Monetary Policy as a Financial Stabilizer : A Simple Discrete-Time Analysis*

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Abstract

This note offers a simple discrete-time framework which provides an illustration for the more general results provided in Lee and Carreras (2021), a much richer continuous-time environment.[1]

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[1] Still, we recommend readers to read Lee and Carreras (2021) for the more detailed treatment of the issue.

1 Model

1.1 Setting

Discrete-time framework offers a simpler description of key important mechanisms and the similar policy prescriptions to the fully fledged continuous-time model of Lee and Carreras (2021), at the expense of rigor and with the help of approximation techniques.[2] As in Lee and Carreras (2021), our discrete-time model features a possible indeterminacy that arises when central bank does not target the endogenous second-order variable (such as financial volatility and risk-premium). Finally, we analyze various fiscal-monetary policies that help to mitigate the crisis caused by zero lower bound (ZLB) in conjunction with our own mechanism.

Here, time is discrete, starting from $t = 0$, and a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}}, \mathbb{P})$ is fixed with a single aggregate technology shock that drives business cycle fluctuation. There are a measure 1 of capitalists and the same measure of hand-to-mouth workers, and agents consume a single final good.[3] For simplicity, we assume inflation away and thus, a final good producer takes the rigid price as given. It obviously is not realistic, but allows us to focus on new economic channels we emphasize through our model. With the perfect nominal rigidity imposed, the economy is perfectly demand-determined.

1.2 Workers, Firms, and the Financial Market

There are a measure 1 of identical firms that produce a single final good. Each firm faces a perfect nominal rigidity in price setting, taking the rigid price $p = 1$[4] as given. N_t labors hired by firms produce $Y_t = A_t N_t$ outputs, where A_t is the aggregate productivity and a martingale, of which the process is given as

$$\mathbb{E}_t(A_{t+1}) = A_t, \ \ \mathrm{Var}_t\Big(\frac{A_{t+1}}{A_t}\Big) = \sigma_t^2. \tag{1}$$

In most cases, we assume σ_t to be constant, but will assume the exogenous change in σ_t when we analyze the dynamics later.

Labor is demand-determined in equilibrium as the output is demand-determined, but we assume for simplicity that the wage w_t is determined to be proportional to the aggregate productivity A_t with $w_t = \tilde{w} A_t$ for some $\tilde{w} < 1$.[5] This assumption ensures that the wage is flexible and procyclical and would be microfounded in Lee and Carreras (2021). Since workers are hand-to-mouth, they supply demand-determined labors, get the paychecks from firms, and consume all of them in

[2]Here the possible complications including monopolistic competition among firms and the optimal consumption-labor decisions of workers are abstracted away, for simplicity.

[3]In Lee and Carreras (2021), we adapt the monopolistic competition structure among intermediate good firms. Here we assume that the final good equals an intermediate good as each producer (firm) faces a perfect rigidity (with the same price) in price setting.

[4]Therefore, there is no distinction between nominal and the real variables in this version.

[5]Therefore, our economy in this version is not Walrasian and features an unemployment, as seen below.

every period.

We assume the full-employment (first-best) level of employment at every period t is given as $\bar{N}$.[6] A full-employment output level at period t is therefore $\overline{Y}_t = A_t\bar{N}$, which is proportional to the technology A_t.

Firms' profits are pooled[7] and securitized in the financial (stock) market. We let Q_t and Q_{t+1} be the entire stock market (after-dividend) wealth at time t and $t+1$, respectively, with the following relation:[8]

$$\mathrm{Var}_t\Big(\frac{Q_{t+1}}{Q_t}\Big) = (\sigma_t^q)^2, \tag{2}$$

where σ_t^q is an 'endogenous' asset price (financial) volatility at time t, given the time t's information, which is one of the most crucial variables in this paper. A stock market investment at time t yields the one-period stochastic return r_{t+1}, whose expected value and variance are given as $\mathbb{E}_t(r_{t+1})$ and $(\sigma_t^q)^2$, respectively. The return r_{t+1} is composed of the dividend yield from firms' profits and capital gain from stock price changes.

Finally, there is a zero-net supplied risk-free bond market at each period t, where the one-period risk-free rate i_t is determined by the central bank through monetary policy, and constrained by zero lower bound (ZLB). Thus we assume that $i_t \geq 0$ holds for every period t.

1.3 Capitalists

At each period t, each capitalist takes price variables i_t, $\mathbb{E}_t(r_{t+1})$, and σ_t^q as given and choose their portfolio decision. Each capitalist with wealth W_t solves the following optimization at time t .

$$\begin{aligned} &\max_{\{C_{t+s},\theta_{t+s}\}_{s\geq 0}} \mathbb{E}_t \sum_{s=0}^{\infty} \phi_q^s \log C_{t+s} \ \text{ s.t.} \\ &W_{t+1} = (W_t - C_t)(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t})), \forall t, \end{aligned} \tag{3}$$

where the gross risk-free rate and the risky rate are $R_{f,t} = 1 + i_t$ and $R'_{t+1} = 1 + r_{t+1}$, respectively, and C_t and θ_t are period-t consumption and the share of (after-consumption) wealth invested in the stock market.

With some approximation[9], we get the following mean-variance optimal portfolio share demand

[6] There is no population growth in this version as $\bar{N}$ is constant over time. In Lee and Carreras (2021), the flexible price equilibrium is microfounded and we get to endogenize the first-best $\bar{N}$ and $\overline{Y}_t$.

[7] Since all firms are identical in terms of price, wage, and the production amount, we can regard the entire firms as one big firm that produces the final good and earns profits.

[8] Q_{t+1} is random from the perspective of period t information, and the asset price growth from Q_t to Q_{t+1} has the volatility σ_t^q.

[9] In the continuous time model of Lee and Carreras (2021), we do not rely on similar approximation tecniques. In particular, here we assume away multiples of first-order variables, which naturally disappears in the continuous time model as $(dt)^2 \to 0$, and keep quadratic variation related terms $(\sigma_t^q)^2$ be alive.

for the stock.

$$\theta_t = \frac{\mathbb{E}_t(r_{t+1}) - i_t}{(\sigma_t^q)^2}, \tag{4}$$

where, as derived in Merton (1971), we can interpret the above equation in the following way.

$$\underbrace{\frac{\mathbb{E}_t(r_{t+1}) - i_t}{\sigma_t^q}}_{\text{Sharpe ratio}} = \underbrace{\theta_t \sigma_t^q}_{\text{Price of risk}}. \tag{5}$$

A log-preference of capitalists makes a price of risk and its quantity they bear equal, where the quantity of risk is given as $\theta_t \sigma_t^q$, when they invest θ_t portion of wealth into the risky stock. Thus, the optimal portfolio share θ_t is the one that equalizes price of risk with the market Sharpe ratio. We can observe that optimal share in the stock market θ_t increases with the risk premium $\mathbb{E}_t(r_{t+1}) - i_t$ and decreases with the volatility σ_t^q. Given the value of risk premium, higher financial volatility makes risk-averse capitalists less willing to bear a risk, thus inducing them to invest less in the stock market.

With the approximation $\phi_q = e^{-\rho} \simeq 1 - \rho$, we show that each capitalist with wealth W_t chooses her consumption at optimum as

$$C_t = \underbrace{\rho}_{\text{Discount rate}} W_t, \tag{6}$$

where ρ is capitalists' time discount rate. The optimal consumption is proportional to the wealth level of capitalists, and the less patient they are, they consume more out of their financial wealth. Financial markets in our paper affect the real economy through this wealth channel, by changing consumption level of capitalists as in Caballero and Simsek (2020).[10]

Next we combine the above conditions to characterize equilibria in both good market and financial market.

1.4 Good Market Equilibrium

Since a risk-free bond is zero-net supplied[11] and all capitalists are identical both ex-ante and ex-post, each one holds exactly the same amount of financial wealth $W_t = Q_t$ and therefore, consumes $C_t = \rho Q_t$ in equilibrium.

As we already discussed, all firms are identical, taking the rigid price $p = 1$ as given, and the

[10] Bernanke and Gertler (2000), based on the previous work (Bernanke et al. (1999)), considered two channels through which financial markets affect the economy: wealth channel and financial accelerator channel in which external finance premia depend inversely on the net financial worth of entrepreneurs, to analyze how monetary policy should respond to stock market fluctuations. They relied on the result of Ludvigson and Steindel (2007) that elasticity of entrepreneurial consumption to the stock market wealth is about 0.04. Recently Chodorow-Reich et al. (2021), through the local labor market approach, found the 3.2 cents increase in consumption out of 1 dollar increase in stock market wealth.

[11] It simplifies the analysis and allows us to focus on the asset price dynamics in tandem with the business cycle fluctuation.

wage is determined to be $w_t = \tilde{w} A_t$, where A_t is the aggregate technology level. At period t, each firm hires N_t labors to satisfy the good demand for good, as the economy is demand-determined. Since all workers are hand-to-mouth, they provide N_t labors, get $w_t N_t$ amount of paychecks, and spend all on consumption. Therefore, good market equilibrium condition at period t can be written as

$$\underbrace{\rho Q_t}_{\text{Capitalists' demand}} + \underbrace{w_t N_t}_{\text{Workers' demand}} = A_t N_t (= Y_t), \tag{7}$$

where we plug $w_t = \tilde{w} A_t$ into the above equation and get the following relation between Y_t and Q_t:

$$Y_t = A_t N_t = \underbrace{\frac{\rho}{1-\tilde{w}}}_{\equiv \alpha > 0} Q_t = \alpha \cdot Q_t. \tag{8}$$

We see an asset price Q_t acts as an endogenous aggregate demand shifter in the economy where the rigid price faced by firms ensures that the equilibrium output is demand-determined. A higher asset price Q_t raises the consumption demand by capitalists, which increases firms' labor demand, which again raises the consumption demand of workers, and ad infinitum. More specifically, if Q_t rises by 1, C_t will go up by ρ, which raises N_t by $\frac{\rho}{A_t}$ and workers' consumption by $\tilde{w}\rho$, which again increases N_t by $\frac{\tilde{w}\rho}{A_t}$ and workers' consumption by $\tilde{w}^2\rho$, and ad infinitum. In sum, a unit increase in Q_t results in an α increase in the output, where α is calculated as

$$\rho + \tilde{w}\rho + \tilde{w}^2\rho + ... = \frac{\rho}{1-\tilde{w}} \equiv \alpha. \tag{9}$$

In sum, an endogenous asset price Q_t shifts the level of aggregate demand and affects all levels of economic activities in our economy as our economy is demand-determined.

Since $\bar{N}$ is the full-employment level of employment, $\overline{Y}_t = A_t \bar{N}$ can be regarded an efficient (target) output level. Then from the previous relation between Q_t, and Y_t, we can reverse-engineer the efficient asset price level $\overline{Q}_t$ that satisfies $\overline{Y}_t = \alpha \overline{Q}_t$. An efficient asset price level $\overline{Q}_t$ is given by[12]

$$\overline{Q}_t = \frac{1-\tilde{w}}{\rho} \bar{Y}_t = \frac{1-\tilde{w}}{\rho} \bar{N} \cdot A_t. \tag{10}$$

We see both $\overline{Y}_t$ and $\overline{Q}_t$ are procyclical and proportional to A_t, with following variance relations.

$$\mathrm{Var}_t\Big(\frac{\overline{Y}_{t+1}}{\overline{Y}_t}\Big) = \mathrm{Var}_t\Big(\frac{\overline{Q}_{t+1}}{\overline{Q}_t}\Big) = \mathrm{Var}_t\Big(\frac{A_{t+1}}{A_t}\Big) = \sigma_t^2. \tag{11}$$

For exposition purposes, we define the period-t natural level of stock price volatility $\sigma_t^{q,n}$, a conditional variance of $\frac{\overline{Q}_{t+1}}{\overline{Q}_t}$. We obtain $\sigma_t^{q,n} = \sigma_t$ from equation (11), which is the volatility of A_t growth.

[12] Also we can define $\bar{C}_t = \rho \overline{Q}_t = (1-\tilde{w}) \bar{N} A_t$ as the first-best consumption level of capitalists.

Basically, the first-best economy entails a perfectly procyclical financial market, and its volatility aligns with that of the (real) business cycle. We regard $\overline{Y}_t$ and $\overline{Q}_t$ as the levels policymakers want to achieve using the monetary policy rule. In our continuous time framework of Lee and Carreras (2021), $\overline{Y}_t$ and $\overline{Q}_t$ are microfounded in the flexible price environment. Now that we have necessary building blocks, we move onto the analysis of how asset price Q_t is determined in general when nominal rigidity is present.

1.5 Equilibrium Asset Pricing and Log-Approximation

In equilibrium, capitalists put all of their wealth into the stock market ($\theta_t = 1$), as a net bond supply is 0. With the help of equation (4), we can pin down a stock market's expected return as

$$\mathbb{E}_t(r_{t+1}) = i_t + (\sigma_t^q)^2. \tag{12}$$

We observe that in equilibrium, the higher the financial volatility σ_t^q is, capitalists ask for a higher premium when investing in the stock market. In fact equation (12) implies $(\sigma_t^q)^2$ is an equilibrium level of equity premium.

Profits of firms are distributed as dividends to capitalists, who naturally become stock owners. As Q_t is the endogenous aggregate demand shifter, a total dividend $D(Q_t)$ also depends on Q_t and is calculated from the good market equilibrium condition as

$$D(Q_t) = Y_t - w_t N_t = \rho \cdot Q_t = C_t, \tag{13}$$

where C_t is the capitalist's consumption at t. Since the dividend amount in equilibrium is equal to capitalists' consumption at every t, we can valuate the stock market based on capitalists' stochastic discount factor, which can be written as

$$Q_t = \sum_{s=0}^{\infty} \phi_q^s \frac{C_t}{C_{t+s}} \cdot C_{t+s} = \frac{C_t}{1-\phi_q} \tag{14}$$

which yields the same expression as in equation (6). Thus, the model is still not closed and later, as usual, a monetary policy rule will take a role of closing the model.

We regard $D(Q_t)$ as the rate of dividend from time t to $t+1$, instead of amount of distributed dividends at exact time t. So our convention is that stock market at time $t+1$ distributes $D(Q_t)$ as a dividend to those who invest in the stock market at time t. We adapt this convention since we want our discrete-time framework to align in terms of formulation with the continuous-time model of Lee and Carreras (2021).[13] Interestingly, we see a dividend rate depends on asset price Q_t, while the asset price Q_t is the expected sum of a stream of discounted dividends. $\mathbb{E}_t(1 + r_{t+1}) = 1 + i_t + (\sigma_t^q)^2$

[13] Even if we regard $D(Q_t)$ as an amount of distributed dividends at period t, it does not change our results qualitatively. It complicates some expressions while not providing a new perspective.

allows us to pin down a current asset price Q_t as

$$Q_t \equiv \frac{D(Q_t) + \mathbb{E}_t(Q_{t+1})}{1 + \mathbb{E}_t(r_{t+1})} = \frac{\overbrace{D(Q_t)}^{=\rho Q_t} + \mathbb{E}_t(Q_{t+1})}{1 + i_t + (\sigma_t^q)^2}, \tag{15}$$

where, by plugging $D(Q_t) = \rho Q_t$ and solving for Q_t, we get Q_t as a function of $\mathbb{E}_t(Q_{t+1})$ as

$$Q_t = \frac{\mathbb{E}_t(Q_{t+1})}{1 - \rho + i_t + (\sigma_t^q)^2}, \text{ where } (\sigma_t^q)^2 = \text{Var}_t\Big(\frac{Q_{t+1}}{Q_t}\Big). \tag{16}$$

For the first-best asset price level $\overline{Q}_t$, we know $\sigma_t^{q,n} = \sigma_t$ and $\overline{Q}_t = \mathbb{E}_t(\overline{Q}_{t+1})$ hold.[14] Therefore, we can pin down the 'natural' rate of interest r_t^n, the rate that prevails in the first-best economy, as

$$r_t^n = \rho - \sigma_t^2, \tag{17}$$

where we observe the natural rate r_t^n depends negatively on the variance of A_t growth, σ_t^2, because a higher σ_t induces capitalists to engage more in the precautionary saving, pushing down the interest rate in the first-best economy.

