Justifying the First-Order Approach in Agency Frameworks with the Agent's Possibly Non-Concave Value Function

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Abstract

To justify the first-order approach in principal-agent problems, all the existing results have been found by making the agent's expected monetary utility, i.e., value function, obtained from this approach "concave" in the agent's effort. However, relying on such concavity is sometimes overly sufficient. We propose a new set of conditions for justifying the first-order approach which is derived not from the concavity of the agent's expected monetary utility but directly from the original incentive compatibility constraint. Our suggested examples illustrate that this set of conditions can be applied to a wider range of principal-agent problems (including the one with normal or gamma density function) than the existing sets of conditions.

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1 Introduction

Replacing the agent's original "argmax" incentive compatibility constraint with its first-order condition, which is called the first-order approach, has been typically adopted to solve the principal-agent problems. However, this approach is not always valid even in the standard setting.[1] Thus, identifying conditions under which using the first-order approach is valid has been one of the major issues in the principal-agent literature, and several sets of conditions for justifying this approach have been found in various agency frameworks.[2] Yet, there are many meaningful principal-agent settings in which using the first-order approach cannot be justified by those existing sets of conditions.

The first set of conditions was proposed by Mirrlees (1975) and Rogerson (1985) in the one-signal case. Those conditions are the well-known MLRP (monotone likelihood ratio property) and CDFC (convexity of the distribution function condition) for the distribution function of signals. Later, other conditions were proposed to generalize the one-signal CDFC to multi-signal cases. They include the generalized CDFC (i.e., GCDFC) by Sinclair-Desgagné (1994), the CISP condition (concave increasing set probability condition) by Conlon (2009), and the CDFCL condition (convexity of the distribution function condition for the likelihood ratio) by Jung and Kim (2015) among others.[3] All these conditions contain the property of the CDFC condition.

However, the CDFC condition and its various extensions have a serious limitation in that they are hardly satisfied by most familiar density functions. For instance, consider a one-signal principal-agent problem in which the signal for the agent's hidden effort is generated by a simple functional form such that $x = a + \theta$, where $a \in [0, \infty)$ is the agent's effort level, and $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$ are the realized values of the signal variable, $\tilde{x}$, and the uncertainty variable, $\tilde{\theta} \sim N(0, \sigma^2)$, respectively.[4] It is well known that the density function of the signal conditional on the agent's effort in this case satisfies none of the above CDF-type conditions.

To overcome this drawback, Jewitt (1988) proposed another set of conditions in the one-signal case which does not contain any CDF-type condition, and is thus in general more applicable.[5] Recently, this set of conditions was also extended to the multi-signal case by Jung and Kim (2015).[6] These sets of conditions, however, have a different kind of limitation, and thus cannot be used for many familiar cases including the above normal density example. Although they do not contain any of the troublesome CDF-type conditions,

[1] For the standard principal-agent model, see Spence and Zeckhauser (1971), Ross (1973), Mirrlees (1975), Harris and Raviv (1979), Holmstrom (1979), Shavell (1979), and Grossman and Hart (1983), among others.

[2] For a detailed review of the literature on the first-order approach, see Jung and Kim (2015).

[3] Jewitt (1988) also proposed two sets of conditions for the multi-signal case assuming that the multiple signals are independently distributed.

[4] We use a letter with a tilde (e.g., $\tilde{x}$) to denote a random variable and a letter without it (e.g., x) to denote a specific realized value of that random variable.

[5] See Theorem 1 in Jewitt (1988).

[6] See Proposition 7 in Jung and Kim (2015).

they contain another condition that the agent's indirect utility given the optimal contract must be concave in the signals' likelihood ratio, which cannot be satisfied in many cases, including those that impose the agent's limited liability.

To justify the first-order approach, the existing sets of conditions containing the CDF-type conditions were derived by putting all the requirements only on the density function of the signals. Therefore, those CDF-type conditions are, in general, too restrictive to be satisfied by most familiar density functions. On the other hand, Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015), by placing an additional requirement on the agent's utility function that the agent's indirect utility be concave in the likelihood ratio of the signals, were able to impose weaker requirements on the density function of the signals than the CDF-type conditions. However, there are still many cases in which such an additional requirement cannot be met. For many density functions of the signals (including the normal density function in the above example), their likelihood ratios are unbounded below. In this case, in order to guarantee the existence of the optimal contract, one needs to include the agent's limited liability constraint which requires that the agent's wage not be lower than a certain level under any circumstances.[7] However, when the optimal contract is to be bounded below due to the agent's limited liability constraint, so is the agent's indirect utility given that optimal contract, and thereby it cannot be concave in the likelihood ratio. This is why neither the Jewitt (1988) conditions nor the Jung and Kim (2015) conditions can be used for such cases.

One thing that is common to all the existing results is that, to justify the first-order approach, they were derived to make the agent's expected monetary utility obtained from this approach concave in the agent's effort. However, such concavity is sufficient but not necessary for justifying the first-order approach in two respects. First, since the agent's expected utility (i.e., value function) given an incentive contract is his expected monetary utility from that incentive contract subtracted by the cost of effort, the conditions derived by relying only on the concavity of the agent's expected monetary utility do not take the agent's cost of effort into consideration. Even if the agent's expected monetary utility is not concave in effort, his expected utility can still be concave if the cost of effort is sufficiently convex, and thus using the first-order approach can be justified. Second, but more importantly, even the concavity of the agent's expected utility (i.e., his expected monetary utility minus cost) in effort itself is not necessary for justifying the first-order approach. As shown in Figure 1, the first-order approach can be justified as long as the agent's expected utility from this approach has a maximum value at the target effort level the principal intends to induce (i.e., the original argmax incentive constraint).

In this paper, we derive a new set of conditions that is not obtained by relying on the concavity of the

[7] This is the well-known Mirrlees' unpleasant theorem. For this issue, see Mirrlees (1975)

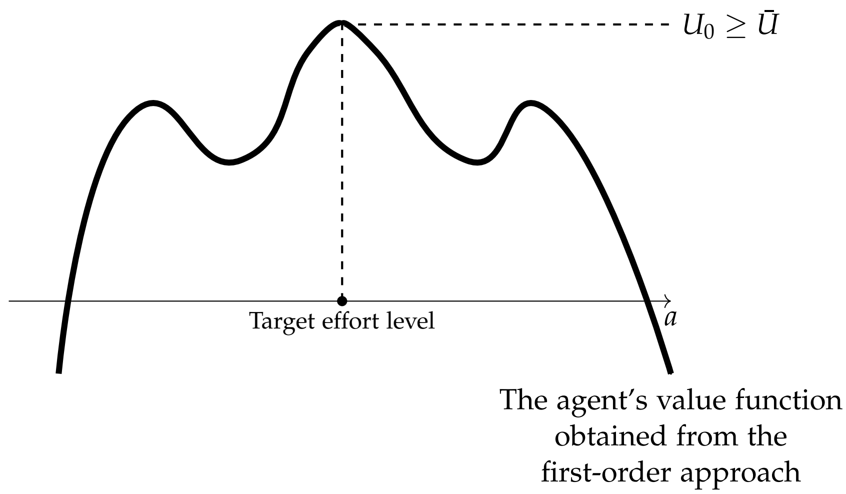


Figure 1: Possibly Non-Concave Indirect Utility of the Agent

agent's expected monetary utility but obtained directly from the original incentive compatibility constraint. This set of conditions contains a statistical condition on the signals' density functions. We show that our statistical condition not only has a very useful implication but also is quite general in that it can be satisfied by the wide range of familiar density functions including the normal distributions and various exponential families of density functions, whereas the existing conditions cannot. We also provide two alternative sets of conditions: one that can be used for the case in which the agent's limited liability constraint is binding for some signal values at the optimum, and the other that can be used for the case in which that constraint is not binding at the optimum, respectively. Also, by using those sets of conditions, we show that the first-order approach can be validly adopted in many useful principal-agent settings (including the above normal density case) in which using the first-order approach has not been able to be justified by the existing sets of conditions.

The remainder of the paper is organized in the following way. In Section 2, we formulate the basic principal-agent framework, and briefly explain the issue of using the first-order approach. In Section 3, we present our main set of conditions that is derived not from the concavity of the agent's expected monetary utility but directly from the original incentive compatibility constraint. Then, in Sections 4, we propose two alternative sets of conditions that are easier to verify than our main set of conditions in both the case where the agent's limited liability constraint is binding for some values of the signals at the optimum and the case where it is not binding at the optimum, respectively. In Section 5, we explain the statistical implications of our conditions, and compare them with the existing conditions. Concluding remarks are given in Section 6, and all formal proofs are relegated to the Appendix.

2 The Basic Model

We consider a one-period standard principal-agent model in which an agent works for a principal by inputting his effort $a \in [0, \infty)$. The principal cannot observe the agent's effort choice directly but can observe signals $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ that are imperfectly correlated with the agent's hidden effort, where $\tilde{x}_i$ is a one-dimensional random variable. By taking the Mirrlees (1975) formulation, we denote $f(\mathbf{x}|a)$ as the joint density function of $\tilde{\mathbf{x}}$ conditional on the agent's effort, a. It is defined from the cumulative distribution function of $\tilde{\mathbf{x}}$ given a, i.e., $F(\mathbf{x}|a) \equiv Pr[\tilde{\mathbf{x}} \leq \mathbf{x}|a]$, where $\mathbf{x} \in \mathbb{R}^n$ is the realized value of signal vector $\tilde{\mathbf{x}}$. We assume that the support of $f(\mathbf{x}|a)$ is independent of a, and both $F(\mathbf{x}|a)$ and $f(\mathbf{x}|a)$ are continuous and differentiable at least twice with respect to a.

When signal $\mathbf{x}$ is realized, the principal obtains $\pi(\mathbf{x})$ as the total value, and she pays to the agent his wage s which depends on $\mathbf{x}$, i.e., $s = s(\mathbf{x})$. The principal is risk-neutral, whereas the agent is risk-averse. It is assumed that the agent's utility function takes an additively separable form such as

$$u(s,a) = u(s) - c(a), \;\; u' > 0, \;\; u'' < 0, \;\; c' > 0, \;\; c'' > 0,$$

where $u(s)$ denotes the agent's utility from monetary payoff s and $c(a)$ denotes his disutility of exerting a.[8] Thus, the agent's expected utility when he takes an effort a under $s(\mathbf{x})$ is

$$U(s(\cdot), a) \equiv \int u(s(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a). \tag{1}$$

We also assume that the agent can get $\overline{U}$ at maximum by working for other principals, thereby $\overline{U}$ is his reservation utility level. Furthermore, there is a limited liability constraint on the agent's side which requires that the agent's wage not be lower than $\underline{s}$ under any circumstances, i.e., $s(\mathbf{x}) \geq \underline{s}$, $\forall \mathbf{x}$, where $\underline{s}$ indicates the agent's subsistence wage level.[9]

So a principal-agent problem can be represented by its characteristic variables $\{\pi(\mathbf{x}), u(s), f(\mathbf{x}|a), c(a), \overline{U}, \underline{s}\}$, and the principal's optimization program given these variables is

$$\begin{aligned} \max_{a, s(\mathbf{x}) \geq \underline{s}} \quad & \int [\pi(\mathbf{x}) - s(\mathbf{x})] f(\mathbf{x}|a) d\mathbf{x} \\ s.t. \quad & (i) \; U(s(\cdot), a) \geq \overline{U}, \end{aligned}$$

[8] We use the prime and the double prime of a function to denote the first and the second derivatives of that function, respectively.

[9] This limited liability constraint of the agent is especially needed to guarantee the existence of the optimal contract in some cases. Also, note that the case in which there is no limited liability constraint on the agent's side is a special case where $\underline{s} = -\infty$. For this 'unpleasantness' issue, see Mirrlees (1975)

$$(ii)\ a \in \arg\max_{a'}\ U(s(\cdot), a').$$

In the above, the constraints are the typical participation and incentive compatibility constraints, respectively. This optimization program indicates that the principal has to decide both the agent's wage scheme, $s(\mathbf{x}) \geq \underline{s}$, $\forall \mathbf{x}$, and the target effort level, a, simultaneously to maximize her expected payoff under the constraints that the self-interested agent actually chooses the target effort level when $s(\mathbf{x})$ is offered and that his expected utility in this case is not lower than $\overline{U}$.

However, the above program is generally not tractable in itself because the incentive constraint is composed of infinitely many inequality constraints. Thus, it has been typically solved by replacing the original incentive constraint with the "relaxed" constraint that the agent's expected utility given $s(\mathbf{x})$ is stationary at that effort level, a, i.e.,

$$\frac{\partial U(s(\cdot), a)}{\partial a} \equiv U_a(s(\cdot), a) = 0, \tag{2}$$

which is known as the first-order approach. The principal's optimization program adopting this approach can be written as

$$\begin{aligned} \max_{a, s(\mathbf{x}) \geq \underline{s}} \quad & \int [\pi(\mathbf{x}) - s(\mathbf{x})] f(\mathbf{x}|a) d\mathbf{x} \\ s.t. \quad & (i)\ U(s(\cdot), a) \geq \overline{U}, \\ & (ii)\ U_a(s(\cdot), a) = \int u(s(\mathbf{x})) f_a(\mathbf{x}|a) d\mathbf{x} - c'(a) = 0, \end{aligned}$$

where $f_a(\mathbf{x}|a) \equiv \frac{\partial f(\mathbf{x}|a)}{\partial a}$ is a partial derivative of the density function $f(\mathbf{x}|a)$ with respect to action a.

Let $(s^o(\mathbf{x}), a^o > 0)$ solve the above optimization program.[10] By solving the Euler equation of the above program, one can derive that the optimal incentive contract, $s^o(\mathbf{x})$, should satisfy

$$\frac{1}{u'(s^o(\mathbf{x}))} = \begin{cases} \lambda + \mu \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}, & \text{if } s^o(\mathbf{x}) \geq \underline{s}, \\ \frac{1}{u'(\underline{s})}, & \text{otherwise,} \end{cases} \tag{3}$$

where λ and μ are the Lagrange multipliers of the participation and "relaxed" incentive compatibility constraints, respectively.

It is well known that the optimal contract, $s^o(\mathbf{x})$ in equation (3), does not always solve the original optimization program. This is because $s^o(\mathbf{x})$ in equation (3) is actually the optimal solution obtained by replacing the original "argmax" incentive constraint with the "relaxed" one, and the principal's opportunity

[10]Thus, the existence of an optimal solution $(s^o(\mathbf{x}), a^o)$ is assumed. We also assume $a^o > 0$ to rule out a trivial case. For the existence of an optimal solution in the principal-agent problem, see Jewitt, Kadan, and Swinkels (2008).

set for $s(\mathbf{x})$ satisfying the relaxed incentive constraint is larger than her true opportunity set for $s(\mathbf{x})$ satisfying the original argmax incentive constraint, and thereby $s^o(\mathbf{x})$ sometimes may not be in her true opportunity set for $s(\mathbf{x})$. Thus, to guarantee that $s^o(\mathbf{x})$ in equation (3) actually solves the original program, it must be ensured that $s^o(\mathbf{x})$ satisfies the original argmax incentive constraint, that is,

$$U(s^o(\cdot), a) \leq U(s^o(\cdot), a^o), \quad \text{for all } \; a. \tag{4}$$

To ensure equation (4), all the existing results for justifying the first-order approach were derived to make the agent's expected utility when he takes a under $s^o(\mathbf{x})$, $U(s^o(\cdot), a)$, concave in a.[11] Obviously, given the concavity of $U(s^o(\cdot), a)$ in a, $U_a(s^o(\cdot), a^o) = 0$ guarantees equation (4), and thus using the first-order approach is valid. However, the concavity of $U(s^o(\cdot), a)$ in a is sufficient but not necessary for ensuring equation (4). As already drawn in Figure 1, even if $U(s^o(\cdot), a)$ is not concave, the first-order approach can still be justified. The main purpose of this paper is to find a new set of conditions which ensures equation (4) without relying on the concavity of $U(s^o(\cdot), a)$, and thus is more general than the existing sets of conditions.

3 Analysis

We start with proving that μ in equation (3) is positive. The basic proof for $\mu > 0$ was already given by Mirrlees (1975) and Holmstrom (1979).[12] But their proofs were proved with the assumption that using the first-order approach is valid. Thus, it is obvious that those proofs cannot be used for finding conditions which justify the first-order approach itself. This is why Jewitt (1988) provided another proof for $\mu > 0$ without such an assumption.[13] But, his proof also has a limitation in that it is valid only for the case in which the agent's limited liability constraint is not binding for all $\mathbf{x}$ at the optimum. However, as will be shown later, there are actually many cases in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum, and one of our main goals is to provide the conditions under which using the first-order approach can be justified even in those cases. Thereby, we provide a different proof for $\mu > 0$ based neither on the assumption that using the first-order approach is valid nor on the assumption that the agent's limited liability constraint is not binding at the optimum.

Lemma 1: $\mu > 0$.

[11] For those results, see Mirrlees (1975), Grossman and Hart (1983), Rogerson (1985), Jewitt (1988), Sinclair-Desgagné (1994), Conlon (2009), and Jung and Kim (2015) among others.

[12] Mirrlees (1975) showed that, in single-signal cases, i.e., $x \in \mathbb{R}$, $\mu > 0$ when $f(x|a)$ satisfies the monotone likelihood ratio property (MLRP), whereas Holmstrom (1979) showed it when $f(x|a)$ satisfies the first-order stochastic dominance (FOSD).

[13] See Lemma 1 in Jewitt (1988).

