), a') = \int u(s(\mathbf{x}))f(\mathbf{x}|a')d\mathbf{x} - c(a') = 0$$

$$(iii)\ \underbrace{s(\mathbf{x}) \geq \underline{s}}_{\text{Limited liability}}$$

**First-Order Approach** (FOA): replace (ii) with its first-order condition (ii)'

$$\max_{a,s(\cdot)} \int \left( \underbrace{\pi(\mathbf{x})}_{\text{Principal's value}} - \underbrace{s(\mathbf{x})}_{\text{Contract}} \right) f(\mathbf{x}|a)d\mathbf{x}$$

$$s.t. \quad (i)\ U(s(\cdot), a) \geq \overline{U}$$

$$(ii)'\ U_a(s(\cdot), a) = \int u(s(\mathbf{x}))f_a(\mathbf{x}|a)d\mathbf{x} - c'(a) = 0$$

$$(iii)\ s(\mathbf{x}) \geq \underline{s}$$

**Note**: limited-liability $s(\mathbf{x}) \geq \underline{s}$ for the solution existence (Mirrlees (1975))

Optimal contract $(s^o(x), a^o)$ based on the first-order approach:

$$\frac{1}{u'(s^o(\mathbf{x}))} = \begin{cases} \lambda + \mu \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}, & \text{if } s^o(\mathbf{x}) \geq \underline{s}, \\ \dfrac{1}{u'(\underline{s})}, & \text{otherwise,} \end{cases}$$

with $\lambda \geq 0$ and $\mu > 0$

- Existence and uniqueness: Jewitt, Kadan, and Swinkels (2008)

If the agent's value function $U(s^o(\cdot), a)$,

$$U(s^o(\cdot), a) = \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a)$$

is concave in $a$, then the first-order approach is valid (e.g., Mirrlees (1975))

**Question (Focus of the literature)**

How can we make $U(s^o(\cdot), a)$ concave in $a$?

**Strategy 1**: put conditions on $f(x|a)$, the technology:

1. One-signal (i.e., $\mathbf{x}$ is scalar): Mirrlees (1975) and Rogerson (1985): MLRP (monotone likelihood ratio property) and CDFC (convexity of the distribution function condition)
2. Multi-signal extension of CDFC: Sinclair-Desgagné (1994, GCDFC: generalized CDFC), Conlon (2009, CISP: concave increasing set property), and Jung and Kim (2015, CDFCL: convexity of the distribution function condition for the likelihood ratio)
3. Too restricted (normal, gamma distributions excluded)

**Strategy 2**: put conditions on both $u(s)$ and $f(x|a)$:

1. Jewitt (1988) and Jung and Kim (2015)
2. Cannot be used with the agent's limited liability $s(\mathbf{x}) \geq \underline{s}$

## Examples show the previous literature is not enough

The first-order approach cannot be justified by the previous literature in:

### Example (Normal distribution)

The agent's utility is $u(s) = \frac{1}{r}s^r$, $r \leq \frac{1}{2}$, The cost function is $c(a) = D(e^{ka} - 1)$, $D > 0$, $k > 0$, and the signal generating function has an additive form $\tilde{x} = a + \tilde{\theta}$, $\tilde{\theta} \sim N(0, \sigma^2)$ thereby

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

- Normal distribution excluded ($\leftarrow$ its likelihood ratio unbounded)

### Example (Gamma distribution)

The agent's utility is $u(s) = \frac{1}{r}s^r$, $r \leq \frac{1}{2}$, Cost function is given by $c(a) = ka$, $k > 0$, and $\tilde{x} \in (0, \infty)$ has the gamma distribution with shape parameter $a$, i.e.,

$$f(x|a) = \frac{x^{a-1}\beta^{-a}}{\Gamma(a)} e^{-\frac{x}{\beta}}. \tag{1}$$

- Gamma distribution excluded ($\leftarrow$ its likelihood ratio unbounded)

The first-order approach cannot be justified by the previous literature in:

