Higher-Order Forward Guidance*

Marc Dordal i Carreras† Seung Joo Lee‡

February 4, 2025

Abstract

This paper introduces a business cycle model that integrates financial markets and endogenous financial volatility at the Zero Lower Bound (ZLB). We derive three key insights: first, central banks can mitigate excess financial volatility at the ZLB by credibly committing to future economic stabilization; second, a commitment to refraining from future stabilization can steer the economy toward more favorable equilibrium paths, thereby revealing a trade-off between future stabilization and reduced financial volatility at the ZLB; third, maintaining uncertainty regarding the timing of future stabilization is strictly superior to alternative forward guidance commitments.

Keywords: Monetary Policy, Forward Guidance, Financial Volatility, Risk Premium
JEL Codes: E32, E43, E44, E52, E62

*We appreciate Yuriy Gorodnichenko for his continuous mentorship. We are grateful to Jordi Galí, Nicolae Gârleanu, Pierre-Olivier Gourinchas, Chen Lian, Yang Lu, Albert Marcet, Maurice Obstfeld, Walker Ray, Alp Simsek, anonymous referees, and participants at the Hong Kong junior macro group meeting, CREi-UPF macroeconomics seminar, and Oxford. This paper was previously circulated with the title 'Monetary Policy as a Financial Stabilizer'.

†Hong Kong University of Science and Technology (marcdordal@ust.hk)
‡Saïd Business School, Oxford University (seung.lee@sbs.ox.ac.uk)

1 Introduction

In the aftermath of the Great Recession and the COVID-19 pandemic, prolonged periods of constrained policy rates at the Zero Lower Bound (ZLB) have underscored the need for alternative monetary interventions, notably forward guidance. ZLB episodes are typically associated with heightened financial market volatility, a phenomenon compounded by the diminished efficacy of conventional monetary policy tools. In this context, forward guidance extends beyond its traditional roles of conveying economic forecasts (Delphic guidance) and issuing policy commitments (Odyssean guidance); it also serves as a mechanism for coordinating market participant actions and mitigating overall economic uncertainty. In this paper, we introduce an analytically tractable framework to assess the effects of forward guidance policies at the ZLB, demonstrating that the central bank can strategically shape the intertemporal evolution of aggregate uncertainty to enhance welfare. For example, by generating controlled uncertainty about future economic conditions, the monetary authority can effectively reduce current market volatility at the ZLB. We refer to such interventions as higher-order guidance, distinguishing them from traditional communication strategies that primarily target the levels or expectations of economic variables.

Our paper extends a Two-Agent New Keynesian (TANK) framework by integrating endogenous financial volatility. The model features a representative stock market index that embodies the ownership rights to firms' profits. Hand-to-mouth workers supply labor to these firms, while capitalists hold the economy's aggregate financial wealth and allocate it between consumption and portfolio investments. In equilibrium, capitalists' wealth is directly influenced by stock market performance. An increase in endogenous financial volatility raises the market risk premium, which depresses asset prices and diminishes capitalists' wealth, thereby reducing aggregate demand. Fluctuations in aggregate demand, in turn, affect financial volatility, creating a feedback loop.[1] This dynamic poses a coordination challenge for economic agents, potentially triggering self-fulfilling shocks in volatility that result in a state of persistently elevated financial volatility.

Our analysis begins by investigating whether financial volatility intensifies when conventional monetary policy is constrained by the ZLB. We show that a credible commitment from the central bank to stabilize the economy *after* the ZLB period can prevent excess volatility *during* the ZLB. This result follows from backward induction: if the monetary

[1] A decline in aggregate demand reduces firm profitability, adversely impacting both stock market capitalization and the aggregate wealth of capitalists. Thus, economic and financial market volatility are closely interlinked in our framework.

authority credibly commits to stabilizing the economy within a finite period, it eliminates the possibility of catastrophic or exuberant outcomes that would otherwise heighten economic volatility. Consequently, such a commitment precludes the emergence of unfavorable coordination equilibrium paths that initially give rise to these scenarios.

We then examine the benefits of various forward guidance strategies. In our framework, traditional forward guidance features an Odyssean component, whereby the central bank credibly commits to maintaining the policy rate at zero for a period longer than the minimum required by economic conditions. After this extended ZLB period, the central bank adopts a policy rule aimed at achieving perfect stabilization once the ZLB constraint is lifted. The outcomes of this policy align with previous research: by committing to a future period of accommodative policy rates, the central bank implicitly accepts a temporary phase of positive excess demand and profits. This effect, driven by the forward-looking behavior of stock markets, elevates current stock values and, in turn, boosts aggregate demand during the ZLB. This approach effectively distributes the costs of the ZLB over time, yielding welfare benefits. Moreover, the commitment to perfect future stabilization continues to mitigate excess financial volatility at the ZLB, as discussed previously.

Next, we introduce a novel strategy that leverages the inherent coordination problem among agents to steer them toward an equilibrium characterized by reduced financial and economic volatility at the ZLB. We refer to this approach as *higher-order forward guidance*. Implementing this policy necessitates that the central bank relinquish its commitment to perfect future stabilization. By pledging not to enforce perfect stabilization at the conclusion of the Odyssean guidance period, the central bank facilitates the emergence of multiple coordinated equilibria that backward induction would otherwise rule out. This strategy enables the central bank to steer agents toward equilibrium paths characterized by low volatility and risk premiums during ZLB episodes, thereby maximizing expected welfare beyond what traditional forward guidance—viewed as a limiting case of our approach—can achieve. However, the intervention entails trade-offs. By foregoing stabilization of the business cycle after the ZLB period, the central bank risks substantial future output gap deviations. Thus, higher-order guidance involves balancing reduced financial volatility at the ZLB against diminished stabilization in the subsequent economy. Moreover, we demonstrate that even a slight indication from the central bank that perfect stabilization is not assured at the conclusion of the Odyssean guidance period renders the higher-order forward guidance strategy viable.[2]

[2] Specifically, we prove that if the central bank guarantees a non-zero probability that the business cycle

Finally, we examine the measures available to the central bank to enforce its preferred equilibrium at the ZLB among the multiple solutions permitted by passive future monetary policy. We present an example in which off-equilibrium threats of fiscal intervention—despite involving zero transfers along the equilibrium path—are sufficient to select the optimal higher-order guidance solution as the model's unique equilibrium. While alternative mechanisms for equilibrium selection may exist, this result underscores the importance of coordination between monetary and fiscal authorities at the ZLB and the public's reliance on the credibility of the central bank's communications and commitments. This exercise illustrates a potential benefit of quasi-fiscal interventions (e.g., large-scale asset purchase programs) in steering coordination toward a higher-welfare equilibrium, thereby offering a potential justification for how Mario Draghi's "whatever it takes" speech lowered market volatility and peripheral bond yields during the European sovereign debt crisis.

Our framework offers a potential theoretical justification for the Federal Reserve's decision during the COVID-19 crisis to deliberately introduce ambiguity regarding the extent and timing of inflation overshoots following ZLB periods.[3] Under its flexible average inflation targeting (FAIT) framework, the Federal Reserve committed to delaying economic stabilization by allowing inflation to "moderately" overshoot its target after periods of persistent undershooting at the ZLB. Although our analysis abstracts from inflation dynamics by assuming perfectly rigid prices,[4] our higher-order forward guidance experiment illustrates one possible rationale behind the Fed's approach: nudging economic agents toward coordination on a more favorable equilibrium and mitigating the recessionary effects associated with the ZLB.

Featuring a demand-driven economy with perfectly rigid prices, our framework underscores the significant influence of stock market performance on aggregate demand.

will not be perfectly stabilized at the end of the Odyssean guidance period, then the higher-order forward guidance strategy becomes viable. In this regard, our model exhibits a novel discontinuity: if the monetary authority achieves perfect stabilization with certainty after the ZLB period, the system reverts to traditional forward guidance, in which the central bank does not affect excess volatility or risk premiums at the ZLB. Conversely, even a slight but credible probability of not attaining perfect stabilization enables the central bank to secure a more favorable equilibrium characterized by lower financial volatility and risk premiums at the ZLB.

[3]The Fed's 2020 Statement on Longer-Run Goals and Monetary Policy Strategy states: "In order to anchor longer-term inflation expectations at this level, the Committee seeks to achieve inflation that averages 2 percent over time, and therefore judges that, following periods when inflation has been running persistently below 2 percent, appropriate monetary policy will likely aim to achieve inflation moderately above 2 percent for some time."

[4]This assumption simplifies the analysis. An extended model with sticky prices à la Calvo (1983) yields qualitatively similar results.

In contrast to previous studies (e.g., Akerlof and Yellen (1985), Blanchard and Kiyotaki (1987), Eggertsson and Krugman (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), and Schmitt-Grohé and Uribe (2016)) that focus on recessions driven by deleveraging borrowers and aggregate demand externalities, our model initiates ZLB episodes through a decline in aggregate demand for risky assets—a factor identified as a key driver of financial recessions by Caballero and Farhi (2017) and Caballero and Simsek (2020). Consistent with Werning (2012), we assume an exogenous and deterministic transition to the ZLB, triggered by a shock that raises the risk premium in financial markets and reduces the demand for risky assets, thereby causing a downward jump in the natural rate of interest into negative territory. Our approach diverges from the existing literature by incorporating an endogenous component of financial volatility influenced by both the ZLB and forward guidance. Prior studies (e.g., Bloom (2009), Basu and Bundick (2017), Bloom et al. (2018)) suggest that uncertainty shocks can drive macroeconomic fluctuations. In particular, Basu and Bundick (2017) examine the stabilizing role of monetary policy in the presence of uncertainty shocks, highlighting how the ZLB exacerbates declines in output and its components during periods of heightened uncertainty. While this literature focuses primarily on exogenous uncertainty, our work emphasizes endogenous volatility. Moreover, at the ZLB, we highlight the strategic creation of future uncertainty as a mechanism to reduce present volatility, thereby demonstrating that central banks can engage in strategic intertemporal uncertainty management through equilibrium selection.

Papers including Eggertsson et al. (2003), Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016), and Caballero and Farhi (2017) explore the implications of forward guidance at the ZLB from both theoretical and empirical perspectives. Our research distinguishes itself by focusing on the impact of forward guidance on higher-order moments, including the endogenous volatility of financial markets and the broader economy.[5] This paper also relates to previous work by Lee and Dordal i Carreras (2025a) on determinacy issues and the multiplicity of solutions in the nonlinear New Keynesian model under conventional monetary policy regimes. In contrast, the present study examines whether central bank forward guidance can steer agents toward equilibrium paths—selected from the potential multiplicity of solutions—that are characterized by lower financial volatility and more rapid economic stabilization at the ZLB.

[5]Our approach, in which central bank communications serve as an equilibrium coordination device, aligns with the concept of 'open-mouth' operations at the ZLB described by Campbell and Weber (2019).

Layout The paper is organized as follows. Section 2 introduces the model. Section 3 explains how the ZLB is incorporated into our framework. Section 4 examines the effectiveness of various forward guidance strategies, and Section 5 concludes. Appendix I details the parameter calibration, and Appendix II contains the derivations and proofs. The Online Appendix provides additional derivations and proofs; in particular, Online Appendix H presents an analysis of the non-linear textbook New Keynesian model and demonstrates that its main equilibrium conditions and results are isomorphic to those of the model with financial markets presented here.

2 The Model

We begin by presenting a theoretical framework that facilitates the analysis of higher-order moments related to the aggregate financial and economic volatility of the economy.[6]

2.1 Setting

We work within a continuous-time framework defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. The economy consists of two equally sized groups: capitalists, modeled as neoclassical agents, and hand-to-mouth workers, modeled as Keynesian agents. Consistent with Greenwald et al. (2014) and Caballero et al. (2024), the model assumes that all financial wealth is held by capitalists, while workers rely exclusively on labor income for consumption. Aggregate technology, denoted by A_t, serves as the sole source of exogenous variation and generates the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. The technology process evolves according to a geometric Brownian motion given by

$$\frac{dA_t}{A_t} = gdt + \sigma_t dZ_t \ ,$$

where g denotes the expected growth rate, while σ_t represents the economy's fundamental risk, treated as exogenous. For simplicity, in Section 2 we assume that σ_t is constant

[6]Our findings, with the exception of those in Section F, also hold in a non-linear version of the standard New Keynesian model (e.g., Woodford (2003) and Galí (2015)). We adopt a Two-Agent New Keynesian (TANK) model because it clarifies the interactions among financial volatility, risk premium, aggregate wealth, and aggregate demand, and allows for a tractable analysis of various macroprudential policies (see Section F). A detailed examination of the standard non-linear New Keynesian model is provided in Online Appendix H.

and equal to σ. In Section 3, we introduce a deterministic shift in σ_t to examine various scenarios involving the ZLB.

