Self-fulfilling Volatility and a New Monetary Policy[*]

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Abstract

We demonstrate that macroeconomic models with nominal rigidities feature multiple global solutions supporting alternative equilibria traditionally overlooked in the literature. In these equilibria, conventional Taylor rules give rise to self-fulfilling aggregate volatility, propelling the economy into crises (booms) characterized by elevated (reduced) aggregate risk. This outcome stems from the inability of conventional rules to target the expected growth rate of output, which is determined not only by the policy rate but also by the strength of the precautionary savings channel. We propose a new policy rule that targets both conventional mandates and aggregate volatility, reestablishing determinacy and achieving full stabilization.

Keywords: Taylor Rules, Self-fulfilling Volatility, Precautionary Savings

JEL codes: E32, E43, E44, E52

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1 Introduction

How should monetary policy respond to fluctuations in aggregate market volatility? The prevailing perspective suggests that central banks require two distinct sets of instruments: macroprudential policies to preserve the stability of markets, that is, maintaining a stable level of market volatility, and monetary and fiscal policies to achieve the conventional goal of macroeconomic stabilization. Nevertheless, the debate surrounding this matter remains unresolved for numerous reasons. For instance, aggregate volatility is inherently endogenous, and integrating its influence into macroeconomic models presents a significant challenge. Mainstream macroeconomic frameworks often rely on approximation techniques that simplify or entirely disregard higher-order terms associated with aggregate volatility. Alternatively, they depend on numerical solution methods that may obscure the underlying economic intuition.

In this paper, we demonstrate that within a macroeconomic model featuring nominal rigidities, Taylor rules, irrespective of their responsiveness to typical business cycle mandates (e.g., output gap), permit aggregate volatility to emerge in a self-fulfilling manner. We illustrate this insight within two macroeconomic models: (i) the standard New-Keynesian model,[1] and (ii) a model incorporating stock markets and portfolio decisions, the latter of which is provided in Online Appendix A. Our continuous-time characterization of the problem allows the models' solutions to remain tractable, yielding closed-form expressions for the time-varying aggregate volatility and business cycle variables, all of which are endogenously determined.

In the standard New-Keynesian model, the economy's time-varying aggregate volatility has a first-order impact on aggregate consumption demand through the precautionary savings channel. More specifically, heightened aggregate volatility leads households to increase their precautionary savings, which reduces aggregate demand and output, while the aggregate volatility itself is determined by fluctuations in output. In this setting, households can generate aggregate volatility through their intertemporal consumption coordination under rational expectations. For instance, consider a scenario where households at time 0 suddenly believe that the economy will be more volatile in the next period. They decrease their current consumption and increase precautionary savings, resulting in a recession at time 0. In period 1, the initial fear at time 0 regarding the volatility of the period 1 economy must be validated. This occurs if, for each possible realization of consumption at period

[1] See, for example, Galí (2015).

1, there exists a corresponding conditional volatility of period 2 consumption justifying them. Specifically, a higher realization of period 1 consumption should be accompanied by a lower conditional volatility of period 2 consumption, leading to a decreased degree of precautionary savings. Essentially, the household's belief in current volatility is shaped by their expectations in the previous period and justified by their actions in future periods. Note that our equilibrium construction with self-fulfilling volatility is made possible due to nominal rigidities: the path-dependent consumption strategy of households determines the stochastic paths of output, as the economy is driven by demand.

In this specific rational expectations equilibrium, which we refer to as the "martingale" equilibrium, the output gap adheres to a local martingale, meaning that, on average, the next period's economy remains at the current level. We prove that in this solution, the stabilized path (i.e., the flexible price economy benchmark) functions as an attractor for all sample paths, and the conditional volatility of the subsequent period's consumption declines as the economy approaches it. Consequently, after the appearance of a self-fulfilling volatility shock, the economy is almost surely stabilized in the long run. However, on the equilibrium path, and until the economy is nearly stabilized following the emergence of the initial self-fulfilling shock, it experiences a prolonged recession accompanied by increased aggregate volatility. We demonstrate that a *probability-zero event*, in which the conditional volatility of the economy ultimately diverges toward infinity, enables the initial appearance of a self-fulfilling shock and ensures that the economy follows a local martingale, even if it is almost surely stabilized in the long run. We relate this property to an endogenously generated rare-disaster event that arises in a self-fulfilling manner.

Additional global solutions exist, which we demonstrate by presenting an alternative class of equilibria with interesting properties. For example, we show that it is possible to sustain stationary aggregate levels of excess volatility in the long run, as well as suboptimal steady states characterized by under- or overproduction. These results suggest that the welfare costs of business cycles or inadequate policy interventions may have been underestimated in the past.

The inability of conventional Taylor rules to prevent the emergence of multiplicity and self-fulfilling shocks arises from their failure to break the feedback loop between endogenous output growth and its volatility, which can be caused by the intertemporal consumption coordination of agents. We demonstrate that central banks can address this issue by either (a) establishing output growth mandates and using the policy rate as an intermediate tool to achieve these targets, or (b) incorporating economic volatility into their interest rate rules. The latter policy requires central banks to measure economic volatility precisely and to target its fluctuations with adequate strength. However, even if they fail to do so perfectly, we show that partial targeting of economic volatility accelerates economic stabilization following a self-fulfilling shock.

Model with Stock Markets Online Appendix A develops an alternative model that incorporates stock markets and optimal portfolio choice within the setting. There, we argue that common measures of financial volatility or risk premium can serve as proxies for aggregate economic volatility, assuming competitive financial markets and fundamental asset valuation. Similarly, shocks to financial volatility can arise in a self-fulfilling manner, leading to fluctuations in aggregate demand under conventional Taylor rules. We argue that an extended Taylor rule that targets the risk premium achieves what we call *ultra-divine coincidence*: the simultaneous stabilization of inflation, the output gap, and the risk premium (equivalently, aggregate stock price volatility). Moreover, a calibrated version of the model in Online Appendix A quantitatively matches the impulse-response functions obtained from an structural vector autoregression in Online Appendix B.

Related literature Our model with stock markets shares similarities with Caballero and Simsek (2020a,b) in terms of incorporating an endogenous asset market interwoven with the fluctuations of the business cycle. However, while their framework focuses on how behavioral biases can generate intriguing crisis dynamics through the feedback loop between asset markets and business cycles,[2] our attention centers on the traditional policy rule under rational expectations and the existence of alternative equilibria arising from higher-order moments.

While Benhabib et al. (2002) study monetary-fiscal regimes in regards to eliminating indeterminacy issues posed by the ZLB, and Obstfeld and Rogoff (2021) show how a probabilistic (and small) fiscal currency backing can rule out speculative hyperinflation in monetary models, our focus is on the self-fulfilling emergence of aggregate volatility outside the ZLB and the exploration of alternative monetary policy rules.

There is a large macro-finance literature on the self-fulfilling nature of real and financial

[2]Caballero and Simsek (2020b) present a model with optimists and pessimists who hold differing beliefs about the probability of an imminent recession or boom. During zero lower bound (ZLB) episodes, an endogenous decline in risky asset valuation, triggered by a reduction in optimists' wealth, leads to a demand recession. We explore related ZLB issues in a separate paper, Dordal i Carreras and Lee (2024).

uncertainty: e.g., Bacchetta et al. (2017), Fajgelbaum et al. (2017),[3] and Benhabib et al. (2019). Bacchetta et al. (2017) characterize an endowment economy where current asset prices are affected by a sunspot that shifts the perceived risk of future asset prices. Benhabib et al. (2019) develop a model of 'mutual learning' between financial markets and the real economy, leading to strategic complementarity and self-fulfilling uncertainty. Benhabib et al. (2024) study a model of aggregate demand externality,[4] where a positive feedback loop between aggregate output and defaults generates a self-fulfilling default cycle.[5] We instead abstract from defaults and focus on the self-fulfilling appearance of aggregate volatility in a model with nominal rigidities and aggregate demand externalities.

Our equilibrium determinacy results resemble those of Acharya and Dogra (2020) and Khorrami and Mendo (2024). While Acharya and Dogra (2020) investigates how determinacy conditions change in the presence of exogenous idiosyncratic risks that are functions of aggregate output, we explore the existence of self-fulfilling aggregate volatility and examine the monetary policy that restores determinacy. Khorrami and Mendo (2024) study similar equilibrium indeterminacy issues around aggregate volatility and propose fiscal rules as an alternative mechanism to determine equilibrium.

2 Standard Non-linear New Keynesian Model

This section illustrates that a *non-linear* characterization of the equilibrium allows higher-order moments associated with aggregate business cycle volatility to have a first-order impact on business cycle dynamics. This feature has important implications for equilibrium determinacy and the appropriate management of monetary policy required to stabilize the economy. Appendix II provides a more detailed characterization of the optimality conditions and the proof of the existence of multiple global solutions.

Households The representative household owns the firms of this economy and receives their profits via lump-sum transfers. For simplicity, we assume a perfectly rigid price level: $p_t = \bar{p}$, $\forall t$, so there is no inflation in the economy. This assumption is not crucial but allows

[3]In Fajgelbaum et al. (2017), higher uncertainty about fundamentals leads to lower investment, slowing down information flows and further discouraging investment. This results in 'uncertainty traps' characterized by self-fulfilling uncertainty and low activity.

[4]For a modern treatment of this issue, see Farhi and Werning (2016).

[5]When aggregate output falls, it raises defaults as firms' revenues and profits decline. As defaults disrupt production, they further decrease aggregate output, ad infinitum.

us to focus on the key mechanism we want to illustrate.[6] The optimization problem of the household is given by

$$\max_{\{B_t,C_t,L_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] dt \quad , \quad \text{s.t. } \dot{B}_t = i_t B_t - \bar{p} C_t + w_t L_t + D_t, \tag{1}$$

where C_t and L_t are its consumption and labor supply, respectively, η is the Frisch elasticity of labor supply, ρ is the time discount rate, B_t is its nominal holding of bonds –which is in zero net supply in equilibrium– and D_t represents the total firms' profits and fiscal transfers from the government. w_t is the equilibrium wage, and i_t is the policy rate set by the central bank. We assume that there is no government spending, and therefore aggregate consumption determines production in this environment with price rigidity.

We obtain the intertemporal optimality condition of (1) as

$$-i_t dt = \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right), \quad \text{where } \xi_t^N = e^{-\rho t} \frac{1}{\bar{p}} \frac{1}{C_t}, \tag{2}$$

with $\frac{d\xi_t^N}{\xi_t^N}$ representing the instantaneous (nominal) stochastic discount factor, whose expected value equals the (minus) nominal risk-free rate, $-i_t dt$.[7] Due to the rigid price assumption, there is no inflation, i.e., $\pi_t = 0, \forall t$. Consequently, the real and nominal risk-free rates of the economy are equal, $r_t = i_t$, where r_t represents the real interest rate.

We can rewrite equation (2) as

$$\mathbb{E}_t \left(\frac{dC_t}{C_t} \right) = (i_t - \rho) dt + \underbrace{\text{Var}_t \left(\frac{dC_t}{C_t} \right)}_{\substack{\text{Endogenous} \\ \text{precautionary savings}}}, \tag{3}$$

where the last term, $\text{Var}_t(\frac{dC_t}{C_t})$, arises from the *endogenous* volatility of the aggregate consumption process. Note that this term is usually a second-order term and is typically dropped in log-linearized models. In contrast, our non-linear characterization of (3) prop-

[6]Online Appendix A relaxes this assumption and introduces price stickiness à la Calvo (1983). In a nutshell, the source of the multiplicity of global solutions in this model arises from the dynamic IS equation —stemming from the household's intertemporal problem— combined with the policy rule set forth by the central bank. By assuming perfectly rigid prices, we derive a trivial expression for the New Keynesian Phillips Curve ($\pi_t = 0$, for all t, where π_t stands for inflation). Therefore, this assumption allows us, for expositional purposes and without loss of generality, to collapse the problem into the two key equations driving our results and to obtain very simple and intuitive analytical expressions for the model's global solutions.

[7]Appendix II provides the Hamilton-Jacobi-Bellman (HJB) equation-based derivation for (2).

erly accounts for consumption volatility and allows it to affect the drift of the aggregate consumption process, where both the volatility and the drift are endogenous variables. This additional term reflects the usual *precautionary savings channel*, in which a more volatile business cycle leads to an increased demand for riskless savings, resulting in a drop in current consumption and a higher expected growth for the consumption process.

Firms We assume the usual Dixit-Stiglitz monopolistic competition among firms, where the demand each firm i faces is given by

$$D_t(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t, \text{ with } p_t = \left(\int_0^1 \left(p_t^i\right)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}},$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. In the assumed rigid price equilibrium, firms never change their prices so $p_t^i = p_t = \bar{p}$ and $D_t(p_t^i, p_t) = D_t(\bar{p}, \bar{p}) = Y_t$ for all $i \in [0, 1]$ and $\forall t$, i.e., each firm i produces to meet the aggregate demand Y_t.

An individual firm i produces with the linear production function: $Y_t^i = A_t L_t^i$, taking the aggregate price p_t, wage w_t, and the aggregate output Y_t as given, where L_t^i is firm i's labor hiring, and A_t is the economy's total factor productivity assumed to be exogenous and to follow a geometric Brownian motion with drift:

$$\frac{dA_t}{A_t} = gdt + \sigma dZ_t, \tag{4}$$

where g is its expected growth rate and σ is what we call 'fundamental' volatility, assumed to be constant over time.[8] It follows that firms' profits to be rebated can be written as $D_t = \bar{p}Y_t - w_t L_t$. We assume that all the aggregate variables are adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ generated by the process in (4) in a given *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$.

Flexible price equilibrium as benchmark With the assumed Dixit-Stiglitz monopolistic competition among firms, we can characterize the flexible price equilibrium where firms can freely choose their prices, in contrast to the fully rigid price, i.e., $p_t = \bar{p}$. The flexible price equilibrium outcomes are termed 'natural' as central banks, in the presence of price rigidity, target these outcomes with their monetary tools. As we prove in Appendix II.2.2,

[8] This assumption is made for simplicity and our analysis can be extended to include cases where σ_t is time-varying and adapted to the Brownian motion Z_t.

the natural output Y_t^n follows

$$\frac{dY_t^n}{Y_t^n} = \Big(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2 \Big) dt + \underbrace{\sigma}_{\text{Natural volatility}} dZ_t, \tag{5}$$

where $r^n = \rho + g - \sigma^2$ is defined as the natural interest rate. From the monetary authority's perspective, the process in (5) is an exogenous process that monetary policy cannot affect or control. Note that natural output Y_t^n follows a geometric Brownian motion with volatility σ, which equals the volatility of the A_t process in (4).

Rigid price equilibrium and the 'gap' economy Going back to the 'rigid' price economy, we introduce σ_t^s as the *excess* volatility of the growth rate of the output process $\{Y_t\}$, compared with the benchmark flexible price economy output in (5). Then:

$$\mathrm{Var}_t \left(\frac{dY_t}{Y_t} \right) = (\sigma + \sigma_t^s)^2 dt, \tag{6}$$

holds by definition. Note that σ_t^s is an *endogenous* volatility term to be determined in equilibrium. By plugging equation (6) into the nonlinear Euler equation (3), we obtain

$$\frac{dY_t}{Y_t} = \left(i_t - \rho + (\sigma + \sigma_t^s)^2 \right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{7}$$

With the usual definition of output gap $\hat{Y}_t = \ln \left(\frac{Y_t}{Y_t^n} \right)$, we obtain[9]

$$d\hat{Y}_t = \left(i_t - \left(r^n \overbrace{- \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2}^{\text{New terms}} \right) \right) dt + \sigma_t^s dZ_t, \tag{8}$$

which features an interesting feedback effect that is omitted in log-linearized equations:[10] Given the policy rate i_t, a rise in the endogenous volatility σ_t^s increases the drift in (8) and lowers the output gap, $\hat{Y}_t$. The intuition follows from households' precautionary behavior observed in (3), whereby households respond to higher economic volatility with increased savings and lower consumption, inducing a recession.

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[9] In (7), we assume that the current output Y_t is adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ generated by the technology process in (4). Therefore, σ_t^s in (7) can be interpreted as a *fundamental* excess volatility.

[10] For illustrative purposes, compare (8) with the conventional IS equation given by $d\hat{Y}_t = (i_t - r^n)\, dt + \sigma_t^s dZ_t$, where the endogenous aggregate volatility σ_t^s has no first-order effect on the drift.

Define the *risk-adjusted* natural rate as

$$r_t^T = r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2, \tag{9}$$

and note that r_t^T is itself endogenous and negatively depends on the endogenous aggregate excess volatility, σ_t^s. This risk-adjusted natural rate can be regarded as a new reference risk-free rate of the economy at which i_t completely eliminates the drift of the output gap.

2.1 Taylor rules and Indeterminacy

Now we study the conventional Taylor rule and its capacity to guarantee model determinacy and economic stabilization. We assume that the central bank sets the risk-free rate i_t to:

$$i_t = r^n + \phi_y \hat{Y}_t, \text{ where } \phi_y > 0. \tag{10}$$

Condition $\phi_y > 0$ is the 'Taylor principle' that guarantees unique equilibrium in conventional log-linearized models that omit the first-order effects of aggregate volatility. Here, we ask whether the policy in (10) retains the capacity to determine a unique equilibrium in our non-linear economy that features the feedback relationship between output gap volatility and its drift explained in (8). Plugging equation (10) into equation (8), we obtain

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t - \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^s)^2}{2}\right) dt + \sigma_t^s dZ_t, \tag{11}$$

as the dynamics for $\hat{Y}_t$. A rational expectations equilibrium in our model is defined as any stochastic path of $\left\{\hat{Y}_t\right\}$ that satisfies equation (11) and the following boundary condition:

$$\lim_{t\to\infty} \mathbb{E}_0 \left|\hat{Y}_t\right| < \infty, \tag{12}$$

preventing divergence in expectations in the long run, and consistent with the traditional definition of a rational expectations equilibrium (see, e.g., Blanchard and Kahn (1980)).

Benchmark equilibrium: no volatility feedback Omitting the new volatility terms from the drift of (11), we obtain the usual log-linear approximation of the $\hat{Y}_t$ dynamics as

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t\right) dt + \sigma_t^s dZ_t. \tag{13}$$

With the local dynamics around the natural steady state described by (13), Blanchard and Kahn (1980) proves the existence of a *unique* linear rational expectations equilibrium when the Taylor principle $\phi_y > 0$ is satisfied in (10),[11] resulting in $\hat{Y}_t = \sigma_t^s = 0$, $\forall t$, which corresponds to a fully stabilized economy.

With volatility feedback We now demonstrate that a variety of global solutions exist, resulting in alternative rational expectations equilibria consistent with the monetary policy described in (10). Specifically, we first show that the feedback between the endogenous output gap volatility σ_t^s and its drift in equation (11) facilitates the emergence of *self-fulfilling* volatility shocks to σ_t^s.

2.1.1 Martingale equilibrium

We begin by presenting a rational expectations equilibrium that supports the emergence of an initial excess volatility $\sigma_0^s > 0$ by explicitly constructing an equilibrium path in which $\hat{Y}_t$ follows a local martingale.[12,13] Our martingale equilibrium construction (i) supports an initial jump in excess volatility, $\sigma_0^s > 0$, which can arise in a self-fulfilling manner; (ii) satisfies the process defined by the dynamic IS equation in (11); and (iii) does not diverge in expectations in the long run, consistent with the boundary condition in (12).[14] We also demonstrate that this specific equilibrium is non-stationary by design. It is explicitly constructed through the following steps, with derivation details provided in Appendix I.

Step 1 Assume that $\hat{Y}_t$ is a local martingale consistent with the dynamics in (11). Therefore, the drift of the $\{\hat{Y}_t\}$ process in (11) must be zero, resulting in:

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}. \tag{14}$$

[11] See Buiter (1984) for an adaptation of the conditions and results from Blanchard and Kahn (1980) to continuous time settings.

[12] This martingale equilibrium represents one of the possible fundamental equilibria consistent with (10). Its construction, however, illustrates how a sudden rise in endogenous volatility interacts with monetary policy and drives business cycles.

[13] The case of negative volatility (i.e., $\sigma_0^s < 0$) can be similarly constructed.

[14] The emergence of the initial volatility σ_0^s is not part of the economy's filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. This can be viewed as a sunspot shock to the excess volatility σ_t^s, with aggregate variables responding to its appearance.

Step 2 Therefore, $\left|\hat{Y}_t\right|$ process becomes a supermartingale, meaning that

$$\lim_{t\to\infty} \mathbb{E}_0 \left|\hat{Y}_t\right| \leq \left|\hat{Y}_0\right| < \infty, \tag{15}$$

which satisfies the boundary conditions imposed by equation (12). Thus, the equilibrium does not diverge in the long run.

Step 3 Finally, we show the existence of a stochastic process for $\{\sigma_t^s\}$ starting from σ_0^s that supports the equilibrium expression for the output gap in (14). Using (11) and (14), we obtain an expression for that process as follows:[15]

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma+\sigma_t^s} dZ_t. \tag{16}$$

Equations (14) and (16) constitute the dynamics of our constructed rational expectations equilibrium supporting initial self-fulfilling volatility $\sigma_0^s > 0$. The following Proposition 1 sheds lights on the behavior of $\left\{\hat{Y}_t, \sigma_t^s\right\}$ under the martingale equilibrium and finds that: even if the economy is hit by an initial self-fulfilling volatility shock $\sigma_0^s > 0$, the business cycle almost surely converges to the perfectly stabilized path in the long run through monetary stabilization based on Taylor rules. Nonetheless, a few sample paths that occur with a *tiny* probability do not converge and explode asymptotically, sustaining the initial volatility $\sigma_0^s > 0$ due to the forward-looking nature of the economy.

Proposition 1 (Taylor Rules and Indeterminacy) *For any value of $\phi_y > 0$:*

1. Indeterminacy*: there is always a rational expectations equilibrium (REE) that supports initial volatility $\sigma_0^s > 0$ and is represented by the $\hat{Y}_t$ dynamics in equation* (14)*, and the σ_t^s process in equation* (16).

2. Properties*: the equilibrium that supports an initial volatility $\sigma_0^s > 0$ satisfies:*

Property 1 *The excess volatility σ_t^s converges to zero almost surely, i.e., $\sigma_t^s \xrightarrow{a.s} \sigma_\infty^s = 0$.*

Property 2 *The output gap $\hat{Y}_t$ converges to zero almost surely, i.e., $\hat{Y}_t \xrightarrow{a.s} \hat{Y}_\infty = 0$.*

[15] When $\sigma = 0$, $\forall t$, equation (16) becomes the following Bessel process:

$$d\sigma_t^s = -\frac{(\phi_y)^2}{2\sigma_t^s} dt - \phi_y dZ_t,$$

which stops when σ_t^s reaches zero. For general properties of Bessel processes, see Lawler (2019).