We define gap variables (from the first-best economy) for asset price and capitalists' consumption: $\hat{Q}_t \equiv \ln \frac{Q_t}{\overline{Q}_t}$, and $\hat{C}_t \equiv \ln \frac{C_t}{\overline{C}_t}$, respectively. With the help of log-linearization,[15] we obtain:

$$\hat{Q}_t = \mathbb{E}_t \hat{Q}_{t+1} - (i_t - (\rho - (\underbrace{\sigma_t^q}_{\text{Endogenous volatility}})^2)), \text{ where } \text{Var}_t(\hat{Q}_{t+1}) = (\sigma_t^q - \sigma_t)^2, \tag{18}$$

which we call the dynamic IS equation for $\hat{Q}_t$ in our framework.[16]

Interestingly, IS equation for the asset price gap $\hat{Q}_t$ contains an endogenous financial volatility term σ_t^q. To be specific, given the expected next-period asset price gap $\mathbb{E}_t \hat{Q}_{t+1}$, a higher σ_t^q results in a higher risk-premium, pushing down the current asset price gap $\hat{Q}_t$ and overall levels of the economy. Since perfect stabilization ($\hat{Q}_t = 0, \forall t$) requires $\sigma_t^q = \sigma_t (= \sigma_t^{q,n})$, as we defined the natural level of interest r_t^n, we can similarly define the natural level of risk-premium $\text{rp}_t^n \equiv \sigma_t^2$, which is the risk premium level that prevails in the first-best economy. If we use rp_t to denote the time t risk-premium, $(\sigma_t^q)^2$, then we can express the term $\rho - (\sigma_t^q)^2$ in the following intuitive way:

$$\rho - (\sigma_t^q)^2 = (\rho - \sigma_t^2) + \text{rp}_t^n - \text{rp}_t = r_t^n + \text{rp}_t^n - \text{rp}_t. \tag{19}$$

[14] Since A_t follows a martingale and $\overline{Q}_t$ is proportional to A_t, the first-best level of asset price $\overline{Q}_t$ also is a martingale.

[15] We preserve the endogenous financial volatility σ_t^q while linearizing the equation (16), while most of the literatures abstract the second-order terms (such as σ_t^q) away whenever relying on the log-linearization.

[16] Actually $\hat{Q}_t = \hat{C}_t$ so it also can be called the dynamic IS equation for $\hat{C}_t$. The IS curve (equation (18)) is derived in Appendix 2.

Plugging equation (19) into equation (18), we get the following dynamic equation for the asset price gap and consumption gap of capitalists:[17]

$$\begin{aligned}\hat{Q}_t &= \mathbb{E}_t\hat{Q}_{t+1} - (i_t - (r^n + \underbrace{\mathrm{rp}^n_t - \mathrm{rp}_t}_{\text{New terms}})) \\ &= \mathbb{E}_t\hat{Q}_{t+1} - (i_t - (r^n_t - \hat{rp}_t)), \text{ where } \hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}^n_t.\end{aligned} \tag{21}$$

In contrast to canonical New-Keynesian frameworks, a risk-premium gap $\hat{rp}_t \equiv \mathrm{rp}_t - \mathrm{rp}^n_t$ enters in the dynamics of gaps in asset price and capitalists' consumption, Given i_t and $\mathbb{E}_t\hat{Q}_{t+1}$, upward divergence of rp_t from its natural level rp^n_t puts a downward pressure on the current asset price $\hat{Q}_t$ and drags the economy into recession, as business cycle is driven by an endogenous asset price fluctuation. One thing to notice is that $\hat{rp}$ is endogenous itself and is determined by how volatile $\hat{Q}_t$ process is. This new feature usually overlooked by previous New-Keynesian literatures allows us to analyze from a different angle how monetary policy should take the financial stability issue into account for a stabilization purpose. Now we move onto an analysis of monetary policy and revisit the classical question about the monetary policy's role of tackling possible financial instabilities.

2 Monetary Policy

This section illustrates through the lens of our framework how a monetary policy framework must changed accordingly, and what happens if the monetary authority sticks to the old rule without consideration of a financial market instability. We also analyze asset price dynamics during zero lower bound (ZLB) episodes and revisit the idea of forward guidance in conjunction with specifics of our model. Throughout Section 2, we assume σ_t to be a constant over time with $\sigma_t = \sigma \leq \sqrt{\rho}$ for $\forall t$. Then $r^n_t = r^n = \rho - \sigma^2 \geq 0$ and $\mathrm{rp}^n_t = \mathrm{rp}^n = \sigma^2$ also become constants.

2.1 Old Monetary Rule and New Indeterminacy

Imagine that monetary authority relies on the usual Taylor rule that targets only an output gap $\hat{Y}_t$:[18]

$$i_t = \underbrace{r^n}_{=\rho-\sigma^2\geq 0} + \underbrace{\kappa}_{>0}\underbrace{\hat{Y}_t}_{=\hat{Q}_t} = r^n + \kappa\hat{Q}_t. \tag{22}$$

[17] We can compare our IS equation with the usual New-Keynesian IS curve with $\hat{C}_t = \hat{Q}_t$ when all firms face perfect nominal rigidity in price setting and thus no inflation arises.

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - (i_t - r^n_t). \tag{20}$$

[18] It becomes the monetary policy rule used by Bernanke and Gertler (2000). For a detailed explanation, see Lee and Carreras (2021).

where we used the property $\hat{Y}_t = \hat{Q}_t = \hat{C}_t = \hat{N}_t$ derived from equation (7) and equation (8) and $\kappa > 0$ implies that the rule in equation (22) satisfies Taylor principle. Plugging equation (22) into IS equation (equation (21))[19] and iterating over time, we get the following recursion for $\hat{Q}_t$:

$$\hat{Q}_t = \frac{1}{1+\kappa}[\mathbb{E}_t\hat{Q}_{t+1} + \sigma^2 - (\sigma_t^q)^2] = \frac{1}{1+\kappa}\mathbb{E}_t\sum_{j=0}^{\infty}\left(\frac{1}{1+\kappa}\right)^j(\sigma^2 - (\sigma_{t+j}^q)^2). \tag{23}$$

Since we know $(\sigma_t^q - \sigma)^2 = \text{Var}_t(\hat{Q}_{t+1})$, clearly $\sigma_t^q = \sigma$, $\hat{Q}_t = 0, \forall t$, which is a perfectly stabilized path, is one possible equilibrium path. However, it is not a unique one and we have sunspots in σ_t^q. Here we present a possible alternative equilibrium path that supports an initial sunspot σ_0^q.

First of all, merging equation (23) with the condition $(\sigma_t^q - \sigma)^2 = \text{Var}_t(\hat{Q}_{t+1})$, we get the following restriction that the stochastic path of $\{\sigma_t^q\}_{t\geq 0}$ must satisfy in equilibrium in expectation:[20]

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+\kappa}\right)^{j+1}\sqrt{\text{Var}_t[\mathbb{E}_{t+1}((\sigma_{t+1+j}^q)^2)]} = |\sigma_t^q - \sigma|, \tag{26}$$

according to which if at time t the economy features $\sigma_t^q \neq \sigma$, we must have $\text{Var}_t[\mathbb{E}_{t+1}((\sigma_{t+1+j^*}^q)^2)] > 0$ for some j^*, which implies $\text{Var}_t((\sigma_{t+1+j^*}^q)^2) > 0$.[21] That we have $\text{Var}_t((\sigma_{t+1+j^*}^q)^2) > 0$ when $\sigma_t^q \neq \sigma$ means we have a stochastic (from time t's perspective) financial volatility at period $t+1+j^*$. On the sample path with $\sigma_{t+1+j^*}^q \neq \sigma$, we apply the same logic and the future stochastic volatility arises again in the farther future.

Equilibrium Indeterminacy and Sunspot It is well known the Taylor rule (equation (22)) excludes a possibility of sunspots in the first-order variable, the output gap $\hat{Y}_t$. The reason is: an unexpected hike in $\hat{Y}_t$ raises the policy rate (that equals the real rate) so that equilibrium path must diverge on expectation, which contradicts the assumptions of linear rational expectation models. However, we

[19] Again we iterate with the transversality assumption $\mathbb{E}_t\hat{Q}_\infty < \infty$.

[20] With respect to the period t filtration $\mathcal{F}_t$, $\mathbb{E}_{t+1}((\sigma_{t+1+j}^q)^2)$ are all (for $\forall j$) perfectly correlated, since we have a unique aggregate shock. Under the martingale restriction ($\mathbb{E}_{t+1}((\sigma_{t+1+j}^q)^2) = (\sigma_{t+1}^q)^2$), equation (26) becomes equation (29) as:

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+\kappa}\right)^{j+1}\sqrt{\text{Var}_t((\sigma_{t+1}^q)^2)} = \frac{1}{\kappa}\sqrt{\text{Var}_t((\sigma_{t+1}^q)^2)} = |\sigma_t^q - \sigma|, \tag{24}$$

and therefore:

$$\text{Var}_t((\sigma_{t+1}^q)^2) = \kappa^2(\sigma_t^q - \sigma)^2. \tag{25}$$

[21] We know $\text{Var}(X) = \mathbb{E}(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$ for random variables X and Y, thus:

$$\underbrace{\text{Var}_t(\mathbb{E}_{t+1}[(\sigma_{t+1+j^*}^q)^2])}_{>0} + \mathbb{E}_t(\underbrace{\text{Var}_{t+1}((\sigma_{t+1+j^*}^q)^2)}_{\geq 0}) = \text{Var}_t((\sigma_{t+1+j^*}^q)^2) > 0. \tag{27}$$

still can have sunspots in σ_t^q, so that σ_0^q can deviate from σ and it can be sustained without violating a long-run convergence.

To gain sharper intuition, we focus on cases in which suddenly, σ_0^q jumps off from σ, and study how it can be sustained rationally on the equilibrium. For this purpose, we just should offer one equilibrium that (i) supports an initial hike $\sigma_0^q > \sigma$, and (ii) on expectation does not diverge in the long-run, following Blanchard and Kahn (1980).

Let us conjecture an equilibrium in which we start from $\sigma_0^q > \sigma$ and $\{(\sigma_t^q)^2\}$ follows a martingale so $\mathbb{E}_t((\sigma_{t+1}^q)^2) = (\sigma_t^q)^2$ holds for $\forall t$. Then $\hat{Q}_t$ only depends on σ_t^q since:

$$\begin{aligned}\hat{Q}_t &= \frac{1}{1+\kappa}\sum_{j=0}^{\infty}\left(\frac{1}{1+\kappa}\right)^j(\sigma^2 - \underbrace{\mathbb{E}_t((\sigma_{t+j}^q)^2)}_{=(\sigma_t^q)^2}) \\ &= \frac{1}{\kappa}(\sigma^2 - (\sigma_t^q)^2),\end{aligned} \tag{28}$$

which satisfies the criterion (ii): long-run convergence in expectation, as $\lim_{t\to\infty}\mathbb{E}_0(\hat{Q}_t) = -\frac{1}{\kappa}(\sigma_0^q)^2$, which is finite. From $\text{Var}_t(\hat{Q}_{t+1}) = (\sigma_t^q - \sigma)^2$, equation (28) satisfies $\text{Var}_t(\hat{Q}_{t+1}) = \frac{1}{\kappa^2}\text{Var}_t((\sigma_{t+1}^q)^2) = (\sigma_t^q - \sigma)^2$ and therefore we obtain

$$\text{Var}_t((\sigma_{t+1}^q)^2) = \kappa^2(\sigma_t^q - \sigma)^2. \tag{29}$$

Basically, the rational and convergent equilibrium in which the stock price variance $(\sigma_t^q)^2$ follows a martingale process and satisfies equation (28) and equation (29) can be one of ways to sustain an initial sunspot $\sigma_0^q > \sigma$. Conversely, when capitalists coordinate an initial sunspot σ_0^q in stock markets, they possibly expect paths of the future economy would be characterized by equation (28) and equation (29). Note that a perfectly stabilized path ($\sigma_t^q = \sigma$ with $\hat{Q}_t = 0$) also satisfies equation (28) and equation (29).

Before we discuss the intuitions in detail, we actually find interesting points that arise from this particular equilibrium. First, for sharper results and mathematical tractability, we divide each time period (between t and $t+1$) by large N and take $N \to \infty$ limit.[22] Then we can simulate the sample paths of $\{\sigma_t^q\}$ using

$$(\sigma_{t+\frac{1}{N}}^q)^2 = (\sigma_t^q)^2 + \kappa(\sigma_t^q - \sigma)\cdot \underbrace{Z_{t,\frac{1}{N}}}_{\underbrace{\sim}_{\text{iid}} N(0,\frac{1}{N})}, \tag{30}$$

which is based on equation (29) and guarantees $(\sigma_t^q)^2 \geq 0$ on any sample path.

With a positive initial sunspot $\sigma_0^q > \sigma$ at $t = 0$, the following Proposition 1 implies if monetary authority uses the Taylor rule (equation (22)) for stabilization and $\{(\sigma_t^q)^2\}$ process follows the con-

[22]This procedure allows us to align our discrete-time model to the continuous-time framework of Lee and Carreras (2021). Without this assumption, we might have a unfavorable situation in which $(\sigma_{t+1}^q)^2$ becomes negative.

jectured process in equation (30), then eventually σ_t^q almost surely converges to σ in the long run, and therefore, an initial financial turmoil ($\sigma_0^q > \sigma$) is mostly driven by a very tiny probability of severe financial crises in the future.

Proposition 1 (The Taylor Rule and Indeterminacy (Financial Volatility Sunspot)) *The rational expectation equilibrium path that supports an initial sunspot $\sigma_0^q > \sigma$, represented by equations $\mathrm{Var}_t((\sigma_{t+1}^q)^2) = \kappa^2(\sigma_t^q - \sigma)^2$ and $\hat{Q}_t = \frac{1}{\kappa}(\sigma^2 - (\sigma_t^q)^2)$, features $\sigma_t^q \xrightarrow{a.s} \sigma_\infty^q = \sigma$ and $\hat{Q}_t \xrightarrow{a.s} 0$. And $\mathbb{E}_0(\max_t(\sigma_t^q)^2) = \infty$ holds.*

The conditions $\sigma_t^q \xrightarrow{a.s} \sigma_\infty^q = \sigma$ and $\hat{Q}_t \xrightarrow{a.s} 0$ mean that in our conjectured equilibrium that supports an initial sunspot $\sigma_0^q > \sigma$, the economy almost surely gets stabilized in the long run, even if we start from $\sigma_0^q > \sigma$. However, with $\mathbb{E}_0(\max_t(\sigma_t^q)^2) = \infty$, we infer that an initial financial crisis with $\sigma_0^q > \sigma$ is sustained by a tiny (0^+) possibility of very severe financial crises in the long run. This result shares a similar intuition to Martin (2012), in a sense that even if our framework does not assume about the existence of specific disasters, our conjectured (and rational) equilibrium that supports an initial crisis features a disaster risk if monetary authority sticks to the Taylor rule that targets output gap only. Martin (2012) showed that the pricing of a broad class of long-dated assets is driven by the possibility of extraordinarily bad news in the future, even without explicit assumptions of those risks.

Figure 1 illustrates a scenario in which $\sigma = 1$ and $(\sigma_0^q)^2 = 10$. The left panel illustrates possible sample paths (the number of which is 1000) with $\kappa = 5$, while the right panel illustrates those with higher value of κ: $\kappa = 20$. We can see that (i) most paths of financial volatility, σ_t^q, converge to σ in the end, (ii) still there arise a small number of big crises with higher excess financial volatilities, and (iii) as we raise the monetary policy's responsiveness to the target, κ, it becomes more likely that the convergence speed of sample paths becomes faster, but there arises a more severe crisis in a given period of time. Many monetary authorities' long-standing view that a so-called conventional monetary policy responds to the financial disruption only when it has an impact on their original mandates (inflation and the employment for the case of U.S.) is thus at odds with our view that without a direct targeting of the financial market instability in the monetary policy framework, monetary policy might fail to achieve perfect stabilization and allows an equilibrium indeterminacy caused by fluctuations in the second-order variable σ_t^q, possibly generating big financial crises to hit the economy. [23]

[23] For example, in a recent FOMC press conference held at September 16, 2020, Fed chair Jerome Powell explicitly said, "We'd be prepared to adjust the stance of monetary policy as appropriate if risks emerge that could impede the attainment of our goals.". About the role of monetary policy in dealing with the financial instability, he argued, "Monetary policy should not be the first line of defense – is not the first line of defense on financial stability. We look to more appropriate tools in the first instance, as a first line of defense. And those would be regulation, supervision, high-capital, high-liquidity stress testing, all of those things, macroprudential tools.". His saying "We will not ignore those kinds of risks, and other kinds of risks more broadly that could impede the attainment of our goals in setting monetary policy" illustrates the long-lasting view of many central banks that the idea of direct targeting of financial stability is not needed or affordable.

Note that the illustrated sample paths just supports an initial sunspot $\sigma_0^q = 10$ as an initial point of the rational expectation equilibrium. Of course at one point during the time, another sunspot might appear and equilibrium might deviate from the prescribed paths.

