Monetary Policy as a Financial Stabilizer*

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Abstract

We develop a New-Keynesian framework with stock markets that features a potential for self-fulfilling financial uncertainty arising from its interaction with risk-premium, wealth, and aggregate demand. Our model remains tractable, providing closed-form expressions for higher-order moments tied to the financial uncertainty and their relations to the rest of the economy. We re-examine the optimality of conventional monetary policy rules and show that the 'Taylor principle' no longer guarantees determinacy, with sunspots in aggregate financial volatility not precluded by aggressive targeting of inflation and output gap alone. We characterize the joint dynamic evolution of financial volatility, risk-premium, asset prices, and the business cycle in a rational expectations equilibrium with sunspots, and uncover that variations in financial uncertainty generate reasonable crises and booms along the business cycle that are consistent with our empirical estimates based on the US data. As this pitfall of the traditional policy rules lies in their inability to target the expected return on aggregate wealth, the relevant rate in stochastic environments, we then propose a 'generalized' Taylor rule that targets risk-premium and asset price, and describe the necessary conditions that restore determinacy and achieve the *ultra-divine coincidence*: the joint stabilization of inflation, output gap, and risk-premium. Finally, we revisit the zero lower bound (ZLB) and show it amplifies the duration, severity, and welfare costs of fluctuations in financial volatility. Alternative policies such as forward guidance reduce these welfare costs on average, but risk worsening economic situations with a non-zero probability, raising interesting trade-offs for policymakers.

Keywords: Monetary Policy, Financial Volatility, Risk-Premium

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1 Introduction

How should monetary policy respond to stock market fluctuations? The current narrative posits that central banks (-governments) need two separate sets of instruments: macroprudential policies and regulations to ensure the stability of financial markets, and monetary (-fiscal) policies to fulfill the traditional objective of macroeconomic stabilization.[1] However, the debate on this issue is far from being settled for many reasons. For example, the stock market plays a dual role: it is a source of business cycle fluctuations (e.g., the Great Depression) and it is a propagation channel itself (e.g., stock prices merely reflect the collective wisdom on expected future business cycle conditions). Relatedly, resolving this debate has proven difficult because mainstream macroeconomic frameworks lack meaningful stock market fluctuations (if there is a stock market in such models) or rely on approximation techniques and numerical methods which can cloud the economic intuition.

In this paper, we shed some lights on this longstanding debate by proposing a New-Keynesian framework with stock markets and optimal portfolio decisions. We incorporate *endogenous* and *time-varying* second-order moments such as stock market volatility and risk-premium. Furthermore, our continuous-time framework allows intuitive analytic expressions which highlight the underlying mechanisms behind our results. The model features an important role of financial volatility and risk-premium for business cycle fluctuations: a more volatile financial market (with higher risk-premia) brings down aggregate financial wealth (through individual investor's portfolio decisions), thereby affecting aggregate demand and output. Because endogenous second-order terms (financial volatility) feed back into the first-order moments (financial wealth and aggregate demand), we explore how monetary policy should be connected to financial stability issues (i.e., financial volatility). We claim that the current monetary policy framework based on two *macroeconomic* mandates (e.g., stable inflation and stable output gap) is not sufficient for macroeconomic stabilization. In addition to these two mandates, we call for targeting time-varying risk premium as a separate policy objective.

Our model solution uncovers that there exists a sunspot equilibrium that arises from aggregate volatility and risk-premium of financial markets[2]: fear of a financial crisis possibly stemming from a rise in risk-premium and stock market volatility, for example, induces investors to reduce their

[1]For example, at the press conference held on September 16, 2020, Federal Reserve chair Powell explicitly mentioned "Monetary policy should not be the first line of defense - is not the first line of defense on financial stability. We look to more appropriate tools in the first instance, as a first line of defense. And those would be regulation, supervision, high-capital, high-liquidity stress testing, all of those things, macroprudential tools.".

[2]Even in the 'textbook' New-Keynesian model without explicit stock markets and portfolio decisions, the economy's *time-varying aggregate risk* can have a first-order impact on the aggregate consumption demand due to the *precautionary savings* channel. Therefore, in Section 2 we provide an alternative standard New-Keynesian model without stock markets and characterize non-linear equilibrium conditions to (i) illustrate the first-order (feedback) effects of endogenous and time-varying aggregate risks on business cycle levels, and (ii) show that sunspot equilibria arise with conventional monetary policy rules. Therefore, most of main results continue to hold in Section 2 including determinacy issues.

demand for the stock market investment, bringing down the current asset price and wealth and thus generating self-fulfilling increases in the expected stock market return and risk-premium. In particular, we characterize rational expectations equilibria that follow self-fulfilling shocks to the financial volatility and risk-premium, where we derive a tractable expression for the joint dynamic evolution of financial volatility, risk-premium, and business cycle variables after those sunspots appear as a function of fundamentals and policy interventions. We prove that under these sunspot equilibria, the financial volatility gets almost surely stabilized in the long run, but a probability-zero event in which this volatility diverges in the long run leading to a severe recession makes the sunspot's initial appearance possible. As it takes time for initial volatility sunspots to be eliminated by monetary policy response, our equilibrium features *crisis* periods (with spikes in stock market volatility and risk-premium and drops in wealth and output) and *boom* phases (with low financial volatility and buoyant wealth and production), depending on the directions of initial sunspots.[3]

We then study conventional monetary policy rules in regard to model determinacy and financial stability. Our analysis shows that traditional Taylor rules that focus on macroeconomic aggregates (i.e., inflation and output gap) cannot fully prevent the appearance of sunspots in aggregate financial volatility, but a stronger targeting of macroeconomic mandates shortens the time it takes for initial volatility sunspot to get stabilized in our rational expectations equilibrium. This stronger responsiveness of monetary policy comes with a side effect, however: a more aggressive targeting of inflation and output gap amplifies the financial market volatility following sunspot shocks, which generates stronger but short-lived boom and bust financial cycles. We argue that the failure of conventional policy rules to restore determinacy lies in their inability to adequately target the expected risky return of financial markets, which governs the agents' intertemporal decision-making.

We then propose a generalized policy reaction function that restores determinacy in our stochastic environment. Specifically, we argue that optimal policy rules should target the risk-premium of financial markets in addition to their usual mandates. Intuitively, agents in our model optimally allocate their wealth between risky and riskless assets, and the return on aggregate financial wealth becomes the relevant rate for their intertemporal consumption smoothing decisions. Therefore, the optimal monetary rule aims to control the return on agents' aggregate wealth, but in order to succeed, it must take into account the risky component of the portfolio return, which is summarized by risk-premium. Thus, our analysis suggests that aggregate wealth should be an *intermediate* target of the central bank for the purpose of macroeconomic stabilization. This new policy rule that targets risk-premium in a specific way achieves what we describe as 'ultra-divine' coincidence: the joint

[3]This result aligns with Basu et al. (2021), where they emphasize roles of fluctuations in risk-premia as a main driver of the business cycle driving movements and comovements among aggregate variables. In Appendix A, we estimate a simple vector autoregression (VAR) with real and financial uncertainty indexes developed by Ludvigson et al. (2015) and uncover that a 1-3% (5-10%) drop in industrial production (S&P-500 index) follows after a one standard deviation shock to financial uncertainty, which our calibrated model replicates.

stabilization of inflation, output gap and risk-premium (equivalently, financial volatility).

Following this rule poses its own challenges though, as the central bank is required to target risk premium with just the right amount of responsiveness. If the policy response is too accommodating or strong, monetary policy is again unable to prevent the appearance of sunspots. Nonetheless, even when the central bank is unable to restore the equilibrium determinacy, targeting financial variables remains an optimal strategy as it enables a faster convergence back to the steady state following a sunspot shock.

We then analyze the effects of the zero lower bound (ZLB) on macroeconomic stabilization. The ZLB in our framework causes stock prices to fall, leading to drops in business cycle variables, as in Caballero and Simsek (2020a).[4] We ask whether we should expect heightened financial instability once the policy tool of central banks is constrained at zero, and find that a credible commitment to economic stabilization upon ZLB-exit is enough to ensure financial stability during ZLB episodes. However in cases where post ZLB or forward guidance exit stability is not guaranteed, ZLB is likely to amplify the duration, severity, and welfare costs of fluctuations in financial market volatility after its sunspots appear.[5]

Related Literature Our paper is related to a broad literature on the intersection between macroeconomics and finance. Our model builds on the idea that changes in financial wealth levels (usually housing and stock) affect aggregate economic outcomes, documented by Mian et al. (2013), Mian and Sufi (2014)[6], Guerrieri and Iacoviello (2017), Berger et al. (2018), Caballero and Simsek (2020a), Caballero and Simsek (2020b), Di Maggio et al. (2020), Caramp and Silva (2020)[7] and Chodorow-Reich et al. (2021), among others. In line with this literature, an endogenous stock price level shifts aggregate demand in our framework through its effect on aggregate financial wealth. In addition, our framework features endogenous risk-premium and financial volatility as key factors that drive fluctuations in financial markets and the business cycle, in line with arguments made by Gilchrist and Zakrajšek (2012), Brunnermeier and Sannikov (2014), Chodorow-Reich (2014), Stein (2014), Cúrdia and Woodford (2016), Guerrieri and Lorenzoni (2017), Di Tella and Hall (2020), and Basu et al.

[4]In Section 5.4, we explore two macroprudential policies at the ZLB that induce investors to bear more risks, thereby raising asset prices and business cycle levels: (i) a tax cut on capital gain taxes and (ii) redistribution across agents.

[5]Even if central bank's post-ZLB stabilization prevents the additional financial instability at the ZLB, business cycle still can feature high levels of volatility and risk-premium due to fundamental risks (e.g., TFP volatility). In Section 5.3, we show that by credibly committing to sacrifice financial stabilization in the future, central bank can attain welfare-enhancing commitment equilibria, in which it boosts asset prices and output, and reduces risk-premium and financial volatility today at the ZLB.

[6]In their works, consumers with a high marginal propensity to consume (MPC) who experience large drops in their housing prices, reduce the consumption amounts due to both wealth effects and a binding credit constraint, the latter of which we do not consider in this paper.

[7]Caramp and Silva (2020) introduced rare-disasters and positive private debt and characterized the roles of time-varying risk-premia and financial wealth in a linearized setting.

(2021)[8] among others, that financial (and in particular, credit) disruptions have large impacts on aggregate demand, especially when monetary policy is constrained. Campbell et al. (2020) points out that New-Keynesian channels, through which a higher inflation pushes down bond returns while propping up aggregate output, dividends, and stock returns, can explain the correlation reversal between bond and stock returns which turned negative in recent years. Our framework shares the same intuitions and sheds lights on how stock market fluctuation can be embedded in conventional New-Keynesian models.[9]

Our result that monetary policy must be systematically concerned with financial markets stability is related to prior literature including Bernanke and Gertler (2000), Stein (2012), Woodford (2012), Cúrdia and Woodford (2016)[10], Caballero and Simsek (2020b), Cieslak and Vissing-Jorgensen (2020), Kekre and Lenel (2021), and Galí (2021)[11]. In contrast to Bernanke and Gertler (2000)'s findings that monetary policy should not target stock prices, which they concluded based on a model with ad-hoc bubbles, bubble components are omitted in our model and thus only the fundamental stock price level serves as the key factor that determines aggregate demand. Therefore, our specification with the stock price as an aggregate demand shifter leads to the equivalence of targeting of stock price 'level' and more conventional mandates such as output gap, and allows us to connect our work with Cieslak and Vissing-Jorgensen (2020) which conclude that stock market performance is a powerful predictor of the policy rate. In particular, Kekre and Lenel (2021) provide a beautiful theoretical framework in which an accommodation shock in monetary policy redistributes toward those with a higher marginal propensity to take risk (MPR), thereby reducing risk-premium levels and amplifying the monetary transmission. While their focus is on how monetary policy following the conventional Taylor rule affects the economy through its impacts on economy-wide risk-premia in the heterogenous agents New Keynesian (HANK) environment, our analytic approach allows us to spot new indeterminacy around the second-order financial variable (aggregate financial volatility) with conventional Taylor rules,[12] thereby allowing us to provide a more generalized Taylor rule

[8] Basu et al. (2021) emphasize roles of fluctuations in risk-premia as a business cycle driver, showing that the shock that explains fluctuations in risk-premia can explain a large fraction of business cycle movements and co-movements. They rely on the third-order perturbation to solve their model. In addition, Kekre and Lenel (2021) provide an elegant framework which illustrates the transmission of monetary policy through its impacts on the equilibrium risk-premium level in the environment that features heterogeneity in households' marginal propensity to take risk (MPR). While their dynamic model relies on global solution methods, their analytic counterpart relies on the third-order approximation.

[9] The previous literature usually focus on channels through which financial wealth and financial market disruptions affect business cycle fluctuations. The other direction, an asset pricing implication of the New-Keynesian model, is also addressed by De Paoli et al. (2010), Weber (2015) and Gorodnichenko and Weber (2016).

[10] Woodford (2012) and Cúrdia and Woodford (2016), in particular, incorporate a friction in financial intermediation between agents with different marginal propensities to consume (MPC) and study how the optimal monetary policy rule must be adjusted.

[11] Galí (2021) introduced rational bubbles in a New-Keynesian model with overlapping generations. He argued that 'leaning against the bubble' policy, if properly specified, insulates the economy from aggregate bubble fluctuations.

[12] Also, we contribute to the literature by providing an exact stochastic process for each business cycle variable after those sunspots appear in the financial market.

that targets risk-premium as a way to facilitate stabilization and (possibly) restore model determinacy. Our approach still aligns with their view in that aggregate wealth is to be managed through monetary policy, and our generalized Taylor rule illustrates that an internal rate of return on aggregate wealth, instead of just the risk-free policy rate, must be responding to fluctuations in business cycle variables for the model to restore determinacy and achieve perfect stabilization.

While Giavazzi and Giovannini (2010), Stein (2012), and Caballero and Simsek (2020b) focus on the preemptive role of monetary policy in avoiding 'future' financial crises, our model features a monetary policy rule targeting the risk-premium of financial markets for the 'current' stabilization purposes, in addition to its traditional inflation and output gap targets. Our result that monetary accommodation props up the business cycle through its effect on the stock market level is in line with evidence provided by Rigobon and Sack (2003), Azali et al. (2013), and Kekre and Lenel (2021).

This paper is also related with literature on New-Keynesian environment and monetary policy at the zero lower bound (ZLB). Due to nominal pricing rigidities à la Calvo (1983), our economy is demand-driven and stock market performance drives the aggregate demand. Thus, in order to characterize monetary policy's stabilization role, endogenous fluctuations in stock markets must be properly taken into account, a topic that has often been overlooked by the previous literature. While several authors focus on demand recessions brought by deleveraging borrowers at the ZLB and aggregate demand externality issues (i.e., Akerlof and Yellen (1985), Blanchard and Kiyotaki (1987), Eggertsson and Krugman (2012), Farhi and Werning (2012), Farhi and Werning (2016), Korinek and Simsek (2016), Schmitt-Grohé and Uribe (2016), and Farhi and Werning (2017)), we turn our attention towards declines in the aggregate demand for risky assets as the key driver behind financial recessions, a channel that has been documented by Caballero and Farhi (2017) and Caballero and Simsek (2020a).

Our paper is similar to Caballero and Simsek (2020a) in terms of how an endogenous asset market is interwoven with business cycle fluctuations. However, while their framework focuses on how behavioral biases can generate interesting crisis dynamics in light with the feedback loop between asset markets and the business cycle[13], our focus is on the traditional monetary policy rule under rational expectations, and the central bank's capacity to intervene in financial markets during crisis caused by the ZLB. Our model's equilibrium determinacy results are similar to Acharya and Dogra (2020) in terms of how countercyclical risks can lead to indeterminacy. While Acharya and Dogra (2020) focus on how determinacy conditions change in the presence of exogenous idiosyncratic risks that are functions of aggregate output, we investigate the existence of sunspots stemming from aggregate financial risk, which is countercyclical in nature and affects both financial markets and

[13]Caballero and Simsek (2020a) features optimists and pessimists who have different beliefs about the probability of an upcoming recession. During ZLB episodes, an endogenous decline in the risky asset valuation generates a demand recession due to a drop in optimists' wealth.

business cycle fluctuations, and study the monetary policy mechanisms that restore determinacy and/or improve economic and financial stability.

Layout In Section 2, we provide a non-linear treatment to the standard New-Keynesian economy, through which we illustrate how non-linearity changes implications about equilibrium determinacy issues and proper monetary policy rules needed for stabilization purposes. In Section 3, we present the model with explicit stock markets and characterize the equilibrium conditions. Section 4 focuses on the proper monetary policy rules in lights with our framework's new features. In Section 5, we analyze zero lower bound (ZLB) crises and possible unconventional fiscal and monetary measures that mitigate recessions, focusing on the derivation of key trade-offs that central banks must take into account. Section 6 concludes.

In Appendix A, we provide evidence on the importance of financial volatility as a driver of business cycle fluctuations, based on a structural Vector Autoregression (VAR) approach. Appendix B contains additional figures and tables. Appendix C contains derivations and proofs. Appendix D derives the quadratic welfare loss function in this framework.

2 Standard Non-linear New Keynesian Model

In this Section 2, we consider a 'standard' New-Keynesian economy[14] where firm profits are transferred in a lump-sum fashion to households. In Section 3, we present the main model of this paper where we instead assume that profits are capitalized into dividend-paying stocks traded in financial markets. Our objective in this Section 2 is to illustrate that a *non-linear* characterization of the equilibrium enables higher-order moments tied to the aggregate business cycle volatility to have a first-order impact on the business cycle dynamics, even when stock markets are absent. This feature will have very important implications for equilibrium determinacy and the proper management of monetary policy needed to stabilize the business cycle.

The representative household owns the firms of this economy and receives the profit stream in a lump-sum fashion. For simplicity, we assume a perfectly rigid price level: $p_t = \bar{p}$, $\forall t$[15] so there is no inflation in the economy. This assumption is not crucial but allows us to focus on the key mechanism we want to illustrate.

The representative household chooses her usual intertemporal consumption-savings decision by

[14] See Woodford (2003) for the standard treatment of a textbook New-Keynesian model.

[15] This assumption can be micro-founded with price stickiness à la Calvo and a price resetting probability of zero.

solving the following optimization problem:

$$\max_{\{B_t, C_t, L_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\log C_t - V(L_t)\right] dt \text{ s.t. } \dot{B}_t = i_t B_t - \bar{p} C_t + w_t L_t + D_t, \tag{1}$$

where C_t and L_t are her consumption and labor supply, $V(L_t)$ is the disutility of labor, B_t is her nominal holding of bonds, and D_t are the entire firms' profits and fiscal transfers from the government. w_t is the equilibrium wage, and i_t is the policy rate set by the central bank. We assume that there is no government spending, and therefore aggregate consumption determines output in this demand-determined environment: thus, $C_t = Y_t$, where Y_t is aggregate output.

The following equation is the optimality condition for the representative household's intertemporal consumption-savings decision:

$$-i_t dt = \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right), \text{ where } \xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{p_t}, \tag{2}$$

where $\frac{d\xi_t^N}{\xi_t^N}$ is the instantaneous (nominal) stochastic discount factor (SDF), and its expectation yields the nominal risk-free rate i_t. Due to the rigid price assumption $\pi_t = 0$, $\forall t$, the real and nominal risk-free rates of the economy are equal $r_t \equiv i_t - \pi_t = i_t$.

We can rewrite equation (2) as

$$\mathbb{E}_t \left(\frac{dC_t}{C_t} \right) = (i_t - \rho) dt + \text{Var}_t \left(\frac{dC_t}{C_t} \right), \tag{3}$$

where the last term $\text{Var}_t(\frac{dC_t}{C_t})$ arises from the endogenous volatility of the aggregate consumption process. Note that this volatility is usually a second-order term and therefore is typically dropped out in log-linearized models. In contrast to those models, our non-linear characterization properly accounts for consumption risk and allows it to affect the drift of the aggregate consumption process, where both aggregate risk and drift are endogenous objects. This additional term reflects the precautionary savings channel in which a more volatile business cycle leads to an increased demand for savings, lower consumption and a higher expected growth for the consumption process.

The 'natural'[16] (benchmark) economy's output Y_t^n follows the stochastic process:

$$\frac{dY_t^n}{Y_t^n} = \left(\underbrace{r_t^n}_{\text{Natural rate}} - \rho + (\sigma_t)^2 \right) dt + \underbrace{\sigma_t}_{\text{Natural volatility}} dZ_t, \tag{4}$$

[16] We define the 'natural' equilibrium of the economy as the equilibrium under fully flexible prices.

where r_t^n is the natural interest rate. Therefore, equation (4) is regarded a real exogenous process[17] that monetary policy cannot affect nor control. Observe that, for a given process of $\{Y_t^n\}$, an increase in the 'natural' volatility σ_t requires a lower natural rate r_t^n as agents' precautionary savings demand increases.

Going back to the 'rigid' economy in equation (2), we define σ_t^s as the 'excess' volatility the rigid price output process $\{Y_t\}$ features compared with the benchmark economy (equation (4)). Then:

$$\mathrm{Var}_t\left(\frac{dY_t}{Y_t}\right) = (\sigma_t + \sigma_t^s)^2 dt \tag{5}$$

holds. Note that σ_t^s is the 'endogenous' volatility to be determined later in equilibrium. By plugging equation (5) into equation (2), we obtain

$$\frac{dY_t}{Y_t} = \left(i_t - \rho + (\sigma_t + \sigma_t^s)^2\right) dt + (\sigma_t + \sigma_t^s) dZ_t. \tag{6}$$

With the usual definition of output gap $\hat{Y}_t = \ln\left(\frac{Y_t}{Y_t^n}\right)$, we obtain the following dynamic IS equation written in $\hat{Y}_t$:

$$d\hat{Y}_t = \left(i_t - \left(r_t^n \overbrace{-\frac{1}{2}(\sigma_t + \sigma_t^s)^2 + \frac{1}{2}(\sigma_t)^2}^{\text{New terms}}\right)\right) dt + \sigma_t^s dZ_t. \tag{7}$$

Equation (7) features an interesting feedback effect that is omitted in log-linearized equations:[18] given the policy rate i_t, a rise in the endogenous volatility σ_t^s pushes up the drift of equation (7) and lowers output gap $\hat{Y}_t$. The intuition for this result follows from the precautionary behavior of households, that respond to higher economic volatility with increased savings and lower consumption, thereby inducing a recession.

Define the risk-adjusted natural rate as

$$r_t^T = r_t^n - \frac{1}{2}(\sigma_t + \sigma_t^s)^2 + \frac{1}{2}(\sigma_t)^2. \tag{8}$$

and note that r_t^T is itself endogenous: it negatively depends on the endogenous aggregate volatility σ_t^s. The risk-adjusted natural rate can be regarded a new reference risk-free rate of the economy at

[17] Given $\{\sigma_t\}$ process, equation (4) is derived from equation (2) with $i_t = r_t^n$ with $Y_t = Y_t^n$. Therefore, we regard dZ_t as an aggregate shock that drives the natural output Y_t^n (e.g., a technology shock).

[18] For illustrative purposes, compare equation (7) with the conventional linearized IS equation given by:

$$d\hat{Y}_t = (i_t - r_t^n)\, dt + \sigma_t^s dZ_t,$$

where the endogenous aggregate volatility σ_t^s has no first-order effect on the drift of output gap.

which i_t completely eliminates the drift of the output gap.

We now turn our attention towards monetary policy and study the implications of following a conventional Taylor rule in this environment, as well as possible alternatives.

2.1 Taylor rules and Indeterminacy

In this section, we study the conventional Taylor rule and its capacity to guarantee model determinacy and economic stabilization. We assume that the central bank sets the risk-free rate i_t of the economy according to:

$$i_t = r_t^n + \phi_y \hat{Y}_t, \text{ where } \phi_y > 0. \tag{9}$$

Condition $\phi_y > 0$ is the 'Taylor principle' that prevents the appearance of sunspot equilibria in the conventionally log-linearized models that omit the first-order effects of volatility. Here, we ask whether this condition retains the capacity to prevent sunspot equilibria in the non-linear economy featuring the feedback relationship between output gap volatility and its drift explained above.

Plugging equation (9) into equation (7), we get the following $\hat{Y}_t$ dynamics.

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t \underbrace{- \frac{(\sigma_t)^2}{2} + \frac{(\sigma_t + \sigma_t^s)^2}{2}}_{\text{New terms}} \right) dt + \sigma_t^s dZ_t. \tag{10}$$

Multiple equilibria Omitting the new volatility terms from the drift of equation (10), we obtain the usual log-linearized version of the $\hat{Y}_t$ dynamics as

$$d\hat{Y}_t = \left(\phi_y \hat{Y}_t \right) dt + \sigma_t^s dZ_t. \tag{11}$$

With dynamics described by equation (11), Blanchard and Kahn (1980) proves the existence of a unique rational expectations equilibrium when the Taylor principle $\phi_y > 0$ is satisfied: $\hat{Y}_t = 0, \forall t$, which corresponds to a fully stabilized economy.

We now claim that this result does <u>not</u> hold in this non-linear version of the $\{\hat{Y}_t\}$ process. The feedback effect from the endogenous volatility σ_t^s of the output gap to its drift (see equation (10)) enables the appearance of multiple sunspot equilibria in σ_t^s. We provide one rational expectations equilibrium that supports an initial sunspot $\sigma_0^s > 0$ in aggregate excess volatility, by constructing an equilibrium path where the $\{\hat{Y}_t\}$ process follows a 'martingale'. The case for negative volatility sunspot ($\sigma_0^s < 0$) can be similarly constructed. This equilibrium path should (i) support an initial sunspot $\sigma_0^s > 0$, and (ii) not diverge on expectation in the long-run (see Blanchard and Kahn (1980)).

Martingale equilibrium We provide the explicit equilibrium in which a sunspot $\sigma_0^s > 0$ appears and $\hat{Y}_t$ follows a martingale process consistent with the dynamics in equation (10). To satisfy the latter, the drift of the $\{\hat{Y}_t\}$ process must be zero, which gives us the following formula for $\hat{Y}_t$:

$$\hat{Y}_t = -\frac{(\sigma_t + \sigma_t^s)^2}{2\phi_y} + \frac{(\sigma_t)^2}{2\phi_y}. \tag{12}$$

The martingale equilibrium guarantees the rationality of the equilibrium, as on average the path of $\{\hat{Y}_t\}$ stays at the same level, satisfying $\mathbb{E}_0(\hat{Y}_t) = \hat{Y}_0$. The last step is to show the existence of a stochastic path for $\{\sigma_t^s\}$ starting from σ_0^s that supports this equilibrium.

Using equation (10) and equation (12), we obtain the stochastic process of σ_t^s starting from σ_0^s as[19]

$$d\sigma_t^s = -(\phi_y)^2 \frac{(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma_t + \sigma_t^s} dZ_t. \tag{13}$$

Therefore, equation (12) and equation (13) constitute the dynamics of this particular rational expectations equilibrium supporting $\sigma_0^s > 0$. The following Proposition 1 sheds light on the behavior of $\{\hat{Y}_t, \sigma_t^s\}$ paths under this equilibrium and finds that the business cycle almost surely converges to the perfectly stabilized path in the long run. Nonetheless, a few paths that occur with tiny probability do not converge and explode asymptotically, sustaining the initial sunspot $\sigma_0^s > 0$ due to the forward-looking nature of the economy.

Proposition 1 (Taylor Rules and Indeterminacy) *For any value of $\phi_y > 0$:*

1. Indeterminacy*: there is always a rational expectations equilibrium (REE) that supports initial sunspot $\sigma_0^s > 0$ and is represented by $\hat{Y}_t$ dynamics in equation* (12)*, and σ_t^s process in equation* (13)
2. Properties*: the rational expectations equilibrium that supports an initial sunspot $\sigma_0^s > 0$ satisfies:*

$$(i)\ \sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0,\ (ii)\ \hat{Y}_t \overset{a.s}{\to} 0,\ \text{and}\ (iii)\ \mathbb{E}_0\left(\max_t (\sigma_t^s)^2\right) = \infty$$

The results that $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$ and $\hat{Y}_t \overset{a.s}{\to} 0$ imply that the equilibrium paths starting from an initial sunspot $\sigma_0^s > 0$ are almost surely stabilized in the long run. Still, almost sure stabilization of paths is compatible with a martingale sunspot equilibrium by the latter result of the Proposition, $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$, which implies that an initial spike in σ_0^s is sustained by a tiny probability of an

[19]When $\sigma_t = 0$, $\forall t$, equation (13) becomes the following Bessel process:

$$d\sigma_t^s = -\frac{(\phi_y)^2}{2\sigma_t^s} dt - \phi_y dZ_t.$$

which stops when σ_t^s hits zero, $\sigma^{q,n} = 0$. For general properties of Bessel process, see Lawler (2019).

∞-large equilibrium volatility in some future paths.

Intuition Here we explain in a detailed manner the intuition for (i) how an initial sunspot σ_0^s in the aggregate volatility can appear, and (ii) the results in Proposition 1. For that purpose, we simplify the economic environment and make the following assumptions:

A.1 A shock dZ_t at each period takes one of two values: $\{+1, -1\}$ with equal probability $\frac{1}{2}$

A.2 Martingale equilibrium: an aggregate demand $\hat{Y}_t$ equals the conditional expected value of the next-period aggregate demand $\hat{Y}_{t+1}$. Therefore, if $\hat{Y}_{t+1}$ takes either $\hat{Y}_{t+1}^{(1)}$ or $\hat{Y}_{t+1}^{(2)}$, then

$$\hat{Y}_t = \frac{1}{2}(\hat{Y}_{t+1}^{(1)} + \hat{Y}_{t+1}^{(2)})$$

A.3 Aggregate demand $\hat{Y}_t$ falls, as the conditional variance of the next-period's $\hat{Y}_{t+1}$ rises (precautionary saving). Both $\{\hat{Y}_t\}$ and $\{\sigma_t^s\}$ are set to be zero on the stabilized path

Since we have two possible realizations of the shock at each period, we can draw a tree diagram as in Figure 1. In Figure 1, the thick vertical line represents the stabilized path, with areas at its left and

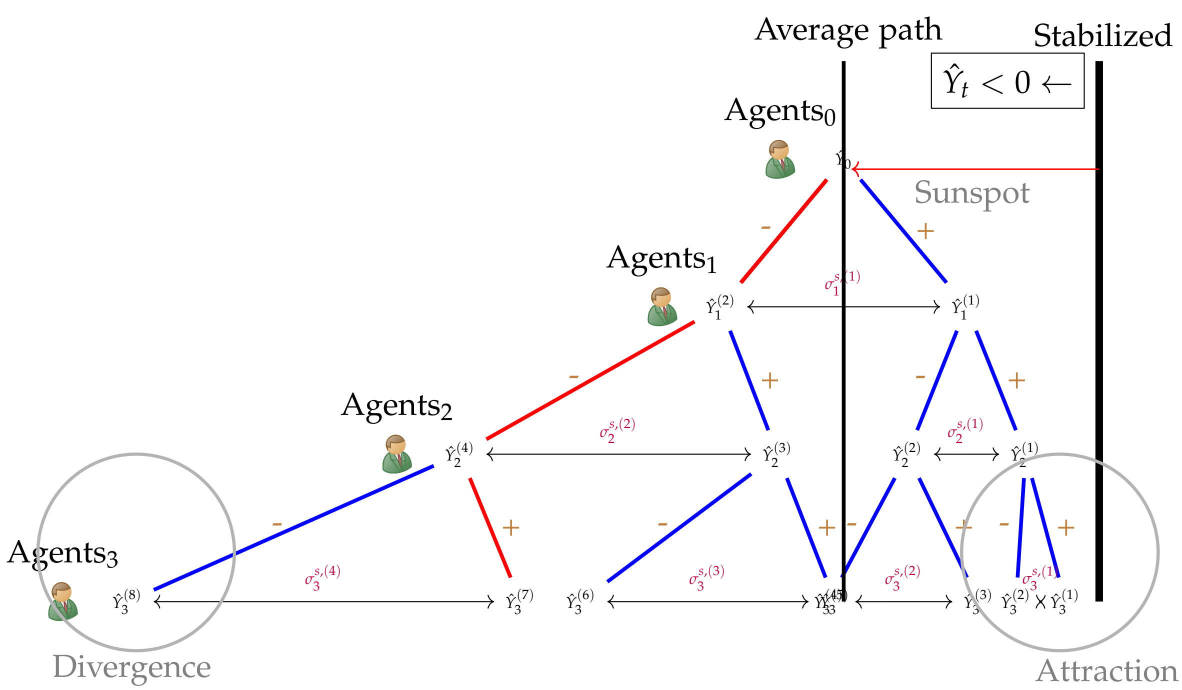


Figure 1: A sunspot in σ_0^s as a rational expectations equilibrium

right representing recessions and booms, respectively. The key to build a rational expectations equilibrium supporting a sunspot $\sigma_0^s > 0$ is to construct the path-dependent consumption strategy for the economy's intertemporal agents. First, let us imagine that the current period agents (Agents$_0$) suddenly believe that future agents will choose the path-dependent consumption demand[20] so that the next-period's $\hat{Y}_1$ becomes $\hat{Y}_1^{(1)}$ after $dZ_0 = +1$ is realized and $\hat{Y}_1^{(2)}$ after $dZ_0 = -1$ is realized, with $\hat{Y}_1^{(1)} > \hat{Y}_1^{(2)}$. Then the current output $\hat{Y}_0$ becomes $\hat{Y}_0 = \frac{1}{2}(\hat{Y}_1^{(1)} + \hat{Y}_1^{(2)})$ with $\hat{Y}_0$ below the stabilized path, as Agents$_0$ believe there exists dispersion in next-period outcomes, which is given as $\sigma_1^{s,(1)} = \hat{Y}_1^{(1)} - \hat{Y}_1^{(2)}$.

Assume that $dZ_0 = -1$ is realized. For Agents$_0$'s belief $\hat{Y}_1 = \hat{Y}_1^{(2)}$ to be consistent, Agents$_1$ must believe that future agents will choose consumption paths in a way that the next period's $\hat{Y}_2$ becomes $\hat{Y}_2^{(3)}$ with $dZ_1 = +1$ and $\hat{Y}_2^{(4)}$ with $dZ_1 = -1$, with conditional volatility $\sigma_2^{s,(2)} = \hat{Y}_2^{(3)} - \hat{Y}_2^{(4)}$ higher than $\sigma_1^{s,(1)}$, since $\hat{Y}_1^{(2)}$ is lower than the initial output $\hat{Y}_0$.