Property 3 *Non-uniform integrability: the total volatility squared* $(\sigma + \sigma_t^s)^2$ *satisfies*

$$\mathbb{E}_0 \left(\sup_{t \geq 0} (\sigma + \sigma_t^s)^2 \right) = \infty, \ \ \textit{and} \ \ \lim_{K \to \infty} \sup_{t \geq 0} \left(\mathbb{E}_0 \left(\sigma + \sigma_t^s\right)^2 \mathbb{1}_{\left\{(\sigma + \sigma_t^s)^2 \geq K\right\}} \right) > 0.$$

Proof. See Appendix I.2. ■

The results that $\sigma_t^s \xrightarrow{a.s} \sigma_\infty^s = 0$ and $\hat{Y}_t \xrightarrow{a.s} \hat{Y}_\infty = 0$ imply that the equilibrium paths starting from an initial volatility $\sigma_0^s > 0$ are almost surely stabilized in the long run. However, almost sure stabilization of paths is compatible with a self-fulfilling appearance of $\sigma_0^s > 0$ by Property 3 of Proposition 1, $\mathbb{E}_0 \left(\sup_{t \geq 0} (\sigma + \sigma_t^s)^2\right) = \infty$, which implies that an initial self-fulfilling shock in σ_0^s is sustained by a *vanishing* probability of an infinitely large equilibrium total volatility in some future paths. Nonetheless, we have $\lim_{t \to \infty} |\mathbb{E}_0(\hat{Y}_t)| \leq |\hat{Y}_0| < \infty$, satisfying the 'convergence in expectations' criteria of Blanchard and Kahn (1980).

Intuition Here we explain in detail the intuition for (i) how an initial aggregate volatility σ_0^s can appear, and (ii) the three Properties in Proposition 1.[16] To this end, we simplify the economic environment and make the following assumptions:

A.1 A shock dZ_t in each period takes one of two possible values: $\{+1, -1\}$, with equal probability.

A.2 Martingale equilibrium: output gap $\hat{Y}_t$ equals the conditional expected value of the next-period output gap, $\hat{Y}_{t+1}$. Thus, if $\hat{Y}_{t+1}$ takes either $\hat{Y}_{t+1}^{(1)}$ or $\hat{Y}_{t+1}^{(2)}$ when $dZ_{t+1} = 1$ or -1, respectively, then

$$\hat{Y}_t = \frac{1}{2} \left(\hat{Y}_{t+1}^{(1)} + \hat{Y}_{t+1}^{(2)} \right).$$

A.3 Aggregate demand (i.e., output gap) $\hat{Y}_t$ falls as the conditional variance of the next period's $\hat{Y}_{t+1}$ rises, due to precautionary savings. Both $\hat{Y}_t$ and σ_t^s are zero on the stabilized path (i.e., flexible-price economy).

Since there are two possible realizations of the shock dZ_t in each period, we can represent this with a tree diagram, as depicted in Figure 1. The thick vertical line represents the stabilized path, with areas to its left and right representing recessions and booms, respectively.

[16] We also provide simulation results from an alternative calibrated model with stock markets in Online Appendix A.5.

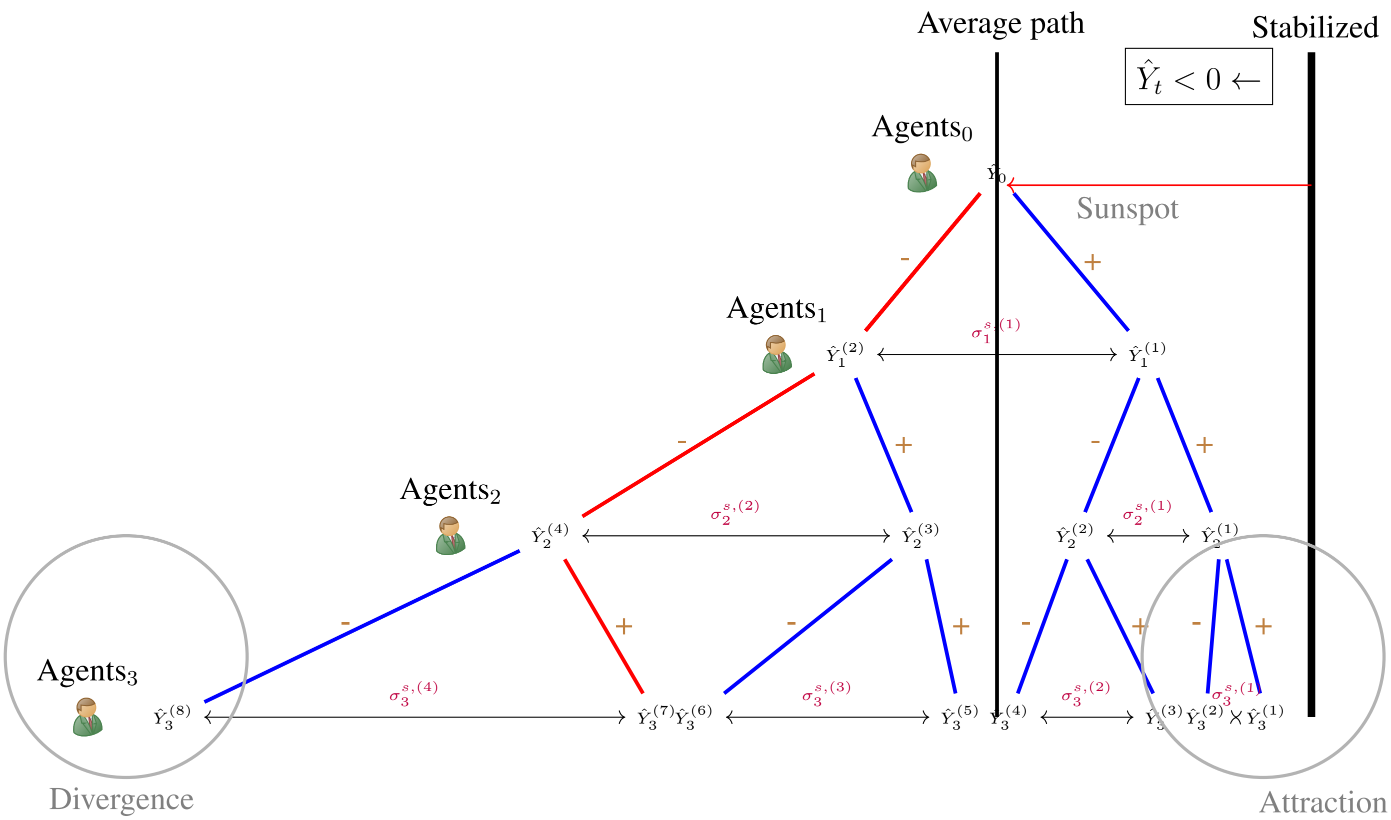


Figure 1: A rise in σ_0^s as a rational expectations equilibrium.

The key to build a rational expectations equilibrium that supports a self-fulfilling increase in volatility $\sigma_0^s > 0$ is to construct a path-dependent consumption strategy for agents with time-varying conditional volatilities.

First, let us imagine that the the current period agents (Agents$_0$) suddenly believe that the future agents will choose the path-dependent consumption demand[17] so that the next-period's $\hat{Y}_1$ becomes $\hat{Y}_1^{(1)}$ after $dZ_1 = +1$ is realized and $\hat{Y}_1^{(2)}$ if $dZ_1 = -1$ is realized, with $\hat{Y}_1^{(1)} > \hat{Y}_1^{(2)}$. Then the current output $\hat{Y}_0$ becomes $\hat{Y}_0 = \frac{1}{2}\left(\hat{Y}_1^{(1)} + \hat{Y}_1^{(2)}\right)$ with $\hat{Y}_0$ below the stabilized path, as Agents$_0$ believe that there exists dispersion in next-period outcomes, which is given by $\sigma_1^{s,(1)} = \frac{\hat{Y}_1^{(1)} - \hat{Y}_1^{(2)}}{2}$, and which leads to lower consumption through precautionary savings at $t = 0$. Now imagine $dZ_1 = -1$ is realized. For Agents$_0$'s belief in $\hat{Y}_1 = \hat{Y}_1^{(2)}$ to be consistent, Agents$_1$ must believe that future agents will choose their consumption in a way that, at time 2, $\hat{Y}_2$ becomes $\hat{Y}_2^{(3)}$ with $dZ_2 = +1$ and $\hat{Y}_2^{(4)}$ with

[17] Note that agents' demand determines output in this environment with rigid prices.

$dZ_2 = -1$, with conditional volatility $\sigma_2^{s,(2)} = \frac{\hat{Y}_2^{(3)} - \hat{Y}_2^{(4)}}{2}$ higher than $\sigma_1^{s,(1)}$, since $\hat{Y}_1^{(2)}$ is lower than the initial output, $\hat{Y}_0$.

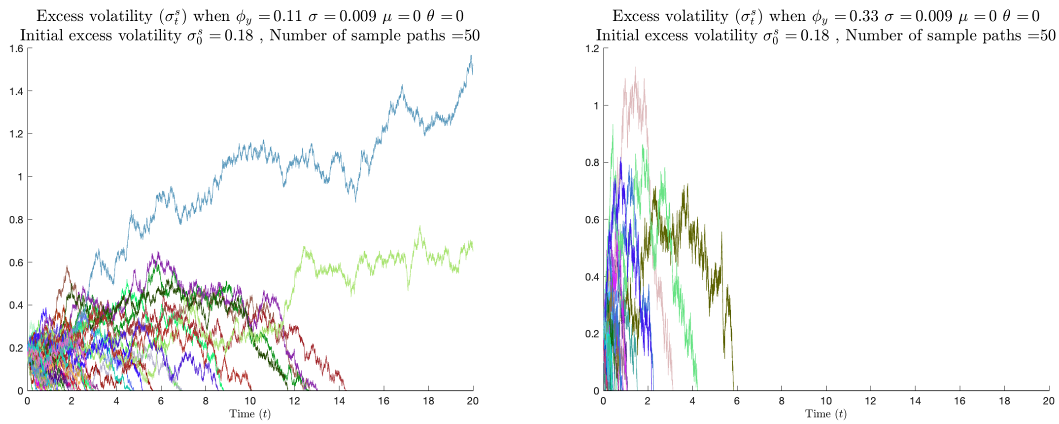
After dZ_2 is realized, Agents$_1$'s belief about $\hat{Y}_2$ can be made consistent through future agents $\{\text{Agents}_{n\geq 2}\}$'s coordination in a forward looking fashion. Observe that all the nodes in Figure 1 satisfy assumptions A.2 and A.3, with distance between adjacent nodes getting progressively narrower (wider) as output gap gets closer (farther) to the stabilization. This results in divergent and attraction paths balancing each other out, and in expectation, output gap $\{\hat{Y}_t\}$ follows a local martingale process. In sum, Agents$_0$'s initial doubt that the next-period's outcome will be volatile is made consistent by coordination between intertemporal agents (i.e., the representative household) at each node.[18]

Note that (i) we obtain an equilibrium with a *stochastic* aggregate volatility: i.e., σ_t^s is dependent on the path of shocks, as output gap $\{\hat{Y}_t\}$ is stochastic and negatively depends on the conditional volatility of its next-period level. Equation (16) specifies the exact stochastic process of $\{\sigma_t^s\}$ starting from $\sigma_0^s > 0$; (ii) Since volatility σ_t^s decreases as output gap $\hat{Y}_t$ approaches the stabilized path, this path becomes an attraction point for the set of alternative paths in its neighborhood, justifying the result of Proposition 1 that σ_t^s almost surely converges to zero over time. Nonetheless, as volatility σ_t^s rises whenever output $\hat{Y}_t$ deviates farther from the stabilized level, this also aligns with the result of Proposition 1 that maximal σ_t^s diverges, $\mathbb{E}_0(\sup(\sigma + \sigma_t^s)^2) = \infty$. However, this divergent behavior only happens with vanishingly small probability as $\sigma_t^s \xrightarrow{a.s} 0$.

The conclusion in terms of monetary policy is that a conventional Taylor rule almost surely stabilizes the disruption caused by an initial volatility shock $\sigma_0^s > 0$ in the long-run, but does not prevent the economy from entering a crisis phase with an excess volatility path $\{\sigma_t^s\}$ starting from σ_0^s.

Simulation Figure 2 illustrates the martingale equilibrium's dynamic paths of $\{\sigma_t^s\}$ supporting $\sigma_0^s = 0.18$, and explores the effects of a change in policy responsiveness to output gap, ϕ_y. The left panel 2a uses the default calibration $\phi_y = 0.11$, while the right panel 2b assumes a more responsive stance, $\phi_y = 0.33$. Given $\sigma_0^s > 0$, we observe that under a higher ϕ_y, sample paths are likely to converge faster towards full stabilization, though this comes at the expense of an increased likelihood of a more severe crisis path within a

[18] Our equilibrium construction is feasible since all future agents share the common knowledge of their consumption strategies and there is no friction in communication among agents in intertemporal periods (i.e., perfect recall). For how limited recall removes indeterminacy, see Angeletos and Lian (2023).



(a) With Taylor coefficient $\phi_y = 0.11$

(b) With Taylor coefficient $\phi_y = 0.33$

Figure 2: Martingale equilibrium with $\phi_y = 0.11$ (Figure 2a), and $\phi_y = 0.33$ (Figure 2b).

given period. The intuition is as follows: for a *given* level of initial volatility $\sigma_0^s > 0$ to be sustained under more responsive monetary policy with higher ϕ_y, it must be true that more amplified endogenous volatility (i.e., high σ_t^s) and severe recession (i.e., low $\hat{Y}_t$) arise with vanishing probability in the future.

Escape clause If the central bank and/or the government *credibly* commit to preventing $\hat{Y}_t$ from falling below a predetermined threshold through interventions,[19] then the equilibria arising from the aggregate volatility σ_0^s and supported by the paths in Figure 1 (i.e., the martingale equilibrium) are no longer sustained as a possible rational expectations equilibrium (REE). This escape clause illustrates that the credible commitment of government entities to intervene whenever the economy probabilistically enters a severe recession actually precludes the possibility of a crisis phase initiated by the positive volatility shock $\sigma_0^s > 0$. The credibility of such a commitment is crucial, as absolute credibility is required to prevent the emergence of an equilibrium with $\sigma_0^s > 0$.

Negative volatility We can similarly construct a rational expectations equilibrium with an initial negative self-fulfilling volatility, $\sigma_0^s < 0$. This equilibrium is characterized by

[19]For example, governments might commit to incurring significant fiscal deficits whenever the economy experiences a severe recession. This approach has similar implications regarding government actions to restore determinate equilibrium as discussed in Benhabib et al. (2002), which focuses on the role of monetary-fiscal regimes in eliminating indeterminacy posed by the ZLB. Similarly, Obstfeld and Rogoff (2021) demonstrate how a probabilistic (and small) fiscal currency backing by the government can rule out speculative hyperinflation in monetary models.

a boom with strong aggregate demand and low volatility.[20] Therefore, we conclude that our non-linear characterization of the model generates the reasonable prediction of (i) the appearance of boom and crisis phases resulting from self-fulfilling volatility shocks, and (ii) the joint evolution of the first (output level) and second (conditional volatility) order moments of the model during crises and booms.

2.1.2 Ornstein-Uhlenbeck equilibria

While the martingale equilibrium discussed in Section 2.1.1 demonstrates the mechanisms behind one potential solution to the model, additional equilibria exist. This section introduces a broader class of such equilibria with several noteworthy properties: (i) initial *excess* volatility σ_0^s adapted to the economy's filtration (i.e., equilibria that do not require initial sunspot volatility shocks in σ_0^s), (ii) non-degenerate stationary stochastic processes for the model variables in the long run,[21] and (iii) the potential for alternative deterministic steady states characterized by under- or overproduction.

For that purpose, we conjecture an alternative class of equilibria where the output gap $\{\hat{Y}_t\}$ dynamics follow a process of the form

$$d\hat{Y}_t = \theta \cdot \left[\mu - \hat{Y}_t\right] \mathrm{d}t + \sigma_t^s \mathrm{d}Z_t, \tag{17}$$

where θ and μ are constant parameters. Note that (17) resembles an Ornstein-Uhlenbeck process, with one major difference: the process features an endogenous volatility σ_t^s, which is determined in equilibrium, whereas typical Ornstein-Uhlenbeck processes have constant volatility associated with the Brownian motion component. When $\theta = 0$, the process becomes the martingale equilibrium studied in Section 2.1.1. To close the model, we equate the drift terms in equation (11) and equation (17) and obtain

$$\hat{Y}_t = \frac{\theta\mu}{\theta + \phi_y} - \frac{1}{2(\theta + \phi_y)} \left[(\sigma + \sigma_t^s)^2 - \sigma^2\right], \tag{18}$$

with

$$\mathrm{d}(\sigma + \sigma_t^s)^2 = -\theta \left[2\mu\phi_y + (\sigma + \sigma_t^s)^2 - \sigma^2\right] \mathrm{d}t - 2(\theta + \phi_y)\sigma_t^s \mathrm{d}Z_t. \tag{19}$$

[20] As seen in equation (6), the actual output Y_t's process features $\sigma + \sigma_t^s$ as its conditional volatility. Thus, a self-fulfilling negative excess volatility $\sigma_0^s < 0$ reduces the volatility of the growth rate of Y_t from σ to $\sigma + \sigma_t^s$.

[21] In our martingale equilibrium, once σ_t^s reaches zero, it remains at zero thereafter, resulting in a non-stationary solution to the model.

The following three cases, depending on the signs of θ and μ, are worth noting:

Case 1 $\mu = 0, \theta > 0$: Observe that when $\sigma_t^s > 0$, the drift of the process (19) now becomes negative, causing $(\sigma + \sigma_t^s)^2$ to be drawn towards σ^2. If $\sigma_0^s > 0$ appears suddenly, as described in Section 2.1.1, we can still apply the technique outlined in Proposition 1 and demonstrate that

$$\sigma_t^s \xrightarrow{a.s} \sigma_\infty^s = 0,$$

since $(\sigma + \sigma_t^s)^2$ remains a supermartingale. Essentially, the behavior of $\{\sigma_t^s\}$ is similar to that described in Section 2.1.1, but the convergence (i.e., the attraction of σ_t^s towards zero) is stronger due to the additional pull exerted by the drift term, as shown in Figure 3.

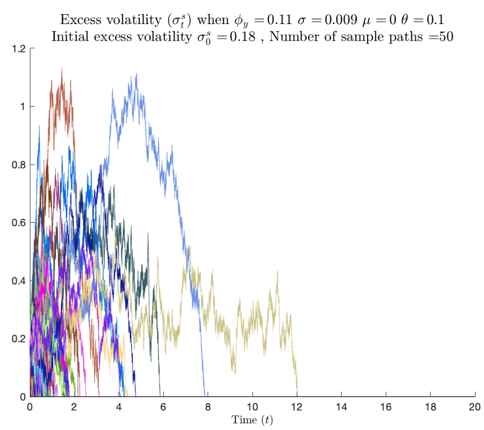


Figure 3: Simulated paths for the excess volatility $\{\sigma_t^s\}$ process under the Ornstein-Uhlenbeck solution with Case 1 calibration, $\theta > 0$, $\mu = 0$.

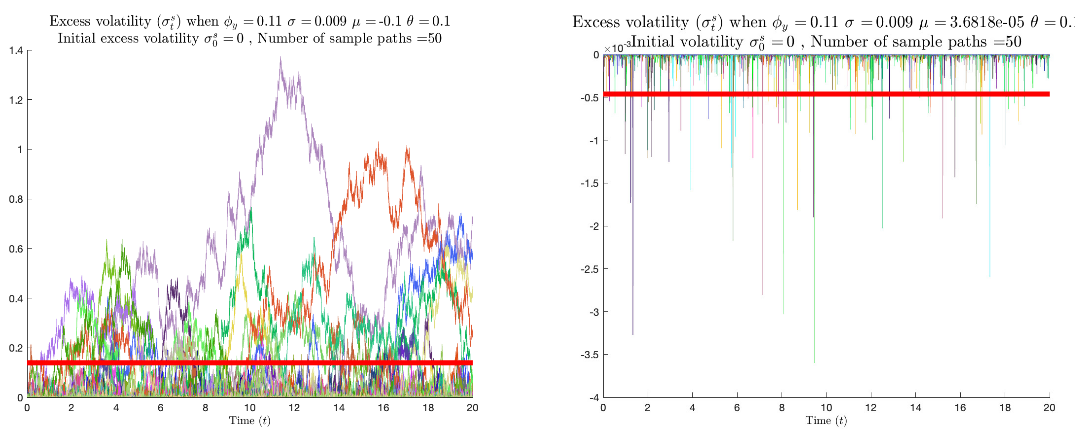
Case 2 $\mu < 0, \theta > 0$: This case is interesting because the drift of (19) pulls σ_t^s towards

$$\sigma^{s,*} = \sqrt{\sigma^2 - 2\mu\phi_y} - \sigma > 0. \tag{20}$$

Since the volatility term of (19) is non-zero at $\sigma^{s,*}$, the process $\{\sigma_t^s\}$ is not permanently trapped at any given value, including zero. For example, even if the economy starts at $\sigma_0^s = 0$, the excess volatility σ_t^s begins to move toward $\sigma^{s,*}$, as the drift of (19) is positive while the volatility is zero at that point. This results in an alternative solution to the model featuring an endogenous and stationary $\{\sigma_t^s\}$ process starting from any arbitrary σ_0^s value and, thus, not reliant on any initial sunspot shock. Figure 4a plots stationary $\{\sigma_t^s\}$ sample paths around $\sigma^{s,*}$, starting from σ_0^s equal to zero.[22]

The solutions within the Ornstein-Uhlenbeck class described in this section feature a deterministic steady state for the output gap $\hat{Y}$ equal to the parameter μ. Therefore, Case 2 equilibria feature a steady state with underproduction, with the output of the economy being $|100 \times \mu|$ percent below the natural level.

Case 3 $0 < \mu < \frac{\sigma^2}{2\phi_y}$, $\theta > 0$: The drift of (19) now pulls σ_t^s towards an attractor point defined as in (20), with $\sigma^{s,*} < 0$. The resulting equilibria share similar properties with Case 2 regarding the stationarity and initial conditions for the $\{\sigma_t^s\}$ process, with simulated sample paths displayed in Figure (4b). However, under Case 3 calibration, the solution results in a steady state with output that is $100 \times \mu$ percent above the natural level.



(a) Case 2

(b) Case 3

Figure 4: Simulated paths for the excess volatility $\{\sigma_t^s\}$ process are presented under the Ornstein-Uhlenbeck solution with different calibrations. In Panel (4a): Case 2 calibration, $\mu < 0$, $\theta > 0$; Panel (4a): Case 3 calibration, $\mu > 0$, $\theta > 0$. The thick red line denotes the attractor points $\sigma^{s,*}$, as defined in equation (20).