**Example (Exponential distribution)**

The agent's utility is $u(s) = \frac{1}{r}s^r$, $r \leq \frac{1}{2}$, and cost $c(a)$ is increasing and convex in $a$. The signal generating function has a multiplicative form, $\tilde{x} = h(a)\tilde{\theta}$, where $h(0) = 0$, $h(a)$ is increasing and convex to a small degree, and $\tilde{\theta}$ is exponentially distributed with mean 1, i.e., the density function of $\tilde{\theta}$ is $p(\theta) = e^{-\theta}$, $\theta \in [0, \infty)$. $\underline{s}$ is low enough. Thereby

$$f(x|a) = \frac{1}{h(a)} e^{-\frac{x}{h(a)}}, \tag{2}$$

- A little convexity of $h(a)$: does not satisfy Jewitt (1988) and Jung and Kim (2015)

## Big Question (Possibly Non-Concave Indirect Utility)

Why should the agent's value function $U(s^o(\cdot), a)$ be concave?

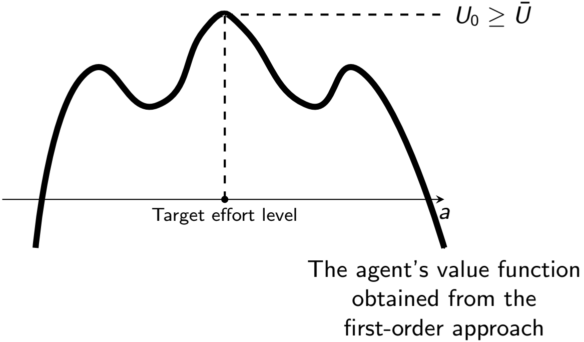
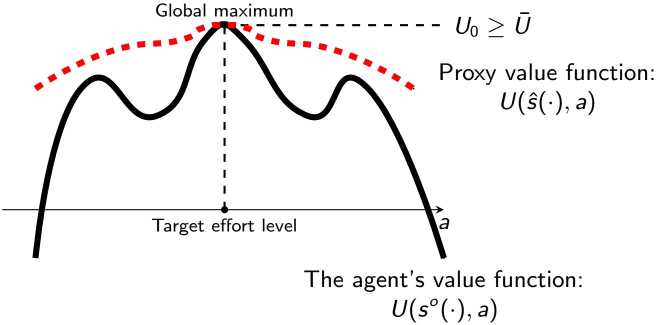



Figure: Possibly Non-Concave Indirect Utility of the Agent




Our approach: justify the first-order approach in all of the above examples

1. Finding a proxy function $\hat{s}(\mathbf{x})$ where the proxy value $U(\hat{s}(\cdot), a)$ is maximized at $a = a^o$, the same target action level
2. Proving $U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a)$, $\forall a$, justifying the first-order approach
3. A proper proxy $\hat{s}(\mathbf{x})$ depends on whether the limited liability binds or not

**Note**: impose additional conditions on the agent's cost function $c(\cdot)$

# Fundamental Lemma

# Change of variables to $q$-space

À la Jung and Kim (2015), define the likelihood ratio

$$\tilde{q} \equiv Q_{a^o}(\tilde{\mathbf{x}}) \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$$

The optimal contract $s^o(x)$ in $q$-space becomes:

$$s^o(x) \equiv w(q) \equiv (u')^{-1}\left(\frac{1}{\lambda + \mu q}\right)$$

The agent's indirect utility (value function) given $s^o(\cdot)$

$$u(s^o(\mathbf{x})) \equiv r(q) = \begin{cases} u(w(q)) \equiv \bar{r}(q), & \text{when } q \geq q_c \\ u(\underline{s}), & \text{when } q < q_c \end{cases}$$

- Threshold $q_c$ solves $u'(\underline{s})^{-1} = \lambda + \mu q_c$: limited liability starts to bind

Distribution function for $q$ given $a$ (possbly different from $a^o$)

$$G(q|a) \equiv Pr\left[Q_{a^o}(\tilde{\mathbf{x}}) \leq q|a\right], \quad dG(q|a) = g(q|a)dq$$

# Properties of a proxy contract

Define $U^o \geq \overline{U}$ at the optimum:

$$U^o = U(s^o(\mathbf{x}), a^o) = \int s^o(\mathbf{x}) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) \tag{3}$$

**Lemma (How to construct a proxy contract $\hat{s}(\cdot)$)**

(1a) $f(\mathbf{x}|a)$ satisfies that $\dfrac{g(q|a)}{g(q|a^o)}$ is convex in $q = \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all $a$