Observe from equation (3) that, especially when the principal is risk-neutral, the agent's optimal contract, $s^o(\mathbf{x})$, depends on signal $\mathbf{x}$ only through $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$. That is, $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ is a sufficient statistic for $\mathbf{x}$ about a^o for designing $s(\mathbf{x})$, implying that what matters to the risk-neutral principal when she designs contracts for her agent is $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ rather than $\mathbf{x}$ itself. In fact, $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ is the information contained in signal $\mathbf{x}$, indicating that how likely it is that the agent has taken a^o rather than some other nearby action when signal $\mathbf{x}$ is realized. Based on this observation, Jung and Kim (2015) showed that analyzing principal-agent problems directly based on $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ generally has an advantage over analyzing them based on the signal vector $\mathbf{x}$. Thus, as in Jung and Kim (2015), we derive conditions under which the first-order approach can be justified based on $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$.

We denote $\tilde{q} \equiv Q_{a^o}(\tilde{\mathbf{x}}) \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ as the information variable, which implies that random variable $\tilde{q}$ and random vector $\tilde{\mathbf{x}}$ have a functional relationship such that $q = Q_{a^o}(\mathbf{x}) \equiv \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$, where q is the realized value of $\tilde{q}$.[14] We also denote $G(q|a)$ as the cumulative distribution function of $\tilde{q}$ given a, i.e.,

$$G(q|a) \equiv Pr[Q_{a^o}(\tilde{\mathbf{x}}) \leq q|a],$$

and $g(q|a)$ as its probability density function.[15]

Based on q, define

$$w(q) \equiv u'^{-1}\left(\frac{1}{\lambda + \mu q}\right) \quad \text{and} \quad \bar{r}(q) \equiv u(w(q)). \tag{5}$$

We see from equation (3) that $w(q)$ denotes the optimal contract defined on the q-space when it is not constrained by the limited liability constraint, i.e., $s^o(\mathbf{x}) \geq \underline{s}$, whereas $\bar{r}(q)$ denotes the agent's indirect utility given the contract also defined on the q-space in that case. Note from equation (5) that the functional forms of $w(\cdot)$ and $\bar{r}(\cdot)$ depend only on the functional form of $u(\cdot)$, and Lemma 1 shows that both $w(q)$ and $r(q)$ are increasing in q.

Thus, the agent's indirect utility given $s^o(\mathbf{x})$ in equation (3), i.e., $u(s^o(\mathbf{x}))$, can be written as

$$u(s^o(\mathbf{x})) \equiv r(Q_{a^o}(\mathbf{x})) = r(q) = \begin{cases} \bar{r}(q), & \text{when } q \geq q_c, \\ u(\underline{s}), & \text{when } q < q_c, \end{cases} \tag{6}$$

where $q_c = Q_{a^o}(\mathbf{x}_c) \equiv \frac{f_a(\mathbf{x}_c|a^o)}{f(\mathbf{x}_c|a^o)}$ solves $\frac{1}{u'(\underline{s})} = \lambda + \mu q_c$. Also, the agent's expected indirect utility when he takes an effort a under $s^o(\mathbf{x})$ can be written as

$$U(s^o(\cdot), a) = \int u(s^o(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a)$$

[14] Note that q is a function of $\mathbf{x}$ but it also depends on a^o. For a more detailed explanation of q, see Jung and Kim (2015).

[15] For a more detailed treatment of distributions $G(q|a)$ and $g(q|a)$, see Jung and Kim (2015).

$$= \int r(q)g(q|a)dq - c(a).$$

It is known that the agent's limited liability constraint may or may not be binding at the optimum depending on the characteristic variables, $\{\pi(\mathbf{x}),u(s),f(\mathbf{x}|a),c(a),\overline{U},\underline{s}\}$, especially on whether the information variable, $\tilde{q} = \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below or not.[16] If it is the case where the agent's limited liability constraint is binding for some $\mathbf{x}$ with the optimal contract, the participation constraint may not be binding (i.e., $\lambda = 0$), and the agent may enjoy some positive rent at the optimum. Thus, to be more general, we denote the agent's expected utility when he takes a^o given $s^o(\mathbf{x})$ as

$$\begin{aligned} U(s^o(\cdot),a^o) &= \int u(s^o(\mathbf{x}))f(\mathbf{x}|a^o)d\mathbf{x} - c(a^o) \\ &= \int r(q)g(q|a^o)dq - c(a^o) = U^o \geq \overline{U}, \end{aligned} \tag{7}$$

where U^o is determined by $\{\pi(\mathbf{x}),u(s),f(\mathbf{x}|a),c(a),\overline{U},\underline{s}\}$. Of course, $U^o = \overline{U}$ if the agent's limited liability constraint is not binding for all $\mathbf{x}$ at the optimum (i.e., $\lambda > 0$).

The following lemma plays the basic role in deriving our main results.

Lemma 2: *For any given a^o, if*

(L1) $f(\mathbf{x}|a)$ *satisfies that* $\frac{g(q|a)}{g(q|a^o)}$ *is convex in q for all a, and $\xi(q)$ satisfies that:*

(L2) $\int \xi(q)g(q|a^o)dq = 0$,

(L3) $\int \xi(q) \cdot q \cdot g(q|a^o)dq = 0$, *and*

(L4) $\xi(q)$ *changes sign twice from negative to positive and then to negative as q increases,*

then, we have

$$\int \xi(q)g(q|a)dq \leq 0, \quad \forall a.$$

Let S_f be the set of contracts that give the agent the same expected utility as U^o in equation (7) when he takes a^o, and satisfy the relaxed incentive constraint at that effort level. That is,

$$S_f \equiv \{s(\mathbf{x})|\ s(\mathbf{x}) \text{ satisfies } U(s(\cdot),a^o) = U^o \text{ and } U_a(s(\cdot),a^o) = 0\}.^{17} \tag{8}$$

Then, using Lemma 2, we obtain the following proposition.

Lemma 3: *For any given a^o, if*

[16] This is associated with what is called the Mirrlees unpleasant theorem.

[17] Not that, since S_f is defined based on given a^o, it varies as a^o changes.

(1a $=$ L1) *$f(\mathbf{x}|a)$ satisfies that $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a*

(2a) *(double crossing property) there exists a contract $\hat{s}(\mathbf{x}) \in S_f$ such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ crosses $r(q) \equiv u(s^o(\mathbf{x}))$ twice starting from above, and*

(3a) *$E[\hat{r}(q)|a]$ is concave in $c(a)$,*

then using the first-order approach is justified.

What Lemma 2 indicates associated with Proposition 1 is the following. Note that, since $s^o(\mathbf{x})$ is the optimal contract obtained from the first-order approach, $s^o(\mathbf{x})$ must be in S_f. Now, consider another contract $\hat{s}(\mathbf{x}) \in S_f$ such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ crosses $r(q) \equiv u(s^o(\mathbf{x}))$ twice starting from above as drawn in Figure 2a.[18] Then, if one thinks of $r(q) - \hat{r}(q)$ as $\xi(q)$ in Lemma 2, it can be easily seen that $\xi(q) = r(q) - \hat{r}(q)$ satisfies condition (L4) in Lemma 2. Furthermore, as shown in the proof of Proposition 1, the fact that both $s^o(\mathbf{x})$ and $\hat{s}(\mathbf{x})$ are in S_f implies

$$\int [r(q) - \hat{r}(q)] g(q|a^o) dq = 0, \tag{9}$$

indicating that condition (L2) in Lemma 2 is satisfied, and

$$\int [r(q) - \hat{r}(q)] q \cdot g(q|a^o) dq = 0, \tag{10}$$

indicating that condition (L3) in Lemma 2 is satisfied. Thus, by using Lemma 2, we derive that, if condition (L1) is satisfied, the agent's expected utility under $s^o(\mathbf{x})$ is lower than that under $\hat{s}(\mathbf{x})$ for all a except for a^o as drawn in Figure 2b, i.e.,

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int [r(q) - \hat{r}(q)] g(q|a) dq \leq 0, \quad \forall a.\text{[19]} \tag{11}$$

Figure 2b is drawn in a way that, although $U(s^o(\cdot), a)$ and $U(\hat{s}(\cdot), a)$ have a stationary point at a^o, they do not have a global maximum point at a^o. This may happen because especially $\hat{s}(\mathbf{x})$ is assumed to be in S_f. Actually, the fact that $s^o(\mathbf{x})$ and $\hat{s}(\mathbf{x})$ are in S_f only ensures that

$$U(s^o(\cdot), a^o) = U(\hat{s}(\cdot), a^o) = U^o, \tag{12}$$

[18]The fact that $\hat{r}(q)$ crosses $r(q)$ twice starting from above is equivalent to that $\hat{s}(\mathbf{x})$ crosses $s^o(\mathbf{x})$ twice starting from above if $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP.

[19]This result can be understood as follows: we know $U(s^o(\cdot), a^o) - U(\hat{s}(\cdot), a^o) = 0$. If $a \neq a^o$, due to the double-crossing in Figure 2a, it is more likely that the distribution $g(q|a)$ puts more weighs on the negative values of $r(q) - \hat{r}(q)$, making its average value negative. Our statistical condition (1a) guarantees that this claim holds.

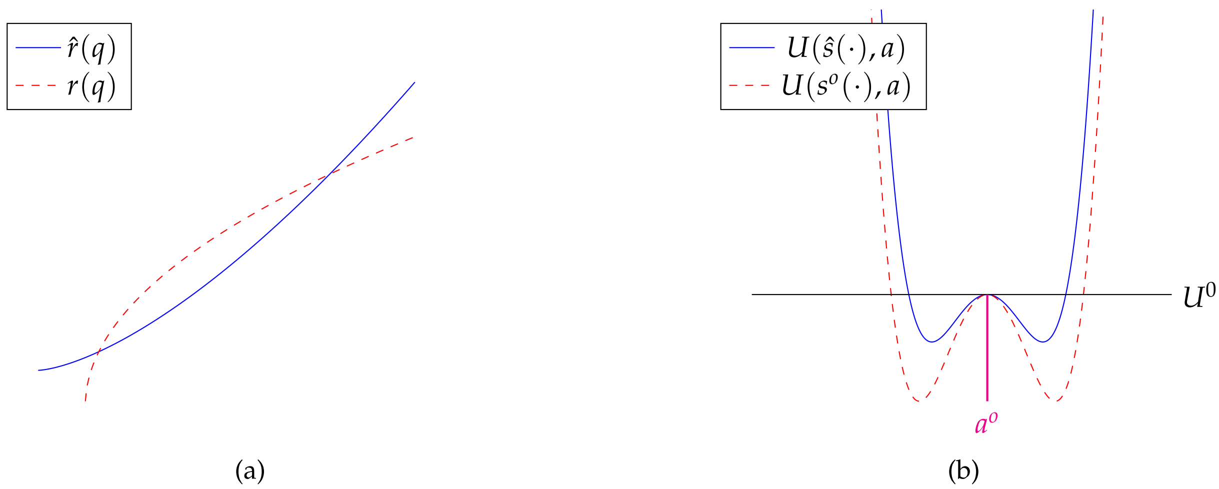


Figure 2: Double Crossing Property

and

$$U_a(s^o(\cdot), a^o) = U_a(\hat{s}(\cdot), a^o) = 0.$$

However, condition (3a) in Lemma 3 guarantees that $U(\hat{s}(\cdot), a)$ actually has a global maximum point at a^o, i.e.,

$$U(\hat{s}(\cdot), a) \leq U(\hat{s}(\cdot), a^o) = U^o, \tag{13}$$

as drawn in Figure 3. Thus, by combining equation (11), equation (12), and equation (13), we have

$$U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a^o) = U^o, \quad \forall a,$$

which justifies the first-order approach.

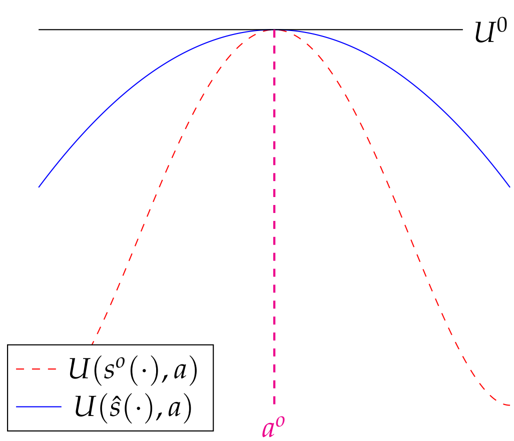


Figure 3: First-Order Approach Justified

Condition (L1) in Lemma 3 is our main statistical condition which is different from the typical statistical conditions in the existing literature such as the CDFC (convexity of the distribution function condition) by Mirrlees (1975) and Rogerson (1985), the CISP condition (concave increasing set probability condition) by Conlon (2009), and the CDFCL (convexity of the distribution function condition for the likelihood ratio) by Jung and Kim (2015). In Section 5, we will explain the statistical implication of (L1) as well as the difference between (L1) and the above existing conditions more precisely. However, it is worth to note that (L1) is much easier to verify than the above conditions and also quite general in that many familiar density functions satisfy it.

For example, in the one-signal case, consider a density function in the exponential family such as

$$f(x|a) = A(a)B(x)e^{\alpha(a)\beta(x)}, \quad x \in \mathbb{R},$$

with $\alpha(a)$ and $\beta(x)$ increasing. Then,

$$\frac{f_a(x|a)}{f(x|a)} = \alpha'(a)\beta(x) + \frac{A'(a)}{A(a)}.$$

Thus, for any given a^o,

$$Q_{a^o}(\tilde{x}) \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \alpha'(a^o)\beta(\tilde{x}) + \frac{A'(a^o)}{A(a^o)}.$$

Since $\alpha(a)$ and $\beta(x)$ are increasing, $f(x|a)$ satisfies the MLRP. Therefore,

$$G(q|a) = Pr[Q_{a^o}(\tilde{x}) \leq q|a] = Pr[\tilde{x} \leq x|a] = F(x|a),$$

where x solves $\frac{f_a(x|a^o)}{f(x|a^o)} = q$,[20] and

$$g(q|a) = f(x|a)\frac{dx}{dq}.$$

By using $\beta(x) = \left(q - \frac{A'(a^o)}{A(a^o)}\right)\frac{1}{\alpha'(a^o)}$ and $\frac{dx}{dq} = \frac{1}{\alpha'(a^o)\beta'(x)}$, we have

$$g(q|a) = \frac{A(a)}{\alpha'(a^o)}\delta(q)\exp\left(\frac{\alpha(a)}{\alpha'(a^o)}\left(q - \frac{A'(a^o)}{A(a^o)}\right)\right),$$

[20] Thus, x which solves $\frac{f_a(x|a^o)}{f(x|a^o)} = q$ is a function of q given a^o, i.e., $x(q; a^o)$.

where $\delta(q) \equiv \frac{B(x)}{\beta'(x)}$.[21] As a result,

$$\frac{g(q|a)}{g(q|a^o)} = \frac{A(a)}{A(a^o)} \exp\left(\frac{\alpha(a)-\alpha(a^o)}{\alpha'(a^o)}\left(q - \frac{A'(a^o)}{A(a^o)}\right)\right),$$

which is convex in q for any given a^o and for all a, satisfying condition (L1).

More generally, even in the multi-signal case (i.e., $\mathbf{x} \in \mathbb{R}^n$), condition (L1) holds if $f(\mathbf{x}|a)$ generates, for any given a^o,

$$\frac{g_a(q|a)}{g(q|a)} = A(a)q + D(a),$$

which is the case for most exponential families of density functions (including normal, gamma, etc.) Note that

$$\frac{g(q|a)}{g(q|a^o)} = \exp\left(\int_{a^o}^{a} \frac{g_a(q|t)}{g(q|t)} dt\right).$$

Thus,

$$\frac{g(q|a)}{g(q|a^o)} = \exp\left(\hat{A}(a)q + \hat{D}(a)\right),$$

where $\hat{A}(a) \equiv \int_{a^o}^{a} A(t)dt$ and $\hat{D}(a) \equiv \int_{a^o}^{a} D(t)dt$. Therefore, one can easily see that condition (L1) is satisfied.

On the other hand, conditions (2a) and (3a) in Lemma 3 need to be elaborated more. In general, directly verifying whether conditions (2a) and (3a) are satisfied or not is not easy. In other words, for a given principal-agent problem, finding a contract $\hat{s}(\mathbf{x})$ which satisfies the double crossing property (i.e., condition (2a)) and condition (3a), if any, is not straightforward. Therefore, in what follows, we investigate the conditions which sufficiently guarantee conditions (2a) and (3a), and are easier to verify.

4 Verifying the Double Crossing Property

To verify the double crossing property between $r(q)$ and $\hat{r}(q)$ (i.e., condition (2a) in Lemma 3), we will take the following steps. First, we introduce an arbitrary contract $\hat{s}(\mathbf{x})$ under which the agent will chooses a^o (i.e., the original "argmax" incentive constraint is satisfied at a^o), and his expected utility in this case is equal to U^o (i.e., the participation constraint is satisfied at a^o). That is,

$$\begin{aligned} U(\hat{s}(\cdot), a) &\equiv \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a) \\ &\leq U(\hat{s}(\cdot), a^o) \equiv \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) = U^o, \quad \forall a. \end{aligned} \tag{14}$$

[21] Since $\beta(x)$ is increasing, we have $x = \beta^{-1}\left(\frac{q - \frac{A'(a^o)}{A(a^o)}}{\alpha'(a^o)}\right) \equiv \Omega(q)$. Then, $\delta(q) \equiv \frac{B(\Omega(q))}{\beta'(\Omega(q))}$.

Then, we derive the conditions under which $r(q) \equiv u(s^o(\mathbf{x}))$ crosses $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ instead of $r(q)$ twice starting from below. Then, we would have:

$$\begin{aligned} U(s^o(\cdot), a) &\equiv \int u(s^o(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a) \\ &= \int r(q) g(q|a) dq - c(a) \\ &\leq \int \hat{r}(q) g(q|a) dq - c(a) \\ &= \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a) \equiv U(\hat{s}(\cdot), a), \quad \forall a. \end{aligned} \tag{15}$$

In (15), the inequality comes from Lemma 2 because both $r(q)$ and $\hat{r}(q)$ satisfy the participation and the relaxed incentive constraints, and $r(q)$ crosses $\hat{r}(q)$ twice starting from below. Consequently, by combining (7), (14), and (15), we derive that

$$U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a) \leq U(\hat{s}(\cdot), a^o) = U(s^o(\cdot), a^o) = U^o, \quad \forall a,$$

which justifies the first-order approach as drawn in Figure 3.