After dZ_1 is realized, Agents$_1$'s belief about $\hat{Y}_2$ can be made consistent by future agents' $\{\text{Agents}_{n\geq 2}\}$ coordination in a forward looking fashion. Observe that all the nodes in Figure 1 satisfy assumptions A.2 and A.3, with distance between adjacent nodes getting progressively narrower (wider) as output gap gets closer (farther) to the stabilized path. This results in divergent and attraction paths balancing each other out, and in expectation, output gap $\{\hat{Y}_t\}$ follows a martingale process.

In sum, Agents$_0$'s initial doubt (sunspot) that the next-period's outcome will be volatile is made consistent by coordination between intertemporal agents (the representative household) at each node.[21]

Note that (i) we obtain an equilibrium with stochastic aggregate volatility: i.e., σ_t^s is dependent on the path of shocks, as output gap $\{\hat{Y}_t\}$ is stochastic and negatively depends on the conditional volatility of its next-period level. Equation (13) specifies the exact stochastic process of $\{\sigma_t^s\}$ starting from $\sigma_0^s > 0$, (ii) since volatility σ_t^s decreases as output gap $\hat{Y}_t$ approaches the stabilized path, this path becomes an attraction point for the set of alternative paths in its neighborhood, justifying the result of Proposition 1 that σ_t^s almost surely converges to zero over time. Nonetheless, as volatility σ_t^s rises whenever output $\hat{Y}_t$ deviates farther from the stabilized level, this also aligns with the result of Proposition 1 that a maximal σ_t^s diverges, $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$.

The conclusion in terms of monetary policy is that a conventional Taylor rule almost surely stabilizes the disruption caused by a $\sigma_0^s > 0$ sunspot in the long-run, but does not prevent the economy from entering a crisis phase with low aggregate demand and higher business cycle volatility.

Escape clause If central bank and/or government credibly commit to prevent $\hat{Y}_t$ from going below

[20] Remember, agents' consumption demand determines output in this demand-determined environment.

[21] This equilibrium is completely feasible since all future agents share a common knowledge of their consumption strategies and there is no behavioral friction blocking communications between agents in intertemporal periods (perfect recall). Our sunspot equilibrium is closely related to notion of 'self-confirming equilibrium'. See Fudenberg and Levine (1993) for this issue. For how limited recall (friction in memory) removes indeterminacy, see Angeletos and Lian (2021).

a predetermined threshold through interventions,[22] these sunspot equilibria arising from the aggregate financial volatility σ_0^q supported by the paths in Figure 1 (martingale equilibrium) are not sustained anymore as a possible rational expectations equilibrium (REE). This escape clause illustrates how the credible commitment of the government entity to intervene whenever the economy (probabilistically) enters a big recession actually precludes a possibility of the crisis phase initiated by the positive sunspot shock $\sigma_0^s > 0$.

Whether this type of commitment from government and central bank is credible is important, as here we need a 100% credibility to kill the sunspot equilibrium supporting $\sigma_0^s > 0$.

Negative sunspot We can similarly construct a rational expectations equilibrium that supports an initial downward sunspot $\sigma_0^s < 0$. This equilibrium features a boom phase with buoyant aggregate demand and low business cycle volatility. Therefore, our non-linear characterization of the model generates a reasonable prediction of (i) appearance of sunspot boom/crisis phases, and (ii) the joint evolution of the first (level) and second (volatility) order moments of the model variables during crisis and booms.[23]

Next, we study a monetary policy rule that restores model determinacy.

2.2 A New Monetary Policy

Let's assume, instead, that the central bank follows this alternative policy rule:

$$i_t = r_t^n + \phi_y \hat{Y}_t - \underbrace{\frac{1}{2}\left((\sigma_t + \sigma_t^s)^2 - (\sigma_t)^2\right)}_{\text{Aggregate volatility targeting}}, \quad \text{where} \quad \phi_y > 0, \tag{14}$$

which, in addition to output gap $\hat{Y}_t$, targets the aggregate volatility of the output gap with a coefficient $\frac{1}{2}$. By plugging the above monetary policy into the IS equation (equation (7)), the volatility feedback terms in the drift part cancel out and therefore, we obtain an expression equal to equation (11), which guarantees model determinacy and ensures $\hat{Y}_t = 0$, $\forall t$ as a unique rational expectations equilibrium when the Taylor principle $\phi_y > 0$ is satisfied.

Interpretation The additional volatility target in the policy rule is necessary to offset the feedback

[22]For example, government might commit to incur huge fiscal deficits whenever the economy undergoes a severe recession. This prescription entails similar implication about what government can do to restore determinate equilibrium to Benhabib et al. (2002). Benhabib et al. (2002) deals with the role of monetary-fiscal regimes in regards to eliminating indeterminacy posed by ZLB. In a similar way, Obstfeld and Rogoff (2021) illustrates how a probabilistic (and small) fiscal backing to the currency by government rules out speculative hyper-inflations in monetary models.

[23]Our sunspot equilibrium can be interpreted as capturing the occurrence of animal spirit shocks. For the neoclassical treatment of this topic, see Angeletos and La'O (2013).

channel between the endogenous volatility of the output gap and its drift. To get a more intuitive interpretation of this result, we can rearrange equation (14) as

$$i_t = r_t^T + \phi_y \hat{Y}_t, \text{ where } \phi_y > 0, \tag{15}$$

where r_t^T is the risk-adjusted natural rate defined in equation (8). Therefore, an alternative interpretation is that monetary policy in a risky environment should target the risk-adjusted, and not simply the natural, interest rate.

A problem with this new policy rule is that it seems very difficult to implement *in practice*, as neither the output volatility components $\{\sigma_t, \sigma_t^s\}$ nor the risk-adjusted rate r_t^T are directly observable. In the next Section 3, we offer an alternative theoretical framework that explicitly incorporates stock markets, and show that commonly observed measures of *financial volatility* or market *risk-premium* serve as a proxy that can be used to effectively implement the rule.

3 The Model with Stock Markets

In this Section 3, we consider a slightly different theoretical framework, which enables us to analyze the effects of higher-order moments tied to the aggregate financial volatility on aggregate demand, and provides us the practical implications about monetary policy rules.

3.1 Setting

Time is continuous, and a *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ is given. The economy consists of a measure one of capitalists, who we regard as neoclassical agents, and the same measure of hand-to-mouth workers, who we regard as Keynesian agents. There is a single source of exogenous variation in the aggregate production technology A_t, which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and evolves according to a geometric process with a possibly time-varying volatility σ_t:

$$\frac{dA_t}{A_t} = \underbrace{g}_{\text{Growth}} dt + \underbrace{\sigma_t}_{\text{Fundamental risk}} dZ_t. \tag{16}$$

We regard the aggregate TFP's volatility σ_t as the economy's 'fundamental' risk. We assume it to be constant in most scenarios, but later, as in Caballero and Simsek (2020a), we will allow σ_t to jump and analyze how it affects the equilibrium dynamics. For convenience, we also assume the average growth rate g to be constant over time.

Finally, there is a standard set of intermediate good producers that face nominal price rigidities, thus making the economy New-Keynesian in nature. Next, we describe roles of each type of agents (capitalists and workers) and firms.

3.1.1 Firms and Workers

There are a measure one of monopolistically competitive firms, each producing a differentiated intermediate good $y_t(i)$, $i \in [0, 1]$. There also exists a competitive representative firm which transforms intermediates into a final consumption good y_t according to a Dixit-Stiglitz aggregator with an elasticity of substitution $\epsilon > 0$ in the following way.

$$y_t = \Big(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \Big)^{\frac{\epsilon}{\epsilon-1}}. \tag{17}$$

Each intermediate good firm i has the same production function $y_t(i) = A_t(N_{W,t})^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the economy's aggregate labor and $n_t(i)$ is the labor demand of an individual firm i at time t. The reason that we introduce a production externality à la Baxter and King (1991) is that it helps us match empirical regularities on asset price and wage co-movements, and it does not affect other qualitative implications of our framework.[24] Each firm i faces the downward-sloping demand curve $y_i(p_t(i) \| p_t, y_t)$, where $p_t(i)$ is the price of its own intermediate good and p_t, y_t are the aggregate price index and output, respectively:

$$y_i(p_t(i) \| p_t, y_t) = y_t \Big(\frac{p_t(i)}{p_t} \Big)^{-\epsilon}. \tag{18}$$

The set of prices charged by intermediate good firms, $\{p_t(i)\}$, is aggregated into the price index p_t as

$$p_t = \Big(\int_0^1 p_t(i)^{1-\epsilon} di \Big)^{\frac{1}{1-\epsilon}}. \tag{19}$$

We also impose a nominal price rigidity à la Calvo (1983), and firms can change prices of their own intermediate goods with δdt probability in a given time interval dt. In the cross-section, this implies that a total δdt portion of firms reset their prices during a given dt time interval.

A representative hand-to-mouth worker supplies labor to intermediate good producers, gets an equilibrium wage income, and spends every dollar he earns on final good consumption. We assume that each worker solves the following optimization at every moment t, where $C_{W,t}$, $N_{W,t}$

[24] In our framework, rising asset prices tend to be correlated with the decreasing wage compensation to workers since firm value (stock price) usually rises if firms can pay less to workers. It violates empirical regularities documented by Chodorow-Reich et al. (2021) in which a rise in stock price tends to push up local aggregate demand variables such as employment and wage. Our production function with externality à la Baxter and King (1991) provides us a reasonable calibration that matches these empirical regularities because higher asset prices and aggregate demand raise the firms' marginal product of labor, thus increasing labor demand and wages. Basically, our externality plays similar roles to the capital in the production function, with a higher degree of tractability.

and w_t are his consumption, labor supply and wage at time t, respectively.

$$\max_{C_{W,t},N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad \text{s.t.} \quad p_t C_{W,t} = w_t N_{W,t}, \tag{20}$$

where χ_0 is the inverse Frisch elasticity of labor supply. Note that we normalize consumption $C_{W,t}$ by technology A_t, which governs the economy's size.[25] As wage w_t is homogeneous across firms, labor demanded by each firm i, $\{n_t(i)\}$, are simply combined into aggregate labor $N_{W,t}$ in a linear manner as

$$N_{W,t} = \int_0^1 n_t(i) di. \tag{21}$$

Final good output y_t can be written as a function of total labor $N_{W,t}$ by the following aggregate production function with price dispersion Δ_t defined below.[26] Due to the Baxter and King (1991) externality, the aggregate production function becomes linear in $N_{W,t}$ as

$$y_t = \frac{A_t N_{W,t}}{\Delta_t}, \text{ where } \Delta_t \equiv \left(\int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} di\right)^{1-\alpha}. \tag{22}$$

3.1.2 Financial Market and Capitalists

Unlike conventional New-Keynesian models where a representative household owns the intermediate goods sector and receives rebated profits in a lump sum way,[27] we assume that firm profits are capitalized in the financial market as a representative stock fund. Capitalist then face an optimal portfolio decision problem involving the allocation of their wealth between a risk-free bond and the risky stock at every instant t.

The total nominal financial wealth of the economy is $p_t A_t Q_t$, where Q_t is the normalized (or TFP detrended) real asset price. Q_t is an endogenous variable adapted to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ and assumed to evolve according to the process in equation (23), with both endogenous drift μ_t^q and volatility σ_t^q terms. In particular, we regard σ_t^q as a measure of financial uncertainty or disruption, as we usually observe spikes in asset price volatility during financial crises. Like Q_t, we assume that the price aggregator p_t follows the general stochastic process in equation (24), in which drift π_t and volatility σ_t^p are endogenous. Thus, it follows that total financial market wealth $p_t A_t Q_t$ evolves as a geometric Brownian motion with volatility $(\sigma_t + \sigma_t^q + \sigma_t^p)$. Intuitively, if some capitalist invests in the stock market, they have to bear all three risks: inflation risk, technology (fundamental) risk,

[25] The qualitative results of the model are not affected by the consumption normalization, which we introduce to simplify the analytic expressions of the model.

[26] See Woodford (2003), Yun (2005), Kaplan et al. (2010) among others for the role of relative price dispersion Δ_t in business cycle fluctuations and economic stabilization issues.

[27] We already studied non-linear implications in the context of standard New-Keynesian models in Section 2.

and (detrended) real asset price risk.

$$\frac{dQ_t}{Q_t} = \mu_t^q dt + \underbrace{\sigma_t^q}_{\text{Financial volatility}} dZ_t, \tag{23}$$

$$\frac{dp_t}{p_t} = \pi_t dt + \underbrace{\sigma_t^p}_{\text{Inflation risk}} dZ_t. \tag{24}$$

Here, σ_t^q is determined in equilibrium and can be either positive or negative. $\sigma_t^q < 0$ corresponds to the case where total real wealth $A_t Q_t$ is less volatile than the TFP process $\{A_t\}$. The nominal price process has inflation rate π_t as its drift, and in general has a volatility part σ_t^p, which we call an inflation risk. In most cases other than the flexible price benchmark, we show that $\sigma_t^p = 0$ holds and we do not need to concern ourselves with this term.

In addition to the stock market, we assume that there is a risk-free bond with an associated nominal rate i_t that is controlled by the central bank. Bonds are in zero net supply in equilibrium because all capitalists are equal. A measure one of identical capitalists chooses the portfolio allocation between a risk-free bond and a risky stock, where in the latter case, they earn the profits of the intermediate goods sector as dividends, as well as the nominal price revaluation of the stock due to changes in p_t, A_t and Q_t. Financial markets are competitive, thus each capitalist takes the nominal risk-free rate i_t, expected stochastic stock market return i_t^m, and the risk level $\sigma_t + \sigma_t^q + \sigma_t^p$ as given when choosing her portfolio decision.[28] If a capitalist invests a share θ_t of her wealth a_t in the stock market, she bears a total risk $\theta_t a_t(\sigma_t + \sigma_t^q + \sigma_t^p)$ between t and $t + dt$. Therefore, the riskiness of her portfolio increases proportionally to the investment share θ_t in the stock. Capitalists are risk-averse, and ask for a risk-premium compensation $i_t^m - i_t$ when they invest in the risky stock, which must also be determined in equilibrium.

Each capitalist with nominal wealth a_t has log-utility and solves the following optimization:

$$\max_{C_t, \omega_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \;\; \text{s.t.} \;\; da_t = (a_t(i_t + \theta_t(i_t^m - i_t)) - p_t C_t)dt + \theta_t a_t(\sigma_t + \sigma_t^q + \sigma_t^p) dZ_t, \tag{25}$$

where ρ is her time discount rate and C_t is final good consumption. At every instant, she earns returns out of both the risk-free bond and the risky stock investments, and spends on final good consumption. From Merton (1971), we know that the solution of the problem features an optimal

[28]This competitive market assumption is related to the reason we initially assume a measure one of identical capitalists. This assumption turns out to be an important aspect of the framework for explaining inefficiencies caused by the aggregate demand externality that individual capitalist's financial investment decision imposes on the aggregate economy.

consumption expenditure rate which is exactly a ρ portion of her wealth a_t, thus satisfying

$$p_t C_t = \rho a_t. \tag{26}$$

Note that a less patient capitalist (higher ρ) increase her instantaneous consumption rate in a proportional manner.

3.2 Equilibrium and Asset Pricing

In equilibrium, every agent with the same type (either worker or capitalist) is identical and chooses the same decisions. Because in equilibrium bonds are in zero net supply, each capitalist's wealth share θ_t in the stock market must satisfy $\theta_t = 1$, which pins down the equilibrium risk-premium value demanded by capitalists. Due to the log-preference of capitalists, risk-premium is given by $(\sigma_t + \sigma_t^p + \sigma_t^q)^2$, as in equation (27). In equilibrium, capitalists hold a wealth amount that equals the total financial market wealth. These equilibrium conditions can be summarized as follows.

$$\mathrm{rp}_t \equiv i_t^m - i_t = \underbrace{(\sigma_t + \sigma_t^q + \sigma_t^p)^2}_{\text{Risk-premium}} \text{ and } a_t = \underbrace{p_t A_t Q_t}_{\text{Market wealth}}, \tag{27}$$

where the risk-premium rp_t demanded by capitalists increases with either of the three volatilities $\{\sigma_t, \sigma_t^q, \sigma_t^p\}$. As the financial volatility σ_t^q is endogenous, the risk-premium rp_t term is endogenous as well and needs to be determined in equilibrium. Note also that by the previous expression, the wealth gain/loss of the capitalist is equal to the nominal revaluation of the stock.

We can characterize the good's market equilibrium and the equilibrium asset pricing condition of the expected stock return i_t^m as follows: Since capitalists spends ρ portion of their wealth a_t on consumption expenditure and they hold the entire wealth, $C_t = \rho A_t Q_t$ holds in equilibrium. Thus we can write the equilibrium condition for the final good market as follows.[29].

$$\rho A_t Q_t + \frac{w_t}{p_t} N_{W,t} = \frac{A_t N_{W,t}}{\Delta_t}. \tag{28}$$

Due to the log-utility of capitalists, their nominal state-price density ξ_t^N[30] is given in the following way, where the stochastic discount factor between time t (now) and s (future) is by definition given as ξ_s^N / ξ_t^N.

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{p_t}. \tag{29}$$

[29]Here $N_{W,t}$ is the solution of the worker's optimization problem in equation (20).

[30]A superscript N means it is a nominal state-price density, where a superscript r means a real state-price density.

Total stock market wealth ($p_t A_t Q_t$) is by definition the sum of discounted profit streams from the intermediate goods sector, which are priced by the above ξ_t^N because capitalists are natural stock market investors in equilibrium. Thus we can price the entire stock market value as in the following relation, where we discount future profits with the stochastic discount factor generated by the state-price density $\{\xi_t^N\}$. We know that the entire profit of the intermediate goods sector is given as:

$$\begin{aligned} D_t &\equiv \int (p_t(i)y_t(i) - w_t n_t(i))di = \underbrace{\int p_t(i)y_t(i)di}_{=p_t y_t} - \underbrace{w_t N_{W,t}}_{=p_t C_{W,t}} \\ &= p_t(y_t - C_{W,t}) = p_t C_t, \end{aligned} \tag{30}$$

where we use the Dixit-Stiglitz aggregator properties (total expenditure equals the sum of expenditures on each good) and linear aggregation of labor (equation (21)). Regardless of price dispersion across firms, the aggregate dividend D_t is equal to the consumption expenditure of capitalists, who are the natural stock investors in equilibrium as hand-to-mouth workers spend all their income on consumption.

Plugging the above expressions into the fundamental asset pricing equation yields the following condition.

$$p_t A_t Q_t = \mathbb{E}_t \frac{1}{\xi_t^N} \int_t^\infty \xi_s^N \Big(\underbrace{D_s}_{=p_s C_s \text{ from equation (28)}} \Big) ds = \frac{p_t C_t}{\rho}, \tag{31}$$

which becomes $C_t = \rho A_t Q_t$, the same expression as capitalist' optimal consumption (equation (26)) when a_t is given by equation (27). Thus, in order to determine the asset price and close the model, we need an additional condition.[31] In general, we can obtain the exact Q_t levels when we have the information about equilibrium levels of labor $N_{W,t}$ from equation (28). As we know that $N_{W,t}$ only depends on time t real wage $\frac{w_t}{p_t}$, it ultimately requires information about the real wage level to pin down an expression for Q_t.

The nominal expected return on the risky stock i_t^m in equilibrium consists of the dividend yield from the intermediate goods sector profits and the nominal stock price re-valuation (capital gain) due to fluctuations in $\{p_t, A_t, Q_t\}$. Within our specifications, the dividend yield always equals ρ, the discount rate of capitalists. Therefore, when i_t^m changes, only nominal stock prices can adjust endogenously, as the dividend yield is fixed.

With $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process, the following equation (32) shows the

[31] Usually, a monetary policy rule takes this role in the New-Keynesian literature

decomposition of i_t^m into dividend yield and stock revaluation:

$$d\mathbf{I}_t^m = \frac{\overbrace{\cancel{p_t}\Big(\underbrace{y_t - \frac{w_t}{p_t}N_{W,t}}_{=C_t}\Big)}^{\text{Nominal dividend}}}{\underbrace{\cancel{p_t}A_tQ_t}_{\text{Total capital market wealth}}}dt + \underbrace{\frac{d(p_tA_tQ_t)}{p_tA_tQ_t}}_{\text{Capital gain}} = \rho dt + \frac{d(p_tA_tQ_t)}{p_tA_tQ_t}$$
$$= \underbrace{[\rho + \underbrace{\pi_t}_{\text{Inflation}} + g + \mu_t^q + \sigma_t^q\sigma_t^p + \sigma_t(\sigma_t^p + \sigma_t^q)]\,dt}_{=i_t^m \text{ (Expected return)}} + \underbrace{(\sigma_t + \sigma_t^q + \sigma_t^p)}_{\text{Risk term}}dZ_t. \tag{32}$$

The equilibrium conditions we have obtained consist of the worker's optimization (solution of equation (20)), labor aggregation (equation (21)), total output (equation (22)), capitalist's optimization (equation (27)), the good market equilibrium (equation (28)), and determination of the risky stock return (equation (32)). To close the model, we also have to derive the supply block of the economy (pricing decisions of intermediate good firms à la Calvo (1983)) and define the monetary policy rule, which is the most important topic of our interest.

Before we characterize the benchmark case without nominal rigidities, the following Lemma 1 adapts the Fisher equation when there is a correlation between the (aggregate) price process and the wealth process. The Lemma 1 shows that the inflation premium should be added to the original Fisher relation.

Lemma 1 (Inflation Premium) *Real interest rate is given by the following variant of the Fisher identity.*

$$r_t = i_t - \pi_t + \underbrace{\sigma_t^p \underbrace{(\sigma_t + \sigma_t^p + \sigma_t^q)}_{\textit{Wealth volatility}}}_{\textit{Inflation premium}} \tag{33}$$

Lemma 1 is useful when we characterize the flexible price equilibrium of the model where the nominal price process is arbitrary and does not affect the real economy.

3.3 Flexible Price Equilibrium

As a benchmark case, we study the flexible price equilibrium. When firms can freely reset their prices ($\delta \to \infty$ case), the real wage becomes proportional to aggregate technology A_t. The following proposition summarizes the real wage, asset price process, natural rate of interest r_t^n (the real, risk-free rate that prevails in this benchmark economy), and consumption process of the capitalist in the flexible price equilibrium. Before we proceed, we define the following parameter, which is the effective labor supply elasticity of workers taking their optimal consumption decision into account.

Definition 1 *Effective labor supply elasticity of workers* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0+\varphi}$

Proposition 2 (Flexible Price Equilibrium) [32] *In the flexible price equilibrium, the following conditions for real wage $\frac{w_t^n}{p_t^n}$, asset price Q_t^n, natural rate of interest r_t^n, and consumption of capitalists C_t^n, hold.*
(i) Every firm charges the same price ($\Delta_t = 1, \forall t$), and the real wage is proportional to aggregate technology A_t.

$$p_t(i) = p_t, \forall i \in [0,1] \text{ and } \frac{w_t^n}{p_t^n} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon}A_t \tag{34}$$

(ii) Equilibrium (detrended) asset price Q_t^n is constant and given as follows.

$$Q_t^n = \frac{1}{\rho}\Big(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\Big)^{\frac{1}{\chi}}\Big(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\Big) \text{ and } \mu_t^{q,n} = \sigma_t^{q,n} = 0 \tag{35}$$

(iii) Natural interest rate r_t^n depends on parameters ρ, g, σ_t in the following way.

$$r_t^n = \rho + g - \sigma_t^2 \tag{36}$$

(iv) Consumption of capitalists evolves with the following stochastic process, which depends on $r^n, \rho, \sigma_t, \chi$.

$$\frac{dC_t^n}{C_t^n} = \underbrace{(r_t^n - \rho + \sigma_t^2)}_{\equiv \mu_t^{c,n}}dt + \underbrace{\sigma_t}_{\equiv \sigma_t^{c.n}}dZ_t \tag{37}$$

In flexible price equilibrium, proposition 2 shows that we can characterize closed-form expressions of the real wage $\frac{w_t^n}{p_t^n}$, (detrended) stock price Q_t^n and natural rate r_t^n. A few points are worth mentioning. In the flexible price economy, $\sigma_t^{q,n} = 0$ holds, which implies that there is no additional financial risk running in the economy, in addition to the TFP risk, σ_t. This feature arises because our economy features no explicit frictions (other than nominal rigidity, which is absent for now) and thus every variable other than the labor supply $N^{w,n}(t)$ becomes proportional to A_t. This means that real wealth $A_tQ_t^n$ has the exact same volatility as A_t itself, and the financial market imposes no additional risk on the economy.

A higher ϵ increases competition among firms, raising the real wage $\frac{w_t^n}{p_t^n}$. It also has two competing effects on the asset price Q_t^n. A higher real wage pushes down the profit of the intermediate sector and reduces the stock price Q_t^n. On the other hand, a higher wage induces workers to supply more labor to firms, raising output and stock price Q_t^n. The effective labor supply elasticity χ^{-1}

[32] We assign a superscript n to denote variables in the flexible price (natural) equilibrium of the economy.

matters in this second effect, thus equation (35) features χ^{-1} exponent on the term that increases with ϵ. As Q_t^n is constant, its drift $\mu_t^{q,n}$ also satisfies $\mu_t^{q,n} = 0$ for all t.

The natural real interest rate r_t^n consists of two parts with countervailing forces. A higher growth rate g induces capitalists to engage in more intertemporal substitution (into both bonds and stocks) and raises the value of r_t^n. A higher σ_t pushes down the natural rate r_t^n in two ways: with higher σ_t, capitalists engage more in precautionary savings, bringing down the natural rate r_t^n. This effect is well documented in the literature.[33] Another channel in which a higher σ_t pushes down r_t^n works through the risk-premium. A higher σ_t raises the equilibrium risk-premium level, inducing capitalists to pull their wealth out of the stock market, forcing r_t^n to go down in order to prevent a fall in the financial wealth. The second channel is present in our framework as we explicitly model the portfolio decision of each capitalist, which collectively pins down the equilibrium wealth and thus the aggregate demand level.

With the flexible price equilibrium as a benchmark, we move on to the sticky price equilibrium and show how our framework differs from the usual New-Keynesian models.

3.4 Sticky Price Equilibrium

When price resetting is sticky à la Calvo (1983), we obtain the Phillips curve that describes inflation dynamics. Since a fixed portion δdt of firms changes their prices on a given infinitesimal interval dt, we have no stochastic fluctuation in the price process in equation (23), thus $\sigma_t^p = 0$ holds. Now, we just need a monetary policy rule to close the model. Before analyzing the proper monetary rule in this framework, we first describe the 'gap' economy, which is defined as the economy where every variable is a log-deviation from the corresponding level in the flexible price economy. That is, we define any business cycle variable x_t's gap, $\hat{x}_t$, to be the log-deviation of x_t from its natural level x_t^n, which is the level of the variable in the flexible price equilibrium.

$$\hat{x}_t \equiv \ln \frac{x_t}{x_t^n}. \tag{38}$$

Because the asset price acts as an endogenous aggregate demand shifter, we first write every other variable's gap in terms of the asset price gap. The following Assumption 1 is the first step.[34]

[33]For example, see Acharya and Dogra (2020) for the recent treatment of precautionary saving in the New-Keynesian environment.

[34]Assumption 1 ensures our framework matches the empirical regularities observed in the data, and holds under a standard calibration of the model (see Table 3). Even without Assumption 1, the main qualitative features of our model remain unchanged.

Assumption 1 (Labor Supply Elasticity) $\chi^{-1} > \dfrac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Assumption 1 is needed to guarantee the positive co-movement between the asset price and business cycle variables (e.g., real wage and consumptions of both capitalists and workers) observed in the data. With a large ϵ, firms' mark-ups decrease as competition between them intensifies, and real wage level rises as a result. This has a negative impact on the stock price as firm profits decrease, making it harder to satisfy a positive co-movement between the asset price and real wage gaps.[35] A larger α amplifies the effect of the Baxter and King (1991) externality, and an increase in asset price gap can result in higher labor demand and real wage. Without Assumption 1, a positive gap in the asset price depresses wages, labor, and consumption of workers, which might explain a portion of the observed long-run trend towards increased wealth inequality and income stagnation.[36]

The following Lemma 2 argues that given Assumption 1, gaps in consumptions of capitalists and workers, asset price, employment, and real wage are all linearly dependent and co-move with one another up to a first-order. Therefore, for stabilization purposes, the central bank only needs to deal with the asset price gap $\hat{Q}_t$.[37] From $C_t = \rho A_t Q_t$, we infer that $\hat{Q}_t = \hat{C}_t$ holds. Thus from now on we can interchangeably use $\hat{Q}_t$ or $\hat{C}_t$ to denote gaps of asset price Q_t and consumption of capitalists C_t.

Lemma 2 (Co-movement) *Given assumption 1, gaps in consumption of capitalists C_t and workers ($C_{W,t}$), employment ($N_{W,t}$), and real wage ($\frac{w_t}{p_t}$) co-move with a positive correlation. Up to a first-order, the following approximation holds.*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \widehat{\frac{w_t}{p_t}} = \underbrace{\frac{\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}}{1+\chi^{-1}}}_{>0} \widehat{C_{W,t}}. \tag{39}$$

[35]When the demand elasticity ϵ is larger, profits of firms per unit revenue decrease, as firms face a fiercer competition. In those cases, a drop in profits cam lead to decreases in both the asset price and capitalists' consumption, while hand-to-mouth workers enjoy a rise in wage income, and hence consumption. A higher χ^{-1} means a higher output elasticity with respect to aggregate technology, which tends to generate a positive correlation between consumption of capitalists and workers.

[36]For example, see Saez and Zucman (2020) for the trend on rising wealth and income inequality in the US. Also, see Autor et al. (2020) for evidence on a decreasing labor share and effects from the rise of market concentration. Especially, growth in pre-tax income for bottom 50% has been only 0.2% on average per year since 1980s, while S&P-500 index has risen almost by 8% per year.

[37]In this demand-determined environment, a positive asset price gap induces stronger economic activities in general, resulting in positive gaps in real wage, employment, and consumption.

Using Lemma 2, we can actually get the following relation between $\hat{Q}_t$ and $\hat{y}_t$.

$$\hat{y}_t = \zeta \hat{Q}_t, \ \text{where } \zeta \equiv \frac{\chi^{-1}}{\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}} > 0, \tag{40}$$

where Assumption 1 implies $\varphi > 0$.[38]

Demand block Now we formulate one of the key building blocks of this paper, a dynamic $\{\hat{Q}_t\}$ process. This $\{\hat{Q}_t\}$ process serves as the demand block of the model, while the Phillips curve will serve as a supply block.

The dynamic IS equation in our model features some important modifications from the canonical New-Keynesian model. Before we characterize it, we define the risk-premium level $\mathrm{rp}_t \equiv (\sigma_t + \sigma_t^q)^2$ and its natural level in the flexible price economy $\mathrm{rp}_t^n \equiv (\sigma_t)^2$ with $\sigma_t^{q,n} = 0$, as we characterized in equation (35). By subtracting rp_t^n from the current risk-premium level rp_t, we define risk-premium gap $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}_t^n$. Basically, as the risk-premium gap rises, capitalists ask for a higher compensation to bear financial risks, which causes asset prices to fall below its natural level. We also define the risk-adjusted natural rate r_t^T as we defined similarly in the standard non-linear New-Keynesian setting (equation (8)), which is related to its natural correspondent as follows.

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t. \tag{41}$$

r_t^T serves as a real rate anchor for monetary policy. A positive risk-premium gap ($\hat{rp}_t > 0$), for example, lowers the demand of capitalists for the risky stock compared with the benchmark economy, and thus decreases the risk-free rate r_t^T that supports the equilibrium dynamics.

In the following proposition, we characterize an asset price gap $\hat{Q}_t$ process, which is similar to the usual dynamic IS equation in textbook New-Keynesian models but different in a very important aspect: the natural rate r_t^n is replaced with the risk-adjusted natural rate r_t^T.

Proposition 3 (Asset Price Gap Process (Dynamic IS Equation)) *With inflation $\{\pi_t\}$, we have the following $\hat{Q}_t$ process, where r_t^T takes the role of r_t^n in the conventional IS equation.*

$$d\hat{Q}_t = (i_t - \pi_t - r_t^T)dt + \sigma_t^q dZ_t. \tag{42}$$

Thus, endogenous financial volatility σ_t^q directly affects the drift of the $\{\hat{Q}_t\}$ process, which governs how all

[38] Since aggregate production is linear in aggregate labor up to a first-order, aggregate mark up gap becomes negation of the real wage gap.

other gap variables fluctuate over time.

With $\sigma_t^p = 0$ due to the nature of staggered pricing à la Calvo (1983), when capitalists invest in the stock market they bear $(\sigma_t + \sigma_t^q)$ amount of risk. We know that the log-preference of capitalists determines the risk-premium level to be $(\sigma_t + \sigma_t^q)^2$. In flexible price equilibrium, the natural rate is given as r_t^n and σ_t^q equals $\sigma_t^{q,n} = 0$. Thus, the level of expected (instantaneous) real return in stock market investment becomes $r_t^n + (\sigma_t)^2 - \frac{1}{2}(\sigma_t)^2$, where the factor $\frac{1}{2}(\sigma_t)^2$ is from the quadratic variation factor that arises from the second-order Taylor expansion. In a sticky price equilibrium with asset price volatility σ_t^q, risk premium changes from $(\sigma_t)^2$ to $(\sigma_t + \sigma_t^q)^2$. Therefore, with monetary policy rate i_t and inflation π_t, the real expected stock market return becomes $i_t - \pi_t + \frac{1}{2}(\sigma_t + \sigma_t^q)^2$. If this value differs from $r_t^n + \frac{1}{2}(\sigma_t)^2$, then asset price gap $\hat{Q}_t$ endogenously adjusts, and this adjustment creates a real distortion from its effect on aggregate demand.

Equation (42) has the same mathematical structure as equation (7) in the standard New-Keynesian model. In Section 2, the endogenous business cycle volatility has a first-order impact on aggregate demand through precautionary savings channel, whereas in the current model with stock markets, an aggregate financial market volatility affects risk-premium and financial wealth, thereby affecting stock prices and aggregate demand. Due to this isomorphic structure between two frameworks, we will show that novel findings in Section 2 continue to hold here, with important implications about monetary policy.