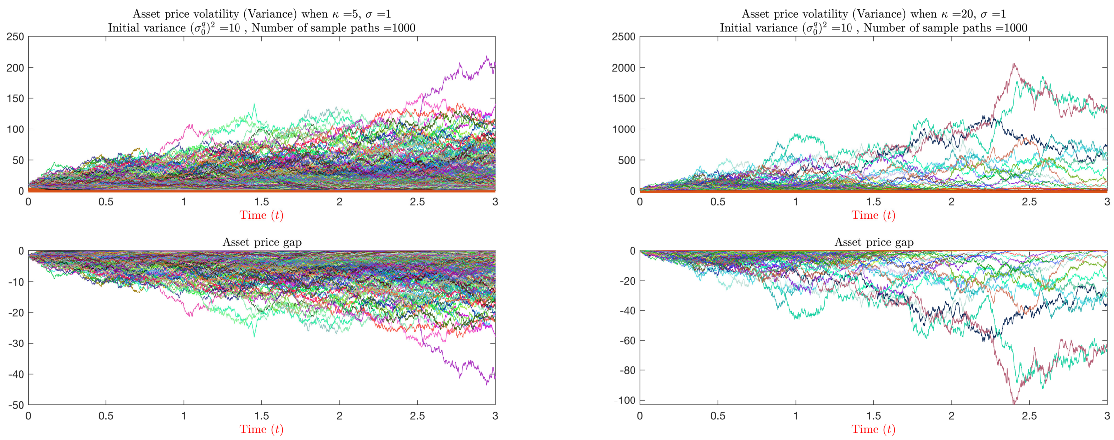

Figure 1: When $\sigma = 1$, $(\sigma_0^q)^2 = 10$, and $\kappa = 5$ (Left) and $\kappa = 20$ (Right)

Intuitions Now we explain in a detailed manner how exactly an initial sunspot σ_0^q can appear in the stock market. For that purpose, we simplify the environment and make the following assumptions:

- **S.1** A shock dZ_t at each period takes one of two: $\{+1, -1\}$ with equal probability $\frac{1}{2}$
- **S.2** A stock price q_t equals a conditional expected value of the next-period stock price q_{t+1} (asset pricing): therefore, if q_{t+1} takes either $q_{t+1}^{(1)}$ or $q_{t+1}^{(2)}$, then $q_t = \frac{1}{2}(q_{t+1}^{(1)} + q_{t+1}^{(2)})$
- **S.3** A stock price q_t falls as a conditional variance of the next-period's q_{t+1} rises (optimal trading strategy). Both $\{q_t\}$ and its conditional volatiliy $\{\sigma_t^q\}$ are set to be 0 on the stabilized path

Since we have only two possible realizations of the shock at each period, we can draw a tree diagram such as follows.

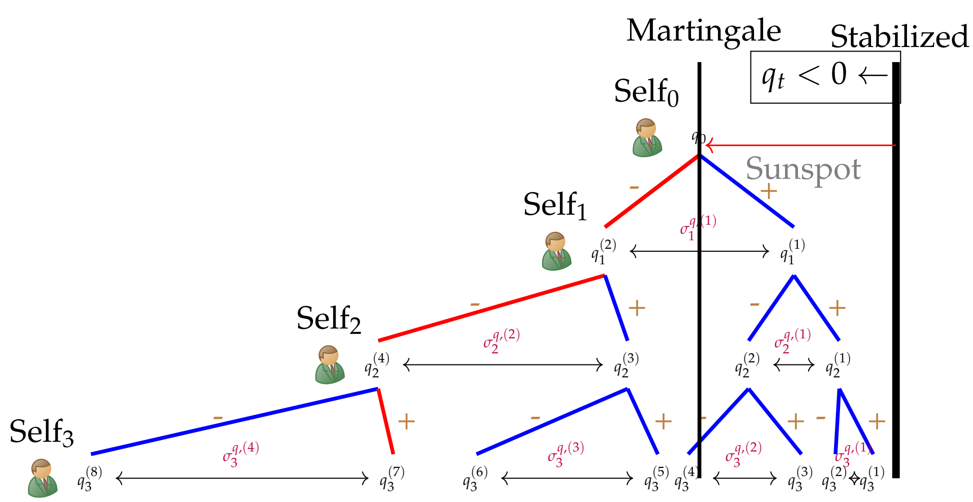

Figure 2: A sunspot in σ_0^q as a rational expectation equilibrium

In Figure 2, a thick vertical line represents the stabilized path, with its left and right representing recessions and booms, respectively. Let us imagine that the current self (Self_0) believes suddenly her future selves will execute path-dependent trading strategies[24] so that the next-period's q_1 becomes $q_1^{(1)}$ after $dZ_0 = +1$ is realized and $q_1^{(2)}$ after $dZ_0 = -1$ is realized, with $q_1^{(1)} > q_1^{(2)}$. Then the current stock price q_0 becomes $q_0 = \frac{1}{2}(q_1^{(1)} + q_1^{(2)})$ with q_0 below the stabilized path, as Self_0 believes there is a dispersion in the next-period stock price, which is given as $\sigma_1^{q,(1)} = q_1^{(1)} - q_1^{(2)}$.

Imagine that $dZ_0 = -1$ is realized. For Self_0's belief that $q_1 = q_1^{(2)}$ to be correct, Self_1 now must believe the future selves will determine stock price paths in a way that the next period's q_2 becomes $q_2^{(3)}$ when $dZ_1 = +1$ is realized and $q_2^{(4)}$ when $dZ_1 = -1$ is realized, with the conditional volatility $\sigma_2^{q,(2)} = q_2^{(3)} - q_2^{(4)}$ higher than $\sigma_1^{q,(1)}$, since $q_1^{(2)}$ is lower than the initial stock price q_0.

After dZ_1 is realized, Self_1's belief about q_2 can be made correct by future selves' coordination and it keeps going on for future selves $\{\text{Self}_{n\geq 2}\}$. We observe that all the nodes in Figure 2 satisfies assumptions S.2 and S.3, with the distance between adjacent nodes getting narrower as the current stock price gets closer to the stabilized path and wider as the stock price deviates more from the stabilized level. Since the stock price process $\{q_t\}$ follows a martingale here, we ensure the economy does not blow up in the long run in expectation. And therefore, Self_0's initial doubt (sunspot) that the next-period asset market would be volatile can be made consistent by coordinations between intertemporal selves at each node.[25]

Note that (i) we have a stochastic volatility in this equilibrium: i.e., σ_t^q is dependent on the path of shocks, as a stock price itself is stochastic and depends negatively on the conditional volatility of its next-period level. Actually, equation (30) in our model specifies the exact stochastic process that volatility $\{\sigma_t^q\}$ follows. (ii) since the volatility σ_t^q gets smaller as the stock price q_t approaches the stabilized path, the economy is likely to stick around the stabilized path if it somehow gets there (therefore, the stabilized path attracting sample paths), justifying the result of Proposition 1 that σ_t^q almost surely converges to σ in the long run. As volatility σ_t^q grows whenever stock price q_t deviates more from the stabilized level, it aligns with the result of Proposition 1 that $\mathbb{E}_0(\max_t(\sigma_t^q)^2) = \infty$.

2.2 Modified Monetary Rule

A modified Taylor rule might include targeting of a risk-premium as[26]

$$i_t = \underbrace{r^n}_{=\rho-\sigma^2\geq 0} - \underbrace{\hat{rp}_t}_{\text{Risk-premium targeting}} + \underbrace{\kappa}_{>0} \underbrace{\hat{C}_t}_{=\hat{Q}_t}. \tag{31}$$

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[24]Their demand for the stock market pins down the stock price's future paths, thus path-dependent trading strategies of capitalists generate a path-dependent stock price process.

[25]It is totally possible since all future selves share a common knowledge of their strategies in the stock market and there is no behavioral friction that blocks communications between intertemporal selves.

[26]Our assumption that $\sigma \leq \sqrt{\rho}$ and $r^n \geq 0$ guarantee that the ZLB constraint does not bind in section 2.2.

One feature of the new monetary policy rule (equation (31)) that stems from our model is that it not only responds to conventional mandates such as $\hat{Y}_t$, which equals $\hat{Q}_t$ in our framework, but also responds in a separate manner to the fluctuations in risk-premium, which usually acts as a proxy for a measure of financial market disruptions.

Plugging equation (31) into IS equation with the transversality condition ($\mathbb{E}_t\hat{Q}_\infty < \infty$) achieves perfect stabilization as the next equation (32) shows:

$$\hat{Q}_t = \frac{1}{1+\kappa}\mathbb{E}_t\hat{Q}_{t+1} = \lim_{s\to\infty}\left(\frac{1}{1+\kappa}\right)^s \underbrace{\mathbb{E}_t\hat{Q}_{t+s}}_{<\infty} = 0. \tag{32}$$

With $\hat{Q}_t \to 0$, we know from $Q_t \to \overline{Q}_t$ that $\sigma_t^q \to \sigma$ and $\text{rp}_t \to \text{rp}^n$. It ensures every variable returns to its first-best level and the realized interest rate i_t becomes $i_t = r^n$. Basically monetary policy's systematic response to fluctuations in the risk-premium, which is determined by the endogenous financial volatility σ_t^q, closes off a possibility of σ_t^q sunspots, and we return to the determinacy and achieves ultra-divine coincidence where one monetary tool (i_t) achieves (i) business cycle stabilization ($\hat{Q}_t = 0$), and (ii) financial market stabilization ($\hat{rp}_t = 0$).

Now we move onto an analysis of the zero lower bound (ZLB) environments and the forward guidance policy through the lens of this framework.

2.3 Zero Lower Bound (ZLB) and the Forward Guidance

We consider a scenario in which σ_t jumps in a deterministic manner from $t = 0$ to T[27]. Specifically, we assume $\sigma_t = \bar{\sigma}$ for $0 \le t < T$ and $\underline{\sigma}$ for $t \ge T$ where $\rho - \bar{\sigma}^2 < 0 < \rho - \underline{\sigma}^2$ ($\underline{\sigma}^2 < \rho < \bar{\sigma}^2$) holds. As $r_t^n < 0$ for $t \le T$, monetary policy loses its ammunition power until T and economy enters ZLB

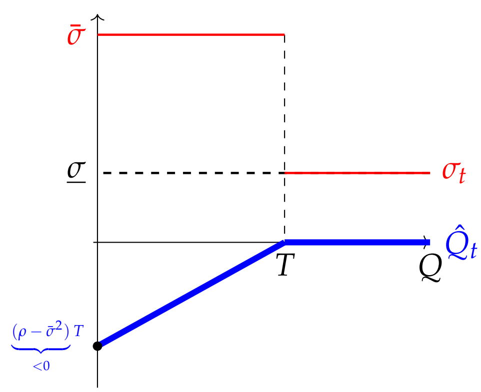


Figure 3: σ jump and the asset price gap $\{\hat{Q}_t\}$ dynamics

episodes for those periods. We solve for the dynamic path of $\{\hat{Q}_t\}$ in the backward manner as usual. First let us study cases after T. Since the economy returns to the normal environment with $\underline{\sigma} < \sqrt{\rho}$,

[27] This deterministic setting is intentional to make our setting as close as possible to that of Werning (2012).

monetary authority can use the modified rule in equation (31): $i_t = r^n(\underline{\sigma}) + (\text{rp}^n(\underline{\sigma}) - \text{rp}_t) + \kappa \hat{Q}_t$[28] and achieves full stabilization, i.e., $\hat{Q}_t = 0$ and $\sigma_t^q = \underline{\sigma}$ for $\forall t \geq T$. As $\hat{Q}_T = 0$ is pinned down, $\hat{Q}_T$ becomes known at time $T-1$, where $\text{Var}_{T-1}(\hat{Q}_T) = (\sigma_{T-1}^q - \bar{\sigma})^2$ implies $\sigma_{T-1}^q = \bar{\sigma}$, which leads to $\hat{rp}_{T-1} = 0$ and $\hat{Q}_{T-1} = r^n(\bar{\sigma}) < 0$. The latter is from the fact that with $r^n(\bar{\sigma}) < 0$, monetary policy is constrained by ZLB at $T-1$ ($i_{T-1} = 0$), and the dynamic IS equation, $\hat{Q}_{T-1} = \mathbb{E}_{T-1}\hat{Q}_T + r^n(\bar{\sigma})$.

Similarly $\text{Var}_{T-2}(\hat{Q}_{T-1}) = 0$ implies $\sigma_{T-2}^q = \bar{\sigma}$ and $\hat{Q}_{T-2} = 2r^n(\bar{\sigma})$. Based on inductive reasoning we get $\sigma_t^q = \bar{\sigma}$ and $\hat{Q}_t = r^n(\bar{\sigma})(T-t) < 0 \; \forall t \leq T$. The equilibrium $\{\hat{Q}_t\}$ path is illustrated in Figure 3. We can see that the path of asset price (or capitalists' consumption) gap process during the ZLB looks exactly like in Werning (2012)'s with zero inflation. However, the main mechanisms are different. Here in our paper, ZLB pushes down the price of a capitalized asset, pushing down the consumption of capitalists, aggregate demand, and the levels of all relevant variables except wage w_t. In Werning (2012) and other canonical New-Keynesian literatures in contrast, ZLB causes household consumption to drop from the usual intertemporal substitution channel combined with general equilibrium features.[29]Also in our economy, the fact that the economy gets stabilized after T due to the modified Taylor rule guarantees that the risk-premium is stabilized during ZLB: $\hat{rp}_t = 0$ even before T.

Now we assume that the central bank commits to keep the interest rate i_t at $i_t = 0$ even after T, until some $\hat{T} > T$. This so-called forward guidance policy has been adapted by many central banks around the world[30] to tackle ZLB constraint and mitigate recessionary pressures. As usual, we assume that after $\hat{T}$, Fed goes back to the modified rule $i_t = r^n(\underline{\sigma}) + (\text{rp}^n(\underline{\sigma}) - \text{rp}_t) + \kappa \hat{Q}_t$, achieving $\hat{Q}_t = 0, \sigma_t^q = \underline{\sigma}$ for $\forall t \geq \hat{T}$. Based on the same reasoning, it can be inferred easily that we have $\sigma_t^q = \underline{\sigma}$ and $\hat{Q}_t = r^n(\underline{\sigma})(\hat{T} - t) > 0$ from $T \leq t \leq \hat{T}$. For $t \leq T$, we have $\sigma_t^q = \bar{\sigma}$ and $\hat{Q}_t = r^n(\bar{\sigma})(T-t) + r^n(\underline{\sigma})(\hat{T} - T)$. We also notice that as in the above case without forward guidance, the fact that the modified Taylor rule achieves perfect stabilization after $\hat{T}$ guarantees that the risk-premium is stabilized during both ZLB and forward guidance period: $\hat{rp}_t = 0$ before $\hat{T}$. The equilibrium path of $\{\hat{Q}_t\}_{t \geq 0}$ is illustrated in Figure 4 as the thick grey line. We still observe that the path of asset price (or capitalists' consumption) gap process during and after ZLB looks exactly like in Werning (2012). A central bank's commitment to keep $i_t = 0$ after ZLB, in our model, raises asset price $\hat{Q}_t$ from T to $\hat{T}$, which raises $\hat{Q}_t$ for $t \leq T$, since asset price is forward-looking and depends positively on the expected future asset price. $\hat{Q}_t$ from T to $\hat{T}$ decreases over time since future stabilization ($\hat{Q}_t = 0$ for $t \geq \hat{T}$) comes closer.

[28]We define r^n and rp^n as functions of σ_t, so $r^n(\bar{\sigma}) = \rho - \bar{\sigma}^2 < 0$ and $r^n(\underline{\sigma}) = \rho - \underline{\sigma}^2 > 0$ by the assumption.

[29]For example, when deleveraging shock hits the economy, households engage in massive deleveraging and reduce their consumption. ZLB prevents monetary policy from responding to this shock and the aggregate demand externality which each household exerts on the entire economy is not alleviated. For this issue, see Eggertsson and Krugman (2012), Korinek and Simsek (2016), Farhi and Werning (2016) among others.

[30]For example, in a recent FOMC statement on September 16, 2020, Fed announced that "The Committee decided to keep the target range for the federal funds rate at 0 to 1/4 percent and expects it will be appropriate to maintain this target range until labor market conditions have reached levels consistent with the Committee's assessments of maximum employment and inflation has risen to 2 percent and is on track to moderately exceed 2 percent for some time.".

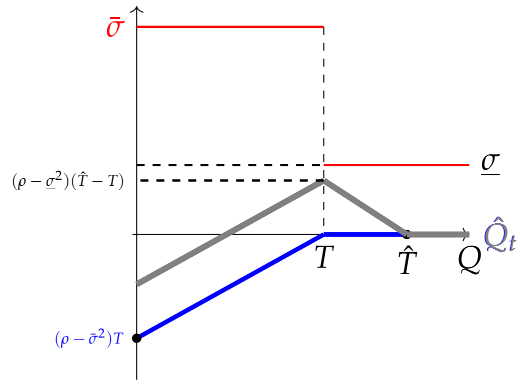


Figure 4: Forward guidance path for $\{\hat{Q}_t\}$

To characterize an optimal value of $\hat{T}$, we need welfare. If we rely on the second-order welfare criterion derived in Lee and Carreras (2021) without the inflation term, we can write a central bank's optimization problem in the following way, where $r^n_s(\sigma_s)$ is the natural rate at time s, depending on whether σ_s at time s is $\bar{\sigma}$ or $\underline{\sigma}$:

$$\min_{\hat{T}} \sum_{s=0}^{\hat{T}} \phi_q^t \hat{Q}_t^2, \text{ s.t. } \hat{Q}_t = \sum_{s=t}^{\hat{T}} r^n_s(\sigma_s). \tag{33}$$

In the continuous-time limit, the optimal forward guidance period $\hat{T}$ satisfies

$$\sum_{t=0}^{\hat{T}} \phi_q^t \hat{Q}_t = 0, \tag{34}$$

which guarantees the optimal $\hat{T}$ is greater than T. Next we move onto the key mechanisms' graphical illustration. It allows us to understand intuitively how various forces interact to drive results in our framework and which types of macroeconomic, both fiscal and monetary, policies are to be used for stabilization purposes.