In Figure 3, $U(\hat{s}(\cdot), a)$ is drawn in a way that the agent will choose a^o given $\hat{s}(\mathbf{x})$ and $U(\hat{s}(\cdot), a^o) = U^o$ (i.e., equation equation (14)), and $U(s^o(\cdot), a)$ is drawn in a way that it lies below $U(\hat{s}(\cdot), a)$ for all a except for a^o since $U(s^o(\cdot), a^o) = U^o$ (i.e., equation (7) and equation (15)).

Thus, the key point is to find an appropriate contract, $\hat{s}(\mathbf{x})$, with which the double crossing property between $r(q)$ and $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ can be easily verified. To do this, however, we have to distinguish the case in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum from the case in which that constraint is not binding for all $\mathbf{x}$ at the optimum. This is because $r(q) \equiv u(s^o(\mathbf{x}))$ in equation (6) has a different functional form depending on whether the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum or not. As shown in equation (6), the agent's indirect utility given $s^o(\mathbf{x})$, i.e., $u(s^o(\mathbf{x})) \equiv r(q)$, must be bounded below by $u(\underline{s})$ for some low values of q (i.e., $q < q_c$) if his limited liability constraint is binding for some $\mathbf{x}$ at the optimum, whereas it is not bounded below by $u(\underline{s})$ (i.e., $r(q) = \overline{r}(q)$, $\forall q$) if that constraint is not binding for all $\mathbf{x}$ at the optimum. This requires us to introduce a different $\hat{s}(\mathbf{x})$ to guarantee the double crossing property between $r(q)$ and $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ in each case.

Associated with the above distinction, it is worth noting that the existing results for the validity of the first-order approach can also be divided into two groups, the results which can be applied even to the case in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum, and the results which can be applied only to the case in which the agent' limited liability constraint is not binding for all $\mathbf{x}$ at the optimum.

As mentioned earlier, all the existing results were derived to make the agent's expected monetary utility given $s^o(\mathbf{x})$, i.e., $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$, concave in a. More precisely, the Mirrlees-Rogerson conditions in the one-signal case (i.e., $f(x|a)$, $x \in \mathbb{R}$, should satisfy the MLRP and the CDFC) were derived based on the fact that $R(a) \equiv \int u(s^o(x))f(x|a)dx$ is concave in a for any "increasing" function $u(s^o(x))$ if $f(x|a)$ satisfies the CDFC. Later, other conditions were found to generalize the CDFC to the multi-signal case They include Sinclair-Desgagné's GCDFC (generalized convexity of the distribution function condition), Conlon's CISP condition (concave increasing set probability condition), and Jung and Kim's CDFCL (convexity of the distribution function condition for the likelihood ratio). All these conditions contain the property of the CDFC because they are just extended versions of the one-signal CDFC to the multi-signal case. Since the results that contain those CDF-type conditions can be used for any "increasing" $u(s^o(\mathbf{x}))$ (or $r(q)$), they can be well applied to the case in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum because $u(s^o(\mathbf{x}))$ is generally increasing even in this case. However, they can be applied only to a limited set of cases in that most familiar density functions of the signals, $f(\mathbf{x}|a)$, hardly satisfy such CDF-type conditions.

To overcome this drawback, Jewitt (1988) proposed another set of conditions in the one-signal case which is not related to CDF-type conditions, and his conditions were generalized to multi-signal cases by Jung and Kim (2015). Both the condition on $f(\mathbf{x}|a)$ in the Jewitt (1988) (i.e., Theorem 1 in Jewitt (1988)) and that in the Jung and Kim (2015) (i.e., Proposition 7 in Jung and Kim (2015)) are weaker than the above CDF-type conditions, and thus can be satisfied by some familiar density functions.[22] This is because they were derived to make $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} = \int r(q)g(q|a)dq$ concave in a not for any "increasing" function but for any "increasing concave" function of $u(s^o(\mathbf{x}))$ (or $r(q)$). However, these conditions, although useful, can also be applied only to another limited set of cases in which the agent's limited liability constraint is not binding at the optimum. When the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum, $u(s^o(\mathbf{x}))$ (or $r(q)$) should be bounded below for those $\mathbf{x}$, and thus cannot be concave. We will elaborate on this issue in the next Section 4.1.

4.1 When the Information Variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, Is Unbounded Below

Whether the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum mainly depends on $f(\mathbf{x}|a)$ and $\underline{s}$. Especially when the density function of the signals has its information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$,

[22]For the examples of such density functions, see Jewitt (1988, p. 1183)

which is not bounded below, i.e., $\tilde{q} \in (-\infty, \overline{q})$,[23] we need to have $\underline{s}$ bounded below, i.e., $\underline{s} > -\infty$, to guarantee the existence of the optimal contract. Then, as shown in equation (3), the agent's limited liability constraint must be binding for some $\mathbf{x}$ for the optimal contract.

In this case, in order to verify the double crossing property between $r(q)$ and $\hat{r}(q)$, we introduce an arbitrary contract $\hat{s}_t(\mathbf{x}), t > 0$, with which the agent's indirect utility has an exponential form such as $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = A \cdot e^{tq} + B$, where A, B, and $t > 0$ are to be set to satisfy both the participation and the relaxed incentive constraints as well as the agent's limited liability constraint, i.e., $\hat{s}_t(\mathbf{x}) \geq \underline{s}$, $\forall \mathbf{x}$. We define $M(a;t)$ as the moment generating function of $g(q|a)$, i.e.,

$$M(a;t) \equiv \int e^{tq} g(q|a) dq.$$

which we assume exists for the distribution $g(q|a)$.

Then, we have the following proposition.

Proposition 1: *Given that the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is unbounded below, if, for any given a^o,*

(1a) $\frac{g(q|a)}{g(q|a^o)}$ *is convex in $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a,*

(2b) *(i) there exists $t > 0$ such that $\frac{c'(a^o)}{M'(a^o;t)} M(a^o;t) - c(a^o) \leq \overline{U} - u(\underline{s})$, and (ii) $c(a)$ is convex in $M(a;t)$ for such t, and*

(3b) *$\overline{r}(q)$ in equation (5) is concave in q,*

then the first-order approach is justified.

The conditions in Proposition 1 sufficiently guarantee conditions ((1a),(2a)) in Lemma 3 in the case where the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)a}$, is unbounded below, and thus the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum. Especially, conditions (2b) and (3b) are given as sufficient conditions for condition (2a) in Lemma 3 (i.e., the double crossing property between $r(q)$ and $\hat{r}(q)$) in this case.

As will be shown in the Section 4.2, to guarantee the double crossing property between $r(q)$ and $\hat{r}(q)$ in the case where the agent's limited liability constraint is not binding at all at the optimum, we require that $r(q)$ in equation (6) be concave in q (i.e., condition (3c) in Proposition 2). However, in this case where the limited liability constraint binds for some x, $r(q)$ in (6) cannot be concave around q_c, which can be seen in Figure 4.

Instead, we require that the agent's indirect utility given $s^o(\mathbf{x})$ before constrained by the limited liability constraint, i.e., $\overline{r}(q)$ in equation (5), be concave in q (i.e., condition (3b)), and introduce an arbitrary contract

[23] This is the case for many familiar density functions of the signals (e.g., normal distribution, gamma distribution, Chi-square distribution, etc.).

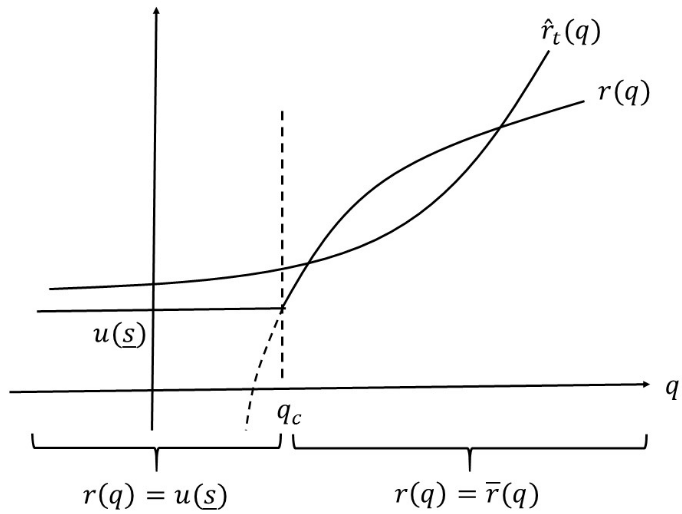


Figure 4: When the Limited Liability Constraint Binds for $q \leq q_c$

$\hat{s}_t(\mathbf{x})$, $t > 0$ such that $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = Ae^{tq} + B$, which is convex.

In fact, condition (2b) (especially, the convexity of $c(a)$ in $M(a;t)$, $t > 0$) guarantees that the agent will choose a^o given $\hat{s}_t(\mathbf{x})$, that is,

$$\begin{aligned} U(\hat{s}_t(\cdot), a) &\equiv \int u(\hat{s}_t(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a) \\ &\leq U(\hat{s}_t(\cdot), a^o) \equiv \int u(\hat{s}_t(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) = U^o, \quad \forall a. \end{aligned} \tag{16}$$

Also, condition (3b) combined with condition (2b) guarantees that $u(s^o(\mathbf{x})) \equiv r(q)$ crosses $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q)$ twice starting from below as shown in Figure 4. Therefore, Lemma 2 implies:

$$\begin{aligned} U(s^o(\cdot), a) &\equiv \int u(s^o(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a) \\ &= \int r(q) g(q|a) dq - c(a) \\ &\leq \int \hat{r}_t(q) g(q|a) dq - c(a) \\ &= \int u(\hat{s}_t(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a) \equiv U(\hat{s}_t(\cdot), a), \quad \forall a. \end{aligned} \tag{17}$$

The inequality in equation equation (17) comes from Lemma 2 because both $r(q)$ and $\hat{r}_t(q)$ satisfy the participation and the relaxed incentive constraints, and $r(q)$ crosses $\hat{r}_t(q)$ twice starting from below.

Then, by combining equation (7), equation (16), and equation (17), we have

$$U(s^o(\cdot), a) \leq U(s^o(\cdot), a^o) = U^o, \ \ \forall a,$$

which justifies the first-order approach.

One thing to note is that, even though condition (2b) looks a bit complicated, it can be reduced to a much simple form especially when there exists $t > 0$ such that the agent's cost function $c(a)$ is linear in $M(a;t)$, i.e., $c(a) = \alpha M(a;t) + \beta$. In this case, condition (2b) holds if $\overline{U} - u(\underline{s}) \geq -\beta$.

We provide the following two examples to show how the conditions in Proposition 1 can be applied to principal-agent problems in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum (i.e., $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ is unbounded below).

Example 1 Consider a one-signal principal-agent problem, $\{\pi(x), u(s), f(x|a), c(a), \overline{U}, \underline{s}\}$, where $\pi(x) = x$ and $u(s) = \frac{1}{r}s^r$, $r \leq \frac{1}{2}$. The agent's cost function is given by $c(a) = D(e^{ka} - 1)$, $D > 0$, $k > 0$, and assume that the signal generating function has a simple additive form such as $\tilde{x} = a + \tilde{\theta}$, $\tilde{\theta} \sim N(0, \sigma^2)$. Then,

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}. \tag{18}$$

Since, given a^o,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \frac{\tilde{x} - a^o}{\sigma^2},$$

we have

$$\tilde{q}|a \sim N\left(\frac{a - a^o}{\sigma^2}, \frac{1}{\sigma^2}\right).$$

Therefore,

$$g(q|a) = \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{2}\left(q - \frac{a - a^o}{\sigma^2}\right)^2\right). \tag{19}$$

Since $\tilde{x} \in (-\infty, \infty)$, $\tilde{q}$ would be unbounded below, and the agent's limited liability constraint must be binding for some low values of $\tilde{q}$ (i.e., for some low values of $\tilde{x}$).

From equation (19), we have

$$\frac{g(q|a)}{g(q|a^o)} = \exp\left((a - a^o)q - \frac{(a - a^o)^2}{2\sigma^2}\right).$$

Thus, one can easily see that condition (1a) is satisfied. Also, from equation (5), $\overline{r}(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$ becomes concave in q when $r \leq \frac{1}{2}$, which implies that condition (3b) is satisfied. Furthermore, since $c(a) = D(e^{ka} - 1)$, $D > 0$, $k > 0$, by picking up $t = k\sigma^2$, we have

$$M(a; t = k\sigma^2) = \int e^{k\sigma^2 q} g(q|a) dq = \exp\left(k(a - a^o) + \frac{k^2\sigma^2}{2}\right) = K \cdot e^{ka},$$

where $K \equiv \exp\left(\frac{k^2\sigma^2}{2} - ka^o\right)$. Therefore, the agent's cost function can be represented as

$$c(a) = \frac{D}{K} M(a; t = k\sigma^2) - D,$$

which implies that condition (2b) holds if $D \le \overline{U} - u(\underline{s})$.

This example clearly shows the advantage of the set of conditions in Proposition 1 over the existing sets of conditions. Using the first-order approach in the above example cannot be justified by any of the existing sets of conditions. First, it is easy to see that neither $f(x|a)$ in equation (18) nor $g(q|a)$ in equation (19) satisfies any of the the CDF-type conditions. Thus, the Mirrless-Rogerson conditions or any extension of those conditions (e.g., the GCDFC by Sinclair-Desgagné (1994), the CISP condition by Conlon (2009), and the CDFCL by Jung and Kim (2015)) cannot be used for justifying the first-order approach in this case. Furthermore, the Jewitt conditions and the extension of those by Jung and Kim cannot be also used for justifying the first-order approach in this case. Both Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015) require that $r(q)$ in equation (6) be concave in q. However, since the agent's limited liability constraint must be binding for some $\mathbf{x}$ at the optimum due to the fact that $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)} \in (-\infty, \infty)$ is unbounded below, $r(q)$ in equation (6) cannot be concave in q. In contrast, using the first-order approach in this case can be actually justified by the conditions in Proposition 1 as long as $\overline{U} - U(\underline{s}) \ge D$.

Actually, the fact that the simple signal generating function with a linear form such as $\tilde{x} = a + \tilde{\theta}$, where $\tilde{\theta}$ is normally distributed, has not been able to be used in the principal-agent problem has been a big obstacle in applying the principal-agent theories to various economic problems. What the above example shows is that the first-order approach can be validly adopted in the principal-agent problem in which the signals for the agent's effort are normally distributed if the agent's cost of effort has a somewhat convex functional form such as an exponential form.

Example 2 Consider a one-signal principal-agent problem, $\{\pi(x), u(s), f(x|a), c(a), \overline{U}, \underline{s}\}$, where $\pi(x) = x$, $u(s) = \frac{1}{r}s^r$, $r \le \frac{1}{2}$, $c(a) = ka^\eta$, $k > 0$, $\eta \ge 1$, and $\tilde{x} \in (0, \infty)$ has the gamma distribution with shape parameter a, i.e.,

$$f(x|a) = \frac{x^{a-1}\beta^{-a}}{\Gamma(a)} e^{-\frac{x}{\beta}}. \tag{20}$$

Since, given a^o,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \log \tilde{x} - K,$$

where $K \equiv \log \beta + \frac{\Gamma'(a^o)}{\Gamma(a^o)}$, $\tilde{q}$ is unbounded below. Thus, the agent's limited liability constraint must be binding for some low values of $\tilde{q}$ (i.e., for some low value of $\tilde{x}$). We have

$$g(q|a) = \frac{\beta^{-a}}{\Gamma(a)} \exp\left(a(q+K) - \frac{1}{\beta}e^{q+K}\right). \tag{21}$$

Thus, we can see from equation (21) that condition (1a) is satisfied. Also, from equation (5), $\bar{r}(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$ is concave in q since $r \leq \frac{1}{2}$, which implies that condition (3b) is satisfied.

Since the agent's cost function is given by $c(a) = ka$, $k > 0$, by picking up $t = 1$, we have

$$M(a; t=1) = \int e^{1 \cdot q} g(q|a) dq = e^{-K} \int x f(x|a) dx = \beta e^{-K} a.$$

Therefore, the agent's cost function can be represented as

$$c(a) = \frac{ke^{\eta K}}{\beta^{\eta}} M(a; t=1)^{\eta},$$

which implies that condition (2b) holds when $(\eta - 1)c(a^0) < \overline{U} - u(\underline{s})$. In the case of $\eta = 1$, this condition must hold as $\overline{U} > u(\underline{s})$.

What Example 2 shows is even more striking. Unlike the case in Example 1, in the case where $f(x|a)$ is a gamma density function with shape parameter a, justifying the first-order approach does not even need a convex cost function of effort. Similar to Example 1, using the first-order approach for the principal-agent problem in the above example cannot be justified by any of the existing sets of conditions. One can see that neither $f(x|a)$ in equation (20) nor $g(q|a)$ in equation (21) satisfies any of the CDF-type conditions. Thus, the Mirrlees-Rogerson conditions and any extension of those conditions (e.g., the GCDFC by Sinclair-Desgagné (1994), the CISP condition by Conlon (2009), and the CDFCL by Jung and Kim (2015)) cannot be used for justifying the first-order approach in this case. Furthermore, since, for any given a^o, $\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \log \tilde{x} - K \in (-\infty, \infty)$, which is unbounded below, the agent's limited liability constraint must be binding for low values of $\tilde{x}$ at the optimum. Thus, by the same reason as in Example 1, Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015) cannot be also used for justifying the first-order approach in this case. However, as shown, using the first-order approach in this case can be actually justified by our new conditions in proposition 1.