Thus we get the lesson that the monetary policy i_t should take deviation in risk-premium from its natural level into account as well as the natural rate of interest r_t^n, since otherwise asset price Q_t will deviate from its natural level and generate business cycle fluctuation. r_t^T can be interpreted as the real risk-free rate that ensures that the real return on stock market investment is equal to its level in the benchmark economy, as shown in the following equation (43).

$$r_t^n + \frac{1}{2}\underbrace{(\sigma_t)^2}_{=\mathrm{rp}_t^n} = r_t^T + \frac{1}{2}\underbrace{(\sigma_t + \sigma_t^q)^2}_{=\mathrm{rp}_t}. \tag{43}$$

When $\sigma_t^q = \sigma_t^{q,n} = 0$ holds, the risk-adjusted rate r_t^T equals the natural rate r_t^n and equation (42) becomes the canonical New-Keynesian IS equation in equation (44).

$$d\hat{C}_t = (i_t - \pi_t - r_t^n)dt. \tag{44}$$

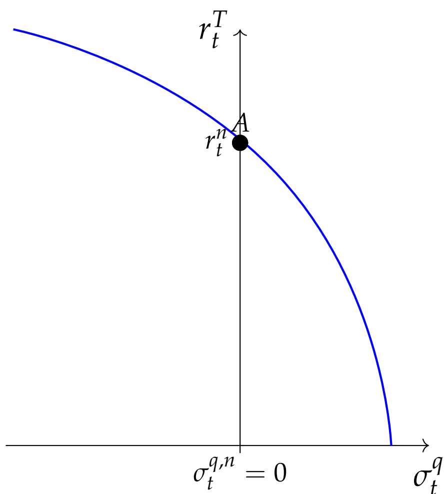
The crux of the problem is that σ_t^q is itself an endogenous variable to be determined in equilibrium, with no guarantee that it will equate its natural level $\sigma_t^{q,n} = 0$.

The endogenous financial volatility σ_t^q can be interpreted a measure of financial disruption, as its rise, given monetary policy rate i_t, reduces stock prices and thus aggregate demand, dragging the economy into recession. This channel has been pointed out by many authors including Gilchrist and Zakrajšek (2012), Stein (2014), Chodorow-Reich (2014), Guerrieri and Lorenzoni (2017), Di Tella and Hall (2020) among others, with different aspects of financial disruption affecting economic activity. Woodford (2012) and Cúrdia and Woodford (2016) especially introduced a friction in credit intermediation between borrowers and savers to the New-Keynesian framework and derived similar dynamics for output gap, but their friction is exogenous and relies on ad-hoc assumptions.

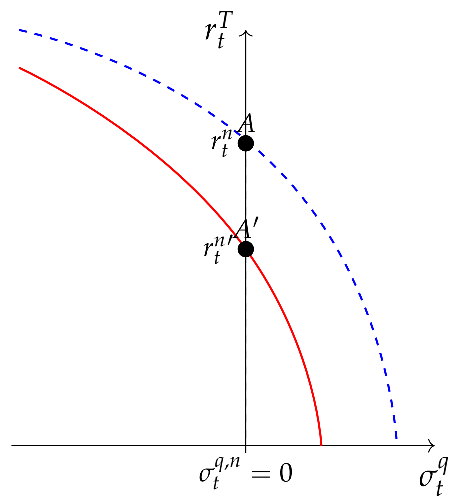
The existence of this new stock market volatility channel invites us to re-think the traditional monetary policy framework, to which we devote Section 4. Before we jump on to the next topic, if we plug equation (36) into equation (41), we get the following expression for r_t^T.

$$r_t^T = \rho + g - \frac{\sigma_t^2}{2} - \frac{(\sigma_t + \sigma_t^q)^2}{2}. \tag{45}$$

Figure 2a represents r_t^T as a function of σ_t^q given σ_t level. Intuitively, when σ_t^q jumps up, a rise in risk-premium rp_t ensues and the rate r_t^T falls. We see r_t^T aligns with the natural rate r_t^n when σ_t^q equals $\sigma_t^{q,n} = 0$. Figure 2b illustrates the effect of a spike in σ_t. When σ_t rises, the curve in Figure 2a uniformly shifts down. The formula $\sigma_t^{q,n} = 0$ in equation (35) implies that $\sigma_t^{q.n}$ remains unchanged, but the natural rate of interest r_t^n still falls due to equation (36).



(a) r^T as a function of σ_t^q given σ_t



(b) σ_t rises to $\sigma_t{}'$ and r_t^n jumps down to $r_t^{n\prime}$

Figure 2: r^T as a function of σ_t^q and σ_t

Supply block We follow the standard literature on pricing à la Calvo (1983) to determine inflation dynamics. The above Lemma 2 allows us to express the firms' aggregate marginal cost gap in terms of the asset price gap up to a first order, as asset price determines aggregate demand, which in turn determines such variables as the aggregate marginal cost.

The following Phillips curve in Proposition 4 describes π_t dynamics, and is of the same form as in many New-Keynesian models.

Proposition 4 (Phillips Curve) *Inflation π_t evolves according to the following stochastic process with $\hat{Q}_t$ entering in the position of output gap in conventional New-Keynesian models.*[39]

$$\mathbb{E}_t d\pi_t = (\rho\pi_t - \frac{\kappa}{\zeta}\hat{y}_t)dt \;\; where, \;\; \kappa \equiv \frac{\delta(\delta+\rho)\Theta}{\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}}, \;\; \Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \tag{46}$$

Plugging equation (40) into the Phillips curve, we get $\mathbb{E}_t d\pi_t = (\rho\pi_t - \kappa\hat{Q}_t)dt$, which is expressed in terms of $\hat{Q}_t$. Under Assumption 1, a higher asset price gap $\hat{Q}_t$ means the economy is over-heated, and thus inflation rates would jump up. Note that: as price resetting probability increases ($\delta \to \infty$), then we have $\kappa \to \infty$ and $\hat{Q}_t = 0$ in equilibrium. Thus, we achieve the flexible price equilibrium when $\delta \to \infty$.

Now that we characterize the model's demand block (the IS equation for $\hat{Q}_t$ (equation (42))) and supply block (Phillips curve in equation (46)), we need to specify the policy reaction function i_t to close the model. Before we move on to the analysis of policy rules, we briefly discuss the traditional approach to the problem of financial and macroeconomic stabilization in the literature.

Macroprudential policies and regulations There are in general two goals in short (and/or medium)-run macroeconomics: *macro-stabilization* and *financial stability*. Many policymakers (including central bankers) and academic economists believe that financial stability should be dealt with by regulations and macroprudential policies imposed on banks and financial institutions, with business cycle stabilization being the sole focus of monetary policy. Because our model is parsimonious and does not include any complex financial market participants, those macroprudential regulations that tackle potential financial instabilities can be regarded as a policy avenue to prevent σ_t^q from deviating from $\sigma_t^{q,n} = 0$. If $\sigma_t^q = \sigma_t^{q,n} = 0$, then as in equation (44), our model features exactly the same dynamics as conventional New Keynesian models. Therefore, in that case a conventional monetary policy rule can solely focus on business cycle stabilization.

One interesting aspect built in our model is that financial stability (volatility and risk-premium) issues are intertwined with macro-stabilization. The more volatile financial markets features higher risk-premium levels, thereby driving down aggregate financial wealth and aggregate demand. Our view is that even without perfect macroprudential policies to guarantee $\sigma_t^q = \sigma_t^{q,n} = 0$, monetary

[39] The coefficient $\chi\delta(\delta+\rho)\Theta$ is attached to the output gap $\hat{y}_t$ in equation (46). In standard New-Keynesian models with a representative agent whose utility is of the same form as our workers', the coefficient becomes $(\chi_0+\varphi)\delta(\delta+\rho)\Theta$, which is different from $\chi\delta(\delta+\rho)\Theta$ as $\chi \neq \chi_0 + \varphi$.

policy might be able to tackle both concerns simultaneously, as stabilization in one dimension might help stabilize the other.

Now we move onto the analysis of distinct monetary policy rules and revisit the classical question on the role of monetary policy as a financial stabilizer.

4 Monetary Policy

In this Section 4, we study the monetary policy's roles of macroeconomic stabilization in the context of our model. First, we analyze conventional Taylor rules with inflation and output gap as policy targets. After showing limitations of such policies and how sunspot equilibria can arise, we propose a generalized version of the Taylor rule for stochastic environments that successfully achieve twin objectives of financial and economic stability.

For simplicity, we assume throughout Section 4 the constant TFP volatility $\sigma_t = \sigma$ for all t such that the real natural rate $r_t^n = \rho + g - \sigma^2 > 0$ and the natural risk-premium $\mathrm{rp}_t^n = \sigma^2$ are constants.

4.1 Old Monetary Rule

4.1.1 Conventional Taylor rule and Bernanke and Gertler (2000) rule

We start with a conventional Taylor rule with a constant intercept equal to the natural rate r^n, and standard inflation and output gap targets.

$$i_t = r^n + \phi_\pi \pi_t + \phi_y \hat{y}_t, \tag{47}$$

where $\hat{y}_t$ is the output gap, π_t inflation and note we implicitly assume a zero trend inflation target, $\bar{\pi} = 0$. As output gap $\hat{y}_t$ is positively correlated with the asset price gap $\hat{Q}_t$ as in equation (40), we can express equation (47) as the monetary policy rule that targets asset price $\hat{Q}_t$ as well as inflation:

$$\begin{aligned} i_t &= r^n + \phi_\pi \pi_t + \underbrace{\phi_y \zeta}_{\equiv \phi_q > 0} \hat{Q}_t \\ &= r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t. \end{aligned} \tag{48}$$

Bernanke and Gertler (2000), by adding stochastic ad-hoc bubbles to the fundamental asset price in a model based on Bernanke et al. (1999), conducted an analysis on whether monetary rules that directly target asset price as in equation (48) can effectively stabilize the economy. They conclude that such rules are undesirable as they deter real economic activity when the 'bubble' appears and bursts.[40] In contrast, our framework features no irrational asset price bubble: here, fluctuations in $\hat{Q}_t$ reflect rational expectation about future business cycle fluctuations, and thus from central bank's perspective, targeting the asset price gap $\hat{Q}_t$ is equivalent to targeting the output gap $\hat{y}_t$, as the two gaps are perfectly correlated up to a first-order. Therefore in our model, a conventional monetary policy rule is equivalent to the rule of Bernanke and Gertler (2000).

Now we study whether equation (48) achieves divine coincidence as in textbook New-Keynesian models. Our objective now is to show that this rule cannot guarantee equilibrium determinacy even if it satisfies the so-called Taylor principle. Let us assume the monetary authority relies on Bernanke and Gertler (2000) rule in equation (48) that targets two factors, π_t and $\hat{Q}_t$. We define the coefficient $\phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0$, which is the total responsiveness of monetary policy to inflation and asset price gap. $\phi > 0$ corresponds to the conventional Taylor principle that excludes the possibility of sunspot in inflation. Thus, i_t follows

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t, \text{ where } \phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0. \tag{49}$$

Plugging equation (49) into equation (42), we get the following $\hat{Q}_t$ dynamics.

$$d\hat{Q}_t = \Big((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t \underbrace{- \frac{\sigma^2}{2} + \frac{(\sigma + \sigma_t^q)^2}{2}}_{\text{New terms}}\Big)dt + \sigma_t^q dZ_t. \tag{50}$$

Multiple equilibria Instead of equation (50), if $\hat{Q}_t$ dynamics is represented by

$$d\hat{Q}_t = \Big((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t\Big)dt + \sigma_t^q dZ_t, \tag{51}$$

then, with the Taylor principle $\phi > 0$ satisfied we achieve divine coincidence: $\hat{Q}_t = \pi_t = 0$ is the unique possible rational expectations equilibrium from the Blanchard and Kahn (1980). In contrast, now that the financial volatility σ_t^q affects the drift of equation (50), we have multiple equilibria and sunspots in σ_t^q can possibly appear. The reason is similar to the reason why we might have sunspots in aggregate business cycle volatility in the standard New-Keynesian model in Section 2. Here, the dynamic IS equation in (50) features a countercyclical financial volatility σ_t^q. Since an increase in σ_t^q raises the risk-premium, it brings down financial wealth and aggregate demand[41] (thus, raising the drift of equation (50)). For example, imagine that capitalists fear of a possible financial crisis arising

[40] Galí (2021) introduces rational bubbles in a New-Keynesian model with overlapping generations. He argues that 'leaning against the bubble' monetary policy, if properly specified, can insulate the economy from the aggregate bubble fluctuations, as only rational bubbles shift the aggregate output in his framework.

[41] Monetary policy in equation (48) responds when its mandates $\hat{Q}_t$ and π_t are affected by a sunspot in σ_t^q, but does not directly target the sunspot or volatility σ_t^q.

from higher levels of risk-premium and financial volatility: they respond by reducing the demand for the risky stock, which leads to the collapse of the asset price, and self-justifies a higher expected return in the stock market investment and a rise in risk-premium. This result is related to Acharya and Dogra (2020)'s findings about equilibrium determinacy issues in models with countercyclical income risks, even though their paper focuses on *idiosyncratic* risks and effects from precautionary savings, while ours centers on the sunspot equilibria stemming from *aggregate* endogenous risk.

We now formalize the multiple equilibrium intuition presented above by constructing a rational expectations equilibrium that supports an initial sunspot σ_0^q. For simplicity, we focus on the case in which σ_0^q jumps off from $\sigma^{q,n} = 0$ (thus, $\sigma_0^q > 0$), and study how the sunspot σ_0^q can be rationally sustained in equilibrium. For that purpose, a rational expectations equilibrium must: (i) support an initial hike $\sigma_0^q > 0$, and (ii) not diverge (on expectation) in the long-run, following Blanchard and Kahn (1980).

Martingale equilibrium[42] In particular, we study one rational expectations equilibrium that supports an initial sunspot σ_0^q: the equilibrium in which asset price gap $\hat{Q}_t$ follows a martingale after the initial sunspot σ_0^q happens. As $\hat{Q}_t$ is martingale, we get the following formula for π_t by iterating equation (46) over time.

$$\pi_t = \kappa \int_t^\infty e^{\rho(s-t)} \underbrace{\mathbb{E}_t(\hat{Q}_s)}_{=\hat{Q}_t} ds = \frac{\kappa}{\rho}\hat{Q}_t, \tag{52}$$

which implies inflation closely follows the trajectory of $\hat{Q}_t$. Plugging equation (52) into equation (50) and imposing a martingale condition, we obtain

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi} + \frac{\sigma^2}{2\phi} \text{ and } \pi_t = \frac{\kappa}{\rho}\Big(-\frac{(\sigma + \sigma_t^q)^2}{2\phi} + \frac{\sigma^2}{2\phi}\Big).^{43} \tag{53}$$

Our martingale equilibrium does not diverge (on expectation) in the long-run, as the paths of $\{\hat{Q}_t, \pi_t\}$ stay, on expectation, at the initial values of the variables, thus satisfying $\mathbb{E}_0(\pi_t) = \pi_0$ and $\mathbb{E}_0(\hat{Q}_t) = \hat{Q}_0$, $\forall t \geq 0$. The last step is to show that there exists a stochastic path of $\{\sigma_t^q\}$ starting from σ_0^q that supports this equilibrium. This equilibrium then both (i) supports an initial sunspot $\sigma_0^q > 0$ and (ii) does not diverge in the long-run. Using equation (50) and equation (53),[44] we obtain

[42] Under some regularity conditions dictating how the expected risk-premium evolves in the long run, our martingale equilibrium becomes a 'unique' rational expectations equilibrium that supports an initial sunspot $\sigma_0^q > 0$. A martingale process for $\hat{Q}_t$ is consistent with the previous findings of the literature on the 'Efficient Market Hypothesis (EMH)' (For example, see Fama (1970)).

[43] Thus in this particular equilibrium, $\sigma_t^q > \sigma^{q,n} = 0$ causes $\hat{Q}_t$ to drop below zero, causing a recession.

[44] Since $\hat{Q}_t$ process is a martingale, the drift part in equation (50) must be 0.

the stochastic process of σ_t^q as[45]

$$d\sigma_t^q = -\frac{\phi^2(\sigma_t^q)^2}{2(\sigma+\sigma_t^q)^3}dt - \phi\frac{\sigma_t^q}{\sigma+\sigma_t^q}dZ_t. \tag{55}$$

Both equation (53) and equation (55) constitute the dynamics of this particular rational equilibrium supporting $\sigma_0^q > 0$. What does this equilibrium look like? The next Proposition 5 sheds light on the behavior of $\hat{Q}_t$ and π_t paths and argues that business cycles almost surely converge to a perfectly stabilized path in the long run. The very few paths that do not converge can blow up asymptotically and, together with the forward-looking nature of the economy, help sustain the initial crisis.

Proposition 5 (Bernanke and Gertler (2000) Rule and Indeterminacy) *For any value of Taylor responsiveness $\phi > 0$:*

1. Indeterminacy*: there is always a rational expectations equilibrium (REE) that supports initial sunspot $\sigma_0^q > 0$ and is represented by $\hat{Q}_t$ and π_t dynamics in equation* (53)*, and σ_t^q process in equation* (55)
2. Properties*: the rational expectations equilibrium that supports an initial sunspot $\sigma_0^q > 0$ satisfies:*

$$(i)\ \sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0,\ \ (ii)\ \hat{Q}_t \overset{a.s}{\to} 0 \text{ and } \pi_t \overset{a.s}{\to} 0,\ \text{ and }\ (iii)\ \mathbb{E}_0\left(\max_t(\sigma_t^q)^2\right) = \infty$$

The conditions $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n}$, $\hat{Q}_t \overset{a.s}{\to} 0$, and $\pi_t \overset{a.s}{\to} 0$ imply that equilibrium paths supporting an initial sunspot $\sigma_0^q > 0$ are almost surely stabilized in the long run. Then, how is it possible for a sunspot $\sigma_0^q > 0$ to appear at first? The finding $\mathbb{E}_0(\max_t(\sigma_t^q)^2) = \infty$ implies that an initial spike in σ_0^q and the ensuing crisis is sustained by the tiny probability of an ∞-severe financial disruption in the future. This result has similar implications to Martin (2012) in a sense that our framework does not assume the existence of specific disasters but disaster risk is always present even if monetary authority satisfies the Taylor principle and actively stabilizes the business cycle. Martin (2012) applied a similar logic to pure asset pricing contexts and showed that the pricing of a broad class of long-dated assets is driven by the possibility of extraordinarily bad news in the future. The intuitions we derived here continue to hold in our simple discrete-time framework in Lee and Carreras (2021).

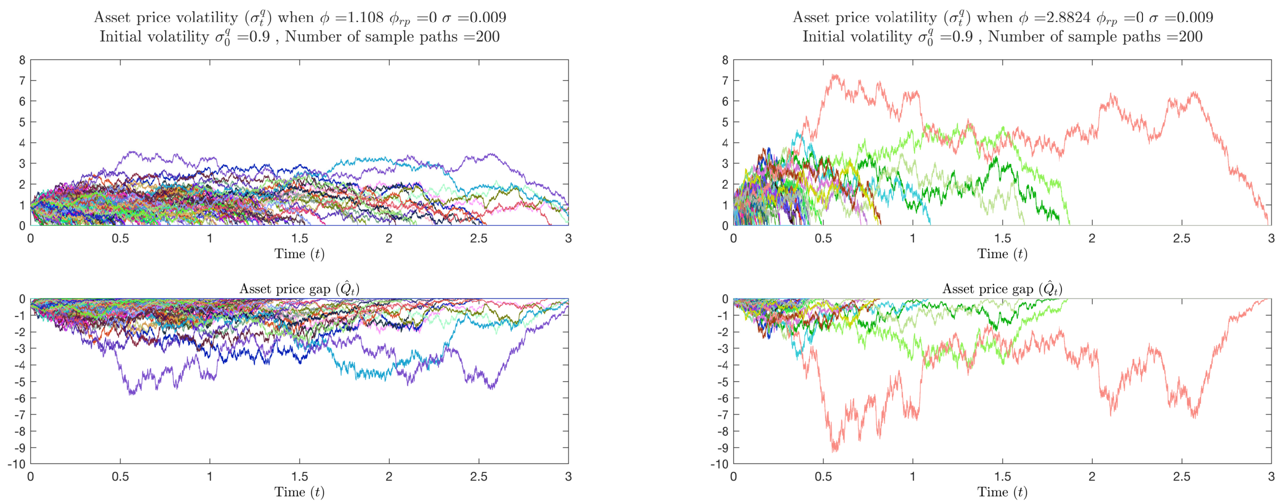
Calibration and Simulation For the rest of the paper, we calibrate the parameters of our model to values commonly found in the literature: see Table 3 in Appendix B for further details. A few

[45]When $\sigma = 0$, this process becomes the following Bessel process:

$$d\sigma_t^q = -\frac{\phi^2}{2\sigma_t^q}dt - \phi dZ_t. \tag{54}$$

which stops when σ_t^q reaches $\sigma^{q,n} = 0$. For general properties of Bessel processes, see Lawler (2019).

points are worth mentioning. For worker's risk-aversion parameter φ, we use $\varphi = 0.2$ following Gandelman and Hernández-Murillo (2014).[46] For individual firm's labor share in production, we use $1 - \alpha = 0.6$ following Alvarez-Cuadrado et al. (2018), as we regard the aggregate labor in the production function as a proxy for the capital in conventional macroeconomic models. With this calibration, our co-movement condition (Assumption 1) is satisfied.



(a) With $\phi_\pi = 1.5$

(b) With $\phi_\pi = 2.5$.

Figure 3: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$, with calibration in Table 3

Figure 3 illustrates the martingale equilibrium's dynamic paths of $\{\sigma_t^q, \hat{Q}_t\}$ supporting $\sigma_0^q = 0.9 > \sigma^{q,n} = 0$. Normalization shows that as σ_0^q jumps off by σ, stock price falls by $2 - 10\%$, which is consistent with our empirical findings in Appendix A (Figure 12b). Figure 3 also explores the effects on the martingale equilibrium of a change in policy responsiveness to inflation ϕ_π. The right panel 3b uses the default calibration value $\phi_\pi = 2.5$, while the left panel 3a assumes a more accomodating stance $\phi_\pi = 1.5$. As we raise ϕ_π, the average sample path converges faster towards full stabilization, but at the expense of an increased likelihood of a more severe crisis path in a given period of time. We obtain similar results when looking at changes in policy responsiveness to the asset price gap ϕ_q (alternatively, output gap ϕ_y), and find that a change in $\phi \equiv \phi_q + (\phi_\pi - 1)\frac{\kappa}{\rho}$, the measure of combined responsiveness of monetary policy i_t, brought by any combination $\{\phi_\pi, \phi_q\}$ follows the same patterns depicted in Figure 3.

Booms In an analogous way, we can construct a rational expectations equilibrium that supports an

[46]Their estimates of φ range between 0.2 and 10. In our environment, a higher risk-aversion of workers makes their labor supply (and therefore, output) less responsive to business cycle fluctuations. In that scenario, a higher asset price tends to translate into less wage income distributed to workers, making it harder to satisfy the co-movement condition (Assumption 1). Thus, we pick a value on the lower end of the acceptable range and set $\varphi = 0.2$.

initial downward sunspot $\sigma_0^q < \sigma_t^{q,n} \equiv 0$. The equilibrium paths feature a boom phase with buoyant production and consumption with lower levels of financial volatility and risk-premium. A higher ϕ value speeds up the stabilization process, but increases the likelihood of an equilibrium path with an overheated economy.[47]

4.2 Modified Monetary Rule

A modified monetary policy rule includes risk-premium as a separate factor in the following way:

$$i_t = \underbrace{r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Bernanke and Gertler (2000)}} - \underbrace{\frac{1}{2}\hat{rp}_t}_{\text{Risk-premium targeting}}, \text{ where } \hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}^n. \tag{56}$$

Thus, the above monetary policy rule contains a 'risk-premium gap term' as a factor in addition to inflation and asset price gap. It also can be written in terms of the risk-adjusted natural rate r_t^T as

$$i_t = r_t^T + \phi_\pi \pi_t + \phi_q \hat{Q}_t, \tag{57}$$

where a higher $\hat{rp}_t$ brings down r_t^T, forcing i_t to fall. The next Proposition 6 establishes that a monetary policy rule consistent with equation (56) and that satisfies the Taylor principle (corresponding to $\phi > 0$) restores equilibrium determinacy and fully stabilizes the economy.

Proposition 6 (Ultra-Divine Coincidence with Risk-Premium Targeting) *The monetary policy rule*

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \frac{1}{2}\hat{rp}_t, \text{ where } \phi \equiv \phi_q + \frac{\kappa(\phi_\pi - 1)}{\rho} > 0, \tag{58}$$

achieves $\hat{Q}_t = \pi_t = \hat{rp}_t = 0$. Therefore, the monetary policy rule in equation (58) attains (i) output (asset price) stabilization, (ii) price level (inflation) stabilization, and (iii) financial market (financial volatility and risk-premium) stabilization. We call it a ultra-divine coincidence.

This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984). The reason central banks must target risk-premium as a separate factor is that this term directly appears in the drift part of our dynamic IS equation (equation (42)). According to the rule in equation (58), a central bank lowers the policy rate i_t when $\mathrm{rp}_t > \mathrm{rp}^n$ to boost $\hat{Q}_t$ and $\hat{C}_t$[48], since a higher risk-premium drags down asset price and business cycle levels. If monetary policy kills an initial excess

[47] We have two singular points in the $\{\sigma_t^q\}$ process (equation (55)): as σ_t^q hits $-\sigma$, the drift and volatility of the process diverge, and $\{\sigma_t^q\}$ process features a jump. When σ_t^q hits 0, it stays there forever. Thus, when σ_0^q is below $-\sigma$, we might end up in paths where we have a jump in σ_t^q to a positive value, which eventually converges to 0.

[48] Even with Bernanke and Gertler (2000) rule, monetary policy responds to a rise in risk-premium since it negatively

volatility (or excess risk-premium) with this additional target in its rule, it precludes the possibility of sunspots in financial volatility that we discussed. Since the Taylor principle ($\phi > 0$) guarantees there is no sunspot inflation, the policy rule in equation (58) restores equilibrium determinacy and achieves both macro stability (with $\hat{Q}_t = \pi_t = 0$) and financial stability (with $\hat{rp}_t = 0$, which implies $\text{rp}_t = \text{rp}^n$ and $\sigma_t^q = \sigma^{q,n} = 0$). The equilibrium interest rate then becomes $i_t = r^n$, which is the same level as in the equilibrium path of a canonical New-Keynesian model. Therefore, the ultra-divine coincidence result implies: one policy tool (i_t rule) achieves an additional objective (financial stability) in addition to the two usual mandates (output gap and inflation stability). This is possible in our framework because financial markets and the business cycle are tightly interwoven and real and financial instabilities are equivalent to each other.

A common view in the literature holds that monetary policy must respond to financial market disruptions only when they affect (or to the degree that they affect) the original central bank mandates of inflation stability and full employment (or full output). This premise is at odds with the results of our paper: the failure to target the risk-premium of financial markets subjects the economy to the apparition of sunspot shocks and the corresponding recessions and overheating episodes that ensue. Only by targeting risk-premium in the particular way characterized in equation (56), the monetary authority can re-establish equilibrium determinacy and achieve the ultra-divine coincidence outlined in the previous paragraphs.

Interpretation We can rewrite our modified Taylor rule (equation (58)) as

$$\underbrace{i_t + \text{rp}_t}_{=i_t^m} - \underbrace{\frac{1}{2}\text{rp}_t}_{\text{Ito term}} = \underbrace{r^n + \text{rp}^n}_{=i_t^{m,n}} - \underbrace{\frac{1}{2}\text{rp}^n}_{\text{Ito term}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}}, \tag{59}$$

or equivalently as

$$\underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\\text{yield}}} + \underbrace{\frac{\mathbb{E}_t\left(d\log a_t\right)}{dt}}_{\substack{\text{Internal rate of return}\\\text{of aggregate wealth}}}}_{\text{Cum-dividend return}} = \underbrace{\underbrace{\rho}_{\substack{\text{Dividend}\\\text{yield}}} + \frac{\mathbb{E}_t\left(d\log a_t^n\right)}{dt}}_{\text{Benchmark cum-dividend return}} + \underbrace{\phi_\pi \pi_t + \phi_q \hat{Q}_t}_{\text{Business cycle targeting}}, \tag{60}$$

where a_t is the economy's aggregate financial wealth and a_t^n is the aggregate wealth of the natural (flexible price) economy. Our modified monetary policy that targets a risk-premium as prescribed in equation (58) thus can be interpreted as the rule on the rate of change of log-aggregate wealth as a function of traditional inflation and output gap (asset price) targets.

affects the asset price gap $\hat{Q}_t$ and inflation π_t. Our point here is that the policy rate must systematically respond to deviations of rp_t from its natural level rp^n given $\hat{Q}_t$ and π_t levels.

In the standard linearized New-Keynesian model (or alternatively, a model under perfect foresight), the economy's risk-free rate (i.e., policy rate) equals the rate of change in log-wealth, whereas the expected stock market return takes that role in our model with risk. Therefore, equation (60) restores determinacy and attains divine coincidence both in the standard linearized model and in our framework where the endogenous volatility of stocks (equivalently, the risk premium) affects expected asset returns. We interpret equation (60) as the *generalized Taylor rule* that holds in both linearized and risk-centric environments. With this rule, the central bank uses the aggregate wealth and its rate of return as *intermediate* targets towards achieving business cycle stabilization, as wealth itself affects aggregate demand, and its internal rate of return changes how a demand-driven economy evolves along the cycle.

Practicality Some issues exist about the feasibility to implement this new policy rule. First, the risk-premium gap $\hat{rp}_t$ in equation (56) depends on the natural risk-premium level, which is a counterfactual variable by definition, and therefore its observation is subject to some form of measurement error. Second, there are multiple kinds of risk-premia in financial markets that can be possibly targeted through monetary policy, and the construction of an aggregate risk-premium index as featured in our model might be subject to error as well.[49,50]

More importantly, and related to the previous two points, the coefficient attached to risk-premium in equation (56) is exactly $\frac{1}{2}$. Given the potential for measurement error in $\hat{rp}_t$, it might be impossible for the central bank to target the risk-premium with the exact strength demanded by equation (56).[51] To understand the consequences of deviating from the $\frac{1}{2}$ risk-premium target, we consider the following alternative rule:

$$i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t - \phi_{\text{rp}} \hat{rp}_t, \tag{61}$$

where ϕ_{rp} is a constant term potentially different from $\frac{1}{2}$. We have the following $\{\hat{Q}_t\}$ process with

[49] Our framework features only an 'index' of the stock market as a feasible vehicle to invest in, but there are multiple risk-premia (including term-premia) covering stocks and bonds in the real world.

[50] There have been long-standing debates about whether monetary authorities should adjust policy rates in response to fluctuations in risk-premia of financial markets. For example, Doh et al. (2015) argued "adjusting short-term interest rates in response to various estimated risk premium levels could be appropriate, especially if the risk premiums are low for a sustained period. In contrast, if policymakers are predominantly concerned about the most likely macroeconomic outcome, monitoring estimated risk premiums and adjusting the monetary policy stance accordingly may be of little benefit.". This argument is based on the fact that information about possible tail risks is summarized by the risk-premia levels in financial markets.

[51] As an example, consider a multiplicative measurement error ε_t such that $\hat{rp}_t^{obs} = \varepsilon_t \cdot \hat{rp}_t$, where $\hat{rp}_t^{obs}$ stands for the observed risk-premium. It is easy to see that the central bank following the policy rule in equation (56) will target the 'true' risk-premium with a coefficient $\neq \frac{1}{2}$.

the policy rule in equation (61):

$$d\hat{Q}_t = \Big((\phi_\pi - 1)\pi_t + \phi_q \hat{Q}_t + \Big(\frac{1}{2} - \phi_{\text{rp}}\Big)\hat{rp}_t\Big)dt + \sigma_t^q dZ_t. \tag{62}$$

With $\phi_{\text{rp}} = \frac{1}{2}$, we return to determinacy (Proposition 6). With $\phi_{\text{rp}} \neq \frac{1}{2}$, the martingale equilibrium reappears and is characterized by[52]

$$\hat{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\phi_{\text{rp}}}} + \frac{\sigma^2}{2\phi_{\phi_{\text{rp}}}} \text{ and } \pi_t = \frac{\kappa}{\rho}\Big(-\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\phi_{\text{rp}}}} + \frac{\sigma^2}{2\phi_{\phi_{\text{rp}}}}\Big) \text{ with } \phi_{\phi_{\text{rp}}} \equiv \frac{\phi}{1 - 2\phi_{\text{rp}}}, \tag{63}$$

where $\{\sigma_t^q\}$'s stochastic process after an initial sunspot σ_0^q appears is given as

$$d\sigma_t^q = -\frac{\phi_{\phi_{\text{rp}}}^2 (\sigma_t^q)^2}{2(\sigma + \sigma_t^q)^3}dt - \phi_{\phi_{\text{rp}}}\frac{\sigma_t^q}{\sigma + \sigma_t^q}dZ_t. \tag{64}$$

When $\phi_{\text{rp}} < \frac{1}{2}$ (including Proposition 5, the case of $\phi_{\text{rp}} = 0$), a rise in ϕ_{rp} leads to an increase in $\phi_{\phi_{\text{rp}}}$ in equation (63). From equation (64) we observe that a higher $\phi_{\phi_{\text{rp}}}$ accelerates the convergence of sample paths while creating more amplified paths after the initial sunspot σ_0^q appears. As far as $\phi_{\text{rp}} < \frac{1}{2}$, a higher ϕ_{rp} means monetary policy responds more strongly to fluctuations in $\hat{rp}_t$, which allows faster stabilization. As ϕ_{rp} goes up from 0 to $\frac{1}{2}$, fluctuations in $\hat{rp}_t$ have less direct effects on dynamics (equation (62)). Thus, the volatility of the $\{\sigma_t^q\}$ process (in equation (64)) must rise to ensure that $\{\hat{Q}_t\}$ eventually is stabilized[53], which results, on average, on shorter but more amplified sample paths.

$\phi_{\text{rp}} < 0$ case is interesting since it implies central bank raises the policy rate when risk-premia rise in financial markets. It is consistent with the "Real Bills Doctrine" which was a popular idea during the first half of the 20th century. Basically, the doctrine advocated for the Fed discount rate to track the average interest rate of the financial markets, as a means of stabilization.[54] In our framework, $\phi_{\text{rp}} < 0$ pushes down $\phi_{\phi_{\text{rp}}}$ from ϕ, which effectively slows down the pace of stabilization after sunspots hit the stock market. Therefore, we see that "Real Bills Doctrine" with $\phi_{\text{rp}} < 0$ is not suitable for stabilization purposes, as empirically documented by Richardson and Troost (2009).