2.4 Graphical Illustration

Let us assume for a moment that the volatility of technology growth σ_t is constant across periods and equals σ. A current asset price Q_t and next period asset price level Q_{t+1} are linked by

$$Q_t \equiv \frac{D(Q_t) + \mathbb{E}_t(Q_{t+1})}{1 + \mathbb{E}_t(r_{t+1})} = \frac{\rho \cdot Q_t + \mathbb{E}_t(Q_{t+1})}{1 + i_t + (\sigma_t^q)^2}. \tag{35}$$

For simplicity, we assume that from period $t+1$ and so on, central bank relies on the modified Taylor rule (equation (31)), so the economy will be fully stabilized. Thus $Q_{t+1} = \overline{Q}_{t+1}$ and $\mathbb{E}_t Q_{t+1} = \overline{Q}_t$ hold, since $\{\overline{Q}_t\}$ process follows a martingale. From $\hat{Q}_{t+1} = 0$ and $\text{Var}_t(\hat{Q}_{t+1}) = (\sigma_t^q - \sigma)^2 = 0$, we know $\sigma_t^q = \sigma$ and $\hat{rp}_t = 0$[31]. Then the above equation (35) becomes

$$Q = \frac{\rho \cdot Q + \overline{Q}}{1 + i + \sigma^2} \equiv \tilde{D}(Q \| i, \sigma), \tag{36}$$

where we eliminate the time subscript t for simplicity. An increase in either real volatility σ or interest rate i reduces both slope and y-intercept of the function $\tilde{D}(Q)$, which changes the equilibrium value of Q. A change in Q leads to change in Y from the relation $Y(Q) = \alpha Q$ with $\alpha = \frac{\rho}{1-\tilde{w}}$ defined in equation (8).

We assume that economy is initially at the efficient level, thus $Q = \overline{Q}$ holds. By solving for Q in equation (36), we get the equilibrium Q as

$$Q = Q(\sigma, i) \equiv \frac{\overline{Q}}{1 - \rho + i + \sigma^2}, \tag{37}$$

where we see that i must be set at $i_0 = \rho - \sigma^2$ to achieve $Q = \overline{Q}$.[32] With $i = i_0$, an equilibrium asset price Q is determined at the point $(\overline{Q}, \overline{Q})$ where 45° line meets $\tilde{D}(Q)$[33], as illustrated in Figure 5a.

Now we consider the case in which σ jumps to $\underline{\sigma' > \sigma}$, while the interest rate is kept at i_0. Since the financial volatility σ_t^q also jumps from σ to σ' and the risk-premium level rp_t rises, each capitalist reduces her investment demand θ_t, which leads to drops in both the discount factor $\frac{1}{1+i_0+(\sigma')^2}$ and the current asset price Q. With dividend rate unchanged from $\rho\overline{Q}$, asset price Q first drops from $\overline{Q}$ to $\frac{\rho\overline{Q}+\overline{Q}}{1+\rho-\sigma^2+(\sigma')^2}$, which would result in a drop in the size of economy. This first channel (illustrated by the path $A \to A_1 \to A_2$ in Figure 5b) arises from the fact that each individual capitalist does not internalize how in aggregation each one's portfolio decision-making affects the equilibrium asset price Q, which drives the business cycle in our model.[34]

It is not the end of the story, for sure. As σ_t^q jumps from σ to σ', an initial drop in Q from the first channel leads to the falls in the wealth and consumption of capitalists, and thus lowers $D(Q)$.[35] This additional round of adjustment in $D(Q)$ drops Q and again causes the aggregate wealth and consumption levels to fall. Adjustments continue to work until the equilibrium reaches the fixed point,

[31] Thus our modified Taylor rule at time t mandates the same level of rate as the old Taylor rule $i_t = r^n + \kappa\hat{Q}_t$.

[32] Under $\hat{Q} = 0$ ($Q = \overline{Q}$), we have $\kappa\hat{Q} = 0$ in the monetary policy rule, so $i_0 = \rho - \sigma^2$ matches the rate implied by the rule: $i_0 = r^n(\sigma) - \hat{rp} + \kappa\hat{Q}$ with $\hat{rp} = 0$.

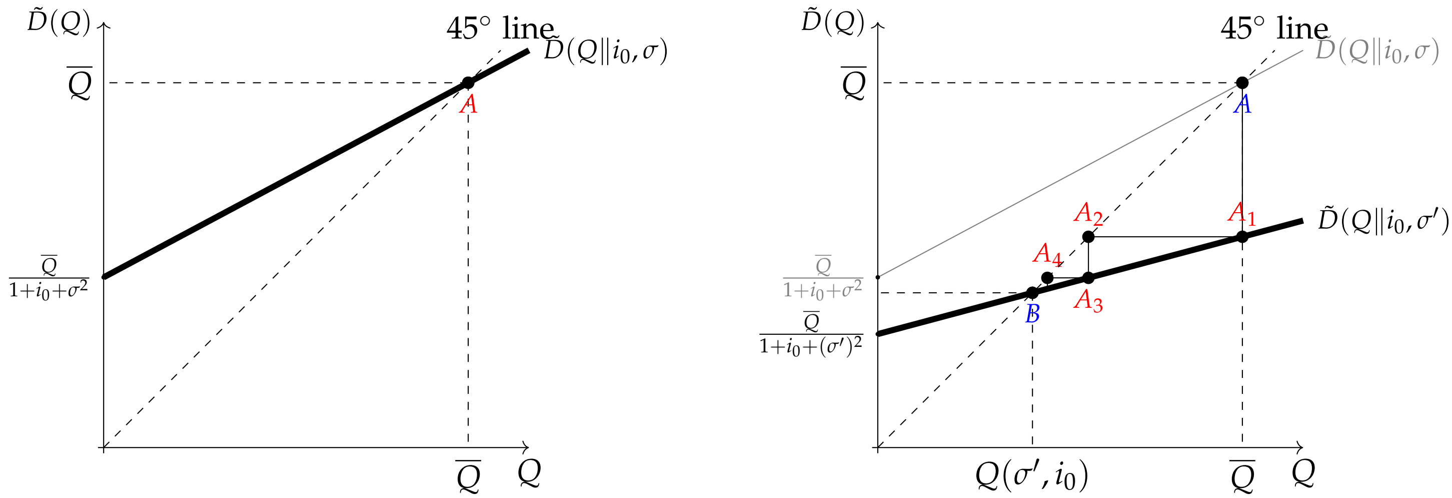
[33] With σ and $i_0 = \rho - \sigma^2$, $\tilde{D}(Q)$'s y-intercept becomes $\frac{\overline{Q}}{1+\rho}$, which is less than $\overline{Q}$, and the slope becomes $\frac{\rho}{1+\rho}$, which is less than 1. So we can guarantee the equilibrium existence.

[34] This can be understood in terms of pecuniary externality. Basically each capitalist reduces her demand for the stock market when σ_t^q jumps, and it reduces the stock price and thus everyone's financial wealth, combined with nominal rigidity hurting their own welfare and the levels of the economy. Ironically when the equilibrium expected return is higher, it is better to have more wealth to invest in the stock market, but the wealth drops endogenously in equilibrium, and there is no insurance market against this unexpected jump in σ_t. For the generic inefficiency arising from pecuniary externality when market is missing (or incomplete), see Geanakoplos and Polemarchakis (1985), Lorenzoni (2008), Farhi and Werning (2016), among others.

[35] A lower aggregate demand from the drop in Q reduces firms' profits, which leads to the lower dividend rate $D(Q)$.

where the dividend yield is ρ. This second channel is illustrated by the path $A_2 \to A_3 \to A_4 \to B$ in Figure 5b, where at point B, asset price level becomes $Q(\sigma', i_0) < Q(\sigma, i_0)$.

Entire story (the first and second channels) can be understood in terms of the aggregate-demand externality. When capitalists lowers their stock market demand with a higher σ_t^q, each of them does not consider how their financial decisions affect the aggregate market outcome in the presence of nominal rigidity. Thus when they collectively reduce consumption out of their financial wealth, it starts a negative spiral in which a lower aggregate demand drops the asset price, while in turn the asset price fall leads to an output drop through the aggregate demand channel.[36]



(a) Asset Price Determination

(b) When σ jumps to $\sigma' > \sigma$

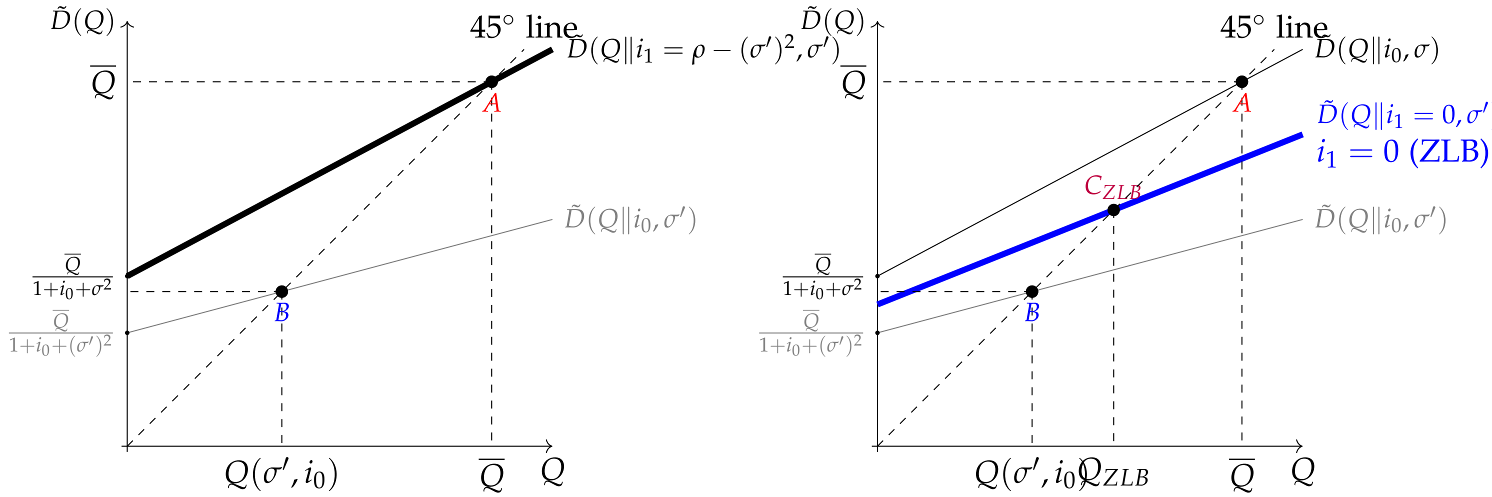
Figure 5: Asset Price Determination and Effect of $\sigma_t \uparrow$

In this case, central banks lowers the interest rate i to boost Q and thus Y. A reduced safe-rate induces capitalists to shift their investment more into a risky one, raising Q and the aggregate demand level.[37] Specifically when $\sigma' < \sqrt{\rho}$, central bank lowers the interest rate to $i_1 = \rho - (\sigma')^2 > 0$[38]. It reduces the required expected return for stock (from $i_0 + (\sigma')^2$ to $i_1 + (\sigma')^2$) and the economy returns to the normal $Q = \overline{Q}$ as illustrated in Figure 6a.

[36]In canonical New-Keynesian models, a collective reduction in the household consumption lowers aggregate output, which pushes down incomes of the household and forces them to further reduce their consumption, ad infinitum. For aggregate demand externality, see Eggertsson and Krugman (2012), Korinek and Simsek (2016), Farhi and Werning (2016) among others. In our model, aggregate demand externality works through the stock market from the relation between asset price Q and dividend $D(Q)$. Caballero and Simsek (2020) focused on similar mechanisms.

[37]Therefore, we see monetary policy response counters the effects of aggregate demand externality arising from the upward jump in σ_t. Farhi and Werning (2016) documents macroprudential policies are needed when the monetary policy is constrained, since in their environment, each investor's financial decision affects the financial wealth distribution, which is relevant for the aggregate demand level since everyone has different spending propensities. In particular, their setting assumes exogenous payoffs of the financial market, while in our framework, asset market is endogenous, and each capitalist's financial decision has relevance to the aggregate demand from their effects on the endogenous asset market. We will revisit the zero lower bound (ZLB) episodes and how macroprudential policies boost a recovery.

[38]It achieves $\hat{Q} = 0$ and we have $\kappa\hat{Q} = 0$ in the monetary policy rule, thus $i_0 = r^n(\sigma') - \hat{rp} + \kappa\hat{Q} = r^n(\sigma') > 0$.



(a) Monetary Policy Response

(b) Zero Lower Bound Crisis

Figure 6: Monetary Policy Response in a Recession from $\sigma_t \uparrow$

However, when $\sigma' > \sqrt{\rho}$, monetary policy is constrained by zero lower bound (ZLB) with $r^n(\sigma') < 0$ and central bank lowers the interest rate up to $i_1 = 0$. In Figure 6b, asset price Q bounces back to point C_{ZLB} but cannot reach the original full-employment point, since we have $Q_{ZLB} = \frac{\overline{Q}}{1-\rho+(\sigma')^2} < \overline{Q}$, and $Y_{ZLB} = \alpha Q_{ZLB} < \alpha \overline{Q} \equiv \overline{Y}$. Under ZLB episodes, the risk-free rate i_t is set too high, depressing stock markets and the overall business cycle.

A Forward guidance can be understood more naturally with the equation (35) and our graphical illustration. From equation (35) we can solve for Q_t with arbitrary $\mathbb{E}_t(Q_{t+1})$ as

$$Q_t = \frac{\mathbb{E}_t(Q_{t+1})}{1 - \rho + i_t + (\sigma')^2}, \tag{38}$$

where in the case of $\mathbb{E}_t(Q_{t+1}) = \overline{Q}_t$[39], we concluded above that when $\sigma' > \sqrt{\rho}$, $i_t = 0$ is set, and we have $Q_t = \frac{\overline{Q}_t}{1-\rho+(\sigma')^2} < \overline{Q}_t$.

Through the forward guidance program, central bank commits not to raise i_t after t, keeping it at 0. It raises the expected future asset price $\mathbb{E}_t(Q_{t+1})$, driving up the current Q_t. If the duration for which forward guidance policy lasts can be fine-tuned to attain $\mathbb{E}_t(Q_{t+1}) = \overline{Q}_t(1 - \rho + (\sigma')^2) > \overline{Q}_t$, then the economy returns to the first best $Q_t = \overline{Q}_t$.[40] However it entails a deviation of Q_{t+1} from

[39] It is from the assumption that the after t, central bank sticks to our modified Taylor rule (equation (31)), which ensures full stabilization and $\hat{Q}_{t+1} = 0$.

[40] For example, Fed chair Jerome Powell's explicit message at the FOMC press conference held on September 16, 2020 emphasizes "The Committee decided to keep the target range for the federal funds rate at 0 to 1/4 percent and expects it will be appropriate to maintain this target range until labor market conditions have reached levels consistent with the Committee's assessments of maximum employment and inflation has risen to 2 percent and is on track to moderately exceed 2 percent for some time." It is a kind of rhetoric that boosts the market confidence and the market expectation about the future economy, which raises the current levels of the economy.

the first-best level $\overline{Q}_{t+1}$ and thus a future welfare loss.

When monetary policy is constrained by zero lower bound (ZLB) constraint and the unconventional monetary policies (including forward guidance) are of limited power, still forward guidance can raise Q_t and levels of the business cycle, as illustrated in Figure 7[41] (equilibrium point moving from C_{ZLB} to C_{FG}).

Even in the case where the unconventional monetary policy cannot attain $\overline{Q}$, still we can come up with clever macroprudential policies[42] to boost asset price Q up until it reaches $\overline{Q}$ and therefore achieve the first-best economy, which is our next topic.

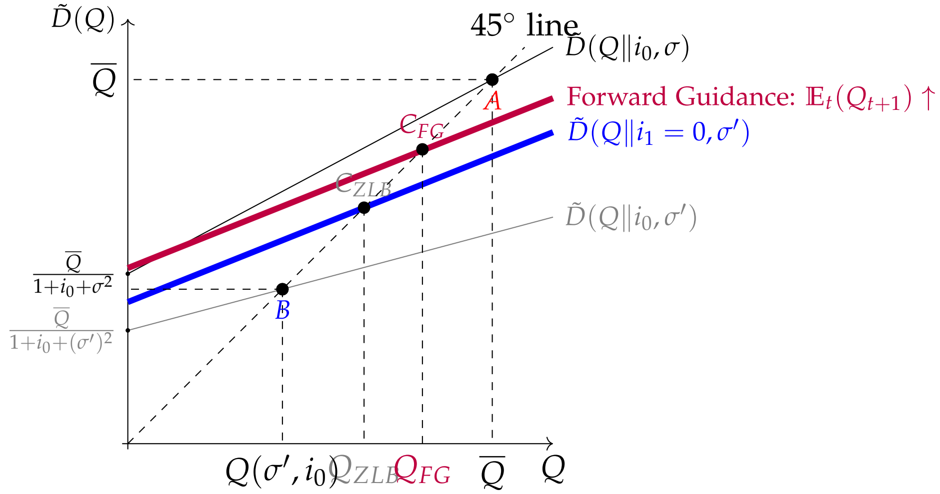


Figure 7: Limited Forward Guidance

3 Unconventional Policies

3.1 Fiscal Redistribution

Assume Q_t is lower than the efficient level, $\overline{Q}_t$, and thus recession comes in with the ZLB-constrained interest rate. In this case, we can rely on fiscal redistribution from capitalists to the hand-to-mouth workers to tackle recessionary pressures. Since workers have a higher marginal propensity to consume (MPC) than capitalists, this type of redistribution raises the consumption demand and profits of firms, which would raise Q and help mitigate the recession. Effectively it raises ρ, the dividend yield, in the equation (37) as we show more in detail below.