4.2 When the Information Variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, Is Bounded Below

We now consider the case in which the agent's limited liability constraint is not binding for all $\mathbf{x}$ at the optimum, and provide an alternative set of conditions that is easier to verify than ((1a),(2a)) in Lemma 3. The agent's limited liability constraint will not be binding for all $\mathbf{x}$ at the optimum if the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)} \in [\underline{q}, \overline{q}]$, has a lower bound, i.e., $\underline{q} > -\infty$, and $\underline{s}$ is low enough such that $\bar{r}(\underline{q})$ in equation (5) is greater than $u(\underline{s})$.[24] Thus, to guarantee that the agent's limited liability constraint is not binding for all $\mathbf{x}$ at the optimum as long as the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, we will assume that that $\underline{s}$ is low enough.

When the agent's limited liability constraint is not binding for all $\mathbf{x}$ at the optimum, equation equation (3) reduces to

$$\frac{1}{u'(s^o(\mathbf{x}))} = \lambda + \mu \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} = \lambda + \mu q, \qquad \text{for all } q, \tag{22}$$

and equation equation (6) reduces to

$$u(s^o(\mathbf{x})) \equiv r(Q_{a^o}(\mathbf{x})) = r(q) = \bar{r}(q), \qquad \text{for all } q. \tag{23}$$

Furthermore, since the agent's participation constraint must be binding (i.e., $\lambda > 0$) in this case, equation equation (7) also reduces to

$$U(s^o(\cdot), a^o) = \int u(s^o(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) = \int r(q) g(q|a^o) dq - c(a^o) = \overline{U}. \tag{24}$$

Then, based on equations equation (22), equation (23) and equation (24), we have the following proposition.

Proposition 2: *Given that the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, if, for any given a^o,*
(1a) $\frac{g(q|a)}{g(q|a^o)}$ *is convex in* $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ *for all* a,
(2c) $c(a)$ *is convex in* $m(a) \equiv \int q g(q|a) dq$, *and*
(3c) $\bar{r}(q)$ *is concave in* q,
then the first-order approach is justified.

The conditions in Proposition 2 sufficiently guarantee conditions ((1a),(2a)) in Lemma 3 in the case where the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, and thereby the agent's limited liability constraint is not binding at all at the optimum. Especially, conditions (2c) and (3c) are given as sufficient conditions for condition (2a) in Lemma 3 (i.e., the double crossing property between $r(q)$ and $\hat{r}(q)$) in this case.

Unlike the previous case in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the

[24] This is the case to which Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015) can be applied.

optimum, to guarantee the double crossing property between $r(q)$ and $\hat{r}(q)$ in this case, we can require that $r(q)$ in equation (6) be concave in q (i.e., condition (3c) in Proposition 2). This is because $r(q) = \overline{r}(q)$, $\forall q$ in this case as seen in Figure 5.

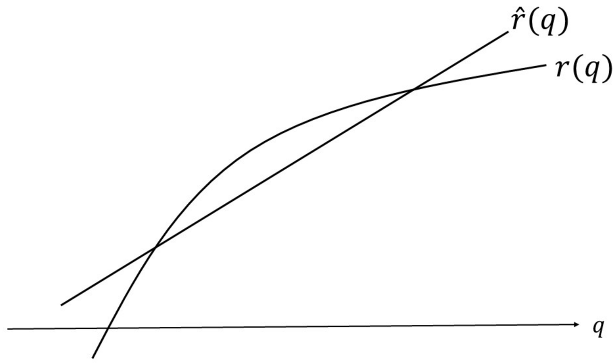


Figure 5: When the Limited Liability Constraint Does Not Bind

Also, we introduce an arbitrary contract, $\hat{s}(\mathbf{x})$, with which the agent's indirect utility is linear in q, i.e., $u(\hat{s}(\mathbf{x})) \equiv \hat{r}(q) = Aq + B$, where A and B are to be set to satisfy both the participation and the relaxed incentive constraints at a^o. Condition (2c) is given to ensure that the the agent will actually choose a^o given $\hat{s}(\mathbf{x})$, satisfying equation (14) with $U^o = \overline{U}$. Also, condition (3c) is given to guarantee the double crossing property between $r(q)$ and $\hat{r}(q)$, satisfying equation (15). Thus, one can easily see that

$$U(s^o(\cdot), a) \leq U(s^o(\cdot), a^o) = \overline{U}, \quad \forall a,$$

which justifies the first-order approach.

The following example explains how the conditions in Proposition 2 can be applied to principal-agent problems where the agent's limited liability constraint is not binding for all $\mathbf{x}$ at the optimum.

Example 3 Consider a one-signal principal-agent problem, $\{\pi(x), u(s), f(x|a), c(a), \overline{U}, \underline{s}\}$, where $\pi(x) = x$, $u(s) = \frac{1}{r}s^r$, $r \leq \frac{1}{2}$, and $c(a)$ is increasing and convex in a. We assume that the signal generating function has a simple multiplicative form, $\tilde{x} = h(a)\tilde{\theta}$, where $h(0) = 0$, $h(a)$ is increasing, and $\tilde{\theta}$ is exponentially distributed with mean 1, i.e., the density function of $\tilde{\theta}$ is $p(\theta) = e^{-\theta}$, $\theta \in [0, \infty)$. We also assume that $\underline{s}$ is low enough. Then,

$$f(x|a) = \frac{1}{h(a)} e^{-\frac{x}{h(a)}}, \tag{25}$$

where $E[x|a] = h(a)$. Since, given a^o,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \frac{h'(a^o)}{[h(a^o)]^2}[\tilde{x} - h(a^o)],$$

$f(x|a)$ satisfies MLRP. Since q is bounded below, i.e., $q \geq -\frac{h'(a^o)}{h(a^o)}$, and since $\underline{s}$ is assumed to be low enough, the agent's limited liability constraint is not binding at all at the optimum. Using $g(q|a)dq = f(x|a)dx$, we derive

$$g(q|a) = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp\left(-\frac{1}{h(a)}\left(\frac{[h(a^o)]^2}{h'(a^o)}q + h(a^o)\right)\right). \tag{26}$$

Thus, it is easily seen that condition (1a) holds. Furthermore, from (22) and (23), $r(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$ is concave in q since $r \leq \frac{1}{2}$, which implies that condition (3c) is satisfied. Now, since

$$m(a) \equiv \int q g(q|a) dq = \frac{h'(a^o)}{[h(a^o)]^2}[h(a) - h(a^o)],$$

condition (2c) will hold if $c(a)$ is convex in $h(a)$, i.e., $c(a) = \phi(h(a))$, $\phi'' \geq 0$. As a result, the only condition that is needed to justify the first-order approach in this case is that $c(a)$ is convex in $h(a)$. Of course when $h(a)$ is concave in a, this condition trivially holds.

Using the first-order approach for the principal-agent problem in the above example cannot be justified by any of the existing sets of conditions if $h(a)$ is not concave in a. First, note that neither $f(x|a)$ in equation (25) nor $g(q|a)$ in equation (26) satisfies any of the CDF-type conditions. This indicates that the Mirrlees-Rogerson conditions and any existing conditions of those conditions (i.e., GCDFC by Sinclair-Desgagné (1994), the CISP condition by Conlon (2009), and the CDFCL by Jung and Kim (2015)) cannot be used for justifying the first-order approach in this case. On the other hand, Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015), which do not contain any CDF-type condition, cannot be satisfied if $h(a)$ is not concave in a. This is because the concavity of $E[x|a]$ is necessary for Theorem 1 in Jewitt (1988), whereas the concavity of $m(a)$ is necessary for Proposition 7 in Jung and Kim (2015). Therefore, those conditions cannot be also used for justifying the first-order approach in this case if $h(a)$ is not concave in a. Of course, if $h(a)$ is concave in a, condition (2c) will also be satisfied because $c(a)$ is increasing and convex in a. Thus, what Example 3 shows is that, even if $h(a)$ is not concave in a, using the first-order approach in this case can still be justified by the conditions in Proposition 2 as long as $c(a)$ is convex in $h(a)$.

One shortcoming of Proposition 2 is that it still has condition (3c), following Jewitt (1988) and Jung and Kim (2015). For example, it cannot be used in the case of $r > \frac{1}{2}$ in Example 3. However, even if the agent's monetary utility in q-space, $r(q)$, is convex, the next Proposition 2' justifies the first-order approach if some conditions (i.e., (2c′) and (3c′) in Proposition 2′) are satisfied.

Proposition 2′: *Given that $u(s) > 0$ for $\forall s$, and the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, if, for any given a^o,*

(1a) $\frac{g(q|a)}{g(q|a^o)}$ *is convex in* $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ *for all* a,

(2c′) *(i) there exists* $t > 0$ *such that* $\frac{c'(a^o)}{M'(a^o;t)}M(a^o;t) - c(a^o) = \overline{U}$, *and (ii)* $c(a)$ *is convex in* $M(a;t) \equiv \int e^{tq} g(q|a)dq$ *for such* $t > 0$, *and*

(3c′) $\ln \overline{r}(q)$ *is concave in* q,

then the first-order approach is justified.

In contrast to Proposition 2, where we relied on the double-crossing property between $r(q)$ and $\hat{r}(q)$ as in Figure 5, we now construct an *exponential* $\hat{r}(q)$ that double-crosses $\overline{r}(q)$ from above even if $\overline{r}(q)$ is convex, as in Figure 6a. This is possible when $\ln \overline{r}(q)$, instead of $\overline{r}(q)$, is concave as we can construct the linear $\ln \hat{r}(q)$ that double-crosses $\ln \overline{r}(q)$ from above, as in Figure 6b. Condition (2c′) guarantees that under the contract $\hat{r}(q)$, the agent would voluntarily participate and choose $a = a^o$, and Lemma 3 with the double-crossing between $\hat{r}(q)$ and $\overline{r}(q)$ justifies the first-order approach in this case. Since we rely on double-crossing of log-utilities, we assume $u(s) > 0$ for $\forall s$. It turns out that in Example 3, we can justify the first-order approach even in the case of $\frac{1}{2} < r < 1$, as shown in the next Example 4 when $h(a)$ is concave enough.

Example 4 Consider the same one-signal principal-agent problem as Example 3, except here we have $\frac{1}{2} < r < 1$ so $\overline{r}(q)$ is convex. From

$$g(q|a) = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp\left(-\frac{1}{h(a)}\left(\frac{[h(a^o)]^2}{h'(a^o)}q + h(a^o)\right)\right), \tag{27}$$

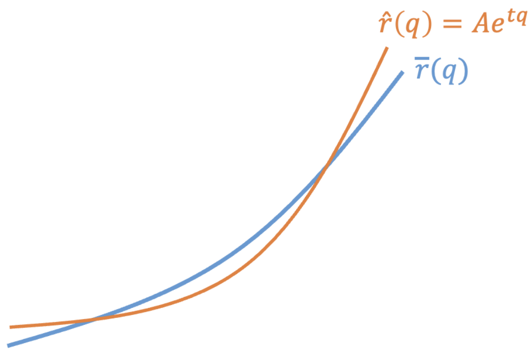
its moment generating function is given by

$$M(a;t) = \frac{h(a^o)^2}{h(a^o)^2 - h'(a^o)h(a)t} \exp\left(-t\frac{h'(a^o)}{h(a^o)}\right), \tag{28}$$

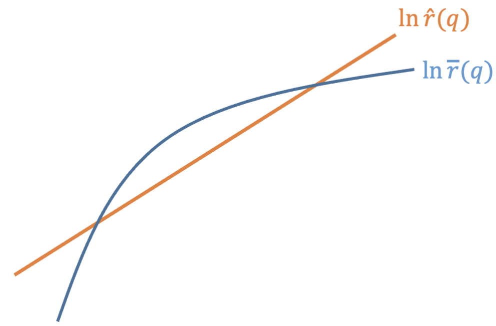
which is defined when $t < \frac{h(a^o)^2}{h(a)h'(a^o)}$. If $c(a)$ becomes convex in $M(a;t)$ in (28) for all feasible $t > 0$, then we satisfy (ii) of condition (2c′). For example, $M(a;t)$ becomes concave in a when $h(a)$ is concave enough, i.e.,

$$h''(a) < -\frac{2th'(a^o)h'(a)^2}{h(a^o)^2 - h'(a^o)h(a)t},$$

and in this case $c(a)$ is convex in $M(a;t)$. With $\overline{r}(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$, $\ln \overline{r}(q) = \frac{r}{1-r}\ln(\lambda + \mu q) - \ln r$ becomes



(a) Double-Crossing between $\hat{r}(q)$ and $\bar{r}(q)$



(b) Double-Crossing between $\ln \hat{r}(q)$ and $\ln \bar{r}(q)$

Figure 6: When $\bar{r}(q)$ is Convex while $\ln \bar{r}(q)$ is Concave

concave in q, satisfying condition (3c′). For (i) of condition (2c′), we can set t as

$$\frac{h(a^o)}{h'(a^o)} > t = \frac{h(a^o)^2 c'(a^o)}{[\overline{U} + c(a^o)]h'(a^o)^2 + h(a^o)h'(a^o)c'(a^o)} > 0$$

that satisfies $\frac{c'(a^o)}{M'(a^o;t)} M(a^o;t) - c(a^o) = \overline{U}$. Therefore, we conclude that we can justify the first-order approach in Example 3 even when $\bar{r}(q)$ is convex. This is a completely different result from the previous literature.[25]

5 The Comparison with the Existing Conditions

As explained, our conditions in Propositions 1 and 2 are different from the existing conditions in that they are directly derived to satisfy equation (4), whereas the existing conditions were derived to make the agent's expected monetary utility given $s^o(\mathbf{x})$ (i.e., $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$) concave in a.

As previously explained, the existing results can be categorized into two groups, the results that contain the CDF-type conditions (i.e., Mirrlees-Rogerson's CDFC, Sinclair-Desgagné's GCDFC, Conlon's CISP condition, and Jung and Kim's CDFCL), and the results that do not contain any CDF-type condition but instead contain a restriction on the agent's utility function (i.e., Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015)). The CDF-type conditions on $f(\mathbf{x}|a)$ in the first group, which were given to make $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ concave in a for any "increasing" $u(s^o(\mathbf{x}))$, can be applied even to the case in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum because $u(s^o(\mathbf{x}))$ is increasing even in this case. However, these conditions have a limitation in that they are hardly satisfied by most familiar density functions.

[25] The detailed derivation of Example 4 is provided in Appendix.

On the other hand, the conditions on $f(\mathbf{x}|a)$ that appear in the results of the second group were given to make $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ concave in a for any "increasing concave" $u(s^o(\mathbf{x}))$. These conditions must be weaker than the above CDF-type conditions because they require that the concavity of $R(a)$ be satisfied for a smaller set of $u(s)$.[26] However, the results in the second group have another limitation in that they cannot be used for the case in which the density function of the signals has its information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, unbounded below, and thus the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum. Unfortunately, many familiar density functions of the signals (e.g., normal, gamma, etc.) belong to this case. In fact, to match with for any "increasing concave" $u(s^o(\mathbf{x}))$, the results in the second group contain another restriction on the agent's utility function $u(s)$ such that $u(s^o(\mathbf{x})) \equiv r(q)$ is concave in q. However, this restriction cannot be satisfied in this case due to the agent's binding limited liability constraint.

As a result, it is rather clear that our conditions in Proposition 1, which can be used even for the case in which the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum, have the advantage over the existing results. When the agent's limited liability constraint is binding for some $\mathbf{x}$ at the optimum, among the existing results, only the results that contain the CDF-type conditions can be used. However, the CDF-type conditions are too restrictive to be satisfied by most familiar density functions of the signals. In contrast, our conditions in Proposition 1 (especially condition (1a)) can be satisfied by the wide range of density functions including the normal and the gamma density functions, as shown in Examples 1 and 2.

On the other hand, the advantage of our conditions in Proposition 2, which can be used only for the case in which the agent's limited liability constraint is not binding at all, over the existing results needs to be explained more carefully. This is because the results which do not contain the CDF-type conditions (i.e., Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015)) can still be used for this case. Thus, it will be interesting to compare our conditions in Proposition 2 with the conditions in Proposition 7 in Jung and Kim (2015) which are up to now the most general conditions in the second group.[27] In this Section 5, we first investigate the statistical implications of condition (1a) and then, based on these statistical implications, compare our new conditions in Proposition 2 with Proposition 7 in Jung and Kim (2015). Also, we illustrate the fundamental differences between our conditions directly derived to satisfy equation (4) and the existing conditions derived to make $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ concave in a.

Proposition 7 in Jung and Kim (2015) states that, if, for any given a^o,

(1J-1) $\int^z G(q|a)dq$ is convex in a for all z,

[26] These conditions are satisfied by some familiar density functions (e.g., Chi-square, Poisson etc.).

[27] Jung and Kim (2015) show that their conditions in Proposition 7 are are more general than the conditions in Theorem 1 in Jewitt (1988).

(1J-2) $m(a) \equiv \int qg(q|a)dq$ is concave in a,[28] and

(2J) $r(q)$ is concave in q,

then the first-order approach is justified.

Notice that (2J) above is identical to (3c) in Proposition 2. Thus, to compare our conditions in Proposition 2 with the conditions in Proposition 7 in Jung and Kim (2015), comparing conditions ((1a),(2c)) with conditions ((1J-1),(1J-2)), given that (2J) (or equivalently (3c)) is satisfied, will be enough. To put forth the conclusion first, the direct comparison between ((1a),(2c)) and ((1J-1),(1J-2)) is not possible because one does not imply the other. This is basically because condition ((1a),(2c)) are derived to directly satisfy equation (4), whereas conditions ((1J-1),(1J-2)) are derived to make $R(a)$ concave in a. However, an indirect comparison between ((1a),(2c)) and ((1J-1),(1J-2)) is possible. Conditions ((1J-1),(1J-2)) are, in general, hard to verify. So our strategy here is as follows:

Step 1: First, we introduce another set of conditions called TP_3-based conditions.

Step 2: We show that (TP_3,(1J-2)) conditions are sufficient for ((1a),(2c)) and easier to verify.