Implications This additional class of alternative solutions highlights several implications for the welfare analysis in New Keynesian models. First, by commonly evaluating the economy around the log-linear solution presented in (13), traditional welfare accounting likely

[22]The processes in both Case 2 and Case 3 satisfy covariance stationarity and reject the null hypothesis in an augmented Dickey-Fuller unit root test.

overlooks the additional losses stemming from the existence of a non-zero excess volatility process, $\{\sigma_t^s\}$. Second, and potentially more important, traditional welfare evaluations omit the capacity of monetary policy interventions to generate first-order gains by moving the economy away from steady states featuring suboptimal production levels.

The next section presents and discusses the implementation details of a new monetary policy capable of restoring model determinacy and uniqueness while bringing the economy to its constrained efficient equilibrium.

2.2 A New Monetary Policy

Constrained Efficiency The flexible price equilibrium characterized by equation (5) and Appendix II.2.2 is a constrained-efficient allocation,[23] see, e.g., Woodford (2003) and Galí (2015). Therefore, in light of the previous analysis in Section 2.1, the monetary authority aims to achieve the flexible price allocation as the *unique* equilibrium path, if possible. In this section, we provide a new monetary policy that allows the central bank to accomplish this goal.

Volatility Targeting Instead of the Taylor rule in (10), let us assume that the central bank follows an alternative policy rule given by

$$i_t = r^n + \phi_y \hat{Y}_t - \underbrace{\frac{1}{2}\left((\sigma + \sigma_t^s)^2 - \sigma^2\right)}_{\text{Aggregate volatility targeting}}, \quad \text{where } \phi_y > 0, \tag{21}$$

which targets not only the output gap $\hat{Y}_t$, but also the total excess volatility of the economy with a coefficient of $\frac{1}{2}$. By substituting the policy rule (21) into the IS equation (8), the volatility feedback terms in the drift part cancel out. Consequently, we can obtain dynamics represented by equation (13), ensuring model determinacy and guaranteeing $\hat{Y}_t = 0$ for all t as the *unique* rational expectations equilibrium when the Taylor principle, $\phi_y > 0$, is satisfied. Therefore, we conclude that monetary policy following equation (21) eliminates the potential for self-fulfilling aggregate volatility and multiple equilibria.

[23]With a proper production subsidy that eliminates the real distortion generated by monopolistic competition, the flexible price equilibrium allocation becomes the first best. For more on this issue, see Woodford (2003).

Interpretation The additional volatility target in the policy rule (21) is necessary to offset the feedback channel between the endogenous volatility of the output gap process and its drift. To gain a more intuitive understanding of this result, we can rearrange equation (21) as $i_t = r_t^T + \phi_y \hat{Y}_t$, where r_t^T is the risk-adjusted natural rate defined in equation (9). This implies that monetary policy in a risky environment should target the risk-adjusted, rather than the simple natural, interest rate. It is important to note that r_t^T in our environment is time-varying, as it depends on the endogenous excess volatility σ_t^s.

Interestingly, following the policy rule in (21) guarantees the elimination of any *excess* volatility, i.e., $\sigma_t^s = 0, \forall t$. This indicates that, on the equilibrium path, a central bank adhering to the policy rule in (21) behaves in an observationally equivalent manner to a bank following the traditional policy rule in (10). The key difference then is that (21) includes an *off-equilibrium* threat to target *excess* volatility, should it arise. A corollary to this result is that, in practice, differences in the perceived credibility of central banks in following through with such threats may explain their varied success in economic stabilization under observationally similar monetary policy regimes.

Practicality A potential issue with the policy rule (21) is its lack of robustness in practical implementations. Specifically, the coefficient attached to the volatility term, which represents the strength of the policymakers' response to deviations in aggregate volatility, must be precisely $\frac{1}{2}$. If the central bank's response to economic volatility is either too strong or too weak, due to policy mistakes or measurement errors,[24] the rule in (21) cannot effectively counteract the precautionary savings feedback loop present in the non-linear IS equation (8). To examine the consequences of deviating from the $\frac{1}{2}$ volatility target, we consider the following alternative rule within the context of the previously discussed martingale equilibrium of Section 2.1.1:

$$i_t = r^n + \phi_y \hat{Y}_t - \phi_{\text{vol}} \left((\sigma + \sigma_t^s)^2 - \sigma^2 \right), \tag{22}$$

[24]In practice, the components of output volatility $\{\sigma, \sigma_t^s\}$ and the risk-adjusted natural rate r_t^T may not be directly observable or may be observed with errors. For instance, assume a multiplicative measurement error for the volatility gap $\equiv (\sigma + \sigma_t^s)^2 - \sigma^2$, such that volatility $\text{gap}_t^{\text{obs}} = \varepsilon_t \cdot$volatility gap_t, where volatility $\text{gap}_t^{\text{obs}}$ represents the measured volatility gap. In these cases, even with the precise targeting strength of $\frac{1}{2}$ on the observed volatility gap, i.e., volatility $\text{gap}_t^{\text{obs}}$, central banks effectively deviate from the $\frac{1}{2}$ response strength on the true volatility gap. Conversely, additive measurement errors result in standard monetary policy shocks.

where ϕ_{vol} is a constant term, which may differ from $\frac{1}{2}$. With the policy rule in (22), we obtain

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t + \left(\frac{1}{2} - \phi_{\text{vol}}\right)\left((\sigma + \sigma_t^s)^2 - \sigma^2\right)\right) dt + \sigma_t^s dZ_t,$$

as the new $\{\hat{Y}_t\}$ dynamics. When $\phi_{\text{vol}} \neq \frac{1}{2}$, the martingale equilibrium with self-fulfilling volatility σ_t^s reappears and is characterized by[25]

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_{\text{total}}} + \frac{\sigma^2}{2\phi_{\text{total}}}\ , \ \text{ with } \ \phi_{\text{total}} \equiv \frac{\phi_y}{1 - 2\phi_{\text{vol}}}, \tag{23}$$

where the $\{\sigma_t^s\}$'s process after an initial volatility shock σ_0^s appears is given by

$$d\sigma_t^s = -\frac{\phi_{\text{total}}^2 (\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3} dt - \phi_{\text{total}} \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t. \tag{24}$$

Note that $\phi_{\text{vol}} \to \frac{1}{2}$, given $\phi_y > 0$, is equivalent to $\phi_y \to \infty$ with $\phi_{\text{vol}} = 0$, both of which lead to $\phi_{\text{total}} \to \infty$ and ensure determinacy. Therefore, there exists an alternative—albeit impractical—stabilization rule that involves an infinitely aggressive *off-equilibrium* threat to output gap deviations.[26] In Online Appendix A, we study how the relative size of the coefficients ϕ_y and ϕ_{vol} affects the pace of stabilization after a self-fulfilling volatility shock σ_0^s occurs. We find that any combination of parameters that drives the value of ϕ_{total} towards infinity results in faster convergence to full stabilization. However, this comes at the cost of an increased likelihood of a more severe crisis path within a given time period, as discussed previously in relation to Figure 2.

Comparison Woodford (2001, 2003) study the Taylor rule in a log-linearized New Keynesian model, summarized by[27,28]

$$\begin{aligned} \mathbb{E}_t(d\hat{Y}_{t+1}) &= (i_t^m - r^n)\, dt, \\ i_t &= i_t^* + \phi_y \hat{Y}_t, \quad \phi_y > 0, \end{aligned} \tag{25}$$

where i_t^m is the interest rate governing the household's intertemporal consumption smoothing, and i_t^* is the central bank's target for the policy rate, i_t. They uncover that:

[25] Equations (23) and (24) are easily derived similarly to Proposition 1.

[26] See Cochrane (2007) for a comprehensive discussion on this topic in traditional New-Keynesian frameworks.

[27] For comparison, inflation is abstracted away in equation (25).

[28] We thank an anonymous referee for suggesting this comparison.

B.1 When i_t^m equals i_t, then $i_t^* = r^n$ guarantees that $\hat{Y}_t = 0$ for all t, as a unique equilibrium. Even if $i_t^* \neq r^n$, there is still a unique equilibrium, but $\hat{Y}_t \neq 0$ on the equilibrium path.

B.2 When $i_t^m \neq i_t$, setting $i_t^* = r^n + (i_t - i_t^m)$ achieves $\hat{Y}_t = 0$ for all t, as a unique equilibrium. If $i_t - i_t^m$ is an exogenous process, then even when $i_t^* \neq r^n + (i_t - i_t^m)$, there is still a unique equilibrium, but $\hat{Y}_t \neq 0$ on the equilibrium path.

What we do corresponds to neither case: in our model, $i_t - i_t^m$ depends on the endogenous volatility of the $\{\hat{Y}_t\}$ process, with $r_t^T \equiv r^n + (i_t - i_t^m)$ in equation (9). We show that

C.1 If $i_t^* = r_t^T$, we achieve $\hat{Y}_t = 0$ for all t, as a unique equilibrium. In this case, the policy rule corresponds to the new rule proposed in (21).

C.2 In contrast to Woodford (2001, 2003), where $i_t - i_t^m$ is exogenous, if $i_t^* \neq r_t^T$, we cannot guarantee a unique equilibrium, and the martingale equilibrium of Section 2.1.1 with a self-fulfilling initial volatility σ_0^s or the Ornstein-Uhlenbeck equilibrium of Section 2.1.2 may potentially appear.

C.3 $i_t - i_t^m$ depends only on the volatility gap, i.e., $(\sigma + \sigma_t^s)^2 - \sigma^2$. Thus, in a knife-edge case where $i_t^* - (i_t - i_t^m)$ does not contain any multiple of the volatility gap (or more generally, is not a function of the *excess* volatility σ_t^s), even if $i_t^* - (i_t - i_t^m) \neq r^n$, we have a unique equilibrium, but $\hat{Y}_t \neq 0$ along the equilibrium path.

Policy reformulation and growth mandates We can rewrite the policy rule in (21) as

$$\underbrace{\frac{\mathbb{E}_t\left(d\log Y_t\right)}{dt}}_{\text{Growth rate}} = \underbrace{\frac{\mathbb{E}_t\left(d\log Y_t^n\right)}{dt}}_{\substack{\text{Benchmark}\\ \text{growth rate}}} + \underbrace{\phi_y \hat{Y}_t}_{\substack{\text{Business cycle}\\ \text{targeting}}}$$
$$= \left(g - \frac{1}{2}\sigma^2\right) + \phi_y \hat{Y}_t.$$

Thus, an output growth rule centered around the natural growth rate can restore model determinacy and stabilize the economy. From a practical perspective, such a policy reformulation has several advantages, as it does not require the monetary authority to measure or target deviations in the aggregate volatility with precise strength. Forecast errors in the output growth rate or its natural counterpart are actually more forgiving in this implementation, resulting in traditional monetary policy shocks instead of multiple equilibria.

To understand the intuition behind this result, recall that the source of equilibrium multiplicity lies in the feedback loop between the endogenous components of the economy's (expected) growth rate and its conditional volatility, generated by the intertemporal consumption decisions of agents and captured by the drift and volatility components of equation (I.3). To break this loop, the monetary authority must establish a (direct or indirect) tight grip over at least one of these components. Examining the definition of the expected growth rate and its first-order linear approximation, i.e., $\frac{\mathbb{E}_t(d\log Y_t)}{dt} = i_t - \rho + \frac{1}{2}(\sigma + \sigma_t^s)^2 \approx i_t - \rho$, we observe that a traditional Taylor rule in a log-linearized framework can exert the same degree of control over the endogenous components of economic growth as a direct growth mandate. However, this statement is no longer true when considering the global solution of the model, which properly accounts for economic risk, $(\sigma + \sigma_t^s)^2$.

Therefore, to avoid sunspot shocks and equilibrium multiplicity, the monetary authority faces the dilemma of either: (a) establishing clear economic growth mandates,[29] using the policy rate i_t as an intermediate tool toward attaining these objectives, or (b) precisely targeting deviations in the aggregate volatility of the economy when following an interest rate rule.

Model with Stock Markets In Online Appendix A, we provide a model that incorporates stock markets and inflation, demonstrating that commonly observed measures of *financial volatility* or *risk premium* can serve as a proxy for the implementation of our new policy rule in (21). The intuition behind this result is that in competitive asset markets, the fundamental valuation of stocks is linked to the cash flows generated by the underlying productive activities. Hence, the aggregate risk premium of the stock market might serve as a proxy for the aggregate volatility of output.

The model highlights a similar role for aggregate stock price volatility and risk premium in business cycle fluctuations: a more volatile stock market with a higher risk premium reduces aggregate financial wealth through individual investors' portfolio decisions, thereby diminishing aggregate demand and output in the presence of nominal rigidities. Due to the analogous mathematical structure concerning the influence of aggregate volatility on aggregate demand, we can construct an equilibrium in which aggregate stock price volatility is generated in a self-fulfilling manner under conventional Taylor rules. We then argue that

[29]That is, in addition to any inflation mandates dictated by traditional considerations.

a *generalized* Taylor rule with the same form as (21) that targets the risk premium achieves what we call *ultra-divine coincidence*: the simultaneous stabilization of inflation, the output gap, and the risk premium (equivalently, aggregate stock price volatility). Furthermore, the impulse responses simulated from a calibrated version of the model, as detailed in Online Appendix A, quantitatively align with the empirical responses derived from a structural Vector Autoregression analysis presented in Online Appendix B.

3 Conclusion

Conventional Taylor rules, even with the aggressive targeting of traditional macroeconomic indicators, cannot guarantee equilibrium determinacy, allowing self-fulfilling aggregate volatility to surface and influence the business cycle. This failure of conventional rules to ensure determinacy arises from their inability to properly target the *expected growth rate of output*, which is pivotal for households' precautionary behavior in making their intertemporal substitution decisions. We propose an alternative monetary rule that restores determinacy by targeting not only the conventional mandates, such as the output gap, but also the volatility of the economy in a specific manner, thus effectively managing the expected growth rate of aggregate output. This new policy rule facilitates the joint stabilization of the output gap and aggregate volatility.

I Proofs and Derivations

I.1 Derivations in Section 2

Derivation of Equation (3) From the definition of (nominal) state-price density $\xi_t^N = e^{-\rho t}\frac{1}{C_t}\frac{1}{p_t}$, we obtain

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} - \frac{dp_t}{p_t} + \left(\frac{dC_t}{C_t}\right)^2 + \left(\frac{dp_t}{p_t}\right)^2 + \frac{dC_t}{C_t}\frac{dp_t}{p_t}. \tag{I.1}$$

Since we have a perfectly rigid price (i.e., $p_t = \bar{p}$ for all t), the above (I.1) becomes

$$\begin{aligned} \frac{d\xi_t^N}{\xi_t^N} &= -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \\ &= -\rho dt - \frac{dC_t}{C_t} + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right). \end{aligned} \tag{I.2}$$

Plugging equation (I.2) into equation (2), we obtain

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)\, dt + \mathrm{Var}_t\left(\frac{dC_t}{C_t}\right).$$

Derivation of Equation (8) From equation (7), we obtain

$$d\ln Y_t = \left(i_t - \rho + \frac{1}{2}(\sigma + \sigma_t^s)^2\right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{I.3}$$

From (5), we obtain

$$d\ln Y_t^n = \left(r^n - \rho + \frac{1}{2}\sigma^2\right) dt + \sigma dZ_t. \tag{I.4}$$

Therefore, by subtracting equation (I.4) from equation (I.3), we obtain equation (8):

$$d\hat{Y}_t = \left(i_t - \left(r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2\right)\right) dt + \sigma_t^s dZ_t.$$

I.2 Proofs of Section 2.1.1

The Construction of the Martingale Equilibrium in Section 2.1.1 and the Proof of Proposition 1. Setting the drift of the $\hat{Y}_t$ process in (11) to zero, i.e.,

$$d\hat{Y}_t = \left(\underbrace{\phi_y \hat{Y}_t - \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^s)^2}{2}}_{=0} \right) dt + \sigma_t^s dZ_t = \sigma_t^s dZ_t, \tag{I.5}$$

leads to

$$\hat{Y}_t = -\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}, \tag{I.6}$$

proving equation (14). From equations (I.5) and (I.6), we obtain

$$d\hat{Y}_t = -\frac{1}{\phi_y}(\sigma + \sigma_t^s) d\sigma_t^s - \frac{1}{2\phi_y} (d\sigma_t^s)^2 = \sigma_t^s dZ_t,$$

which leads to

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t,$$

proving equation (16).

From equations (I.5) and (I.6), it is evident that $\mathcal{E}_t \equiv (\sigma + \sigma_t^s)^2 - \sigma^2$ has no drift, thereby qualifying as a local martingale. This can also be demonstrated as follows:

$$\begin{aligned} d\mathcal{E}_t &= 2\left(\sigma + \sigma_t^s\right) d\sigma_t^s + \left(d\sigma_t^s\right)^2 \\ &= 2(\sigma + \sigma_t^s) \left(\cancel{-\frac{(\phi_y)^2 (\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3}} dt - \phi_y \frac{\sigma_t^s}{\sigma + \sigma_t^s} dZ_t \right) + \cancel{(\phi_y)^2 \frac{(\sigma_t^s)^2}{(\sigma + \sigma_t^s)^2}} dt \\ &= -2\phi_y(\sigma_t^s) dZ_t = 2\phi_y \left(\sigma - \sqrt{\sigma^2 + \mathcal{E}_t}\right) dZ_t. \end{aligned} \tag{I.7}$$

Our logic proceeds as follows:

Step 1 Starting from $\mathcal{E}_0 > 0$, $\mathcal{E}_t > 0$ for all t almost surely, i.e., 0 is a natural boundary of the $\mathcal{E}_t$ process.

Proof. We use the methodology of Linetsky (2007): by defining a function $m(\mathcal{E})$ that measures the speed of convergence in the process (I.7) as follows:

$$m\left(\mathcal{E}\right) \equiv \frac{1}{\left(\sqrt{\sigma^2 + \mathcal{E}} - \sigma\right)^2},$$

which determines the behavior of $\mathcal{E}_t$ process.

For small $\Delta > 0$, we calculate the following two integrals:

$$\begin{aligned} I_0 &\equiv \int_0^{\Delta} \mathcal{E} \cdot m(\mathcal{E})\, d\mathcal{E} = \int_0^{\Delta} \mathcal{E} \cdot \frac{1}{\left(\sqrt{\sigma^2 + \mathcal{E}} - \sigma\right)^2} d\mathcal{E} \\ &= \int_0^{\sqrt{\Delta+\sigma^2}-\sigma} \left(\frac{t+2\sigma}{t}\right) \cdot 2(t+\sigma) dt \to \infty, \end{aligned} \tag{I.8}$$

with $t \equiv \sqrt{\mathcal{E} + \sigma^2} - \sigma$ as a change of variable. Similarly,

$$\begin{aligned} J_0 &\equiv \int_0^{\Delta} (\Delta - \mathcal{E}) \cdot m(\mathcal{E})\, d\mathcal{E} = \int_0^{\Delta} (\Delta - \mathcal{E}) \cdot \frac{1}{\left(\sqrt{\sigma^2 + \mathcal{E}} - \sigma\right)^2} d\mathcal{E} \\ &= \int_0^{\sqrt{\Delta+\sigma^2}-\sigma} \left[\frac{(\Delta+\sigma^2) - (t+\sigma)^2}{t^2}\right] \cdot 2(t+\sigma) dt \to \infty. \end{aligned} \tag{I.9}$$

With $I_0 = \infty$ and $J_0 = \infty$, process $\mathcal{E}_t$ has zero as a natural boundary, i.e., $\mathcal{E}_t$ never reaches the boundary at zero if it starts in the interior of the state space. In other words, if $\mathcal{E}_0 > 0$, then $\mathcal{E}_t \geq 0$ almost surely. ■

Step 2 As $\mathcal{E}_t$ is a local martingale that is non-negative due to Step 1, it becomes a supermartingale: see e.g., Le Gall (2016) about how to use Fatou's lemma in proving this statement.[1] Thus, equation (15) is proved.

Step 3 Since $\mathcal{E}_t$ is a supermartingale that is non-negative (or more generally, bounded from below), we can apply the famous martingale convergence theorem (see e.g., Williams (1991) and Le Gall (2016)), that implies:

$$\mathcal{E}_t \xrightarrow{a.s} \mathcal{E}_\infty,$$

point-wise, where $\mathcal{E}_\infty$ exists and is finite almost surely.

Step 4 Now, we define a function that is globally concave:[2]

$$\Phi(x) \equiv 4\left[\sigma \log\left(\sqrt{\sigma^2 + x} - \sigma\right) + \sqrt{\sigma^2 + x}\right],$$

[1] $\mathcal{E}_t$ might not be a true martingale as $\sigma - \sqrt{\sigma^2 + \mathcal{E}_t}$ in equation (I.7) is not bounded.

[2] We appreciate Victor Kleptsyn at CNRS à Institut de Recherche Mathématique de Rennes for suggesting function $\Phi(x)$.

which yields

$$\Phi'(x) = \frac{1}{\sqrt{\sigma^2 + x} - \sigma},$$

and $\Phi(x) \to -\infty$ as $x \to 0$. Additionally, we can obtain

$$\begin{aligned} d\left(\Phi\left(\mathcal{E}_t\right)\right) &= \Phi'(\mathcal{E}_t)d\mathcal{E}_t + \frac{1}{2}\Phi''(\mathcal{E}_t)(d\mathcal{E}_t)^2 \\ &= -\phi_y^2 \frac{1}{\sqrt{\sigma^2 + \mathcal{E}_t}}dt - 2\phi_y dZ_t. \end{aligned} \tag{I.10}$$

From Step 3, we know that $\mathcal{E}_\infty$ is finite with probability one, which implies that the drift $\frac{1}{\sqrt{\sigma^2+\mathcal{E}_\infty}}$ of (I.10) is finite and positive as well. Then, the only way to satisfy (I.10) in the long run is for $\Phi(\mathcal{E}_t) \to -\infty$, implying $\mathcal{E}_\infty = 0$.

Step 5 Finally, $\mathcal{E}_t \to 0$ implies that $\sigma_t^s \to 0$ almost surely. It can be easily shown that this satisfies our stochastic process (16) as follows:

$$\underbrace{d\sigma_t^s}_{\xrightarrow{a.s} 0} = -\underbrace{\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma + \sigma_t^s)^3}}_{\xrightarrow{a.s} 0} dt - \phi_y \underbrace{\frac{\sigma_t^s}{\sigma + \sigma_t^s}}_{\xrightarrow{a.s} 0} dZ_t. \tag{I.11}$$

Step 6 Finally from Le Gall (2016), we know that if

$$\mathbb{E}_0\left(\sup_{t\geq 0} |\mathcal{E}_t|\right) < \infty, \quad \text{or} \quad \lim_{K\to\infty} \sup_{t\geq 0} \left(\mathbb{E}_0\left(|\mathcal{E}_t| \mathbb{1}_{\{|\mathcal{E}_t|\geq K\}}\right)\right) > 0,$$

then $\mathcal{E}_t$, which is a local martingale, becomes an uniformly integrable martingale. But then, if $\mathcal{E}_t$ is an uniformly integrable martingale,

$$0 < \mathcal{E}_0 = \lim_{t\to\infty} \mathbb{E}_0\mathcal{E}_t = \mathbb{E}_0 \underbrace{\mathcal{E}_\infty}_{=0} = 0,$$

which is a contradiction. Therefore, Property 3 of Proposition 1 is proved.