[41]When $\mathbb{E}_t(Q_{t+1})$ increases, we have a parallel shift in $\tilde{D}(Q_t\|i_t,\sigma')$ curve (with Q_t in the x-axis).

[42]A number of papers have been written over relations between externalities (pecuniary or aggregate-demand) and macroprudential policies. See Caballero and Krishnamurthy (2001), Lorenzoni (2008), Farhi et al. (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010), Stein (2012), Dàvila and Korinek (2018) among others.

To be specific, we assume that government[43] imposes a lump-sum tax which amounts to a φ_t portion of capitalists' wealth in order to finance the transfer[44]. In that case[45], dividends of the entire stock market (or profits of firms) will increase, effectively raising the value of ρ by $\varphi_t(1-\rho)$ in our formula for the dividend $D_t(Q_t)$ as seen in

$$D_t \equiv Y_t - w_t N_t = (\rho + \varphi_t(1-\rho))Q_t. \tag{41}$$

It can be understood in terms of marginal propensity to consume (MPC) differential between two groups of agents. As φ_t share of financial wealth is transferred from those with MPC=ρ (capitalists) to those with unit MPC (workers), dividend yield is effectively raised by $\varphi_t(1-\rho)$.

In our framework, benefits of a fiscal transfer toward workers are two-fold. First it raises aggregate consumption demand and boosts output immediately as our economy is demand-determined. Second, as asset price Q rises due to the higher dividend paid out of stock market, this additional channel further raises aggregate demand, starting a positive spiral between Q and output Y. Thus, compared to the usual New-Keynesian frameworks where a fiscal lump-sum transfer toward those with higher MPC is effective boosting the output, fiscal redistribution becomes more effective in alleviating the recession.

Due to equation (41), equation (37) with time subscript t is changed to:

$$Q_t = \frac{\overline{Q}_t}{1-(\rho+\varphi_t(1-\rho))+i_t+(\sigma')^2} \underbrace{=}_{i_t=0 \text{ (ZLB)}} \frac{\overline{Q}_t}{1-(\rho+\varphi_t(1-\rho))+(\sigma')^2}. \tag{42}$$

[43] In our framework, the government's only role is to redistribute across agents (capitalists and workers) and it is not allowed to issue government (safe) bonds.

[44] Using taxation to subsidize firms' payroll ($w_t N_t$ in our model) yields the same dividend formula as equation (41). As capitalists subsidize firms' profits, a dividend yield rises, raising asset price and aggregate demand. However, this policy is inferior to the direct redistribution policy, since the first round effect of an increase in workers' consumption from a direct transfer of resources is absent. To be specific, with the same $T_t = \varphi_t Q_t$ subsidizing firm's payroll, good market equilibrium becomes

$$\rho(1-\varphi_t)Q_t + w_t N_t = Y_t, \tag{39}$$

while dividends of the firms become $D_t = Y_t - (w_t N_t - T_t) = (\rho(1-\varphi_t)+\varphi_t)Q_t$, which is the same as equation (41). From equation (39), equation (41), and equation (42), the equilibrium Y_t in the payroll subsidy case becomes

$$Y_t = \frac{\rho(1-\varphi_t)}{1-\tilde{w}} \frac{\overline{Q}_t}{1-(\rho+\varphi_t(1-\rho))+(\sigma')^2}. \tag{40}$$

which is less than the output we achieve when we implement a direct transfer from capitalists to workers (equation (69)). In equation (40), a negative $\varphi_t = 1 - \frac{(\sigma')^2}{\rho} < 0$ achieves the first-best output level $Y_t = \overline{Y}_t$, which seems a bit unintuitive. It is because in our discrete-time model, capitalists consume less than ρ portion of their wealth Q_t as they pay lump-sum taxes T_t every period. If this channel that lowers their consumption and thus aggregate demand is stronger than the original channel in which a higher dividend yield drives asset price and aggregate demand up, then at optimum we need to tax, not subsidize, the payroll of firms to give more resource to capitalists for their own consumption. However in our continuous-time model of Lee and Carreras (2021), capitalists consume exactly ρ portion of their total wealth Q_t as tax amount T_t is a flow variable that only affects effective rate of return, and we still have $\varphi_t > 0$ at optimum.

[45] The entire derivation of the model when we introduce φ_t are provided in Appendix 2.

Without fiscal transfer, we were originally trapped in ZLB, i.e., $i_t = 0$ with $\sigma' > \sqrt{\rho}$. With following $\varphi_t = \varphi_t^*$ in equation (43), however, even at ZLB, the output level returns to $Y_t = \overline{Y}_t$ (thus $N_t = \bar{N}$) and the economy reaches its first-best. It turns out capitalists reduce their consumption whereas workers consume more after redistribution is implemented.[46]

$$\varphi_t^* = \frac{\rho}{1-\rho^2}((\sigma')^2 - \rho) > 0, \;\; Y_t = \overline{Y}_t, \;\; Q_t = \frac{1+\rho}{1+(\sigma')^2}\overline{Q}_t < \overline{Q}_t. \tag{43}$$

With $\varphi_t = \varphi_t^*$, we only need $Q_t < \overline{Q}_t$[47] to achieve the first-best $Y_t = \overline{Y}_t$. It's because after the transfer is implemented, we have a higher overall aggregate demand given the same asset price level. Even if there is an institutional restriction on the magnitude of fiscal redistribution φ_t thus we can only implement $\varphi_t < \varphi_t^*$, we can still raise the value of Y_t, but not achieve $\overline{Y}_t$.[48] It can similarly be represented in our diagram to Figure 7. In contrast to the forward guidance case where an increase in $\mathbb{E}_t(Q_{t+1})$ causes a parallel shift in $\tilde{D}(Q_t \| i_t, \sigma_t)$ curve, an increase in ρ raises both the level and the slope of the $\tilde{D}(Q_t \| i_t, \sigma_t)$ curve.

3.2 Fiscal Distortion in the Portfolio Decision

A marginal subsidy that induces capitalists to bear more risks in the stock market would raise asset price Q_t and levels of the economy at the ZLB. In particular, we consider the policy environment in which when a capitalist invests θ_t share of after-consumption wealth $(W_t - C_t)$ in the stock with the expected return $\mathbb{E}_t r_{t+1}$ at time t, she additionally gets the subsidy from the government in the next period $t+1$ that amounts to $\tau_t \theta_t (W_t - C_t)\mathbb{E}_t r_{t+1}$. Under this policy, government pays additional \$$\tau_t$ per expected (ex-ante) unit \$ capitalists make out of the stock market.[49] We assume that this subsidy is financed by a wealth tax imposed on capitalists, whose rate is taken as constant by capitalists.[50] In that case, each capitalist's share θ_t of wealth invested in the stock market is modified to:

$$\theta_t = \frac{(1+\tau_t)\mathbb{E}_t(r_{t+1}) - i_t}{(\sigma')^2} \underbrace{=}_{\text{In equilibrium}} 1. \tag{44}$$

[46] In our continuous-time model of Lee and Carreras (2021), consumptions of capitalists and workers both increase, in contrast to the result here, since lump-sum tax T_t is a flow variable while the stock price Q_t is a stock variable. In our continuous-time model, T_t affects effective portfolio returns and at optimum capitalists' consumption rate becomes exactly (consumption is a flow variable) ρ portion of wealth. With discrete-time, there is no distinction between stock and flow variables and capitalists consume ρ share of 'after-tax' wealth, as illustrated in Appendix 2.

[47] Still an asset price Q_t is greater than $\frac{\overline{Q}_t}{1-\rho+(\sigma')^2}$, the level at the ZLB when macroprudential policy is absent.

[48] Also Q_t is less than $\overline{Q}_t$ since $\varphi_t < \varphi_t^*$, implies $\rho + \varphi_t(1-\rho) < (\sigma')^2$ and equation (42) implies $Q_t < \overline{Q}_t$.

[49] The reason we assume a form of τ_t dollar per expected (ex-ante) unit dollar earned from the stock market, instead of per a realized (ex-post) unit dollar is that we want to keep the risk amount unchanged while incentivizing capitalists to take more risks in their portfolio decision problem.

[50] In our continuous-time model of Lee and Carreras (2021), we analyze also the other case where the subsidy is financed by lump-sum taxation on workers, instead of capitalists. It turns out to be disastrous, since hand-to-mouth workers (with high MPC) reduce their consumption, leading to falls in dividend yield and thus asset price Q_t.

We observe $\tau_t > 0$ alleviates a drop in asset price Q_t and mitigates recessionary pressures during ZLB episodes, as it reduces the required return $\mathbb{E}_t(r_{t+1})$, thus pushing the current stock price Q_t up from its ZLB value, $Q_{t,ZLB}$, as seen in

$$Q_t = \frac{\overline{Q}_t}{1-\rho+\dfrac{(\sigma')^2}{1+\tau_t}} > \frac{\overline{Q}_t}{1-\rho+(\sigma')^2} \equiv Q_{t,ZLB}. \tag{45}$$

In this case, each capitalist consumes ρ portion of her stock market wealth at optimum and the economy returns to the first-best ($Y_t = \overline{Y}_t$) when $Q_t = \overline{Q}_t$. It can be easily shown that the following

$$\tau_t^* = \frac{(\sigma')^2}{\rho} - 1 > 0 \tag{46}$$

achieves $Q_t = \overline{Q}_t$, where τ_t^* increases with the volatility σ', a measure of the severity of ZLB crisis.

In sum, macroprudential policies help to mitigate recessionary forces in ZLB crisis and level up the economy through their impacts on financial markets and especially, the asset price Q_t. Still there are other types of market interventions that are able to substitute monetary policy's lack of ammunition power during ZLB and stimulate the economy through their impacts on asset markets. In the next Section 3.3, we analyze possible commitment path engineered by central banks and look into subtle issues that arise when central bank engages in distorting stock market to prop up the economy.

3.3 Intertemporal Stabilization Trade-off with Commitment

As our model's <u>Ricardian</u> structure does not allow the central bank's balance sheet quantities to affect the equilibrium asset price and business cycle, in Section 3.3 we turn our eyes to a different type of policy that can prop up the asset market and business cycle: central bank committing to bring the future (after ZLB) stabilities to today (during ZLB) by committing to act irresposibly, and mitigating a recession by lowering risk-premium and pushing asset price up.

3.3.1 General Idea

In this section, we discuss a possible commitment path engineered by the central bank at the ZLB, which brings a higher welfare than our previous forward guidance path. In the forward guidance path (Figure 4), the fact that central bank achieves perfect stabilization after $\hat{T}$ determines financial volatility levels before $\hat{T}$. To be specific, we derived $\sigma_t^q = \underline{\sigma}$ for $T \le t \le \hat{T}$ (forward guidance period) and $\sigma_t^q = \bar{\sigma}$ for $t < T$ (ZLB period). Thus risk-premium levels before $\hat{T}$ are completely determined to be the same as its natural correspondent rp_t^n and we have $\hat{rp}_t = 0$ for $t \le \hat{T}$. This logic can be illustrated by the following Figure 8.

$$\boxed{\text{1. Central bank achieves perfect stabilization after } \hat{T} \ (\hat{Q}_t = 0, \forall t \geq \hat{T})}$$

$$\downarrow$$

$$\boxed{\text{2. } \hat{Q}_{\hat{T}} = 0 \text{ guarantees } \sigma_t^q = \sigma_t, \ \mathrm{rp}_t = \mathrm{rp}_t^n \text{ for } t \leq \hat{T}}$$

Figure 8: Mechanism under the Forward Guidance

We ask the following question: What if central bank forgoes the full stabilization after the forward guidance (after $\hat{T}$ in Figure 4) while squashing levels of the financial volatility σ_t^q[51] both during ZLB (until T) and during forward guidance period (from T to $\hat{T}$)? Specifically, from central bank's point of view, a risk premium level, which is determined by the volatility of TFP process, is set too high during the ZLB episodes so that it causes an asset price (or financial markets in general) to drop too much, leading to a very harsh recession. Thus, central bank might conclude that pushing down σ_t^q levels (or equivalently, risk-premium levels $(= (\sigma_t^q)^2)$) will make the crisis less severe during ZLB by propping up asset price levels, which eventually raises the level of aggregate demand and mitigates recession.

A possibility that σ_t^q differs from σ_t ($\bar{\sigma}$ for $t \leq T$ and $\underline{\sigma}$ for $T \leq t \leq \hat{T}$) causes an asset price gap $\hat{Q}_t$ to fluctuate in a stochastic manner until $\hat{T}$[52] and thus central bank cannot attain $\hat{Q}_{\hat{T}} = 0$, which must be satisfied if it yields a full stabilization after $\hat{T}$. Thus the above conjectured path above will be successful only if central bank commits ex-ante not to pursue perfect stabilization even after all the forward guidance policy is over and fundamental goes back to normal.[53,54] This logic can be interpreted as a contrapositive to the one in Figure 8 and is illustrated by Figure 9. In other words, when engineering a new equilibrium path in which risk-premium and financial volatility deviate from the levels in the natural economy, central bank should juggle between boosting economy now at the ZLB and perfectly stabilizing the economy after the ZLB.

One thing to notice is central bank would like to reduce the levels of financial volatility and risk-premium even after ZLB (from T to $\hat{T}$) while it still follows the forward guidance rate prescription

[51] We hypothetically assume that central bank can choose σ_t^q path in cases it is not determined.

[52] $\mathrm{Var}_t(\hat{Q}_{t+1}) = (\sigma_t^q - \sigma_t)^2$ means $\sigma_t^q \neq \sigma_t$ creates a probabilistic movement in the asset price gap $\{\hat{Q}_t\}$ until $\hat{T}$.

[53] It can be understood in terms of following: σ_t^q being different from σ_t until $\hat{T}$ means that stock market is separated from what is stipulated by the real economy. It will eventually lead to a failure for monetary authority to stabilize the economy at $\hat{T}$, the exact moment when forward guidance policy ends. If we interpret it in a backward way, it means that when central bank tries to manipulate financial markets and especially push down the risk-premium it deems 'too high' during the ZLB, it must commit to the public that it will not seek strict stabilization after $\hat{T}$. This feature arises as financial market and business cycle are interwoven with each other in our model.

[54] For example, after $\hat{T}$, the central bank uses a passive interest rate rule with $i_t = r^n(\underline{\sigma})$. It opens up the possibility of multiple equilibria after $\hat{T}$ as we already discussed, but in this section, we select one equilibrium in which we have $\sigma_t^q = \underline{\sigma}$ after the forward guidance ends (after $\hat{T}$) in Figure 4. In that particular equilibrium, $\hat{Q}_t$ will remain unchanged at $\hat{Q}_{\hat{T}}$ forever after $t \geq \hat{T}$ and $\hat{Q}_{\hat{T}}$ will not drop to 0 immediately.

$\boxed{\neg 2.\ \sigma_t^q < \sigma_t,\ \mathrm{rp}_t < \mathrm{rp}_t^n \text{ for } t \leq \hat{T}}$

$\downarrow$

$\boxed{\neg 1.\ \hat{Q}_{\hat{T}} \neq 0. \text{ Central bank cannot perfectly stabilize the economy after } \hat{T}}$

Figure 9: Financial Market Intervention and Stabilization

($i_t = 0$). It is because it will push up $\hat{Q}_t$ at $T \leq t \leq \hat{T}$, which further drives up the asset price gap $\hat{Q}_t$ during ZLB ($t \leq T$) as $\hat{Q}_t$ is forward-looking. As we have introduced other ways (manipulating the financial market) to boost asset price and levels of the economy during and after ZLB, forward guidance is not as necessary to prop up $\hat{Q}_t$ as before, thus its duration must decrease with intervention.