With $\phi_{\text{rp}} > \frac{1}{2}$, monetary policy responds too strongly to fluctuations in risk-premium, thus with

[52]The equations (equation (61) and equation (63)) are easily derived in a similar way to the proof of Proposition 5 in Appendix C.

[53]Here with the monetary policy in equation (61), $(\frac{1}{2} - \phi_{\text{rp}})\hat{rp}_t$ appears in the drift of the $\{\hat{Q}_t\}$ process (equation (62)). When $\phi_{\text{rp}} < \frac{1}{2}$, a higher ϕ_{rp} implies that $\{\sigma_t^q\}$ path, on average, features more volatility (of $\{\sigma_t^q\}$ path itself) to raise $\hat{rp}_t$ given the levels of $\hat{Q}_t$ and π_t, as $\hat{rp}_t$ is a convex function of σ_t^q. Eventually, $\hat{Q}_t$ and π_t adjust as they are jump variables.

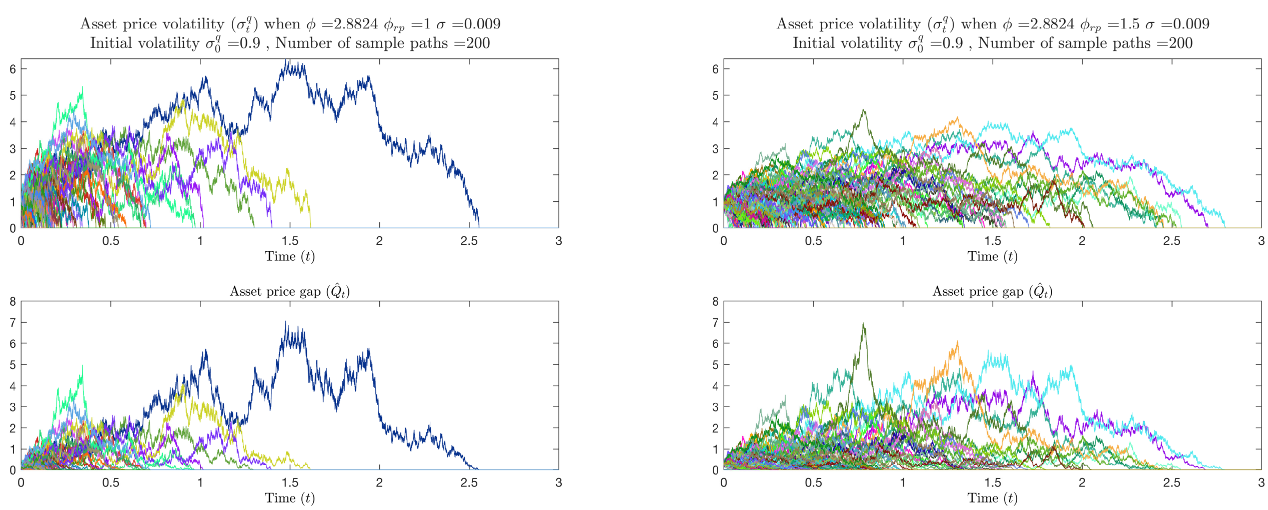
[54]Richardson and Troost (2009) studied the effects of such policy during the Great Depression era, exploiting the fact that the state of Mississippi is divided by the Federal Reserve act between the 6th (Atlanta) and 8th (St. Louis) districts which had different approaches to the economy-wide banking panics and depressions.

an initial positive sunspot $\sigma_0^q > 0$, policy rate drops excessively and creates an artificial boom instead of a crisis.[55] A higher ϕ_{rp} reduces $|\phi_{\phi_{\text{rp}}}|$ and slows down stabilization since a higher ϕ_{rp} means monetary policy deviates more from determinacy (the case of $\phi_{\text{rp}} = \frac{1}{2}$), and therefore becomes less efficient at stabilization. Figure 4 illustrates that with $\phi_{\text{rp}} > \frac{1}{2}$, a spike in financial volatility, $\sigma_t^q > 0$, actually acts as a boon to the economy, as we have $\hat{Q}_t > 0$ and $\pi_t > 0$ along sample paths. Moreover, with $\phi_\pi = 2.5$ fixed, as we raise ϕ_{rp} (from 1 to 1.5), stabilization slows down[56] as we further deviate from the determinacy case ($\phi_{\text{rp}} = \frac{1}{2}$).

These results are summarized in Table 1.

$\boldsymbol{\phi_{\text{rp}} < 0}$ **(Real Bills Doctrine)**	$\boldsymbol{0 \le \phi_{\text{rp}} < \frac{1}{2}}$
(i) With $\phi_{\text{rp}} \downarrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)	(i) With $\phi_{\text{rp}} \uparrow$, convergence speed$\uparrow$ and more amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t < 0$ and $\pi_t < 0$)
$\boldsymbol{\phi_{\text{rp}} = \frac{1}{2}}$	$\boldsymbol{\phi_{\text{rp}} > \frac{1}{2}}$
No sunspot (ultra-divine coincidence)	(i) With $\phi_{\text{rp}} \uparrow$, convergence speed$\downarrow$ and less amplified paths (ii) $\sigma_t^q > \sigma_t^{q,n} = 0$ means a boom ($\hat{Q}_t > 0$ and $\pi_t > 0$)
As $\phi \uparrow$, convergence speed$\uparrow$ and $\exists$ more amplified paths	

Table 1: Effects of different parameters $\{\phi_{\text{rp}}, \phi\}$ on stabilization



(a) With $\phi_{\text{rp}} = 1$

(b) With $\phi_{\text{rp}} = 1.5$.

Figure 4: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$, with varying $\phi_{\text{rp}} > \frac{1}{2}$

[55] With $\phi_{\text{rp}} > \frac{1}{2}$, we have $\phi_{\phi_{\text{rp}}} < 0$ from equation (63), thus $\sigma_t^q > 0$ is equivalent to the boom phase with $\pi_t > 0$ and $\hat{Q}_t > 0$.

[56] Also, a higher ϕ_{rp} causes less amplification from the initial sunspot $\sigma_0^q > 0$.

In Figure 3, we observe that an initial sunspot $\sigma_0^q > 0$ can be amplified endogenously through monetary policy's responses to the business cycle fluctuation, which might drag the economy into zero lower bound (ZLB) episodes when the $\{\sigma_t^q\}$ path hits some threshold from below. When monetary policy is constrained at those episodes, both asset market and business cycle would collapse, which we observed in the 2007-2009 Global Financial Crisis (GFC). In the next Sections, we formally study ZLB issues following the prior literature and discuss possible fiscal-monetary policies that mitigate recessionary pressures and stabilize financial markets and the real economy.

5 Zero Lower Bound (ZLB) and Forward Guidance

The ZLB featured prominently during the Great Financial Crisis, and in this section we will show that it has, indeed, very interesting implications for the costs of financial volatility sunspot shocks. In the previous section, and as depicted in Figure 3, we have shown that a less responsive monetary policy (to π_t and $\hat{Q}_t$) results in persistent sunspot recessions. During a ZLB episode, the policy rate is stuck at zero, and therefore the unresponsiveness of monetary policy can be approximated, to a first pass, as an extreme case of very low monetary responsiveness, in which case ZLB amplifies the duration -and the costs- of positive sunspot shocks. Central banks have developed alternative tools like forward guidance to retain their capacity to intervene in the economy and minimize the costs of ZLB recessions. In this section, we will study the capacity of this tools to stabilize the economy and financial markets, and the potential trade-offs in terms of stabilization that their use entails.

Following Werning (2012), we consider a scenario in which exogenous TFP volatility σ_t jumps in a deterministic manner between $t = 0$ and T. In particular, we consider the case where $\sigma_t = \bar{\sigma}$ for $0 \leq t \leq T$ and $\sigma_t = \underline{\sigma}$ for $t \geq T$. We assume that $\underline{r} \equiv r^n(\bar{\sigma}) < 0$[57] and $\bar{r} \equiv r^n(\underline{\sigma}) > 0$, thus monetary policy is constrained by the ZLB until $t = T$.

5.1 Perfect Stabilization after ZLB or Forward Guidance

Perfect stabilization after ZLB We assume that after $t = T$, monetary policy follows the modified Taylor rule (equation (58)) and achieves perfect stabilization, satisfying $\pi_t = \hat{Q}_t = 0$ for $t \geq T$.[58] From equation (42) and equation (46), the fact that $\hat{Q}_T = \pi_T = 0$ is pinned down implies that there is no volatility for both $\hat{Q}_t$ and π_t processes before T thus $\sigma_t^q = \sigma_t^{q,n} = 0$ and $\sigma_{\pi,t} = 0$ for $t \leq T$.

[57] From equation (36) we can express natural rate r_t^n as a function of fundamental volatility σ_t only. Throughout this Section 5.1 we assume $r^n(\bar{\sigma}) < 0$ and $r^n(\underline{\sigma}) > 0$.

[58] Basically, we assume that monetary policy returns to the modified Taylor rule that includes a risk-premium factor as one of targets as in equation (56) after $t = T$. For $t \leq T$, since $\sigma_t = \bar{\sigma}$ satisfies $r^n(\bar{\sigma}) < 0$, monetary authority cannot implement this rule and is constrained by the ZLB.

Therefore, in this case the risk-adjusted natural rate r_t^T equals the natural rate $\underline{r} = r^n(\bar{\sigma})$ for $t \leq T$ and we get exactly the same dynamics for π_t and $\hat{Q}_t$ as in Werning (2012) and Cochrane (2017). For a reasonable calibration,[59] Figure 5 (dashed black for $\{\hat{Q}_t\}$ and dashed gray for $\{\pi_t\}$) illustrates $\hat{Q}_t$ and π_t dynamics during ZLB crises. Both variables are negative until T and become stabilized after T due to our generalized Taylor rule that targets risk-premium.

Notice that even though we have similar dynamics for $\{\hat{Q}_t, \pi_t\}$ to the ones in Werning (2012) and Cochrane (2017), the forces that drive our results are different. Here, the ZLB constraint causes asset price $\hat{Q}_t$ to fall, as the ZLB is higher than the risk-free rate that is needed for full stabilization, which reduces capitalists' demand for stock market investment. This eventually translates into a reduction of aggregate financial wealth and demand.[60] In canonical New-Keynesian models, on the other hand, the ZLB induces agents to engage in deleveraging and reduce consumption, which collectively lowers the aggregate output through the aggregate-demand externality. Note also that the dynamics in Figure 5 depend on the perfect stabilization after T due to our modified Taylor rule. We get an important lesson from Figure 5: central banks can prevent the appearance of sunspot equilibria at the ZLB by credibly committing to stabilize financial markets at some future date $T < +\infty$. Therefore, even if the monetary authority is unable to temporarily follow the modified Taylor rule in equation (58) due to a binding ZLB, the additional financial stability costs of policy inaction can be contained (indeed, eliminated) by a credible commitment to stabilization upon-ZLB exit. A corollary of this result is that the costs of the ZLB might be highly heterogeneous across countries: countries with a monetary authority committed to financial stabilization will 'only' experience the demand-driven recession outlined above, while countries without that capacity (or willingness) to stabilize financial markets might suffer additional costs from long-lived sunspot shocks to financial volatility.

Now we turn our eyes to the forward guidance policy, in which central banks commit to keep the policy rate at zero for a longer duration than T. In our framework, forward guidance is a powerful tool as in Werning (2012) and Cochrane (2017), with the premise that after the forward guidance is over, central banks return to our modified monetary policy rule that stabilizes the financial market.

Perfect stabilization after forward guidance Now central bank keeps i_t at zero until $\hat{T} > T$. After $\hat{T}$, central bank fully stabilizes the economy with $\hat{Q}_t = \pi_t = 0$ for $t \geq \hat{T}$ with the generalized Taylor rule in equation (56). Due to the same reason exposed in the analysis of ZLB, we know that $\sigma_t^q = \sigma_t^{q,n} = 0$ and $\sigma_{\pi,t} = 0$ for $t \leq \hat{T}$ and therefore, equation (44) and equation (46) characterize

[59] We use parameters: $T = 3, \underline{\sigma} = 0.0090, \bar{\sigma} = 0.2090 \underline{r} = -1.54\%, \bar{r} = 2.82\%$ in simulating our equilibrium throughout this Section 5. The ZLB can be created by not only a spike in σ_t, but also downward jump in g.

[60] This is the other side of Caballero and Farhi (2017). While Caballero and Farhi (2017) showed that a high demand for safe assets drags the economy into recession when monetary policy is constrained, we argue that it causes marginal investors to pull their wealth out of the stock market, thereby reducing stock market wealth and aggregate demand.

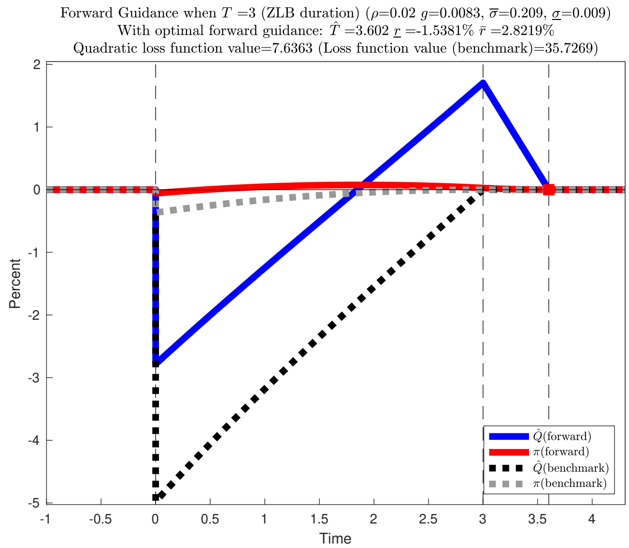


Figure 5: Zero lower bound (ZLB) crisis and forward guidance: $\{\hat{Q}_t, \pi_t\}$ dynamics

$\{\hat{Q}_t, \pi_t\}$ dynamics until $\hat{T}$ with $\hat{Q}_{\hat{T}} = \pi_{\hat{T}} = 0$. Forward guidance in our framework features similar dynamics to the ones in Werning (2012) and Cochrane (2017), but acts through a different mechanism. In our framework, forward guidance is powerful because it raises asset price $\hat{Q}_t$ from T to $\hat{T}$, leading to a rise in $\hat{Q}_t$ even before T, thus increasing the consumption level of capitalists. In traditional New-Keynesian models, forward guidance is useful as it raises household consumption and thus income from T to $\hat{T}$, leading to a rise in consumption before T due to the usual intertemporal substitution channels and general equilibrium effects.

To characterize optimal $\hat{T}$, we minimize the following quadratic loss function with respect to $\hat{T}$,[61]which is derived in Appendix D.

$$L(\{\hat{Q}_t, \pi_t\}_{t\geq 0}) = \mathbb{E}_0 \int_0^\infty e^{-\rho t}(\hat{Q}_t^2 + \Gamma \pi_t^2)dt. \tag{65}$$

Figure 5 (thick blue for $\{\hat{Q}_t\}$ and red for $\{\pi_t\}$) illustrates $\{\hat{Q}_t, \pi_t\}$ dynamics with the optimal forward guidance duration $\hat{T} = 3.602 > T = 3$. $\hat{Q}_t$ and π_t drop less than in cases without forward guidance (dashed black for $\{\hat{Q}_t\}$ and dashed gray for $\{\pi_t\}$) and $\hat{Q}_t$ is even positive for some periods during the ZLB episode due to forward guidance. After $\hat{T} = 3.602$, $\hat{Q}_t$ and π_t are both stabilized since monetary policy becomes active again and targets risk premium in addition to $\{\hat{Q}_t, \pi_t\}$.

Note that the heterogeneous costs of the ZLB reappear under forward guidance: central banks that can credibly commit to stabilize financial markets after $\hat{T}$ do not face any adverse financial volatility costs from the implementation of such policies, and results are unambiguously positive in

[61]With our calibrated parameters, we have $\Gamma = 161.8876$. A basic intuition is that since $\hat{Q}_t$, the asset price gap, drives fluctuations in other business cycle variables, our loss function (equation (65)) has $\hat{Q}_t$ in the position of the output gap.

terms of welfare and business cycle stabilization. This conclusion changes dramatically when a central bank cannot commit to perfect stabilization after $\hat{T}$, as the economy is subject to the appearance of sunspot shocks (which are specially costly at the ZLB). In that scenario, *voluntarily* lengthening the time spent at the ZLB is a risky business: by keeping a passive monetary policy until $\hat{T}$, a central bank risks aggravating the costs stemming from the endogenous financial volatility of the economy -precisely at the moment in which those costs are higher-.[62] We believe this constitutes a novel result on the trade-offs involved in forward guidance policy, and think that it might be one of the reasons behind the cautiousness with which central bankers around the world approach the implementation of such policies *in practice*. In the next Section 5.2, we look into this case where central banks cannot credibly commit to attain perfect stabilization after ZLB or forward guidance, and instead uses a conventional Taylor rule out of the ZLB, only targeting usual mandates.

5.2 Imperfect Stabilization after ZLB or Forward Guidance

For analytic tractability, we assume in this Section 5.2 that inflation is fixed at zero $\pi_t = 0$ for $\forall t$, which corresponds to a perfectly rigid-price economy. First, we consider the usual ZLB case without forward guidance.

Imperfect stabilization after ZLB In this section, we explore the case in which the central bank cannot achieve full stabilization after T and instead relies on the conventional Taylor rule $i_t = \bar{r} + \phi_q \hat{Q}_t$ for $t \geq T$. Under imperfect financial stabilization, sunspots in σ_t^q can appear, and in this section we provide a rational expectations equilibrium of the economy at the ZLB.

We can tackle the problem based on the standard backward induction logic. Assume that a positive sunspot $\sigma_0^q > 0$ arises, based on the fact that the central bank cannot fully stabilize the economy after T. Then, rational agents expect that upon-ZLB exit for $t \geq T$, the economy follows the martingale equilibrium outlined in Section 4.2, with an initial date $t = T$ endogenous volatility σ_T^q. Therefore, they expected the asset price gap $\hat{Q}_t$ path for $t \geq T$ to evolve according to equation (66).

$$\hat{Q}_t = -\frac{(\underline{\sigma} + \sigma_t^q)^2}{2\phi_q} + \frac{(\underline{\sigma})^2}{2\phi_q} \text{ for } t \geq T, \tag{66}$$

with $\sigma_T^q \geq 0$ being stochastic from a $t = 0$ point of view. The asset price gap $\hat{Q}_t$ process for $t < T$ follows

$$d\hat{Q}_t = \left(\underbrace{-\underline{r}}_{>0} + \frac{1}{2}(\bar{\sigma} + \sigma_t^q)^2 - \frac{1}{2}(\bar{\sigma})^2 \right) dt + \sigma_t^q dZ_t. \tag{67}$$

[62] To be clear, even though the scope for forward guidance policies is reduced when the central bank cannot commit to future stabilization, the gains from the policy are still positive, on expectation.

We can integrate equation (67) and obtain $\hat{Q}_t$ for $t \leq T$ as:

$$\hat{Q}_t = -\frac{\mathbb{E}_t\left((\underline{\sigma} + \sigma_T^q)^2\right)}{2\phi_q} + \frac{(\underline{\sigma})^2}{2\phi_q} + \int_t^T \left(\underline{r} - \frac{1}{2}\mathbb{E}_t\left((\bar{\sigma} + \sigma_s^q)^2\right) + \frac{1}{2}(\bar{\sigma})^2\right) ds. \tag{68}$$

As equation (68) must satisfy the dynamic IS equation (equation (67)), we take a total derivative to equation (68) to obtain:

$$-\frac{d\mathbb{E}_t\left(\left(\underline{\sigma} + \sigma_T^q\right)^2\right)}{2\phi_q} \underbrace{-\frac{1}{2}\int_t^T d\mathbb{E}_t\left(\left(\bar{\sigma} + \sigma_s^q\right)^2\right) ds}_{\text{New ZLB term}} = \sigma_t^q dZ_t. \tag{69}$$

Equation (69) is the stochastic differential equation (SDE) that the $\{\sigma_t^q\}$-path starting from $\sigma_0^q > 0$ follows until T during the ZLB. Unfortunately, this SDE has no know analytic solution. Here, we try to understand heuristically how σ_t^q evolves during the ZLB. Note that in equation (69), without the new ZLB term, equation (69) becomes exactly the same as the SDE that characterizes the martingale equilibrium.

We already know from Section 4.1 that outside ZLB episodes, a lower ϕ_q value slows down the stabilization process of σ_t^q in the martingale equilibrium. Therefore, we guess that ZLB actually is very ineffective in stabilizing $\sigma_0^q > 0$ and σ_T^q can still be large compared to σ_0^q. In other words, under ZLB regimes, a positive sunspot σ_0^q is unlikely to disappear until the economy exits the ZLB at $t = T$ and monetary policy becomes active again.

Therefore, in our framework the ZLB raises the welfare costs of business cycle fluctuations in two ways: (i) it brings down asset prices, financial wealth, and aggregate output (level effect), and (ii) it keeps σ_t^q sunspots alive, making business cycle more volatile (volatility effect). Therefore, the inability of conventional monetary policy to intervene at the ZLB and prevent financial disruption (in terms of endogenous volatility σ_t^q) supposes an additional business cycle cost, in addition to the level effect.

In this case, $\hat{Q}_t$-dynamics at the ZLB can be illustrated as in Figure 6. The solid blue line corresponds to the equilibrium with perfect stabilization after T, while the two dashed lines correspond to the other case where the central bank uses the conventional Taylor rule and $\sigma_0^q > 0$ appears at $t = 0$.

We observe: (i) compared with the perfect stabilization case (solid blue), paths under imperfect future stabilization feature lower $\{\hat{Q}_t\}$ levels, as financial volatility $\{\sigma_t^q > 0\}$ starting from $\sigma_0^q > 0$ raises risk-premium and additionally depresses the levels of $\{\hat{Q}_t\}$. (ii) $\{\hat{Q}_t\}$ is stochastic, and $\{\sigma_t^q\}$ converges slowly to zero during the ZLB and faster after $t \geq T$ as monetary policy becomes active again and responds to $\{\hat{Q}_t\}$ following the conventional Taylor rule. The dashed lines in Figure 6

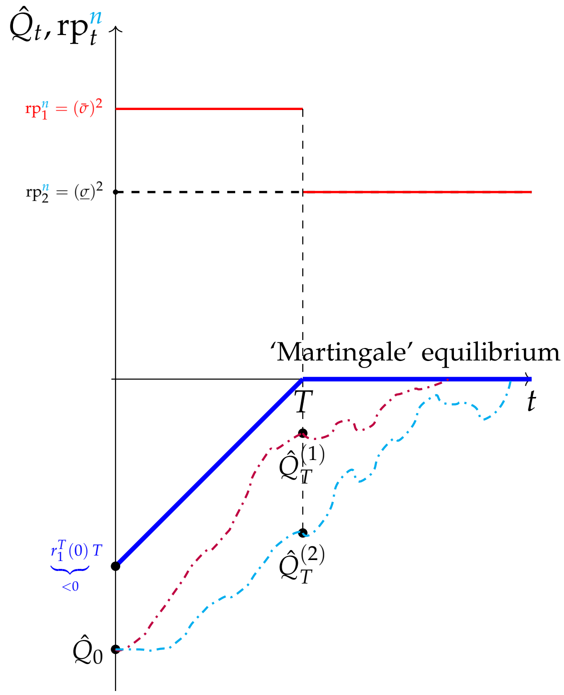


Figure 6: ZLB dynamics (Taylor rule after T) with initial sunspot $\sigma^q_0 > 0$

correspond to two possible sample paths with stochastic $\hat{Q}_T$ ($\hat{Q}^{(1)}_T$ or $\hat{Q}^{(2)}_T$) from $t = 0$ perspective.

Mathematical explanation In the martingale equilibrium out of the ZLB (for $t \geq T$), $\{\sigma^q_t\}$ starting from σ^q_T follows:

$$d\sigma^q_t = -(\phi_q)^2 \frac{(\sigma^q_t)^2}{2(\underline{\sigma} + \sigma^q_t)^3} dt - \phi_q \frac{\sigma^q_t}{\underline{\sigma} + \sigma^q_t} dZ_t, \ \ \forall t \geq T. \tag{70}$$

Here, we provide a very heuristic explanation of why σ^q_t does not fall, in general, during ZLB regimes. Our strategy is to start with the σ^q_t-process in equation (70) and how it should be modified to satisfy equation (69) at the ZLB (for $t < T$).

First, for simplicity, we replace equation (69) with

$$-\frac{d\left(\left(\underline{\sigma} + \sigma^q_t\right)^2\right)}{2\phi_q} - \frac{1}{2}\int_t^T d\left(\left(\bar{\sigma} + \sigma^q_t\right)^2\right) ds = \sigma^q_t dZ_t, \tag{71}$$

where we replace $\mathbb{E}_t((\underline{\sigma} + \sigma^q_T)^2)$ by $(\underline{\sigma} + \sigma^q_t)^2$ (which holds with equation (70)) and $\mathbb{E}_t((\bar{\sigma} + \sigma^q_s)^2)$ by $(\bar{\sigma} + \sigma^q_t)^2$ (which does not hold technically, but allows us to obtain some intuitions as equation (71) is simpler than equation (69)). Since

$$d\left((\bar{\sigma}+\sigma_t^q)^2\right) = \underbrace{\left(\phi_q \frac{\sigma_t^q}{\underline{\sigma}+\sigma_t^q}\right)^2 \left(-\frac{\bar{\sigma}+\sigma_t^q}{\underline{\sigma}+\sigma_t^q}+1\right)}_{<0} dt - 2\phi_q \left(\frac{\bar{\sigma}+\sigma_t^q}{\underline{\sigma}+\sigma_t^q}\right) \sigma_t^q dZ_t, \tag{72}$$

therefore, the $(\bar{\sigma}+\sigma_t^q)^2$-process has a negative drift, which by equation (71) implies that $(\underline{\sigma}+\sigma_t^q)^2$ must have a positive drift because the $\sigma_t^q dZ_t$-process does not contain a drift term. As the $\{\sigma_t^q\}$ process in equation (70) implies $(\underline{\sigma}+\sigma_t^q)^2$ is a martingale without drift, the new process that satisfies equation (71) must have a higher drift than equation (70), which implies a slower stabilization under ZLB than in the martingale equilibrium in equation (70).

Imperfect stabilization after forward guidance With the lesson that keeping the policy rate at zero does not help stabilize the financial volatility sunspot in mind, we consider the case in which central banks return to the conventional Taylor rule $i_t = \bar{r} + \phi_q \hat{Q}_t$ after its forward guidance program ends at $\hat{T}$. Forward guidance has two countervailing effects. (i) It raises asset prices, aggregate financial wealth, and aggregate demand until $\hat{T}$ (level effect). (ii) slows the stabilization of financial volatility σ_t^q between $T \le t \le \hat{T}$ by prolonging the period of policy inaction, thereby generating a welfare loss (volatility effect). Figure 7 illustrates those two opposite forces generated by forward guidance under imperfect stabilization with an initial sunspot $\sigma_0^q > 0$. Here, the solid gray line represents the $\hat{Q}_t$ path with perfect stabilization after $\hat{T}$, while the dashed lines represent stochastic paths under imperfect future stabilization after $\hat{T}$.

In the next Section 5.4, we turn our attention to the study of possible macroprudential interventions from the fiscal side to raise asset prices $\hat{Q}_t$ and inflation π_t during a ZLB crisis.

5.3 Intertemporal Stabilization Trade-off with Commitment

During the Global Financial Crisis and afterwards, central banks around the world purchased large amounts of assets in financial markets, which mitigated collapses in asset prices and brought down levels of risk-premia for a variety of assets.[63] As our framework's Ricardian structure does not allow the central bank's balance sheet quantities to affect the equilibrium, we turn our eyes to a different

[63] See Gagnon et al. (2010), Krishnamurthy and Vissing-Jorgensen (2011), and Gorodnichenko and Ray (2018) for empirical evidence on how the Fed's QE1 and QE2 programs affected asset prices and risk-premia in financial markets. For example, Gagnon et al. (2010) presented evidence that asset purchases led to reductions in interest rates on a range of securities, which reflects reductions in the levels of risk premia. Krishnamurthy and Vissing-Jorgensen (2011) found that QE1, which involved large purchases of agency-backed MBSs, yielded huge reductions in mortgage rates, while QE2, in contrast, involved only Treasury purchases and brought down Treasury bond rates only. Gorodnichenko and Ray (2018) found: the more disrupted financial markets are, asset purchases act more strongly as local demand shocks in financial markets.

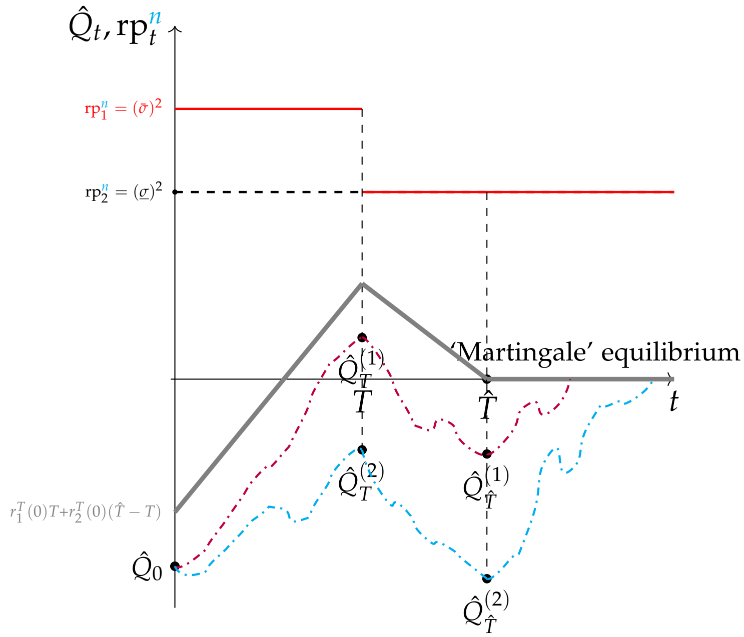


Figure 7: ZLB dynamics with initial sunspot $\sigma^q_0 > 0$ and forward guidance until $\hat{T} > T$

type of policy that can prop up asset markets and the business cycle: a central bank commitment to passive financial stabilization *in the future* in exchange for lower financial volatility *at* the ZLB.

To obtain sharper analytic results, in the following Section 5.3.1 we consider the rigid-price case with no inflation (i.e. $\pi_t = 0$ for $\forall t$). In Section 5.3.2 we consider the 3-equations sticky price model (equation (42), equation (46), equation (56)) and extend our logic from Section 5.3.1. In this Section 5.3, we assume the same ZLB environment as in Section 5.1.

5.3.1 Rigid Price Case

We assume: $\pi_t = 0$, $\forall t$, which corresponds to the limit case with zero price-resetting probability, instead of the Phillips curve (equation (46)). This simplification allows us to derive sharper analytic implications about the optimal commitment path.

General idea We discuss a possible central bank's commitment path at the ZLB which brings the higher expected welfare than the conventional forward guidance studied in the previous sections. For that purpose, we make a strong assumption: *central bank can choose the path of endogenous financial volatility* $\{\sigma^q_t\}$ *if that path is consistent with the dynamic IS equation (equation* (42)*) and the Phillips curve (equation* (46)*).* In the forward guidance path we described in Section 5.1, the fact that the central bank achieves perfect stabilization after $\hat{T}$ (in Figure 5) determines financial volatility levels during the ZLB, including the forward guidance period (between T and $\hat{T}$). To be specific, we derived that $\sigma^q_t = \sigma^{q,n}_t = 0$ for both $T \le t \le \hat{T}$ (forward guidance period) and $t < T$ (ZLB period). Therefore, the risk-premium level before $\hat{T}$ is completely determined to be the same as its natural correspondent rp_t^n and we have $\hat{rp}_t = 0$ for $t \leq \hat{T}$. This logic can be illustrated by the following diagram.

$$\boxed{\text{1. Central bank achieves perfect stabilization after } \hat{T} \ (\hat{Q}_t = 0, \forall t \geq \hat{T})}$$

$$\downarrow$$

$$\boxed{\text{2. } \hat{Q}_{\hat{T}} = 0 \text{ guarantees } \sigma_t^q = \sigma_t^{q,n} = 0,\ \text{rp}_t = \text{rp}_t^n \text{ for } t \leq \hat{T}}$$

Figure 8: Mechanism under the Forward Guidance

We ask the following question: What if central bank can commit to forgo full stabilization after the forward guidance period (after $\hat{T}$ in Figure 5) while engineering an equilibrium path with lower levels of risk-premium rp_t both during the ZLB (until T) and the forward guidance period (from T to $\hat{T}$)? Specifically, from the central bank's point of view, the risk premium level, which is determined by the volatility of $\{A_t\}$ process, is too high during the ZLB episode and it causes the asset price to plummet, bringing the economy into a harsh recession. Thus, central banks might conclude that pushing down σ_t^q levels (or equivalently, risk-premium levels) from $\sigma^{q,n} = 0$ will mitigate the crisis during the ZLB by propping up asset price levels, which eventually stimulate the aggregate demand. This policy can be related to the central bank's efforts to supply liquidity to financial markets through LSAP policies, recapitalization and/or government guarantee of financial entities, and many other bail-out policies aimed (in part) at reducing financial uncertainty, whereas in our model, this reduction in uncertainty comes from the central bank's commitment to deviate from the perfectly stabilized path in the future (after $\hat{T}$).

The possibility that σ_t^q differs from $\sigma_t^{q,n} = 0$ causes asset price gap $\hat{Q}_t$ to fluctuate in a stochastic manner until $\hat{T}$[64] and therefore, the central bank cannot attain $\hat{Q}_{\hat{T}} = 0$ surely. Thus, the conjectured engineered path in which $\{\sigma_t^q\}$ is pushed down below $\{\sigma_t^{q,n} = 0\}$ until $\hat{T}$ for stabilization purpose would be successful only if the central bank commits ex-ante not to pursue perfect stabilization even after the forward guidance period is over and the economy returns to normal.[65,66] This logic can be interpreted as a contrapositive to the one in Figure 8 and is illustrated by Figure 9.

[64] From equation (42), $\sigma_t^q \neq \sigma^{q,n} = 0$ creates a stochastic movement in the asset price gap process $\{\hat{Q}_t\}$ until $\hat{T}$.