Special case With $\sigma = 0$, the stochastic process (I.11) becomes:

$$d\sigma_t^s = -\frac{(\phi_y)^2}{2\sigma_t^s}dt - \phi_y dZ_t, \tag{I.12}$$

which is known as a Bessel process and widely studied in the literature. The process stops when σ_t^s reaches zero. In this case, we can observe that equation (I.8) becomes[3]

$$I_0 \equiv \int_0^{\Delta} \mathcal{E} \cdot \frac{1}{\left(\sqrt{0^2 + \mathcal{E}} - 0\right)^2} d\mathcal{E} = \Delta < \infty,$$

with $J_0 = \infty$. This implies that the $\mathcal{E}_t$ process has zero as an exit boundary, meaning the $\mathcal{E}_t$ process is instantaneously terminated the first time this boundary is reached. The behavior of the Bessel process (I.12) hitting time (i.e., the first time it reaches zero) is well known. For example, its hitting time τ has a well-defined distribution, as derived in Lawler (2019).

■

I.3 Derivations of Section 2.1.2

The equilibrium output gap $\hat{Y}_t$ follows:

$$\mathrm{d}\hat{Y}_t = \left[\phi_y \hat{Y}_t - \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^s)^2}{2}\right] \mathrm{d}t + \sigma_t^s \mathrm{d}Z_t. \tag{I.13}$$

Now we guess that the solution of the model represented by equation (I.13) has the following form:

$$\mathrm{d}\hat{Y}_t = \theta \cdot \left[\mu - \hat{Y}_t\right] \mathrm{d}t + \sigma_t^s \mathrm{d}Z_t, \tag{I.14}$$

where θ and μ are constant parameters. The process I.14 is similar to the Ornstein-Uhlenbeck process, except for the fact that it has an endogenous volatility σ_t^s which is to be determined in equilibrium.

Note that when $\theta = 0$, the process becomes the martingale conjectured in Section 2.1.1. For this new conjectured solution to be valid, the drift term in equation (I.13) and equation (I.14) should be equal, implying that the output gap under this conjecture is:

$$\hat{Y}_t = \frac{\theta\mu}{\theta + \phi_y} - \frac{1}{2(\theta + \phi_y)} \left[(\sigma + \sigma_t^s)^2 - \sigma^2\right]. \tag{I.15}$$

[3] We appreciate an anonymous referee for pointing out this discontinuity.

We know that σ_t^s follows a process of the form:[4]

$$\mathrm{d}\sigma_t^s = \mu_t^\sigma \mathrm{d}t + \tilde{\sigma}_t \mathrm{d}Z_t,$$

where μ_t^σ and $\tilde{\sigma}_t$ are unknown variables. Applying Ito's Lemma to I.15 we obtain:

$$\mathrm{d}\hat{Y}_t = -\left(\frac{1}{\theta+\phi_y}\right)\left[(\sigma+\sigma_t^s)\cdot\mu^\sigma + \frac{\tilde{\sigma}_t^2}{2}\right]\mathrm{d}t - \tilde{\sigma}_t\cdot\left(\frac{\sigma+\sigma_t^s}{\theta+\phi_y}\right)\mathrm{d}Z_t. \tag{I.16}$$

By equating the drift and volatility terms of equations I.16 and I.15, and based on equation (18), we obtain the unknown variables μ_t^σ and $\tilde{\sigma}_t$ consistent with our guessed solution as follows:

$$\begin{aligned}
\tilde{\sigma}_t &= -(\theta+\phi_y)\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right), \\
\mu_t^\sigma &= -\left(\frac{\theta}{\sigma+\sigma_t^s}\right)\left[\mu\phi_y + \frac{1}{2}\left[(\sigma+\sigma_t^s)^2-\sigma^2\right]\right] - (\theta+\phi_y)^2\frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}.
\end{aligned}$$

Therefore, the σ_t^s process consistent with our guessed solution in (17) can be written as:

$$\begin{aligned}
\mathrm{d}\sigma_t^s = &-\left[\left(\frac{\theta}{\sigma+\sigma_t^s}\right)\left[\mu\phi_y + \frac{1}{2}\left[(\sigma+\sigma_t^s)^2-\sigma^2\right]\right] + (\theta+\phi_y)^2\frac{(\sigma_t^s)^2}{2(\sigma+\sigma_t^s)^3}\right]\mathrm{d}t \\
&-(\theta+\phi_y)\left(\frac{\sigma_t^s}{\sigma+\sigma_t^s}\right)\mathrm{d}Z_t,
\end{aligned} \tag{I.17}$$

leading to

$$\mathrm{d}(\sigma+\sigma_t^s)^2 = -\theta\left[2\mu\phi_y + (\sigma+\sigma_t^s)^2 - \sigma^2\right]\mathrm{d}t - 2(\theta+\phi_y)\sigma_t^s\mathrm{d}Z_t.$$

Notice the two following important observations:

Observation 1 When $\theta = 0$, the solution (I.17) becomes the martingale equilibrium solution of Section 2.1.1.

Observation 2 We obtain solutions with different values of θ and μ parameters, so there is a chance that different parametrization of this pair of parameters can be consistent with a rational expectations equilibrium solution: thus, we consider Case 1, Case 2, and Case 3 in Section 2.1.2.

[4] As in Section 2, we assume that all the aggregate variables are adapted to the given filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Brownian motion dZ_t.

II Detailed Derivations in Section 2

II.1 Model Setup

A representative household solves the following intertemporal optimization consumption-savings decision problem:

$$\max_{\{C_s,L_s\}_{s\geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s, \quad \text{s.t.} \quad \mathrm{d}B_t = \left[i_t B_t - p_t C_t + w_t L_t + D_t\right] \mathrm{d}t,$$

where C_t is consumption, L_t aggregate labor, w_t is the equilibrium wage level, B_t are risk-free bonds held by the household at the beginning of t (hence, B_t at t is taken as given for each household), i_t is the nominal interest rate, D_t is a lump-sum transfer of any firm profits/losses towards the household, p_t the nominal price of consumption goods and ρ is the subjective discount rate of the household.

An individual firm i produces in this economy with the following production function:

$$Y_t^i = A_t L_t^i, \text{ with } \frac{\mathrm{d}A_t}{A_t} = g\mathrm{d}t + \underbrace{\sigma}_{\text{Fundamental risk}} \mathrm{d}Z_t,$$

where A_t is the economy's total factor productivity, assumed to be exogenous and to follow a geometric Brownian motion with drift, where g is the expected growth rate of A_t, σ is its volatility, which we assume to be constant over time and which we define as the *fundamental* volatility, and Z_t is a standard Brownian motion process. It follows that firms' profits are defined as:

$$D_t = p_t Y_t - w_t L_t.$$

Finally, we assume bonds are in zero net supply in equilibrium (i.e., $B_t = 0, \ \forall t$), and that there is no government spending, so market clearing in this economy results in $C_t = Y_t$.

II.2 Flexible Price Economy

We first solve the flexible price economy as our benchmark economy. For that purpose, we assume the usual Dixit-Stiglitz monopolistic competition among firms, where the demand each firm i faces is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. Each firm i takes the aggregate price p_t, wage w_t, and the aggregate output Y_t as given.

II.2.1 Household problem

In the flexible price economy, each household takes the $\{A_t, p_t, i_t\}$ processes as given:

$$\frac{dp_t}{p_t} = \pi_t dt + \sigma_t^p dZ_t,$$

and

$$di_t = \mu_t^i dt + \sigma_t^i dZ_t,$$

where π_t, σ_t^p, μ_t^i, and σ_t^i are all endogenous, so the state variables for each household would become $\{B_t, A_t, p_t, i_t\}$.[5]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, p_t, i_t, t\right) = \max_{\{C_s, L_s\}_{s \geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s.$$

The formula for the stochastic HJB equation is given by

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t} \right\}. \tag{II.1}$$

[5]This is a conjectural but correct statement due to the classical dichotomy between real and nominal sectors: output, consumption, and labor in equilibrium turn out to depend on A_t (only), and it turns out that p_t and i_t do not matter for the real economy and the welfare of the households.

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^{\Gamma} dt + \sigma_t^{\Gamma} dZ_t, \tag{II.2}$$

where

$$\begin{aligned}\mu_t^{\Gamma} =& \Gamma_t + \Gamma_B \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_p \cdot p_t \pi_t + \Gamma_i \cdot \mu_t^i \\ &+ \frac{1}{2}\Gamma_{AA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{pp} \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{ii} \cdot (\sigma_t^i)^2 \\ &+ \Gamma_{Ap} \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{Ai} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{pi} \cdot (p_t \sigma_t^p)\sigma_t^i,\end{aligned}$$

and $\sigma_t^{\Gamma} = \Gamma_A(\sigma A_t) + \Gamma_p(p_t \sigma_t^p) + \Gamma_i(\sigma_t^i)$. In the same way, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$, where

$$\begin{aligned}\mu_t^{\Gamma_B} =& \Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - p_t C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{Bp} \cdot p_t \pi_t + \Gamma_{Bi} \cdot \mu_t^i \\ &+ \frac{1}{2}\Gamma_{BAA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{Bpp} \cdot (p_t \sigma_t^p)^2 + \frac{1}{2}\Gamma_{Bii} \cdot (\sigma_t^i)^2 \\ &+ \Gamma_{BAp} \cdot (\sigma A_t)(p_t \sigma_t^p) + \Gamma_{BAi} \cdot (\sigma A_t)\sigma_t^i + \Gamma_{Bpi} \cdot (p_t \sigma_t^p)\sigma_t^i,\end{aligned} \tag{II.3}$$

and $\sigma_t^{\Gamma_B} = \Gamma_{BA}(\sigma A_t) + \Gamma_{Bp}(p_t \sigma_t^p) + \Gamma_{Bi}(\sigma_t^i)$. Note that $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, p, i\}$. Now plug equation (II.2) into equation (II.1) to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma} \right\}. \tag{II.4}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (II.4), we obtain:

$$\Gamma_B = \frac{1}{p_t C_t}, \tag{II.5}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t}. \tag{II.6}$$

Finally, merging (II.5) with (II.6) gives us the intratemporal optimality condition.

State price density and pricing kernel We know the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$, and $dQ_t = \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (II.4) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*}. \tag{II.7}$$

Taking the derivative of both sides of equation (II.7) with respect to B_t, using the envelope theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*}, \tag{II.8}$$

where $\mu_t^{\Gamma_B,*}$ is from equation (II.3) and it is evaluated at the optimum. Plugging (II.8) into the process for Γ_B, we obtain a simplified expression:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA}(A_t\sigma) + \Gamma_{Bp}(p_t\sigma_t^p) + \Gamma_{Bi}\left(\sigma_t^i\right)\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t. \tag{II.9}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then, using equation (II.9) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t.$$

From the definition of $\mathrm{d}Q_t$, we obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t, \tag{II.10}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ follows by taking expectations, which proves (2) in the flexible price equilibrium.

Nominal and real interest rates Prices and consumption should be adapted to the filtration generated by the Brownian motion Z_t process. Let us express the processes for consumption and price as:

$$\begin{aligned} \mathrm{d}p_t &= \pi_t p_t \mathrm{d}t + \sigma_t^p p_t \mathrm{d}Z_t, \\ \mathrm{d}C_t &= g_t^C C_t \mathrm{d}t + \sigma_t^C C_t \mathrm{d}Z_t, \end{aligned} \tag{II.11}$$

where π_t, g_t^C, σ_t^p and σ_t^C are variables to be determined in equilibrium, and which can be interpreted as the inflation rate, the expected consumption growth, and the volatilities of the price and consumption processes, respectively. As the real state density is defined as $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$, the real interest rate r_t is defined by the relation $\mathbb{E}_t\left[\frac{d\zeta_t^r}{\zeta_t^r}\right] = -r_t dt$, similarly to (2).

With (II.11), applying Ito's Lemma to the real state density $\zeta_t^r = e^{-\rho t}\frac{1}{C_t}$ results in

$$\frac{d\zeta_t^r}{\zeta_t^r} = -\underbrace{\left[\rho + g_t^C - \left(\sigma_t^C\right)^2\right]}_{\equiv r_t}\mathrm{d}t - \sigma_t^C \mathrm{d}Z_t, \tag{II.12}$$

which determines the real interest rate $r_t = \rho + g_t^C - (\sigma_t^C)^2$. We also apply Ito's Lemma to $\zeta_t^N = e^{-\rho t}\frac{1}{p_t C_t}$ and use the above processes for p_t and C_t to obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\left[\rho + g_t^C + \pi_t - (\sigma_t^p)^2 - \left(\sigma_t^C\right)^2 - \sigma_t^p\sigma_t^C\right]\mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right]\mathrm{d}Z_t,$$

which can be rearranged as:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -\underbrace{\left[r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right)\right]}_{=i_t}\mathrm{d}t - \left[\sigma_t^p + \sigma_t^C\right]\mathrm{d}Z_t. \tag{II.13}$$

Comparing equation (II.10) and equation (II.13), we obtain

$$i_t = r_t + \pi_t - \sigma_t^p\left(\sigma_t^C + \sigma_t^p\right), \text{ where: } r_t = \rho + g_t^C - \left(\sigma_t^C\right)^2.$$

II.2.2 Firm problem and equilibrium

Firm optimization The demand faced by each firm i is given by

$$D(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

where p_t^i is an individual firm's price, p_t is the price aggregator, and Y_t is the aggregate output. Each firm i solves the following problem:

$$\max_{p_t^i} \quad p_t^i \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t - \frac{w_t}{A_t}\left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

which results in the following first-order condition for the firm:[6]

$$p_t = \left(\frac{\varepsilon}{\varepsilon - 1}\right)\frac{w_t}{A_t}, \tag{II.14}$$

which is intuitive as it tells us that in equilibrium, price is equal to the marginal cost of production multiplied by the constant mark-up, due to the constant elasticity of demand $\varepsilon > 1$. Using equation (II.14) and the equilibrium condition $C_t = Y_t = A_t L_t$ in the first-order condition of the household in (II.5) and (II.6), we obtain $L_t^n = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$,[7] which is a constant. This implies that in the flexible price equilibrium, we have $C_t^n = Y_t^n = A_t\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$. It follows that the stochastic process for Y_t^n is the same as that for A_t, as follows:

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \frac{\mathrm{d}C_t^n}{C_t^n} = g\mathrm{d}t + \sigma\mathrm{d}Z_t. \tag{II.15}$$

Equation (II.15) implies that the growth rate of consumption and its volatility are $g_t^C = g$ and $\sigma_t^C = \sigma$, so the real interest rate in the flexible price economy, i.e., the natural rate of interest, can be expressed as $r_t^n \equiv r^n = \rho + g - \sigma^2$ from (II.12), which finally gives

$$\frac{\mathrm{d}Y_t^n}{Y_t^n} = \left(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}Z_t,$$

which proves equation (5).

II.3 Rigid Price Economy

We now solve the equilibrium of the rigid price economy with $p_t = \bar{p}$ for all t. The rigid price economy's consumption volatility, which we define as σ_t^C, is given by $\sigma_t^C = \sigma + \sigma_t^s$

[6] In equilibrium, $p_t^i = p_t$ as every firm chooses the same price level.

[7] We impose the superscript n (i.e., natural) in variables to denote that those are the equilibrium values in the flexible price economy.

(i.e. volatility of the flexible price equilibrium in (II.15), plus excess volatility of rigid price equilibrium). Therefore, the consumption process can be written as:

$$\mathrm{d}C_t = g_t^C C_t \mathrm{d}t + (\sigma + \sigma_t^s) C_t \mathrm{d}Z_t. \tag{II.16}$$

Let us conjecture that this endogenous 'excess' volatility σ_t^s, which is one of the state variables in the rigid price economy, follows the process $d\sigma_t^s = \mu_t^\sigma dt + \sigma_t^\sigma dZ_t$. With price rigidity (i.e., $p_t = \bar{p}$ for all t), the agent takes the $\{A_t, \sigma_t^s\}$ processes as given, so the state variables for each household become $\{B_t, A_t, \sigma_t^s\}$.[8]

Hamilton-Jacobi-Bellman (HJB) formulation of the households' problem We define the value function as:

$$\Gamma \equiv \Gamma\left(B_t, A_t, \sigma_t^s, t\right) = \max_{\{C_s, L_s\}_{s \geq t}} \mathbb{E}_t \int_s^\infty e^{-\rho(s-t)} \left[\log C_s - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}\right] \mathrm{d}s.$$

The formula for the stochastic HJB equation is:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \frac{\mathbb{E}_t\left[\mathrm{d}\Gamma\right]}{\mathrm{d}t} \right\}, \tag{II.17}$$

Using Ito's Lemma, we compute:

$$d\Gamma = \mu_t^\Gamma dt + \sigma_t^\Gamma dZ_t, \tag{II.18}$$

where

$$\begin{aligned} \mu_t^\Gamma =& \Gamma_t + \Gamma_B \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_A \cdot A_t g + \Gamma_\sigma \cdot \mu_t^\sigma \\ &+ \frac{1}{2}\Gamma_{AA} \cdot (A_t \sigma)^2 + \frac{1}{2}\Gamma_{\sigma\sigma} \cdot (\sigma_t^\sigma)^2 + \Gamma_{A\sigma} \cdot (A_t\sigma)(\sigma_t^\sigma), \end{aligned}$$

[8] This is a conjectural (but correct) statement as the actual output (thereby, consumption and other variables including inflation, nominal interest rate (that follows the Taylor rule), etc) would turn out to only depend on A_t and σ_t^s under our equilibrium construction.

and $\sigma_t^\Gamma = \Gamma_A(\sigma A_t) + \Gamma_\sigma(\sigma_t^\sigma)$. Applying Ito's Lemma to Γ_B, we compute $d\Gamma_B = \mu_t^{\Gamma_B} dt + \sigma_t^{\Gamma_B} dZ_t$, where

$$\begin{aligned}\mu_t^{\Gamma_B} =& \Gamma_{Bt} + \Gamma_{BB} \cdot (i_t B_t - \bar{p} \cdot C_t + w_t L_t + D_t) + \Gamma_{BA} \cdot A_t g + \Gamma_{B\sigma} \cdot \mu_t^\sigma \\ &+ \frac{1}{2}\Gamma_{BAA} \cdot (A_t\sigma)^2 + \frac{1}{2}\Gamma_{B\sigma\sigma} \cdot (\sigma_t^\sigma)^2 + \Gamma_{BA\sigma} \cdot (A_t\sigma)(\sigma_t^\sigma),\end{aligned} \tag{II.19}$$

and $\sigma_t^{\Gamma_B} = \Gamma_{BA} \cdot (\sigma A_t) + \Gamma_{B\sigma} \cdot \sigma_t^\sigma$. Note that $\Gamma_\Delta = \frac{\partial \Gamma}{\partial \Delta}$ is defined as the derivative with respect to any subindex variable $\Delta = \{t, B, A, \sigma_t^s\}$. Now plug equation (II.18) into equation (II.17) to obtain:

$$\rho \cdot \Gamma = \max_{C_t, L_t} \left\{ \log C_t - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^\Gamma \right\}. \tag{II.20}$$

Households' first-order conditions (FOC) Computing the first-order conditions with respect to C_t and L_t from equation (II.20), we obtain:

$$\Gamma_B = \frac{1}{\bar{p} C_t}, \tag{II.21}$$

$$\Gamma_B = \frac{L_t^{\frac{1}{\eta}}}{w_t}. \tag{II.22}$$

Finally, merging (II.21) with (II.22) gives us the intratemporal condition of the problem.

State price density and pricing kernel We know that the state price density and the stochastic discount factor between two adjacent periods are given by $\zeta_t^N = e^{-\rho t} \frac{1}{\bar{p} C_t}$, and $dQ_t = \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N}$, respectively. Let us use a star superscript to denote the choice variables evaluated at the optimum, that is C_t^* and L_t^*. Then, we can express equation (II.20) as:

$$\rho \cdot \Gamma = \log C_t^* - \frac{(L_t^*)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} + \mu_t^{\Gamma,*}. \tag{II.23}$$

Taking the derivative of both sides of equation (II.23) with respect to B_t, using the envelop theorem and rearranging, we obtain:

$$(\rho - i_t) \cdot \Gamma_B = \mu_t^{\Gamma_B,*}, \tag{II.24}$$

where $\mu_t^{\Gamma_B,*}$ follows from equation (II.19) evaluated at the optimum. Plugging equation (II.24) into the process for Γ_B, we obtain a simplified expression at the optimum:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t + \underbrace{\left(\Gamma_{BA} \cdot (A_t\sigma) + \Gamma_{B\sigma} \cdot (\sigma_t^{\sigma})\right)}_{\equiv \sigma_t^{\Gamma_B}} \mathrm{d}Z_t. \tag{II.25}$$

Notice that $\zeta_t^N = e^{-\rho t}\Gamma_B$, then using equation (II.25) and applying Ito's Lemma, we obtain:

$$\mathrm{d}\zeta_t^N = -\,\zeta_t^N \cdot i_t \mathrm{d}t + \zeta_t^N \cdot \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t.$$

From the previous equation, we obtain:

$$dQ_t \equiv \frac{\mathrm{d}\zeta_t^N}{\zeta_t^N} = -i_t \mathrm{d}t + \left[\frac{\sigma_t^{\Gamma_B}}{\Gamma_B}\right] \mathrm{d}Z_t, \tag{II.26}$$

and $\mathbb{E}_t\left[\mathrm{d}Q_t\right] = -i_t \mathrm{d}t$ also follows in the rigid price economy by taking conditional expectations.