Step 3: We show that (TP_3,(1J-2)) conditions are sufficient for ((1J-1),(1J-2)) and easier to verify.
And we explain ((1a),(2c)) and ((1J-1),(1J-2)) do not imply each other. The TP_3-based conditions are, for any given a^o, given by (TP_3,(1J-2)) as follows.

(TP_3) $g(q|a)$ is totally positive of degree 3 (i.e., TP_3), and

(1J-2) $m(a) \equiv \int qg(q|a)dq$ is concave in a.[29]

Definition 1 (Total Positivity): A function $f(x,a)$, $f : \mathbb{R}^2 \to \mathbb{R}$, is totally positive of degree n (i.e., TP_n) if, for every $x_1 < x_2 < \cdots < x_n$ and $a_1 < a_2 < \cdots < a_n$,

$$T(f,k) \equiv \begin{vmatrix} f(x_1,a_1) & \cdots & f(x_1,a_k) \\ \vdots & \vdots & \vdots \\ f(x_k,a_1) & \cdots & f(x_k,a_k) \end{vmatrix} \geq 0, \qquad \text{for all } k = 1,2,\cdots,n.$$ [30]

To show that conditions ((1a),(2c)) are more general than the above TP_3-based conditions, we first consider

[28]Condition (1J-2) is generally implied by (1J-1) when z goes to infinity. But the reason we list (1J-2) as a separate condition is because $\int G(q|a)dq$ sometimes may not exist.

[29]Since the TP_3-based conditions are given to sufficiently guarantee ((1J-1),(1J-2)), the first-order approach can be justified if conditions (TP_3), (1J-2), and (2J) are satisfied.

[30]For a detailed explanation of "total positivity", see Karlin (1968)

the case in which there is a single signal, i.e., $x \in \mathbb{R}$, and $f(x|a)$ satisfies the MLRP, and thus there is a 1:1 relation between $\tilde{q}$ and $\tilde{x}$. Then, we consider the case in which either there are multiple signals, i.e., $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, or $f(x|a)$, $x \in \mathbb{R}$, doest not satisfy the MLRP, and thus there is no 1:1 relation between $\tilde{q}$ and $\tilde{\mathbf{x}}$.

5.1 When $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP

When there is a single signal, i.e., $x \in \mathbb{R}$, and its density function, $f(x|a)$, satisfies the MLRP, there is a 1:1 relation between $\tilde{q}$ and $\tilde{x}$ for any given a^o. Then, as previously shown,

$$G(q|a) = F(x|a) \qquad \text{for all } a, \tag{29}$$

where x solves $\frac{f_a(x|a^o)}{f(x|a^o)} = q$, and

$$g(q|a)Q'_{a^o}(x) = f(x|a). \tag{30}$$

Note that $Q'_{a^o}(x) \equiv \frac{dQ_{a^o}(x)}{dx}$ is independent of a. Thus, we have

$$\frac{g(q|a)}{g(q|a^o)} = \frac{f(x|a)}{f(x|a^o)} \quad \text{and} \quad \frac{g_a(q|a)}{g(q|a)} = \frac{f_a(x|a)}{f(x|a)} \text{ for all } a. \tag{31}$$

From equation (31), we obtain an interesting result that, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, condition (1a) reduces to

(1d) $\frac{f(x|a)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)} = \frac{g_a(q|a^o)}{g(q|a^o)}$ for any given a^o and for all a.

In other words, if $f(x|a)$ satisfies MLRP for $x \in \mathbb{R}$, condition (1a) can be replaced by condition (1d), which is much easier to verify because it does not require us to explicitly calculate $g(q|a)$ from $f(x|a)$. For instance, consider the case in Example 1. Note that, since $\tilde{x} = a + \tilde{\theta}$, $\tilde{\theta} \sim N(0, \sigma^2)$, $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP. Thus, condition (1a) can be easily verified by using condition (1d) without even calculating $g(q|a)$ from $f(x|a)$. From equation (18), one can derive that

$$q = \frac{f_a(x|a^o)}{f(x|a^o)} = \frac{x - a^o}{\sigma^2},$$

and

$$\frac{f(x|a)}{f(x|a^o)} = \exp\left(-\frac{1}{2\sigma^2}\left(2(a^o - a)x + a^2 - (a^o)^2\right)\right).$$

Since q is linear in x, and $\frac{f(x|a)}{f(x|a^o)}$ is convex in x for any given a^o, it is easy to confirm that condition (1a) is satisfied.

Lemma 4: *Given two functions, $\phi(x)$ and $\psi(x)$, $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$, where $\phi(x)$ is increasing in x,*

$$\begin{vmatrix} 1 & \phi(x_1) & \psi(x_1) \\ 1 & \phi(x_2) & \psi(x_2) \\ 1 & \phi(x_3) & \psi(x_3) \end{vmatrix} \underset{(\leq)}{\geq} 0, \quad \text{for every } x_1 < x_2 < x_3,$$

if and only if $\psi(x)$ is convex (concave) in $\phi(x)$.

Using Lemma 4, we derive the following lemma and corollary.

Lemma 5: *Given $x \in \mathbb{R}$, $f(x|a)$ is TP_3 if and only if*

(i) $f(x|a)$ satisfies the MLRP, and

(ii) $\frac{f(x|a)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$ for any given a^o and for all a (i.e., condition (1d)).

Corollary 1: *Given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^o.*

One thing to note is that Corollary 1 should not be read as "Given that $x \in \mathbb{R}$, $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^o". The statement, "Given that $x \in \mathbb{R}$, $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^o", is true only when $f(x|a)$ satisfies the MLRP. This is because, although, as can be seen from equation (31), the MLRP for $f(x|a)$ (i.e., $T(f,2) \geq 0$ for every $x_1 < x_2$ and $a_1 < a_2$) implies the MLRP for $g(q|a)$ (i.e., $T(g,2) \geq 0$ for every $q_1 < q_2$ and $a_1 < a_2$ for any given a^o), the converse is not always true.

For instance, consider a class of density functions which is a convex mixture of two probability density functions $p_H(x)$ and $p_L(x)$ such that

$$f(x|a) = \alpha(a)p_H(x) + (1 - \alpha(a))p_L(x),$$

where $p_H(x) = -6x^2 + 6x$ and $p_L(x) = 1$ with $x \in [0,1]$, and $\alpha(a) \in [0,1]$ is increasing in a.[31] Then,

$$f(x|a) = \alpha(a)(-6x^2 + 6x - 1) + 1,$$

and

$$\frac{f_a(x|a)}{f(x|a)} = \frac{(-6x^2 + 6x - 1)\alpha'(a)}{1 + (-6x^2 + 6x - 1)\alpha(a)}.$$

Thus, it is easy to see that $f(x|a)$ does not satisfy the MLRP.

[31] This example is from Jung and Kim (2015).

Define, for any given a^o,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \frac{(-6\tilde{x}^2+6\tilde{x}-1)\alpha'(a^o)}{1+(-6\tilde{x}^2+6\tilde{x}-1)\alpha(a^o)}.$$

Then, $\tilde{q}$ is distributed with support $[\frac{\alpha'(a^o)}{\alpha(a^o)-1}, \frac{\alpha'(a^o)}{\alpha(a^o)+2}]$. Thus,

$$G(q|a) = Pr[\tilde{q} \le q|a] = Pr\left[-6\tilde{x}^2+6\tilde{x}-1 \le \frac{q}{\alpha'(a^o)-\alpha(a^o)q}\Big|a\right].$$

If $x_1(q)$ and $x_2(q)$ be two roots of $-6x^2+6x-1 = \frac{q}{\alpha'(a^o)-\alpha(a^o)q}$, where $x_1(q) \le x_2(q)$, then $x_2(q) = 1 - x_1(q)$, where $x_1(q) \in [0, \frac{1}{2}]$ is increasing in q, whereas $x_2(q) \in [\frac{1}{2}, 1]$ is decreasing in q. Therefore,

$$G(q|a) = Pr[\tilde{x} \le x_1(q)|a] + Pr[\tilde{x} \ge x_2(q)|a].$$

Since $f(x|a)$ is symmetric around $x = \frac{1}{2}$,

$$Pr[\tilde{x} \le x_1(q)|a] = Pr[\tilde{x} \ge x_2(q)|a].$$

Thus,

$$G(q|a) = 2Pr[\tilde{x} \le x_1(q)|a] = 2F(x_1(q)|a),$$

and

$$g(q|a) = 2f(x_1(q)|a)x_1'(q).$$

Consequently, we have

$$\frac{g_a(q|a)}{g(q|a)} = \frac{f_a(x_1(q)|a)}{f(x_1(q)|a)}.$$

Since $\frac{f_a(x|a)}{f(x|a)}$ is increasing in $x \in [0, \frac{1}{2}]$, we finally have $\frac{g_a(q|a)}{g(q|a)}$ is increasing in q, indicating that $g(q|a)$ satisfies the MLRP.

Based on Lemma 5 and Corollary 1, we easily see that, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, the following three statements are equivalent.

(1a) For any given a^o, $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)} = \frac{g_a(q|a^o)}{g(q|a^o)}$ for all a.

(1d) For any given a^o, $\frac{f(x|a)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$ for all a

(i.e., $f(x|a)$ is TP_3 given the MLRP for $f(x|a)$).

(1e) For any given a^o, $\frac{g(q|a)}{g(q|a_t)}$ is convex in $\frac{g_a(q|a_t)}{g(q|a_t)}$ for all a, a_t

(i.e., $g(q|a)$ is TP_3 given the MLRP for $g(q|a)$).[32]

Thus, given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, condition (1a) indicates that $g(q|a)$ is TP_3 for any given a^o.

Therefore, we have the following proposition.

Proposition 3: *Given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, conditions* ((TP_3),(1J-2)) *imply conditions* ((1a),(2c)), *but the converse is not true.*

As explained above, condition (1a) is equivalent to condition (TP_3) provided that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP. However, condition (2c) is obviously more general than condition (1J-2) because $c(a)$ is increasing and convex in a. In other words, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, our conditions ((1a),(2c)) require that $g(q|a)$ be TP_3 for any given a^o (or equivalently that $f(x|a)$ be TP_3) but do not necessarily require that $m(a) \equiv \int qg(q|a)dq$ be concave in a (i.e., (1J-2)).

To understand why conditions ((1a),(2c)) are more general than conditions ((TP_3),(1J-2)) intuitively, the following characteristics of a density function with TP_3 will be useful.

Lemma 6: *If a density function $f(x|a)$ is TP_3, then, for any increasing concave function $u(x)$,*

$$u^*(a) \equiv \int u(x)f(x|a)dx \textit{ is increasing concave in } \mu(a) \equiv \int xf(x|a)dx.$$

Lemma 6 shows one of the most interesting characteristics of a density function, $f(x|a)$, with TP_3. When the density function with TP_3 is combined with any increasing concave function, the expected value of that function is increasing concave in the density's mean value. As already explained in Lemma 5 and Corollary 1, given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, condition (1a) is equivalent to that $g(q|a)$ is TP_3 for any given a^o. Thus, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, another interpretation for the conditions in Proposition 2 is possible based on Lemma 6.

Denote

$$U(s^o(\cdot), a) \equiv \int r(q)g(q|a)dq - c(a) = R(a) - c(a) \equiv \xi(m(a)) - \phi(m(a)),$$

where $m(a) \equiv \int qg(q|a)dq$, i.e., the mean value of $g(q|a)$. Then, conditions (1a) and (3c) in Proposition 2 imply that $R(a)$ is concave in $m(a)$ (i.e., $\xi'' \leq 0$) by Lemma 6. Furthermore, since $c(a)$ is convex in $m(a)$ (i.e., $\phi'' \geq 0$) by condition (2c), conditions ((1a),(2c),(3c)) in Proposition 2 ensure that, when $x \in \mathbb{R}$ and

[32] It is worth noting that (1a) is weaker than (1e) in general. This is because (1a) requires that (1e) hold only for $a_t = a^o$ but not for all a_t. However, (1a) and (1e) become equivalent when $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP.

$f(x|a)$ satisfies MLRP, $U(s^o(\cdot), a)$ is concave in $m(a)$, which sufficiently guarantees equation (4) since $m(a)$ is increasing in a. To justify the first-order approach, all the existing results were derived to make $U(s^o(\cdot), a)$ concave in a (to be more precisely, $R(a)$ concave in a). However, the first-order approach can be more generally justified by showing that there exists an increasing function of a, such as $m(a)$, in which $U(s^o(\cdot), a)$ is concave. This is actually what the conditions in Proposition 2 entails.

We now explain why there is no direct comparison available between conditions ((1J-1),(1J-2)) and conditions ((1a),(2c)). First, note that our conditions ((1a),(2c)) do not imply conditions ((1J-1),(1J-2)). For instance, consider the case in Example 3 where $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP. Assume that $h(a)$ is not concave in a. Then, since $m(a) \equiv \int qg(q|a)dq = \frac{h'(a^o)}{[h(a^o)]^2}(h(a) - h(a^o))$, neither (1J-1) nor (1J-2) is satisfied. The concavity of $m(a) \equiv \int qg(q|a)dq$ is essential not only for (1J-2) itself but also for (1J-1). However, conditions ((1a),(2c)) will be satisfied as long as $c(a)$ is convex in $h(a)$.

Conditions ((1J-1),(1J-2)) do not imply conditions ((1a),(2c)), either. To see this, it is worth to note that conditions ((TP_3),(1J-2)) are sufficient but not necessary for conditions ((1J-1),(1J-2)). As shown in Jewitt (1988), conditions ((1J-1),(1J-2)) are necessary and sufficient for $R(a) \equiv \int r(q)g(q|a)dq$ to be increasing and concave in a for any increasing concave function $r(q)$. On the other hand, conditions ((TP_3),(1J-2)) are sufficient but not necessary for $R(a) \equiv \int r(q)g(q|a)dq$ to be increasing and concave in a for any increasing concave function $r(q)$. The sufficient part comes from that $R(a) \equiv \int r(q)g(q|a)dq \equiv \xi(m(a))$ is increasing concave in $m(a)$ by (TP_3), and $m(a)$ is concave in a by (1J-2). However, even if $R(a) \equiv \int r(q)g(q|a)dq \equiv \xi(m(a))$ is increasing concave in a for any increasing concave function $r(q)$ given that $m(a) \equiv \int qg(q|a)dq$ is concave in a (i.e., (1J-1) and (1J-2)), it does not necessarily mean that $R(a)$ is increasing concave in $m(a)$, i.e., $\xi'' \leq 0$. Furthermore, even if $R(a)$ is increasing concave in $m(a)$ for any increasing concave function $r(q)$, $g(q|a)$ may not be always TP_3.[33] Therefore, condition (1J-1) is more general than condition (TP_3) given that condition (1J-2) is satisfied.[34] This tells that conditions ((1J-1),(1J-2)) do not imply conditions ((1a),(2c)) because (1a) is equivalent to (TP_3) provided that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP. Consequently, there is no relation of inclusion between conditions ((1J-1),(1J-2)) and conditions((1a),(2c)).

In fact, Figure 7 below demonstrates the relation among ((1a),(2c)), ((1J-1),(1J-2)), and ((TP_3),(1J-2)) more clearly. Given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, ((TP_3),(1J-2)) is equivalent to ((1a),(1J-2)) due to Lemma 5. In turn, ((1a),(1J-2)) implies ((1a),(2c)) because the concave $m(a)$ implies that our $c(a)$ is convex in $m(a)$. Also ((TP_3),(1J-2)) implies ((1J-1),(1J-2)) from Lemma 6. Still ((1J-1),(1J-2)) and ((1a),(2c)) do not include each

[33] In fact, the above argument indicates that conditions ((TP_3),(1J-2)) implies conditions ((1J-1),(1J-2)) given that condition (2J) is satisfied. However, as shown in our another paper, Jung and Kim (2015), conditions ((TP_3),(1J-2)) actually imply conditions ((1J-1),(1J-2)) even without condition (2J) being satisfied.

[34] Without condition (1J-2) being satisfied, condition (TP_3) does not always imply condition (1J-1).

other.

$$
\begin{tikzcd}
((TP_3),(1J-2)) \arrow[r, leftrightarrow, "\text{Lemma 5}", "\text{Given MLRP}"'] \arrow[dr] & ((1a),(1J-2)) \arrow[r] & ((1a),(2c)) \arrow[dl, leftrightarrow, "\text{No inclusion}"'] \\
& ((1J-1),(1J-2)) &
\end{tikzcd}
$$

Figure 7: Relation Diagram between Conditions

As a result, direct comparison between our conditions ((1a),(2c)) and conditions ((1J-1),(1J-2)) is not possible. However, our conditions ((1a),(2c)) are at least more general than TP_3-based conditions, ((TP_3),(1J-2)), when $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP.

5.2 When $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, or $f(x|a)$, $x \in \mathbb{R}$, does not satisfy the MLRP

When there are multiple signals, i.e., $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, or density function $f(x|a)$ does not satisfy the MLRP even if $x \in \mathbb{R}$, there is no 1:1 relation between $\tilde{q}$ and $\tilde{\mathbf{x}}$. Thus, some of the results that are derived when there is a 1:1 relation between $\tilde{q}$ and $\tilde{x}$ in the previous subsection may not hold. However, the main result in the previous subsection still holds even in this case. That is, although there is no relation of inclusion between our conditions ((1a),(2c)) and conditions ((1J-1),(1J-2)), conditions ((1a),(2c)) are more general than the TP_3-based conditions, ((TP_3),(1J-2)). Nevertheless, it is worth noting that there are two non-trivial differences in this case compared with the previous case.