[65] It can be understood as follows: a σ_t^q different from $\sigma^{q,n} = 0$ until $\hat{T}$ means that the stock market becomes separated from what is stipulated by the real economy. It will eventually lead to a failure of the monetary authority to stabilize the economy at $\hat{T}$, the moment when forward guidance ends. If we interpret it in a backward way, it implies: when the central bank tries to engineer business cycle paths while bringing down the risk-premium it deems 'too high' during ZLB, it must consider how these paths would trouble the economy after forward guidance is over. This feature arises as financial market and the real economy are connected with each other in our model.

[66] For example, we assume that after $\hat{T}$ the central bank uses the passive policy rule with just $i_t = r^n(\underline{\sigma})$, creating a possibility of multiple equilibria after $\hat{T}$. In this Section 5.3.1, we select one equilibrium in which we have $\sigma_t^q = \sigma^{q,n} = 0$ after forward guidance ends (after $\hat{T}$ in Figure 5). Therefore, $\hat{Q}_t$ will remain at $\hat{Q}_{\hat{T}}$ after $t \geq \hat{T}$ with $i_t = r^n(\underline{\sigma})$.

In other words, when engineering an equilibrium path, the central bank must juggle between boosting the economy during ZLB by lowering the financial volatility $\{\sigma_t^q\}$ and risk-premium rp_t, and the perfect stabilization after the fundamentals return to normal, thus effectively future stability for present stabilization at the ZLB.

$$\boxed{\neg 2.\ \sigma_t^q < \sigma_t^{q,n} = 0,\ \mathrm{rp}_t < \mathrm{rp}_t^n \text{ for } t \leq \hat{T}}$$

$$\downarrow$$

$$\boxed{\neg 1.\ \hat{Q}_{\hat{T}} \neq 0. \text{ Central bank commits not to perfectly stabilize the economy after } \hat{T}}$$

Figure 9: Financial Market Intervention and Stabilization

One thing to notice is that the central bank would like to reduce the financial volatility level and the risk-premium even after the TFP volatility σ_t returns to $\underline{\sigma}$ (from T to $\hat{T}$) while it still follows the forward guidance rate prescription ($i_t = 0$). It is because it would push up $\hat{Q}_t$ between $T \leq t \leq \hat{T}$, which further raises the asset price $\hat{Q}_t$ during high-TFP volatility periods ($\sigma_t = \bar{\sigma}$ for $t \leq T$) as $\hat{Q}_t$ is forward-looking. Since we have introduced another way (manipulating financial market volatility) to stimulate the economy during and after ZLB, forward guidance is not as necessary to prop up $\hat{Q}_t$ as in Section 5.1, thus its duration must decrease with this intervention.

To be more formal, we define $\mathrm{rp}_1^n \equiv (\bar{\sigma})^2$, $\mathrm{rp}_2^n \equiv (\underline{\sigma})^2$, and $\mathrm{rp}_3^n \equiv (\underline{\sigma})^2$, the risk-premium levels (i.e., $(\sigma_t + \sigma_t^q)^2$) in time-intervals $t \leq T$, $T < t \leq \hat{T}$, and $t \geq \hat{T}$, respectively when the central bank perfectly stabilizes the economy after ZLB. We define $\hat{T}'$ to be the new forward guidance duration under the newly engineered path, which is possibly different from $\hat{T}$, the original forward guidance duration. For simplicity, we assume that the central bank maintains the same financial volatility and risk-premium levels in the same regime: specifically, financial volatility σ_t^q is $\sigma_1^{q,L}$ from 0 to T (High TFP volatility region), $\sigma_2^{q,L}$ from T to $\hat{T}'$ (Low TFP volatility region with the forward guidance), and 0 after $\hat{T}'$ (Low TFP volatility region). The assumption that $\sigma_t^q = 0$ after $\hat{T}'$ means that the central bank does not manipulate financial markets after $\hat{T}'$ when the forward guidance ends.

Therefore, under this newly engineered path, the risk-premium levels $\mathrm{rp}_t = (\sigma_t + \sigma_t^q)$ in each time-interval become $\mathrm{rp}_1 \equiv (\bar{\sigma} + \sigma_1^{q,L})^2 < \mathrm{rp}_1^n$ for $t \leq T$, $\mathrm{rp}_2 \equiv (\underline{\sigma} + \sigma_2^{q,L})^2 < \mathrm{rp}_2^n$ for $T \leq t \leq \hat{T}'$, and $\mathrm{rp}_3 \equiv (\underline{\sigma})^2$ after $\hat{T}'$.[67] Since the policy intervention lowers the economy's total risk, risk-premium levels fall and both asset price and business cycle levels rise in response. From equation (45), we can express the risk-adjusted natural rates r_1^T and r_2^T that enter the dynamic IS equation (equation (42))

[67]We will eventually prove $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma^{q,n} = 0$ at optimum. For illustration purposes, we assume these conditions are satisfied in the rest of the argument in Section 5.3.1.

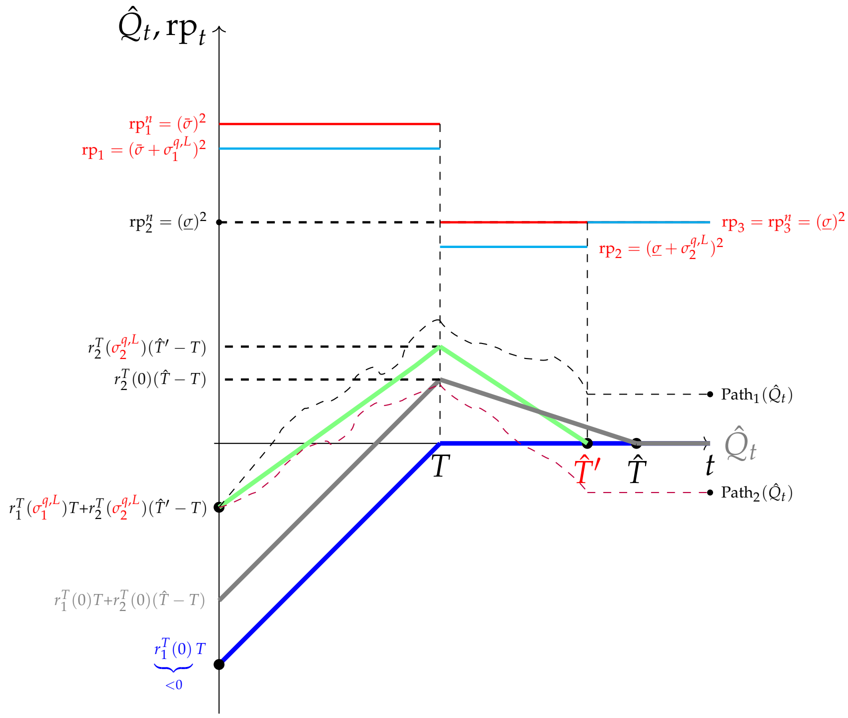


Figure 10: Possible Intervention Dynamics of $\{\hat{Q}_t\}$ with $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$

as functions of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, respectively, as

$$\begin{aligned} r_1^T(\sigma_1^{q,L}) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} > \underline{r} \equiv r^n(\bar{\sigma}) = r_1^T(0) \text{ when } \sigma_1^{q,L} < 0, \\ r_2^T(\sigma_2^{q,L}) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma} + \sigma_2^{q,L})^2}{2} > \bar{r} \equiv r^n(\underline{\sigma}) = r_2^T(0) \text{ when } \sigma_2^{q,L} < 0. \end{aligned} \tag{73}$$

We observe $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ imply $r_1^T > \underline{r}$ and $r_2^T > \bar{r}$, respectively, which yield higher levels of $\{\hat{Q}_t\}$ during and after the ZLB on average. That would be the reason the central bank wants to push down $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ from zero, the natural level of financial volatility, but from equation (42) we see that it creates a stochastic fluctuation of $\hat{Q}_t$, which brings additional costs in terms of stabilization. Therefore, central bank faces a trade-off between future and current stability when it engineers the new commitment path.

To pin down equilibrium paths (there are multiple equilibria and we need to select one equilibrium), we assume that, at $t = 0$, the monetary authority anchors an expected value of $\hat{Q}_{\hat{T}'}$, the asset price gap level when the forward guidance is over, at zero, i.e., $\mathbb{E}_0 \hat{Q}_{\hat{T}'} = 0$.[68] In Figure 10, the gray line is the original forward guidance path, while the green one is the path of average (or determin-

[68]It does not have to be the asset price gap, since the other gap variables are all proportional to $\hat{Q}_t$.

istic component of) $\{\hat{Q}_t\}$ when the central bank engineers a new path with $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$.[69] As we now have $\sigma_1^{q,L} \neq 0$ and $\sigma_2^{q,L} \neq 0$, stochastic fluctuations of $\{\hat{Q}_t\}$ around the deterministic path are generated and illustrated by two possible sample paths (dashed lines) in Figure 10. These stochastic fluctuations bring the welfare loss. We also observe that: with $\sigma_2^{q,L} < 0$ and $\mathbb{E}_0 \hat{Q}_{\hat{T}'} = 0$, the average level of $\hat{Q}_t$ becomes higher from T to $\hat{T}'$ than in the gray forward guidance path. The rises in $\hat{Q}_t$ from T to $\hat{T}'$ bring up $\hat{Q}_t$ levels before T. In addition, $\sigma_1^{q,L} < 0$ for $t \leq T$ further props up the average level of $\hat{Q}_t$ during the high TFP volatility period (before T) and thus $\hat{Q}_0$ falls less than in the (gray) forward guidance path in Figure 10.

In sum, this type of financial market intervention must exploit a trade-off between higher asset price and output levels before $\hat{T}'$ and the future stabilization after the ZLB. Central banks should balance this trade-off when manipulating $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}')$. Now, the next step is to check whether our conjecture-based analysis in Section 5.3.1 indeed holds as the optimal commitment solution.

Central bank's optimal commitment path The central bank chooses optimal $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}'$ to minimize the loss function in equation (65), with functions $r_1^T(\cdot)$ and $r_2^T(\cdot)$ defined in equation (73). We assume that the central bank keeps $i_t = 0$ until $\hat{T}'$. Therefore, $\hat{T}'$ is our new forward guidance period as we explained above. After $\hat{T}'$, it implements a *passive monetary policy* (interest rate anchoring) with $i_t = r_2^T(0) = r^n(\underline{\sigma})$, not seeking to stabilize the economy. In sum, central bank solves the following optimization problem:

$$\min_{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}'} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left(\hat{Q}_t\right)^2 dt, \;\; \text{s.t.} \;\; \begin{cases} d\hat{Q}_t = -(\underbrace{r_1^T(\sigma_1^{q,L})}_{<0})dt + (\sigma_1^{q,L})dZ_t, \;\; \text{for } t < T, \\ d\hat{Q}_t = -(\underbrace{r_2^T(\sigma_2^{q,L})}_{>0})dt + (\sigma_2^{q,L})dZ_t, \;\; \text{for } T \leq t < \hat{T}', \\ d\hat{Q}_t = 0, \;\; \text{for } t \geq \hat{T}', \end{cases} \tag{74}$$
$$\text{with } \hat{Q}_0 = \underbrace{r_1^T(\sigma_1^{q,L})}_{<0} T + r_2^T(\underbrace{\sigma_2^{q,L}}_{>0})(\hat{T}' - T).$$

The forward guidance path (Section 5.1 with $\pi_t \equiv 0$) corresponds to $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}') = (0, 0, \hat{T})$ case, and thus optimal $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}')$ combination yields the lower level of (quadratic) loss function. It turns out our conjecture in Section 5.3.1 that we have $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$ at optimum is correct, as summarized by the next Proposition 7.

Proposition 7 (Optimal Commitment Path) *The solution of the central bank's optimization program in equation (74) features $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$ hold.*

[69] When we draw the deterministic path in Figure 10, we ignore the fact that we have stochastic fluctuations of $\hat{Q}_t$.

We solve equation (74) with $T = 4.5$ and parameters in Table 3, and simulate optimal commitment paths. We calculate a sample estimate of the loss function with:

$$\int_0^\infty e^{-\rho t}\mathbb{E}_0(\hat{Q}_t^2)dt \simeq \int_0^\infty e^{-\rho t}\frac{1}{s}\sum_{i=1}^{s}(\hat{Q}_t^{(i)})^2 dt. \tag{75}$$

where $\hat{Q}_t^{(i)}$ is the i^{th} realized sample path.[70] Our result reveals when $\sigma_1^{q,L}$, $\sigma_2^{q,L}$ and $\hat{T}$ are chosen optimally, the loss value is reduced by 0.4239%, which constitutes a moderate gain. In our simulation, we observe $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}' < \hat{T}$ all hold at optimum,[71] which aligns with Proposition 7, but with optimal volatilities of very small magnitudes. The reason that the optimal commitment path features very small degrees of financial volatility (and risk-premium) manipulation is because after T, there is no ZLB possibility at all, which raises the cost of destabilization when central bank's monetary policy becomes passive after $\hat{T}'$. In a more realistic setting, there exists the possibility that the economy gets trapped at the ZLB in a stochastic way, which can raise the degree to which the central bank manipulates financial market volatilities and risk-premia. This exercise clearly illustrates concerns that a central bank must consider when it tries to change the business cycle path by manipulating financial market variables at the ZLB.

5.3.2 Sticky Price Case

In cases where inflation π_t evolves according to the Phillips curve (equation (46)), we still preserve the logic of intervention from Section 5.3.1. Since we assume that central bank does not pursue full stabilization after $\hat{T}'$ based on interest rate anchoring, there arise multiple equilibria in $\{\hat{Q}_t, \pi_t\}$ and we focus on the particular equilibrium where we have $\sigma_{\pi,t} = 0$, $\forall t$ in equation (46). As in Section 5.3.1, we assume that the central bank manipulates the financial volatility levels, and $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ are those levels at the ZLB ($t \leq T$) and in the forward guidance ($T \leq t \leq \hat{T}'$) region, respectively, that central bank targets while it tries to engineer a new commitment path.

Before we state the central bank's optimization problem, we first consider the dynamics of $\{\hat{Q}_t\}$ without its stochastic component ($\sigma_t^q dZ_t$ in equation (42)), where we assign superscript c to denote it is a counterfactual. We do so in order to identify the pure benefits that central bank's manipulation of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ brings in terms of welfare. In this particular counterfactual path, we assume that after $\hat{T}'$, the economy is fully stabilized so $\hat{Q}_t^c = \pi_t^c = 0$ for $\forall t \geq \hat{T}'$. Then we have the following IS equations, which together with our Phillips curve (equation (46) with $\sigma_{\pi,t} = 0$ for $\forall t$), characterize

[70] We use $s = 1000$ number of sample paths in our simulation.

[71] Our simulation yields $\hat{T}' = 5.612 < \hat{T} = 5.614$, $\sigma_1^{q,L} = -1.4325 \times 10^{-4} < 0$, and $\sigma_2^{q,L} = -1.0467 \times 10^{-6} < 0$.

the complete $\{\hat{Q}^c_t, \pi^c_t\}$ dynamics.

$$d\hat{Q}^c_t = \begin{cases} -(r^T_1(\sigma^{q,L}_1) + \pi^c_t)dt & \text{for } t < T, \\ -(r^T_2(\sigma^{q,L}_2) + \pi^c_t)dt & \text{for } T \leq t < \hat{T}', \end{cases} \quad \text{and } \hat{Q}^c_t = \pi^c_t = 0, \text{ for } t \geq \hat{T}'. \tag{76}$$

Observe $\sigma^{q,L}_1 < 0$ and $\sigma^{q,L}_2 < 0$ imply $r^T_1(\sigma^{q,L}_1) > r^T_1(0) = r^n(\bar{\sigma})$ and $r^T_2(\sigma^{q,L}_2) > r^T_2(0) = r^n(\underline{\sigma})$. Therefore, the above counterfactual path in general yields a higher welfare than the forward guidance path in Section 5.1, as falls in financial volatilities and risk-premia mitigate the severity of ZLB. This welfare gain constitutes the benefit that the central bank's manipulation of asset price volatilities $(\sigma^{q,L}_1, \sigma^{q,L}_2)$ brings to the economy. We express the path $\{\hat{Q}^c_t(\sigma^{q,L}_1, \sigma^{q,L}_2), \pi^c_t(\sigma^{q,L}_1, \sigma^{q,L}_2)\}$ satisfying equation (76) as a function of $\sigma^{q,L}_1$ and $\sigma^{q,L}_2$, as equation (76) is deterministic. The more $\sigma^{q,L}_1$ and $\sigma^{q,L}_2$ are reduced from their natural levels, the more the levels of $\hat{Q}^c_t(\sigma^{q,L}_1, \sigma^{q,L}_2)$ and $\pi^c_t(\sigma^{q,L}_1, \sigma^{q,L}_2)$ rise, in a similar manner to Cochrane (2017).

Now, we consider the case where there is a stochastic fluctuation term ($\sigma^q_t dZ_t$ in equation (42)). We assume that the path starts at $\{\hat{Q}^c_0(\sigma^{q,L}_1, \sigma^{q,L}_2), \pi^c_0(\sigma^{q,L}_1, \sigma^{q,L}_2)\}$ given the financial volatility levels $\sigma^{q,L}_1$ and $\sigma^{q,L}_2$ that central bank manipulates.[72] And we assume that after $\hat{T}'$ central bank follows a passive monetary policy rule (interest rate anchoring) $i_t = r^n(\underline{\sigma}) = r^T_2(0)$, committing not to seek to stabilize the economy, as in Section 5.3.1,

Therefore, we can write the central bank's optimization problem in the following way.

$$\min_{\sigma^{q,L}_1, \sigma^{q,L}_2, \hat{T}'} \mathbb{E}_0 \int_0^\infty e^{-\rho t}(\hat{Q}^2_t + \Gamma \pi^2_t)dt, \text{ s.t. } \begin{cases} d\hat{Q}_t = -(\underbrace{r^T_1(\sigma^{q,L}_1)}_{<0} + \pi_t)dt + \sigma^{q,L}_1 dZ_t, & \text{for } t < T, \\ d\hat{Q}_t = -(\underbrace{r^T_2(\sigma^{q,L}_2)}_{>0} + \pi_t)dt + \sigma^{q,L}_2 dZ_t, & \text{for } T \leq t < \hat{T}', \\ d\hat{Q}_t = (-\pi_t)dt, & \text{for } t \geq \hat{T}', \\ d\pi_t = (\rho \pi_t - \kappa \hat{Q}_t)dt, & \text{for } \forall t, \end{cases}$$
$$\text{with } \hat{Q}_0 = \hat{Q}^c_0(\sigma^{q,L}_1, \sigma^{q,L}_2) \text{ and } \pi_0 = \pi^c_0(\sigma^{q,L}_1, \sigma^{q,L}_2). \tag{77}$$

As it is not possible to analytically characterize optimal $\sigma^{q,L}_1$, $\sigma^{q,L}_2$, and $\hat{T}'$, we rely on numerical simulation[73] to check whether our intuitions in Section 5.3.1 still hold in the sticky price case. The

[72]It is similar to our assumption in Section 5.3.1 that central bank anchors $\mathbb{E}_0 \hat{Q}_{\hat{T}'} = 0$. Here, initial levels of $\hat{Q}_0, \pi_0$ are those that arise when there is no stochastic fluctuation in both variables. This way of selecting an equilibrium helps us to separate benefits and costs that the optimal commitment path brings, and deal with each part carefully.

[73]In particular, we use the following approximation similar to equation (75):

$$\int_0^\infty e^{-\rho t}\mathbb{E}_0(\hat{Q}^2_t + \Gamma \pi^2_t)dt \simeq \int_0^\infty e^{-\rho t}\frac{1}{s}\sum_{i=1}^{s}((\hat{Q}^{(i)}_t)^2 + \Gamma(\pi^{(i)}_t)^2)dt, \tag{78}$$

simulation result confirms our intuition that at the optimal commitment equilibrium central bank is better off by pushing down $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ from zero, which is the natural level of financial volatility, and the forward guidance duration $\hat{T}'$ is shortened from $\hat{T}$, the original duration.[74,75] Basically, when the central bank tries to engineer a new commitment path by controlling the financial volatility level and the endogenous risk-premium, it must trade-off some future stability (after the ZLB) to stabilize the present economy (during the ZLB), and accept the future destabilizing effects that the policy entails.

5.4 Macroprudential Policies

In Section 5.4, we analyze two types of stimulative macroprudential policies at the ZLB. First, we introduce a fiscal subsidy that incentivizes capitalists to bear higher levels of risks, effectively raising asset price and other real activities. Second, we study direct fiscal redistribution from capitalists to hand-to-mouth workers whose marginal propensity to consume (MPC) is much higher than former. We show that this policy raises total dividends of the stock market and eventually, asset price $\hat{Q}_t$ and consumption. To isolate the effects of macroprudential policies on the business cycle, forward guidance policy is not considered in this Section 5.4.

5.4.1 Fiscal Subsidy on Stock Market Investment

For $t \in [0, T]$, monetary policy is constrained by the ZLB, and the risk-premium level demanded by capitalists puts a downward pressure on $\hat{Q}_t$ and π_t through the aggregate demand externality that financial decisions of individual capitalist exert.[76] In this section, we develop a subsidy policy that induces capitalists to increase their demand for risky stocks, which raises the aggregate asset price level Q_t and corrects the aggregate demand externalities affecting the economy.

We start by considering a government subsidy for the purchase of risky stock market assets[77], and which is funded by imposing a lump-sum tax on capitalists. In specific, instead of the original

where $\hat{Q}_t^{(i)}, \pi_t^{(i)}$ are i'th sample paths of $\{\hat{Q}_t, \pi_t\}$. We use $s = 1000$ for the number of sample paths.

[74] With parameters in Section 5.1, the optimal forward guidance duration becomes $\hat{T}' = 3.601 < \hat{T} = 3.602$, and we have $\sigma_1^{q,L} = -1.0491 \times 10^{-5} < 0$ and $\sigma_2^{q,L} = -9.7365 \times 10^{-6} < 0$. The loss function value drops by 0.03%.

[75] As we emphasize, the whole point of this exercise is not to accurately quantify the actual benefits of this type of stock market intervention as is done in quantitative DSGE models, but to convey the key intuitions that must be taken into account in conjunction with our monetary policy framework.

[76] A number of papers have been written over relations between externalities (either pecuniary or aggregate-demand) and macroprudential policies. See Caballero and Krishnamurthy (2001), Lorenzoni (2008), Farhi et al. (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010), Stein (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), Dàvila and Korinek (2018) among others.

[77] In our model, a stock market subsidy is equivalent to a tax break on capital income, which is most often the policy implemented *in practice* by governments. We model this policy using the 'subsidy' version in order to economize on notation.

expected return i_t^m, capitalists get $(1+\tau_t)i_t^m$ in expectation out of a 1$ stock market investment. If the government imposes a T_t lump-sum tax on capitalists to finance this subsidy, then each capitalist with nominal wealth a_t solves the following optimization problem.

$$\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \text{ s.t. } da_t = (a_t(i_t + \theta_t((1+\tau_t)i_t^m - i_t)) - p_t C_t - T_t)dt + \theta_t a_t(\sigma_t + \sigma_t^q)dZ_t. \tag{79}$$

In equilibrium, capitalists end up paying a tax amount $T_t = \tau_t a_t \theta_t i_t^m$ to finance the subsidy on their own stock market investment. Imposing $\theta_t = 1$ as the equilibrium condition, the equilibrium stock market's expected return can be expressed as

$$i_t^m = \frac{i_t + (\sigma_t + \sigma_t^q)^2}{1+\tau_t} = \underbrace{\rho}_{\text{Dividend yield}} + \underbrace{g + \mu_t^q + \sigma_t \sigma_t^q}_{\text{Capital gain}}. \tag{80}$$

Note that given $i_t = 0$ (ZLB), the presence of subsidy $\tau_t > 0$ pushes down the equilibrium levels of i_t^m and μ_t^q, boosting the current $\hat{Q}_t$ until T. Therefore, this policy mitigates the severity of recessions, as summarized by the following Proposition 8. Notice that when $\tau_t \to \infty$, we immediately escape from the ZLB crisis and return to the fully stabilized economy.[78]

Proposition 8 (Fiscal Subsidy on Stock Market (Expected) Return) *Under the ZLB environment of 5.1 with the fiscal subsidy $\tau_t > 0$ on the expected stock market return, a dynamic IS equation for $\hat{Q}_t$ can be written as*

$$d\hat{Q}_t = -\Big(\underbrace{\underline{r}}_{\equiv r^n(\bar{\sigma})<0} + \underbrace{\frac{\tau_t}{1+\tau_t}(\bar{\sigma})^2}_{>0} + \pi_t \Big) dt. \tag{81}$$

Since the central bank fully stabilizes the economy after T, we have $\sigma_t^q = \sigma^{q,n} = 0$ for all t.

In equation (81), a positive $\tau_t > 0$ raises the effective natural rate from $\underline{r}$ to $\underline{r} + \frac{\tau_t}{1+\tau_t}(\bar{\sigma})^2$, reducing the gap between the ZLB and the effective natural rate. The following Figure 11 confirms recessionary pressures at the ZLB are alleviated and both $\hat{Q}_t$ and π_t drop by less upon entering the ZLB, as we raise the subsidy rate τ_t.

A subsidy on risky asset holdings (or equivalently, a tax cut on capital gains) effectively raises capitalists' stock market demand, stimulating both financial markets and real activity at the ZLB.

Tax on whom? What if government instead imposes a lump-sum tax T_t on hand-to-mouth workers?

[78] Since we always have $\underline{r} + (\bar{\sigma})^2 > 0$, with $\tau_t \to \infty$, the effective natural rate becomes positive, and therefore, central bank can attain full stabilization.

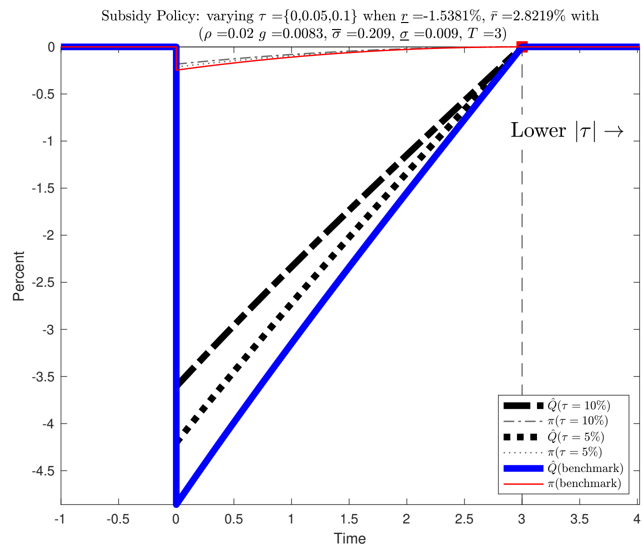


Figure 11: Zero lower bound (ZLB) crisis with varying τ_t rates: $\{\hat{Q}_t, \pi_t\}$ dynamics

In this case, each worker's budget constraint changes as

$$\frac{w_t}{p_t} N_{W,t} = C_{W,t} + \frac{T_t}{p_t}. \tag{82}$$

As workers are hand-to-mouth, this taxation reduces their consumption one-by-one and negatively impacts the stock market's total dividend amount and stock price $\hat{Q}_t$. Therefore, with a lump-sum tax on workers the stock market's expected return i_t^m is represented as

$$i_t^m = \frac{y_t - \frac{w_t}{p_t} N_{W,t}}{A_t Q_t} + \mathbb{E}_t \left(\frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \frac{1}{dt} \right) = \rho - \tau_t i_t^m + \mathbb{E}_t \left(\frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \frac{1}{dt} \right). \tag{83}$$

where we plug $T_t = \tau_t i_t^m p_t A_t Q_t$ into the worker's budget constraint (equation (82)).[79] As it will turn out, the negative effect on $\hat{Q}_t$ from drops in workers' consumption (from the tax) exactly cancels out with the positive effect of the stock subsidies on capitalists. Therefore, there is no net effect (beyond pure redistribution from workers to capitalists) of the subsidy policy on $\{\hat{Q}_t, \pi_t\}$ dynamics during a ZLB episode when it is financed by a tax on workers. The following Proposition 9 summarizes this point.

Proposition 9 (Fiscal Subsidy and Tax on Workers) *The policy of Section 5.4.1 that subsidizes the expected return on risky stocks, if financed through a lump-sum taxation on workers, would have no effect on $\{\hat{Q}_t, \pi_t\}$ dynamics during the ZLB. Such policy features the same dynamics as in Figure 5.*

[79] Therefore in equation (83), the dividend yield jumps down from ρ to $\rho - \tau i_t^m$.

5.4.2 Fiscal Redistribution

A direct fiscal transfer $T_t > 0$ from capitalists to hand-to-mouth workers can raise the total amount of dividends in the financial market, thereby pushing up the current $\hat{Q}_t$ at the ZLB.[80] The formula for the expected stock market return i_t^m in this case can be written as

$$i_t^m = \frac{y_t - \frac{w_t}{p_t} N_{W,t}}{A_t Q_t} + \mathbb{E}_t \left(\frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \frac{1}{dt} \right) = \rho + \underbrace{\frac{T_t}{p_t A_t Q_t}}_{>0} + \mathbb{E}_t \left(\frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \frac{1}{dt} \right). \tag{84}$$

If we assume capitalists pay φ_t portion of their wealth to finance this transfer, we will have $T_t = \varphi_t p_t A_t Q_t$ and dividend yield becomes $\rho + \varphi_t$ instead of just ρ. This raises the effective natural rate of interest from $\underline{r}$ to $\underline{r} + \varphi_t$ during the ZLB, thereby increasing $\hat{Q}_t$ and π_t. The following Proposition 10 summarizes this result.

Proposition 10 (Direct Redistribution) *Under the ZLB environment of Section 5.1 and a direct (instantaneous) transfer of φ_t portion of capitalists' aggregate financial wealth towards hand-to-mouth workers, the dynamic IS equation for $\hat{Q}_t$ can be expressed as*

$$d\hat{Q}_t = -(\underbrace{\underline{r}}_{\equiv r^n(\bar{\sigma})<0} + \underbrace{\varphi_t}_{>0} + \pi_t) dt. \tag{85}$$

It is worth mentioning that from the perspective of capitalists with a nominal wealth a_t, paying $T_t = \varphi_t a_t$ effectively lowers their equilibrium return by φ_t. The equilibrium return jumps down by φ_t because the asset price value $\hat{Q}_t$ rises so that the capital gain is reduced also by φ_t. Thus, a transfer to workers with a higher marginal propensity to consume (MPC) brings an additional stabilizing effect other than just pushing up the current consumption and aggregate demand. This policy also raises the dividend yield, boosting $\hat{Q}_t$ and therefore increasing aggregate demand through a wealth effect.[81]

[80] If the dividend yield increases, the required capital gain for given policy rate falls and the current asset price $\hat{Q}_t$ jumps up at the ZLB.

[81] The policy that subsidizes firms' payroll based on tax imposed on capitalists achieves exactly the same dynamic IS equation as in equation (85). To be precise, if firms pay $w_t N_{W,t} - T_t$ instead of $w_t N_{W,t}$ as the total payroll to workers, where T_t is financed by capitalists in a lump-sum way, this policy props up the profit amounts that are to be distributed as dividends to capitalists, raising stock prices and business cycle. One difference is that this policy lowers the effective marginal cost for firms, which affects firms' pricing decisions and distorts the Phillips curve as

$$d\pi_t = (\rho \pi_t + \underbrace{\delta(\delta + \rho)\Theta \tau_t}_{>0} - \kappa \hat{Q}_t) dt, \tag{86}$$

where this additional term would negatively affect π_t and $\hat{Q}_t$ during a ZLB crisis.

In sum, macroprudential policies help to mitigate recessionary forces during ZLB crises through their impact on financial markets and asset price Q_t. However, there exist other types of financial market interventions that are able to substitute monetary policy's lack of ammunition power during the ZLB and stimulate the economy through their impact on asset markets. In the next Section 5.3, we analyze possible financial market interventions and look into subtle issues that arise when the central bank distorts the stock market and manipulates its risk in order to prop up the economy.

6 Conclusion

In this paper, we illustrate that properly accounting for higher-order moments related to the business cycle and stock markets changes the business cycle dynamics of the New-Keynesian framework and provides new implications about monetary policy. To that end, we develop a model with stock markets that features higher-order stock market variables (time-varying aggregate financial volatility and risk-premium). This setup allows a tractable analytical characterization of the equilibrium conditions and uncovers interesting dynamics stemming from the role of aggregate financial volatility: a rise in aggregate financial volatility raises the risk-premium, reducing wealth and aggregate demand. This feedback structure from higher-order terms (aggregate volatility and risk-premium) to first-order ones (wealth and aggregate demand) opens up the possibility of second-order sunspot equilibria, which require a different set of monetary policy rules for stabilization purposes.

Our analysis reveals that conventional monetary policy rules, even with aggressive targeting of traditional macroeconomic measures, cannot guarantee determinacy. This failure of conventional rules in ensuring determinacy lies in their inability to adequately target the 'expected risky return' of financial markets, the relevant rate in a stochastic environment. We then propose a generalized Taylor rule that restores determinacy, with which the central bank targets not only conventional mandates (inflation and output gap), but also the risk-premium in a specific way, thus effectively managing the expected rate of return on aggregate financial wealth. This new policy rule achieves what we describe as the *ultra-divine coincidence*: joint stabilization of inflation, output gap and risk-premium. Finally, we study various policy options when the policy rate is constrained by the zero lower bound (ZLB).

Our framework opens new avenues for future research focused on understanding the interaction of the real economy and the higher-order variables of financial markets. For example, in our model we largely abstract from wealth inequality and potentially heterogenous sensitivity of economic players to financial volatility. We view future work aiming to incorporate these realistic features as a particularly fruitful direction to pursue.

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Appendices

A Suggestive Evidence

Stock market volatility is commonly viewed in the literature as a proxy of financial and economic uncertainty, which Bloom (2009) and later Gilchrist and Zakrajšek (2012), Bachmann et al. (2013), Jurado et al. (2015), Caldara et al. (2016), Baker et al. (2020), Coibion et al. (2021) further studied as a driving force behind business cycles fluctuations. In this Section, we will evaluate these claims and present interesting empirical results. Figure 14 provides the first piece of supportive evidence in that direction. Panel 14a depicts several variables commonly used in the literature to measure financial uncertainty. The correlation between series is remarkably high and they all display positive spikes at the beginning and/or initial months following an NBER-dated recession, which is consistent with the evidence that many of these episodes were financial in nature.[82] Panel 14b plots Ludvigson et al. (2015) (henceforth, LMN) financial and real (i.e. non-financial) uncertainty series. These variables are positively correlated and display a similar propensity to increase around recessions, though a different type of crisis (e.g. financial or not) is correlated with a different type of uncertainty playing the dominant role. For example, the massive spike in real vis-à-vis financial uncertainty following the recent Covid-19 recession, which initially was a health crisis that spilled into the real economy, can be observed in Panel 14b.