Verification of the Martingale Equilibrium Now let us verify that our martingale equilibrium, characterized by equations (14) and (16), satisfies the equilibrium conditions derived above. From (14) and (16),

$$\hat{Y}_t = -\frac{\left(\sigma + \sigma_t^s\right)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}, \tag{II.27}$$

$$\mathrm{d}\sigma_t^s = \underbrace{-(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3}}_{=\mu_t^{\sigma}} \mathrm{d}t \underbrace{-\phi_y \left(\frac{\sigma_t^s}{\sigma_t + \sigma_t^s}\right)}_{=\sigma_t^{\sigma}} \mathrm{d}Z_t. \tag{II.28}$$

These equations will be a solution to the model, as long as there is no contradiction with the equilibrium conditions. In order to check if (II.27) and (II.28) satisfy the equilibrium conditions, first, the output gap is defined as:

$$\hat{Y}_t = \log\left(\frac{Y_t}{Y_t^n}\right) = \log\left(\frac{C_t}{C_t^n}\right) = \log\left(\frac{C_t}{A_t}\right) - \frac{\eta}{\eta+1}\log\left(\frac{\varepsilon - 1}{\varepsilon}\right), \tag{II.29}$$

where the last equality follows from $C_t^n = A_t \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}}$, as shown above for the flexible price equilibrium. Combining (II.27) and (II.29), we obtain:

$$C_t = A_t \left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\frac{\eta}{\eta+1}} \cdot \exp\left\{-\frac{(\sigma + \sigma_t^s)^2}{2\phi_y} + \frac{\sigma^2}{2\phi_y}\right\},$$

which is a function of A_t and σ_t^s. We can now compute the derivative of equation (II.21) with respect to A_t and σ_t^s as:

$$\Gamma_{BA} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial A_t}, \tag{II.30}$$

$$\Gamma_{B\sigma} = -\frac{\Gamma_B}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s}. \tag{II.31}$$

Plugging equations (II.30) and (II.31) into equation (II.25), we obtain:

$$\mathrm{d}\Gamma_B = (\rho - i_t) \cdot \Gamma_B \mathrm{d}t - \Gamma_B \left[\frac{A_t}{C_t} \cdot \frac{\partial C_t}{\partial A_t} \cdot \sigma + \frac{1}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s} \cdot \sigma_t^\sigma\right] \mathrm{d}Z_t. \tag{II.32}$$

Using Ito's Lemma in equation (II.21) together with equation (II.16), we obtain

$$\mathrm{d}\Gamma_B = -\Gamma_B \left(g_t^C - (\sigma_t^C)^2\right) \mathrm{d}t - \Gamma_B(\sigma + \sigma_t^s)\mathrm{d}Z_t. \tag{II.33}$$

Comparing the volatility terms in (II.32) and (II.33) (i.e., the terms multiplying $\mathrm{d}Z_t$), it must follow that:

$$\sigma + \sigma_t^s = \frac{A_t}{C_t} \cdot \frac{\partial C_t}{\partial A_t} \cdot \sigma + \frac{1}{C_t} \cdot \frac{\partial C_t}{\partial \sigma_t^s} \cdot \sigma_t^\sigma. \tag{II.34}$$

We can now compute the derivative of C_t with respect to A_t and σ_t^s as:

$$\frac{\partial C_t}{\partial A_t} = \frac{C_t}{A_t},$$

and

$$\frac{\partial C_t}{\partial \sigma_t^s} = C_t \cdot \left(\frac{-(\sigma + \sigma_t^s)}{\phi_y}\right) = C_t \cdot (\sigma_t^\sigma)^{-1} \cdot \sigma_t^s,$$

which satisfies (II.34). Therefore, the conjectured martingale solution is verified as a valid equilibrium of the model.

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Online Appendix: For Online Publication Only

A The Model with Stock Markets

We now consider a different theoretical framework with explicit stock markets: Two-Agent New-Keynesian model (TANK) based on Dordal i Carreras and Lee (2024), which enables us to analyze the higher-order moments tied to the *aggregate financial volatility*, and provides us the practical implications about monetary policy rules.[1]

A.1 Setting

Time is continuous, and a *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ is given as in Section 2. The economy consists of a measure one of capitalists, regarded as neoclassical agents, and the same measure of hand-to-mouth workers, regarded as Keynesian agents. All of the financial wealth is concentrated in the hands of capitalists, while hand-to-mouth workers finance their consumption out of labor income in a similar way to Greenwald et al. (2014).[2] There is a single source of exogenous variation in the aggregate production technology A_t, which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and evolves according to a geometric process with volatility σ_t:

$$\frac{dA_t}{A_t} = \underbrace{g}_{\text{Growth}} dt + \underbrace{\sigma}_{\text{Fundamental risk}} dZ_t.$$

We regard the aggregate TFP's volatility σ as the economy's *fundamental* risk, which we take as exogenous. We assume both g and σ to be constant.[3]

A.1.1 Firms and Workers

There are a measure one of monopolistically competitive firms, each producing a differentiated intermediate good $y_t(i)$, $i \in [0,\, 1]$. The final good y_t is constructed

[1] All the detailed derivations and proofs are provided in Online Appendix D.

[2] Greenwald et al. (2014) focus on redistributive shock that shift the share of income between labor and capital as a systemic risk for cross-sectional asset pricing. We instead introduce price nominal rigidities in the framework and analyze monetary policy implications.

[3] This assumption is made for simplicity and our analysis can be extended to include cases where σ_t is time-varying and adapted to the Brownian motion Z_t.

via a Dixit-Stiglitz aggregator with elasticity of substitution $\epsilon > 0$ as in

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

An intermediate firm i has the same production function $y_t(i) = A_t(N_{W,t})^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the economy's aggregate labor, and $n_t(i)$ is the labor demand of an individual firm i at time t. The reason that we introduce a production externality à la Baxter and King (1991) is that it helps us match empirical regularities on asset price and wage co-movements, and it does not affect other qualitative implications of our model.[4] Firm i faces the downward-sloping demand curve $y_i(p_t(i)|p_t, y_t)$, where $p_t(i)$ is the price of its own intermediate good and p_t, y_t are the aggregate price index and output, respectively:

$$y_i(p_t(i)|p_t, y_t) = y_t \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon}.$$

The set of prices charged by intermediate good firms, $\{p_t(i)\}$, is aggregated into the price index p_t as $p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$. In contrast to our Section 2 and Appendix II where we assume perfectly rigid prices, we impose a price stickiness à la Calvo (1983), and firms can change prices of their own intermediate goods with δdt probability in a given time interval dt. In the cross-section, this implies that a total δdt portion of firms reset their prices during a given dt time interval.

A representative hand-to-mouth worker supplies labor to intermediate good producers, receives the equilibrium wage income, and spends every dollar he earns on

[4] In our model, rising asset prices tend to be correlated with the decreasing wage compensation to workers since firm value usually rises if firms can pay less to workers. It violates empirical regularities documented by Chodorow-Reich et al. (2021) in which an increase in stock price tends to push up local aggregate demand variables such as employment and wage. The externality à la Baxter and King (1991) provides us a reasonable calibration that matches these empirical regularities because higher asset prices and aggregate demand raise the firms' marginal product of labor, thus increasing labor demand and wages. Basically, our externality plays similar roles to the capital in the production function.

final good consumption. Each worker solves

$$\max_{C_{W,t},N_{W,t}} \frac{\left(\dfrac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0}, \quad \text{s.t.} \quad p_t C_{W,t} = w_t N_{W,t}, \tag{A.1}$$

at every moment t, where $C_{W,t}$, $N_{W,t}$ and w_t are his consumption, labor supply, and equilibrium wage at time t, respectively, and χ_0 is the inverse Frisch elasticity of labor supply. In a similar manner to Mertens and Ravn (2011), we normalize consumption $C_{W,t}$ by technology A_t, allowing us to work with a simple linearly additive utility consistent with a balanced growth path.[5] As wage w_t is homogeneous across firms, labor demanded by each firm i, $\{n_t(i)\}$, are simply combined into aggregate labor $N_{W,t}$ in a linear manner, i.e., $N_{W,t} = \int_0^1 n_t(i)di$.

Final good output y_t can be written as

$$y_t = \frac{A_t N_{W,t}}{\Delta_t}, \quad \text{where} \quad \Delta_t \equiv \left(\int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} di\right)^{1-\alpha}. \tag{A.2}$$

where Δ_t is defined as the price dispersion measure. Due to the externality à la Baxter and King (1991), the aggregate production function becomes linear in $N_{W,t}$.

A.1.2 Financial Market and Capitalists

Unlike conventional New-Keynesian models where a representative household owns the firms and receives rebated profits in a lump sum manner, we assume firm profits are capitalized in stock markets as a representative index fund. Capitalists face an optimal portfolio allocation problem involving the allocation of their wealth between the risk-free bond and the stock index at every instant t.

Nominal aggregate financial wealth is $p_t A_t Q_t$, where Q_t is the normalized (or TFP detrended) real asset price. Q_t and p_t are endogenous variables adapted to

[5] We introduce the consumption normalization by the aggregate TFP due to the economy's growth. The qualitative results of the model are not affected by this consumption normalization.

filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and assumed to evolve according to

$$\frac{dQ_t}{Q_t} = \mu_t^q dt + \underbrace{\sigma_t^q}_{\text{Financial volatility}} dZ_t, \text{ and } \frac{dp_t}{p_t} = \pi_t dt + \underbrace{\sigma_t^p}_{\text{Inflation risk}} dZ_t,$$

with endogenous drift μ_t^q and volatility σ_t^q. In particular, we interpret σ_t^q as a measure of financial uncertainty or disruption, as spikes in σ_t^q are empirically observed during a financial crisis. Like Q_t, we assume that the price aggregator p_t follows a geometric Brownian motion with drift π_t and volatility σ_t^p. The total financial market wealth $p_t A_t Q_t$ evolves with a geometric Brownian motion with total volatility $\sigma + \sigma_t^q + \sigma_t^p$. Intuitively, if a capitalist invests in the stock market, they have to bear all three risks: inflation risk, technology risk, and (detrended) real asset price risk.

Note that σ_t^q is determined in equilibrium and can be either positive or negative, i.e., $\sigma_t^q < 0$ corresponds to the case where total real wealth $A_t Q_t$ is less volatile than the TFP process $\{A_t\}$.

In addition to the stock market, we assume that there is a risk-free bond with an associated nominal rate i_t that is controlled by the central bank. Bonds are in zero net supply in equilibrium since all capitalists are equal. A measure one of identical capitalists chooses the portfolio allocation between a risk-free bond and a risky index stock, where in the latter case, they earn the profits of the intermediate goods sector as dividends, as well as the nominal price revaluation of the index due to changes in p_t, A_t and Q_t. Financial markets are competitive, thus each capitalist takes the nominal risk-free rate i_t, the expected stochastic stock market return i_t^m, and the total risk $\sigma + \sigma_t^q + \sigma_t^p$ as given when choosing her portfolio decision.[6] If a capitalist invests a share θ_t of her wealth a_t in the stock market, she bears a total risk $\theta_t a_t(\sigma + \sigma_t^q + \sigma_t^p)$ between t and $t + dt$. Therefore, the riskiness of her portfolio increases proportionally to the investment share θ_t in the index. Capitalists are risk-averse, and ask for a risk-premium compensation $i_t^m - i_t$ when they invest in the risky index market, which is to be determined in equilibrium.

[6]This competitive market assumption turns out to be an important aspect of our model for explaining inefficiencies caused by aggregate demand externality that individual capitalist's financial investment decision imposes on the aggregate economy. For this issue, see Farhi and Werning (2016).

Each capitalist with nominal wealth a_t has $\log$-utility and solves

$$\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} \log C_t dt, \text{ s.t. } da_t = \left(a_t \left(i_t + \theta_t(i_t^m - i_t)\right) - p_t C_t\right) dt + \theta_t a_t \left(\sigma + \sigma_t^q + \sigma_t^p\right) dZ_t, \tag{A.3}$$

where ρ, C_t are her discount rate and final good consumption, respectively. At every instant, she earns returns out of both the risk-free bond and the risky stock investments, and spends on final good consumption.

A.2 Equilibrium and Asset Pricing

Due to the $\log$-utility of capitalists, their nominal state price density ξ_t^N[7] is given by

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{p_t}, \text{ with } \mathbb{E}_t\left(\frac{d\xi_t^N}{\xi_t^N}\right) = -i_t dt, \tag{A.4}$$

where the stochastic discount factor between time t (now) and s (future) is by definition given as $\frac{\xi_s^N}{\xi_t^N}$. Aggregate stock market wealth, $p_t A_t Q_t$, is by definition the sum of discounted profit streams from the intermediate goods sector, priced by the above ξ_t^N, as capitalists are the marginal stock market investors in equilibrium. We know that at time t, the entire profit of the intermediate goods sector is given by

$$D_t \equiv \int (p_t(i) y_t(i) - w_t n_t(i)) di = \underbrace{\int p_t(i) y_t(i) di}_{= p_t y_t} - \underbrace{w_t N_{W,t}}_{= p_t C_{W,t}} = p_t(y_t - C_{W,t}) = p_t C_t,$$

where we use the Dixit-Stiglitz aggregator properties that the total expenditure equals the sum of expenditures on intermediate goods and the linear aggregation of labor. Regardless of the price dispersion across firms, the aggregate dividend D_t is equal to the consumption expenditure of capitalists, as workers spend all of their income on consumption.

Plugging the above expressions into the fundamental asset pricing equation

[7] A superscript N means a nominal state-price density, where a superscript r implies a real one.

yields

$$p_t A_t Q_t = \mathbb{E}_t \frac{1}{\xi_t^N} \int_t^{\infty} \xi_s^N \left(\underbrace{D_s}_{=p_s C_s \text{ from (A.7)}} \right) ds = \frac{p_t C_t}{\rho}, \tag{A.5}$$

so $p_t C_t = \rho\,(p_t A_t Q_t)$, which is equal to ρa_t in equilibrium with $a_t = p_t A_t Q_t$. That is, in equilibrium, capitalists hold a wealth amount that equals the total financial market wealth.

Every agent with the same type (i.e., worker or capitalist) is identical and chooses the same decisions in equilibrium. As bonds are in zero net supply, each capitalist's wealth share in the stock market satisfies $\theta_t = 1$, which pins down the equilibrium risk-premium demanded by capitalists. Using (A.3), (A.4), and (A.5), risk-premium is given by

$$\mathrm{rp}_t \equiv i_t^m - i_t = \underbrace{(\sigma + \sigma_t^q + \sigma_t^p)^2}_{\text{Risk-premium}}, \tag{A.6}$$

where rp_t demanded by capitalists increases with either of the three volatilities $\{\sigma_t, \sigma_t^q, \sigma_t^p\}$. As the financial volatility σ_t^q is endogenous, risk-premium rp_t term is endogenous as well and needs to be determined in equilibrium. Note that the wealth gain/loss of the capitalist is equal to the nominal revaluation of the stock. Also note that our equilibrium conditions in (A.5) and (A.6) align with Merton (1971).

We characterize the good's market equilibrium and the equilibrium asset pricing condition of the expected stock return i_t^m as follows: Since capitalists spends ρ portion of their wealth a_t on consumption expenditure and they hold the entire wealth, $C_t = \rho A_t Q_t$ holds in equilibrium. Thus, we can write the equilibrium condition for the final good market as

$$\rho A_t Q_t + \frac{w_t}{p_t} N_{W,t} = \frac{A_t N_{W,t}}{\Delta_t} = y_t. \tag{A.7}$$

The nominal expected return on stock markets i_t^m consists of the dividend yield from the firm profits and the nominal stock price re-valuation (i.e., capital gain) due to fluctuations in $\{p_t, A_t, Q_t\}$. Within our specifications, the dividend yield always is equal to ρ, the discount rate of capitalists. Therefore, when i_t^m changes, only nominal stock prices can adjust endogenously, as the dividend yield is fixed. If we define $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process with $\mathbb{E}_t\left(dI_t^m\right) = i_t^m dt$, the following (A.8) shows the decomposition of i_t^m into dividend yield and stock revaluation in equilibrium:

$$
\begin{aligned}
d\mathbf{I}_t^m &= \frac{\overbrace{\cancel{p_t}\left(\underbrace{y_t - \frac{w_t}{p_t}N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\cancel{p_t}A_tQ_t}_{\text{Total capital market wealth}}}dt + \underbrace{\frac{d\left(p_tA_tQ_t\right)}{p_tA_tQ_t}}_{\text{Capital gain}} = \rho \cdot dt + \frac{d\left(p_tA_tQ_t\right)}{p_tA_tQ_t} \\
&= \underbrace{\left(\rho + \underbrace{\pi_t}_{\text{Inflation}} + g + \mu_t^q + \sigma_t^q\sigma_t^p + \sigma(\sigma_t^p + \sigma_t^q)\right)}_{=i_t^m} dt + \underbrace{\left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{\text{Risk term}} dZ_t.
\end{aligned}
\tag{A.8}
$$

The equilibrium conditions we have obtained consist of the worker's optimization (i.e., solution of (A.1)), labor aggregation, output formula (i.e., (A.2)), capitalist's optimization (i.e., (A.5) and (A.6)), the good market equilibrium (i.e., (A.7)), and determination of the risky stock return (i.e., (A.8)). To close the model, we also have to derive the supply block of the economy (i.e., pricing decisions of intermediate good firms à la Calvo (1983)) and define the monetary policy rule, which is our most important topic of interest.

The following Lemma A.1 re-derives the Fisher equation when there is a correlation between the (aggregate) price process and the wealth process.

Lemma A.1 (Inflation Premium) *Real interest rate is given by*

$$
r_t = i_t - \pi_t + \sigma_t^p \underbrace{\left(\sigma + \sigma_t^p + \sigma_t^q\right)}_{\textit{Wealth volatility}}. \tag{A.9}
$$

A.3 Flexible Price Equilibrium

As a benchmark case, we study the flexible price economy. When firms freely reset their prices (i.e., $\delta \to \infty$ case), the real wage $\frac{w_t}{p_t}$ becomes proportional to aggregate technology A_t. The following proposition summarizes real wage, asset price, natural rate of interest r_t^n (i.e., the real risk-free rate that prevails in the benchmark economy), and consumption process of the capitalist in the flexible price equilibrium. Before we proceed, we define the following parameter, which is the effective labor supply elasticity of workers taking their optimal consumption decision into account.

Definition A.1 *Effective labor supply elasticity of workers:* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0+\varphi}$.

Proposition A.1 (Flexible Price Equilibrium) *In the flexible price equilibrium,*[8] *we obtain the analytic characterization of real wage* $\frac{w_t^n}{p_t^n}$*, asset price* Q_t^n*, natural rate of interest* r_t^n*, and consumption of capitalists* C_t^n *as given below:*

1. *The real wage is proportional to aggregate technology* A_t*, and given by*

$$\frac{w_t^n}{p_t^n} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon} A_t.$$

2. *The equilibrium detrended asset price* Q_t^n *is constant and given by*

$$Q_t^n = \frac{1}{\rho}\left(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right), \ \ \textit{and} \ \ \mu_t^{q,n} = \sigma_t^{q,n} = 0. \tag{A.10}$$

3. *The natural rate* r_t^n *is constant, and given by* $r_t^n \equiv r^n = \rho + g - \sigma^2$*, and consumption of capitalists evolves with*

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \left(\underbrace{r^n - \rho + \sigma^2}_{\equiv \mu_t^{c,n}}\right) dt + \underbrace{\sigma}_{\equiv \sigma_t^{c.n}} dZ_t.$$

In a flexible price equilibrium, Proposition A.1 shows that we can characterize closed-form expressions of the real wage $\frac{w_t^n}{p_t^n}$, detrended stock price Q_t^n, and the natural rate r_t^n. First, $\sigma_t^{q,n} = 0$ holds, implying that there is no additional financial volatility running in the economy, in addition to the TFP risk, σ. This feature arises

[8] We assign superscript n to denote variables in the flexible price (i.e., natural) equilibrium.

because our economy features no explicit frictions (other than nominal rigidity, which is absent for now) and thus every variable other than the labor supply $N_{W,t}^n$ becomes proportional to A_t. This means that real wealth $A_t Q_t^n$ has the exactly same volatility as A_t itself, and the financial market imposes no additional risk on the economy. A higher ϵ, the elasticity of substitution, raises the real wage $\frac{w_t^n}{p_t^n}$. It has two competing effects on asset price Q_t^n. A higher real wage reduces the firms' profit as well as the stock price Q_t^n. On the other hand, a higher wage yields a higher labor supply, raising output and stock price Q_t^n. The effective labor supply elasticity χ^{-1} matters in this second effect, thus (A.10) features χ^{-1} exponent on the term that increases with ϵ.

We observe that the natural real interest rate r_t^n is of the same form as (5) in Section 2. Here, a rise in σ raises the stock market's risk-premium level, given by $\mathrm{rp}_t^n \equiv \sigma^2$, which induces capitalists to reduce their portfolio demand for the stock index, thereby forcing r_t^n to go down in order to prevent a fall in their financial wealth and aggregate demand.

A.4 Sticky Price Equilibrium

When price resetting is sticky à la Calvo (1983), we obtain the Phillips curve that describes inflation dynamics. Since a fixed portion δdt of firms can change their prices on a given infinitesimal time interval dt, we have no stochastic fluctuation in the price process up to a first order, thus $\sigma_t^p = 0$ holds.[9]

A monetary policy rule closes the model. Before analyzing the proper monetary policy rule in this framework, we first describe the 'gap' economy, which is defined as the economy where every variable is in $\log$-deviation from the corresponding level in the flexible price economy. That is, we define any business cycle variable x_t's gap, $\hat{x}_t$, to be the $\log$-deviation of x_t from its natural level x_t^n, which is the level of the variable in the flexible price equilibrium, i.e., $\hat{x}_t \equiv \ln \frac{x_t}{x_t^n}$.

Because the asset price acts as an endogenous aggregate demand shifter, we

[9] As in Section 2, we globally characterize the model's demand block, accounting for time-varying higher-order terms. To simplify the analysis, here we linearize the supply block, following Woodford (2003).

write every other variable's gap in terms of the asset price gap.[10]

Assumption A.1 (Labor Supply Elasticity) $\chi^{-1} > \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Assumption A.1 guarantees the positive co-movement between asset price and other business cycle variables (e.g., real wage and consumptions of capitalists and workers) observed in the data. With large ϵ, firms' mark-ups decrease and real wage rises as a result. It has a negative impact on the stock price as firm profits decrease, making it harder to satisfy a positive co-movement between gaps in asset price and real wage. A rise in α amplifies the effect of Baxter and King (1991)'s externality, raising labor demand: so that a rise in asset price can result in higher labor demand and real wages. Without Assumption A.1, a positive gap in the asset price depresses wages, labor, and consumption of workers, which can be regarded as a redistributive shock from labor to capital, or in the longer-run, might explain a portion of the observed trend towards increased wealth inequality and income stagnation.[11]

The following Lemma A.2 implies given Assumption A.1, gaps in consumptions of capitalists and workers, asset price, employment, and real wage all co-move with one another up to a first-order. For stabilization purposes, the central bank only needs to deal with the asset price gap $\hat{Q}_t$. From $C_t = \rho A_t Q_t$, we infer that $\hat{Q}_t = \hat{C}_t$. Thus we can interchangeably use $\hat{Q}_t$ or $\hat{C}_t$ to denote gaps of asset price Q_t and consumption of capitalists C_t.

Lemma A.2 (Co-movement) *Given Assumption A.1, gaps in consumption C_t of capitalists, and $C_{W,t}$ of workers, labor supply $N_{W,t}$, and real wage $\frac{w_t}{p_t}$ are positively correlated. Up to a first-order, the following approximation holds:*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{\frac{w_t}{p_t}} = \frac{1}{1+\chi^{-1}} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right) \widehat{C_{W,t}}.$$

[10] Assumption A.1 allows our model to match empirical regularities, and holds under a standard calibration in Table C.1 of Appendix C. Even without Assumption A.1, main features of our model in Appendix A remain unchanged.

[11] For instance, Greenwald et al. (2014) interpret redistributive shocks that shift the share of income between labor and capital as a systemic risk to explain various asset pricing phenomena.