Consider a multi-signal case where there is a random vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \cdots, \tilde{x}_n)$, $n \geq 2$, with a density $f(\mathbf{x}|a)$. Although there are multiple $\mathbf{x}$ satisfying $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} = q$, for any given q, one can calculate $g(q|a)$ from $f(\mathbf{x}|a)$ by using the transformation method of random variables.[35] In order to use the transformation method, we introduce a random vector $\tilde{\mathbf{y}} = (\tilde{y}_1, \cdots, \tilde{y}_n)$ with a density function $\hat{f}(\mathbf{y}|a)$ such that $\tilde{y}_j = \tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ and $\tilde{y}_i = \tilde{x}_i$ for all $i = 1, \cdots, n$, $i \neq j$. If there exists a coordinate x_j in which $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ is increasing for any $\mathbf{x}_{-j}$, the density function of $\tilde{\mathbf{y}}$ can be expressed as

$$\hat{f}(\mathbf{y}|a) = f(\mathbf{x}|a) \cdot |J| = f(x_j(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \times \left| \frac{\partial x_j(q, \mathbf{x}_{-j})}{\partial q} \right|,$$

[35] In one-signal cases in which $f(x|a)$, $x \in \mathbb{R}$, does not satisfy MLRP, one easily obtains $g(q|a) = \sum_k f(x_k(q)|a) \left| \frac{dx_k(q)}{dq} \right|$ where $x_k(q) \in X(q) \equiv \left\{ x | \frac{f_a(x|a^o)}{f(x|a^o)} = q \right\}$.

where J is the transformation's Jacobian and $x_j(q, \mathbf{x}_{-j})$ solves $\frac{f_a(x_j, \mathbf{x}_{-j}|a^o)}{f(x_j, \mathbf{x}_{-j}|a^o)} = q$ for given $\mathbf{x}_{-j}$.[36] Note that, in this case, $|J| = \left|\frac{\partial x_j(q, \mathbf{x}_{-j})}{\partial q}\right|$, which is independent of a because $x_j(q, \mathbf{x}_{-j})$ is independent of a. Then, we have:

$$g(q|a) = \int \hat{f}(q, \mathbf{y}_{-j}|a) d\mathbf{y}_{-j} = \int f(x_j(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \cdot |J| d\mathbf{x}_{-j} = \int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j},$$

where $X(q) \equiv \left\{\mathbf{x} | \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} = q\right\}$. Thus, although

$$\frac{g_a(q|a^o)}{g(q|a^o)} = \frac{\int_{\mathbf{x} \in X(q)} f_a(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}} = \frac{\int_{\mathbf{x} \in X(q)} q f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}} = q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} \tag{32}$$

for any given a^o even in this case, it is generally true that

$$\frac{g(q|a)}{g(q|a^o)} = \frac{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}} \neq \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^o)}, \text{ and } \frac{g_a(q|a)}{g(q|a)} = \frac{\int_{\mathbf{x} \in X(q)} f_a(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}} \neq \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)}, \tag{33}$$

where $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$.

For instance, consider multi-signal cases where $\tilde{\mathbf{x}} \sim N(\mu(a), \Sigma)$ where $\mu(a) = [\mu_1(a), \cdots, \mu_n(a)]'$ and Σ is a covariance matrix. Then,

$$f(\mathbf{x}|a) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}[\mathbf{x} - \mu(a)]' \Sigma^{-1} [\mathbf{x} - \mu(a)]\right),$$

and

$$\frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)} = [\mu'(a)]' \Sigma^{-1} [\mathbf{x} - \mu(a)].$$

Thus, we have

$$\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)} \sim N\left(m(a), \sigma_q^2\right),$$

[36] Even in multi-signal cases where there is no x_j in which $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ is increasing, (32) and (33) still hold. For example, if the support $\{\mathbf{x}|f(\mathbf{x}|a) > 0\}$ can be decomposed into subsets $X_1, \cdots, X_m$ such that $\tilde{\mathbf{y}}$ is a 1:1 transformation of X_k onto a subset of the support $\left\{\mathbf{y}|\hat{f}(\mathbf{y}|a) > 0\right\}$, the density function of $\tilde{\mathbf{y}}$ can be expressed by $\hat{f}(\mathbf{y}|a) = \sum_k f(x_j^k(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \cdot |J_k|$, where $x_j^k(q, \mathbf{x}_{-j})$ solves $\frac{f_a(x_j, \mathbf{x}_{-j}|a^o)}{f(x_j, \mathbf{x}_{-j}|a^o)} = q$ on X_k for given $\mathbf{x}_{-j}$ and J_k is the Jacobian of the transformation on X_k, from which one can easily obtain $g(q|a) = \int \sum_k f(x_j^k(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \cdot |J_k| d\mathbf{x}_{-j}$.

where $m(a) \equiv [\mu'(a^o)]'\Sigma^{-1}[\mu(a) - \mu(a^o)]$ and $\sigma_q^2 \equiv [\mu'(a^o)]'\Sigma^{-1}\mu'(a^o)$, and

$$\frac{g_a(q|a)}{g(q|a)} = \frac{q - [\mu'(a^o)]'\Sigma^{-1}[\mu(a) - \mu(a^o)]}{[\mu'(a^o)]'\Sigma^{-1}\mu'(a^o)}[\mu'(a^o)]'\Sigma^{-1}\mu'(a).$$

Therefore, one easily sees: $\frac{g(q|a)}{g(q|a^o)} \neq \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^o)}$ and $\frac{g_a(q|a)}{g(q|a)} \neq \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)}$ for $\forall a \neq a^o$ where $q = [\mu'(a^o)]'\Sigma^{-1}[\mathbf{x} - \mu(a^o)]$. This shows that, when there is no 1:1 relation between $\tilde{\mathbf{x}}$ and $\tilde{q}$, equation (33) is generally true.

From equation (33), one sees two non-trivial differences between this case and the previous case in which $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP. First, condition (1a) cannot be reduced to condition (1d). This indicates that, to verify condition (1a) in this case, one should explicitly calculate $g(q|a)$ from $f(\mathbf{x}|a)$. Second, condition (1a) is not equivalent to that $g(q|a)$ is TP_3 for any given a^o. In fact, the condition that $g(q|a)$ is TP_3 for any given a^o is stronger than condition (1a) in this case. In order for $g(q|a)$ to be TP_3 for any given a^o, it is needed that, for any given a^o,

(1) $g(q|a)$ satisfies the MLRP, and
(2)(=(1e)) $\frac{g(q|a)}{g(q|a_t)}$ is convex in $\frac{g_a(q|a_t)}{g(q|a_t)}$ for all a, a_t.

However, condition (1a) requires neither that $g(q|a)$ satisfy the MLRP nor that condition (1e) hold for all a_t.[37] Thus, in contrast with the previous case, when there is no 1:1 relation between $\tilde{q}$ and $\tilde{\mathbf{x}}$, condition (1a) is more general than condition (TP_3). As a result, our conditions ((1a),(2c)) are even more general than conditions ((TP_3),(1J-2)) in this case.

However, there is a meaningful exception even in this case. Consider a density function $f(\mathbf{x}|a)$, $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, which generates, for any given a^o,

$$\frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)} = \alpha(a) \cdot q + \beta(a), \tag{34}$$

where $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$, $\alpha(a^o) = 1$, $\beta(a^o) = 0$, and $\alpha(a) \geq 0$. Note that most exponential family density functions with an appropriate parameterization satisfy equation (34).

[37] When there is no 1:1 relation between $\tilde{q}$ and $\tilde{\mathbf{x}}$, MLRP for $f(\mathbf{x}|a)$ does not always imply MLRP for $g(q|a)$ for a given a^o. For example, consider a discrete case where $f(x_1, x_2|a) = e^{2\sqrt{a}x_1 + ax_2 - K(a)}$, $x_i \in \{0, 1\}$, $i = 1, 2$, $a > 0$, where $K(a) = \log[(1 + e^{2\sqrt{a}})(1 + e^a)]$. $\frac{f_a(x_1,x_2|a)}{f(x_1,x_2|a)} = \frac{x_1}{\sqrt{a}} + x_2 - K'(a)$ implies that $f(x_1, x_2|a)$ satisfies MLRP. Define $\tilde{q} = \frac{\tilde{x}_1}{\sqrt{a^o}} + \tilde{x}_2 - K'(a^o)$ and let $a^o < 1$. Then, $g(q_1|a) = f(0, 0|a) = e^{-K(a)}$, $g(q_2|a) = f(0, 1|a) = e^{1-K(a)}$, $g(q_3|a) = f(1, 0|a) = e^{2\sqrt{a}-K(a)}$ and $g(q_4|a) = f(1, 1|a) = e^{2\sqrt{a}+a-K(a)}$, where $q_1 = -K'(a^o)$, $q_2 = 1 - K'(a^o)$, $q_3 = \frac{1}{\sqrt{a^o}} - K'(a^o)$, and $q_4 = 1 + \frac{1}{\sqrt{a^o}} - K'(a^o)$ so $q_1 < q_2 < q_3 < q_4$. Thus, $\frac{g_a(q_1|a)}{g(q_1|a)} = -K'(a)$, $\frac{g_a(q_2|a)}{g(q_2|a)} = 1 - K'(a)$, $\frac{g_a(q_3|a)}{g(q_3|a)} = \frac{1}{\sqrt{a}} - K'(a)$, and $\frac{g_a(q_4|a)}{g(q_4|a)} = \frac{1}{\sqrt{a}} + 1 - K'(a)$. Since, when $a > 1$, $\frac{g_a(q_2|a)}{g(q_2|a)} = 1 - K'(a) > \frac{1}{\sqrt{a}} - K'(a) = \frac{g_a(q_3|a)}{g(q_3|a)}$, $g(q|a)$ does not satisfy MLRP.

Since

$$\frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^o)} = \exp\left(\int_{a^o}^{a} \frac{f_a(\mathbf{x}|t)}{f(\mathbf{x}|t)} dt\right) = \exp\left(\left(A(a) - A(a^o)\right) q + B(a) - B(a^o)\right),$$

where $A(a) \equiv \int_0^a \alpha(t)dt$ and $B(a) \equiv \int_0^a \beta(t)dt$, one can see that, for any given q, $\frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^o)}$ has the same value for all $\mathbf{x} \in X(q) \equiv \left\{\mathbf{x}|\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} = q\right\}$. Thus,

$$\begin{aligned}
\frac{g(q|a)}{g(q|a^o)} &= \frac{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}} = \frac{\int_{\mathbf{x}\in X(q)} \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^o)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}} \\
&= \frac{\exp\left(\left(A(a) - A(a^o)\right) q + B(a) - B(a^o)\right) \cancel{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}}}{\cancel{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a^o) \cdot |J| d\mathbf{x}_{-j}}} = \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^o)}.
\end{aligned}$$

Furthermore, from equation (34), we also have

$$\begin{aligned}
\frac{g_a(q|a)}{g(q|a)} &= \frac{\int_{\mathbf{x}\in X(q)} f_a(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}} = \frac{\int_{\mathbf{x}\in X(q)} \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}} \\
&= \frac{(\alpha(a)q + \beta(a)) \cancel{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}}{\cancel{\int_{\mathbf{x}\in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}} = \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)}.
\end{aligned}$$

Therefore, even if there is no 1:1 relation between $\tilde{q}$ and $\tilde{\mathbf{x}}$, all the results in the previous Section 5.1 equally hold in this case. That is, condition (1a) reduces to condition (1d) and thus verifying (1a) can be replaced by verifying (1d) which does not require to calculate $g(q|a)$ from $f(\mathbf{x}|a)$. Furthermore, since $\alpha(a) \geq 0$, $g(q|a)$ satisfies the MLRP from equation (35). Therefore, based on equation (34) and equation (35), one can also see that condition (1a) is equivalent to that $g(q|a)$ is TP_3 for any given a^o.

6 Conclusion

All the existing results for justifying the first-order approach in the principal-agent problems have been derived to make the agent's expected monetary utility obtained from that approach "concave" in the agent's effort. However, to justify the first-order approach, relying on such concavity is sometimes overly sufficient. We have proposed a new set of conditions which is derived not from the concavity of the agent's expected monetary utility but directly from the original "argmax" incentive constraint, and shown that it can be applied to a wider range of principal-agent problems than the existing results. The examples we suggested (i.e., Examples 1, 2, and 3) illustrate cases where the previous literature cannot justify their use of the first-order approach while our new sets of conditions, in contrast, can.

This set of conditions contains of a statistical condition on the density function of the signals, which is quite general and easy to verify. We also have provided two alternative sets of conditions that are derived to be applied specifically to the case in which the agent's limited liability constraint is binding for some values of the signal vector at the optimum and to the case in which it is not binding at the optimum, respectively. Then, statistical implications of these two sets of conditions have been explored, and the comparison between these conditions and the existing conditions has been provided.

When the agent's limited liability constraint is binding for some values of the signal vector at the optimum, among the existing results in the literature, only the sets of conditions containing the CDF-type conditions can be applied. While the CDF-type conditions are hardly satisfied by most familiar density functions, our corresponding set of conditions (i.e., the conditions in Proposition 1) can be used for many useful density functions including the normal density function and gamma density function. On the other hand, when the agent's limited liability constraint is not binding at all at the optimum, Jewitt (1988) conditions (i.e., Theorem 1 in Jewitt (1988)) and Jung and Kim (2015) conditions (i.e., Proposition 7 in Jung and Kim (2015)) which do not contain any CDF-type conditions can still be applied. We have shown that there is no direct comparison available between our corresponding set of conditions (i.e., the conditions in Proposition 2) and the conditions in Proposition 7 in Jung and Kim (2015) in the sense that one does not imply the other. This is because Jung and Kim (2015) conditions, like other existing sets of conditions, were derived to make the agent's expected monetary utility concave in effort, whereas our conditions are derived to directly satisfy the original "argmax" incentive constraint without relying on such concavity. Nevertheless, we have shown that our conditions are more general than the TP_3-based conditions which were proposed to verify Jung and Kim (2015) conditions.

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7 Appendix

Proof of Lemma 1. Using equation (2), the relaxed incentive constraint at a^o is

$$\begin{aligned} c'(a^o) &= \int u(s^o(\mathbf{x}))f_a(\mathbf{x}|a^o)d\mathbf{x} \\ &= \int u(s^o(\mathbf{x}))\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}f(\mathbf{x}|a^o)d\mathbf{x} \\ &= \int \left(u(s^o(\mathbf{x})) - E[u(s^o(\mathbf{x}))]\right)\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}f(\mathbf{x}|a^o)d\mathbf{x} \\ &= Cov\left(u(s^o(\mathbf{x})), \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}\right). \end{aligned}$$

In the above equation, the third equality comes from the fact that $\int E[u(s^o(\mathbf{x}))]f_a(\mathbf{x}|a^o)d\mathbf{x} = 0$ since $E[u(s^o(\mathbf{x}))]$ is constant, and the last equality comes from the fact that $E\left[\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}|a^o\right] = \int f_a(\mathbf{x}|a^o)d\mathbf{x} = 0$. Suppose to the contrary that $\mu \leq 0$. Then, $Cov\left(u(s^o(\mathbf{x})), \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}\right) \leq 0$ from equation (3), which contradicts $c'(a^o) > 0$. Therefore, μ must be positive.

■

Proof of Lemma 2. Let $\psi(c) \equiv \xi(c)g(c|a^o)$, and define $\psi^{(1)}(q) \equiv \int^q \xi(c)g(c|a^o)dc = \int^q \psi(c)dc$ and $\psi^{(2)}(q) \equiv \int^q \int^c \xi(t)g(t|a^o)dtdc = \int^q \psi^{(1)}(c)dc$. Then, we have $\psi^{(1)}(\overline{q}) = 0$ from (L2). Furthermore, since

$$\begin{aligned} \int \xi(q)q \cdot g(q|a^o)dq = q\int^q \xi(c)g(c|a^o)dc\Big|_{\underline{q}}^{\overline{q}} - \int\int^q \xi(c)g(c|a^o)dcdq \\ = -\int\int^q \xi(c)g(c|a^o)dcdq = 0, \end{aligned}$$

where the second equality comes from (L2), and the last equality is from (L3), we have $\psi^{(2)}(\overline{q}) = 0$. Note from (L4) that $\psi(c)$ changes sign twice from negative to positive and to negative as c increases. Thus, $\psi^{(1)}(q)$ changes sign once from negative to positive as q increases since $\psi^{(1)}(\overline{q}) = 0$. Since $\psi^{(1)}(q)$ changes sign once from negative to positive as q increases and since $\psi^{(2)}(\overline{q}) = \int \psi^{(1)}(q)dq = 0$, we have

$$\psi^{(2)}(q) = \int^q \int^c \xi(t)g(t|a^o)dtdc \leq 0, \quad \forall q. \tag{A.1}$$

Denote

$$\int \xi(q)g(q|a)dq = \int \xi(q)\frac{g(q|a)}{g(q|a^o)}g(q|a^o)dq = \int \xi(q)\Gamma(q,a)g(q|a^o)dq, \tag{A.2}$$

where $\Gamma(q,a) \equiv \frac{g(q|a)}{g(q|a^o)}$. Then, by taking integration by parts twice, we have

$$\begin{aligned}\int \xi(q)\Gamma(q,a)g(q|a^o)dq &= \Gamma(q,a)\int^q \xi(c)g(c|a^o)dc\Big|_{\underline{q}}^{\overline{q}} - \int\left(\int^q \xi(c)g(c|a^o)dc\right)\Gamma_q(q,a)dq \\ &= -\Gamma_q(q,a)\int^q\int^c \xi(t)g(t|a^o)dtdc\Big|_{\underline{q}}^{\overline{q}} + \int\left(\int^q\int^c \xi(t)g(t|a^o)dtdc\right)\Gamma_{qq}(q,a)dq \\ &= \int\left(\int^q\int^c \xi(T)g(t|a^o)dtdc\right)\Gamma_{qq}(q,a)dq,\end{aligned} \tag{A.3}$$

where $\Gamma_q(q,a) \equiv \frac{\partial}{\partial q}\Gamma(q,a)$ and $\Gamma_{qq}(q,a) \equiv \frac{\partial^2}{\partial q^2}\Gamma(q,a)$. In equation equation (A.3), the second equality comes from the fact that $\psi^{(1)}(\overline{q}) = 0$, and the last equality is from the fact that $\psi^{(2)}(\overline{q}) = 0$. Thus, by using equation (A.1), equation (A.2), equation (A.3), and the fact that $\Gamma_{qq}(q,a) \geq 0$ (i.e., $\frac{g(a|a)}{g(q|a^o)}$ is convex in q), we finally have

$$\int \xi(q)g(q|a)dq = \int\left(\int^q\int^c \xi(t)g(t|a^o)dtdc\right)\Gamma_{qq}(q,a)dq \leq 0, \quad \forall a.$$