(2a) (DOUBLE-CROSSING PROPERTY) $\exists$a contract $\hat{s}(\mathbf{x})$ satisfying

$$(i) \quad \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) = U^o \tag{4}$$

$$(ii) \quad \int u(\hat{s}(\mathbf{x})) f_a(\mathbf{x}|a^o) d\mathbf{x} - c'(a^o) = 0 \tag{5}$$

such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ crosses $r(q) \equiv u(s^o(\mathbf{x}))$ twice starting from above

(3a) $E[\hat{r}(q)|a]$ is concave in $c(a)$

then using the first-order approach is justified

(1a) and (2a) jointly imply:

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int (r(q) - \hat{r}(q))\, g(q|a) dq \leq 0, \quad \forall a$$

**Why?**: we know that $U(s^o(\cdot), a^o) = U(\hat{s}(\cdot), a^o)$ when $a = a^o$

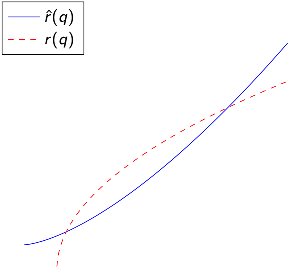



Figure: $r(q)$ and $\hat{r}(q)$: double-crossing

As $a\uparrow$ from $a^o$: $g(q|a)$ moves toward higher $q$, where $r(q) - \hat{r}(q)$ is more likely to be negative. When $a\downarrow$ from $a^o$, the same

- (1a) condition operationalizes this intuition

(1a) and (2a) jointly imply:

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int (r(q) - \hat{r}(q))\, g(q|a) dq \leq 0, \quad \forall a$$

**But**: It might be the following case

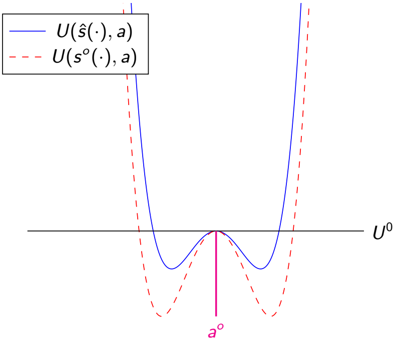



Figure: First-order approach not justified?

(3a) makes sure that $U(\hat{s}(\cdot), a)$ is maximized at $a = a^o$, therefore:

# Intuition

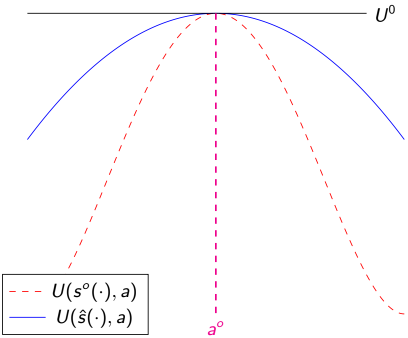



Figure: First-Order Approach Justified

So $U(s^o(\cdot), a)$ must be maximized at $a = a^o$

- The first-order approach (FOA) justified

# When the Limited Liability (LL) Binds

# Finding a proxy contract when (LL) binds for $q \leq q_c$

Define the moment generating function (MGF) of $g(q|a)$:

$$M(a;t) \equiv \int e^{tq} g(q|a) dq.$$

**Proposition (When (LL) binds for $q \leq q_c$)**

Given that the likelihood ratio, $\tilde{q} \equiv \dfrac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is unbounded below, given $a^o$,

(1a) $\dfrac{g(q|a)}{g(q|a^o)}$ is convex in $q = \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all $a$

(2b) (i) there exists $t > 0$ such that

$$\frac{c'(a^o)}{M'(a^o;t)} M(a^o;t) - c(a^o) \leq \overline{U} - u(\underline{s})$$

and (ii) $c(a)$ is convex in $M(a;t)$ for such $t$, and

(3b) $\overline{r}(q)$ is concave in $q$

then the first-order approach is justified

**Note**: Concave $\overline{r}(q) \overset{X}{\rightarrow}$ concave $r(q)$ due to the kink generated by (LL)

**Intuition**: a proxy contract $\hat{s}(\mathbf{x})$ must respect the limited liability constraint (LL). We use the following $t$-dependent contract

$$u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = Ae^{tq} + B$$

which has a good property: $\hat{r}_t(q) \longrightarrow \underbrace{B \geq u(\underline{s})}_{\text{by (i) of (2b)}}$ as $q \longrightarrow -\infty$