To be more formal, we assume $\sigma_1^{q,L}$[55] is the stock price volatility during the ZLB (before T) which central bank targets below $\bar{\sigma}$, $\sigma_2^{q,L}$ is the one from T to $\hat{T}'$ which it targets below $\underline{\sigma}$, and $\hat{T}' < \hat{T}$ is the new period when forward guidance ends. To pin down equilibrium paths, we assume that at

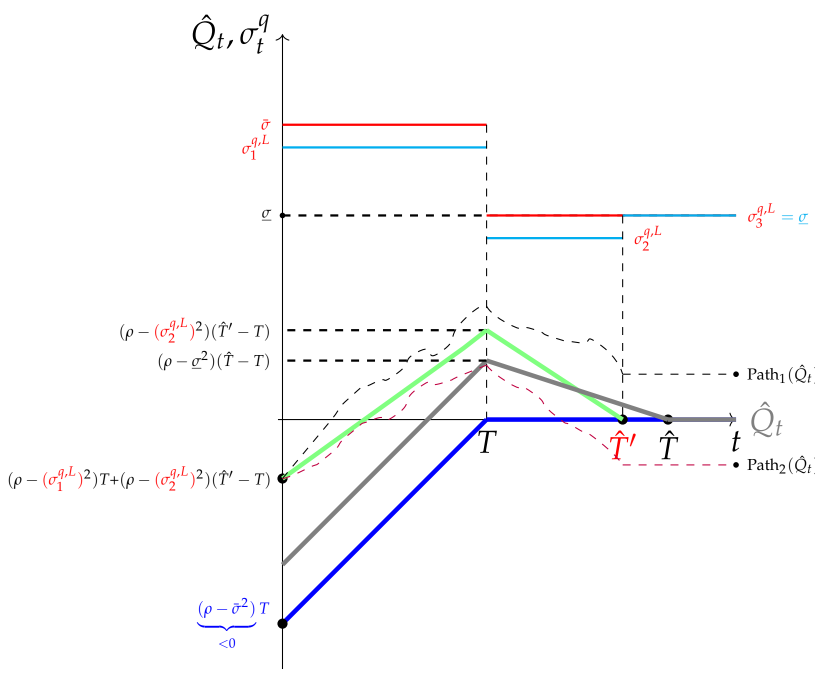


Figure 10: Possible Intervention Dynamics of $\{\hat{Q}_t\}$ with $\sigma_1^{q,L} < \bar{\sigma}$, $\sigma_2^{q,L} < \underline{\sigma}$, and $\hat{T}' < \hat{T}$

$t = 0$, monetary authority anchors the expected value of $\hat{Q}_{\hat{T}'}$, the stock price gap when the forward guidance is over, at 0, i.e., $\mathbb{E}_0 \hat{Q}_{\hat{T}'} = 0$. In Figure 10, the grey line is our original forward guidance path, while the green one is a path of average (or deterministic component of) $\{\hat{Q}_t\}$ process under these paths engineered by the central bank. With $\sigma_1^{q,L} < \bar{\sigma}$ and $\sigma_2^{q,L} < \underline{\sigma}$, a stochastic fluctuation of

[55] We assume that the central bank keeps the same σ_t^q level in the same regime. Specifically, a stock market volatility σ_t^q becomes $\sigma_1^{q,L}$ from 0 to T (ZLB), $\sigma_2^{q,L}$ from T to $\hat{T}'$ (forward guidance), $\underline{\sigma}$ after $\hat{T}'$.

$\{\hat{Q}_t\}$ around the deterministic path is generated and illustrated by two sample paths (dashed lines) in Figure 10. It creates additional welfare loss. We observe that with $\sigma_2^{q,L} < \underline{\sigma}$ and $\mathbb{E}_0\hat{Q}_{\hat{T}'} = 0$, the average level of $\hat{Q}_t$ gets higher from T to $\hat{T}'$ than in the forward guidance path, which would raise its level during the ZLB (before T). Also, $\sigma_1^{q,L} < \bar{\sigma}$ further props up the average level of $\hat{Q}_t$ at the ZLB (before T) and thus $\hat{Q}_0$ falls less in the green path than in the (grey) forward guidance path in Figure 10. For simplicity, we assume central bank commits to use a passive monetary rule after $\hat{T}'$.

In sum, central bank can engineer an equilibrium path that features lower levels of risk-premium and financial volatility during the ZLB by committing to act irresponsibly (giving up the stability) in the future (after ZLB) to raise the levels of financial market and the business cycle now (at the ZLB), when manipulating $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}')$.

3.3.2 Central Bank's Optimization with Commitment

If central bank picks optimal $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}'$, then it solves the following optimization program.

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}'} \mathbb{E}_0 \sum_{s=0}^{\infty} \phi_q^t \hat{Q}_t^2, \text{ s.t. } \hat{Q}_t = \mathbb{E}_t\hat{Q}_{t+1} - (i_t - (\rho - (\sigma_t^q)^2)),$$
$$\text{where } (i_t, \sigma_t^q) = \begin{cases} (0, \sigma_1^{q,L}) \text{ for } t \le T, \\ (0, \sigma_2^{q,L}) \text{ for } T < t \le \hat{T}', \\ (r^n(\underline{\sigma}), \underline{\sigma}) \text{ for } t > \hat{T}', \end{cases} \quad \mathrm{Var}_t(\hat{Q}_{t+1}) = \begin{cases} (\sigma_1^{q,L} - \bar{\sigma})^2 \text{ for } t \le T, \\ (\sigma_2^{q,L} - \underline{\sigma})^2 \text{ for } T < t \le \hat{T}', \\ 0 \text{ for } t \ge \hat{T}', \end{cases} \tag{47}$$
$$\text{and } \hat{Q}_0 = (\rho - \left(\sigma_1^{q,L}\right)^2)T + (\rho - \left(\sigma_2^{q,L}\right)^2)(\hat{T}' - T).$$

The forward guidance path in Figure 4 corresponds to the case of $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}') = (\bar{\sigma}, \underline{\sigma}, \hat{T})$, and the optimal $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}')$ yields a higher welfare. It turns out that our conjecture $\sigma_1^{q,L} < \bar{\sigma}$, $\sigma_2^{q,L} < \underline{\sigma}$, and $\hat{T}' < \hat{T}$ at optimum is correct, as summarized by the next proposition, in which we rely again on continuous-time approximation.[56]

Proposition 2 (Optimal commitment path) *At optimum, $\sigma_1^{q,L} < \bar{\sigma}$, $\sigma_2^{q,L} < \underline{\sigma}$, and $\hat{T}' < \hat{T}$ hold.*

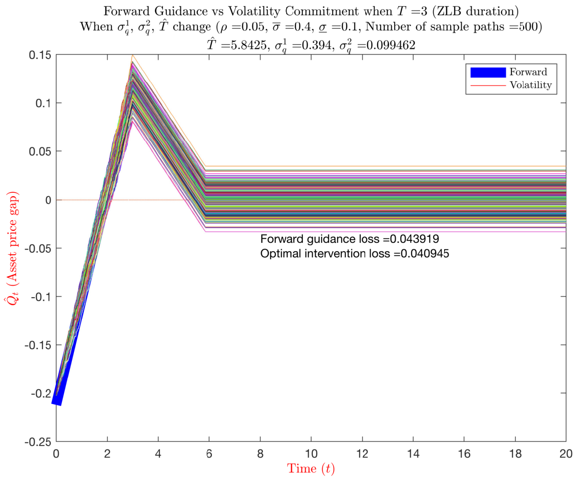
We use following parameters:

$$\rho = -\log \phi_q = 0.05, \bar{\sigma} = 0.4, \underline{\sigma} = 0.1, T = 3$$

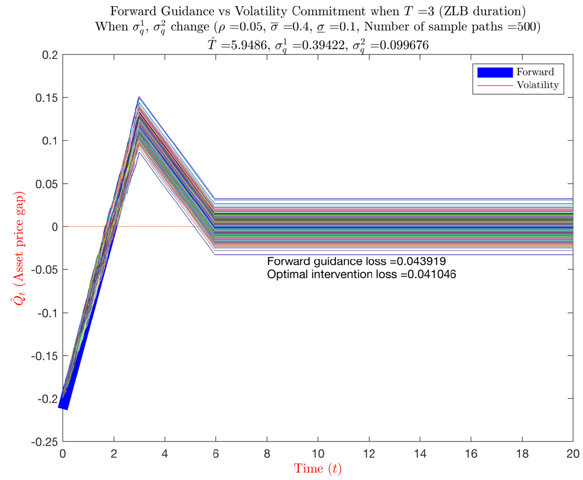
[56] We use the continuous-time approximation in proving the proposition, since it allows us to use a Wiener process to simultaneously characterize both the IS equation and the variance restriction $\mathrm{Var}_t(\hat{Q}_{t+1}) = (\sigma_t^q - \sigma_t)^2$. Our objective (loss) function in equation (47) in this case becomes:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \phi_q^t \hat{Q}_t^2 \simeq \int_0^{\infty} e^{-\rho t} \mathbb{E}_0(\hat{Q}_t^2) dt. \tag{48}$$

In our simulation below (in Figure 11), we use $dt = \frac{1}{N}$ with $N = 1000$.



(a) When $\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}$ are controlled.



(b) When $\sigma_1^{q,L}, \sigma_2^{q,L}$ are controlled.

Figure 11: Optimal Intervention Path of $\hat{Q}_t$

for simulation exercises. Figure 11a assumes that central bank controls all $\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}$ as prescribed in Proposition 2. Figure 11b solves only for $\sigma_1^{q,L}, \sigma_2^{q,L}$ while keeping $\hat{T}' = \hat{T}$. We calculate a sample estimate of the loss function based on

$$\int_0^\infty e^{-\rho t}\mathbb{E}_0(\hat{Q}_t^2)dt \simeq \int_0^\infty e^{-\rho t}\frac{1}{s}\sum_{i=1}^{s}(\hat{Q}_t^{(i)})^2 dt, \tag{49}$$

where $\hat{Q}_t^{(i)}$ stands for i^{th} realized sample path.[57] Our results imply when the central bank chooses $\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}$ optimally, a loss estimate is reduced by 6.77%, whereas if it manipulates stock market volatilities $\sigma_1^{q,L}, \sigma_2^{q,L}$ only, while fixing $\hat{T}' = \hat{T}$, then the loss value falls only by 6.54%. In both cases, $\sigma_1^{q,L} < \bar{\sigma}$, $\sigma_2^{q,L} < \underline{\sigma}$ hold and $\hat{T}' < \hat{T}$ also is satisfied in the first case.[58] Thus most of the welfare gain comes from changes in endogenous financial volatilities $\sigma_1^{q,L}, \sigma_2^{q,L}$, and not from a change in $\hat{T}$.

Even though our simple discrete-time framework based on the rigid-price assumption illustrates key points we want to convey, it still is ad-hoc and relies a lot on continuous-time approximations. That is why in Lee and Carreras (2021), we constructed the continuous-time full-fledged model in which capitalists and workers both optimize, and firms charge sticky prices à la Calvo (1983) instead of the rigid price. It turns out that we can strip down the model into 3-equations (dynamic IS equation, Phillips curve, and the monetary policy) system as we do in conventional New-Keynesian models, with a very important modification from the role of σ_t^q.

[57] We use $s = 500$ number of sample paths in our simulation.
[58] $\hat{T}' = 5.8425 < \hat{T} = 5.9486$ in the first case.

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Appendices: Derivations and Proofs

A. Derivation in Section 1.1

We start by solving each capitalist's optimal consumption-portfolio choice problem.
Capitalist's Optimization : Each capitalist with wealth W_t solves the following optimization.

$$\max_{\{C_{t+s},\theta_{t+s}\}_{s\geq 0}} \mathbb{E}_t \sum_{s=0}^{\infty} \phi_q^s \log C_{t+s} \text{ s.t. } W_{t+1} = (W_t - C_t)(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t})), \forall t. \tag{50}$$

The value function has the following form $\left(\underline{s_t} \text{ contains } \left\{ \mathbb{E}_t i_{t+s}, \underbrace{\mathbb{E}_t \left(\frac{(\mathbb{E}_{t+s}(r_{t+s+1}) - i_{t+s})^2}{(\sigma^q_{t+s})^2} \right)}_{\text{(Expected) squared Sharpe ratio } (t+s)} \right\}_{s\geq 0} \right)$:

$$V_t(W_t, \cdot) = t \cdot \log W_t + K_t(\underline{s_t}). \tag{51}$$

Therefore, each capitalist solves the following Bellman equation:

$$V_t \equiv \max_{C_t,\theta_t} \log C_t + t \cdot \phi_q \log(W_t - C_t) + t \cdot \phi_q \mathbb{E}_t \log(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t})) + \phi_q \mathbb{E}_t K_{t+1}. \tag{52}$$

Using the properties $\text{Var}_t(r_{t+1}) = (\sigma_t^q)^2$ and $\log(1+x) \simeq x - \frac{x^2}{2}$ when $x \simeq 0$, we approximate $\mathbb{E}_t \log(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t}))$ as in equation (53). Especially we ignore multiples of the first-order return variables i_t and $\mathbb{E}_t r_{t+1}$. (For example, $i_t \mathbb{E}_t r_{t+1} \simeq 0$ in our approximation)[59,60]

$$\begin{aligned}
\mathbb{E}_t \log(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t})) &= \mathbb{E}_t \log(1 + i_t + \theta_t(r_{t+1} - i_t)) \\
&= i_t + \theta_t(\mathbb{E}_t r_{t+1} - i_t) - \frac{1}{2}\mathbb{E}_t[((1-\theta_t)i_t + \theta_t r_{t+1})^2] \\
&= i_t + \theta_t(\mathbb{E}_t r_{t+1} - i_t) - \frac{1}{2}\mathbb{E}_t[(1-\theta_t)^2 i_t^2 + 2\theta_t(1-\theta_t)i_t r_{t+1} + \theta_t^2 r_{t+1}^2] \\
&\simeq i_t + \theta_t(\mathbb{E}_t r_{t+1} - i_t) - \frac{\theta_t^2}{2}\underbrace{\text{Var}_t(r_{t+1})}_{=(\sigma_t^q)^2} = i_t + \theta_t(\mathbb{E}_t r_{t+1} - i_t) - \frac{\theta_t^2}{2}(\sigma_t^q)^2.
\end{aligned} \tag{53}$$

The first-order conditions for C_t and θ_t can be characterized as

$$C_t = \frac{W_t}{1 + t\phi_q}, \;\; W_t - C_t = \frac{t\phi_q}{1 + t\phi_q} W_t, \;\; \theta_t = \frac{\mathbb{E}_t r_{t+1} - i_t}{(\sigma_t^q)^2} \underbrace{=}_{\text{In equilibrium}} 1. \tag{54}$$

[59] These terms naturally disappear in our continuous-time model as $(dt)^2 \to 0$ relative to the quadratic variation. For more detailed description of how an expected utility framework with log-utility can be approximated by mean-variance preference, see Pulley (1983).

[60] We depend on the approximation $\mathbb{E}_t[(1-\theta_t)^2 i_t^2 + 2\theta_t(1-\theta_t)i_t r_{t+1} + \theta_t^2 r_{t+1}^2] \simeq \theta_t^2 \mathbb{E}_t(r_{t+1}^2) = \theta_t^2(\text{Var}_t(r_{t+1}) + (\mathbb{E}_t r_{t+1})^2) \simeq \theta_t^2 \text{Var}_t(r_{t+1}) = (\theta_t)^2(\sigma_t^q)^2$.

To get the value of the constant t, we plug above first-order conditions into the original Bellman equation and collect terms with the same arguments to compare coefficients:

$$\begin{aligned} t\log W_t + K_t &= \log\frac{W_t}{1+t\phi_q} + t\phi_q \log\left(\frac{t\phi_q}{1+t\phi_q}W_t\right) + t\phi_q\left(i_t + \theta_t(\mathbb{E}_t r_{t+1} - i_t) - \frac{\theta_t^2}{2}(\sigma_t^q)^2\right) + \phi_q \mathbb{E}_t K_{t+1} \\ &= (1+t\phi_q)\log W_t - \log(1+t\phi_q) + t\phi_q\log\left(\frac{t\phi_q}{1+t\phi_q}\right) + t\phi_q\left(i_t + \frac{(\mathbb{E}_t r_{t+1}-i_t)^2}{2(\sigma_t^q)^2}\right) + \phi_q\mathbb{E}_t K_{t+1}. \end{aligned} \tag{55}$$

Comparing coefficients for $\log W_t$, we get $t = 1 + t\phi_q$ and $t = \frac{1}{1-\phi_q}$. Thus, in optimum, using approximation $\phi_q = e^{-\rho} \simeq 1-\rho$, the optimal consumption C_t becomes

$$C_t = \frac{1}{1+t\phi_q}W_t = (1 - \underbrace{\phi_q}_{\simeq e^{-\rho}})W_t \simeq \rho W_t. \tag{56}$$

Collecting the remaining terms, we get the formula for K_t:

$$\begin{aligned} K_t &= -\log\frac{1}{1-\phi_q} + \frac{\phi_q}{1-\phi_q}\log\phi_q + \frac{\phi_q}{1-\phi_q}\left(i_t + \frac{(\mathbb{E}_t r_{t+1}-i_t)^2}{2(\sigma_t^q)^2}\right) + \phi_q\mathbb{E}_t K_{t+1} \\ &= \frac{1}{1-\phi_q}\left(\log(1-\phi_q) + \frac{\phi_q}{1-\phi_q}\log\phi_q\right) + \frac{\phi_q}{1-\phi_q}\mathbb{E}_t\sum_{s=0}^{\infty}\phi_q^s\underbrace{\left(i_{t+s} + \frac{(\mathbb{E}_{t+s}r_{t+s+1}-i_{t+s})^2}{2(\sigma_{t+s}^q)^2}\right)}_{\underbrace{=}_{\text{In equilibrium}} \frac{\mathbb{E}_{t+s}r_{t+s+1}+i_{t+s}}{2}} \\ &\underbrace{=}_{\text{In equilibrium}} \frac{1}{1-\phi_q}\left(\log(1-\phi_q) + \frac{\phi_q}{1-\phi_q}\log\phi_q\right) + \frac{\phi_q}{2(1-\phi_q)}\sum_{s=0}^{\infty}\mathbb{E}_t(r_{t+s+1}+i_{t+s}). \end{aligned} \tag{57}$$

Now we derive dynamic IS curve for the asset price gap $\hat{Q}_t$ using the log-linearization technique. **Derivation of IS equation**: We start from the original asset pricing equation:

$$Q_t = \frac{\mathbb{E}_t(Q_{t+1})}{1-\rho+i_t+(\sigma_t^q)^2}. \tag{58}$$

Let Q^{SS} be steady-state asset price value, with fixed value of A. If we define a new gap $\ln\frac{Q_t}{Q^{SS}} \equiv \tilde{Q}_t$, then:

$$(\sigma_t^q)^2 = \mathrm{Var}_t\left(\frac{Q_{t+1}}{Q_t}\right) = \mathrm{Var}_t(e^{\tilde{Q}_{t+1}-\tilde{Q}_t}) \simeq \mathrm{Var}_t(\tilde{Q}_{t+1}-\tilde{Q}_t), \tag{59}$$

of which the log-linearization achieves:[61]

$$\tilde{Q}_t = \mathbb{E}_t\tilde{Q}_{t+1} - (i_t - (\rho - (\sigma_t^q)^2)) \tag{60}$$

[61] Thus we do not eliminate $(\sigma_t^q)^2$ term even in linearization procedures.