The patterns displayed in Figure 14 do not yet constitute a proof of the importance of financial uncertainty as a driver of the business cycle, as we still should worry about the possibility of reverse causation running from unfavorable economic conditions towards uncertainty. We tackle this issue by proposing a simple Vector Autoregression (VAR) with the structural identification strategy based on the timing of macroeconomic shocks similar to Bloom (2009). Equation (87) presents the variables considered and their ordering, with non-financial series first and financial variables last.[83]

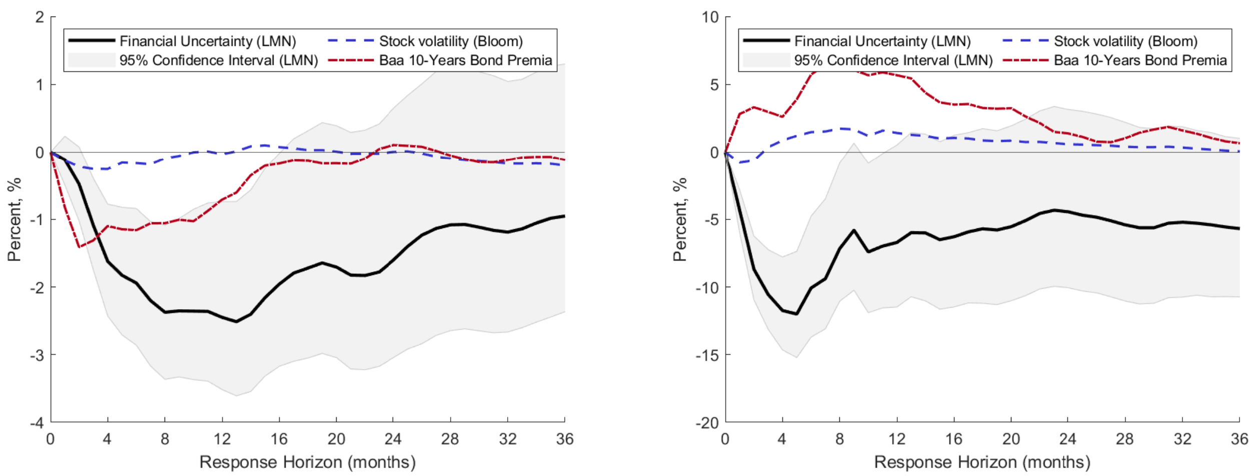
$$\text{VAR-11 order:} \quad \begin{bmatrix} \log(\text{Industrial Production}) \\ \log(\text{Employment}) \\ \log(\text{Real Consumption}) \\ \log(\text{CPI}) \\ \log(\text{Wages}) \\ \text{Hours} \\ \text{Real Uncertainty (LMN)} \\ \text{Fed Funds Rate} \\ \log(\text{M2}) \\ \log(\text{S\&P-500 Index}) \\ \text{Financial Uncertainty (LMN)} \end{bmatrix} \tag{87}$$

Both LMN real and financial uncertainty measures are included to differentiate the effects of financial volatility shocks from the effects from real uncertainty. For similar reasons, we include the S&P-500

[82] See Reinhart and Rogoff (2009) and Romer and Romer (2017) for classification of the past recessions. Their analysis showed many recessions had roots in financial markets.

[83] The ordering is also used by Ludvigson et al. (2015), which, using identification strategy based on event constraints, find that the uncertainty of financial markets tends to be an exogenous source of business cycle fluctuations, while real uncertainty tends to be an endogenous response to the business cycle fluctuations. We also have considered alternative specifications and orderings that produced qualitatively similar results (not reported, provided upon request).

index in our VAR to empirically distinguish between shocks affecting the level of financial markets and shocks affecting their volatility. In order to ameliorate possible concerns about the validity of the structural identification strategy, we estimate our VAR using monthly data, where the identification assumptions are more likely to hold. Figure 12 presents the impulse responses to the orthogonalized financial uncertainty shock. Panel 12a plots the response of industrial production, which falls by up to 2.5% and displays moderate persistence following a one standard deviation shock to financial uncertainty. Panel 12b plots the response of the S&P-500 Index, which drops up to 12% within the first four months before gradually recovering. Together, both pictures imply that an increase of financial uncertainty tends to depress both industrial activity and financial markets.



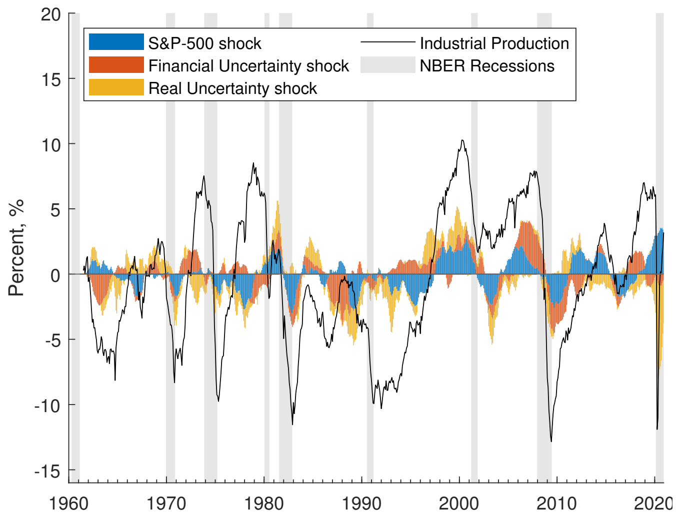
(a) Response: Industrial Production

(b) Response: S&P-500 Index

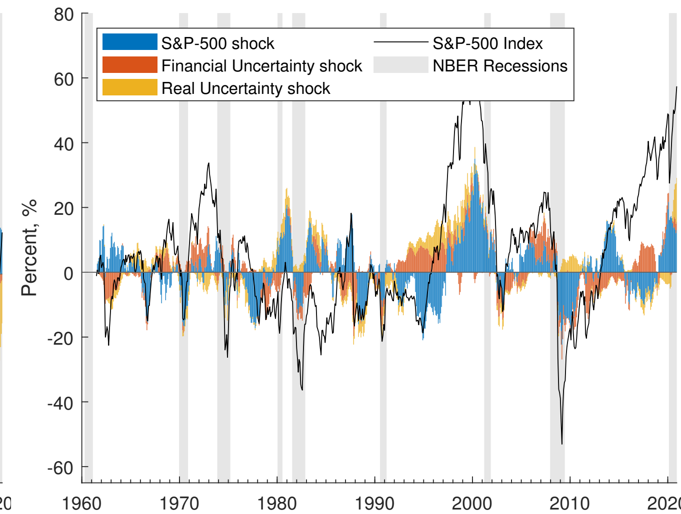
Figure 12: Impulse Response Functions (IRFs), selected series. Figures 12a and 12b display the response to a one standard deviation financial uncertainty shock of monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (87) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.

Figure 12 also features alternative estimates using common financial uncertainty proxies such as Bloom (2009) stock market volatility index and 10-years premium on Baa-rated corporate bonds. The responses are generally more muted, and take the opposite sign in the case of the S&P Index. These results can be explained by the fact that standard proxies contain information unrelated to financial uncertainty that distorts our estimates (see Jurado et al. (2015) for a discussion), and therefore we choose LMN as our preferred financial uncertainty measure. In Appendix B, we report additional impulse response estimates. Especially, the Figure 16 shows that monetary authorities respond with accommodating interest rate movements to financial uncertainty shocks, while real uncertainty has no statistically significant effect on either interest rates or stock market fluctuations. We will further discuss optimal monetary policy response to financial volatility shocks in Section 4.

Finally, we can further explore the contribution of financial uncertainty to business cycles fluctuations by looking at Table 2 in Appendix B, which reports the Forecast Error Variance Decomposition (FEVD) of Industrial Production and the S&P-500 Index. Financial uncertainty shocks explain close to 5% of the fluctuations in both series, while real uncertainty explains an additional 2-4% of movements in industrial activity in the medium run. Figure 13 provides a more graphical illustration of these results by plotting the historical decomposition of the series. We observe the contribution of financial uncertainty rivals that of shocks to the level of financial variables captured by the S&P-500 shock, and is especially important in driving industrial production boom-bust patterns during and in the preceding months of recessionary episodes, as it can be seen during the Global Financial Crisis (2007).



(a) Industrial Production

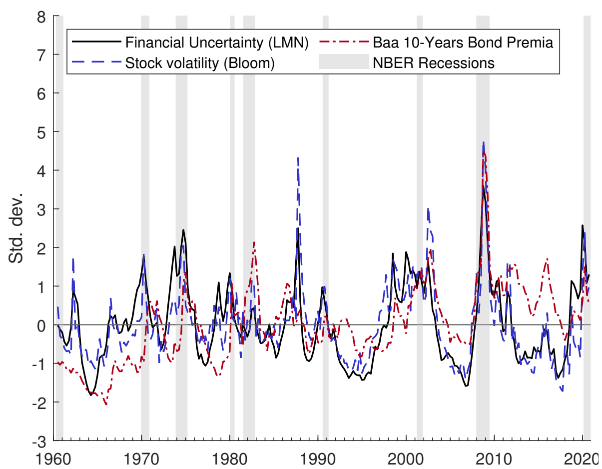


(b) S&P-500 Index

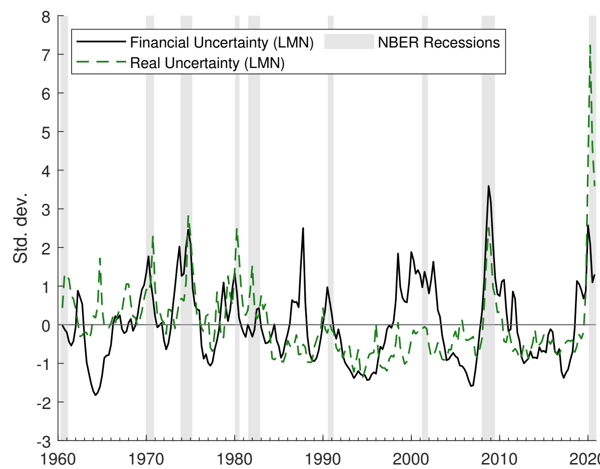
Figure 13: Historical Decomposition, selected series. Figures 13a and 13b display historical decomposition of monthly Industrial Production and S&P-500 Index series, respectively, based on the VAR-11 with equation (87) variable composition and ordering. Shaded areas indicate NBER dated recessions (peak trough the through). Variables of interest are de-trended by subtracting the contribution of initial conditions and constant terms after series decomposition. Columns report a contribution of each shock to the fluctuations around trend of the variable considered.

In this Appendix A, we have revisited the empirical evidence on financial market volatility and shown that it acts as a major driving force of the business cycle.

B Additional Figures and Tables



(a) Financial Uncertainty series



(b) Financial vs. Real Uncertainty

Figure 14: Uncertainty series. Figure 14a displays common measures of financial uncertainty. Figure 14b displays Ludvigson et al. (2015) (henceforth, LMN) measures of financial and real economic uncertainty. Shaded areas indicate NBER dated recessions (peak trough the through). LMN financial and real economic uncertainty series are constructed as the average volatility of the residuals from predictive regressions on financial and real economic variables, respectively (See Ludvigson et al. (2015) for the series construction). Bloom (2009)'s stock market volatility variable is constructed using VXO data from 1987 onward and the monthly volatility of the S&P 500 index normalized to the same mean and variance in the overlapping interval for the 1960-1987 period (See Bloom (2009) for the series construction). The bond risk-premia series is the Moody's seasoned Baa corporate bond yield relative to the yield on a 10-year treasury bond at constant maturity. For graphical comparison purposes, the depicted series have a normalized zero mean and one standard deviation.

(i) Industrial Production

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0	0.30	0.21	0.12
h=6	1.27	3.37	2.98	1.36
h=12	4.28	4.38	3.16	1.94
h=36	3.24	1.67	1.98	0.64

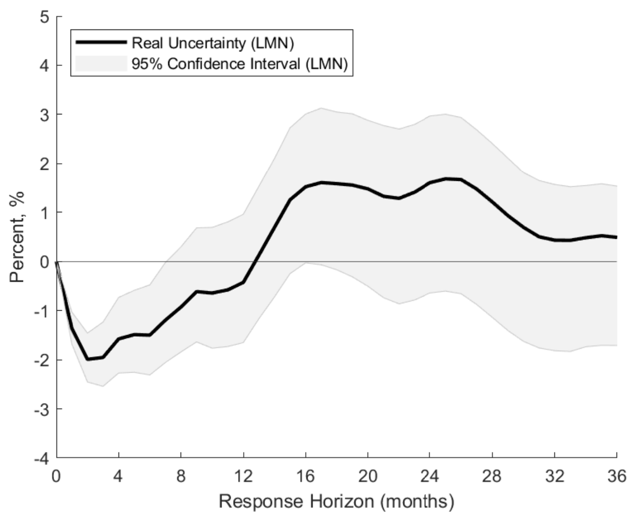
(ii) S&P-500 Index

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.11	0.08	0.39	0.06
h=6	3.30	0.25	3.26	0.62
h=12	4.77	0.54	10.03	2.16
h=36	6.50	0.91	12.16	2.40

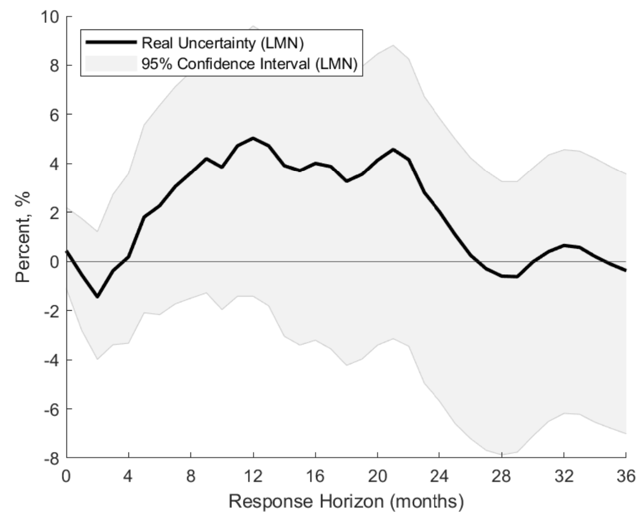
(iii) Fed Funds Rate

Horizon	Fin. Uncert. (LMN)	Real Uncert. (LMN)	Stock Vol. (Bloom)	Baa 10-Yr Premia
h=1	0.01	0.98	0	0.08
h=6	0.42	0.84	3.11	1.66
h=12	1.47	0.91	4.69	2.30
h=36	2.81	2.05	5.02	3.17

Table 2: Forecast Error Variance Decomposition (FEVD). The table presents the variance contribution (in percentage) of financial and real uncertainty shocks to selected series at different time horizons (in months). The FEVD is constructed using a VAR-11 with equation (87) variable composition and ordering. The first two columns report the contribution of LMN financial and real uncertainty shocks, respectively. The last two columns report alternative VAR specifications where the preferred LMN financial uncertainty measure (column one) is replaced by common proxies employed in the literature, either Bloom (2009) stock market volatility measure or the Baa 10-years corporate bond premia, respectively.

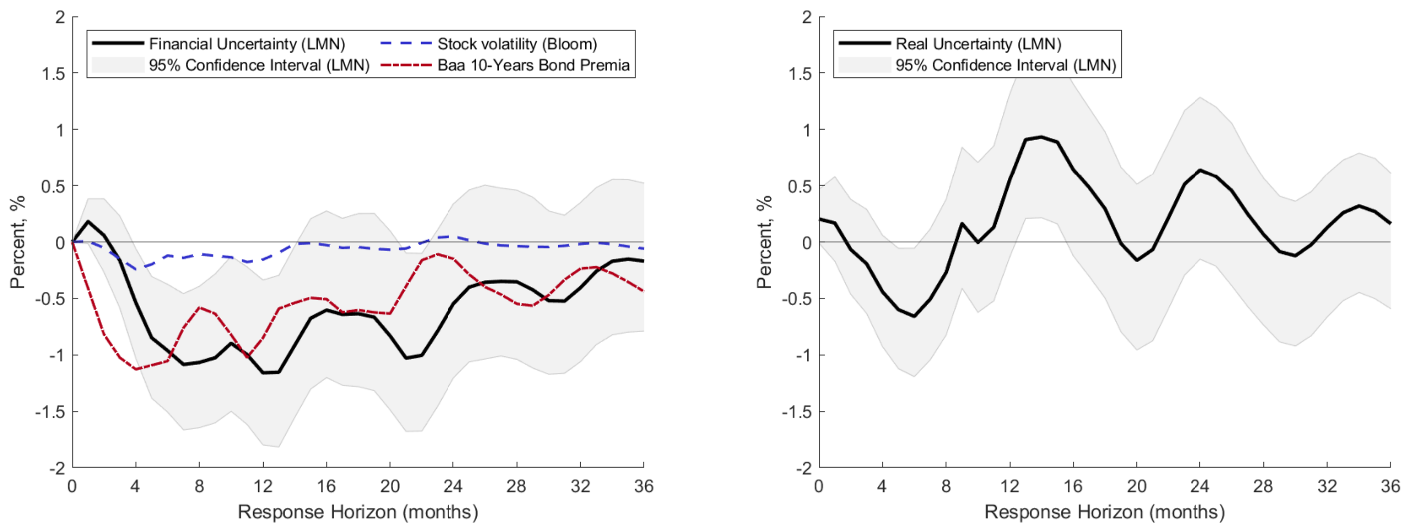


(a) Response: Industrial Production



(b) Response: S&P-500 Index

Figure 15: Impulse Response Functions (IRFs), selected series. Figures 15a and 15b display the response to one standard deviation real uncertainty shock by monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (87) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.



(a) Shock: Financial Uncertainty

(b) Shock: Real Uncertainty

Figure 16: Impulse Response Functions (IRFs), Fed Funds Rate. This Figure displays the response to a one standard deviation uncertainty (financial or real) shock by monthly Fed Funds Rate series, using a VAR-11 with equation (87) variable composition and ordering. Panel 16a plots the response to a financial uncertainty shock, and Panel 16b to a real uncertainty shock. Shaded area indicates 95% confidence interval around preferred financial/real uncertainty measure computed using standard bootstrap techniques. Additional lines display alternative impulse responses obtained by substituting preferred LMN financial uncertainty measure with common proxies employed in the literature.

Parameter	Value	Description
φ	0.2	Relative Risk Aversion
χ_0	0.25	Inverse Frisch labor supply elasticity
ρ	0.020	Subjective time discount factor
σ	0.0090	TFP volatility
g	0.0083	TFP growth rate
α	0.4	$1 -$ Labor income share
ϵ	7	Elasticity of substitution intermediate goods
δ	0.45	Calvo price resetting probability
ϕ_π	2.50	Policy rule inflation response
ϕ_y	0.11	Policy rule output gap response
ϕ_{rp}	0	Policy rule risk premium response
$\bar{\pi}$	0	Steady state trend inflation target

Table 3: The table presents the baseline parameter calibration used in Sections 4 and 5.3 of the paper.

C Derivations and Proofs for Sections 2, 3, 4, and 5.3

B.0. Section 2

Derivation of equation (3): From the definition of (nominal) state-price density $\xi_t^N = e^{-\rho t}\frac{1}{C_t}\frac{1}{p_t}$, we get:

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} - \frac{dp_t}{p_t} + \left(\frac{dC_t}{C_t}\right)^2 + \left(\frac{dp_t}{p_t}\right)^2 + \frac{dC_t}{C_t}\frac{dp_t}{p_t}. \tag{I.1}$$

Since we have a perfectly rigid price ($p_t = \bar{p}$ for $\forall t$), the above expression becomes:

$$\frac{d\xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \tag{I.2}$$

$$= -\rho dt - \frac{dC_t}{C_t} + \text{Var}_t\left(\frac{dC_t}{C_t}\right). \tag{I.3}$$

Plugging equation (I.2) into equation (2), we get the following equation (3).

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)dt + \text{Var}_t\left(\frac{dC_t}{C_t}\right). \tag{I.4}$$

Derivation of equation (7): From equation (6), we obtain

$$d\ln Y_t = \left(i_t - \rho + \frac{1}{2}(\sigma_t + \sigma_t^s)^2\right)dt + (\sigma_t + \sigma_t^s)dZ_t. \tag{I.5}$$

From equation (4), we obtain

$$d\ln Y_t^n = \left(r_t^n - \rho + \frac{1}{2}(\sigma_t)^2\right)dt + \sigma_t dZ_t. \tag{I.6}$$

Therefore, by subtracting equation (I.6) from equation (I.5), we obtain

$$d\hat{Y}_t = \left(i_t - \left(r_t^n - \frac{1}{2}(\sigma_t + \sigma_t^s)^2 + \frac{1}{2}(\sigma_t)^2\right)\right)dt + \sigma_t^s dZ_t, \tag{I.7}$$

which is equation (7).

Proof of Proposition 1. From equation (13), $\{\sigma_t^s\}$ process can be written as

$$d\sigma_t^s = -(\phi_y)^2\frac{(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3}dt - \phi_y\frac{\sigma_t^s}{\sigma_t + \sigma_t^s}dZ_t. \tag{I.8}$$

Using Ito's lemma, we get the process for $(\sigma + \sigma_t^s)^2$ which is a martingale, as

$$
\begin{aligned}
d(\sigma_t+\sigma_t^s)^2 &= 2(\sigma_t+\sigma_t^s)d\sigma_t^s + (d\sigma_t^s)^2 \\
&= 2(\sigma_t+\sigma_t^s)\left(-\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma_t+\sigma_t^s)^3}dt - \phi_y\frac{\sigma_t^s}{\sigma_t+\sigma_t^s}dZ_t\right) + (\phi_y)^2\frac{(\sigma_t^s)^2}{(\sigma_t+\sigma_t^s)^2}dt \\
&= -2\phi_y(\sigma_t^s)dZ_t.
\end{aligned} \tag{I.9}
$$

Therefore, we have $\mathbb{E}_0((\sigma+\sigma_t^s)^2) = (\sigma+\sigma_0^s)^2$. By Doob's martingale convergence theorem (as $(\sigma+\sigma_t^s)^2 \geq 0, \forall t$), we know $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$ since:

$$
\underbrace{d\sigma_t^s}_{\overset{a.s}{\to}0} = -\underbrace{\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma_t+\sigma_t^s)^3}}_{\overset{a.s}{\to}0}dt - \phi_y\underbrace{\frac{\sigma_t^s}{\sigma_t+\sigma_t^s}}_{\overset{a.s}{\to}0}dZ_t. \tag{I.10}
$$

Thus equation (I.10) proves $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^s = 0$. From equation (12) $\sigma_t^s \overset{a.s}{\to} \sigma_\infty^q = 0$ leads to $\hat{Y}_t \overset{a.s}{\to} 0$. Finally, we must have $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$, since otherwise the uniform integrability says $\mathbb{E}_0((\sigma+\sigma_\infty^s)^2) = (\sigma+\sigma_0^s)^2$, which is a contradiction to our earlier result $\sigma_t^s \overset{a.s}{\to} 0$ since $\sigma_\infty^s = 0$ and $\sigma_0^s > 0$ by assumption in Proposition 1.

■

B.1. Section 3

B.1.1. Section 3.1

Here we solve the optimization problems of workers (equation (20)) and capitalists (equation (25)).
Worker's Optimization : Workers solve the following optimization problem in equation (20).

$$
\max_{C_{W,t},N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \quad \text{s.t.} \quad p_tC_{W,t} = w_tN_{W,t}. \tag{I.11}
$$

If we let $\lambda_t A_t^{\varphi-1}$ be the multiplier on the budget constraint, then solution is easy to compute as follows.

$$
\begin{aligned}
&C_{W,t}^{-\varphi} = \lambda_t p_t, \;\; A_t^{1-\varphi}(N_{W,t})^{\chi_0} = \lambda_t w_t = \frac{w_t}{p_t}C_{W,t}^{-\varphi} = \left(\frac{w_t}{p_t}\right)^{1-\varphi}N_{W,t}^{-\varphi}, \\
&\therefore N_{W,t} = \left(\frac{w_t}{p_t}\right)^{\frac{1-\varphi}{\chi_0+\varphi}}\frac{1}{A_t^{\frac{1-\varphi}{\chi_0+\varphi}}}1 \equiv \left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}}\frac{1}{A_t^{\frac{1}{\chi}}} \text{ with } \chi \equiv \frac{\chi_0+\varphi}{1-\varphi}, \; C_{W,t} = \frac{w_t}{p_t}N_{W,t} = \left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}}\frac{1}{A_t^{\frac{1}{\chi}}}.
\end{aligned} \tag{I.12}
$$

Capitalist's Optimization : Each capitalist with wealth a_t solves the following optimization in equation (25).

$$
\max_{C_t,\omega_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\log C_t dt \;\; \text{s.t.} \;\; da_t = (a_t(i_t+\theta_t(i_t^m - i_t)) - p_tC_t)dt + \theta_t a_t(\sigma_t+\sigma_t^q+\sigma_t^p)dZ_t. \tag{I.13}
$$

Putting all the state variables $(i_t, p_t, i_t^m, \sigma_t, \sigma_t^q, \sigma_t^p)$ into the vector S_t, then Hamilton-Jacobi-Bellman (HJB) equation can be written in the following way.

$$\begin{aligned}\rho V(a_t, S_t, t) = \max_{C_t,\theta_t} \log C_t + \frac{\partial V}{\partial a_t}(a_t(i_t + \theta_t(i_t^m - i_t)) - p_t C_t) + \frac{1}{2}\frac{\partial^2 V}{\partial a^2}\theta_t^2 a_t^2(\sigma_t + \sigma_t^q + \sigma_t^p)^2 + \frac{\partial V}{\partial t} \\ + \frac{\partial V}{\partial S_t}\frac{\mathbb{E}_t(dS_t)}{dt} + \frac{1}{2}\mathrm{Tr}\Big(\frac{\partial^2 V}{\partial S_t \partial S_t'}\frac{dS_t dS_t'}{dt}\Big).\end{aligned} \tag{I.14}$$

Following Merton (1971), we know the value function has the following form.

$$V(a_t, S_t, t) = \frac{1}{\rho}\log a_t + f(S_t, t). \tag{I.15}$$

The first-order conditions for C_t and θ_t are easy to compute as follows.

$$p_t C_t = \rho a_t \ \text{ and } \ \underbrace{\frac{i_t^m - i_t}{\sigma_t + \sigma_t^q + \sigma_t^p}}_{\text{Sharpe ratio}} = \theta_t \underbrace{(\sigma_t + \sigma_t^q + \sigma_t^p)}_{\text{Price of risk}}. \tag{I.16}$$

If we plug the guessed value function form (equation (I.15)) into HJB equation, we get the following partial differential equation (PDE) for the function $f(S_t, t)$, verifying our form in equation (I.15) is a reasonable guess.

$$\begin{aligned}\rho f(S_t, t) = &\log \frac{\rho}{p_t} + \frac{1}{\rho}(i_t + \theta_t(i_t^m - i_t) - \rho) - \frac{1}{2\rho}\theta_t^2(\sigma_t + \sigma_t^q + \sigma_t^p)^2 + \frac{\partial f}{\partial t} \\ &+ \frac{\partial f}{\partial S_t'}\frac{\mathbb{E}_t(dS_t)}{dt} + \frac{1}{2}\mathrm{Tr}\Big(\frac{\partial^2 f}{\partial S_t \partial S_t'}\frac{dS_t dS_t'}{dt}\Big) \text{ with } \theta_t = \frac{i_t^m - i_t}{(\sigma_t + \sigma_t^q + \sigma_t^p)^2}.\end{aligned} \tag{I.17}$$

Thus solving the partial differential equation in equation (I.17) restores the functional form $f(S_t, t)$.

B.1.2. Section 3.2

We can easily derive the equilibrium condition in equation (27) by plugging in $\theta_t = 1$ to equation (I.16). $a_t = p_t A_t Q_t$ holds since all capitalists are identical both ex-ante and ex-post. Now we prove Lemma 1.

Proof of Lemma 1. First we start by stating capitalist's nominal state-price density ξ_t^N and real state-price density ξ_t^r. Nominal state-price density will be relevant to the nominal interest rate, while real state-price density matters when we calculate the real interest rate.

$$\xi_t^N = e^{-\rho t}\frac{1}{C_t}\frac{1}{p_t}, \ \ \xi_t^r = e^{-\rho t}\frac{1}{C_t} = p_t \xi_t^N. \tag{I.18}$$

If λ_t is price of risk ($(\sigma_t + \sigma_t^q + \sigma_t^p)$ in this model), the nominal pricing kernel evolves with the following process.

$$\frac{d\xi_t^N}{\xi_t^N} = -i_t dt - \lambda_t dZ_t, \ \ \xi_t^N = \exp\Big(- \int_0^t \Big(i_s + \frac{1}{2}\lambda_s^2\Big) ds - \int_0^t \lambda_s dZ_s \Big). \tag{I.19}$$

If we apply Ito's lemma to the relation $\xi_t^r = p_t \xi_t^N$ in equation (I.18), we get the following process for real pricing kernel ξ_t^r.

$$\frac{d\xi_t^r}{\xi_t^r} = (\underbrace{\pi_t - i_t - \sigma_t^p \lambda_t}_{=-r_t}) dt - (\sigma + \sigma_t^q) dZ_t. \tag{I.20}$$

Thus we get the following Fisher identity with the inflation premium in equation (33).

$$r_t = i_t - \pi_t + \sigma_t^p(\sigma_t + \sigma_t^q + \sigma_t^p). \tag{I.21}$$

■

B.1.3. Section 3.3

Here we prove the Proposition 2 based on the results above.

Proof of Proposition 2. We start from the pricing decision of intermediate good firms. Since we have an externality à la Baxter and King (1991), we need to have additional step to aggregate across each firm. Let firm i take his demand as given and choose the optimal price $p_t(i)$ at given moment t. With $E_t \equiv (N_{W,t})^\alpha$, we can get the following conditions for $\{n_t(i), y_t(i)\}$.

$$n_t(i) = \Big(\frac{y_t(i)}{A_t E_t}\Big)^{\frac{1}{1-\alpha}}, \ \ y_t(i) = y_t \Big(\frac{p_t(i)}{p_t}\Big)^{-\epsilon}. \tag{I.22}$$

Each firm i chooses p_i that maximizes its profit, solving the following optimization.

$$\max_{p_t(i)} p_t(i) \Big(\frac{p_t(i)}{p_t}\Big)^{-\epsilon} y_t - w_t \Big(\frac{y_t}{A_t E_t}\Big)^{\frac{1}{1-\alpha}} \Big(\frac{p_t(i)}{p_t}\Big)^{-\frac{\epsilon}{1-\alpha}}. \tag{I.23}$$

Here all firms charge the same price ($p_t(i) = p_t$ holds for $\forall i$). The solution of equation (I.23) combined with this condition yields the following solution. In equilibrium, we also know $n_t(i) = N_{W,t}$ for $\forall i$.

$$\begin{aligned} \frac{w_t^n}{p_t^n} &= \frac{\epsilon - 1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t E_t)^{\frac{1}{1-\alpha}} \\ &= \frac{\epsilon - 1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} \\ &= \frac{\epsilon - 1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} \Big(\frac{w_t^n}{p_t^n}\Big)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}}. \end{aligned} \tag{I.24}$$

Thus we get the following condition for the real wage.

$$\frac{w_t^n}{p_t^n} = \Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}}. \tag{I.25}$$

And then we know the aggregate production is linear, thus $y_t = A_t N_{W,t}$ due to the externality. Thus we have:

$$\begin{aligned} y_t = A_t N_{W,t} &= A_t \Big(\frac{w_t^n}{p_t^n}\Big)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} \\ &= A_t \Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}}. \end{aligned} \tag{I.26}$$

Thus we get the natural level of output y_t^n and the natural level of real wage w_t^n / p_t^n.

$$y_t^n = \Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{\frac{1}{\chi}} A_t, \;\; \frac{w_t^n}{p_t^n} = \frac{\epsilon - 1}{\epsilon}(1-\alpha) A_t, \tag{I.27}$$

from which we get the following consumption and labor for workers.

$$N_{W,t}^n = \Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{\frac{1}{\chi}}, \;\; C_{W,t}^n = \Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{1+\frac{1}{\chi}} A_t. \tag{I.28}$$

In equilibrium, consumptions of capitalists and workers add up to the amount of final good output. If we plug the real wage in equation (I.24) into workers' consumption and the labor supply decision in equation (I.12), we get the following good-market equilibrium condition, where we define Q_t^n to be the natural level of detrended stock price. Also from equation (I.16), we see the consumption of capitalists would be $C_t = \rho A_t Q_t$ in equilibrium.

$$\rho A_t Q_t^n + \Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{1+\frac{1}{\chi}} A_t = \Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{\frac{1}{\chi}} A_t. \tag{I.29}$$

Thus we get the following expression for Q_t^n and C_t^n, a natural asset price level and capitalists' consumption in the flexible price equilibrium.

$$\begin{aligned} Q_t^n &= \frac{1}{\rho}\Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{\frac{1}{\chi}} \Big(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\Big), \\ C_t^n &= A_t\Big(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\Big)^{\frac{1}{\chi}} \Big(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\Big). \end{aligned} \tag{I.30}$$

Since Q_t^n is constant, there should be no drift and volatility for its process in the flexible price economy, thus we have $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To calculate the natural interest rate r_t^n, we start from the capital gain component in equation (32). If we apply Ito's lemma, we get the following capital gain formula.

$$\mathbb{E}\frac{d(p_t A_t Q_t)}{p_t A_t Q_t}\frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0} \sigma_t^p + \sigma_t(\sigma_t^p + \underbrace{\sigma_t^q}_{=0}). \tag{I.31}$$

As dividend yield is always ρ, imposing expectation on both sides of equation (32) and combining with the equilibrium condition in equation (27) yield the following relation.

$$\mathbb{E}(i_t^m) = \rho + \pi_t + g + \sigma_t \sigma_t^p = i_t + (\sigma_t + \sigma_t^p)^2. \tag{I.32}$$

Using Lemma 1, we finally express natural rate of interest r_t^n as a function of structural parameters and σ_t, which proves (iii) of Proposition 2.

$$r_t^n = i_t - \pi_t + \sigma_t^p(\sigma_t + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p) = \rho + g - \sigma_t^2. \tag{I.33}$$

For the capitalist's consumption process in the flexible price case, Since their consumption C_t^n is directly proportional to TFP A_t, we know

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma_t dZ_t = \left(r_t^n - \rho + \sigma_t^2\right)dt + \sigma_t dZ_t, \tag{I.34}$$

where we use $r_t^n - \rho + \sigma_t^2 = g$ from equation (I.33). ■

B.1.4. Section 3.4

Proof of Lemma 2. First from $C_t = \rho A_t Q_t$, we get $\hat{C}_t = \hat{Q}_t$. We start from the flexible price case's good market equilibrium condition, where we use equation (I.12). Here $\frac{w_t^n}{p_t^n}$ is the real wage level in the flexible price economy.

$$A_t\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}\frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}\frac{1}{A_t^{\frac{1}{\chi}}}. \tag{I.35}$$

We subtract equation (I.35) from the same good market equilibrium condition in sticky price economy.

$$A_t\left(\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}\right)\frac{1}{A_t^{\frac{1}{\chi}}} = C_t - C_t^n + \left(\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}\right)\frac{1}{A_t^{\frac{1}{\chi}}}, \tag{I.36}$$

where we divide both sides of equation (I.36) by $A_t^{1-\frac{1}{\chi}}(\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$ and obtain

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\widehat{\frac{w_t}{p_t}}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}}(\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}(1+\frac{1}{\chi})\widehat{\frac{w_t}{p_t}}}. \tag{I.37}$$

Thus equation (I.37) can be written in the following way, which proves Lemma 2.

$$\frac{1}{\chi}\widehat{\frac{w_t}{p_t}} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\underbrace{\left(1+\frac{1}{\chi}\right)\widehat{\frac{w_t}{p_t}}}_{=\hat{C^w}(t)}. \tag{I.38}$$

We finally obtain

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\frac{\widehat{w_t}}{p_t} = \underbrace{\frac{\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}}{1+\chi^{-1}}}_{>0}\widehat{C_{W,t}}. \tag{I.39}$$

We see that Assumption 1 guarantees that all gaps (asset price, consumptions for both capitalists and workers, employment, and real wage) co-move with positive correlations. Now we can use $\hat{Q}_t$ and $\hat{C}_t$ interchangeably, and if one gap variable becomes 0, then all other gap variables become also stabilized and 0.