2.1.1 Firms

The economy comprises a unit measure of monopolistically competitive firms, each producing a distinct intermediate good $y_t(i)$ for $i \in [0, 1]$. These intermediate goods are aggregated into the final output y_t via a Dixit-Stiglitz aggregation function characterized by a substitution elasticity $\epsilon > 0$, as follows:

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}.$$

Each intermediate firm i employs a production function $y_t(i) = A_t \left(N_{W,t}\right)^{\alpha} n_t(i)^{1-\alpha}$, where $N_{W,t}$ denotes the total labor in the economy and $n_t(i)$ represents the labor demand of firm i at time t. Incorporating a production externality à la Baxter and King (1991) aligns our model with observed asset price and wage co-movements, without altering its other qualitative outcomes.[7]

Intermediate firms face a downward-sloping demand curve $y_i(p_t(i) \mid p_t, y_t)$, where $p_t(i)$ is the price of an individual firm's good, p_t is the aggregate price index, and y_t is aggregate output. Specifically, demand is given by

$$y_i(p_t(i) \mid p_t, y_t) = y_t \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon},$$

with the aggregate price index defined as $p_t = \left(\int_0^1 p_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}}$. For tractability, we assume perfect price rigidity, so that $p_t(i) = p_t = \bar{p}$ for all i and t.[8] As a result, each firm produces an identical level of output $y_t(i) = y_t$, determined by aggregate demand.

[7] In models without the Baxter and King (1991) externality, rising asset prices are typically associated with lower wages, which contradicts empirical evidence (Chodorow-Reich et al., 2021) on the effects of stock price increases on aggregate demand, employment, and wages. Incorporating the Baxter and King (1991) externality allows our calibration to capture these empirical trends by linking higher asset prices and aggregate demand with increased labor demand and wages.

[8] An alternative assumption of sticky price resetting à la Calvo (1983) does not significantly alter the model's dynamics or its qualitative results.

2.1.2 Workers

A representative hand-to-mouth worker supplies labor to intermediate firms, earning wage income $w_t N_{W,t}$ that is entirely devoted to final good consumption. The worker maximizes

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \ , \quad \text{s.t.} \quad \bar{p} C_{W,t} = w_t N_{W,t} \ , \tag{1}$$

where $C_{W,t}$ denotes consumption, $N_{W,t}$ denotes labor supply, w_t represents the wage, and χ_0 is the inverse Frisch elasticity of labor supply. Following Mertens and Ravn (2011), we normalize consumption by technology A_t to allow for a linearly additive utility specification consistent with a balanced growth path.[9]

Under the assumption of perfectly rigid prices, equilibrium labor demand by each firm i aggregates directly into total labor $N_{W,t}$, implying that $n_t(i) = N_{W,t}$ for all i. Substituting this result into the production function yields equilibrium output as a linear function of total labor, $y_t = A_t N_{W,t}$.

2.1.3 Financial Market and Capitalists

Unlike conventional New Keynesian models in which a representative household owns firms and receives lump-sum rebated profits, we assume that firm profits are capitalized in the stock market via a representative index fund. Capitalists face an optimal portfolio allocation problem at each moment t, choosing between investment in a risk-free bond and the stock index.

The aggregate nominal value of the stock index fund is given by $\bar{p} A_t Q_t$, where Q_t is the normalized real index price. The price Q_t is determined endogenously and evolves with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ as follows:

$$\frac{dQ_t}{Q_t} = \mu_t^q \, dt + \sigma_t^q \, dZ_t,$$

where μ_t^q and σ_t^q represent the endogenous drift and volatility of the process, respectively. We interpret σ_t^q as a measure of financial uncertainty or disruption. Consequently, aggregate financial wealth $\bar{p} A_t Q_t$ evolves according to a geometric Brownian motion with a combined volatility of $\sigma + \sigma_t^q$. Notably, since σ_t^q is determined in equilibrium and may

[9] This normalization simplifies the analysis without altering the qualitative results of our model.

be either positive or negative, the volatility of the aggregate real stock market value $A_t Q_t$ may exceed or fall below the volatility of the technology process $\{A_t\}$. In particular, when σ_t^q is negative, the overall financial wealth volatility $\sigma + \sigma_t^q$ is lower than the fundamental volatility σ.[10]

In addition to the stock market, we introduce a risk-free bond with a nominal interest rate i_t set by the central bank, assuming that bonds are in zero net supply in equilibrium. A unit measure of identical capitalists allocates their wealth between risk-free bonds and the risky stock index. By holding the stock index, capitalists earn profits from the intermediate goods sector—distributed as dividends—and benefit from stock price revaluations driven by changes in A_t and Q_t. Given the competitive nature of financial markets, each capitalist takes the nominal risk-free rate i_t, the expected stochastic stock market return i_t^m, and the total risk level $\sigma + \sigma_t^q$ as given when making portfolio decisions.[11] If a capitalist allocates a fraction θ_t of their nominal wealth a_t to the stock market, the risk borne over the interval $[t, t+dt]$ is $\theta_t a_t(\sigma + \sigma_t^q)$; thus, the portfolio's riskiness is directly proportional to the investment share θ_t. Being risk-averse, capitalists require a risk premium $i_t^m - i_t$ for investing in the risky asset, which is determined in equilibrium.

A representative capitalist solves the following problem:

$$\begin{aligned} &\max_{C_t, \theta_t} \quad \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dt, \\ &\text{s.t.} \quad da_t = \Big(a_t \big(i_t + \theta_t (i_t^m - i_t) \big) - \bar{p} C_t \Big) dt + \theta_t a_t (\sigma + \sigma_t^q) \, dZ_t, \end{aligned} \tag{2}$$

where ρ denotes the subjective discount rate and C_t represents the final good consumption of capitalists. At each moment, the capitalist receives returns from both risk-free bonds and risky stock investments and allocates these returns to consumption.

2.2 Equilibrium and Asset Pricing

The nominal state price density of capitalists, denoted by ξ_t^N, is given by

$$\xi_t^N = e^{-\rho t} \frac{1}{C_t} \frac{1}{\bar{p}} \ , \ \text{ with } \ \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right) = -i_t dt \ . \tag{3}$$

[10] When $\sigma_t^q < 0$, we have $\text{Cov}_t\,(dA_t, dQ_t) = \sigma \sigma_t^q A_t Q_t \, dt < 0$, implying a negative covariance between total factor productivity and asset prices.

[11] The competitive market assumption is essential in our model to account for inefficiencies stemming from the aggregate demand externality imposed by each capitalist's investment decision. For further details, see Farhi and Werning (2016).

The stochastic discount factor between time t and a future time s is ξ_s^N/ξ_t^N. Under the assumption that capitalists are the equilibrium marginal investors, the aggregate stock market wealth, $\bar{p}A_tQ_t$, is determined as the sum of discounted profit streams from the intermediate goods sector, using ξ_t^N for discounting.

At time t, the total profit of the intermediate goods sector, D_t, is defined by

$$D_t \equiv \bar{p}y_t - w_t N_{W,t} = \bar{p}(y_t - C_{W,t}) = \bar{p}C_t \ , \tag{4}$$

where $w_t N_{W,t}$ (wage income) equals the consumption expenditure of hand-to-mouth workers, $\bar{p}C_{W,t}$. Hence, total dividends equal the capitalists' aggregate consumption expenditure. Substituting this result into the asset pricing equation yields

$$\bar{p}A_tQ_t = \mathbb{E}_t\left[\frac{1}{\xi_t^N}\int_t^\infty \xi_s^N D_s \, ds\right] = \frac{\bar{p}C_t}{\rho}, \tag{5}$$

which implies $C_t = \rho A_t Q_t$. Thus, in equilibrium, capitalists' consumption is a fixed proportion ρ of their aggregate financial wealth.

Agents of the same type—whether workers or capitalists—are identical and make symmetric decisions in equilibrium. Because bonds are in zero net supply, capitalists invest their entire wealth in the stock market, implying that the wealth share θ_t is equal to one at all times. This condition determines the equilibrium risk premium demanded by capitalists. By combining equations (2), (3), and (5), the risk premium is given by

$$\mathrm{rp}_t \equiv i_t^m - i_t = (\sigma + \sigma_t^q)^2, \tag{6}$$

indicating that rp_t increases with the total volatility $\sigma + \sigma_t^q$ of aggregate financial wealth $\bar{p}A_tQ_t$. Notably, a capitalist's wealth gain or loss corresponds to the nominal revaluation of the stock market index. These equilibrium conditions, as specified in equations (5) and (6), are consistent with Merton (1971).

The equilibrium in the goods market and the expected stock return i_t^m are characterized as follows. Given that in equilibrium capitalists' consumption satisfies $C_t = \rho A_t Q_t$, the final goods market equilibrium condition can be written as

$$\rho A_t Q_t + \frac{w_t}{\bar{p}} N_{W,t} = y_t = A_t N_{W,t} \, . \tag{7}$$

The nominal expected return on stocks, i_t^m, consists of two components: the dividend yield from firm profits and the nominal revaluation of stock prices due to fluctuations in A_t and Q_t. In equilibrium, changes in i_t^m affect only nominal stock prices because the dividend yield remains fixed at ρ. Define $\{\mathbf{I}_t^m\}$ as the cumulative stock market return process, such that $\mathbb{E}_t[d\mathbf{I}_t^m] = i_t^m \, dt$. Equation equation (8) decomposes $\mathbf{I}_t^m$ into its dividend yield and stock revaluation components:

$$d\mathbf{I}_t^m = \frac{\overbrace{\not{p}\left(\underbrace{y_t - \frac{w_t}{\bar{p}} N_{W,t}}_{=C_t}\right)}^{\text{Nominal dividend}}}{\underbrace{\not{p} A_t Q_t}_{\text{Total stock market wealth}}} dt + \underbrace{\frac{d\left(\not{p} A_t Q_t\right)}{\not{p} A_t Q_t}}_{\text{Stock revaluation}} \tag{8}$$
$$= \underbrace{\left(\rho + g + \mu_t^q + \sigma\sigma_t^q\right)}_{=i_t^m} dt + \underbrace{\left(\sigma + \sigma_t^q\right)}_{\text{Risk term}} dZ_t \, .$$

The real stock price Q_t plays a critical role in driving the business cycle in equilibrium. An increase in Q_t raises capitalists' consumption, which in turn leads to higher wages and greater labor demand by firms, ultimately boosting aggregate consumption.

Flexible Price Equilibrium. Consistent with the literature, we adopt the flexible price equilibrium as the benchmark guiding the monetary authority's policy objectives. Details of this equilibrium are provided in Online Appendix A. Additionally, Online Appendix B specifies the conditions necessary for positive co-movement among the gaps in asset prices, wages, labor supply, and consumption for both capitalists and workers. Here, 'gaps' refer to the log-deviations from the flexible price equilibrium. As demonstrated in Online Appendix B, these gaps are proportional; henceforth, we express equilibrium conditions in terms of the asset price gap $\hat{Q}_t$.

In the flexible price equilibrium, denoted by the superscript n (for 'natural'), we have $\mu_t^{q,n} = 0$ and $\sigma_t^{q,n} = 0$, implying a constant natural stock price Q_t^n. The natural interest rate, $r^n \equiv \rho + g - \sigma^2$, represents the real risk-free rate in the flexible price economy and remains constant in equilibrium.