Using Lemma A.2, we can actually get the following relation between $\hat{Q}_t$ and $\hat{y}_t$.

$$\hat{y}_t = \zeta \hat{Q}_t, \quad where \ \zeta \equiv \chi^{-1} \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \right)^{-1} > 0. \tag{A.11}$$

Proof. See Online Appendix D. ■

Demand block The dynamic IS equation of $\{\hat{Q}_t\}$ in our model features some important modifications from the canonical New-Keynesian framework. Before we characterize it, we define the risk-premium level $\mathrm{rp}_t \equiv (\sigma + \sigma_t^q)^2$ and its natural level in the flexible price economy $\mathrm{rp}_t^n \equiv \sigma^2$ with $\sigma_t^{q,n} = 0$, as we characterized in equation (A.10). By subtracting rp_t^n from the current risk-premium level rp_t, we define risk-premium gap $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$. Basically, as the risk-premium gap becomes positive in the absence of monetary policy responses, capitalists ask for a higher compensation in bearing financial market risks, causing asset prices to fall below its natural level. We also define the risk-adjusted natural rate r_t^T in the similar way to (9) in Section 2 as $r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t$. r_t^T serves as a real rate anchor for monetary policy. A positive risk-premium gap (i.e., $\hat{rp}_t > 0$), for example, lowers the portfolio demand of capitalists for the stock market compared with the benchmark economy, and thus decreases the anchor rate r_t^T that monetary policy must target for stabilization.

We next characterize the asset price gap $\hat{Q}_t$'s stochastic process. As in equation (8) of Section 2's standard non-linear New-Keynesian framework, the natural rate r_t^n is replaced by the risk-adjusted natural rate r_t^T.

Proposition A.2 (Asset Price Gap Process: IS Equation) *With inflation $\{\pi_t\}$, we obtain*

$$d\hat{Q}_t = (i_t - \pi_t - r_t^T)dt + \sigma_t^q dZ_t, \tag{A.12}$$

where r_t^T takes the role of the natural rate r_t^n. Thus, the aggregate and endogenous financial volatility σ_t^q directly affects the drift of the $\{\hat{Q}_t\}$ process, governing how all other gap variables fluctuate over time.

Proof. See Online Appendix D. ■

With $\sigma_t^p = 0$, capitalists bear $(\sigma + \sigma_t^q)$ as total risk when investing in the stock market. Due to their $\log$ preference, the risk-premium level rp_t is determined to be $(\sigma + \sigma_t^q)^2$. In the flexible price equilibrium, we have the natural rate given by $r_t^n = r^n = \rho + g - \sigma^2$ and σ_t^q is given by $\sigma_t^{q,n} = 0$. Therefore, the expected real stock market return becomes $r_t^n + \sigma^2 - \frac{1}{2}\sigma^2$, where the factor $\frac{1}{2}\sigma^2$ comes from the quadratic variation factor that arises from the second-order Taylor expansion. In our sticky price equilibrium with endogenous asset price volatility σ_t^q, risk premium changes from σ^2 to $(\sigma + \sigma_t^q)^2$. Thus, with monetary policy rate i_t and inflation π_t, the (real) expected stock market return becomes $i_t - \pi_t + \frac{1}{2}(\sigma + \sigma_t^q)^2$. If this return differs from $r_t^n + \frac{1}{2}\sigma^2$, then $\hat{Q}_t$ endogenously adjusts, and this adjustment creates a real distortion from its effect on aggregate demand.

Equation (A.12) has the same mathematical structure as equation (8) in the standard New-Keynesian model. In Section 2, the endogenous business cycle volatility has a first-order impact on aggregate demand through the household's precautionary savings channel, whereas in the current model with stock markets, the aggregate financial volatility affects risk-premium and financial wealth, determining stock prices and aggregate demand. Due to this isomorphic structure between the two models, we will show that our novel findings in Section 2 continue to hold here, with important implications for monetary policy.

Note that when $\sigma_t^q = \sigma_t^{q,n} = 0$, the risk-adjusted natural rate r_t^T equals the natural rate r_t^n and (A.12) becomes

$$d\hat{C}_t = (i_t - \pi_t - r_t^n)dt, \tag{A.13}$$

which is the IS equation in a standard New-Keynesian model. The crux of the problem is that σ_t^q, which we use as a proxy for financial instability, is itself an endogenous variable to be determined in equilibrium, with no guarantee that it equates its natural level $\sigma_t^{q,n} = 0$.

Supply block We follow the literature on pricing à la Calvo (1983) to determine inflation dynamics. The above Lemma A.2 allows us to express the firms' aggregate marginal cost gap in terms of the asset price gap up to a first order, as asset price determines aggregate demand, which in turn determines variables including aggregate marginal cost.

Proposition A.3 (Phillips Curve) *Inflation π_t evolves according to*

$$\mathbb{E}_t d\pi_t = (\rho\pi_t - \frac{\kappa}{\zeta}\hat{y}_t)dt,$$

$$\textit{where: } \kappa \equiv \delta(\delta+\rho)\Theta\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1}, \ \textit{and} \ \ \Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \tag{A.14}$$

Proof. See Online Appendix D. ■

Plugging equation (A.11) into the Phillips curve, we get $\mathbb{E}_t d\pi_t = \left(\rho\pi_t - \kappa\hat{Q}_t\right)dt$, which is expressed in terms of $\hat{Q}_t$. Under Assumption A.1, i.e., $\kappa > 0$, a higher asset price gap $\hat{Q}_t$ means that the economy is over-heated, and thus inflation increases. Note that when the price resetting probability increases (i.e., $\delta \to \infty$), we have $\kappa \to \infty$ and $\hat{Q}_t = 0$ for all t.

As in Section 2, we study one particular form of rational expectations equilibrium that supports an initial volatility σ_0^q: the equilibrium in which the asset price gap $\hat{Q}_t$ follows a local martingale after σ_0^q appears. As $\hat{Q}_t$ is a martingale for any finite t we obtain

$$\pi_t = \kappa\int_t^\infty e^{\rho(s-t)}\underbrace{\mathbb{E}_t(\hat{Q}_s)}_{=\hat{Q}_t}ds = \frac{\kappa}{\rho}\hat{Q}_t, \tag{A.15}$$

by iterating (A.14) over time, which implies that inflation π_t closely follows the trajectory of $\hat{Q}_t$.

Macroprudential policies There are in general two goals in short (and/or medium)-run macroeconomics: *macro-stabilization* and *financial stability*. Many policymakers believe that financial stability should be dealt with by regulations and macroprudential policies imposed on banks and financial institutions, with business cycle stabilization being the sole focus of monetary policy. Because our model is parsimonious and does not include any complex financial market participants, those macroprudential regulations that tackle potential financial instabilities can be regarded as a policy avenue to prevent σ_t^q from deviating from $\sigma_t^{q,n} = 0$. If $\sigma_t^q = \sigma_t^{q,n} = 0$, then as seen in (A.13), our model features exactly the same dynamics as conventional New-Keynesian models. In that case, a conventional policy rule can solely focus on business cycle stabilization.

One interesting aspect built in our model is that financial stability issues (i.e., volatility and risk-premium) are intertwined with macro-stabilization (i.e., output gap and inflation). Therefore, an important question is whether conventional monetary policy rules can achieve both financial stability as well as macro stabilization.

A.5 Monetary Policy

We now analyze conventional Taylor rules with inflation and output gap as policy targets. After showing limitations of such policies and how a self-fulfilling financial volatility can arise, we propose a generalized version of the Taylor rule for stochastic environments that successfully achieve the twin objectives of financial stability and macroeconomic stability. Note that the natural rate of interest and the natural risk-premium are given by $r_t^n = r^n = \rho + g - \sigma^2 > 0$ and $\text{rp}_t^n = \text{rp}^n = \sigma^2$.

A.5.1 Old Monetary Rule

Conventional Taylor rule and Bernanke and Gertler (2000) rule We start with a conventional Taylor rule with a constant intercept equal to the natural rate r^n, and the standard inflation and output gap targets, given by $i_t = r^n + \phi_\pi \pi_t + \phi_y \hat{y}_t$, where $\hat{y}_t$ and π_t are output gap and inflation, respectively. Note that we assume a zero trend inflation target. We can rewrite it as

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t, \ \text{ with } \ \phi_q \equiv \phi_y \zeta, \tag{A.16}$$

as output gap $\hat{y}_t$ is positively correlated with the asset price gap $\hat{Q}_t$ from (A.11). (A.16) is the policy reaction function that targets asset price $\hat{Q}_t$ as well as inflation π_t. Bernanke and Gertler (2000), by incorporating stochastic ad-hoc bubble components to asset prices in a model based on financial frictions à la Bernanke et al. (1999), study whether the monetary policy rule that directly targets asset price as in (A.16) is an effective business cycle stabilizer. Their conclusion is that such rules are undesirable as they deter real economic activities when bubbles appear and burst.[12] In contrast, our framework features no *irrational* asset price bubble: fluctuations in $\hat{Q}_t$ reflect the *rational expectations* about business cycle fluctuations, and thus from the monetary authority's perspective, targeting the stock price gap $\hat{Q}_t$ becomes equivalent to targeting the output gap $\hat{y}_t$, as the two gaps are perfectly correlated up to a first-order. Thus in our framework, a conventional monetary policy rule becomes equivalent to the rule of Bernanke and Gertler (2000).

As we did in Section 2, now we study whether equation (A.16) achieves divine coincidence as in textbook New-Keynesian models. Our objective now is to show that (i) this rule cannot guarantee equilibrium determinacy even if it satisfies the so-called Taylor principle; (ii) the aggregate financial volatility σ_t^q can be created in a self-fulfilling way as in Section 2. We first define $\phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0$, which is the total responsiveness of monetary policy to inflation and asset price gap. $\phi > 0$ corresponds to the conventional Taylor principle that guarantees the uniqueness of equilibriun in log-linearized models. Plugging equations (A.15) and (A.16) into (A.12), we obtain

$$d\hat{Q}_t = \left((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t \underbrace{- \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^q)^2}{2}}_{\text{New terms}} \right) dt + \sigma_t^q dZ_t. \tag{A.17}$$

Multiple equilibria Omitting the volatility feedback terms in equation (A.17), we obtain the usual log-linearized version of the $\hat{Q}_t$ dynamics as

$$d\hat{Q}_t = \left((\phi_\pi - 1)\, \pi_t + \phi_q \hat{Q}_t \right) dt + \sigma_t^q dZ_t,$$

with which the Taylor principle $\phi > 0$ ensures that we achieve the famous *divine coincidence*: $\hat{Q}_t = \pi_t = 0$ for all t is the unique possible rational expectations equilibrium from Blanchard and Kahn (1980). In contrast, now that the aggregate

[12] Recently, Galí (2021) introduces rational bubbles in a New-Keynesian model with overlapping generations, arguing 'leaning against the bubble' monetary policy, if properly specified, can insulate the economy from the aggregate fluctuations due to rational bubbles.

financial volatility σ_t^q affects the drift of equation (A.17), we have multiple equilibria, and σ_t^q can possibly appear in a self-fulfilling way. The reason is similar to why we have self-fulfilling endogenous volatility in Section 2, i.e., σ_t^s.[13] Here, the dynamic IS equation (A.17) features the countercyclical financial volatility σ_t^q: an increase in σ_t^q raises the risk-premium. It in turn brings down the financial wealth and aggregate demand, thus raising the drift of (A.17).

Here is an intuitive way to think about the core reason why the financial volatility σ_t^q is created in a self-fulfilling manner. Imagine that capitalists in our model suddenly fear of a potential financial crisis that features higher levels of risk-premium and financial volatility: they respond by reducing their portfolio demand for the stock market, which leads to the collapse of the asset price, and self-justifies a higher expected return in the stock market and a rise in the risk-premium. This result is related to Acharya and Dogra (2020)'s findings about equilibrium determinacy in models with countercyclical income risks, even though their paper focuses on *idiosyncratic* risks and the effects from precautionary savings, while ours centers on the alternative equilibria stemming from self-fulfilling *aggregate* endogenous risk.

We now formalize the multiple equilibrium intuition presented above by constructing a rational expectations equilibrium that supports an initial volatility σ_0^q. For simplicity, we focus on the case in which σ_0^q jumps off from $\sigma_0^{q,n} = 0$ (i.e., $\sigma_0^q > 0$).

Martingale equilibrium Plugging (A.15) into (A.17) and imposing the local martingale condition with no drift, we obtain

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi} + \frac{\sigma^2}{2\phi}. \tag{A.18}$$

Our martingale equilibrium trajectory does not diverge on expectation in a similar manner to our equilibrium construction in Section 2. The last step is to show that there exists a stochastic path of $\{\sigma_t^q\}$ starting from σ_0^q that supports this equilibrium.

[13]Due to the isomorphic mathematical structure between the dynamics in (A.12) and (8), we easily predict that σ_t^q can arise similarly to the ways σ_t^s arise in a self-fulfilling way in Section 2.

This equilibrium then both (i) supports an initial volatility $\sigma_0^q > 0$ and (ii) does not diverge in the long-run. Using equations (A.17) and (A.18),[14] we obtain the stochastic process of σ_t^q as given by

$$d\sigma_t^q = -\frac{\phi^2 \left(\sigma_t^q\right)^2}{2\left(\sigma+\sigma_t^q\right)^3}dt - \phi\frac{\sigma_t^q}{\sigma+\sigma_t^q}dZ_t. \tag{A.19}$$

Both (A.18) and (A.19) constitute the dynamics of $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ in this particular rational expectations equilibrium supporting $\sigma_0^q > 0$. What does this equilibrium look like? Proposition A.4, like Proposition 1 of Section 2, sheds light on the behavior of $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ paths and argues that similarly to Section 2, $\{\hat{Q}_t, \pi_t, \sigma_t^q\}$ almost surely converge to a perfectly stabilized path (i.e., $\hat{Q}_t = \pi_t = \sigma_t^q = 0$) in the long run. Few paths that do not converge blow up asymptotically with vanishing probability and together with the forward-looking nature of the economy, help sustain the initial crisis.

Proposition A.4 (Bernanke and Gertler (2000) Rule and Indeterminacy) *For any value of Taylor responsiveness $\phi > 0$:*

1. Indeterminacy: *there is always a rational expectations equilibrium (REE) that supports initial volatility $\sigma_0^q > 0$ and is represented by $\hat{Q}_t$ and π_t dynamics in (A.18), and σ_t^q process in (A.19).*

2. Properties*: the equilibrium that supports an initial volatility $\sigma_0^q > 0$ satisfies:*

Property 1 *The endogenous financial volatility σ_t^q converges to zero almost surely, i.e., $\sigma_t^s q \xrightarrow{a.s} \sigma_\infty^q = 0$.*

Property 2 *The asset price gap $\hat{Q}_t$ converges to zero almost surely, i.e., $\hat{Q}_t \xrightarrow{a.s} \hat{Q}_\infty = 0$.*

Property 3 *Non uniform integrability: the total volatility squared $\left(\sigma+\sigma_t^q\right)^2$ satisfies*

$$\mathbb{E}_0\left(\sup_{t\geq 0}\left(\sigma+\sigma_t^q\right)^2\right) = \infty,$$

[14] Since $\hat{Q}_t$ process is a martingale, the drift part in equation (A.17) must be 0 almost surely.

and

$$\lim_{K\to\infty}\sup_{t\geq 0}\left(\mathbb{E}_0\left(\sigma+\sigma_t^q\right)^2\mathbf{1}_{\left\{\left(\sigma+\sigma_t^q\right)^2\geq K\right\}}\right)>0.$$

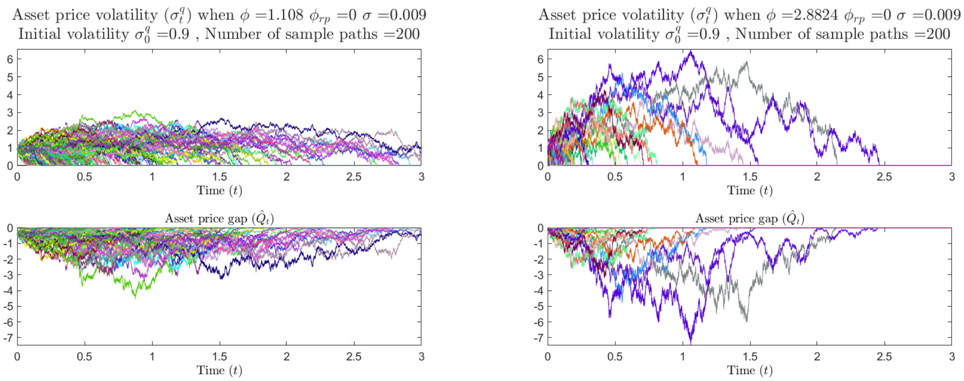
Proposition A.4 is similar to Proposition 1 due to the isomorphic equilibrium structure between Section 2 and Online Appendix A.5.[15] The conditions $\sigma_t^q \xrightarrow{a.s} \sigma_\infty^q = \sigma^{q,n} = 0$, $\hat{Q}_t \xrightarrow{a.s} 0$, and $\pi_t \xrightarrow{a.s} 0$ imply that equilibrium paths supporting an initial volatility $\sigma_0^q > 0$ are almost surely stabilized in the long run. Then, how is it possible for $\sigma_0^q > 0$ to appear at first? The finding $\mathbb{E}_0(\sup_t (\sigma+\sigma_t^q)^2) = \infty$ implies that an initial self-fulfilling shock σ_0^q and the ensuing crisis can be sustained by the vanishing probability of an ∞-severe financial disruption in the far future. This result has similar implications to Martin (2012) in a sense that our framework does not assume the existence of specific disasters but disaster risk is always present even if monetary authority satisfies the Taylor principle and actively stabilizes the business cycle. Martin (2012) applied a similar logic to asset pricing contexts and showed that the pricing of a broad class of long-dated assets is driven by the possibility of extraordinarily bad news in the future.

Calibration and Simulation For the rest of the paper, we calibrate our model parameters based on values commonly found in the previous literature: see Table C.1 in Appendix C for details. A few points are worth mentioning. For worker's risk-aversion parameter φ, we use $\varphi = 0.2$ following Gandelman and Hernández-Murillo (2014).[16] For an individual firm's labor share in production, we use $1-\alpha = 0.6$ following Alvarez-Cuadrado et al. (2018), as we regard the aggregate labor in the production function as a proxy for the capital in conventional macroeconomic models. With this calibration, our co-movement condition (i.e., Assumption A.1) is satisfied.

Figure A.1 illustrates the martingale equilibrium's dynamic paths of $\{\sigma_t^q, \hat{Q}_t\}$

[15] Even with the presence of nontrivial inflation π_t, Figure 1 illustrates the construction of the martingale equilibrium in Proposition A.4.

[16] Gandelman and Hernández-Murillo (2014)'s estimates of φ range between 0.2 and 10. In our environment, a higher risk aversion of workers makes their labor supply (and therefore, output) less responsive to business cycle fluctuations. In that scenario, a higher asset price tends to translate into less wage income distributed to workers, making it harder to satisfy the co-movement condition (i.e., Assumption A.1). Thus, we pick a value on the lower end of the acceptable range and set $\varphi = 0.2$.



(a) With $\phi_\pi = 1.5$

(b) With $\phi_\pi = 2.5$.

Figure A.1: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$.

supporting $\sigma_0^q = 0.9 > \sigma^{q,n} = 0$. Our normalization shows that as σ_0^q jumps off by σ, stock price falls by $2 - 10\%$, which is consistent with our empirical findings in Online Appendix B. Figure A.1 also explores the effects on the martingale equilibrium paths of a change in policy responsiveness to inflation ϕ_π. The right panel A.1b uses the default calibration value $\phi_\pi = 2.5$, while the left panel A.1a assumes a more accommodating stance $\phi_\pi = 1.5$. As we raise ϕ_π, we observe that sample paths are likely to converge faster towards full stabilization at the expense of an increased likelihood of a more severe crisis path in a given period of time. The intuition is simple: for a *given* level of initial volatility $\sigma_0^q > 0$ to be sustained under a more responsive policy rate with higher ϕ_π, it must be true that more amplified endogenous volatility (i.e., high σ_t^q) and severe recession (i.e., low $\hat{Q}_t$) arise with vanishing probability in the future.

Booms In an analogous way, we can construct a rational expectations equilibrium that supports a negative volatility $\sigma_0^q < \sigma_t^{q,n} \equiv 0$. The equilibrium paths feature a boom phase with buoyant production and consumption and with lower levels of financial volatility and risk-premium. A higher ϕ value speeds up the stabilization process, but increases the likelihood of an equilibrium path with an overheated economy.[17]

A.6 Modified Monetary Rule

A modified monetary policy rule includes risk-premium as a separate factor as in

$$i_t = \underbrace{r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Bernanke and Gertler (2000)}} - \underbrace{\frac{1}{2}\hat{rp}_t}_{\text{Risk-premium targeting}}, \quad \text{where } \hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}^n. \tag{A.20}$$

The above monetary policy rule in (A.20) contains a 'risk-premium gap term' as a factor in addition to inflation and asset price gap. It also can be written in terms of the risk-adjusted natural rate r_t^T as

$$i_t = r_t^T + \phi_\pi \pi_t + \phi_q \hat{Q}_t,$$

where a higher $\hat{rp}_t$ brings down r_t^T, forcing i_t to fall. The following Proposition A.5 establishes that a monetary policy rule following (A.20) and that satisfies the Taylor principle, i.e., $\phi > 0$ restores equilibrium determinacy and fully stabilizes the economy.

Proposition A.5 (Risk-Premium Targeting and Ultra-Divine Coincidence) *The monetary policy rule*

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \frac{1}{2}\hat{rp}_t, \quad \textit{where } \phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0, \tag{A.21}$$

achieves $\hat{Q}_t = \pi_t = \hat{rp}_t = 0$ as the unique rational expectations equilibrium. Therefore, monetary policy rule (A.21) attains stabilization in (i) output and asset price, (ii) inflation, and (iii) financial markets (i.e., financial volatility and risk-premium). We call it the ultra-divine coincidence.

This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984). The reason that central banks target risk-premium as a separate factor is

[17] Two singular points exist in the $\{\sigma_t^q\}$ process in (A.19): as σ_t^q hits $-\sigma$, both drift and volatility diverge, and $\{\sigma_t^q\}$ process features a jump. When σ_t^q hits 0, it stays there forever so $\sigma_t^q = 0$ thereafter.

that this term directly appears in the drift of our dynamic IS equation (i.e., (A.12)). According to the policy rule in (A.21), central banks lower the policy rate i_t when $\mathrm{rp}_t > \mathrm{rp}^n$ to boost $\hat{Q}_t$ and $\hat{C}_t$,[18] since a higher risk-premium drags down asset price and business cycle levels. If monetary policy offsets effects of the excess volatility (or excess risk-premium) with this additional target in its rule, it precludes the possibility of a self-fulfilling rise in financial volatility. Combined with the Taylor principle (i.e., $\phi > 0$) that guarantees unique equilibrium in a log-linearized setting, the policy rule in equation (A.21) restores equilibrium determinacy and achieves both macro stability (with $\hat{Q}_t = \pi_t = 0$) and financial stability (with $\hat{rp}_t = 0$, which implies $\mathrm{rp}_t = \mathrm{rp}^n$ and $\sigma_t^q = \sigma_t^{q,n} = 0$). The interest rate on the equilibrium path then becomes $i_t = r^n$, which is the same level as in the equilibrium path of a canonical New-Keynesian model. Therefore, the ultra-divine coincidence result implies: one policy tool (i_t rule) achieves an additional objective (financial stability) in addition to the two usual mandates (output gap and inflation stability). This is possible in our framework because financial markets and the business cycle are tightly interwoven and real and financial instabilities are equivalent to each other.