■

Proof of Proposition 3. In sticky price equilibrium, we have $\sigma_t^p \equiv 0$, as over the small time period dt a δdt portion of firms get to change their prices and there is no stochastic change in aggregate price level p_t. Thus capitalist's consumption C_t has the following process, where we use the equilibrium condition $i_t^m = i_t + (\sigma + \sigma_t^q)^2$.

$$\begin{aligned}\frac{dC_t}{C_t} &= (i_t^m - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dZ_t \\ &= (i_t + (\sigma_t + \sigma_t^q)^2 - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dZ_t.\end{aligned} \tag{I.40}$$

Thus we have the following two process for $\ln C_t$ and $\ln C_t^n$:

$$d\ln C_t = \left(i_t - \pi_t + \frac{(\sigma_t + \sigma_t^q)^2}{2} - \rho\right)dt + (\sigma_t + \sigma_t^q)dZ_t, \;\; d\ln C_t^n = \left(r_t^n - \rho + \frac{\sigma_t^2}{2}\right)dt + \sigma_t dZ_t, \tag{I.41}$$

of which the latter is from equation (I.34). We get the following $\hat{C}_t = \hat{Q}_t$ gap from equation (I.41):

$$\begin{aligned}d\hat{Q}_t = d\hat{C}_t &= \Bigg(i_t - \pi_t - \underbrace{\left(r_t^n - \frac{(\sigma_t + \sigma_t^q)^2}{2} + \frac{\sigma_t^2}{2}\right)}_{\equiv r_t^T}\Bigg)dt + \sigma_t^q dZ_t \\ &= (i_t - \pi_t - r_t^T)dt + (\sigma_t^q - \sigma_t^{q,n})dZ_t.\end{aligned} \tag{I.42}$$

As we have risk-premium levels $\mathrm{rp}_t = (\sigma_t + \sigma_t^q)^2$ in the sticky price case and $\mathrm{rp}_t^n = (\sigma_t)^2$ in the flexible price economy, we can express r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}(\mathrm{rp}_t - \mathrm{rp}_t^n) = r_t^n - \frac{1}{2}\hat{rp}_t, \tag{I.43}$$

where we know that when $\sigma_t^q = \sigma_t^{q,n} = 0$ holds, then we have $\hat{rp}_t = 0$ and $r_t^T = r_t^n$.

■

Proof of Proposition 4. We assume that firms change their prices with instantaneous probability δdt

à la Calvo (1983). If there is price dispersion Δ_t, as defined in equation (22), across intermediate goods firms, then labor market equilibrium condition can be written as follows.

$$N_{W,t} = \int_0^1 n_t(i)di = \Big(\frac{y_t}{A_t(N_{W,t})^\alpha}\Big)^{\frac{1}{1-\alpha}} \underbrace{\int_0^1 \Big(\frac{p_t(i)}{p_t}\Big)^{-\frac{\epsilon}{1-\alpha}}di}_{\equiv \Delta_t^{\frac{1}{1-\alpha}}},\ \ y(t) = \frac{A_t N_{W,t}}{\Delta_t} = C_t + C_{W,t}. \tag{I.44}$$

Plugging equation (I.12) and equation (I.16) (optimal consumption decisions of workers and capitalists) into equation (I.44), we get the following equilibrium condition.

$$\rho A_t Q_t + A_t\Big(\frac{w_t}{p_t A_t}\Big)^{1+\frac{1}{\chi}} = A_t\Big(\frac{w_t}{p_t A_t}\Big)^{\frac{1}{\chi}}\frac{1}{\Delta_t}. \tag{I.45}$$

Since a price level (nominal side) does not matter for the allocation of resources in the flexible price economy, we can regard $\hat{x}_t$ to be the log-deviation of x_t from the constant price flexible price equilibrium value of itself. From price aggregator in equation (19), we get the log-linearize version easily.

$$p_t^{1-\epsilon} = \int_0^1 p_t(i)^{1-\epsilon}di \ \text{ thus } \ \hat{p}_t = \int_0^1 \widehat{p_t(i)}di. \tag{I.46}$$

To get a sense of price dispersion Δ_t, we illustrate Woodford (2003)'s treatment of Δ_t up to first-order. Up to the first-order we can regard $\Delta_t \simeq 1$ because Δ_t is in nature the second order variable, as the following relation shows.

$$\begin{aligned}\frac{1}{1-\alpha}\hat{\Delta}_t &= \frac{1}{1-\alpha}\ln\frac{\Delta_t}{\Delta_t^n(=1)} = \ln\int_0^1 \exp[-\frac{\epsilon}{1-\alpha}(\widehat{p_t(i)}-\hat{p}_t)]di] \\ &= \ln\int_0^1[1-\frac{\epsilon}{1-\alpha}\cancel{(\widehat{p_t(i)}-\hat{p}_t)}+\frac{1}{2}\Big(\frac{\epsilon}{1-\alpha}\Big)^2(\widehat{p_t(i)}-\hat{p}_t)^2]di \simeq \frac{1}{2}\Big(\frac{\epsilon}{1-\alpha}\Big)^2 Var_i(\widehat{p_t(i)}).\end{aligned} \tag{I.47}$$

Pricing is standard, except that our model is in continuous time. With δdt probability at time t, individual firm can change the price instantaneously from t to $t+dt$. From time-0 perspective, a probability that firm can reset its price for the first time at time t is $\delta e^{-\delta t}dt = \underbrace{\delta dt}_{\text{Change now}} \cdot \underbrace{e^{-\delta t}}_{\text{No change until t}}$.

At time t, price-changing firm i solves the following optimization to choose p_{it}:

$$\begin{aligned}&\max_{p_t(i)}\frac{1}{\xi_t^N p_t}\mathbb{E}_t\int_t^\infty e^{-\delta(s-t)}\xi_s^N p_s\Big(\frac{p_t(i)}{p_s}y_i(s|t)-\frac{1}{p_s}C(y_i(s|t))\Big)ds, \ \text{ where } \ y_i(s|t) = \Big(\frac{p_t(i)}{p_s}\Big)^{-\epsilon}y_s, \\ &=\frac{1}{\xi_t^N p_t}\mathbb{E}_t\int_t^\infty e^{-\delta(s-t)}\xi_s^N p_s\Big[\Big(\frac{p_t(i)}{p_s}\Big)^{1-\epsilon}y_s - \frac{1}{p_s}C\Big(\Big(\frac{p_t(i)}{p_s}\Big)^{-\epsilon}y_s\Big)\Big]ds,\end{aligned} \tag{I.48}$$

where $C(\cdot)$ is the nominal cost function for each firm. Let $MC_{s|t}$ and $\varphi_{s|t}$ be the nominal and real marginal cost at time s conditional on price resetting at prior time t. The nominal pricing kernel has following simple formula due to log-preference of capitalists.

$$\xi_s^N = e^{-\rho s}\frac{1}{p_s C_s}, \quad \frac{\xi_s^N p_s}{\xi_t^N p_t} = e^{-\rho(s-t)}\frac{C_t}{C_s}. \tag{I.49}$$

Thus optimal adjusted price $p_i^*(t)$ is given as the following first-order condition. Here $\varphi_{s|t}$, a real marginal cost of firms at time s given time t price resetting, appears, where $\bar{\varphi}$ is flexible-price (natural) equilibrium level of real marginal cost, which is $\frac{\epsilon-1}{\epsilon}$.

$$p_i^*(t) = \frac{\mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)(s-t)}\frac{y_s}{C_s}\frac{\varphi_{s|t}}{\bar{\varphi}}p_s^\epsilon ds}{\mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)(s-t)}\frac{y_s}{C_s}p_s^{\epsilon-1} ds}. \tag{I.50}$$

If we log-linearize this equation around the steady state equilibrium with the constant price as in equation (I.46), we obtain the following log-linearized $\widehat{p_t^*}$ expressed as

$$\widehat{p_t^*} = (\delta+\rho)\mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)(s-t)}(\hat{\varphi}_{s|t} + \hat{p}_s)ds. \tag{I.51}$$

We know that the total conditional real cost and real marginal cost can be written as

$$\frac{1}{p_s}C(y_{s|t}) = \frac{w_s}{p_s}\Big(\frac{y_{s|t}}{A_s(N_{W,s})^\alpha}\Big)^{\frac{1}{1-\alpha}}, \quad \varphi_{s|t} = \frac{w_s}{p_s}\Big(\frac{y_{s|t}}{A_s(N_{W,s})^\alpha}\Big)^{\frac{\alpha}{1-\alpha}}\frac{1}{A_s(N_{W,s})^\alpha}. \tag{I.52}$$

A conditional real marginal cost gap at time s conditional on price resetting at time t can be written as

$$\hat{\varphi}_{s|t} = \underbrace{\frac{\hat{w}_s}{p_s}}_{\equiv\hat{\varphi}_s} - \frac{\alpha\epsilon}{1-\alpha}(\widehat{p_t^*} - \hat{p}_s) = \hat{\varphi}_s - \frac{\alpha\epsilon}{1-\alpha}(\widehat{p_t^*} - \hat{p}_s). \tag{I.53}$$

Thus $\hat{\varphi}_s$ is the aggregate marginal cost index, and since production becomes linear in aggregate level, it equals the real wage gap. We then characterize the change in aggregate price gap $\hat{p}_t$, using equation (I.46).

$$d\hat{p}_t = \delta dt(\widehat{p_t^*} - \hat{p}_t) = \delta dt(\delta+\rho)\mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t)ds, \text{ where } \Theta \equiv \frac{1-\alpha}{1-\alpha+\alpha\epsilon}. \tag{I.54}$$

As we log-linearize around the steady state equilibrium with constant price, $\hat{p}_t$ changes with a rate of π_t,[84] we have

$$\pi_t = \frac{d\hat{p}_t}{dt} = \delta(\delta+\rho)\mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s + \hat{p}_s - \hat{p}_t)ds. \tag{I.55}$$

Since we now have equation (I.55) for instantaneous inflation π_t. we manipulate this equation as:

$$\begin{aligned}\pi_t + \delta\hat{p}_t &= \delta(\delta+\rho)\mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s + \hat{p}_s)ds = \delta(\delta+\rho)e^{(\delta+\rho)t}\mathbb{E}_t \int_t^\infty e^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds \\ &= \delta(\delta+\rho)(\Theta\hat{\varphi}_t + \hat{p}_t)dt + \delta(\delta+\rho)e^{(\delta+\rho)t}\mathbb{E}_t \int_{t+dt}^\infty e^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s + \hat{p}_s)ds,\end{aligned} \tag{I.56}$$

[84] According to Woodford (2003) and Yun (2005), this assumption is reasonable as it becomes a part of optimal monetary policy in the presence of price dispersion Δ_t. In the case of positive inflation targets, see Coibion et al. (2012).

where we can rewrite the first line of equation (I.56) at time $t+dt$ instead of t as

$$\begin{aligned}\pi_{t+dt}+\delta\hat{p}_{t+dt} &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)(t+dt)}\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds \\ &= \delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}(1+(\delta+\rho)dt)\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds.\end{aligned} \tag{I.57}$$

Due to the martingale representation theorem (see Oksendal (1995)), there exists a measurable process H_t such that following holds.

$$\mathbb{E}_{t+dt}\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds = \mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds + H_t dZ_t, \tag{I.58}$$

which we plug into equation (I.57), to obtain

$$\begin{aligned}\pi_{t+dt}+\delta\hat{p}_{t+dt} = \delta(\delta+\rho)\Big(&\mathrm{e}^{(\delta+\rho)t}\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds+\mathrm{e}^{(\delta+\rho)t}H_t dZ_t \\ &+\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds\Big).\end{aligned} \tag{I.59}$$

We subtract equation (I.56) from equation (I.59) to get the following expression. We use $dZ_t dt = 0$ to get the second equality. Also $\sigma_{\pi,t}$ is defined as an instantaneous volatility of inflation fluctuation.

$$\begin{aligned}d\pi_t+\cancel{\delta}\pi_t dt &= \delta(\delta+\rho)\Big(\mathrm{e}^{(\delta+\rho)t}(\delta+\rho)dt\cdot\mathbb{E}_t\int_{t+dt}^{\infty}\mathrm{e}^{-(\delta+\rho)s}(\Theta\hat{\varphi}_s+\hat{p}_s)ds+\mathrm{e}^{(\delta+\rho)t}H_t dZ_t-(\Theta\hat{\varphi}_t+\hat{p}_t)dt\Big) \\ &= \underbrace{\delta(\delta+\rho)\mathrm{e}^{(\delta+\rho)t}H_t}_{\equiv\sigma_{\pi,t}}dZ_t-\delta(\delta+\rho)\Theta\hat{\varphi}_t dt \\ &\quad+\underbrace{\delta(\delta+\rho)\Big((\delta+\rho)dt\mathbb{E}_t\int_t^{\infty}\mathrm{e}^{-(\delta+\rho)(s-t)}(\Theta\hat{\varphi}_s+\hat{p}_s-\hat{p}_t)ds\Big)}_{=(\cancel{\delta}+\rho)\pi_t dt}.\end{aligned} \tag{I.60}$$

Thus from equation (I.60) we get the following continuous time version of New Keynesian Phillips curve (NKPC).[85]

$$d\pi_t = \rho\pi_t dt-\delta(\delta+\rho)\Theta\hat{\varphi}_t dt+\sigma_{\pi,t}dZ_t. \tag{I.61}$$

We know in flexible price equilibrium, a real marginal cost is given as $\bar{\varphi}$, thus $\hat{\varphi}_t$ can be thought of log-deviation of the marginal cost from the flexible price case, which equals the log-deviation of real wage from the flexible price real wage. Therefore, we obtain:[86]

[85] This form is exactly the same as the Phillips curve in Werning (2012) and Cochrane (2017) if we take expectation.

[86] Here we use log-linearization result of Lemma 2 to represent the real marginal cost gap $\frac{\hat{w}_t}{p_t}$ as a function of capitalists' consumption gap $\hat{C}_t = \hat{Q}_t$.

$$\hat{\varphi}_t = \frac{\widehat{w_t}}{p_t} = \frac{\hat{Q}_t}{\chi^{-1} - \dfrac{\underline{(\epsilon-1)(1-\alpha)}}{\dfrac{\epsilon}{1-\dfrac{(\epsilon-1)(1-\alpha)}{\epsilon}}}} \equiv \frac{\kappa}{\delta(\delta+\rho)\Theta}\hat{Q}_t. \tag{I.62}$$

Finally plugging equation (I.62) into equation (I.61), we represent New-Keynesian Phillips curve in terms of asset price gap $\hat{Q}_t$. We know $\kappa > 0$ due to the Assumption 1.

$$d\pi_t = (\rho\pi_t - \kappa\hat{Q}_t)dt + \sigma_{\pi,t}dZ_t, \text{ and } \mathbb{E}_t d\pi_t = (\rho\pi_t - \kappa\hat{Q}_t)dt, \tag{I.63}$$

which proves the proposition 4.[87]

■

B.2. Section 4

B.2.1. Section 4.2

Proof of Proposition 6. This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984). ■

B.2.2. Section 4.1

Proof of Proposition 5. From equation (55), $\{\sigma_t^q\}$ process can be written in the following way.

$$d\sigma_t^q = -\frac{\phi^2(\sigma_t^q)^2}{2(\sigma+\sigma_t^q)^3}dt - \phi\frac{\sigma_t^q}{\sigma+\sigma_t^q}dZ_t. \tag{I.64}$$

Using Ito's lemma, we get the process for $(\sigma+\sigma_t^q)^2$ which is a martingale, as seen below.

$$\begin{aligned} d(\sigma+\sigma_t^q)^2 &= 2(\sigma+\sigma_t^q)d\sigma_t^q + (d\sigma_t^q)^2 \\ &= 2(\sigma+\sigma_t^q)\Big(-\frac{\phi^2(\sigma_t^q)^2}{2(\sigma+\sigma_t^q)^3}dt - \phi\frac{\sigma_t^q - \sigma^{q,n}}{\sigma+\sigma_t^q}dZ_t\Big) + \phi^2\frac{(\sigma_t^q)^2}{(\sigma+\sigma_t^q)^2}dt \\ &= -2\phi(\sigma_t^q)dZ_t. \end{aligned} \tag{I.65}$$

Therefore, we would have $\mathbb{E}_0((\sigma+\sigma_t^q)^2) = (\sigma+\sigma_0^q)^2$ where $\mathbb{E}_0$ is an expectation operator with respect to the $t=0$ filtration. By the famous Doob's martingale convergence theorem (as $(\sigma+\sigma_t^q)^2 \geq 0, \forall t$), we know $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma^{q,n} = 0$ since:

$$\underbrace{d\sigma_t^q}_{\overset{a.s}{\to}0} = -\underbrace{\frac{\phi^2(\sigma_t^q)^2}{2(\sigma+\sigma_t^q)^3}}_{\overset{a.s}{\to}0}dt - \phi\underbrace{\frac{\sigma_t^q}{\sigma+\sigma_t^q}}_{\overset{a.s}{\to}0}dZ_t. \tag{I.66}$$

[87]Since $\hat{y}_t = \zeta\hat{Q}_t$, Phillips curve can be represented in terms of output gap $\hat{y}_t$ as in Proposition 4.

Therefore, equation (I.66) proves $\sigma_t^q \xrightarrow{a.s} \sigma_\infty^q = \sigma^{q,n} = 0$. From equation (53) $\sigma_t^q \xrightarrow{a.s} \sigma_\infty^q = \sigma^{q,n} = 0$ leads to $\hat{Q}_t \xrightarrow{a.s} 0$ and $\pi_t \xrightarrow{a.s} 0$. Finally, we must have $\mathbb{E}(\max_t(\sigma_t^q)^2) = \infty$, otherwise the uniform integrability says $\mathbb{E}((\sigma + \sigma_\infty^q)^2) = (\sigma + \sigma_0^q)^2$, which is a contradiction to our earlier result $\sigma_t^q \xrightarrow{a.s} \sigma^{q,n}$ since $\sigma_\infty^q = \sigma^{q,n} = 0$ and $\sigma_0^q > \sigma^{q,n} = 0$ by assumption in Proposition 5.

■

B.2.4. Section 5.4

B.2.4.1. Section 5.4.1

Proof of Proposition 8. First we solve the capitalist's problem in equation (79) with τ_t subsidy rate on stock market investment.

$$\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \text{ s.t. } da_t = (a_t(i_t + \theta_t((1+\tau_t)i_t^m - i_t)) - p_t C_t - T_t)dt + \theta_t a_t(\sigma_t + \sigma_t^q)dZ_t. \quad \text{(I.67)}$$

Putting all the relevant state variables $(i_t, \tau_t, T_t, p_t, i_t^m, \sigma_t, \sigma_t^q)$ into vector S_t, then Hamilton-Jacobi-Bellman (HJB) equation can be written in the following way.

$$\begin{aligned} \rho V(a_t, S_t, t) = \max_{C_t,\theta_t} \log C_t &+ \frac{\partial V}{\partial a_t}(a_t(i_t + \theta_t((1+\tau_t)i_t^m - i_t)) - p_t C_t - T_t) + \frac{1}{2}\frac{\partial^2 V}{\partial a^2}\theta_t^2 a_t^2(\sigma_t + \sigma_t^q)^2 + \frac{\partial V}{\partial t} \\ &+ \frac{\partial V}{\partial S_t}\frac{\mathbb{E}_t(dS_t)}{dt} + \frac{1}{2}\mathrm{Tr}\Big(\frac{\partial^2 V}{\partial S_t \partial S_t{}'}\frac{dS_t dS_t{}'}{dt}\Big). \end{aligned} \quad \text{(I.68)}$$

Following Merton (1971), we know the value function has the following form.

$$V(a_t, S_t, t) = \frac{1}{\rho}\log a_t + f(S_t, t). \quad \text{(I.69)}$$

The first-order conditions for C_t and θ_t are easy to compute as follows.

$$p_t C_t = \rho a_t \text{ and } \underbrace{\frac{(1+\tau_t)i_t^m - i_t}{\sigma_t + \sigma_t^q}}_{\text{Sharpe ratio}} = \underbrace{\theta_t(\sigma_t + \sigma_t^q)}_{\text{Price of risk}}. \quad \text{(I.70)}$$

Thus compared to the case without τ_t (equation (I.16)), each capitalist invests more in the stock market, as she gets a higher expected return per unit risk she bears. By plugging the guessed value function form (equation (I.69)) into equation (I.68), we get a partial differential equation (PDE) for the function $f(S_t, t)$, verifying our form in equation (I.69) is a reasonable guess. Here we plug $T_t = a_t \tau_t \theta_t i_t^m$ that holds in equilibrium into equation (I.68).

$$\begin{aligned} \rho f(S_t, t) = &\log \frac{\rho}{p_t} + \frac{1}{\rho}(i_t + \theta_t(i_t^m - i_t) - \rho) - \frac{1}{2\rho}\theta_t^2(\sigma_t + \sigma_t^q)^2 + \frac{\partial f}{\partial t} \\ &+ \frac{\partial f}{\partial S_t{}'}\frac{\mathbb{E}_t(dS_t)}{dt} + \frac{1}{2}\mathrm{Tr}\Big(\frac{\partial^2 f}{\partial S_t \partial S_t{}'}\frac{dS_t dS_t{}'}{dt}\Big) \text{ with } \theta_t = \frac{(1+\tau_t)i_t^m - i_t}{(\sigma_t + \sigma_t^q)^2}. \end{aligned} \quad \text{(I.71)}$$

Thus solving the partial differential equation in equation (I.71) restores the functional form $f(S_t, t)$. In equilibrium, $\theta_t = 1$ holds and it pins down the risk-premium level as follows.

$$i_t^m = \frac{i_t + (\sigma_t + \sigma_t^q)^2}{1 + \tau_t}, \tag{I.72}$$

which is equation (80). A consumption for capitalists C_t thus evolves with the following process.

$$\begin{aligned} \frac{dC_t}{C_t} &= (i_t^m - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dZ_t \\ &= \Big(\frac{i_t + (\sigma_t + \sigma_t^q)^2}{1 + \tau_t} - \pi_t - \rho\Big)dt + (\sigma_t + \sigma_t^q)dZ_t, \end{aligned} \tag{I.73}$$

with which we obtain,

$$d \ln C_t = \Big(\frac{i_t + (\sigma_t + \sigma_t^q)^2}{1 + \tau_t} - \pi_t - \frac{(\sigma_t + \sigma_t^q)^2}{2} - \rho\Big)dt + (\sigma_t + \sigma_t^q)dZ_t, \tag{I.74}$$

from which we subtract the second equation (the process for $\ln C_t^n$) in equation (I.41) and get the following $\hat{C}_t$ process.

$$d\hat{Q}_t = d\hat{C}_t = \Big(\frac{i_t + (\sigma_t + \sigma_t^q)^2}{1 + \tau_t} - \pi_t - \frac{(\sigma_t + \sigma_t^q)^2}{2} - r_t^n - \frac{(\sigma_t)^2}{2}\Big)dt + \sigma_t^q dZ_t. \tag{I.75}$$

In the ZLB situation described in Section 5.1, $i_t = 0$ for $t \leq T$ holds (ZLB) but since the economy gets after T, we have $\sigma_t^q = \sigma_t^{q,n} = 0$ for $t \leq T$. Plugging these conditions into equation (I.75) with $\sigma_t = \bar{\sigma}$ and $r_t^n = r^n(\bar{\sigma}) = \underline{r}$ for $t \leq T$ yields equation (81), thus proving Proposition 8.
■

Proof of Proposition 9. We start from equation (83), the condition that characterizes equilibrium stock market return i_t^m.

$$i_t^m = \frac{y_t - \overbrace{\frac{w_t}{p_t}}^{=C_{W,t} + \frac{T_t}{p_t}} N_{W,t}}{A_t Q_t} + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t}\frac{1}{dt} = \rho - \tau_t i_t^m + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t}\frac{1}{dt}. \tag{I.76}$$

Thus we get $(1 + \tau_t)i_t^m = \rho + \pi_t + g + \mu_t^q + \sigma_t \sigma_t^q$. Due to $(1 + \tau_t)i_t^m = i_t + (\sigma_t + \sigma_t^q)^2$ with the subsidy rate τ_t, we can predict that μ_t^q remains unchanged compared with no subsidy case, given the levels of i_t, σ_t^q. Thus the policy does not change $\{\hat{Q}_t\}$ process. To connect this intuition with the formulas, we start from the following process for C_t, which is different from equation (I.73) since here capitalists do not pay $T_t = a_t \theta_t \tau_t i_t^m$ amount of lump-sum taxes.

$$\begin{aligned} \frac{dC_t}{C_t} &= ((1 + \tau_t)i_t^m - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dZ_t \\ &= (i_t + (\sigma_t + \sigma_t^q)^2 - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dZ_t, \end{aligned} \tag{I.77}$$

where we use the equilibrium condition $(1+\tau_t)i_t^m = i_t + (\sigma_t + \sigma_t^q)^2$ in equation (I.72). Since equation (I.77) is the same as equation (I.40), the dynamics of C_t with $\tau_t = 0$, the policy that subsidizes stock market investment with the lump-sum tax imposed on workers does not have any effect on $\{\hat{Q}_t, \pi_t\}$ and $\{\hat{Q}_t, \pi_t\}$ process remains the same as the economy without τ_t.

■

B.2.4.2. Section 5.4.2

Proof of Proposition 10. A direct fiscal transfer $T_t > 0$ from capitalists to hand-to-mouth workers raises the total amount of dividends in the financial market, leading to a lower required capital gain and a higher $\hat{Q}_t$ at the ZLB. Then stock market return i_t^m in this case can be written in the following way. Here we use the fact that $T_t = \varphi_t p_t A_t Q_t$ holds in equilibrium.

$$
\begin{aligned}
i_t^m &= \frac{A_t N_{W,t} - \overbrace{\frac{w_t}{p_t}}^{=C_{W,t}-\frac{T_t}{p_t}} N_{W,t}}{A_t Q_t} + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t}\frac{1}{dt} = \rho + \underbrace{\frac{T_t}{p_t A_t Q_t}}_{>0} + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t}\frac{1}{dt} \\
&\underbrace{=}_{T_t=\varphi_t p_t A_t Q_t} \rho + \varphi_t + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t}\frac{1}{dt}.
\end{aligned} \tag{I.78}
$$

Thus we see that a dividend yield rises by φ_t, which leads to the case in which capital gain is reduced by φ_t given the level of i_t and σ_t^q. Thus it can mitigate the recession during the ZLB as the asset price Q_t drops less. To derive equation (85), we start from the following capitalist's optimization problem.

$$
\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \ \text{ s.t. } \ da_t = (a_t(i_t + \theta_t(i_t^m - i_t)) - p_t C_t - T_t)dt + \theta_t a_t(\sigma_t + \sigma_t^q)dZ_t, \tag{I.79}
$$

where $T_t = \varphi_t a_t$ holds in equilibrium. The equilibrium conditions for C_t and θ_t are exactly the same as equation (I.16) with $\sigma_t^p = 0$, thus we would have $C_t = \rho p_t A_t Q_t$ and $i_t^m = i_t + (\sigma_t + \sigma_t^q)^2$ in equilibrium. Thus in equilibrium, we get the following wealth process for capitalists.

$$
\frac{da_t}{a_t} = (i_t^m - \rho - \varphi_t)dt + (\sigma_t + \sigma_t^q)dZ_t, \tag{I.80}
$$

which leads to the following consumption process for C_t.

$$
\begin{aligned}
\frac{dC_t}{C_t} &= (i_t^m - \pi_t - \varphi_t - \rho)dt + (\sigma_t + \sigma_t^q)dZ_t \\
&= (i_t + (\sigma_t + \sigma_t^q)^2 - \pi_t - \varphi_t - \rho)dt + (\sigma_t + \sigma_t^q)dZ_t,
\end{aligned} \tag{I.81}
$$

with which we derive

$$
d\ln C_t = \left(i_t + \frac{(\sigma_t + \sigma_t^q)^2}{2} - \pi_t - \varphi_t - \rho\right)dt + (\sigma_t + \sigma_t^q)dZ_t. \tag{I.82}
$$

If we subtract the process for C_t^n in equation (I.41), we get the following $\hat{C}_t$ process.

$$\begin{aligned} d\hat{Q}_t = d\hat{C}_t &= \left(i_t - \pi_t - \varphi_t - \left(r_t^n - \frac{(\sigma_t + \sigma_t^q)^2}{2} + \frac{(\sigma_t)^2}{2}\right)\right)dt + \sigma_t^q dZ_t \\ &= (i_t - \pi_t - \varphi_t - r_t^T)dt + \sigma_t^q dZ_t. \end{aligned} \tag{I.83}$$

In the ZLB situation described in Section 5.1, $i_t = 0$ for $t \le T$ holds (ZLB) but since the economy gets stabilized after T, we have $\sigma_t^q = \sigma_t^{q,n} = 0$ for $t \le T$. Plugging these conditions into equation (I.75) with $\sigma_t = \bar{\sigma}$ and $r_t^n = r^n(\bar{\sigma}) = \underline{r}$ for $t \le T$ yields equation (85), thus proving Proposition 10.
■

B.3. Section 5.3

Proof of Proposition 7. Central bank solves the following problem in the environment in Section 5.3.[88]

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}'} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2 dt, \text{ s.t. } \begin{cases} d\hat{Q}_t = -(\underbrace{r_1^T(\sigma_1^{q,L})}_{<0})dt + (\sigma_1^{q,L})dZ_t, & \text{for } t < T, \\ d\hat{Q}_t = -(\underbrace{r_2^T(\sigma_2^{q,L})}_{>0})dt + (\sigma_2^{q,L})dZ_t, & \text{for } T \le t < \hat{T}', \\ d\hat{Q}_t = 0, & \text{for } t \ge \hat{T}', \\ r_1^T(\sigma_1^{q,L}) \equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} < 0, \\ r_2^T(\sigma_2^{q,L}) \equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma} + \sigma_2^{q,L})^2}{2} > 0, \end{cases} \tag{I.84}$$

with $\hat{Q}_0 = r_1^T(\sigma_1^{q,L})T + r_2^T(\sigma_2^{q,L})(\hat{T}' - T)$.

With $r_1^T(\sigma_1^{q,L}) < 0, r_2^T(\sigma_2^{q,L}) > 0$, a gap process is represented in the following ways (for $\hat{C}_t = \hat{Q}_t$).

$$d\hat{C}_t = \begin{cases} \underbrace{-r_1^T(\sigma_1^{q,L})}_{>0} dt + (\sigma_1^{q,L})dZ_t, & \text{for } t \le T, \\ \underbrace{-r_2^T(\sigma_2^{q,L})}_{<0} dt + (\sigma_2^{q,L})dZ_t, & \text{for } T \le t \le \hat{T}', \\ 0, & \text{for } t \ge \hat{T}'. \end{cases} \tag{I.85}$$

After $\hat{T}$, there is no movement of $\hat{C}_t$ at all. If we let r_s^T be $r_1^T(\sigma_1^{q,L})$ for $s < T$ and $r_2^T(\sigma_2^{q,L})$ for $T \le s \le \hat{T}'$, then gap process can be written in the following integral form. Here Z_t, W_{t-T}, and $U_{\hat{T}-T}$ are independent brownian motion.