2.3 Gap Economy

We define the risk-premium gap as $\hat{rp}_t \equiv \text{rp}_t - \text{rp}_t^n$, where rp_t^n denotes the risk premium in the natural (flexible-price) equilibrium. We then introduce the risk-adjusted natural rate, r_t^T, defined by

$$r_t^T \equiv r_t^n - \frac{1}{2}\hat{rp}_t \, . \tag{9}$$

This rate adjusts the natural rate of return to account for the risk differential between rigid-price and flexible-price economies, serving as an anchor for monetary policy in our model. For example, a positive risk-premium gap ($\hat{rp}_t > 0$) reduces the demand for the stock market portfolio by capitalists relative to the benchmark economy, potentially triggering a recession. This effect is formalized in Proposition 1, which shows that a decline in r_t^T relative to the risk-free policy rate i_t fosters expectations of future asset price revaluations. These expectations translate into lower current asset prices and a widening output gap.

Proposition 1 (Dynamic IS Equation) *The dynamic IS equation of the model, expressed in terms of the asset price gap, is given by:*[12]

$$d\hat{Q}_t = (i_t - r_t^T)dt + \sigma_t^q dZ_t. \tag{10}$$

Proof. See Online Appendix C.

■

Notably, in a linearized approximation of the New Keynesian model, the natural rate r_t^n replaces r_t^T in equation (10), muting the impact of risk on asset prices.

2.4 Monetary Policy and Equilibrium Uniqueness

We complete the model by incorporating a monetary policy rule, which, together with the dynamic IS equation in equation (10) and the implementation of forward guidance or other macroprudential measures, determines the model's solution. The baseline policy rule is given by

$$i_t = \max\left\{r_t^T + \phi_q \hat{Q}_t, \ 0\right\}, \tag{11}$$

[12] A conventional formulation using the output gap yields a similar expression in our model, as both variables are proportional in equilibrium.

where $\phi_q > 0$ satisfies the Taylor principle when the ZLB is not binding.[13] When the ZLB is not binding, combining equations (10) and (11) yields

$$\mathbb{E}_t \, d\hat{Q}_t = \phi_q \, \hat{Q}_t,$$

which implies perfect stabilization of the asset price gap ($\hat{Q}_t = 0$ for all t), corresponding to the unique rational expectations equilibrium outside the ZLB.[14] Section 3 discusses the stabilization and uniqueness properties of the model when the ZLB is binding, while Section 4 considers alternative forward guidance strategies that deviate from equation (11) by temporarily committing to a different set of passive policy rules (Odyssean guidance), whose stabilization and uniqueness properties are examined later.

3 The Zero Lower Bound

ZLB Recession. Following Werning (2012), we consider a scenario in which the interest rate reaches the ZLB as a result of a deterministic shift in the natural rate of interest, r_t^n. Specifically, we assume that the TFP volatility is $\sigma_t = \bar{\sigma}$ for $0 \leq t \leq T$ and $\sigma_t = \underline{\sigma}$ (with $\underline{\sigma} < \bar{\sigma}$) for $t \geq T$. These volatility regimes imply that the natural rate satisfies

$$\underline{r} \equiv r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2 < 0 \quad \text{and} \quad \bar{r} \equiv r^n(\underline{\sigma}) = \rho + g - \underline{\sigma}^2 > 0,$$

so that the ZLB binds in the first period. Without loss of generality—and as evident from the expression for r_t^n—one may alternatively consider shocks to the growth rate g or the discount rate ρ as drivers of the ZLB spell. Our results also hold if T follows a stochastic distribution, as illustrated in Online Appendix E; therefore, we focus here on the simplest case in which T is deterministic.

Recovery Without Guidance. We begin our analysis of ZLB recessions by examining the benchmark scenario—economic recovery in the absence of forward guidance or other macroprudential policies. In this baseline, after period T, the monetary authority adheres to the Taylor rule in equation (11), achieving perfect economic stabilization, so that $\hat{Q}_t = 0$ for

[13] In addition to the Taylor principle ($\phi_q > 0$), Lee and Dordal i Carreras (2025a) show that targeting the risk-adjusted natural rate or its risk-premium component is also necessary for ensuring equilibrium uniqueness in models incorporating higher-order terms in the dynamic IS equation.

[14] See Blanchard and Kahn (1980) and Buiter (1984) for a detailed presentation of the necessary conditions for this uniqueness result.

$t \geq T$. By backward induction from equation (10), perfect stabilization at T necessarily implies the absence of volatility in the asset price gap $\hat{Q}_t$ for $t < T$.[15] Consequently, whenever the monetary authority can credibly commit to the Taylor rule for $t \geq T$, it follows that $\sigma_t^q = 0$ and $r_t^T = \underline{r} < 0$ for $t < T$. Under these conditions, the dynamics of $\hat{Q}_t$ simplify to

$$d\hat{Q}_t = -\underline{r}\, dt \;, \qquad \text{for } t < T \;, \tag{12}$$

with the boundary condition $\hat{Q}_T = 0$ and an initial asset price gap $Q_0 = \underline{r}\, T$. The resulting trajectory of $\{\hat{Q}_t\}$ is illustrated in Figure 1.

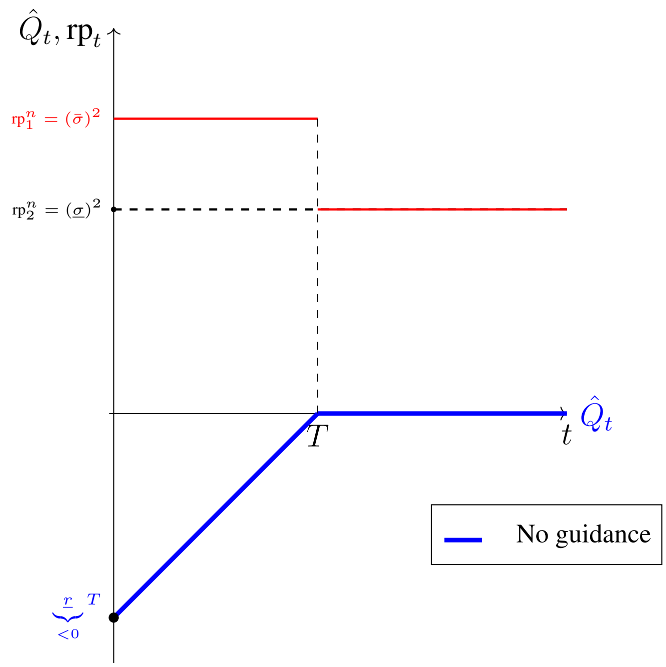

Figure 1: ZLB dynamics, economic recovery without guidance (Benchmark).

The initial increase in σ_t from $\underline{\sigma}$ to $\bar{\sigma}$ raises the natural risk premium from $\mathrm{rp}_2^n = (\underline{\sigma})^2$ to $\mathrm{rp}_1^n = (\bar{\sigma})^2$. Since the ZLB prevents the risk-free rate from declining into negative territory, the increased risk premium induces a decline in asset prices, $\hat{Q}_t$. This reduction in asset prices diminishes capitalists' appetite for stock market investments, leading to lower aggregate financial wealth and reduced consumption demand.[16] These dynamics align with

[15]For example, at $T - \Delta$, where Δ is an infinitesimal time interval, the only rational solution to equation (10) consistent with $\hat{Q}_T = 0$ for any realization of $dZ_{T-\Delta}$ is $\sigma_{T-\Delta}^q = 0$. This deterministically pins down $\hat{Q}_{T-\Delta}$ and, by backward induction, implies $\sigma_t^q = 0$ for all $t \leq T$.

[16]While Caballero and Farhi (2017) show that an increased demand for safe assets under ZLB constraints

the findings of Werning (2012) and Cochrane (2017), even though our model incorporates a distinct IS equation (10) with endogenous volatility σ_t^q affecting the drift of $\hat{Q}_t$—a departure from traditional New Keynesian models. In essence, the credible commitment to future stabilization for $t \geq T$ eliminates excess endogenous volatility during the ZLB episode.

Remarks. Central banks can avert the emergence of endogenous volatility, σ_t^q, at the ZLB by making a credible commitment to stabilize the business cycle by a predetermined future date $T < +\infty$. Even if the monetary authority is constrained by the ZLB and unable to fully implement the policy rule in equation (10)—which directly targets the risk premium—the additional financial stability costs associated with policy inaction can be effectively managed, or even entirely eliminated, by pledging to stabilize the economy upon exiting the ZLB. One implication of this result is that the impact of the ZLB may vary considerably across countries. Monetary authorities committed to post-ZLB stabilization are likely to experience only the demand-driven recession described in this Section, whereas countries that lack either the capacity or the willingness to stabilize in the future might incur additional costs due to increases in σ_t^q during a ZLB episode. Further exploration of these scenarios is left for future research.

4 Forward Guidance

This section analyzes two forward guidance strategies and examines the stabilization tradeoffs inherent in employing these policy tools.

4.1 Traditional Forward Guidance

We define traditional forward guidance as the communication strategy in which the central bank credibly commits[17] to maintaining a zero policy rate for a period $\hat{T}^{\text{TFG}}$ that exceeds the initial interval T of high fundamental volatility. After the forward guidance period ends,

can drive a recession, our analysis suggests that investors withdraw from the stock market, thereby reducing both stock market value and aggregate demand, consistent with Caballero and Simsek (2020).

[17]For analytical tractability, we assume the central bank has the technology for credibly committing to its policy promises. However, Camous and Cooper (2019) (in the context of debt monetization) demonstrates that "grim-trigger strategies" in repeated monetary policy games could sustain coordinated equilibria even in the absence of such commitment. We conjecture that a similar approach could support the equilibria based on commitments presented here in an economy lacking central bank commitment capacity, but we leave a detailed exploration of this issue to future work.

the central bank reverts to the policy rule in equation (11), achieving perfect stabilization of both the business cycle and financial markets for $t \geq \hat{T}^{\text{TFG}}$. By backward induction (see Section 3), this certainty of future stabilization implies that endogenous financial volatility is absent ($\sigma_t^q = 0$) for $t < \hat{T}^{\text{TFG}}$.

Under these conditions, the dynamics of the asset price gap $\hat{Q}_t$ are given by

$$d\hat{Q}_t = \begin{cases} -\underline{r}\, dt, & \text{for } t < T, \\ -\bar{r}\, dt, & \text{for } T \leq t < \hat{T}^{\text{TFG}}, \end{cases} \tag{13}$$

with the boundary condition $\hat{Q}_{\hat{T}^{\text{TFG}}} = 0$. This yields an initial asset price gap of $\hat{Q}_0 = \underline{r}\, T + \bar{r}\, (\hat{T}^{\text{TFG}} - T)$. Figure 2 illustrates the dynamics of $\{\hat{Q}_t\}$ as described by equation (13).

Traditional forward guidance induces an artificial economic boom between T and $\hat{T}^{\text{TFG}}$, thereby mitigating recessionary pressures during $0 \leq t < T$. Specifically, by increasing asset prices between T and $\hat{T}^{\text{TFG}}$, this policy reduces the initial asset price gap $\hat{Q}_0$ owing to the forward-looking behavior of stock markets.

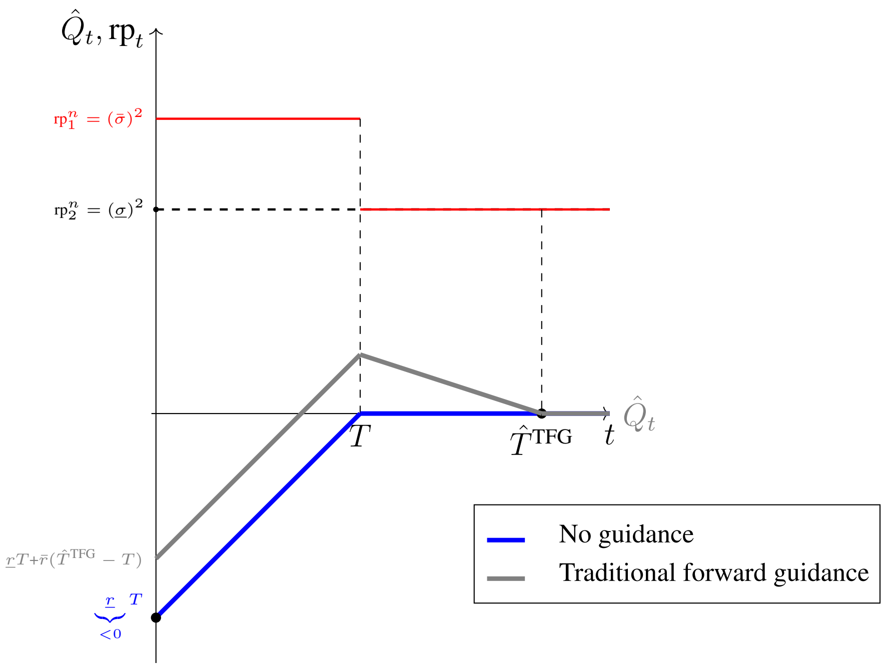

Figure 2: ZLB dynamics under traditional forward guidance.