A common view holds that monetary policy should respond to financial market disruptions only when they affect (or to the degree that they affect) the original mandates (i.e., inflation stability and full employment). This premise is at odds with our results: the failure to target the risk-premium of financial markets subjects the economy to the apparition of self-fulfilling financial volatility and risk-premium, and the corresponding recessions and overheating episodes that ensue. Only by targeting risk-premium in the particular way characterized in (A.20), the monetary authority can re-establish equilibrium determinacy and achieve the ultra-divine coincidence outlined in the previous paragraphs.

[18] Even with Bernanke and Gertler (2000) rule, monetary policy responds to a rise in risk-premium since it negatively affects the asset price gap $\hat{Q}_t$ and inflation π_t. Our point is that the policy rate must systematically respond to deviations of rp_t from its natural level rp^n given $\hat{Q}_t$ and π_t levels.

Interpretation We can rewrite our modified Taylor rule in (A.21) as

$$\underbrace{i_t + \mathrm{rp}_t}_{=i_t^m} - \underbrace{\frac{1}{2}\mathrm{rp}_t}_{\text{Ito term}} = \underbrace{r^n + \mathrm{rp}^n}_{=i_t^{m,n}} - \underbrace{\frac{1}{2}\mathrm{rp}^n}_{\text{Ito term}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}} \ ,$$

or equivalently as

$$\underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\ \text{yield}}} + \underbrace{\frac{\mathbb{E}_t\left(d\log a_t\right)}{dt}}_{\substack{\text{Internal rate of return}\\ \text{of aggregate wealth}}}}_{\text{Cum-dividend stock return}} = \underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\ \text{yield}}} + \frac{\mathbb{E}_t\left(d\log a_t^n\right)}{dt}}_{\text{Benchmark cum-dividend stock return}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}} \ , \qquad \text{(A.22)}$$

where a_t is the economy's aggregate financial wealth, i.e., $p_t A_t Q_t$, and a_t^n is the aggregate wealth of the natural (i.e., flexible price) economy. Our modified monetary policy rule that targets a risk-premium as prescribed in equation (A.21) thus can be interpreted as a rule on the rate of change of $\log$-aggregate wealth as a function of traditional inflation and output gap (asset price) targets. Basically, the rate that determines the households' intertemporal substitution should be the expected return on stock markets, instead of just the risk-free policy rate, and therefore in order to achieve determinacy as well as stabilization in our model, the expected return on stock markets must target business cycles.

We interpret the rule in (A.22) as the *generalized Taylor rule*. With this rule, the central bank uses the aggregate wealth and its rate of return as *intermediate* targets towards achieving business cycle stabilization.

Practicality Some issues still remain about the feasibility to implement this new policy rule in (A.21). First, the risk premium gap $\hat{\mathrm{rp}}_t$ in (A.20) depends on the natural level of risk-premium, rp^n, which is a counterfactual variable by definition, and therefore its observation is subject to some form of measurement error. Second, there are multiple kinds of risk-premia in financial markets that can be possibly targeted through monetary policy, and the construction of the aggregate risk-premium index as featured in our model might be subject to error as well. More importantly, and related to the previous two points, the coefficient attached to the risk-premium in (A.20) is exactly $\frac{1}{2}$. Given the potential for measurement error in $\hat{rp}_t$, it might be impossible for the central bank to target the risk-premium with the exact strength demanded by (A.20).[19] To understand the consequences of deviating from the $\frac{1}{2}$ risk-premium target, we consider the following alternative rule:

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \phi_{\text{rp}} \hat{rp}_t, \tag{A.23}$$

where ϕ_{rp} is a constant term potentially different from $\frac{1}{2}$. With the policy rule in (A.23), we obtain

$$d\hat{Q}_t = \left((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t + \left(\frac{1}{2} - \phi_{\text{rp}} \right) \hat{rp}_t \right) dt + \sigma_t^q dZ_t, \tag{A.24}$$

as $\{\hat{Q}_t\}$ dynamics. With $\phi_{\text{rp}} = \frac{1}{2}$, we return to determinacy (i.e., Proposition A.5). With $\phi_{\text{rp}} \neq \frac{1}{2}$, the martingale equilibrium with self-fulfilling volatility σ_t^q reappears and is characterized by[20]

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\text{total}}} + \frac{\sigma^2}{2\phi_{\text{total}}} \quad \text{with} \quad \phi_{\text{total}} \equiv \frac{\phi}{1 - 2\phi_{\text{rp}}}, \tag{A.25}$$

where $\{\sigma_t^q\}$'s stochastic process after an initial volatility σ_0^q appears is given as

$$d\sigma_t^q = -\frac{\phi_{\text{total}}^2 \left(\sigma_t^q\right)^2}{2\left(\sigma + \sigma_t^q\right)^3} dt - \phi_{\text{total}} \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t. \tag{A.26}$$

When $\phi_{\text{rp}} < \frac{1}{2}$, including the case of $\phi_{\text{rp}} = 0$ in Proposition A.4, an increase in ϕ_{rp} leads to an increase in ϕ_{total} from (A.25). From (A.26), we observe that a higher ϕ_{total} accelerates the convergence of sample paths while creating more amplified ones given initial volatility σ_0^q. As far as $\phi_{\text{rp}} < \frac{1}{2}$, a higher ϕ_{rp} means monetary policy responds more strongly to fluctuations in $\hat{rp}_t$, which allows for faster stabilization. As ϕ_{rp} goes up from 0 to $\frac{1}{2}$, fluctuations in $\hat{rp}_t$ have a weaker direct effect on the drift of (A.24). Thus, the volatility of $\{\sigma_t^q\}$ process in (A.26) must rise to

[19] As an example, consider a multiplicative measurement error ε_t such that $\hat{rp}_t^{obs} = \varepsilon_t \cdot \hat{rp}_t$, where $\hat{rp}_t^{obs}$ is the measured premium.

[20] Equations (A.23) and (A.25) are easily derived in a similar way to Proposition A.4.

ensure that the initial volatility σ_0^q is supported, as on average the economy is better stabilized with a higher ϕ_{rp}. $\{\hat{Q}_t\}$ is eventually stabilized, which results, on average, on shorter but more amplified sample paths.

$\phi_{\text{rp}} < 0$ case is interesting since it implies that the central bank raises the policy rate in response to an increase of the risk premia. It is consistent with the *Real Bills Doctrine* which was a popular idea during the first half of the 20th century. Basically, the doctrine advocated for the Fed discount rate to track the average interest rate of the financial markets, as a means of stabilization. In our framework, $\phi_{\text{rp}} < 0$ pushes down ϕ_{total} from ϕ, thereby effectively slowing down the pace of stabilization after self-fulfilling σ_0^q arises. So this confirms that the *Real Bills Doctrine* with $\phi_{\text{rp}} < 0$ is not suitable for stabilization purposes, as empirically documented by Richardson and Troost (2009).

With $\phi_{\text{rp}} > \frac{1}{2}$, monetary policy responds too strongly to fluctuations in risk-premium, thus with an initial positive volatility $\sigma_0^q > 0$, the policy rate drops excessively and creates an artificial boom instead of a crisis.[21] A higher ϕ_{rp} reduces $|\phi_{\text{total}}|$ and slows down stabilization since it means that monetary policy deviates more from determinacy (the case of $\phi_{\text{rp}} = \frac{1}{2}$), and thus becomes less efficient at stabilization. Table C.3 and Figure C.7 in Online Appendix C summarize our discussion and provides simulation results, respectively.

B Suggestive Evidence

Purpose of this section In this section, we provide the empirical evidence that financial volatility is an important driver of the business cycle. The impulse-response function results in this section provide moments to match with our model with stock markets in Online Appendix A, as seen in Figure A.1.

Stock market volatility is commonly viewed in the literature as a proxy of financial and economic uncertainty, which Bloom (2009) and later Gilchrist and Zakrajšek (2012), Bachmann et al. (2013), Jurado et al. (2015), Caldara et al. (2016), Baker et al. (2020), Coibion et al. (2021) further studied as a driving force behind

[21] With $\phi_{\text{rp}} > \frac{1}{2}$, $\phi_{\text{total}} < 0$ from (A.25). Therefore $\sigma_t^q > 0$ is equivalent to the boom phase with $\pi_t > 0$ and $\hat{Q}_t > 0$.

business cycles fluctuations. In this Section, we evaluate these claims and present interesting empirical results. Figure C.4 in Online Appendix C provides the first piece of supportive evidence in that direction. Panel C.4a depicts several variables commonly used in the literature to measure financial uncertainty. The correlation between series is remarkably high and they all display positive spikes at the beginning and/or initial months following an NBER-dated recession, which is consistent with the evidence that many of these episodes were financial in nature.[22] Panel C.4b plots Ludvigson et al. (2021) (henceforth, LMN) financial and real (i.e. non-financial) uncertainty series. These variables are positively correlated and display a similar propensity to increase around recessions, though a different type of crisis (e.g. financial or not) is correlated with a different type of uncertainty playing the dominant role. For example, the massive spike in real vis-à-vis financial uncertainty following the recent Covid-19 recession, which initially was a health crisis that spilled into the real economy, can be observed in Panel C.4b.

The patterns displayed in Figure C.4 do not yet constitute a proof of the importance of financial market uncertainty as a driver of the business cycle, as we should worry about the possibility of reverse causation running from unfavorable economic conditions towards uncertainty. We tackle this issue by proposing a simple Vector Autoregression (VAR) with the structural identification strategy based on the timing of macroeconomic shocks similar to Bloom (2009). Equation (B.1) presents the variables considered and their ordering, with non-financial series first and financial variables last.[23]

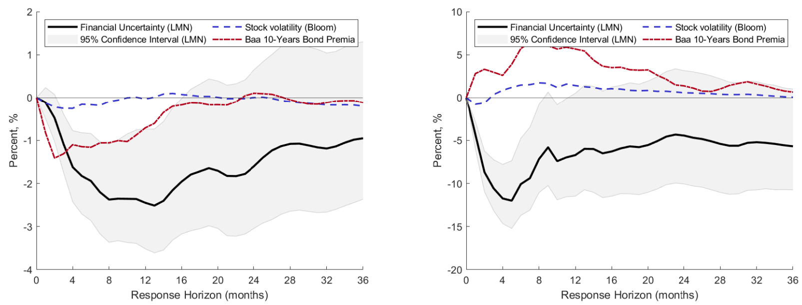
[22]See Reinhart and Rogoff (2009) and Romer and Romer (2017) for the classification of the past recessions. Their analysis showed many recessions had roots in financial markets.

[23]The ordering is used by Ludvigson et al. (2021), who, using identification strategy based on event constraints, find that the uncertainty of financial markets tends to be an exogenous source of business cycle fluctuations, while the real uncertainty is more likely an endogenous response to the business cycle fluctuations. We also have implemented alternative specifications and orderings that produced qualitatively similar results (not reported, provided upon request).

$$\text{VAR-11 order:} \quad \begin{bmatrix} \log(\text{Industrial Production}) \\ \log(\text{Employment}) \\ \log(\text{Real Consumption}) \\ \log(\text{CPI}) \\ \log(\text{Wages}) \\ \text{Hours} \\ \text{Real Uncertainty (LMN)} \\ \text{Fed Funds Rate} \\ \log(\text{M2}) \\ \log(\text{S\&P-500 Index}) \\ \text{Financial Uncertainty (LMN)} \end{bmatrix} \tag{B.1}$$

Both LMN real and financial uncertainty measures are included to differentiate the effects of financial volatility shocks from the effects from real uncertainty. For similar reasons, we include the S&P-500 index in our VAR to empirically distinguish between shocks affecting the level of financial markets and shocks affecting their volatility. In order to ameliorate possible concerns about the validity of the structural identification strategy, we estimate our VAR using monthly data, where the identification assumptions are more likely to hold. Figure B.2 presents the impulse responses to the orthogonalized financial uncertainty shock. Panel B.2a plots the response of industrial production, which falls by up to 2.5% and displays moderate persistence following a one standard deviation shock to financial uncertainty. Panel B.2b plots the response of the S&P-500 Index, which drops up to 12% within the first four months before gradually recovering. Together, both pictures imply a rise of financial uncertainty depresses both industrial activity and financial markets.

Figure B.2 also features alternative estimates using common financial uncertainty proxies such as Bloom (2009) stock market volatility index and 10-years premium on Baa-rated corporate bonds. The responses are generally more muted, and take the opposite sign in the case of the S&P Index. These results can be explained by the fact that standard proxies contain information unrelated to financial uncertainty that distorts our estimates (see Jurado et al. (2015) for a discussion),



(a) Response: Industrial Production (b) Response: S&P-500 Index

Figure B.2: Impulse Response Functions (IRFs), selected series. Figures B.2a and B.2b display the response to a one standard deviation financial uncertainty shock of monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with the variable composition and ordering given in (B.1). Shaded area indicates 95% confidence interval around financial uncertainty measure computed using standard bootstrap techniques.

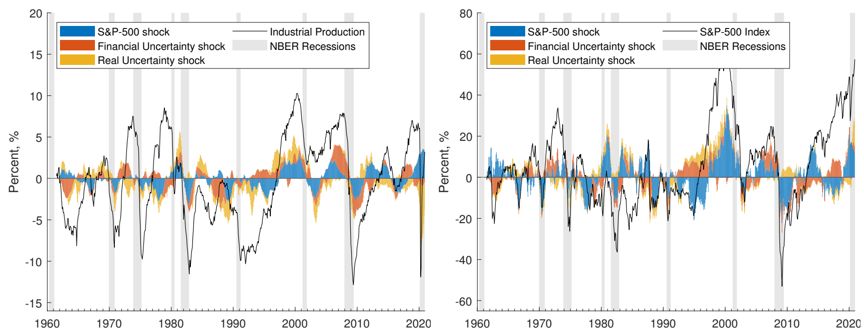
and therefore we choose LMN as our preferred financial uncertainty measure. In Online Appendix C, we report additional impulse response estimates. Especially, the Figure C.6 in Online Appendix C shows that monetary authorities respond with accommodating interest rate movements to financial uncertainty shocks, while real uncertainty has no statistically significant effect on either interest rates or stock market fluctuations.

Finally, we can further explore the contribution of financial uncertainty to business cycle fluctuations by looking at Table C.2 in Online Appendix C, which reports the Forecast Error Variance Decomposition (FEVD) of Industrial Production and the S&P-500 Index. Financial uncertainty shocks explain close to 5% of the fluctuations in both series, while real uncertainty explains an additional 2-4% of movements in industrial activity in the medium run. Figure C.3 provides a more graphical illustration of these results by plotting the historical decomposition of the series. We observe that the contribution of financial uncertainty rivals that of shocks to the level of financial variables captured by the S&P-500 shock, and is especially important in driving industrial production boom-bust patterns during and in the preceding months of recessionary episodes.

C Additional Figures and Tables

Parameter	Value	Description
φ	0.2	Relative Risk Aversion
χ_0	0.25	Inverse Frisch labor supply elasticity
ρ	0.020	Subjective time discount factor
σ	0.0090	TFP volatility
g	0.0083	TFP growth rate
α	0.45	$1 -$ Labor income share
ϵ	7	Elasticity of substitution intermediate goods
δ	0.45	Calvo price resetting probability
ϕ_π	2.50	Policy rule inflation response
ϕ_y	0.11	Policy rule output gap response
ϕ_{rp}	0	Policy rule risk premium response
$\bar{\pi}$	0	Steady state trend inflation target

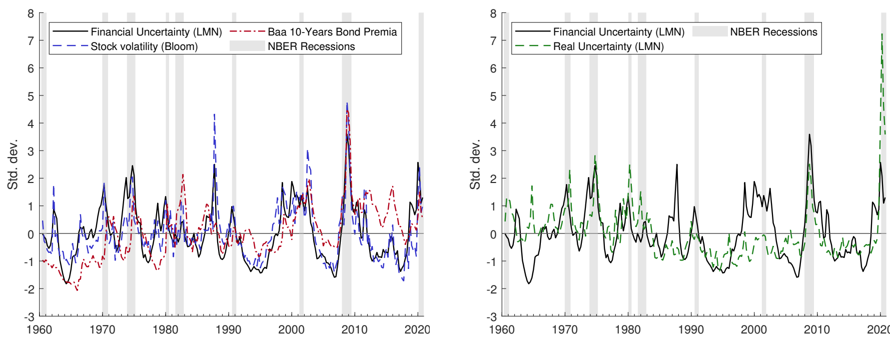
Table C.1: Baseline parameter calibration used in Online Appendix A.5.



(a) Industrial Production

(b) S&P-500 Index

Figure C.3: Historical Decomposition, selected series. Figures C.3a and C.3b display the historical decomposition of monthly Industrial Production and S&P-500 Index series, respectively, based on the VAR-11 with variable composition and ordering in (B.1). Variables are de-trended by subtracting the contribution of initial conditions and constant terms after series decomposition. Columns report a contribution of each shock to the fluctuations around trend of the variable considered.



(a) Financial Uncertainty series (b) Financial vs. Real Uncertainty

Figure C.4: Uncertainty series. Figure C.4a displays common measures of financial uncertainty. Figure C.4b displays Ludvigson et al. (2021) (henceforth, LMN) measures of financial and real economic uncertainty. LMN financial and real economic uncertainty series are constructed as the average volatility of the residuals from predictive regressions on financial and real economic variables, respectively (See Ludvigson et al. (2021)). Bloom (2009)'s stock market volatility is constructed using VXO data from 1987 onward and the monthly volatility of the S&P 500 index normalized to the same mean and variance in the overlapping interval for the 1960-1987 period (See Bloom (2009)). The bond risk-premia series is the Moody's seasoned Baa corporate bond yield relative to the yield on a 10-year treasury bond at constant maturity. The depicted series have a normalized zero mean and one standard deviation.

Online Appendix: For Online Publication Only

(i) Industrial Production

Horizon	Fin. Uncert (LMN)	Real Uncert. (LMN)	Stock Vol (Bloom)	Baa 10-Yr Premia
h=1	0	0.30	0.21	0.12
h=6	1.27	3.37	2.99	1.35
h=12	4.28	4.38	3.15	1.93
h=36	3.24	1.67	1.98	0.64

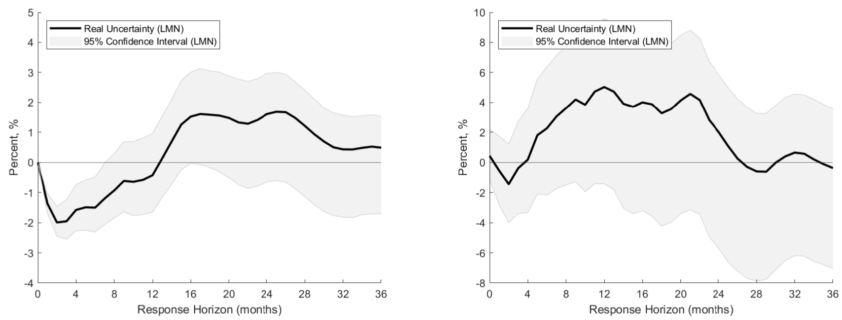
(ii) S&P-500 Index

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.11	0.07	0.39	0.06
h=6	3.29	0.25	3.25	0.62
h=12	4.76	0.54	10.05	2.16
h=36	6.49	0.9	12.21	2.40

(iii) Fed Funds Rate

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.01	0.98	0	0.08
h=6	0.42	0.84	3.14	1.65
h=12	1.47	0.91	4.71	2.30
h=36	2.81	2.05	5.02	3.16

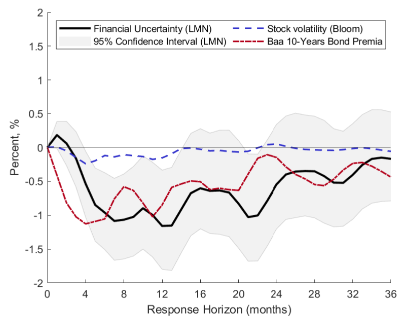
Table C.2: Forecast Error Variance Decomposition (FEVD). The table presents the variance contribution (in percentage) of financial and real uncertainty shocks to selected series at different time horizons (in months). The FEVD is constructed using a VAR-11 with equation (B.1) variable composition and ordering. The first two columns report the contribution of LMN financial and real uncertainty shocks, respectively. The last two columns report alternative VAR specifications where the preferred LMN financial uncertainty measure (column one) is replaced by common proxies employed in the literature, either Bloom (2009) stock market volatility measure or the Baa 10-years corporate bond premia.



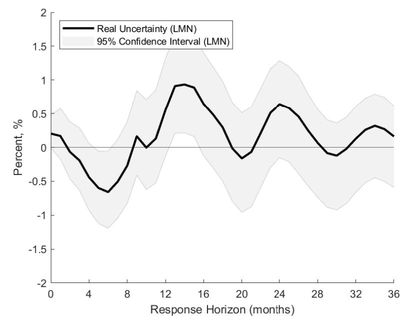
(a) Response: Industrial Production

(b) Response: S&P-500 Index

Figure C.5: Impulse Response Functions (IRFs), selected series. Figures C.5a and C.5b display the response to one standard deviation real uncertainty shock by monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (B.1) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.



(a) Shock: Financial Uncertainty

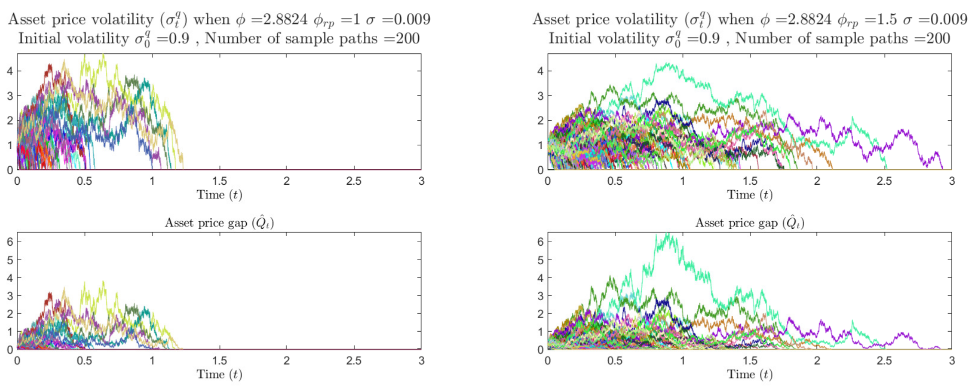


(b) Shock: Real Uncertainty

Figure C.6: Impulse Response Functions (IRFs), Fed Funds Rate. This Figure displays the response to a one standard deviation uncertainty (financial or real) shock by monthly Fed Funds Rate series, using a VAR-11 with equation (B.1) variable composition and ordering. Panel C.6a plots the response to a financial uncertainty shock, and Panel C.6b to a real uncertainty shock. Shaded area indicates 95% confidence interval around preferred financial/real uncertainty measure computed using standard bootstrap techniques. Additional lines display alternative impulse responses obtained by substituting preferred LMN financial uncertainty measure with common proxies employed in the literature.