■

Proof of Lemma 3. Since $s^o(\mathbf{x})$ is the optimal contract obtained under the first-order approach, we know $s^o(\mathbf{x}) \in S_f$. Thus, from, equation (8),

$$\begin{aligned}U(s^o(\cdot),a^o) &= \int u(s^o(\mathbf{x}))f(\mathbf{x}|a^o)d\mathbf{x} - c(a^o) \\ &= \int r(q)g(q|a^o)dq - c(a^o) = U^o,\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}U_a(s^o(\cdot),a^o) &= \int u(s^o(\mathbf{x}))f_a(\mathbf{x}|a^o)d\mathbf{x} - c'(a^o) \\ &= \int u(s^o(\mathbf{x}))\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}f(\mathbf{x}|a^o)d\mathbf{x} - c'(a^o) = \int r(q)\cdot q\cdot g(q|a^o)dq - c'(a^o) = 0.\end{aligned} \tag{A.5}$$

Also, since $\hat{s}(\mathbf{x}) \in S_f$,

$$U(\hat{s}(\cdot),a^o) = \int u(\hat{s}(\mathbf{x}))f(\mathbf{x}|a^o)d\mathbf{x} - c(a^o) = \int \hat{r}(q)g(q|a^o)dq - c(a^o) = U^o, \tag{A.6}$$

and

$$U_a(\hat{s}(\cdot),a^o) = \int u(\hat{s}(\mathbf{x}))f_a(\mathbf{x}|a^o)d\mathbf{x} - c'(a^o) = \int \hat{r}(q)\cdot q\cdot g(q|a^o)dq - c'(a^o) = 0. \tag{A.7}$$

Thus, from equation (A.4) and equation (A.6),

$$\int (r(q) - \hat{q}(q))g(q|a^o)dq = 0, \tag{A.8}$$

and, from equation (A.5) and equation (A.7),

$$\int (r(q) - \hat{r}(q))q \cdot g(q|a^o)dq = 0. \tag{A.9}$$

By substituting $r(q) - \hat{r}(q)$ for $\xi(q)$ in Lemma 2, one can easily see that $\xi(q) = r(q) - \hat{r}(q)$ satisfies conditions (L2) and (L3) in Lemma 2. Furthermore, as $\hat{r}(q)$ crosses $r(q)$ twice starting from above, $\xi(q) = r(q) - \hat{r}(q)$ satisfies (L4) in Lemma 2. Therefore, given that (L1) is satisfied, we derive from Lemma 2 that

$$\int (r(q) - \hat{r}(q))g(q|a)dq \leq 0, \quad \forall a, \tag{A.10}$$

implying that

$$U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a), \quad \forall a. \tag{A.11}$$

Let $E[\hat{r}(q)|a] \equiv \phi(c(a))$. Then,

$$\begin{aligned} U(\hat{s}(\cdot), a) &= \int \hat{r}(q)g(q|a)dq - c(a) \\ &= E[\hat{r}(q)|a] - c(a) = \phi(c(a)) - c(a). \end{aligned}$$

Since $\phi'(c(a^o)) = 1$ from equation (A.7) and since $\phi'' < 0$, $U(\hat{s}(\cdot), a)$ has a global maximum point at a^o because

$$\phi'(c(a)) \geq 1, \quad \forall a \leq a^o, \quad \text{and} \quad \phi'(c(a)) < 1, \quad \forall a > a^o.$$

Therefore, we have

$$U(\hat{s}(\cdot), a) \leq U(\hat{s}(\cdot), a^o) = U^o, \quad \forall a. \tag{A.12}$$

As a result, by combining equation (A.11) and equation (A.12), we have

$$U(s^o(\cdot), a) \leq U(s^o(\cdot), a^o) = U^o, \quad \forall a,$$

which justifies the first-order approach.

■

Proof of Proposition 1. For any given a^o, consider an arbitrary contract $\hat{s}_t(\mathbf{x})$ such that $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = Ae^{tq} + B$, $t > 0$, which satisfies both the participation constraint and the "relaxed" incentive constraint at a^o. Therefore, since

$$\hat{R}_t(a) \equiv \int \hat{r}_t(q) g(q|a) dq = A \cdot M(a;t) + B, \tag{A.10}$$

from the participation constraint in equation (7), $\hat{r}_t(q)$ should satisfy

$$\hat{R}_t(a^o) - c(a^o) = A \cdot M(a^o;t) + B - c(a^o) = U^o \geq \overline{U}. \tag{A.11}$$

Also, from the relaxed incentive constraint, $\hat{r}_t(q)$ should satisfy

$$\hat{R}'_t(a^o) - c'(a^o) = A \cdot M'(a^o;t) - c'(a^o) = 0. \tag{A.12}$$

Then, solving equation (A.11) and equation (A.12) gives

$$A = \frac{c'(a^o)}{M'(a^o;t)}, \tag{A.13}$$

and

$$B = U^o + c(a^o) - c'(a^o)\frac{M(a^o;t)}{M'(a^o;t)}. \tag{A.14}$$

Furthermore, since

$$\hat{r}_t(-\infty) = A \cdot e^{t(-\infty)} + B = B = U^o + c(a^o) - c'(a^o)\frac{M(a^o;t)}{M'(a^o;t)},$$

there exists $t > 0$ such that

$$\begin{aligned} \hat{r}_t(-\infty) &\geq \overline{U} + c(a^o) - c'(a^o)\frac{M(a^o;t)}{M'(a^o;t)} \\ &\geq u(\underline{s}), \end{aligned}$$

where the first inequality comes from that $U^o \geq \overline{U}$ and the second inequality comes from (2b). This indicates that $\hat{s}_t(\mathbf{x})$ also satisfies the agent's limited liability constraint. Now, using equation (A.10), equation (A.13), and equation (A.14), we have

$$\begin{aligned} \hat{R}_t(a) - c(a) &= A \cdot M(a;t) + B - c(a) \\ &= \frac{c'(a^o)}{M'(a^o;t)}[M(a;t) - M(a^o;t)] + U^o + c(a^o) - c(a). \end{aligned}$$

Thus, $\hat{R}_t(a) - c(a)$ has a maximum value at a^o if

$$\begin{aligned}
\hat{R}_t'(a) - c'(a) \geq 0 &\iff \frac{M'(a;t)}{M'(a^o;t)} \geq \frac{c'(a)}{c'(a^o)}, \quad \forall a \leq a^o, \\
\hat{R}_t'(a) - c'(a) \leq 0 &\iff \frac{M'(a;t)}{M'(a^o;t)} \leq \frac{c'(a)}{c'(a^o)}, \quad \forall a > a^o.
\end{aligned} \tag{A.15}$$

Note that, since $g(q|a)$ exhibits the FOSD, $M'(a;t) \geq 0$ for all a. Thus, by defining $c(a) \equiv \phi(M(a;t))$, one can easily see that equation (A.15) holds if (2b) is satisfied (i.e., $\phi'' \geq 0$). Therefore,

$$\hat{R}_t(a) - c(a) \leq \hat{R}_t(a^o) - c(a^o) = U^o, \quad \text{for any given } a^o \text{ and for all } a, \tag{A.16}$$

implying that $\hat{s}_t(\mathbf{x})$ actually satisfies the original "argmax" incentive constraint.

Since $\bar{r}(q)$ is concave in q by (3b), and since $r(q) \equiv u(s^o(\mathbf{x}))$ also satisfies both the participation constraint and the "relaxed" incentive constraint, $r(q)$ must cross $\hat{r}_t(q)$ twice starting from below as drawn in Figure 4. In fact, $r(q)$ and $\hat{r}_t(q)$ must cross because they both satisfy the same participation constraint, and they must cross twice because $\bar{r}(q)$ is concave in q whereas $\hat{r}_t(q)$ is convex in q with its minimum value higher than $u(\underline{s})$.[38]

The fact that both $r(q)$ and $\hat{r}_t(q)$ satisfy the same participation constraint at a^o gives

$$\int (r(q) - \hat{r}_t(q)) g(q|a^o) dq = 0, \tag{A.17}$$

and the fact that they also satisfy the relaxed incentive constraint at a^o gives

$$\int (r(q) - \hat{r}_t(q)) q \cdot g(q|a^o) dq = 0. \tag{A.18}$$

Thus, by combining (1a) with (A.17), (A.18), and the double crossing property between $r(q)$ and $\hat{r}_t(q)$, we have from Lemma 2 that

$$\int (r(q) - \hat{r}_t(q)) g(q|a) dq \leq 0, \quad \forall a. \tag{A.19}$$

Therefore, from equation (A.16) and equation (A.19), we finally have

$$\begin{aligned}
U(s^o(\cdot), a) = R(a) - c(a) &\leq \hat{R}_t(a) - c(a) \\
&\leq \hat{R}_t(a^o) - c(a^o) = U^o = U(s^o(\cdot), a^o), \quad \text{for any given } a^o \text{ and for all } a,
\end{aligned}$$

[38]If $r(q)$ crosses $\hat{r}_t(q)$ only once, then equation (A.18) below is not possible. For detailed discussion for this, see Lemma 1 in Innes (1990).

which justifies the first-order approach.

■

Proof of Proposition 2. Given a^o, consider an arbitrary contract $\hat{s}(\mathbf{x})$ such that $u(\hat{s}(\mathbf{x})) \equiv \hat{r}(q) = Aq + B$ which satisfies both the participation and the "relaxed" incentive constraints at a^o. Thus, since

$$\hat{R}(a) \equiv \int \hat{r}(q) g(q|a) dq = A \cdot m(a) + B,$$

from the participation constraint in equation (24), $\hat{r}(q)$ should satisfy

$$\hat{R}(a^o) - c(a^o) = A \cdot m(a^o) + B - c(a^o) = \overline{U}. \tag{A.20}$$

Also, from the relaxed incentive constraint, $\hat{r}(q)$ should satisfy

$$\hat{R}'(a^o) - c'(a^o) = A \cdot m'(a^o) - c'(a^o) = 0. \tag{A.21}$$

Note that

$$m(a^o) = \int q g(q|a^o) dq = \int \frac{f_a(\mathbf{x}|a^o)}{\cancel{f(\mathbf{x}|a^o)}} \cancel{f(\mathbf{x}|a^o)} d\mathbf{x} = \int f_a(\mathbf{x}|a^o) d\mathbf{x} = 0,$$

which indicates that the expected value of information is always zero (i.e., no information ex-ante).[39] Thus, by solving equation (A.20) and equation (A.21), we obtain

$$A = \frac{c'(a^o)}{m'(a^o)}, \tag{A.22}$$

and

$$B = \overline{U} + c(a^o). \tag{A.23}$$

Then, using equation (A.22) and equation (A.23), we have

$$\hat{R}(a) - c(a) = \frac{c'(a^o)}{m'(a^o)} m(a) + \overline{U} + c(a^o) - c(a).$$

[39] Also, note that

$$m'(a^o) = \int q g_a(q|a^o) dq = \int \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} f_a(\mathbf{x}|a^o) d\mathbf{x} = \int \left(\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} \right)^2 f(\mathbf{x}|a^o) d\mathbf{x} = Var(q|a^o).$$

Thus, $\hat{R}(a) - c(a)$ has a maximum value at a^o if

$$\begin{aligned}
\hat{R}'(a) - c'(a) \geq 0 \quad &\Longleftrightarrow \quad \frac{m'(a)}{m'(a^o)} \geq \frac{c'(a)}{c'(a^o)}, \quad \forall a \leq a^o, \\
\hat{R}'(a) - c'(a) \leq 0 \quad &\Longleftrightarrow \quad \frac{m'(a)}{m'(a^o)} \leq \frac{c'(a)}{c'(a^o)}, \quad \forall a > a^o.
\end{aligned} \tag{A.24}$$

Note that, since $g(q|a)$ exhibits the FOSD, $m'(a) \geq 0$ for all a. Thus, by defining $c(a) \equiv \phi(m(a))$, one can easily see that equation (A.24) holds if (2c) is satisfied (i.e., $\phi'' \geq 0$). Therefore,

$$\hat{R}(a) - c(a) \leq \hat{R}(a^o) - c(a^o) = \overline{U}, \quad \text{for any given } a^o \text{ and for all } a, \tag{A.25}$$

implying that the arbitrary contract, $\hat{s}(\mathbf{x})$, also satisfies the original "argmax" incentive constraint.

Since $r(q) \equiv u(s^o(\mathbf{x}))$ is concave in q by (3c), and since it also satisfies both the participation and the relaxed incentive constraints, $r(q)$ must cross $\hat{r}(q)$ twice starting from below as drawn in Figure 5. Actually, $r(q)$ and $\hat{r}(q)$ must cross because they both satisfy the same participation constraint, and they must cross twice because $r(q)$ is (increasing and) concave in q whereas $\hat{r}(q)$ is (increasing and) linear in q.[40]

The fact that both $r(q)$ and $\hat{r}(q)$ satisfy the participation constraint at a^o gives

$$\int (r(q) - \hat{r}(q)) g(q|a^o) dq = 0, \tag{A.26}$$

and the fact that they also satisfy the relaxed incentive constraint at a^o gives

$$\int (r(q) - \hat{r}(q)) q g(q|a^o) dq = 0. \tag{A.27}$$

Thus, by combining (1a) with equation (A.26), equation (A.27), and the double crossing property between $r(q)$ and $\hat{r}(q)$, we have from Lemma 2 that

$$\int (r(q) - \hat{r}(q)) g(q|a) dq \leq 0, \quad \forall a. \tag{A.28}$$

Therefore, from equation (A.25) and equation (A.28), we finally derive

$$\begin{aligned}
U(s^o(\cdot), a) &= R(a) - c(a) \\
&\leq \hat{R}(a) - c(a)
\end{aligned}$$

[40]They should not cross once by the same reason as in Proposition 1.

$$\leq \hat{R}(a^o) - c(a^o) = \overline{U} = U(s^o(\cdot), a^o), \quad \text{for any given } a^o \text{ and for all } a,$$

which justifies the first-order approach.

■

Proof of Proposition 2′. As we already explained, let $\hat{r}(q) = Ae^{tq}$ for some $t > 0$. Then $\ln \hat{r}(q) = tq + \ln A$ becomes linear in q. For $\hat{r}(q)$ to satisfy the agent's participation constraint, we need to have

$$\hat{R}(a^o) = AM(a^o; t) = \overline{U} + c(a^o).$$

For $\hat{r}(q)$ to satisfy the agent's incentive compatibility constraint, we need to have

$$\hat{R}'(a^o) = AM'(a^o; t) = c'(a^o), \;\; A = \frac{c'(a^o)}{M'(a^o; t)}.$$

Therefore, t must be chosen such that

$$M(a^o; t)\frac{c'(a^o)}{M'(a^o; t)} = \overline{U} + c(a^o).$$

to satisfy the agent's participation constraint. Under $\hat{r}(q)$, the agent's indirect utility function becomes $\hat{R}(a) = AM(a; t) - c(a)$, which achieves its maximum at $a = a^o$ if $c(a)$ is convex in $M(a; t)$ for such t. As the linear $\hat{r}(q)$ induces the agent to participate and voluntarily choose $a = a^o$, it must double-cross $\overline{r}(q)$ as explained in Innes (1990). Following Lemma 3, the first-order approach is justified.

■

Proof of Lemma 4. Note that

$$\begin{vmatrix} 1 & \phi(x_1) & \psi(x_1) \\ 1 & \phi(x_2) & \psi(x_2) \\ 1 & \phi(x_3) & \psi(x_3) \end{vmatrix} = (\phi(x_3) - \phi(x_2))\,(\phi(x_2) - \phi(x_1))\left(\frac{\psi(x_3) - \psi(x_2)}{\phi(x_3) - \phi(x_2)} - \frac{\psi(x_2) - \psi(x_1)}{\phi(x_2) - \phi(x_1)}\right).$$

Since $\phi(x)$ is increasing in x, $\phi(x_1) \leq \phi(x_2) \leq \phi(x_3)$ for every $x_1 < x_2 < x_3$. Therefore, we have, for every $x_1 < x_2 < x_3$,

$$\begin{vmatrix} 1 & \phi(x_1) & \psi(x_1) \\ 1 & \phi(x_2) & \psi(x_2) \\ 1 & \phi(x_3) & \psi(x_3) \end{vmatrix} \underset{(\leq)}{\geq} 0 \quad \Longleftrightarrow \quad \frac{\psi(x_3) - \psi(x_2)}{\phi(x_3) - \phi(x_2)} - \frac{\psi(x_2) - \psi(x_1)}{\phi(x_2) - \phi(x_1)} \underset{(\leq)}{\geq} 0,$$

which indicates that $\psi(x)$ is convex (concave) in $\phi(x)$.

■

Proof of Lemma 5. Assume that $x_1 < x_2 < x_3$ and $a_1 < a_2 < a_3$, and, without lost of generality, let $a^o = a_2$.