Optimal Traditional Forward Guidance. To determine the optimal forward guidance duration $\hat{T}^{\text{TFG}}$, we minimize the quadratic welfare loss function

$$\mathbb{L}^Q \left(\{\hat{Q}_t\}_{t\geq 0} \right) = \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2 \, dt \ , \tag{14}$$

subject to the dynamics specified in equation (13).[18] The first-order condition with respect to $\hat{T}^{\text{TFG}}$ yields

$$\int_0^\infty e^{-\rho t} \hat{Q}_t \, dt = 0 \ . \tag{15}$$

Section 4.5 summarizes the key statistics and welfare gains achieved under the optimal traditional forward guidance policy.

In the next section, we argue that central banks might deliberately forgo perfect future stabilization to reduce financial volatility at the ZLB, thereby attaining higher welfare compared to the traditional forward guidance approach. We refer to this alternative method as *higher-order* forward guidance.

4.2 Higher-Order Forward Guidance

The principal cause of ZLB recessions in our model is an excessively high risk premium driven by increased fundamental volatility σ_t. Consequently, central banks might consider an alternative approach that focuses on mitigating financial risk by guiding agents toward a favorable trajectory for asset price volatility, $\{\sigma_t^q\}$, during the ZLB period, thereby supporting asset prices and consumption demand.[19]

Context. In the traditional forward guidance policy discussed earlier, the central bank's commitment to perfect stabilization at $\hat{T}^{\text{TFG}}$ ensures a smooth transition to economic recovery. However, this commitment constrains σ_t^q to remain at its natural level of zero during the ZLB period, as illustrated in Figure 3. To sustain alternative equilibria where σ_t^q deviates from zero, the central bank must refrain from promising perfect stabilization upon exiting the ZLB at $\hat{T}^{\text{TFG}}$, as shown in Figure 4.

[18]The derivation of the quadratic welfare loss function in equation (14) is provided in Online Appendix I.

[19]The risk premium, rp_t, is defined as $\text{rp}_t = (\bar{\sigma} + \sigma_t^q)^2$ for $t < T$ and $\text{rp}_t = (\underline{\sigma} + \sigma_t^q)^2$ for $T \leq t < \hat{T}^{\text{TFG}}$. A negative σ_t^q reduces the risk premium below its natural level, thereby improving asset prices and aggregate demand at the ZLB.

1. Central bank achieves perfect stabilization with certainty after $\hat{T}^{\text{TFG}}$ (i.e., $\hat{Q}_t = 0$, for $t \geq \hat{T}^{\text{TFG}}$)

$\downarrow$

2. $\hat{Q}_{\hat{T}^{\text{TFG}}} = 0$ guarantees $\sigma_t^q = \sigma_t^{q,n} = 0$, $\text{rp}_t = \text{rp}_t^n$ for $t < \hat{T}^{\text{TFG}}$

Figure 3: Mechanism under traditional forward guidance.

$\neg$2. $\sigma_t^q < \sigma_t^{q,n} = 0$, $\text{rp}_t < \text{rp}_t^n$ for $t < \hat{T}^{\text{TFG}}$

$\downarrow$

$\neg$1. $\hat{Q}_{\hat{T}^{\text{TFG}}} \neq 0$: central bank commits not to perfectly stabilize the economy after $\hat{T}^{\text{TFG}}$

Figure 4: Mechanism under higher-order forward guidance.

Implementation. We define $\hat{T}^{\text{HOFG}}$ as the duration during which the policy rate remains zero under our higher-order forward guidance policy. In our framework (see Figure 4), the central bank commits to a forward guidance regime with $i_t = 0$ until $\hat{T}^{\text{HOFG}}$. After this period, the monetary authority adopts a *passive policy rule* that fixes i_t at $\bar{r}$, thereby allowing for multiple equilibria. Under this approach, the central bank coordinates agents onto an optimal equilibrium path, subject to the constraints $\sigma_t^q = 0$ for $t \geq \hat{T}^{\text{HOFG}}$ and $\mathbb{E}_0 \hat{Q}_\infty = 0$. The latter constraint ensures the transversality condition is met, while the former simplifies the optimization by ending the central bank's influence on financial market volatility at the conclusion of the forward guidance period. Together with the dynamic IS equation in (10), these constraints imply that the asset price gap is expected to close by $\hat{T}^{\text{HOFG}}$ on average (i.e., $\mathbb{E}_0 \hat{Q}_{\hat{T}^{\text{HOFG}}} = 0$). In Section 4.3, we relax these constraints by assuming that, after $\hat{T}^{\text{HOFG}}$, the central bank permanently reverts to the active Taylor rule in equation (11) with a probability less than one.

Formalism. We denote the natural risk premiums as follows: $\text{rp}_1^n \equiv \bar{\sigma}^2$ for $t < T$ (the high fundamental volatility region), $\text{rp}_2^n \equiv \underline{\sigma}^2$ for $T \leq t < \hat{T}^{\text{HOFG}}$ (the low fundamental volatility region), and $\text{rp}_3^n \equiv \underline{\sigma}^2$ for $t \geq \hat{T}^{\text{HOFG}}$ (the low fundamental volatility region after the forward guidance period).[20]

[20] The risk premium is defined as $\text{rp}_t = (\sigma_t + \sigma_t^q)^2$, and the natural level follows from the assumption of zero endogenous financial volatility in a flexible-price economy, where $\sigma_t^{q,n} = 0$ for all t.

We simplify the optimization problem by assuming that the central bank maintains constant levels of financial volatility and risk premia within each regime. Specifically, we set the financial volatility σ_t^q to $\sigma_1^{q,L}$ for $t < T$, $\sigma_2^{q,L}$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and 0 for $t \geq \hat{T}^{\text{HOFG}}$. Accordingly, the risk premia in each period are defined as $\text{rp}_1 \equiv (\bar{\sigma} + \sigma_1^{q,L})^2 < \text{rp}_1^n$ for $t < T$, $\text{rp}_2 \equiv (\underline{\sigma} + \sigma_2^{q,L})^2 < \text{rp}_2^n$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and $\text{rp}_3 \equiv (\underline{\sigma})^2$ for $t \geq \hat{T}^{\text{HOFG}}$.[21] This simplified setup is depicted in Figure 5.

Moreover, the risk-adjusted natural rate in equation (9) is expressed as r_1^T for $t < T$ and r_2^T for $T \leq t < \hat{T}^{\text{HOFG}}$, where these rates are functions of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, respectively:

$$\begin{aligned} r_1^T\left(\sigma_1^{q,L}\right) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{\left(\bar{\sigma} + \sigma_1^{q,L}\right)^2}{2} > \underline{r} \equiv r_1^T(0) \quad \text{if } \sigma_1^{q,L} < 0, \\ r_2^T\left(\sigma_2^{q,L}\right) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{\left(\underline{\sigma} + \sigma_2^{q,L}\right)^2}{2} > \bar{r} \equiv r_2^T(0) \quad \text{if } \sigma_2^{q,L} < 0. \end{aligned} \tag{16}$$

Equation (16) shows that lower risk premia during the forward guidance period (up to $\hat{T}^{\text{HOFG}}$) lead to higher risk-adjusted natural rates and, consequently, higher values of the asset price gap $\{\hat{Q}_t\}$ along the expected equilibrium path compared to a traditional forward guidance policy of the same duration. This, in turn, reduces the expected quadratic loss function in (14). However, as indicated by the dynamic IS equation (10), a nonzero σ_t^q introduces stochastic fluctuations in the trajectory of $\hat{Q}_t$, potentially incurring additional stabilization costs in the future.

Figure 6 illustrates this framework: the green line represents the expected trajectory of $\{\hat{Q}_t\}$ under our higher-order forward guidance policy, while the dashed lines depict two possible sample paths arising from stochastic variations.

In summary, central banks employing higher-order forward guidance with commitment face a trade-off between achieving lower risk premia and higher asset price levels prior to $\hat{T}^{\text{HOFG}}$ and incurring subsequent stabilization costs. This balancing act requires a careful

$$\boxed{\neg 2.\ \sigma_t^q = \sigma_1^{q,L} < 0 \text{ for } t < T;\ \sigma_2^{q,L} < 0 \text{ for } T \leq t \leq \hat{T}^{\text{HOFG}};\ \sigma_t^{q,n} = 0 \text{ for } t > \hat{T}^{\text{HOFG}}}$$

$$\downarrow$$

$$\boxed{\neg 1.\ \hat{Q}_{\hat{T}^{\text{HOFG}}} \neq 0\text{: central bank pegs its policy rate } i_t = \bar{r} \text{ after } \hat{T}^{\text{HOFG}}}$$

Figure 5: Simplified higher-order forward guidance.

[21] Proposition 2 later demonstrates that $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ at the optimum. For illustration purposes, we assume these conditions hold for the remainder of this section.

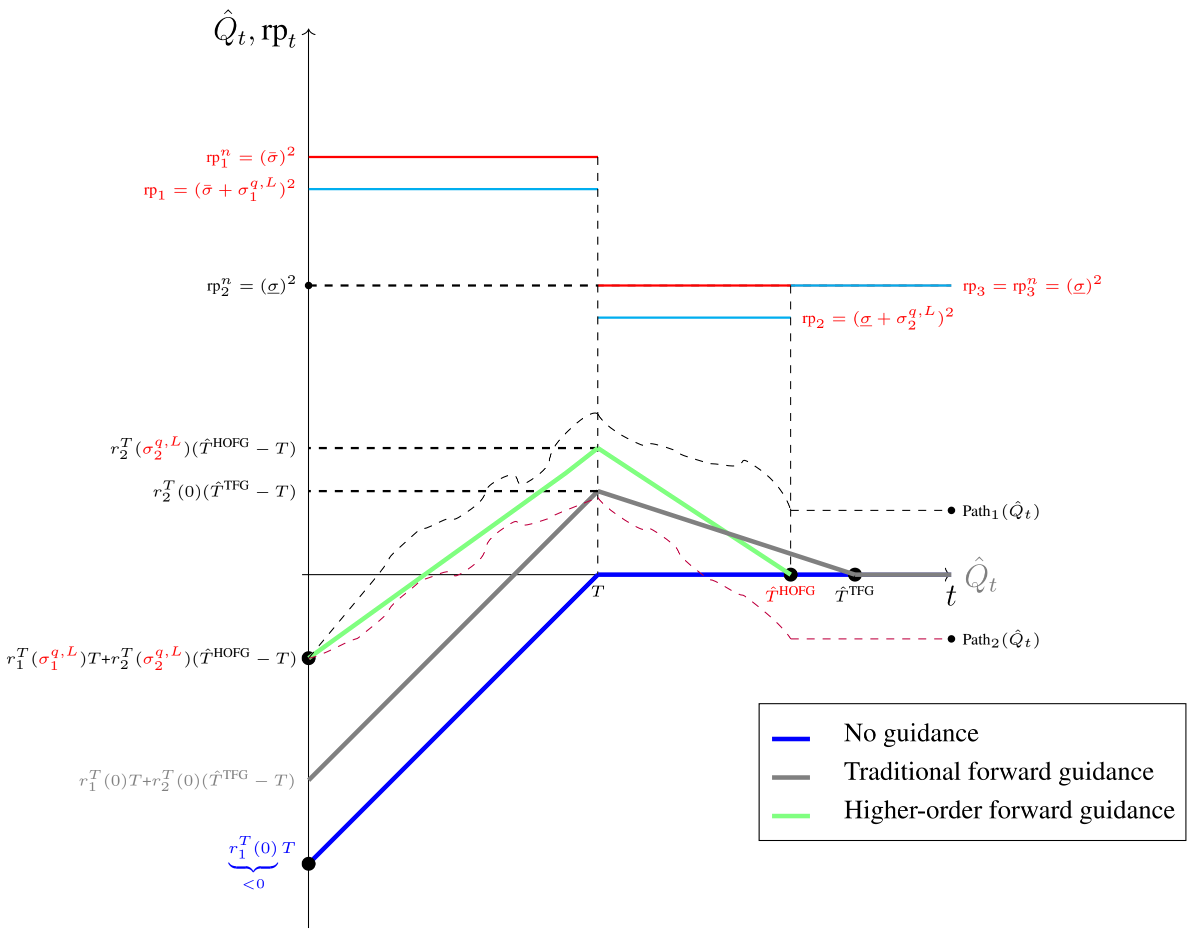

Figure 6: Intervention dynamics of $\{\hat{Q}_t\}$ with $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$.

choice of $\left(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}\right)$. Ultimately, owing to the additional stabilization effects induced by negative $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, the optimal duration of the zero policy rate period, $\hat{T}^{\text{HOFG}}$, becomes shorter than $\hat{T}^{\text{TFG}}$, as proven in Proposition 2.