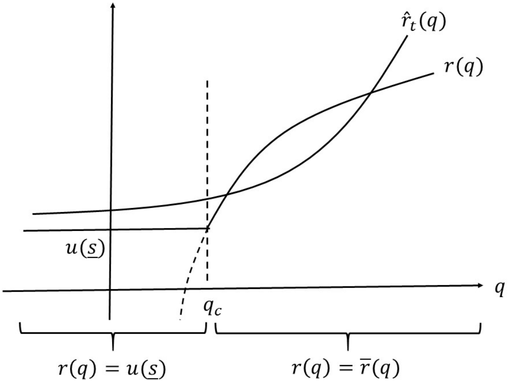



Figure: When the Limited Liability Constraint Binds for $q \leq q_c$

# Finding a proxy contract when (LL) binds for $q \leq q_c$

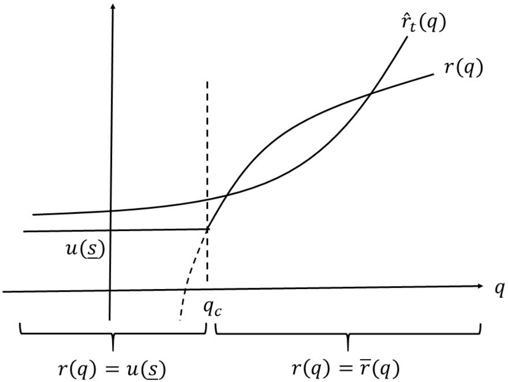



Figure: When the Limited Liability Constraint Binds for $q \leq q_c$

**Note**: earlier examples (cases of Normal and Gamma distributions) can be justified of their use of the first-order approach

- Both distribution features unbounded likelihood ratio (thus we need (LL))
- Jewitt (1988) and Jung and Kim (2015) assume away (LL)

# When the Limited Liability (LL) Not Binds

**Proposition (When (LL) does not bind)**

Given that the likelihood ratio, $\tilde{q} \equiv \dfrac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, given $a^o$,

(1a) $\dfrac{g(q|a)}{g(q|a^o)}$ is convex in $q = \dfrac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all $a$

(2c) $c(a)$ is convex in $m(a) \equiv \displaystyle\int qg(q|a)dq$, and

(3c) $r(q) = \overline{r}(q)$ is concave in $q$

then the first-order approach is justified

**Note**: Now $\overline{r}(q) = r(q)$ due to the nonbinding (LL)

- In this case, finding a proxy contract $\hat{s}(\mathbf{x})$ is easier (no need to respect the limited liability (LL))

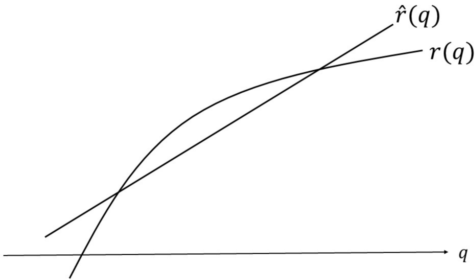



Figure: When the Limited Liability Constraint Does Not Bind

**Simplest case**: our proxy contract $\hat{r}(q)$ is linear in q

- (2c) makes sure under $\hat{r}(q)$, the agent will choose $a = a^o$
- (1c) and (3c) allow us to apply the lemma above (double-crossing)
- This case justifies the last example (the exponential distribution case)

# Comparison with the earlier literature

To compare with Jung and Kim (2015)'s conditions (1J-1) and (1J-2):

- We introduce the total positivity of degree 3 ($TP_3$) (Karlin (1968))
- Our (1a) condition is related to this ($TP_3$) condition

$$
\begin{tikzcd}
((TP_3),(1J-2)) \arrow[r, leftrightarrow, "\text{Lemma 5}", "\text{Given MLRP}"'] \arrow[dr] & ((1a),(1J-2)) \arrow[r] & ((1a),(2c)) \arrow[dl, leftrightarrow, "\text{No inclusion}"'] \\
& ((1J-1),(1J-2)) &
\end{tikzcd}
$$

Figure: Relation Diagram between Conditions

So no direct inclusion between our paper and Jung and Kim (2015)