Since the first best $\overline{Q}_t$ satisfies $\overline{Q}_t = \mathbb{E}_t \overline{Q}_{t+1}$ (as A_t is a martingale), if we define $\ln \frac{\overline{Q}_t}{Q^{SS}} \equiv \tilde{\overline{Q}}_t$, then $\tilde{\overline{Q}}_t = \mathbb{E}_t \tilde{\overline{Q}}_{t+1}$ holds in the first-order approximation. If we define $\hat{Q}_t = \tilde{Q}_t - \tilde{\overline{Q}}_t$ as the log-gap between the current asset price Q_t and the first-best $\overline{Q}_t$, we obtain

$$\hat{Q}_t = \mathbb{E}_t \hat{Q}_{t+1} - (i_t - (\rho - (\sigma_t^q)^2)). \tag{61}$$

To calculate the conditional variance $\mathrm{Var}_t(\hat{Q}_{t+1})$ of asset price gap $\hat{Q}_{t+1}$, we use the fact that every shock is correlated (since there is an only aggregate shock) and $\mathrm{Var}_t(\tilde{\overline{Q}}_{t+1} - \tilde{\overline{Q}}_t) = \sigma_t^2$.

$$\begin{aligned} \mathrm{Var}_t(\hat{Q}_{t+1}) &= \mathrm{Var}_t(\hat{Q}_{t+1} - \hat{Q}_t) = \mathrm{Var}_t((\tilde{Q}_{t+1} - \tilde{Q}_t) - (\tilde{\overline{Q}}_{t+1} - \tilde{\overline{Q}}_t)) \\ &= (\sigma_t^q)^2 + \sigma_t^2 - 2\sigma_t^q \sigma_t = (\sigma_t^q - \sigma_t)^2. \end{aligned} \tag{62}$$

In sum, our dynamic IS equation becomes

$$\hat{Q}_t = \mathbb{E}_t \hat{Q}_{t+1} - (i_t - (\rho - (\sigma_t^q)^2)), \text{ where } \mathrm{Var}_t(\hat{Q}_{t+1}) = (\sigma_t^q - \sigma_t)^2. \tag{63}$$

We can check that if the economy is perfectly stabilized so $\hat{Q}_t = 0$, $\hat{Q}_{t+1} = 0$, then $\mathrm{Var}_t(\hat{Q}_{t+1}) = 0$, σ_t^q becomes σ_t, and the interest rate i_t naturally aligns with the natural rate of interest $r_t^n = \rho - \sigma_t^2$.

A.1. Proof in Section 2

Proof of Proposition 1. From $\mathbb{E}_t((\sigma_{t+1}^q)^2) = (\sigma_t^q)^2$, we know that $\mathbb{E}((\sigma_t^q)^2) = (\sigma_0^q)^2$ where $\mathbb{E}$ is the population expectation. By Doob's martingale convergence theorem (as $(\sigma_t^q)^2 \geq 0\ \forall t$) we know $\sigma_t^q \overset{a.s}{\to} \sigma_\infty^q = \sigma$ since:

$$\underbrace{(\sigma_{t+\frac{1}{N}}^q)^2}_{\overset{a.s}{\to}\sigma} = (\underbrace{\sigma_t^q}_{\overset{a.s}{\to}\sigma})^2 + \kappa(\underbrace{\sigma_t^q - \sigma}_{\overset{a.s}{\to}0}) \cdot \underbrace{Z_{t,\frac{1}{N}}}_{\underbrace{\sim}_{\text{iid}} N(0,\frac{1}{N})}. \tag{64}$$

Finally we must have $\mathbb{E}(\max_t (\sigma_t^q)^2) = \infty$, otherwise the uniform integrability says $\mathbb{E}((\sigma_\infty^q)^2) = (\sigma_0^q)^2$, contradiction to $\sigma_t^q \overset{a.s}{\to} \sigma$. ■

B. Derivation in Section 3

B.1. Fiscal Redistribution

Now we modify our basic discrete-time framework by adding a direct fiscal transfer from capitalists to workers and derive equation (41) and equation (42). Lump-sum tax (T_t) from capitalists with $T_t = \varphi_t W_t$, $\varphi_t \geq 0$ modifies the capitalists' budget constraint as in[62]

$$W_{t+1} = (W_t - T_t - C_t)(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t})). \tag{65}$$

[62]Thus, we solve equation (50) replacing the budget constraint with equation (65).

Then consumptions of capitalists (C_t) and workers (C_t^W) become[63]

$$C_t = \rho(W_t - T_t) = \rho(1-\varphi_t)W_t,\ \ C_t^W = T_t + w_t N_t.^{64} \tag{66}$$

A good market equilibrium (demand-determined economy) condition can be written as

$$\underbrace{\rho(1-\varphi_t)W_t}_{=C_t} + \underbrace{(\underbrace{\varphi_t W_t}_{=T_t} + w_t N_t)}_{=C_t^W} = A_t N_t = Y_t. \tag{67}$$

From equation (67), we can write the amount of dividends (profits) of the firm sector as

$$D_t \equiv Y_t - w_t N_t = Y_t(1-\tilde{w}) = \underbrace{(\rho + \varphi_t(1-\rho))}_{\equiv \rho'} \underbrace{W_t}_{=Q_t}. \tag{68}$$

which is equation (41). Plugging equation (68) into equation (36) with time t subscript, we finally obtain equation (42). From equation (68) and equation (42) with $i_t = 0$, we obtain

$$Y_t = \frac{\rho + \varphi_t(1-\rho)}{1-\tilde{w}} \frac{\overline{Q}_t}{1-(\rho+\varphi_t(1-\rho)) + (\sigma')^2}. \tag{69}$$

which increases with φ_t. Since $\overline{Y}_t = \frac{\rho}{1-\tilde{w}}\overline{Q}_t$, to achieve $Y_t = \overline{Y}_t$, the first-best economy, it must be $\varphi_t = \varphi_t^*$ such that

$$\varphi_t^* = \frac{\rho}{1-\rho^2}((\sigma')^2 - \rho) > 0. \tag{70}$$

From $\varphi_t^*(1-\rho) = \frac{\rho}{1+\rho}((\sigma')^2-\rho)$, $\rho + \varphi_t^*(1-\rho) = \rho(1+\frac{(\sigma')^2-\rho}{1+\rho}) = \rho\frac{1+(\sigma')^2}{1+\rho}$, $1-(\rho+\varphi_t^*(1-\rho)) + (\sigma')^2 = \frac{1+(\sigma')^2}{1+\rho}$, Q_t with the optimal $\varphi_t = \varphi_t^*$ can be written as

$$Q_t = \frac{\overline{Q}_t}{1-(\rho+\varphi_t^*(1-\rho)) + (\sigma')^2} = \frac{1+\rho}{1+(\sigma')^2}\overline{Q}_t < \overline{Q}_t. \tag{71}$$

We observe $\frac{\overline{Q}_t}{1-\rho+(\sigma')^2} < Q_t < \overline{Q}_t$ as $\rho < (\sigma')^2$, despite $Y_t = \overline{Y}_t$. To calculate the consumption of capitalists C_t with $\varphi_t = \varphi_t^*$, we use $C_t = \rho(1-\varphi_t^*)Q_t$, which with equation (70) and equation (71) becomes

$$C_t = \rho \underbrace{\frac{1-\rho(\sigma')^2}{1-\rho^2}}_{=1-\varphi_t^*} \underbrace{\frac{1+\rho}{1+(\sigma')^2}\overline{Q}_t}_{=Q_t} = \rho\frac{1-\rho(\sigma')^2}{1-\rho}\frac{\overline{Q}_t}{1+(\sigma')^2} = \rho\frac{1-\rho(\sigma')^2}{1-\rho(\sigma')^2-\rho+(\sigma')^2}\overline{Q}_t, \tag{72}$$

which happens to be smaller than $\rho\frac{\overline{Q}_t}{1-\rho+(\sigma')^2}$, the consumption level at the ZLB without any additional

[63]We invoke the property that at optimum (with the help of a log-linear approximation), the capitalist consumes ρ portion of (after-tax) wealth.

[64]Using T_t to subsidize the payroll of the firms yields the equivalent result

fiscal policy. As output is restored to normal, $Y_t = \overline{Y}_t$, with $\varphi_t = \varphi_t^*$, and $\overline{Y}_t$ is obviously larger than its corresponding level at the ZLB, $\frac{\rho}{1-\tilde{w}}\frac{\overline{Q}_t}{1-\rho+(\sigma')^2}$, the fact that capitalists consume less implies workers must consume more to satisfy a good market equilibrium condition.

B.2. Fiscal Distortion in the Portfolio Decision

We assume that government imposes a wealth tax on (after-consumption) wealth $W_t - C_t$ with the rate ϕ_t to finance subsidizing the expected (ex-ante) returns capitalists earn from the stock market. In specific, each capitalist, taking ϕ_t as given, faces the following budget dynamics:

$$\begin{aligned} W_{t+1} &= (W_t - C_t)(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t}) + \underbrace{\tau_t\theta_t\mathbb{E}_t(r_{t+1})}_{\text{(Ex-ante) return subsidy}}) - \underbrace{(W_t - C_t)\phi_t}_{\text{Taxation}}, \forall t \\ &= (W_t - C_t)(R_{f,t} + \theta_t(R'_{t+1} - R_{f,t}) + \tau_t\theta_t\mathbb{E}_t(r_{t+1}) - \phi_t) \\ &= (W_t - C_t)(i_t + \theta_t(r_{t+1} - i_t) + \tau_t\theta_t\mathbb{E}_t(r_{t+1}) - \phi_t), \end{aligned} \tag{73}$$

where in equilibrium $\phi_t = \tau_t\theta_t\mathbb{E}_t(r_{t+1})$ holds. Using the same technique as in equation (56), at optimum we get $C_t = \rho W_t$ (in equilibrium $W_t = Q_t$). With the same approximation technique, we obtain

$$\begin{aligned} &\mathbb{E}_t \log(1 + i_t + \theta_t(r_{t+1} - i_t) + \tau_t\theta_t\mathbb{E}_t(r_{t+1}) - \phi_t) \\ &= i_t + \theta_t((1+\tau_t)\mathbb{E}_t r_{t+1} - i_t) - \phi_t - \frac{1}{2}\mathbb{E}_t[((1-\theta_t)i_t + \theta_t r_{t+1} + \underbrace{\tau_t\theta_t\mathbb{E}_t(r_{t+1}) - \phi_t}_{\equiv\xi_t})^2] \\ &= i_t + \theta_t((1+\tau_t)\mathbb{E}_t r_{t+1} - i_t) - \phi_t - \frac{1}{2}\mathbb{E}_t[(1-\theta_t)^2 \underbrace{i_t^2}_{\simeq 0} + 2\theta_t(1-\theta_t)\underbrace{i_t r_{t+1}}_{\simeq 0} \\ &\qquad + \theta_t^2 r_{t+1}^2 + \underbrace{\xi_t^2}_{\simeq 0} + 2(1-\theta_t)\underbrace{\xi_t i_t}_{\simeq 0} + 2\theta_t \underbrace{\xi_t r_{t+1}}_{\simeq 0}] \\ &\simeq i_t + \theta_t((1+\tau_t)\mathbb{E}_t r_{t+1} - i_t) - \phi_t - \frac{\theta_t^2}{2}\underbrace{\mathrm{Var}_t(r_{t+1})}_{=(\sigma_t^q)^2} \\ &= i_t + \theta_t((1+\tau_t)\mathbb{E}_t r_{t+1} - i_t) - \phi_t - \frac{\theta_t^2}{2}(\sigma_t^q)^2. \end{aligned} \tag{74}$$

The first-order condition for θ_t yields equation (44). And equation (45) follows from equation (44). In this case, good market equilibrium stays the same as $C_t = \rho Q_t = Y_t - w_t N_t = (1-\tilde{w})Y_t$ so $Y_t = \overline{Y}_t$ condition is satisfied when we have $Q_t = \overline{Q}_t$.

C. Proof and Derivation in Section 3.3

Proof of Proposition 2. We just follow the same strategy as in our continuous-time framework of Lee and Carreras (2021). First we map the central bank's optimization program (equation (47)) into its continuous-time correspondent. Central bank solves the following problem in continuous-time environment (with $\underline{\sigma}^2 < \rho < \bar{\sigma}^2$):[65]

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}'} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2 dt, \text{ s.t. } \begin{cases} d\hat{Q}_t = -(\underbrace{\rho - (\sigma_1^{q,L})^2}_{<0})dt + (\sigma_1^{q,L} - \bar{\sigma})dZ_t, & \text{for } t < T, \\ d\hat{Q}_t = -(\underbrace{\rho - (\sigma_2^{q,L})^2}_{>0})dt + (\sigma_2^{q,L} - \underline{\sigma})dZ_t, & \text{for } T \le t < \hat{T}', \\ d\hat{Q}_t = 0, & \text{for } t \ge \hat{T}', \end{cases} \tag{75}$$

where $\hat{Q}_0 = (\rho - (\sigma_1^{q,L})^2)T + (\rho - (\sigma_2^{q,L})^2)(\hat{T}' - T)$.

With $r_1 \equiv \rho - (\sigma_1^{q,L})^2 < 0, r_2 \equiv \rho - (\sigma_2^{q,L})^2 > 0$, a gap process is represented as (with $\hat{C}_t = \hat{Q}_t$)

$$d\hat{C}_t = \begin{cases} \underbrace{-r_1}_{>0} dt + (\sigma_1^{q,L} - \bar{\sigma})dZ_t & \text{for } t \le T, \\ \underbrace{-r_2}_{<0} dt + (\sigma_2^{q,L} - \underline{\sigma})dZ_t & \text{for } T \le t \le \hat{T}', \\ 0 & \text{for } t \ge \hat{T}'. \end{cases} \tag{76}$$

After $\hat{T}$, there is no movement of $\hat{C}_t$ at all. If we let risk-adjusted natural rate at time s, r_s^T, be r_1 for $s < T$ and r_2 for $T \le s \le \hat{T}'$, then gap process can be written as

$$\hat{C}_t = \begin{cases} \underbrace{\int_t^{\hat{T}'} r_s^T ds}_{\equiv \hat{C}_{det}(t;\hat{T}')} + (\sigma_1^{q,L} - \bar{\sigma}) \underbrace{Z_t}_{\sim N(0,t)} & \text{for } t \le T, \\ \underbrace{\int_t^{\hat{T}'} r_s^T ds}_{\equiv \hat{C}_{det}(t;\hat{T}')} + (\sigma_1^{q,L} - \bar{\sigma})Z_T + (\sigma_2^{q,L} - \underline{\sigma}) \underbrace{[W_{t-T}]}_{\sim N(0,t-T)} & \text{for } T < t \le \hat{T}', \\ \hat{C}_{\hat{T}'} = (\sigma_1^{q,L} - \bar{\sigma})Z_T + (\sigma_2^{q,L} - \underline{\sigma}) \underbrace{[W_{\hat{T}-T}]}_{\sim N(0,\hat{T}-T)} & \text{for } \hat{T}' < t. \end{cases} \tag{77}$$

where Z_t, W_{t-T}, and $U_{\hat{T}-T}$ are independent brownian motions. We square each term in equation (77) and take an expectation operator with respect to the information at time 0, the time central bank solves its commitment problem. Then, we obtain

$$\mathbb{E}_0\hat{C}_t^{\,2} = \begin{cases} \hat{C}_{det}(t;\hat{T}')^2 + (\sigma_1^{q,L} - \bar{\sigma})^2 t & \text{for } t \le T, \\ \hat{C}_{det}(t;\hat{T}')^2 + (\sigma_1^{q,L} - \bar{\sigma})^2 T + (\sigma_2^{q,L} - \underline{\sigma})^2(t - T) & \text{for } T < t \le \hat{T}', \\ (\sigma_1^{q,L} - \bar{\sigma})^2 T + (\sigma_2^{q,L} - \underline{\sigma})^2(\hat{T}' - T) & \text{for } \hat{T}' < t. \end{cases} \tag{78}$$

[65] In the proof, we implicitly assume that $\rho - (\sigma_1^{q,L})^2 < 0$ and $\rho - (\sigma_2^{q,L})^2 > 0$ so ZLB binds until T. It is a natural assumption, given that we are interested in central bank's financial market intervention in the ZLB environment. Our simulation exercises support that even if a central bank controls $\sigma_1^{q,L}$, and $\sigma_2^{q,L}$, still we have $(\sigma_1^{q,L})^2 > \rho > (\sigma_2^{q,L})^2$ at optimum.