Optimal Higher-Order Forward Guidance. The initial asset price gap, $\hat{Q}_0$, is determined by the boundary condition $\mathbb{E}_0[\hat{Q}_{\hat{T}^{\text{HOFG}}}] = 0$ and the dynamic IS equation in (10), yielding

$$\hat{Q}_0 = r_1^T(\sigma_1^{q,L})\,T + r_2^T(\sigma_2^{q,L})\left(\hat{T}^{\text{HOFG}} - T\right). \tag{17}$$

The central bank minimizes the quadratic loss function defined in (14) by optimally choosing $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$. The optimization problem is formulated as follows:

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t}\, \hat{Q}_t^2\, dt\ ,$$

$$\text{s.t.} \quad d\hat{Q}_t = \begin{cases} -r_1^T(\sigma_1^{q,L})\, dt + \sigma_1^{q,L}\, dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L})\, dt + \sigma_2^{q,L}\, dZ_t, & \text{for } T \le t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \ge \hat{T}^{\text{HOFG}}, \end{cases} \tag{18}$$

with $\hat{Q}_0$ given by equation (17).

Proposition 2 (Optimal Commitment Path) *The solution to the central bank's higher-order forward guidance optimization problem in (18) yields an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. Moreover, the optimal higher-order forward guidance policy results in an equal or lower expected quadratic loss than the traditional forward guidance policy discussed in Section 4.1.*

Proof. See Appendix II. The latter part follows from the observation that when $(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}) = (0, 0, \hat{T}^{\text{TFG}})$, the trajectory of the asset price gap $\{\hat{Q}_t\}$ coincides with that of a traditional forward guidance policy of duration $\hat{T}^{\text{TFG}}$. Thus, any optimal choice of these parameters yields an equal or lower value of the quadratic loss function in equation (14).

■

4.3 Higher-Order Forward Guidance *with* Stochastic Stabilization

In the previous section, we assumed that once the forward guidance regime ends at $\hat{T}^{\text{HOFG}}$, the monetary authority passively pegs the policy rate i_t to the natural rate $\bar{r}$ and sets $\sigma_t^q = 0$ indefinitely. This arrangement permits σ_t^q to deviate from zero during the ZLB period, as illustrated in Figure 6. In this section, we relax those assumptions while preserving the framework's support for multiple equilibria. Specifically, we now assume that after forward guidance ends, the central bank follows the passive rule but also commits to a stochastic return to the perfect stabilization policy in equation (11). This commitment is modeled as a constant probability event governed by a Poisson process. Consequently, for $t \ge \hat{T}^{\text{HOFG}}$, the asset price gap evolves according to

$$d\hat{Q}_t = -\hat{Q}_t \, d\Pi_t, \quad \text{with} \quad d\Pi_t = \begin{cases} 1, & \text{with probability } \nu \, dt, \\ 0, & \text{with probability } 1 - \nu \, dt, \end{cases}$$

where $d\Pi_t$ is a Poisson random variable with rate parameter $\nu \geq 0$.[22]

The central bank's optimization problem is then formulated as follows:

$$\begin{aligned} \min_{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}} \quad & \mathbb{E}_0 \left[\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \, \hat{Q}_t^2 \, dt + \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t} \, e^{-\nu\left(t - \hat{T}^{\text{HOFG}}\right)} \hat{Q}_t^2 \, dt \right], \\ \text{s.t.} \quad d\hat{Q}_t = & \begin{cases} -r_1^T(\sigma_1^{q,L}) \, dt + \sigma_1^{q,L} \, dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{q,L}) \, dt + \sigma_2^{q,L} \, dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} \end{aligned} \tag{19}$$

with $\hat{Q}_0$ determined by equation (17).

Proposition 3 (Optimal Commitment Path with Stochastic Stabilization) *The solution to the central bank's optimization problem in (19) yields an optimal commitment path characterized by $\sigma_1^{q,L} < 0$, $\sigma_2^{q,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. Moreover, the optimal higher-order forward guidance policy with a stochastic stabilization probability produces an expected quadratic loss that is equal to or lower than that under the traditional forward guidance policy discussed in Section 4.1.*

Furthermore, an increased stabilization probability (i.e., higher ν) leads to smaller optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, thereby reducing risk premia at the ZLB.

Proof. See Online Appendix D. The first part of the proposition extends the results of Proposition 2 to the stochastic stabilization context, while the second part follows from the reduced costs of a more aggressive countercyclical policy when future stabilization is more likely.

■

Finally, Corollary 1 shows that introducing even a minimal degree of uncertainty about the timing of future stabilization is always optimal for the central bank. This uncer-

[22] Here, ν is treated as an exogenous parameter determined by external factors. If the central bank could choose ν, it would set $\nu \to +\infty$, but $\nu \neq \infty$, as shown in Online Appendix D.

tainty enables private agents to coordinate on a stochastic equilibrium in which σ_t^q deviates from zero during the ZLB, as depicted in Figure 6. Consequently, higher-order forward guidance yields equilibrium paths that are strictly superior—in terms of the quadratic loss—compared to those under traditional forward guidance.

Corollary 1 (Discontinuity at the Limit) *When the stabilization parameter ν equals $+\infty$, the problem reduces to the traditional forward guidance case described in Section 4.1. As ν approaches $+\infty$ from below, the central bank's expected quadratic loss function exhibits a discontinuity. In particular, the expected quadratic loss is always lower when there is a non-zero probability of not achieving immediate stabilization at the end of the forward guidance period, $\hat{T}$. Formally:*

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu = \infty\right) ,$$

where $\mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right)$ denotes the quadratic loss function defined in equation (14), evaluated at its optimum for an economy characterized by a Poisson rate ν.

Proof. See Online Appendix D. The intuition is that when $\nu = +\infty$, the probability of immediate stabilization at $\hat{T}^{\text{HOFG}}$ becomes one, which is equivalent to the traditional forward guidance policy discussed in Section 4.1. Moreover, Proposition 3 shows that higher-order guidance always achieves an equal or lower expected quadratic loss compared to traditional guidance, regardless of the value of ν.

■

COVID-19 and the Federal Reserve. Proposition 3 and Corollary 1 theoretically justify the Federal Reserve's deliberate decision during the COVID-19 recession to maintain ambiguity in its communications regarding the extent and timing of inflation stabilization following ZLB episodes. Under its flexible average inflation targeting (FAIT) framework, the Federal Reserve committed to allowing inflation to overshoot its target "moderately" after periods of persistent undershooting at the ZLB. Although our framework abstracts from inflation, the corresponding moderate overshooting of the business cycle is captured by $0 < \nu < \infty$ in Proposition 3, which is preferable to the case $\nu = \infty$ associated with traditional forward guidance.

This analysis offers a rationale for the Fed's policy adjustment: nudging[23] economic agents toward coordination on a more favorable equilibrium while mitigating the recessionary effects associated with the ZLB.

Equilibrium Selection. We next present an example demonstrating how the model's optimal higher-order solution can be implemented in practice among the multiple existing ZLB equilibria through coordinated fiscal policy measures.

4.4 Fiscal Policy Coordination

Fiscal intervention, alongside the forward guidance policies discussed earlier, can enforce the optimal higher-order equilibrium as the unique solution of the model. Although various forms of fiscal coordination are feasible, we focus on a fiscal subsidy (or tax) that adjusts stock returns relative to the risk-free rate.[24] Alternatively, these transfers can be interpreted as a stylized representation of the impact of large-scale asset purchase (LSAP) programs on the relative returns of risky assets. The potential for balance sheet losses (and gains) from such policies—as well as other direct or indirect effects—has led many authors to highlight the quasi-fiscal nature of these interventions (see, e.g., Woodford (2016), Chionis et al. (2021), and Lee and Dordal i Carreras (2025b)).

We consider a subsidy scheme financed by withdrawals from the fiscal authority's monetary reserves, denoted F_t. The fiscal authority provides funds in response to unexpected shocks in stock returns, and for simplicity, we assume that the reserves are affected solely by these transfers. The process evolves as follows:

$$\begin{aligned} dF_t &= -\theta_t \, a_t \, \tau_t \, dZ_t \ , \\ F_0 &= F_{0-} - \chi \, \theta_{0-} \, a_{0-} \ , \end{aligned} \tag{20}$$

where τ_t and χ are parameters determined by the fiscal authority. The subscript $0-$ denotes the values immediately prior to the ZLB shock, allowing for an initial jump in the subsidy

[23] For the empirical or experimental evidence on the effect of nudging in the presence of multiple equilibria, see e.g., Barron and Nurminen (2020), Battiston and Harrison (2024).

[24] In our model, a subsidy (or tax) on stock investments functions similarly to a tax break or hike on capital income, a policy commonly employed by governments. We choose the subsidy formulation for notational simplicity.

process of magnitude $\chi\,\theta_{0-}\,a_{0-}$. The flow budget constraint for capitalists becomes:

$$\begin{aligned} da_t &= \left(a_t\left(i_t + \theta_t\left(i_t^m - i_t\right)\right) - \bar{p}\,C_t\right)dt + \theta_t\,a_t\left[\left(\sigma_t + \sigma_t^q\right) + \tau_t\right]dZ_t\ , \\ \Delta a_0 &= \chi\,\theta_{0-}\,a_{0-} + \bar{p}\,A_{0-}\,\Delta Q_0\ , \end{aligned} \tag{21}$$

where $\Delta x_t \equiv x_t - x_{t-}$ represents the initial jump in the corresponding variable following the ZLB shock. The second equation in (21) indicates that the change in initial capitalist wealth equals the change in the value of stocks (with full allocation in equilibrium at time $0-$) plus the transfer from the fiscal authority.

Proposition 4 describes the implementation of the subsidy scheme that enforces the optimal equilibrium paths detailed in Sections 4.2 and 4.3 as the unique equilibrium outcome.

Proposition 4 (Fiscal Coordination and Unique Optimal Equilibrium) *In the ZLB environment described in Section 3, and under the forward guidance policies of Sections 4.2 and 4.3, if the subsidy scheme is governed by the variables $\{\tau_t, \chi\}$ defined below, the optimal higher-order solution is the unique equilibrium of the model:*

$$\tau_t = \left(\sigma_t^{q,*} - \sigma_t^q\right)\ , \quad \text{and} \quad \chi = \bar{p}A_{0-}\frac{Q_0^* - Q_0}{\theta_{0-}a_{0-}}\ , \tag{22}$$

where the star superscript denotes variables along the optimal higher-order forward guidance path. Thus, $\hat{Q}_t = \hat{Q}_t^*$ and $\sigma_t^q = \sigma_t^{q,*}$ for all $t \geq 0$ in equilibrium.

Proof. See Appendix II. The intuition is that the transfer schedule in equation (22) targets two sources of deviation from the optimal path: (i) the initial stock price response Q_0 to the shock that leads to the ZLB, and (ii) the sensitivity σ_t^q of agents' responses to subsequent stochastic shocks.

■

Under our proposed policy, fiscal intervention acts as an off-equilibrium threat by linking subsidies to deviations from the optimal path, resulting in zero transfers in equilibrium (see Corollary 2). We conjecture that alternative coordination mechanisms may exist—potentially involving elements beyond fiscal intervention—but as long as equilibrium selection is achieved through off-equilibrium threats, the properties and welfare benefits of higher-order guidance can be assessed independently of the specific coordination mechanism employed by the monetary authority.