$\phi_{\mathbf{rp}} < 0$ **(Real Bills Doctrine)**	$0 \leq \phi_{\mathbf{rp}} < \frac{1}{2}$
(i) With $\phi_{\mathrm{rp}} \downarrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)	(i) With $\phi_{\mathrm{rp}} \uparrow$, convergence speed$\uparrow$ and more amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)
$\phi_{\mathbf{rp}} = \frac{1}{2}$	$\phi_{\mathbf{rp}} > \frac{1}{2}$
No sunspot (ultra-divine coincidence)	(i) With $\phi_{\mathrm{rp}} \uparrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a boom ($\hat{Q}_t > 0$ and $\pi_t > 0$)
As $\phi \uparrow$, convergence speed$\uparrow$ and $\exists$ more amplified paths	

Table C.3: Effects of different parameters $\{\phi_{\mathrm{rp}},\ \phi\}$ on stabilization in Section A.6.



(a) With $\phi_{\text{rp}} = 1$

(b) With $\phi_{\text{rp}} = 1.5$.

Figure C.7: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$, with varying $\phi_{\text{rp}} > \frac{1}{2}$.

D Derivations and Proofs for Online Appendix A

Worker's optimization At each time t, each hand-to-mouth worker solves

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0}, \quad \text{s.t.} \quad p_t C_{W,t} = w_t N_{W,t}. \tag{D.1}$$

Solving (D.1) is trivial, resulting in

$$N_{W,t} = \left(\frac{w_t}{p_t}\right)^{\frac{1-\varphi}{\chi_0+\varphi}} \frac{1}{A_t^{\frac{1-\varphi}{\chi_0+\varphi}}} = \left(\frac{w_t}{p_t A_t}\right)^{\frac{1}{\chi}}, \quad C_{W,t} = \frac{w_t}{p_t} N_{W,t} = \left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} A_t^{-\frac{1}{\chi}}, \tag{D.2}$$

where we use $\chi \equiv \frac{\chi_0+\varphi}{1-\varphi}$ from Definition A.1.

Capitalist's optimization In equilibrium, each capitalist chooses $\theta_t = 1$ as bond markets are in zero net supply. Using $\rho a_t = p_t C_t$ from (A.5), the capitalists' budget flow constraint in (A.3) becomes:

$$\frac{da_t}{a_t} = (i_t^m - \rho)\, dt + (\sigma + \sigma_t^q + \sigma_t^p)\, dZ_t. \tag{D.3}$$

The capitalist's state price density in equilibrium is thereby given by

$$\xi_t^N = e^{-\rho t}\frac{1}{p_t C_t} = e^{-\rho t}\frac{1}{\rho a_t},$$

on which we can apply Ito's Lemma and obtain

$$\begin{aligned} -\frac{d\xi_t^N}{\xi_t^N} &= \frac{da_t}{a_t} - \left(\frac{da_t}{a_t}\right)^2 + \rho dt \\ &= \underbrace{\left(i_t^m - (\sigma + \sigma_t^q + \sigma_t^p)^2\right)}_{=i_t} dt + (\sigma + \sigma_t^q + \sigma_t^p)\, dZ_t = i_t dt + (\sigma + \sigma_t^q + \sigma_t^p) dZ_t, \end{aligned} \tag{D.4}$$

which results in $i_t + (\sigma + \sigma_t^q + \sigma_t^q)^2 = i_t^m$ (i.e., equation (A.6)) from $\mathbb{E}_t\left(-\frac{d\xi_t^N}{\xi_t^N}\right) = i_t dt$. Note that (A.5) and (D.4) are the same conditions as in Merton (1971).

Proof of Lemma A.1. We know that in equilibrium, each capitalist holds financial wealth $a_t = p_t A_t Q_t$, since all of them are identical both ex-ante and ex-post. We start by stating capitalist's nominal state-price density ξ_t^N and real state-price density ξ_t^r. The nominal state-price density is relevant to the nominal interest rate, while the real state-price density matters when we calculate the real interest rate. The real state price density ξ_t^r is given by

$$\xi_t^r = e^{-\rho t}\frac{1}{C_t} = p_t \xi_t^N. \tag{D.5}$$

Using (D.4), we can apply Ito's Lemma to (D.5) and obtain

$$\frac{d\xi_t^r}{\xi_t^r} = \left(\underbrace{\pi_t - i_t - \sigma_t^p\left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{=-r_t}\right) dt - (\sigma + \sigma_t^q) dZ_t, \tag{D.6}$$

from which we obtain the Fisher identity with the inflation premium in equation (A.9):

$$r_t = i_t - \pi_t + \sigma_t^p\left(\sigma + \sigma_t^q + \sigma_t^p\right).$$

■

Proof of Proposition A.1. We start from the intermediate firms' optimization problem. As we have the externality à la Baxter and King (1991), we need to go through additional steps in aggregaing individual decisions across firms. Let firm i take its demand function as given and choose the optimal price $p_t(i)$ at any t. With $E_t \equiv (N_{W,t})^{\alpha}$, from the production function, we have

$$n_t(i) = \left(\frac{y_t(i)}{A_t E_t}\right)^{\frac{1}{1-\alpha}}.$$

Then each firm i chooses p_i that maximizes its profit, solving

$$\max_{p_t(i)} p_t(i) \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} y_t - w_t \left(\frac{y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}}. \tag{D.7}$$

In the flexible price economy, all firms charge the same price (i.e., $p_t(i) = p_t$, $\forall i$) and hire the same amount of labor (i.e., $n_t(i) = N_{w,t}$, $\forall i$). From (D.7), we obtain

$$\frac{w_t^n}{p_t^n} = \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} = \frac{\epsilon-1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}},$$

from which we obtain the following equilibrium real wage:

$$\frac{w_t^n}{p_t^n} = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}}.$$

In flexible price equilibrium, we know the aggregate production is linear, i.e., $y_t = A_t N_{W,t}$. Therefore, we obtain

$$y_t = A_t \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}}. \tag{D.8}$$

From (D.8), we write the natural level of output y_t^n and the natural real wage $\frac{w_t^n}{p_t^n}$ as

$$y_t^n = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \text{ and } \frac{w_t^n}{p_t^n} = \frac{\epsilon-1}{\epsilon}(1-\alpha) A_t,$$

from which in equilibrium, we obtain

$$N_{W,t}^{n} = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \text{ and } C_{W,t}^{n} = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t. \tag{D.9}$$

In equilibrium, consumption of capitalists and workers add up to the final output produced (i.e., equation (A.7)). Based on (D.9), we obtain

$$\rho A_t Q_t^n + \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t.$$

where we define Q_t^n to be the natural level of detrended stock price. Therefore we obtain Q_t^n and C_t^n, given by

$$Q_t^n = \frac{1}{\rho}\left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}}\left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right),$$

and $C_t^n = \rho A_t Q_t^n$. Since Q_t^n is constant, there is no drift and volatility for its process in the flexible price economy, thus we have $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To calculate the natural interest rate r_t^n, we start from the capital gain component in equation (A.8). By applying Ito's lemma, we obtain

$$\mathbb{E}\frac{d\left(p_t A_t Q_t\right)}{p_t A_t Q_t}\frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0}\sigma_t^p + \sigma\left(\sigma_t^p + \underbrace{\sigma_t^q}_{=0}\right).$$

As the dividend yield is always ρ, imposing expectation on both sides of (A.8) and combining with the equilibrium condition in equation (A.6) yields

$$i_t^m = \rho + \pi_t + g + \sigma\sigma_t^p = i_t + (\sigma + \sigma_t^p)^2. \tag{D.10}$$

Plugging (D.10) into the real interest rate formula in Lemma A.1, we express the natural rate of interest r_t^n as

$$r_t^n = i_t - \pi_t + \sigma_t^p\left(\sigma + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p\right) = \rho + g - \sigma^2, \tag{D.11}$$

which is a function of structural parameters including σ, proving (iii) of Proposition A.1. Since capitalists' consumption C_t^n is directly proportional to TFP A_t, we know

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \left(r_t^n - \rho + \sigma^2\right) dt + \sigma dZ_t,$$

where we use $r_t^n - \rho + \sigma^2 = g$ from equation (D.11).

■

Proof of Proposition A.2. In the sticky price equilibrium, we would have $\sigma_t^p \equiv 0$, since over the small time period dt, a δdt portion of firms get to change their prices and there is no stochastic change in aggregate price level p_t up to a first-order. With (D.3) and (A.5), the capitalist's consumption C_t follows

$$\frac{dC_t}{C_t} = \left(i_t + (\sigma + \sigma_t^q)^2 - \pi_t - \rho\right) dt + (\sigma_t + \sigma_t^q) dZ_t,$$

where we use the equilibrium condition in (A.6): $i_t^m = i_t + (\sigma + \sigma_t^q)^2$. Thus, we obtain

$$\begin{aligned} d\hat{Q}_t = d\hat{C}_t &= \left(i_t - \pi_t - \underbrace{\left(r_t^n - \frac{(\sigma + \sigma_t^q)^2}{2} + \frac{\sigma^2}{2} \right)}_{\equiv r_t^T} \right) dt + \sigma_t^q dZ_t \\ &= \left(i_t - \pi_t - r_t^T\right) dt + \sigma_t^q dZ_t. \end{aligned}$$

Since we have risk-premium levels $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$ in the sticky price economy and $\text{rp}_t^n = \sigma^2$ in the flexible price economy, we can express our risk-adjusted natural rate r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}\left(\text{rp}_t - \text{rp}_t^n\right) = r_t^n - \frac{1}{2}\hat{rp}_t.$$

■

Proof of Proposition A.5. This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984).

■

Proof of Proposition A.4. The proof strategy is isomorphic to Proposition 1 in the main body, with ϕ taking a role of ϕ_y in Section 2 and σ_t^q taking a role of σ_t^s of Section 2.

■

Proof of Lemma A.2. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, where we use equation (D.2). Here $\frac{w_t^n}{p_t^n}$ is the real wage level in the flexible price economy. The good market equilibrium condition can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}}. \tag{D.12}$$

We subtract equation (D.12) from the same good market condition in the sticky price economy to obtain

$$A_t \left(\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}} = (C_t - C_t^n) + \left(\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}\right) \frac{1}{A_t^{\frac{1}{\chi}}}, \tag{D.13}$$

where we divide both sides of equation (D.13) by $y_t^n \equiv A_t^{1-\frac{1}{\chi}} (\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$ and obtain

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\frac{\widehat{w_t}}{p_t}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}} \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\frac{\widehat{w_t}}{p_t}},$$

which can be written as:

$$\frac{1}{\chi}\frac{\widehat{w_t}}{p_t} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon} \underbrace{\left(1 + \frac{1}{\chi}\right) \frac{\widehat{w_t}}{p_t}}_{=\hat{C^w}(t)}. \tag{D.14}$$

Equation (D.14) with $\hat{C}_t = \hat{Q}_t$ leads to

$$\hat{Q}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{\frac{w_t}{p_t}} = \underbrace{\frac{1}{1+\chi^{-1}}\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{C_{W,t}}.$$

We observe that Assumption A.1 guarantees that gaps of asset price, consumption of capitalists and workers, employment, and real wage all co-move with positive correlations. Now we can use $\hat{Q}_t$ and $\hat{C}_t$ interchangeably, and if one gap variable becomes 0, then all other gap variables become also stabilized to 0, up to a first order.

■

Proof of Proposition A.3. Firms change their prices with instantaneous probability δdt à la Calvo (1983). If there is price dispersion Δ_t, as defined in (A.2), across intermediate goods firms, then labor market equilibrium condition can be written as

$$N_{W,t} = \int_0^1 n_t(i)di = \left(\frac{y_t}{A_t\left(N_{W,t}\right)^\alpha}\right)^{\frac{1}{1-\alpha}} \underbrace{\int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} di}_{\equiv \Delta_t^{\frac{1}{1-\alpha}}},$$

where

$$y(t) = \frac{A_t N_{W,t}}{\Delta_t} = C_t + C_{W,t}.$$

We know that the good market equilibrium condition in (A.7) can be written as

$$\rho A_t Q_t + A_t \left(\frac{w_t}{p_t A_t}\right)^{1+\frac{1}{\chi}} = A_t \left(\frac{w_t}{p_t A_t}\right)^{\frac{1}{\chi}} \frac{1}{\Delta_t}.$$

Since a price process does not affect the resource allocation in the flexible price economy, we can regard $\hat{x}_t$ to be the $\log$-deviation of x_t from the flexible price economy *where the price is constant*. From the price aggregator, we obtain up to a first order

$$\hat{p}_t = \int_0^1 \widehat{p_t(i)} di. \tag{D.15}$$

To study price dispersion Δ_t up to a first-order, we illustrate Woodford (2003)'s treatment of Δ_t up to a second-order. From

$$\frac{1}{1-\alpha}\hat{\Delta}_t = \ln \int_0^1 \left(1 - \frac{\epsilon}{1-\alpha}\left(\cancel{\widehat{p_t(i)} - \hat{p}_t}\right) + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \left(\widehat{p_t(i)} - \hat{p}_t\right)^2\right) di + \text{h.o.t.}$$
$$= \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 Var_i\left(\widehat{p_t(i)}\right) + \text{h.o.t}, \tag{D.16}$$

where h.o.t stands for higher-order terms. We observe that $\Delta_t \simeq 1$ up to a first-order because Δ_t is in nature the second order as (D.16) suggests. Pricing à la Calvo (1983) is standard, except that our model is in continuous time. For a dt period from t to $t + dt$, an individual firm i changes the price with δdt probability. From time-0's perspective, the probability that a firm resets its price for the first time at time t is

$$\delta \mathrm{e}^{-\delta t} dt = \underbrace{\delta dt}_{\text{Change now}} \cdot \underbrace{\mathrm{e}^{-\delta t}}_{\text{No change until t}} .$$

At time t, a price-changing firm i chooses $p_t(i)$ to solve

$$\max_{p_t(i)} \frac{1}{\xi_t^N p_t}\mathbb{E}_t \int_t^\infty \mathrm{e}^{-\delta(s-t)} \xi_s^N p_s \left(\frac{p_t(i)}{p_s} y_{s|t}(i) - \frac{1}{p_s} C(y_{s|t}(i))\right) ds,$$
$$= \frac{1}{\xi_t^N p_t}\mathbb{E}_t \int_t^\infty \mathrm{e}^{-\delta(s-t)} \xi_s^N p_s \left(\left(\frac{p_t(i)}{p_s}\right)^{1-\epsilon} y_s - \frac{1}{p_s} C\left(\left(\frac{p_t(i)}{p_s}\right)^{-\epsilon} y_s\right)\right) ds, \tag{D.17}$$

where $y_{s|t}(i) = \left(\frac{p_t(i)}{p_s}\right)^{-\epsilon} y_s$ and $C(\cdot)$ is defined as an individual firm's nominal production cost as a function of its output produced, which is to be written explicitly. Let $MC_{s|t}$ and $\varphi_{s|t}$ be the nominal and real marginal cost at time s conditional on the last price resetting at prior time t. Using the nominal pricing kernel ξ_s^N formula in (A.4), we obtain

$$\frac{\xi_s^N p_s}{\xi_t^N p_t} = \mathrm{e}^{-\rho(s-t)} \frac{C_t}{C_s}. \tag{D.18}$$

By plugging (D.18) into (D.17) and solving (D.17), the optimal adjusted price p_t^*[24]

[24] We use the property that every price-setting firm at any time t chooses the same price, so we drop the firm index i in $p_t^*(i)$ and use p_t^*.

is given as

$$p_t^* = \frac{\mathbb{E}_t \displaystyle\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} \frac{\varphi_{s|t}}{\bar{\varphi}} p_s^\epsilon ds}{\mathbb{E}_t \displaystyle\int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \frac{y_s}{C_s} p_s^{\epsilon-1} ds}, \tag{D.19}$$

where $\varphi_{s|t}$ appears, and $\bar{\varphi}$ is its level in the flexible-price equilibrium, which is $\frac{\epsilon-1}{\epsilon}$. If we log-linearize (D.19) around the flexible price equilibrium with a constant price as in (D.15), we can express $\widehat{p_t^*}$ as

$$\widehat{p_t^*} = (\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\hat{\varphi}_{s|t} + \hat{p}_s\right) ds.$$

We know the conditional real production cost and the conditional real marginal cost can be written as

$$\frac{1}{p_s} C(y_{s|t}) = \frac{w_s}{p_s} \left(\frac{y_{s|t}}{A_s (N_{W,s})^\alpha}\right)^{\frac{1}{1-\alpha}},$$

and

$$\varphi_{s|t} \equiv \frac{1}{p_s} C'(y_{s|t}) = \frac{w_s}{p_s} \left(\frac{y_{s|t}}{A_s (N_{W,s})^\alpha}\right)^{\frac{\alpha}{1-\alpha}} \frac{1}{A_s (N_{W,s})^\alpha}, \tag{D.20}$$

From equation (D.20)), we obtain the conditional real marginal cost gap at time s conditional on price resetting at time t, which is given by

$$\hat{\varphi}_{s|t} = \underbrace{\frac{\hat{w}_s}{p_s}}_{\equiv \hat{\varphi}_s} - \frac{\alpha\epsilon}{1-\alpha} \left(\widehat{p_t^*} - \hat{p}_s\right) = \hat{\varphi}_s - \frac{\alpha\epsilon}{1-\alpha} \left(\widehat{p_t^*} - \hat{p}_s\right).$$

where $\hat{\varphi}_s$ is defined as the aggregate marginal cost index: as production is linear in aggregate level, $\hat{\varphi}_s$ should be equal to the real wage gap. Using (D.15), we then characterize the change in aggregate price gap $\hat{p}_t$ as

$$\begin{aligned} d\hat{p}_t &= \delta dt \left(\widehat{p_t^*} - \hat{p}_t\right) \\ &= \delta dt (\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t\right) ds, \quad \text{where } \Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \end{aligned}$$

As we log-linearize our economy around the flexible price equilibrium with constant price (i.e., $\pi_t = \sigma_t^p = 0$), $\hat{p}_t$ changes with an inflation rate π_t given by[25]

$$\pi_t = \frac{d\hat{p}_t}{dt} = \delta(\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)} \left(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t\right) ds. \tag{D.21}$$

Now that we have (D.21) for the instantaneous inflation rate π_t, we manipulate (D.21) as:

$$\begin{aligned}
\pi_t + \delta\hat{p}_t &= \delta(\delta+\rho)\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s + \hat{p}_s)ds \\
&= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t \int_t^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds \\
&= \delta(\delta+\rho)(\Theta\hat{\varphi}_t + \hat{p}_t)dt + \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds,
\end{aligned} \tag{D.22}$$

where we can rewrite the first line of equation (D.22) at time $t+dt$ instead of t as

$$\begin{aligned}
\pi_{t+dt} + \delta\hat{p}_{t+dt} &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)(t+dt)}\mathbb{E}_{t+dt} \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds \\
&= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t} \left(1 + (\delta+\rho)dt\right) \mathbb{E}_{t+dt} \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s} \left(\Theta\hat{\varphi}_s + \hat{p}_s\right) ds.
\end{aligned} \tag{D.23}$$

Due to the *martingale representation theorem* (see e.g., Oksendal (1995)), there exists a measurable H_t such that

$$\mathbb{E}_{t+dt} \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds = \mathbb{E}_t \int_{t+dt}^\infty \mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds + H_t dZ_t, \tag{D.24}$$

[25] In the case of positive inflation targets, see e.g., Coibion et al. (2012).

holds. We plug (D.24) into equation (D.23) to obtain[26]

$$\pi_{t+dt} + \delta\hat{p}_{t+dt} = \delta(\delta+\rho)\Bigg(\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds + \mathrm{e}^{(\delta+\rho)t}H_t dZ_t$$
$$+\,\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds\Bigg). \tag{D.25}$$

We subtract (D.22) from (D.25) to obtain

$$\begin{aligned}
&d\pi_t + \cancel{\delta}\pi_t dt\\
&= \delta(\delta+\rho)\left(\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds + \mathrm{e}^{(\delta+\rho)t}H_t dZ_t - (\Theta\hat{\varphi}_t+\hat{p}_t)dt\right)\\
&= \underbrace{\delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}H_t}_{\equiv\sigma_{\pi,t}} dZ_t - \delta(\delta+\rho)\Theta\hat{\varphi}_t dt\\
&\quad \underbrace{+\,\delta(\delta+\rho)\left((\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds\right)}_{=(\cancel{\delta}+\rho)\pi_t dt},
\end{aligned} \tag{D.26}$$

where we use

$$(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds = (\delta+\rho)dt\cdot\mathbb{E}_t\int_{t}^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds,$$

which holds from the property $(dt)^2 = 0$. Note that in (D.26), we define $\sigma_{\pi,t}$ as an instantaneous volatility of the inflation process. Finally from equation (D.26) we get the continuous time version of the New Keynesian Phillips curve (NKPC), written as[27]

$$d\pi_t = \rho\pi_t dt - \delta(\delta+\rho)\Theta\hat{\varphi}_t dt + \sigma_{\pi,t}dZ_t. \tag{D.27}$$

Due to the linear aggregate production function up to a first-order, we obtain:[28]

[26] We use the property that $dt\cdot dZ_t = 0$.

[27] Our continuous-time version of the Phillips curve in (D.26) is of the same form as in Werning (2012) and Cochrane (2017) after taking expectation on both sides.

[28] We use Lemma 2's log-linearization result to represent the real aggregate marginal cost gap $\frac{\hat{w}_t}{p_t}$

$$\hat{\varphi}_t = \frac{\widehat{w_t}}{p_t} = \left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \right)^{-1} \hat{Q}_t \equiv \frac{\kappa}{\delta(\delta+\rho)\Theta} \hat{Q}_t. \tag{D.28}$$

Finally plugging equation (D.28) into equation (D.27), we represent the New-Keynesian Phillips curve in terms of asset price gap $\hat{Q}_t$ in the following way:

$$d\pi_t = \left(\rho\pi_t - \kappa\hat{Q}_t\right) dt + \sigma_{\pi,t} dZ_t, \text{ and } \mathbb{E}_t d\pi_t = \left(\rho\pi_t - \kappa\hat{Q}_t\right) dt,$$

which proves the proposition A.3.[29] We know $\kappa > 0$ due to Assumption A.1. ■

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as a function of capitalists' consumption gap $\hat{C}_t = \hat{Q}_t$.

[29]Since $\hat{y}_t = \zeta\hat{Q}_t$, Phillips curve can be represented in terms of output gap $\hat{y}_t$ as in Proposition A.3.

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