(i) the "if" part: Since

$$T(f,2) \equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) \\ f(x_2|a_1) & f(x_2|a_2) \end{vmatrix} = f(x_1|a_1)f(x_2|a_1) \times \begin{vmatrix} 1 & \frac{f(x_1|a_2)}{f(x_1|a_1)} \\ 1 & \frac{f(x_2|a_2)}{f(x_2|a_1)} \end{vmatrix}$$
$$= f(x_1|a_1)f(x_2|a_1)\left(\frac{f(x_2|a_2)}{f(x_2|a_1)} - \frac{f(x_1|a_2)}{f(x_1|a_1)}\right), \tag{A.29}$$

the MLRP for $f(x|a)$ implies $T(f,2) \geq 0$. Also, given the MLRP for $f(x|a)$, condition (ii) in Lemma 4 means that $\frac{f(x|a_1)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$, $\forall a_1 < a^o$ as well as that $\frac{f(x|a_3)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$, $\forall a_3 > a^o$. Thus, we have

$$\frac{\frac{f(x_2|a_1)}{f(x_2|a^o)} - \frac{f(x_1|a_1)}{f(x_1|a^o)}}{q_2 - q_1} \leq \frac{\frac{f(x_3|a_1)}{f(x_3|a^o)} - \frac{f(x_2|a_1)}{f(x_2|a^o)}}{q_3 - q_2} \leq 0 \tag{A.30}$$

and

$$0 \leq \frac{\frac{f(x_2|a_3)}{f(x_2|a^o)} - \frac{f(x_1|a_3)}{f(x_1|a^o)}}{q_2 - q_1} \leq \frac{\frac{f(x_3|a_3)}{f(x_3|a^o)} - \frac{f(x_2|a_3)}{f(x_2|a^o)}}{q_3 - q_2}, \tag{A.31}$$

where $q_i = \frac{f_a(x_i|a^o)}{f(x_i|a^o)}$. By combining equation (A.30) and equation (A.31), we derive

$$\frac{\frac{f(x_2|a_1)}{f(x_2|a^o)} - \frac{f(x_1|a_1)}{f(x_1|a^o)}}{\frac{f(x_2|a_3)}{f(x_2|a^o)} - \frac{f(x_1|a_3)}{f(x_1|a^o)}} \leq \frac{\frac{f(x_3|a_1)}{f(x_3|a^o)} - \frac{f(x_2|a_1)}{f(x_2|a^o)}}{\frac{f(x_3|a_3)}{f(x_3|a^o)} - \frac{f(x_2|a_3)}{f(x_2|a^o)}} \leq 0. \tag{A.32}$$

Since

$$T(f,3) \equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a^o) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a^o) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a^o) & f(x_3|a_3) \end{vmatrix} = \left(\prod_{i=1}^{3} f(x_i|a^o)\right) \times \begin{vmatrix} \frac{f(x_1|a_1)}{f(x_1|a^o)} & 1 & \frac{f(x_1|a_3)}{f(x_1|a^o)} \\ \frac{f(x_2|a_1)}{f(x_2|a^o)} & 1 & \frac{f(x_2|a_3)}{f(x_2|a^o)} \\ \frac{f(x_3|a_1)}{f(x_3|a^o)} & 1 & \frac{f(x_3|a_3)}{f(x_3|a^o)} \end{vmatrix}$$
$$= \left(\prod_{i=1}^{3} f(x_i|a^o)\right) \times \begin{vmatrix} 1 & \frac{f(x_1|a_3)}{f(x_1|a^o)} & \frac{f(x_1|a_1)}{f(x_1|a^o)} \\ 1 & \frac{f(x_2|a_3)}{f(x_2|a^o)} & \frac{f(x_2|a_1)}{f(x_2|a^o)} \\ 1 & \frac{f(x_3|a_3)}{f(x_3|a^o)} & \frac{f(x_3|a_1)}{f(x_3|a^o)} \end{vmatrix},$$

one can check that equation (A.32) implies $T(f,3) \geq 0$.

(ii) the "only if" part: If $f(x|a)$ is TP_3, then, by definition, $T(f,2) \geq 0$ and $T(f,3) \geq 0$. First, from equation (A.29), it is obvious that $T(f,2) \geq 0$ implies the MLRP for $f(x|a)$. Second, notice that

$$\begin{aligned}
T(f,3) &\equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a^o) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a^o) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a^o) & f(x_3|a_3) \end{vmatrix} \\
&= (a^o - a_1) \times \begin{vmatrix} f(x_1|a_1) & \frac{f(x_1|a^o)-f(x_1|a_1)}{a^o-a_1} & f(x_1|a_3) \\ f(x_2|a_1) & \frac{f(x_2|a^o)-f(x_2|a_1)}{a^o-a_1} & f(x_2|a_3) \\ f(x_3|a_1) & \frac{f(x_3|a^o)-f(x_3|a_1)}{a^o-a_1} & f(x_3|a_3) \end{vmatrix} \\
&= (a^o - a_1) \times \left\{ \prod_{i=1}^{3} f(x_i|a_1) \right\} \times \underbrace{\begin{vmatrix} 1 & \frac{f(x_1|a^o)-f(x_1|a_1)}{(a^o-a_1)f(x_1|a_1)} & \frac{f(x_1|a_3)}{f(x_1|a_1)} \\ 1 & \frac{f(x_2|a^o)-f(x_2|a_1)}{(a^o-a_1)f(x_2|a_1)} & \frac{f(x_2|a_3)}{f(x_2|a_1)} \\ 1 & \frac{f(x_3|a^o)-f(x_3|a_1)}{(a^o-a_1)f(x_3|a_1)} & \frac{f(x_3|a_3)}{f(x_3|a_1)} \end{vmatrix}}_{\equiv A}.
\end{aligned}$$

Since $a^o > a_1$ and since

$$\lim_{a_1 \to a^o} A = \begin{vmatrix} 1 & \frac{f_a(x_1|a^o)}{f(x_1|a^o)} & \frac{f(x_1|a_3)}{f(x_1|a^o)} \\ 1 & \frac{f_a(x_2|a^o)}{f(x_2|a^o)} & \frac{f(x_2|a_3)}{f(x_2|a^o)} \\ 1 & \frac{f_a(x_3|a^o)}{f(x_3|a^o)} & \frac{f(x_3|a_3)}{f(x_3|a^o)} \end{vmatrix},$$

$T(f,3) \geq 0$ implies that $\frac{f(x|a_3)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$, $\forall a_3 > a^o$ by Lemma 4.

Similarly, notice that

$$\begin{aligned}
T(f,3) &= (a_3 - a^o) \times \begin{vmatrix} f(x_1|a_1) & f(x_1|a^o) & \frac{f(x_1|a_3)-f(x_1|a^o)}{a_3-a^o} \\ f(x_2|a_1) & f(x_2|a^o) & \frac{f(x_2|a_3)-f(x_2|a^o)}{a_3-a^o} \\ f(x_3|a_1) & f(x_3|a^o) & \frac{f(x_3|a_3)-f(x_3|a^o)}{a_3-a^o} \end{vmatrix} \\
&= (a_3 - a^o) \times \left(\prod_{i=1}^{3} f(x_i|a^o) \right) \times \begin{vmatrix} \frac{f(x_1|a_1)}{f(x_1|a^o)} & 1 & \frac{f(x_1|a_3)-f(x_1|a^o)}{(a_3-a^o)f(x_1|a^o)} \\ \frac{f(x_2|a_1)}{f(x_2|a^o)} & 1 & \frac{f(x_2|a_3)-f(x_2|a^o)}{(a_3-a^o)f(x_2|a^o)} \\ \frac{f(x_3|a_1)}{f(x_3|a^o)} & 1 & \frac{f(x_3|a_3)-f(x_3|a^o)}{(a_3-a^o)f(x_3|a^o)} \end{vmatrix} \\
&= (a_3 - a^o) \times \left(\prod_{i=1}^{3} f(x_i|a^o) \right) \times \underbrace{\begin{vmatrix} 1 & \frac{f(x_1|a_3)-f(x_1|a^o)}{(a_3-a^o)f(x_1|a^o)} & \frac{f(x_1|a_1)}{f(x_1|a^o)} \\ 1 & \frac{f(x_2|a_3)-f(x_2|a^o)}{(a_3-a^o)f(x_2|a^o)} & \frac{f(x_2|a_1)}{f(x_2|a^o)} \\ 1 & \frac{f(x_3|a_3)-f(x_3|a^o)}{(a_3-a^o)f(x_3|a^o)} & \frac{f(x_3|a_1)}{f(x_3|a^o)} \end{vmatrix}}_{\equiv B}.
\end{aligned}$$

Since $a_3 > a^o$ and since

$$\lim_{a_3 \to a^o} B = \begin{vmatrix} 1 & \frac{f_a(x_1|a^o)}{f(x_1|a^o)} & \frac{f(x_1|a_1)}{f(x_1|a^o)} \\ 1 & \frac{f_a(x_2|a^o)}{f(x_2|a^o)} & \frac{f(x_2|a_1)}{f(x_2|a^o)} \\ 1 & \frac{f_a(x_3|a^o)}{f(x_3|a^o)} & \frac{f(x_3|a_1)}{f(x_3|a^o)} \end{vmatrix},$$

$T(f,3) \geq 0$ also implies that $\frac{f(x|a_1)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$, $\forall a_1 < a^o$ by Lemma 4. Consequently, $T(f,3) \geq 0$ implies that, for any given a^o, $\frac{f(x|a)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$ for all a.

■

Proof of Corollary 1:. Using equation (30), we have, for any given a^o,

$$\begin{aligned} T(f,2) \equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) \\ f(x_2|a_1) & f(x_2|a_2) \end{vmatrix} &= Q'_{a^o}(x_1)Q'_{a^o}(x_2) \begin{vmatrix} g(q_1|a_1) & g(q_1|a_2) \\ g(q_2|a_1) & g(q_2|a_2) \end{vmatrix} \\ &= Q'_{a^o}(x_1)Q'_{a^o}(x_2)T(g,2), \end{aligned}$$

where $q_i = Q_{a^o}(x_i) \equiv \frac{f_a(x_i|a^o)}{f(x_i|a^o)}$, $i = 1,2$. Since $f(x|a)$ satisfies MLRP, we have $Q'_{a^o}(x) \geq 0$, $\forall x$. Thus, given MLRP for $f(x|a)$,

$$\begin{aligned} & T(f,2) \geq 0, \quad \text{for every } x_1 < x_2 \text{ and } a_1 < a_2 \\ \Longleftrightarrow\ & T(g,2) \geq 0, \quad \text{for every } q_1 < q_2 \text{ and } a_1 < a_2. \end{aligned} \tag{A.33}$$

Likewise, we have, for any given a^o,

$$\begin{aligned} T(f,3) &\equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix} \\ &= Q'_{a^o}(x_1)Q'_{a^o}(x_2)Q'_{a^o}(x_3) \begin{vmatrix} g(q_1|a_1) & g(q_1|a_2) & g(q_1|a_3) \\ g(q_2|a_1) & g(q_2|a_2) & g(q_2|a_3) \\ g(q_3|a_1) & g(q_3|a_2) & g(q_3|a_3) \end{vmatrix} \\ &= Q'_{a^o}(x_1)Q'_{a^o}(x_2)Q'_{a^o}(x_3)T(g,3), \end{aligned}$$

where $q_i = Q_{a^o}(x_i) \equiv \frac{f_a(x_i|a^o)}{f(x_i|a^o)}$, $i = 1,2,3$. Therefore, by the same way, we derive that, given MLRP for $f(x|a)$,

$$\begin{aligned} & T(f,3) \geq 0, \quad \text{for every } x_1 < x_2 < x_3 \text{ and } a_1 < a_2 < a_3 \\ \Leftrightarrow\ & T(g,3) \geq 0, \quad \text{for every } q_1 < q_2 < q_3 \text{ and } a_1 < a_2 < a_3 \end{aligned} \tag{A.34}$$

Thus, by combining (A.33) and (A.34), we have that $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^o given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP.

■

Proof of Proposition 3. From Lemma 5 and Corollary 1, it is shown that, given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, condition (TP_3) is equivalent to condition (1a). Furthermore, condition (1J-2) implies condition (2c) since $c(a)$ is increasing convex in a. However, condition (2c) does not imply condition (1J-2).

■

Proof of Lemma 6. Define, for a given density function $f(x|a)$,

$$\psi(a,k) \equiv \int \phi(x,k) f(x|a) dx,$$

where $x \in \mathbb{R}$, $a \in \mathbb{R}$, and k is the parameter that determines the functional form of $\phi(x,k)$. Then, by the "basic composition formula" by Karlin (1968),[41] we have

$$\begin{vmatrix} \psi(a_1,k_1) & \psi(a_1,k_2) & \psi(a_1,k_3) \\ \psi(a_2,k_1) & \psi(a_2,k_2) & \psi(a_2,k_3) \\ \psi(a_3,k_1) & \psi(a_3,k_2) & \psi(a_3,k_3) \end{vmatrix} = \iiint_{x_1<x_2<x_3} \begin{vmatrix} \phi(x_1,k_1) & \phi(x_1,k_2) & \phi(x_1,k_3) \\ \phi(x_2,k_1) & \phi(x_2,k_2) & \phi(x_2,k_3) \\ \phi(x_3,k_1) & \phi(x_3,k_2) & \phi(x_3,k_3) \end{vmatrix} \times \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix} dx_1 dx_2 dx_3. \quad \text{(A.35)}$$

Let $\phi(x,k_1) \equiv 1$, $\phi(x,k_2) \equiv x$, and $\phi(x,k_3) \equiv u(x)$. We have $\psi(a,k_1) = 1$, $\psi(a,k_2) = \mu(a) \equiv \int x f(x|a) dx$, and $\psi(a,k_3) = u^*(a) \equiv \int u(x) f(x|a) dx$. Thus, by using equation (A.35),

$$\begin{vmatrix} 1 & \mu(a_1) & u^*(a_1) \\ 1 & \mu(a_2) & u^*(a_2) \\ 1 & \mu(a_3) & u^*(a_3) \end{vmatrix} = \iiint_{x_1<x_2<x_3} \begin{vmatrix} 1 & x_1 & u(x_1) \\ 1 & x_2 & u(x_2) \\ 1 & x_3 & u(x_3) \end{vmatrix} \times \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix} dx_1 dx_2 dx_3. \quad \text{(A.36)}$$

The fact that $f(x|a)$ is TP_3 implies

$$T(f,2) = \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) \\ f(x_2|a_1) & f(x_2|a_2) \end{vmatrix} \geq 0, \quad \text{for every } x_1 < x_2 \text{ and } a_1 < a_2,$$

[41] For the detailed proof of the basic composition formula, see Karlin (1968) p. 17. The equation (A.35) is a direct extension of the famous Cauchy-Binet theorem.

which is equivalent to the MLRP for $f(x|a)$. Thus, both $\mu(a)$ and $u^*(a)$ are increasing in a. Since $u(x)$ is concave in x, we have from Lemma 4 that

$$\begin{vmatrix} 1 & x_1 & u(x_1) \\ 1 & x_2 & u(x_2) \\ 1 & x_3 & u(x_3) \end{vmatrix} \leq 0, \quad \text{for every } x_1 < x_2 < x_3.$$

Also, the fact that $f(x|a)$ is TP_3 implies

$$T(f,3) = \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix} \geq 0, \quad \text{for every } x_1 < x_2 < x_3 \text{ and } a_1 < a_2 < a_3.$$

Thus, from equation (A.36), we have

$$\begin{vmatrix} 1 & \mu(a_1) & u^*(a_1) \\ 1 & \mu(a_2) & u^*(a_2) \\ 1 & \mu(a_3) & u^*(a_3) \end{vmatrix} \leq 0, \quad \text{for every } a_1 < a_2 < a_3,$$

which indicates that $u^*(a)$ is increasing concave in $\mu(a) \equiv \int x f(x|a)dx$, given that $\mu(a)$ is increasing in a.

■

Derivation of Example 4. Given

$$g(q|a) = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp\left(-\frac{1}{h(a)}\left(\frac{[h(a^o)]^2}{h'(a^o)}q + h(a^o)\right)\right),$$

the moment generating function for arbitrary a will be given as

$$\begin{aligned} M(a;t) &= \int e^{tq} g(q|a)dq = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \int_{-\frac{h'(a^o)}{h(a^o)}}^{\infty} \exp\left[-\frac{1}{h(a)}\left(\frac{h(a^o)^2}{h'(a^o)}q + h(a^o)\right) + tq\right] dq \\ &= \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp\left(-\frac{h(a^o)}{h(a)}\right) \int_{-\frac{h'(a^o)}{h(a^o)}}^{\infty} \exp\left[-\left(\frac{h(a^o)^2}{h(a)h'(a^o)} - t\right) q\right] dq \\ &= \frac{[h(a^o)]^2}{h'(a^o)h(a)} \frac{1}{\dfrac{h(a^o)^2}{h(a)h'(a^o)} - t} \exp\left(-t\frac{h'(a^o)}{h(a^o)}\right) = \frac{h(a^o)^2}{h(a^o)^2 - h'(a^o)h(a)t} \exp\left(-t\frac{h'(a^o)}{h(a^o)}\right), \end{aligned}$$

which is equation (28). And then we can calculate its derivative $M'(a;t)$ as:

$$M'(a;t) = \frac{-h(a^o)^2}{\left[h(a^o)^2 - h'(a^o)h(a)t\right]^2} \left(-h'(a^o)h'(a)t\right) \exp\left(-t\frac{h'(a^o)}{h(a^o)}\right),$$

thereby

$$\frac{M(a;t)}{M'(a;t)} = \frac{h(a^o)^2 - h'(a^o)h(a)t}{h'(a^o)h'(a)t}.$$

Therefore, we obtain

$$\frac{M(a^o;t)}{M'(a^o;t)}c'(a^o) - c(a^o) = \frac{h(a^o)^2 - h'(a^o)h(a^o)t}{[h'(a^o)]^2 t}c'(a^o) - c(a^o) = \overline{U},$$

which is satisfied when

$$t = \frac{h(a^o)^2 c'(a^o)}{[\overline{U} + c(a^o)]h'(a^o)^2 + h(a^o)h'(a^o)c'(a^o)} > 0.$$

It is clear that the above t is lower than $\frac{h(a^o)}{h'(a^o)}$.