Corollary 2 (Zero Equilibrium Transfers) *Under the scheme described in Proposition 4, subsidies and tax transfers are zero in equilibrium:*

$$\begin{aligned} \tau_t = \chi = 0 \ , \\ F_t = F_{0-} \ , \end{aligned} \quad \forall t \geq 0 \ .$$

Proof. This follows directly from Proposition 4 since $Q_t = Q_t^*$ and $\sigma_t^q = \sigma_t^{q,*}$ for all $t \geq 0$, together with the definitions in equation (22).

■

Note that these results do not preclude the fiscal authority from implementing conventional fiscal transfer schemes to address output gap deviations caused by the ZLB. The effects of such policies and related implementation details are discussed in Online Appendix F.

European Sovereign Debt Crisis and "Whatever It Takes". Our analysis provides a rationale for Mario Draghi's famous "whatever it takes" speech in 2012, delivered during the European sovereign debt crisis when he was president of the European Central Bank. Given the quasi-fiscal nature of monetary interventions such as large-scale asset purchase (LSAP) programs,[25] his comment that "the ECB is ready to do whatever it takes to preserve the euro" helped market participants coordinate on an equilibrium characterized by lower volatility. This coordination contributed to a significant reduction in Italian and Spanish government bond yields (Acharya et al., 2019). Our results indicate that central banks can guide private agents toward a more favorable equilibrium—achieving higher welfare—by announcing interventions with fiscal characteristics, even if these actions do not involve immediate expenditures.

4.5 Welfare Comparison

For the quantitative evaluation of the forward guidance policies discussed in this paper, we simulate optimal commitment paths at the ZLB under three scenarios: (i) no forward guidance; (ii) traditional forward guidance; and (iii) higher-order forward guidance with varying stabilization probabilities. The initial ZLB duration, T, is set to 20 quarters, reflecting the extended ZLB periods following the global financial crisis. The Poisson rate parameter, ν, in the higher-order forward guidance policy is calibrated first to zero—indicating a zero

[25] For the European Central Bank, we refer to the Outright Monetary Transactions (OMT) program.

probability of reverting to an active policy rule—and then to one, indicating an expectation of resuming an active policy rule one quarter after the guidance period. All remaining model parameters are set to values commonly used in the literature (see Appendix Table I.1).

We define the loss function $\mathbb{L}$ as the per-period quadratic output loss and approximate it by

$$\mathbb{L}^{Y}_{\text{Per-period}} \equiv \rho \int_0^\infty e^{-\rho t}\mathbb{E}_0\left(\hat{Y}_t^2\right) dt \approx \zeta^2 \cdot \rho \int_0^\infty e^{-\rho t} \frac{1}{s}\sum_{i=1}^{s}\left(\hat{Q}_t^{(i)}\right)^2 dt \,,$$

where the constant $\zeta > 0$ is defined by the relationship $\hat{Y}_t = \zeta \hat{Q}_t$ (see equation (B.1) in Online Appendix B). Here, $\hat{Q}_t^{(i)}$ denotes the i^{th} simulated stochastic sample path of the asset price gap.[26] We consider a scenario in which a one-time ZLB recession begins at period zero with no expectation of recurrence; thus, $\mathbb{L}$ is interpreted as the expected conditional loss from a single ZLB episode.

Policy	**No guidance**	**Traditional**	**Higher-Order** (no stochastic stabilization)	**Higher-Order** (*with* stoch. stab., $\nu = 1$)
$\sigma_1^{q,L}$	0	0	-1.27%	-4.13%
$\sigma_2^{q,L}$	0	0	-0.24%	-3.79%
$\hat{T}^{\text{HOFG}}$	20	25.27	25.09	24.68
$\mathbb{L}^{Y}_{\text{Per-period}}$	7%	1.93%	1.81%	1.69%

Table 1: Policy comparisons.

Table 1 presents our simulation results. In this table, $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ are reported as percentages of the fundamental volatilities $\bar{\sigma}$ and $\underline{\sigma}$, respectively. The initial columns assess the effectiveness of traditional forward guidance, showing that the central bank extends the ZLB period by just over one year and reduces total loss by approximately five percentage points. These findings are consistent with the existing literature (e.g., Campbell et al. (2012, 2019), Del Negro et al. (2013), McKay et al. (2016)).[27] The final two columns provide summary statistics on the optimal implementation of higher-order guidance under the

[26] We simulate $s = 10^4$ sample paths to approximate the quadratic loss under higher-order forward guidance.

[27] These studies also note that traditional forward guidance can be overly potent in plain-vanilla New Keynesian models relative to empirical estimates. This paper does not incorporate the quantitative adjustments proposed to address that discrepancy, focusing instead on the differences between traditional and higher-order forward guidance.

two stabilization regimes discussed above. Consistent with Propositions 2 and 3, higher-order guidance further reduces ZLB costs by 0.12–0.24 percentage points per quarter via lower financial market volatility during the guidance period, and permits an earlier exit from the ZLB. Moreover, gains from higher-order guidance double when there is a positive probability of returning to full stabilization in the future.

Standard New Keynesian Model. Our results in Sections 3 and 4 also hold in a non-linear version of the standard New Keynesian model (e.g., Woodford (2003); Galí (2015)). In this model, higher aggregate endogenous volatility increases precautionary savings, depresses consumption demand, and induces a recession. Consequently, the central bank has an incentive to select an equilibrium with lower aggregate volatility during ZLB periods by adopting our higher-order forward guidance policy. A detailed analysis of this framework is provided in Online Appendix H.

5 Conclusion

This paper investigates the potential for increased financial volatility at the ZLB and demonstrates that a credible commitment to future economic stabilization can prevent the emergence of excess volatility. We subsequently examine the effects of traditional forward guidance—defined as the monetary authority's promise to maintain a zero policy rate for an extended period. This commitment generates expectations of higher future asset prices and aggregate demand, thereby enhancing the market valuation of households' financial wealth and, consequently, their aggregate consumption at the ZLB.

Our findings indicate that a central bank may not always find it optimal to commit to perfectly stabilizing the business cycle in the future. By refraining from such a commitment, the central bank allows for alternative equilibrium paths characterized by lower financial volatility at the ZLB and higher expected welfare. Although this strategy is welfare-improving, it entails trade-offs: a lack of commitment, or uncertainty about the timing of future stabilization, reduces financial volatility at the ZLB but may result in large and costly output gap deviations later.

This paper offers valuable insights for academics and policymakers interested in the interplay between financial uncertainty and unconventional policies at the ZLB, particularly forward guidance. Future research should explore central banks' communication strategies under alternative scenarios, such as those involving private information about the state of the economy.

References

Acharya, Viral V, Tim Eisert, Christian Eufinger, and Christian Hirsch, "Whatever It Takes: The Real Effects of Unconventional Monetary Policy," *Review of Financial Studies*, 2019, *32* (9), 3366–3411.

Akerlof, George and Janet L. Yellen, "A Near-Rational Model of the Business Cycle, With Wage and Price Inertia," *Quarterly Journal of Economics*, 1985, *100*, 823–838.

Alvarez-Cuadrado, Francisco, Ngo Van Long, and Markus Poschke, "Capital-labor substitution, structural change and the labor income share," *Journal of Economic Dynamics and Control*, 2018, *87*, 206–231.

Barron, Kai and Tuomas Nurminen, "Nudging cooperation in public goods provision," *Journal of Behavioral and Experimental Economics*, 2020, *88.*

Basu, Susanto and Brent Bundick, "Uncertainty Shocks in a Model of Effective Demand," *Econometrica*, 2017, *85* (3), 937–958.

Battiston, Pietro and Sharon G. Harrison, "Believe it or not: Experimental evidence on sunspot equilibria with social networks," *Games and Economic Behavior*, 2024, *143*, 223–247.

Baxter, Marianne and Robert King, "Productive externalities and business cycles," *Working Paper*, 1991.

Blanchard, Olivier Jean and Charles M. Kahn, "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 1980, *48* (5), 1305–1311.

_ **and Nobuhiro Kiyotaki**, "Monopolistic Competition and the Effects of Aggregate Demand," *American Economic Review*, 1987, *77* (4), 647–666.

Bloom, Nicholas, "The impact of uncertainty shocks," *Econometrica*, 2009, *77* (3), 623–685.

_ **, Max Floetotto, Nir Jaimovich, Itay Saporta-Eksten, and Stephen J. Terry**, "Really Uncertain Business Cycles," *Econometrica*, 2018, *86* (3), 1031–1065.

Buiter, Willem H., "Saddlepoint Problems in Continuous Time Rational Expectations Models: A General Method and Some Macroeconomic Examples," *NBER Working Paper*, 1984.

Caballero, Ricardo J and Alp Simsek, "A Risk-centric Model of Demand Recessions and Speculation," *Quarterly Journal of Economics*, 2020, *135* (3), 1493–1566.

Caballero, Ricardo J. and Emmanuel Farhi, "The Safety Trap," *Review of Economic Studies*, 2017, *85* (1), 223–274.

Caballero, Ricardo, Tomas Caravello, and Alp Simsek, "Financial Conditions Targeting," *Working Paper*, 2024.

Calvo, Guillermo, "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 1983, *12* (3), 383–398.

Camous, Antoine and Russell Cooper, ""Whatever it takes" is all you need: Monetary policy and debt fragility," *American Economic Journal: Macroeconomics*, 2019, *11* (4), 38–81.

Campbell, Jeffrey R. and Jacob P. Weber, "Open Mouth Operations," *FRB of Chicago Working Paper No. WP-2018-3*, 2019.

_ , Charles L. Evans, Jonas DM Fisher, Alejandro Justiniano, Charles W. Calomiris, and Michael Woodford, "Macroeconomic effects of federal reserve forward guidance [with comments and discussion]," *Brookings papers on economic activity*, 2012, pp. 1–80.

_ , Filippo Ferroni, Jonas D. M. Fisher, and Leonardo Melosi, "The limits of forward guidance," *Journal of Monetary Economics*, 2019, *108*, 118—-134.

Chionis, Dionysios, Fotios Mitropoulos, and Antonios Sarantidis, "The Impact of Quantitative Easing Policy on the Government Debt and the NPLs of the Eurozone Periphery Countries," *Debt in Times of Crisis: Does Economic Crisis Really Impact Debt?*, 2021, pp. 55–76.

Chodorow-Reich, Gabriel, Plamen Nenov, and Alp Simsek, "Stock Market Wealth and the Real Economy : A Local Labor Market Approach," *American Economic Review*, 2021, *115* (5), 1613–57.

Cochrane, John, "The new-Keynesian liquidity trap," *Journal of Monetary Economics*, 2017, *92*, 47–63.

Comin, Diego A.and Javier Quintana, Tom G. Schmitz, and Antonella Trigari, "Revisiting Productivity Dynamics in Europe: A New Measure of Utilization-Adjusted TFP Growth," Technical Report, National Bureau of Economic Research 2023.

Dordal i Carreras, Marc, Olivier Coibion, Yuriy Gorodnichenko, and Johannes Wieland, "Infrequent but Long-Lived Zero-Bound Episodes and the Optimal Rate of Inflation," *Annual Review of Economics*, 2016, *8*, 497–520.

Eggertsson, Gauti and Paul Krugman, "Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo approach," *Quarterly Journal of Economics*, 2012, *127* (3), 1469–1513.

Eggertsson, Gauti B et al., "Eggertsson, Gauti and Michael Woodford," *Brookings papers on economic activity*, 2003, *2003* (1), 139–233.

Farhi, Emmanuel and Iván Werning, "Dealing with the Trilemma: Optimal Capital Controls with Fixed Exchange Rates," *Working Paper*, 2012.

_ **and** _ , "A Theory of Macroprudential Policies in the Presence of Nominal Rigidities," *Econometrica*, 2016, *84* (5), 1645–1704.

_ **and** _ , "Fiscal Unions," *American Economic Review*, 2017, *107* (12), 3788–3834.

Galí, Jordi, *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications - Second Edition*, Princeton University Press, 2015.

Gandelman, Néstor and Rubén Hernández-Murillo, "Risk Aversion at the Country Level," *Federal Reserve Bank of St.Louis Working Paper*, 2014.