If we plug these expressions into central bank's loss function, then central bank's commitment problem can be represented as

$$\begin{aligned}
&\min_{i_t \geq 0, \sigma_1^{q,L}, \sigma_2^{q,L}} \quad \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{C}_t^2 dt \\
&= \min_{\hat{T}', \sigma_1^{q,L}, \sigma_2^{q,L}} \int_0^{\hat{T}} e^{-\rho t} \hat{C}_{det}(t;\hat{T}')^2 dt + (\sigma_1^{q,L} - \bar{\sigma})^2 \underbrace{\int_0^T t e^{-\rho t} dt}_{=\frac{1}{\rho^2} - \frac{1}{\rho^2}e^{-\rho T} - \cancel{\frac{T}{\rho}e^{-\rho T}}} + (\sigma_1^{q,L} - \bar{\sigma})^2 T \underbrace{\int_T^\infty e^{-\rho t} dt}_{=\cancel{\frac{1}{\rho}e^{-\rho T}}} \\
&\quad + (\sigma_2^{q,L} - \underline{\sigma})^2 \underbrace{\int_T^{\hat{T}'} e^{-\rho t}(t-T) dt}_{=-\cancel{\frac{1}{\rho}(\hat{T}-T)e^{-\rho \hat{T}'}} + \frac{e^{-\rho T} - e^{-\rho \hat{T}'}}{\rho^2}} + (\sigma_2^{q,L} - \underline{\sigma})^2 (\hat{T}' - T) \underbrace{\int_{\hat{T}'}^\infty e^{-\rho t} dt}_{=\cancel{\frac{1}{\rho}e^{-\rho \hat{T}'}}} \\
&= \min_{\hat{T}, \sigma_1^{q,L}, \sigma_2^{q,L}} \underbrace{\int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t;\hat{T}')^2 dt}_{\text{From deterministic fluctuation}} + \underbrace{(\sigma_1^{q,L} - \bar{\sigma})^2 \frac{1}{\rho^2}(1 - e^{-\rho T}) + (\sigma_2^{q,L} - \underline{\sigma})^2 \left(\frac{e^{-\rho T} - e^{-\rho \hat{T}'}}{\rho^2} \right)}_{\text{From stochastic fluctuation}}. \quad (79)
\end{aligned}$$

where now central bank can control $\sigma_1^{q,L}, \sigma_2^{q,L}$ in addition to its conventional monetary policy tool $\{i_t\}$ (including $\hat{T}'$). First we get the following first-order condition for $\hat{T}'$:

$$2 \underbrace{r_2}_{>0} \int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t;\hat{T}') dt + (\sigma_2^{q,L} - \underline{\sigma})^2 \frac{1}{\rho} e^{-\rho \hat{T}'} = 0. \quad (80)$$

Therefore, we have

$$\int_0^\infty e^{-\rho t} \hat{C}_{det}(t;\hat{T}') dt = \int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t;\hat{T}' \| \sigma_1^{q,L} < \bar{\sigma}, \sigma_2^{q,L} < \underline{\sigma}) dt < 0. \quad (81)$$

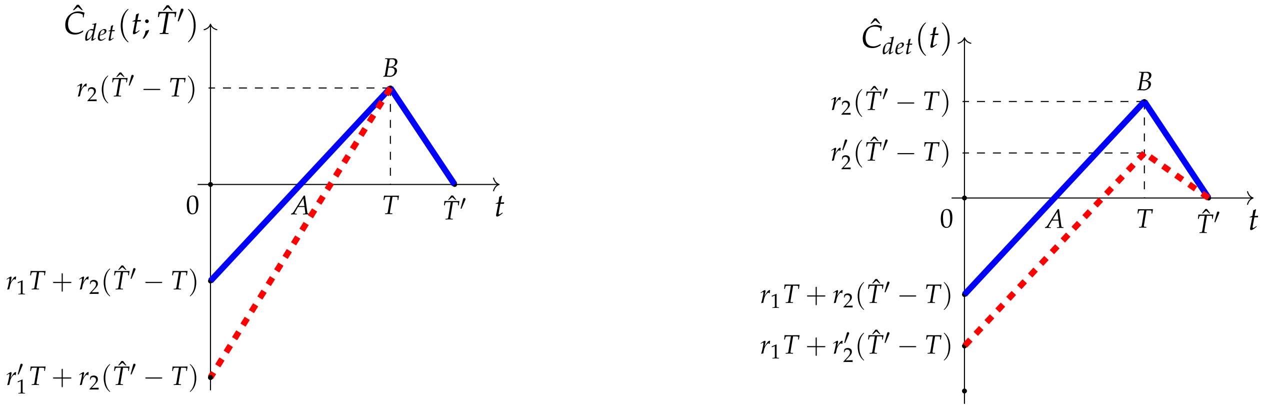
The above first-order condition for $\hat{T}'$ shows central bank lowers the value of $\hat{T}'$ in the optimum, compared to $\hat{T}$, the duration for which it implements forward guidance only, thus $\hat{T}' < \hat{T}$. The reason is that in the case central bank only implements a forward guidance without financial market intervention, we had the following optimization condition from equation (34).

$$\int_0^{\hat{T}} e^{-\rho t} \hat{C}_{det}(t;\hat{T} \| \sigma_1^{q,L} = \bar{\sigma}, \sigma_2^{q,L} = \underline{\sigma}) dt = 0. \quad (82)$$

Due to $\hat{C}_{det}(t;\hat{T}' \| \sigma_1^{q,L} = \bar{\sigma}, \sigma_2^{q,L} = \underline{\sigma}) < \hat{C}_{det}(t;\hat{T}' \| \sigma_1^{q,L} < \bar{\sigma}, \sigma_2^{q,L} < \underline{\sigma})$, from equation (81) and equation (82), we infer $\hat{T}' < \hat{T}$ at optimum. To find the optimal $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, we need **variational argument**, as $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ affect levels of r_1, r_2, and thus $\hat{C}_{det}(t;\hat{T}')$. In specific, we obtain

$$\frac{\partial r_1}{\partial \sigma_1^{q,L}} = -2\sigma_1^{q,L} < 0, \quad \frac{\partial r_2}{\partial \sigma_2^{q,L}} = -2\sigma_2^{q,L} < 0. \quad (83)$$

<u>**Finding** $\sigma_1^{q,L}$</u> : If $\sigma_1^{q,L}$ rises, then r_1 falls, changing $\hat{C}_{det}(t;\hat{T}')$ path as the following figure indicates (from thick blue to dashed red in Figure 12a).



(a) Variation along $\sigma_1^{q,L}$ increase

(b) Variation along $\sigma_2^{q,L}$ increase

Figure 12: Variation along $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ increase

When we differentiate $\hat{C}_{det}(t;\hat{T}')$ with respect to $\sigma_1^{q,L}$, we obtain

$$\hat{C}_{det}(t;\hat{T}') = \int_t^{\hat{T}'} r_s^T ds, \ \frac{\partial \hat{C}_{det}}{\partial \sigma_1^{q,L}} = \int_t^T -2\sigma_1^{q,L} ds = -2\sigma_1^{q,L}(T-t), \forall t \le T. \tag{84}$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function by $\sigma_1^{q,L}$ and get

$$2\sigma_1^{q,L}\int_0^T e^{-\rho t}\hat{C}_{det}(t;\hat{T}')(T-t)dt = (\sigma_1^{q,L}-\bar{\sigma})\frac{1-e^{-\rho T}}{\rho^2}. \tag{85}$$

Finally, we obtain the following expression

$$\int_0^T e^{-\rho t}\hat{C}_{det}(t;\hat{T}')(T-t)dt = \underbrace{\int_0^t e^{-\rho s}\hat{C}_{det}(s;\hat{T}')ds \cdot (T-t)\Big|_0^T}_{=0} + \int_0^T \Big(\underbrace{\int_0^t e^{-\rho s}\hat{C}_{det}(s;\hat{T}')ds}_{<0}\Big)dt < 0, \tag{86}$$

which clearly shows $\sigma_1^{q,L} < \bar{\sigma}$ must be satisfied at optimum from equation (85).

<u>**Finding** $\sigma_2^{q,L}$</u> : If $\sigma_2^{q,L}$ rises, r_2 falls, changing $\hat{C}_{det}(t;\hat{T}')$ shape as in Figure 12b). By differentiating $\hat{C}_{det}$ with respect to $\sigma_2^{q,L}$, we obtain

$$\frac{\partial \hat{C}_{det}}{\partial \sigma_2^{q,L}} = \begin{cases} \int_T^{\hat{T}'} -2\sigma_2^{q,L} ds = -2\sigma_2^{q,L}(\hat{T}'-T), & \text{for } t < T, \\ \int_t^{\hat{T}'} -2\sigma_2^{q,L} ds = -2\sigma_2^{q,L}(\hat{T}'-t), & \text{for } T < \forall t < \hat{T}'. \end{cases} \tag{87}$$

To find optimal $\sigma_2^{q,L}$, we differentiate the objective function by $\sigma_2^{q,L}$ and obtain

$$2\sigma_2^{q,L}\Big(\int_0^T e^{-\rho t}\hat{C}_{det}(t;\hat{T}')(\hat{T}'-T)dt+\int_T^{\hat{T}'} e^{-\rho t}\underbrace{\hat{C}_{det}(t;\hat{T}')}_{>0}(\hat{T}'-t)dt\Big)=(\sigma_2^{q,L}-\underline{\sigma})\frac{e^{-\rho T}-e^{-\rho\hat{T}}}{\rho^2}. \tag{88}$$

The following expression with the above equation (88) shows $\sigma_2^{q,L}<\underline{\sigma}$ at the optimum.

$$\begin{aligned}&\int_0^T e^{-\rho t}\hat{C}_{det}(t;\hat{T}')(\hat{T}'-T)dt+\int_T^{\hat{T}'} e^{-\rho t}\underbrace{\hat{C}_{det}(t;\hat{T}')}_{>0}(\hat{T}'-t)dt\\&<\int_0^T e^{-\rho t}\hat{C}_{det}(t;\hat{T}')(\hat{T}'-T)dt+\int_T^{\hat{T}'} e^{-\rho t}\underbrace{\hat{C}_{det}(t;\hat{T}')}_{>0}(\hat{T}'-T)dt=(\hat{T}'-T)\underbrace{\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{det}(t;\hat{T}')dt}_{<0}<0.\end{aligned} \tag{89}$$

We proved during both ZLB and the forward guidance period, a central bank wants to target financial volatility levels lower than the volatility of TFP process. It lowers the required risk-premium and boost the asset price level $\hat{Q}_t$, thus raising the output. ■

First-order conditions for $\hat{T}',\sigma_1^{q,L},\sigma_2^{q,L}$:[66] A deterministic part of the asset price gap $\hat{Q}_t$, $\hat{C}_{\text{det}}(t;\hat{T}')$, is given as (with $r_1=\rho-(\sigma_1^{q,L})^2, r_2=\rho-(\sigma_2^{q,L})^2$):

$$\hat{C}_{\text{det}}(t;\hat{T}')=\int_t^{\hat{T}'} r_s^T ds=\begin{cases}\underbrace{r_1}_{<0}(T-t)+\underbrace{r_2}_{>0}(\hat{T}'-T), & \text{for } \forall t\le T,\\ r_2(\hat{T}'-t), & \text{for } T\le\forall t<\hat{T}'.\end{cases} \tag{90}$$

Thus we obtain

$$\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt=\int_0^T e^{-\rho t}[r_1(T-t)+r_2(\hat{T}'-T)]dt+\int_T^{\hat{T}'} e^{-\rho t}r_2(\hat{T}'-t)dt. \tag{91}$$

The first condition (first-order condition for $\hat{T}'$) can be written as:

$$2r_2\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt+(\sigma_2^{q,L}-\underline{\sigma})^2\frac{e^{-\rho\hat{T}'}}{\rho}=0, \tag{92}$$

where

$$\int_0^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt=r_1\Big[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\Big]+r_2(\hat{T}'-T)\frac{1-e^{-\rho T}}{\rho}+r_2\Big[\frac{e^{-\rho\hat{T}'}}{\rho^2}+\frac{\hat{T}'-T}{\rho}e^{-\rho T}-\frac{1}{\rho^2}e^{-\rho T}\Big]. \tag{93}$$

[66] These explicit first-order conditions facilitate the computation of optimal values of $\hat{T}',\sigma_1^{q,L},\sigma_2^{q,L}$ in our simulations. The second-order conditions hold.

We plug all the integration and $r_2 = \rho - (\sigma_2^{q,L})^2$ into the above equation and obtain

$$2(\rho - (\sigma_2^{q,L})^2)\Big\{(\rho - (\sigma_1^{q,L})^2)\Big[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\Big] + (\rho - (\sigma_2^{q,L})^2)(\hat{T}' - T)\frac{1 - e^{-\rho T}}{\rho} + (\rho - (\sigma_2^{q,L})^2)\Big[\frac{e^{-\rho \hat{T}'}}{\rho^2} + \frac{\hat{T}' - T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\Big]\Big\} + (\sigma_2^{q,L} - \underline{\sigma})^2\frac{e^{-\rho\hat{T}'}}{\rho} = 0. \tag{94}$$

The second condition (first-order condition for $\sigma_1^{q,L}$ can be written as:

$$2\sigma_1^{q,L}\int_0^T e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')(T-t)dt = (\sigma_1^{q,L} - \bar{\sigma})\frac{1 - e^{-\rho T}}{\rho^2}, \tag{95}$$

into which we plug equation (90) and obtain the first-order condition for $\sigma_1^{q,L}$ as

$$2\sigma_1^{q,L}\Big\{(\rho - (\sigma_1^{q,L})^2)\Big[-\frac{2}{\rho^3}e^{-\rho T} + \frac{T^2}{\rho} - \frac{2T}{\rho^2} + \frac{2}{\rho^3}\Big] + (\rho - (\sigma_2^{q,L})^2)(\hat{T}' - T)\Big[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\Big]\Big\} = (\sigma_1^{q,L} - \bar{\sigma})\frac{1 - e^{-\rho T}}{\rho^2}. \tag{96}$$

The first-order condition for the $\sigma_2^{q,L}$ becomes:

$$2\sigma_2^{q,L}\Big((\hat{T}' - T)\int_0^T e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt + \int_T^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')(\hat{T}' - t)dt\Big) = (\sigma_2^{q,L} - \underline{\sigma})\frac{e^{-\rho T} - e^{-\rho\hat{T}'}}{\rho^2}, \tag{97}$$

where

$$\int_0^T e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')dt = r_1\Big[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\Big] + r_2(\hat{T}' - T)\frac{1 - e^{-\rho T}}{\rho}, \tag{98}$$

and

$$\int_T^{\hat{T}'} e^{-\rho t}\hat{C}_{\text{det}}(t;\hat{T}')(\hat{T}' - t)dt = r_2\Big[-\frac{2}{\rho^3}e^{-\rho\hat{T}'} + \frac{(\hat{T}' - T)^2}{\rho}e^{-\rho T} - \frac{2(\hat{T}' - T)}{\rho^2}e^{-\rho T} + \frac{2}{\rho^3}e^{-\rho T}\Big]. \tag{99}$$

Thus the first-order condition for the $\sigma_2^{q,L}$ can be written as:

$$2\sigma_2^{q,L}\Big\{\Big[(\rho - (\sigma_1^{q,L})^2)\Big[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\Big] + (\rho - (\sigma_2^{q,L})^2)(\hat{T}' - T)\frac{1 - e^{-\rho T}}{\rho}\Big](\hat{T}' - T) + (\rho - (\sigma_2^{q,L})^2)\Big[-\frac{2}{\rho^3}e^{-\rho\hat{T}'} + \frac{(\hat{T}' - T)^2}{\rho}e^{-\rho T} - \frac{2(\hat{T}' - T)}{\rho^2}e^{-\rho T} + \frac{2}{\rho^3}e^{-\rho T}\Big]\Big\} = (\sigma_2^{q,L} - \underline{\sigma})\frac{e^{-\rho T} - e^{-\rho\hat{T}'}}{\rho^2}. \tag{100}$$