[88] In the proof, we implicitly assume that $r_1^T(\sigma_1^{q,L}) < 0$ and $r_2^T(\sigma_2^{q,L}) > 0$ hold for optimal $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ so ZLB binds until T.

$$\hat{C}_t = \begin{cases} \underbrace{\int_t^{\hat{T}'} r_s^T ds}_{\equiv \hat{C}_{det}(t;\hat{T}')} + (\sigma_1^{q,L}) \underbrace{Z_t}_{\sim N(0,t)}, & \text{for } t \leq T, \\ \underbrace{\int_t^{\hat{T}'} r^T(s) ds}_{\equiv \hat{C}_{det}(t;\hat{T}')} + (\sigma_1^{q,L}) Z_T + (\sigma_2^{q,L}) \underbrace{[W_{t-T}]}_{\sim N(0,t-T)}, & \text{for } T < t \leq \hat{T}', \\ \hat{C}_{\hat{T}'} = (\sigma_1^{q,L}) Z_T + (\sigma_2^{q,L}) \underbrace{[W_{\hat{T}-T}]}_{\sim N(0,\hat{T}-T)}, & \text{for } \hat{T}' < t. \end{cases} \tag{I.86}$$

We square each term and take the expectation operator with respect to the information at $t = 0$, when central bank solves its commitment problem. We get the following expressions.

$$\mathbb{E}_0 \hat{C}_t^{\,2} = \begin{cases} \hat{C}_{det}(t;\hat{T}')^2 + (\sigma_1^{q,L})^2 t, & \text{for } t \leq T, \\ \hat{C}_{det}(t;\hat{T}')^2 + (\sigma_1^{q,L})^2 T + (\sigma_2^{q,L})^2 (t-T), & \text{for } T < t \leq \hat{T}', \\ (\sigma_1^{q,L})^2 T + (\sigma_2^{q,L})^2 (\hat{T}' - T), & \text{for } \hat{T}' < t. \end{cases} \tag{I.87}$$

If we plug these expressions into central bank's loss function, then central bank's commitment problem can be represented by the following optimization. Now central bank can control $\sigma_1^{q,L}, \sigma_2^{q,L}$ in addition to its conventional monetary policy tool $\{i_t\}$ (including $\hat{T}'$).

$$\begin{aligned} &\min_{i_t \geq 0, \sigma_1^{q,L}, \sigma_2^{q,L}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{C}_t^2 dt \\ = &\min_{\hat{T}', \sigma_1^{q,L}, \sigma_2^{q,L}} \int_0^{\hat{T}} e^{-\rho t} \hat{C}_{det}(t;\hat{T}')^2 dt + (\sigma_1^{q,L})^2 \underbrace{\int_0^T t e^{-\rho t} dt}_{= \frac{1}{\rho^2} - \frac{1}{\rho^2} e^{-\rho T} - \cancel{\frac{T}{\rho} e^{-\rho T}}} + (\sigma_1^{q,L})^2 T \underbrace{\int_T^\infty e^{-\rho t} dt}_{= \cancel{\frac{1}{\rho} e^{-\rho T}}} \\ &+ (\sigma_2^{q,L})^2 \underbrace{\int_T^{\hat{T}'} e^{-\rho t} (t-T) dt}_{= -\cancel{\frac{1}{\rho} (\hat{T}-T) e^{-\rho \hat{T}'}} + \frac{e^{-\rho T} - e^{-\rho \hat{T}'}}{\rho^2}} + (\sigma_2^{q,L})^2 (\hat{T}' - T) \underbrace{\int_{\hat{T}'}^\infty e^{-\rho t} dt}_{= \cancel{\frac{1}{\rho} e^{-\rho \hat{T}'}}} \\ = &\min_{\hat{T}, \sigma_1^{q,L}, \sigma_2^{q,L}} \underbrace{\int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t;\hat{T}')^2 dt}_{\text{From deterministic fluctuation}} + \underbrace{(\sigma_1^{q,L})^2 \frac{1}{\rho^2} (1 - e^{-\rho T}) + (\sigma_2^{q,L})^2 \left(\frac{e^{-\rho T} - e^{-\rho \hat{T}'}}{\rho^2} \right)}_{\text{From stochastic fluctuation}}. \end{aligned} \tag{I.88}$$

First we get the first-order condition for $\hat{T}'$.

$$2 \underbrace{r_2^T (\sigma_2^{q,L})}_{>0} \int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t;\hat{T}') dt + (\sigma_2^{q,L})^2 \frac{1}{\rho} e^{-\rho \hat{T}'} = 0, \tag{I.89}$$

from which we have:

$$\int_0^\infty e^{-\rho t}\hat{C}_{det}(t;\hat{T}')dt = \int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{det}(t;\hat{T}'\|\sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0)dt < 0. \tag{I.90}$$

The above first-order condition for $\hat{T}'$ shows that the central bank lowers the value of $\hat{T}'$ in the optimum, compared to $\hat{T}$, the duration for which it implements forward guidance only (with $\sigma_1^{q,L} = \sigma_1^{q,n}$ and $\sigma_2^{q,L} = \sigma_2^{q,n}$), thus we would have $\hat{T}' < \hat{T}$ at optimum. The reason is that in the case central bank only implements a forward guidance without financial market intervention, we have the following optimization condition, which is derived by plugging $\sigma_1^{q,L} = 0$ and $\sigma_2^{q,L} = 0$ into equation (I.89).

$$\int_0^{\hat{T}} e^{-\rho t}\hat{C}_{det}(t;\hat{T}\|\sigma_1^{q,L} = \sigma_1^{q,n}, \sigma_2^{q,L} = \sigma_2^{q,n})dt = 0. \tag{I.91}$$

Since we know $\hat{C}_{det}(t;\hat{T}'\|\sigma_1^{q,L} = 0, \sigma_2^{q,L} = 0) < \hat{C}_{det}(t;\hat{T}'\|\sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0)$, from equation (I.84), we infer $\hat{T}' < \hat{T}$ at optimum by comparing equation (I.91) with equation (I.90).
To characterize optimal $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, we need **variational argument**, as $\sigma_1^{q,L}$ and $\sigma_{q,2}$ affects the level of $r_1^T(\sigma_1^{q,L})$, $r_2^T(\sigma_2^{q,L})$, and $\hat{C}_{det}(t;\hat{T}')$ eventually. In specific, we have the following conditions.

$$\frac{\partial r_1^T(\sigma_1^{q,L})}{\partial \sigma_1^{q,L}} = -(\bar{\sigma} + \sigma_1^{q,L}) < 0, \quad \frac{\partial r_2^T(\sigma_2^{q,L})}{\partial \sigma_2^{q,L}} = -(\underline{\sigma} + \sigma_2^{q,L}) < 0. \tag{I.92}$$

Finding $\sigma_1^{q,L}$: If $\sigma_1^{q,L}$ increases, then $r_1^T(\sigma_1^{q,L})$ falls, changing $\hat{C}_{det}(t;\hat{T}')$'s path as the following Figure I.1 indicates (from thick blue to dashed red in Figure I.1). When we differentiate $\hat{C}_{det}(t;\hat{T}')$ with respect to

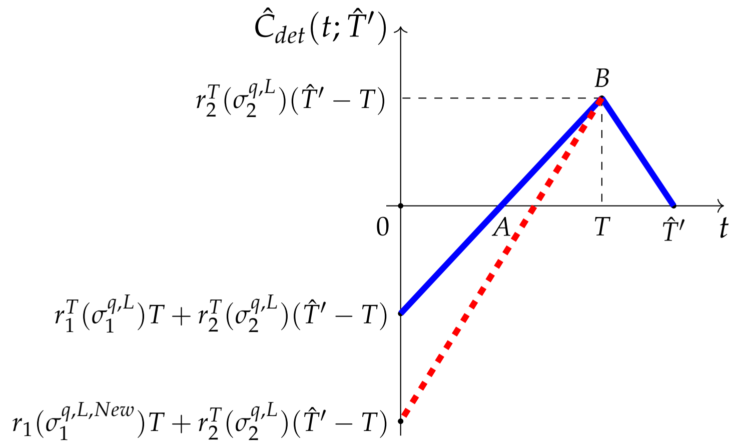


Figure I.1: Variation along $\sigma_1^{q,L}$ Increase to $\sigma_1^{q,L,New} > \sigma_1^{q,L}$

$\sigma_1^{q,L}$, we get the following conditions.

$$\hat{C}_{det}(t;\hat{T}') = \int_t^{\hat{T}'} r_s^T ds, \quad \frac{\partial \hat{C}_{det}}{\partial \sigma_1^{q,L}} = \int_t^T -(\bar{\sigma} + \sigma_1^{q,L})ds = -(\bar{\sigma} + \sigma_1^{q,L})(T - t), \forall t \leq T. \tag{I.93}$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function by $\sigma_1^{q,L}$ and get the following condition.

$$(\bar{\sigma} + \sigma_1^{q,L}) \int_0^T e^{-\rho t} \hat{C}_{det}(t;\hat{T}')(T-t)dt = (\sigma_1^{q,L}) \frac{1-e^{-\rho T}}{\rho^2}, \tag{I.94}$$

from which we can show $\sigma_1^{q,L} < 0$ must be satisfied at optimum, since:

$$\int_0^T e^{-\rho t} \hat{C}_{det}(t;\hat{T}')(T-t)dt = \underbrace{\int_0^t e^{-\rho s} \hat{C}_{det}(s;\hat{T}')ds \cdot (T-t)\Big|_0^T}_{=0} + \int_0^T \Big(\underbrace{\int_0^t e^{-\rho s} \hat{C}_{det}(s;\hat{T}')ds}_{<0} \Big) dt < 0. \tag{I.95}$$

And equation (I.94) implies $\sigma_1^{q,L} < 0$ at optimum.

<u>**Finding** $\sigma_2^{q,L}$</u> : If $\sigma_2^{q,L}$ increases, then $r_2^T(\sigma_2^{q,L})$ falls, changing $\hat{C}_{det}(t;\hat{T}')$'s shape as the following Figure I.2 indicates (from thick blue to dashed red in Figure I.2). When we differentiate $\hat{C}_{det}(t;\hat{T}')$ with respect to $\sigma_2^{q,L}$, we get the following conditions.

$$\frac{\partial \hat{C}_{det}}{\partial \sigma_2^{q,L}} = \begin{cases} \int_T^{\hat{T}'} -(\underline{\sigma} + \sigma_2^{q,L})ds = -(\underline{\sigma} + \sigma_2^{q,L})(\hat{T}' - T), \ \forall t < T, \\ \int_t^{\hat{T}'} -(\underline{\sigma} + \sigma_2^{q,L})ds = -(\underline{\sigma} + \sigma_2^{q,L})(\hat{T}' - t), \ T < \forall t < \hat{T}'. \end{cases} \tag{I.96}$$

To find optimal $\sigma_2^{q,L}$, we differentiate the objective function by $\sigma_2^{q,L}$ and get the following conditions.

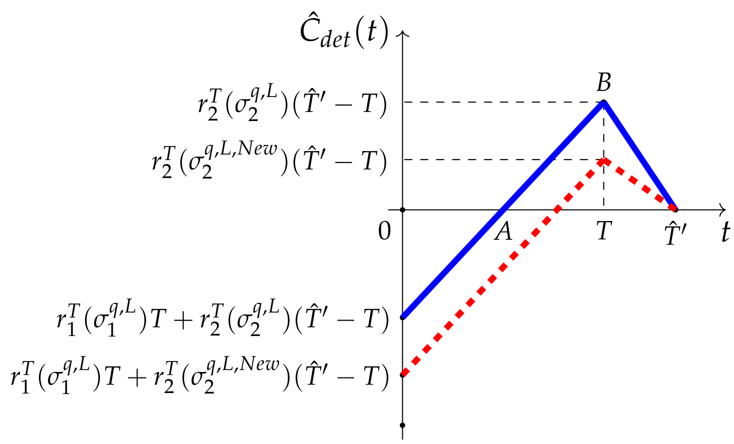


Figure I.2: Variation along $\sigma_2^{q,L}$ Increase to $\sigma_2^{q,L,New} > \sigma_2^{q,L}$

$$(\underline{\sigma} + \sigma_2^{q,L}) \Big(\int_0^T e^{-\rho t} \hat{C}_{det}(t;\hat{T}')(\hat{T}'-T)dt + \int_T^{\hat{T}'} e^{-\rho t} \underbrace{\hat{C}_{det}(t;\hat{T}')}_{>0}(\hat{T}'-t)dt \Big) = (\sigma_2^{q,L}) \frac{e^{-\rho T} - e^{-\rho \hat{T}}}{\rho^2}, \tag{I.97}$$

with which the following equation (I.98) shows $\sigma_2^{q,L} < 0$ holds at the optimum.

$$\begin{aligned}
&\int_0^T e^{-\rho t}\hat{C}_{det}(t;\hat{T}')(\hat{T}'-T)dt + \int_T^{\hat{T}'} e^{-\rho t}\underbrace{\hat{C}_{det}(t;\hat{T}')}_{>0}(\hat{T}'-t)dt \\
&< \int_0^T e^{-\rho t}\hat{C}_{det}(t;\hat{T}')(\hat{T}'-T)dt + \int_T^{\hat{T}'} e^{-\rho t}\underbrace{\hat{C}_{det}(t;\hat{T}')}_{>0}(\hat{T}'-T)dt = (\hat{T}'-T)\underbrace{\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{det}(t;\hat{T}')dt}_{<0} < 0.
\end{aligned} \tag{I.98}$$

Thus we proved that during high TFP volatility period ($t \le T$) and the low TFP volatility period with the forward guidance ($T \le t \le \hat{T}'$), a central bank wants to target financial volatility levels lower than their levels in flexible price economy ($\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma_2^{q,n} = 0$). This intervention lowers the required risk-premium and boost asset price level $\hat{Q}_t$, thus raising the output.
■

<u>**First-order conditions for $\hat{T}', \sigma_1^{q,L}, \sigma_2^{q,L}$**</u> : A deterministic component of capitalists' consumption gap $\hat{C}_t$ process, $\hat{C}_{\text{det}}(t;\hat{T}')$, is given as (with $r_1^T(\sigma_1^{q,L})$ and $r_2^T(\sigma_2^{q,L})$ given in equation (73)):

$$\hat{C}_{\text{det}}(t;\hat{T}') = \int_t^{\hat{T}'} r_s^T ds = \begin{cases} \underbrace{r_1^T(\sigma_1^{q,L})}_{<0}(T-t) + \underbrace{r_2^T(\sigma_2^{q,L})}_{>0}(\hat{T}'-T), & \text{for } \forall t \le T, \\ r_2^T(\sigma_2^{q,L})(\hat{T}'-t), & \text{for } T \le \forall t < \hat{T}', \end{cases} \tag{I.99}$$

based on which we obtain the following formula.

$$\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt = \int_0^T e^{-\rho t}[r_1^T(\sigma_1^{q,L})(T-t) + r_2^T(\sigma_2^{q,L})(\hat{T}'-T)]dt + \int_T^{\hat{T}'} e^{-\rho t} r_2^T(\sigma_2^{q,L})(\hat{T}'-t)dt. \tag{I.100}$$

And we will use the following integration results in this part.

$$\begin{cases} \int_0^T e^{-\rho t}(T-t)dt = \frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}, \quad \int_T^{\hat{T}'} e^{-\rho t}(\hat{T}'-t)dt = \frac{e^{-\rho\hat{T}'}}{\rho^2} + \frac{\hat{T}'-T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}, \\ \int_0^T e^{-\rho t}(T-t)^2 dt = -\frac{2}{\rho^3}e^{-\rho T} + \frac{T^2}{\rho} - \frac{2T}{\rho^2} + \frac{2}{\rho^3}, \\ \int_T^{\hat{T}'} e^{-\rho t}(\hat{T}'-t)^2 dt = -\frac{2}{\rho^3}e^{-\rho\hat{T}'} + \frac{(\hat{T}'-T)^2}{\rho}e^{-\rho T} - \frac{2(\hat{T}'-T)}{\rho^2}e^{-\rho T} + \frac{2}{\rho^3}e^{-\rho T}. \end{cases} \tag{I.101}$$

The first condition (first-order condition for $\hat{T}'$) can be written as:

$$2r_2^T(\sigma_2^{q,L})\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt + (\sigma_2^{q,L})^2\frac{e^{-\rho\hat{T}'}}{\rho} = 0, \tag{I.102}$$

where

$$\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt = r_1^T(\sigma_1^{q,L})\Big[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\Big]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho} \\ + r_2^T(\sigma_2^{q,L})\Big[\frac{e^{-\rho\hat{T}'}}{\rho^2}+\frac{\hat{T}'-T}{\rho}e^{-\rho T}-\frac{1}{\rho^2}e^{-\rho T}\Big]. \tag{I.103}$$

We plug all the integration and get the following expression for the first-order condition for $\hat{T}'$.

$$2r_2^T(\sigma_2^{q,L})\Big\{r_1^T(\sigma_1^{q,L})\Big[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\Big]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho} \\ + r_2^T(\sigma_2^{q,L})\Big[\frac{e^{-\rho\hat{T}'}}{\rho^2}+\frac{\hat{T}'-T}{\rho}e^{-\rho T}-\frac{1}{\rho^2}e^{-\rho T}\Big]\Big\}+(\sigma_2^{q,L})^2\frac{e^{-\rho\hat{T}'}}{\rho}=0. \tag{I.104}$$

The above equation (I.104) has all of $\{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}'\}$ as $r_1^T(\sigma_1^{q,L})$ and $r_2^T(\sigma_2^{q,L})$ are functions of $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, respectively.
The second condition (first-order condition for $\sigma_1^{q,L}$ can be written as:

$$(\bar{\sigma}+\sigma_1^{q,L})\int_0^T e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')(T-t)dt = (\sigma_1^{q,L})\frac{1-e^{-\rho T}}{\rho^2}, \tag{I.105}$$

where

$$\int_0^T e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')(T-t)dt = r_1^T(\sigma_1^{q,L})\Big[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\Big]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\Big[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\Big]. \tag{I.106}$$

Plugging equation (I.106) into equation (I.105), we get the following first-order condition for the $\sigma_1^{q,L}$.

$$(\bar{\sigma}+\sigma_1^{q,L})\Big\{r_1^T(\sigma_1^{q,L})\Big[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\Big]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\Big[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\Big]\Big\}=(\sigma_1^{q,L})\frac{1-e^{-\rho T}}{\rho^2}. \tag{I.107}$$

Finally, the first-order condition for the $\sigma_2^{q,L}$ becomes:

$$(\underline{\sigma}+\sigma_2^{q,L})\Big((\hat{T}'-T)\int_0^T e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt+\int_T^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')(\hat{T}'-t)dt\Big)=(\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}'}}{\rho^2}, \tag{I.108}$$

where

$$\int_0^T e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt = r_1^T(\sigma_1^{q,L})\Big[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\Big]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho}, \tag{I.109}$$

and

$$\int_T^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')(\hat{T}'-t)dt = r_2^T(\sigma_2^{q,L})\Big[-\frac{2}{\rho^3}e^{-\rho\hat{T}'}+\frac{(\hat{T}'-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}'-T)}{\rho^2}e^{-\rho T}+\frac{2}{\rho^3}e^{-\rho T}\Big]. \tag{I.110}$$

Thus the first-order condition for the $\sigma_2^{q,L}$ can be written as:

$$\begin{aligned}
&(\underline{\sigma}+\sigma_2^{q,L})\Big\{\Big[r_1^T(\sigma_1^{q,L})\Big[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\Big]+r_2^T(\sigma_2^{q,L})(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho}\Big](\hat{T}'-T) \\
&\qquad\qquad +r_2^T(\sigma_2^{q,L})\Big[-\frac{2}{\rho^3}e^{-\rho\hat{T}'}+\frac{(\hat{T}'-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}'-T)}{\rho^2}e^{-\rho T}+\frac{2}{\rho^3}e^{-\rho T}\Big]\Big\} \qquad \text{(I.111)} \\
&=(\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}'}}{\rho^2}.
\end{aligned}$$

D Welfare Derivation

D.1 Efficient steady State (Efficient Flexible Price Equilibrium) with a production subsidy

C.1.1. First-Best Allocation

A first-best allocation must be the solution of the following optimization problem.

$$\max_{C_t, N_{W,t}, C_{W,t}} \omega_1 \log \frac{C_t}{A_t} + \omega_2 \left(\frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \right) \text{ s.t } C_t + C_{W,t} = A_t N_{W,t}, \tag{I.112}$$

where $\omega_1 > 0$ and $\omega_2 > 0$ are two welfare weights attached to capitalists and workers, respectively, and we assume no price dispersion, thus $\Delta_t = 1$. For the expositional purposes, let us define $x_t \equiv N_{W,t}$ and $y_t \equiv \frac{C_{W,t}}{A_t}$, then the first-order condition for equation (I.112) can be written as

$$y_t^{-\varphi} = x_t^{\chi_0}, \ \frac{\omega_1}{\omega_2} = x_t^{\chi_0}(x_t - y_t). \tag{I.113}$$

C.1.2. Workers' and Firms' Problem

Now we introduce a production subsidy $\tau > 0$ given to the firms, assuming that it is financed through a lump-sum tax on workers. Our objective is to make sure our flexible price equilibrium (or steady-state) allocation $(N^n_{W,t}, \frac{C^n_{W,t}}{A_t}, \frac{C^n_t}{A_t})$ is efficient and satisfies equation (I.113). With the subsidy τ, workers solve the following problem.

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \text{ s.t. } p_t C_{W,t} = w_t N_{W,t} - p_t T_t, \tag{I.114}$$

where $T_t = \tau y_t$ is the (real) lump-sum tax amount imposed on workers. The equation (I.114)'s first order condition is written as:

$$(N_{W,t})^{\chi_0+\varphi} \left(\frac{w_t}{p_t A_t} - \tau \right)^{\varphi} = \frac{w_t}{p_t A_t}, \tag{I.115}$$

where we express the $N_{W,t}$ that satisfies equation (I.115) as a function of a normalized real wage $\frac{w_t}{p_t A_t}$, assuming $N_{W,t} \equiv f_N(\frac{w_t}{p_t A_t})$. When $\tau = 0$, it returns to equation (I.12). Due to τ, $N_{W,t}$ rises, compared to the amount implied by equation (I.12), since (i) workers feel poorer due to the lump-sum tax T_t, thus a higher marginal utility of consumption induces them to work more. (ii) eventually firms' labor demand would rise, which raises the labor supply of workers.
Since we are dealing with the flexible price economy benchmark, each firm's optimization is changed with the introduction of τ in the following way, with $E_t = (N_{W,t})^{\alpha}$.

$$\max_{p_t(i)} (1+\tau) p_t(i) \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon} y_t - w_t \left(\frac{y_t}{A_t E_t} \right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t} \right)^{\frac{-\epsilon}{1-\alpha}}, \tag{I.116}$$

where $p_t(i) = p_t$ for $\forall i$ at optimum and the solution of equation (I.116) features

$$\frac{w_t^n}{p_t^n A_t} = \frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}, \tag{I.117}$$

where we plug equation (I.117) into equation (I.115) and obtain

$$\begin{aligned} N_{W,t}^n &= f_N\Big(\frac{w_t^n}{p_t^n A_t}\Big) = f_N\Big(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\Big), \\ \frac{C_{W,t}^n}{A_t} &= \frac{w_t^n}{p_t^n A_t} f_N\Big(\frac{w_t^n}{p_t^n A_t}\Big) - \tau f_N\Big(\frac{w_t^n}{p_t^n A_t}\Big) = \Big[\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau\Big] f_N\Big(\frac{w_t^n}{p_t^n A_t}\Big). \end{aligned} \tag{I.118}$$

Since our goal is to align the allocation implied by equation (I.118) with the first-best allocation implied by equation (I.113), $N_{W,t}^n$ and $\frac{C_{W,t}^n}{A_t}$ in equation (I.118) must satisfy equation (I.113) as follows.

$$\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau = f_N\Big(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\Big)^{-\frac{\chi_0+\varphi}{\varphi}}. \tag{I.119}$$

Plugging equation (I.117) into equation (I.115), we get:

$$(N_{W,t}^n)^{\chi_0+\varphi}\Big(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau\Big)^{\varphi} = \frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}. \tag{I.120}$$

Solving jointly equation (I.119) and equation (I.120), we conclude the optimal τ^* must satisfies the following familiar condition.

$$\frac{(1+\tau^*)(\epsilon-1)(1-\alpha)}{\epsilon} = 1.^{89} \tag{I.121}$$

Therefore, the optimal $\tau^* > 0$ eliminates mark-up of firms and restores efficiency. With $\tau = \tau^*$, normalized real wage becomes 1 and we get the following benchmark efficient allocation from equation (I.118).

$$N_{W,t}^n \equiv \bar{x} = (1-\tau^*)^{-\frac{\varphi}{\chi_0+\varphi}}, \quad \frac{C_{W,t}^n}{A_t} \equiv \bar{y} = (1-\tau^*)^{\frac{\chi_0}{\chi_0+\varphi}}, \quad \frac{C_t^n}{A_t} = \bar{x} - \bar{y} = (1-\tau^*)^{-\frac{\varphi}{\chi_0+\varphi}}\tau^*. \tag{I.122}$$

The last step is to ensure the welfare weights $\omega_1 > 0$ and $\omega_2 > 0$ satisfy equation (I.113).[90] By plugging equation (I.122) into the second condition of equation (I.113), we obtain

$$\frac{\omega_1}{\omega_2} = (N_{W,t}^n)^{\chi_0}\Big(N_{W,t}^n - \frac{C_{W,t}^n}{A_t}\Big) = (1-\tau^*)^{-\frac{(\chi_0+1)\varphi}{\chi_0+\varphi}} \cdot \tau^*. \tag{I.123}$$

Thus, with $\omega_1 > 0$ and $\omega_2 > 0$ satisfying equation (I.123), our allocation with $\tau = \tau^*$ is efficient. The next step is to approximate a joint welfare in equation (I.112) with those ω_1, ω_2 up to a second-order and express it in terms of gap variables and the price dispersion.

[89] Therefore, τ^* is a function of primitive parameters ϵ and α.

[90] Since ω_1 and ω_2 are chosen arbitrarily, we make sure that our allocation with a production subsidy can be on the efficient frontier, which is generated by a varying set of $\{\omega_1, \omega_2\}$.

D.2 Derivation of a Quadratic Loss Function

As we previously defined, let $x_t \equiv N^W(t)$, $y_t \equiv \frac{C_{W,t}}{A_t}$. Their steady-state values (flexible price equilibrium values) are the ones in equation (I.122). From the economy-wide resource constraint, we express

$$\frac{C_t}{A_t} = \frac{N_{W,t}}{\Delta_t} - \frac{C_{W,t}}{A_t} = \frac{x_t}{\Delta_t} - y_t, \text{ where } \Delta = \left(\int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} di\right)^{1-\alpha}. \tag{I.124}$$

With equation (I.124), we express our social welfare in equation (I.112) with ω_1 and ω_2 satisfying equation (I.123) as follows.

$$U(x_t, y_t, \Delta_t) \equiv \omega_1 \log\left(\frac{x_t}{\Delta_t} - y_t\right) + \omega_2\left(\frac{y_t^{1-\varphi}}{1-\varphi} - \frac{x_t^{1+\chi_0}}{1+\chi_0}\right), \tag{I.125}$$

which achieves its maximum value $\bar{U}$ when $x_t = \bar{x}, y_t = \bar{y}, \Delta_t = 1$.[91] A second-order approximation of equation (I.125) around the efficient benchmark allocation $(\bar{x}, \bar{y}, 1)$ in equation (I.122) results in the following expression.

$$U_t - \bar{U} = U_\Delta \cdot \bar{\Delta} \cdot \hat{\Delta}_t + \frac{1}{2} U_{xx} \cdot \bar{x}^2 \cdot (\hat{x}_t)^2 + \frac{1}{2} U_{yy} \cdot \bar{y}^2 \cdot (\hat{y}_t)^2 + U_{xy} \cdot \bar{x} \cdot \bar{y} \cdot \hat{x}_t \cdot \hat{y}_t + h.o.t, \tag{I.126}$$

where $\bar{\Delta} = 1$ since we do not have a steady-state inflation and all partial derivatives $(U_\Delta, U_{xx}, U_{yy}, U_{xy})$ are evaluated at the benchmark point $(\bar{x}, \bar{y}, 1)$ as follows.

$$\begin{aligned}
U_\Delta &= -\omega_2 (1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}, \\
U_{xx} &= -\omega_2 (1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \left(\frac{1}{\tau^*} + \chi_0\right), \\
U_{yy} &= -\omega_2 (1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \left(\frac{1}{\tau^*} + \frac{\varphi}{1-\tau^*}\right), \\
U_{xy} &= \omega_2 (1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \frac{1}{\tau^*},
\end{aligned} \tag{I.127}$$

where we use the relation between ω_1 and ω_2 in equation (I.123) in the process of derivation. Since ω_2 can be regarded a free parameter, we set $\omega_2 \equiv 1$.

Log-Linearization With the price dispersion Δ_t, the hand-to-mouth worker's problem with τ^* features the following solution.

$$\begin{aligned}
(N_{W,t})^{\chi_0+\varphi}\left(\frac{w_t}{p_t A_t} - \frac{\tau^*}{\Delta_t}\right)^{\varphi} &= \frac{w_t}{p_t A_t}, \\
\frac{C_{W,t}}{A_t} &= \left(\frac{w_t}{p_t A_t} - \frac{\tau^*}{\Delta_t}\right) N_{W,t}.
\end{aligned} \tag{I.128}$$

Linearizing the first equation around the benchmark allocation yields

$$\widehat{N_{W,t}} = \frac{1 - \frac{\varphi}{1-\tau^*}}{\chi_0 + \varphi} \widehat{\left(\frac{w_t}{p_t}\right)} - \frac{\frac{\varphi\tau^*}{1-\tau^*}}{\chi_0+\varphi} \widehat{\Delta}_t. \tag{I.129}$$

[91] We have $U_x = U_y = 0$ at $(\bar{x}, \bar{y}, 1)$, where U_x and U_y are the partial derivatives with respect to x_t and y_t, respectively.

Linearizing the second consumption equation yields

$$\widehat{C_{W,t}} = \frac{1+\frac{\chi_0}{1-\tau^*}}{\chi_0+\varphi}\widehat{\left(\frac{w_t}{p_t}\right)} + \frac{\tau^*}{1-\tau^*}\frac{\chi_0}{\chi_0+\varphi}\widehat{\Delta}_t. \tag{I.130}$$

Finally, by linearizing the economy-wide resource constraint (equation (I.124)) with $\hat{Q}_t = \hat{C}_t$ and solving jointly with equation (I.129) and equation (I.130), we can express gaps in real wage, labor supply, and workers' consumption as functions of gaps in asset price and price dispersion as follows.

$$\begin{aligned}
\widehat{\left(\frac{w_t}{p_t}\right)} &= \frac{\tau^*(\chi_0+\varphi)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\hat{Q}_t + \frac{\tau^*\left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\widehat{\Delta}_t, \\
\hat{x}_t \equiv \widehat{N_{W,t}} &= \frac{\tau^*\left(1 - \frac{\varphi}{1-\tau^*}\right)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\hat{Q}_t + \frac{\tau^*}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\widehat{\Delta}_t, \\
\hat{y}_t \equiv \widehat{C_{W,t}} &= \frac{\tau^*\left(1 + \frac{\chi_0}{1-\tau^*}\right)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\hat{Q}_t + \frac{\frac{\tau^*}{1-\tau^*}}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}.\widehat{\Delta}_t
\end{aligned} \tag{I.131}$$

Plugging equation (I.127) into the second-order approximation equation (equation (I.126)), we get the following expression

$$\begin{aligned}
U_t - \bar{U} &= -(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\widehat{\Delta}_t - \frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)(1-\tau^*)^{\frac{-2\varphi}{\chi_0+\varphi}}(\hat{x}_t)^2 \\
&\quad - \frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\frac{\varphi}{1-\tau^*}\right)(1-\tau^*)^{\frac{2\chi_0}{\chi_0+\varphi}}(\hat{y}_t)^2 + (1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\frac{1}{\tau^*}(1-\tau^*)^{\frac{\chi_0-\varphi}{\chi_0+\varphi}}\hat{x}_t\hat{y}_t \\
&= -(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\widehat{\Delta}_t - \frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)(\hat{x}_t)^2 \\
&\quad - \frac{1}{2}(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\left(\frac{1-\tau^*}{\tau^*}+\varphi\right)(\hat{y}_t)^2 + (1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\frac{1}{\tau^*}\hat{x}_t\hat{y}_t.
\end{aligned} \tag{I.132}$$

Finally by plugging equation (I.131) into equation (I.132), we get the following expression for $U_t - \bar{U}$:

$$U_t - \bar{U} = \tilde{\gamma}_\Delta\widehat{\Delta}_t + \tilde{\gamma}_q(\hat{Q}_t)^2 + h.o.t \text{ with } \tilde{\gamma}_\Delta = -(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}} \text{ and,} \tag{I.133}$$

$$\begin{aligned}
\tilde{\gamma}_q &= -\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)\left(\frac{\tau^*\left(1-\frac{\varphi}{1-\tau^*}\right)}{\tau^* - \left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)^2 - \frac{1}{2}(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\left(\frac{1-\tau^*}{\tau^*}+\varphi\right)\left(\frac{\tau^*\left(1+\frac{\chi_0}{1-\tau^*}\right)}{\tau^* - \left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)^2 \\
&\quad + (1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\frac{1}{\tau^*}\left(\frac{\tau^*\left(1-\frac{\varphi}{1-\tau^*}\right)}{\tau^* - \left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)\left(\frac{\tau^*\left(1+\frac{\chi_0}{1-\tau^*}\right)}{\tau^* - \left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right),
\end{aligned} \tag{I.134}$$

where $\tilde{\gamma}_\Delta < 0$ and $\tilde{\gamma}_q < 0$ with our calibrated parameters.

Loss function Finally, we express dynamic loss function as

$$L_0(\{\hat{Q}_t, \Delta_t\}_{t\geq 0}) = \mathbb{E}_0 \int_0^\infty \left(\hat{Q}_t^2 + \frac{\tilde{\gamma}_\Delta}{\tilde{\gamma}_q}\Delta_t\right) dt, \tag{I.135}$$

and furthermore, we know the following relation from Woodford (2003).

$$\widehat{\Delta}_t = \frac{\epsilon}{2\Theta}\mathrm{Var}_i(p_t(i)), \text{ and } \int_0^\infty e^{-\rho t}\mathrm{Var}_i(p_t(i))dt = \frac{1}{\delta(\delta+\rho)}\int_0^\infty e^{-\rho t}\pi_t^2 dt. \tag{I.136}$$

By plugging equation (I.136) into equation (I.135) and expressing the loss function L as a function of $\{\hat{Q}_t\}$ and $\{\pi_t\}$, we finally obtain

$$L((\{\hat{Q}_t, \pi_t\}_{t\geq 0}) = \mathbb{E}_0 \int_0^\infty \left(\hat{Q}_t^2 + \Gamma\pi_t^2\right) dt, \text{ with } \Gamma \equiv \frac{1}{\delta(\delta+\rho)}\frac{\epsilon}{2\Theta}\frac{\tilde{\gamma}_\Delta}{\tilde{\gamma}_q} > 0, \tag{I.137}$$

which is equation (65).