Greenwald, Daniel L., Martin Lettau, and Sydney C. Ludvigson, "Origins of Stock Market Fluctuations," *Working Paper*, 2014.

King, Robert G. and Sergio T. Rebelo, "Resuscitating Real Business Cycles," *Handbook of Macroeconomics, ed. John B. Taylor and Michael Woodford*, 1999, pp. 927–1007.

Korinek, Anton. and Alp Simsek, "Liquidity Trap and Excessive Leverage," *American Economic Review*, 2016, *106* (3), 699–738.

Lee, Seung Joo and Marc Dordal i Carreras, "Self-fulfilling Volatility and a New Monetary Policy," *Working Paper*, 2025.

_ and _ , "A Unified Theory of the Term-Structure and Monetary Stabilization," *Working Paper*, 2025.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson, "The Power of Forward Guidance Revisited," *American Economic Review*, 2016, *106* (10), 3133—-3158.

Mertens, Karel and Morten O Ravn, "Understanding the aggregate effects of anticipated and unanticipated tax policy shocks," *Review of Economic dynamics*, 2011, *14* (1), 27–54.

Merton, Robert C, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, 1971, *3* (4), 373–413.

Negro, Marco Del, Marc Giannoni, and Christina Patterson, "The Forward Guidance Puzzle," *Federal Reserve Bank of New York Staff Report 574*, 2013.

Schmitt-Grohé, Stephanie and Martin Uribe, "Downward Nominal Wage Rigidity, Currency Pegs, and Involuntary Unemployment," *Journal of Political Economy*, 2016, *124* (5), 1466–1514.

Werning, Iván, "Managing a Liquidity Trap: Monetary and Fiscal Policy," *Working Paper*, 2012.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, 2003.

_ , "Quantitative easing and financial stability," Technical Report, National Bureau of Economic Research 2016.

I Parameter Calibration

	Parameter Description	Value	Source
φ	Relative Risk Aversion	0.2	Within the admissible calibration ranges specified by Gandelman and Hernández-Murillo (2014).
χ_0	Inverse Frisch labor supply elasticity	0.25	See King and Rebelo (1999).
ρ	Subjective time discount factor	0.020	Target 2.8% natural rate.
g	TFP growth rate	0.0083	Annual growth rate of 3.3%, which corresponds to the US TFP growth rate from 2009 to 2020, as detailed in Table 8 of Comin et al. (2023).
$\underline{\sigma}$	TFP volatility, low volatility regime	0.009	See Dordal i Carreras et al. (2016).
$\bar{\sigma}$	TFP volatility, high volatility regime	0.209	Target -1.5% natural rate (ZLB recession).
T	ZLB duration (quarters)	20	A five-year ZLB duration, consistent with periods such as the Global Financial Crisis and the Great Recession. See Dordal i Carreras et al. (2016).
ν	Stabilization probability parameter	1	Target average duration $1/\nu$ of one quarter before returning to stabilization.
α	$1-$ Labor income share	0.4	See Alvarez-Cuadrado et al. (2018).
ϵ	Elasticity of substitution intermediate goods	7	Target steady-state mark-up of 16.7%. See Galí (2015).

Table I.1: Parameter calibration used in Section 4.

II Proofs and Derivations

Proof of Proposition 2. In the context outlined in Section 4.2, the central bank solves the following problem:[1]

$$\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2 dt \;, \quad \text{s.t. } d\hat{Q}_t = \begin{cases} -\underbrace{r_1^T(\sigma_1^{q,L})}_{<0} dt + \sigma_1^{q,L} dZ_t \;, & \text{for } t < T \;, \\ -\underbrace{r_2^T(\sigma_2^{q,L})}_{>0} dt + \sigma_2^{q,L} dZ_t \;, & \text{for } T \le t < \hat{T}^{\text{HOFG}} \;, \\ 0 \;, & \text{for } t \ge \hat{T}^{\text{HOFG}} \;, \end{cases}$$

$$\text{with } \hat{Q}_0 = r_1^T(\sigma_1^{q,L})T + r_2^T(\sigma_2^{q,L})(\hat{T}^{\text{HOFG}} - T) \;, \tag{II.1}$$

where

$$r_1^T(\sigma_1^{q,L}) \equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} < 0 \;, \; r_2^T(\sigma_2^{q,L}) \equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{(\underline{\sigma} + \sigma_2^{q,L})^2}{2} > 0 \;.$$

After $\hat{T}^{\text{HOFG}}$, there are no additional fluctuation in $\hat{Q}_t$. Defining r_s^T as $r_1^T(\sigma_1^{q,L})$ for $s < T$ and as $r_2^T(\sigma_2^{q,L})$ for $T \le s \le \hat{T}^{\text{HOFG}}$, the process of $\hat{Q}_t$ can be articulated as follows:

$$\hat{Q}_t = \begin{cases} \underbrace{\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds}_{\equiv \hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})} + \sigma_1^{q,L} \underbrace{Z_t}_{\sim N(0,t)} \;, & \text{for } t \le T \;, \\ \underbrace{\int_t^{\hat{T}^{\text{HOFG}}} r_s^T ds}_{\equiv \hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})} + \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{t-T}}_{\sim N(0,t-T)} \;, & \text{for } T < t \le \hat{T}^{\text{HOFG}} \;, \\ \sigma_1^{q,L} Z_T + \sigma_2^{q,L} \underbrace{W_{\hat{T}^{\text{HOFG}}-T}}_{\sim N(0,\hat{T}^{\text{HOFG}}-T)} \;, & \text{for } \hat{T}^{\text{HOFG}} < t \;. \end{cases} \tag{II.2}$$

where it is assumed that after $\hat{T}^{\text{HOFG}}$, central banks maintain $\sigma_t^q = \sigma_t^{q,n} = 0$. In this equation, Z_t, W_{t-T}, and $U_{\hat{T}-T}$ are independent Brownian motions. If we square each term in equation (II.2) and apply the expectation operator with respect to the information

[1] For this proof, it is implicitly assumed that $r_1^T(\sigma_1^{q,L}) < 0$ and $r_2^T(\sigma_2^{q,L}) > 0$ hold for the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, ensuring that the ZLB remains effective up to time T.

available at $t = 0$, we obtain:

$$
\mathbb{E}_0\, \hat{Q}_t^2 = \begin{cases} \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 + \left(\sigma_1^{q,L}\right)^2 t\,, & \text{for } t \leq T\,, \\ \hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 + \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (t-T)\,, & \text{for } T < t \leq \hat{T}^{\mathrm{HOFG}}\,, \\ \left(\sigma_1^{q,L}\right)^2 T + \left(\sigma_2^{q,L}\right)^2 (\hat{T}^{\mathrm{HOFG}} - T)\,, & \text{for } \hat{T}^{\mathrm{HOFG}} < t\,. \end{cases} \tag{II.3}
$$

If we substitute equation (II.3) into the central bank's loss function (14), the central bank's commitment problem can be expressed as follows:

$$
\begin{aligned}
&\min_{\hat{T}^{\mathrm{HOFG}},\sigma_1^{q,L},\sigma_2^{q,L}} \quad \mathbb{E}_0 \int_0^\infty e^{-\rho t}\hat{Q}_t^2\, dt \\
&= \min_{\hat{T}^{\mathrm{HOFG}},\sigma_1^{q,L},\sigma_2^{q,L}} \quad \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 dt + \left(\sigma_1^{q,L}\right)^2 \underbrace{\int_0^T te^{-\rho t}dt}_{=\frac{1}{\rho^2}-\frac{1}{\rho^2}e^{-\rho T}-\cancel{\frac{T}{\rho}e^{-\rho T}}} + \left(\sigma_1^{q,L}\right)^2 T \underbrace{\int_T^\infty e^{-\rho t}dt}_{=\cancel{\frac{1}{\rho}e^{-\rho T}}} \\
&\quad + \left(\sigma_2^{q,L}\right)^2 \underbrace{\int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}(t-T)dt}_{=-\cancel{\frac{1}{\rho}(\hat{T}^{\mathrm{HOFG}}-T)e^{-\rho\hat{T}^{\mathrm{HOFG}}}}+\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2}} + \left(\sigma_2^{q,L}\right)^2 (\hat{T}^{\mathrm{HOFG}} - T) \underbrace{\int_{\hat{T}^{\mathrm{HOFG}}}^\infty e^{-\rho t}dt}_{=\cancel{\frac{1}{\rho}e^{-\rho\hat{T}^{\mathrm{HOFG}}}}} \\
&= \min_{\hat{T},\sigma_1^{q,L},\sigma_2^{q,L}} \quad \underbrace{\int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})^2 dt}_{\text{Deterministic fluctuations}} + \underbrace{\left(\sigma_1^{q,L}\right)^2 \frac{1}{\rho^2}(1-e^{-\rho T}) + \left(\sigma_2^{q,L}\right)^2 \left(\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2}\right)}_{\text{Stochastic fluctuations}}\,.
\end{aligned} \tag{II.4}
$$

The central bank now has control over $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\mathrm{HOFG}}$, in addition to its conventional monetary policy tool $\{i_t\}$. Initially, we derive the first-order condition for $\hat{T}^{\mathrm{HOFG}}$, which is as follows:

$$
2\cdot \underbrace{r_2^T(\sigma_2^{q,L})}_{>0} \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt + \left(\sigma_2^{q,L}\right)^2 \frac{1}{\rho}e^{-\rho\hat{T}^{\mathrm{HOFG}}} = 0\,, \tag{II.5}
$$

from which we obtain

$$
\int_0^\infty e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}\|\sigma_1^{q,L}<0,\sigma_2^{q,L}<0)dt < 0\,. \tag{II.6}
$$

The first-order condition for $\hat{T}^{\text{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\text{HOFG}}$ compared to $\hat{T}^{\text{TFG}}$ (traditional forward guidance), as discussed in Section 4.1. This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$, the expression above becomes

$$\int_0^{\hat{T}^{\text{TFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T} \| \sigma_1^{q,L} = \sigma_1^{q,n} = 0, \sigma_2^{q,L} = \sigma_2^{q,n} = 0) dt = 0 \ , \tag{II.7}$$

which is derived by plugging $\sigma_1^{q,L} = 0$ and $\sigma_2^{q,L} = 0$ into equation (II.5).

Given that at the optimum, $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ (which we will demonstrate),

$$\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} = 0, \sigma_2^{q,L} = 0) < \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) \ .$$

Therefore, we deduce from equation (II.1) that at the optimum, $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$, as evidenced by comparing (II.7) with (II.6).

To characterize the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, a **variational argument** is required. This is because $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ influence the levels of $r_1^T(\sigma_1^{q,L})$, $r_2^T(\sigma_2^{q,L})$, and $\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})$. Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{q,L})}{\partial \sigma_1^{q,L}} = -\left(\bar{\sigma} + \sigma_1^{q,L}\right) < 0, \ \ \frac{\partial r_2^T(\sigma_2^{q,L})}{\partial \sigma_2^{q,L}} = -\left(\underline{\sigma} + \sigma_2^{q,L}\right) < 0 \ .$$

Determining $\sigma_1^{q,L}$ An increase in $\sigma_1^{q,L}$ leads to a decrease in $r_1^T(\sigma_1^{q,L})$, which alters the trajectory of $\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})$. This change is illustrated in Figure II.1, as depicted by the transition from the thick blue line to the dashed red line.

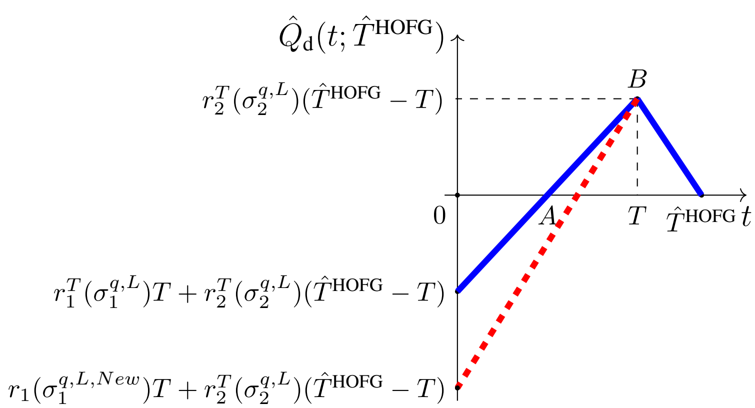

Figure II.1: Variation along $\sigma_1^{q,L}$. Increase to $\sigma_1^{q,L,New} > \sigma_1^{q,L}$.

Differentiating $\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG}) = \int_t^{\hat{T}^{\rm HOFG}} r_s^T ds$ with respect to $\sigma_1^{q,L}$, we obtain:

$$\frac{\partial \hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})}{\partial \sigma_1^{q,L}} = \int_t^T -\left(\bar{\sigma}+\sigma_1^{q,L}\right) ds = -\left(\bar{\sigma}+\sigma_1^{q,L}\right)(T-t), \;\; \forall t \leq T \,.$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function in (II.4) by $\sigma_1^{q,L}$ and obtain the following condition:

$$\left(\bar{\sigma}+\sigma_1^{q,L}\right)\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(T-t)dt = \left(\sigma_1^{q,L}\right)\frac{1-e^{-\rho T}}{\rho^2}\,,$$

from which we can prove that $\sigma_1^{q,L}<0$ must be satisfied at the optimum, given that

$$\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(T-t)dt = \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\rm d}(s;\hat{T}^{\rm HOFG})ds(T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\rm d}(s;\hat{T}^{\rm HOFG})ds}_{<0}\,dt < 0\,,$$

where $\int_0^t e^{-\rho s}\hat{Q}_{\rm d}(s;\hat{T}^{\rm HOFG})ds < 0$ for $t\leq T$, as derived in equation (II.6).

Determining $\sigma_2^{q,L}$ An increase in $\sigma_2^{q,L}$ leads to a decrease in $r_2^T(\sigma_2^{q,L})$, which alters the shape of $\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})$. This effect is illustrated in Figure II.2 by the transition from the thick blue line to the dashed red line. To further analyze this, we differentiate $\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})$ with respect to $\sigma_2^{q,L}$ and obtain:

$$\frac{\partial \hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})}{\partial \sigma_2^{q,L}} = \begin{cases} \int_T^{\hat{T}^{\rm HOFG}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right) ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\rm HOFG}-T)\,, & t<T\,, \\ \int_t^{\hat{T}^{\rm HOFG}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right) ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\rm HOFG}-t)\,, & T\leq t\leq \hat{T}^{\rm HOFG}\,. \end{cases}$$

To find the optimal $\sigma_2^{q,L}$, we differentiate the objective function in (II.4) by $\sigma_2^{q,L}$ and obtain

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left(\int_0^T e^{-\rho t}\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(\hat{T}^{\rm HOFG}-T)dt + \int_T^{\hat{T}^{\rm HOFG}} e^{-\rho t}\underbrace{\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})}_{>0}(\hat{T}^{\rm HOFG}-t)dt\right) = (\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}}}{\rho^2}\,,$$

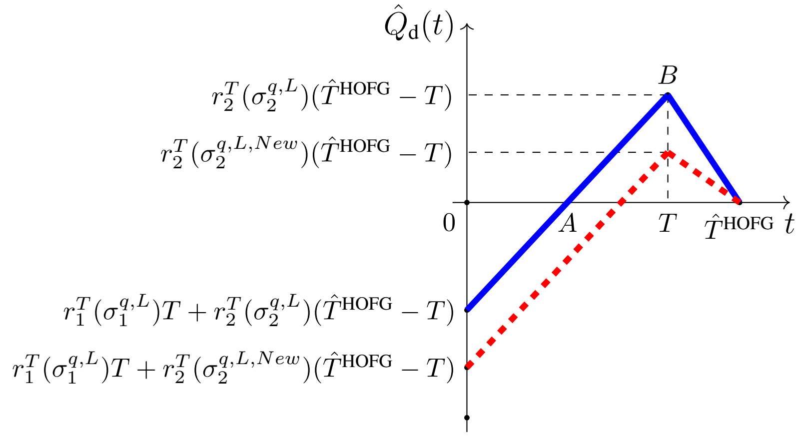

Figure II.2: Variation along $\sigma_2^{q,L}$. Increase to $\sigma_2^{q,L,New} > \sigma_2^{q,L}$.

from which we can demonstrate that at the optimum, $\sigma_2^{q,L} < 0$ must be satisfied, given that

$$
\begin{aligned}
&\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}} - T)dt + \int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}_{>0}(\hat{T}^{\mathrm{HOFG}} - t)dt \\
&< \int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}} - T)dt + \int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}_{>0}(\hat{T}^{\mathrm{HOFG}} - T)dt \\
&= (\hat{T}^{\mathrm{HOFG}} - T)\underbrace{\int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt}_{<0} < 0\,,
\end{aligned}
$$

where the final inequality is derived from equation (II.6). Hence, during periods of high TFP volatility (i.e., $t < T$) and low TFP volatility with forward guidance (i.e., $T \leq t \leq \hat{T}^{\mathrm{HOFG}}$), a central bank aims to target financial volatility levels below those in a flexible price economy: $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma_2^{q,n} = 0$. Such intervention reduces the required risk premium and raises the asset price level $\hat{Q}_t$, thereby increasing output.

■

First-Order Conditions for $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\mathrm{HOFG}}$ The deterministic component of the capitalists' asset gap process $\hat{Q}_t$, denoted as $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$, is defined as follows (with

$r_1^T(\sigma_1^{q,L})$ and $r_2^T(\sigma_2^{q,L})$ specified in equation (16)):

$$\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}}) = \int_t^{\hat{T}^{\mathrm{HOFG}}} r_s^T ds = \begin{cases} \underbrace{r_1^T(\sigma_1^{q,L})}_{<0}(T-t) + \underbrace{r_2^T(\sigma_2^{q,L})}_{>0}(\hat{T}^{\mathrm{HOFG}}-T), & \text{for } \forall t \leq T \ , \\ r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-t), & \text{for } T \leq \forall t < \hat{T}^{\mathrm{HOFG}} \ , \end{cases}$$

from which we derive the following:

$$\begin{aligned} \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = & \int_0^T e^{-\rho t}\left[r_1^T(\sigma_1^{q,L})(T-t) + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\right] dt \\ & + \int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t} r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-t)dt \ . \end{aligned} \tag{II.8}$$

The first condition for $\hat{T}^{\mathrm{HOFG}}$ can be written as

$$2\cdot r_2^T(\sigma_2^{q,L})\int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho} = 0 \ , \tag{II.9}$$

where

$$\begin{aligned} \int_0^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = & r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\frac{1-e^{-\rho T}}{\rho} \\ & + r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2} + \frac{\hat{T}^{\mathrm{HOFG}}-T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right] \ , \end{aligned}$$

follows from equation (II.8). Combined with equation (II.9), the first-order condition for $\hat{T}^{\mathrm{HOFG}}$ is expressed as follows:

$$\begin{aligned} 2\cdot r_2^T(\sigma_2^{q,L}) & \left[r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\frac{1-e^{-\rho T}}{\rho} \right. \\ & \left. + r_2^T(\sigma_2^{q,L})\left[\frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2} + \frac{\hat{T}^{\mathrm{HOFG}}-T}{\rho}e^{-\rho T} - \frac{1}{\rho^2}e^{-\rho T}\right]\right] + \left(\sigma_2^{q,L}\right)^2 \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho} = 0 \ . \end{aligned}$$

The first-order condition for $\sigma_1^{q,L}$ is expressed as

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right)\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = \left(\sigma_1^{q,L}\right)\frac{1-e^{-\rho T}}{\rho^2} \ , \tag{II.10}$$

where

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right] + r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]. \tag{II.11}$$

Substituting equation (II.11) into equation (II.10), we arrive at:

$$(\bar{\sigma}+\sigma_1^{q,L})\left[r_1^T(\sigma_1^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho T}+\frac{T^2}{\rho}-\frac{2T}{\rho^2}+\frac{2}{\rho^3}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]\right] = (\sigma_1^{q,L})\frac{1-e^{-\rho T}}{\rho^2},$$

as the first-order condition for $\sigma_1^{q,L}$. Finally, the first-order condition for $\sigma_2^{q,L}$ is as follows:

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left((\hat{T}^{\mathrm{HOFG}}-T)\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt+\int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}}-t)dt\right) = (\sigma_2^{q,L})\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2},$$

Therefore, the first-order condition for $\sigma_2^{q,L}$ is expressed as:[2]

$$\left(\underline{\sigma}+\sigma_2^{q,L}\right)\left[\left[r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\frac{1-e^{-\rho T}}{\rho}\right](\hat{T}^{\mathrm{HOFG}}-T) + r_2^T(\sigma_2^{q,L})\left[-\frac{2}{\rho^3}e^{-\rho\hat{T}^{\mathrm{HOFG}}}+\frac{(\hat{T}^{\mathrm{HOFG}}-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}^{\mathrm{HOFG}}-T)}{\rho^2}e^{-\rho T}+\frac{2}{\rho^3}e^{-\rho T}\right]\right] = \left(\sigma_2^{q,L}\right)\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^2}.$$

[2] We use the following properties of $\hat{Q}_d\left(t;\hat{T}^{\mathrm{HOFG}}\right)$:

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})dt = r_1^T(\sigma_1^{q,L})\left[\frac{e^{-\rho T}}{\rho^2}+\frac{T}{\rho}-\frac{1}{\rho^2}\right]+r_2^T(\sigma_2^{q,L})(\hat{T}^{\mathrm{HOFG}}-T)\frac{1-e^{-\rho T}}{\rho},$$

and

$$\int_T^{\hat{T}^{\mathrm{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(\hat{T}^{\mathrm{HOFG}}-t)dt = r_2^T(\sigma_2^{q,L})\left[-\frac{2e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho^3}+\frac{(\hat{T}^{\mathrm{HOFG}}-T)^2}{\rho}e^{-\rho T}-\frac{2(\hat{T}^{\mathrm{HOFG}}-T)}{\rho^2}e^{-\rho T}+\frac{2e^{-\rho T}}{\rho^3}\right].$$

Fiscal Policy Coordination - Derivations Integrating equation (20), we obtain an expression for the monetary reserves as:

$$F_t = \underbrace{F_{0-} - \chi\theta_{0-}a_{0-}}_{\equiv F_0} - \int_0^t \theta_s a_s \left(\sigma_s^{q,*} - \sigma_s^q\right) dZ_s \ . \tag{II.12}$$

From the first order condition of capitalists, and setting $\theta_t = 1$ in equilibrium, we can express the stock market's expected returns for $t \geq 0$ as:

$$i_t^m = i_t + \left[(\sigma_t + \sigma_t^q) + \tau_t\right]^2 \ . \tag{II.13}$$

Using equations (5), (22), (II.13) and the equilibrium condition $\theta_t = 1$ into the equations in (21), we obtain:

$$\frac{da_t}{a_t} = \left(i_t + (\sigma_t + \sigma_t^{q,*})^2 - \rho\right) dt + (\sigma_t + \sigma_t^{q,*})\, dZ_t \ , \tag{II.14}$$

$$a_0 = a_{0-} + \bar{p}A_{0-}\Delta Q_0^* = \underbrace{\bar{p}A_0Q_0^*}_{=a_0^*} \ , \tag{II.15}$$

where the last equality in equation (II.15) follows from the fact that wealth is fully allocated into stocks in equilibrium, $a_{0-} = \bar{p}A_{0-}Q_{0-}$, and that TFP does not experience any jump at the limit, $A_0 = A_{0-}$. From equation (II.14) we obtain an expression for the ouput gap process as:

$$d\hat{Q}_t = \left(i_t - r_t^{T,*}\right) dt + \sigma_t^{q,*} dZ_t \ , \tag{II.16}$$

which aligns with the process of the output gap under the optimal forward guidance solution. We previously defined the volatility of the asset price gap as $\sigma_t^q = Var_t\left(\frac{dQ_t}{Q_t}\right)$, so equation (II.16) demonstrates that $\sigma_t^q = \sigma_t^{q,*}$ in equilibrium under the subsidy scheme. Similarly, equation (II.15) implies that $Q_0 = Q_0^*$, and threfore $\hat{Q}_0 = \hat{Q}_0^*$ in equilibrium. Finally, substituting these findings into equations (22) and (II.12), we establish the statements in Corollary 2.

A Flexible Price Equilibrium

This section derives the flexible price equilibrium of the model, establishing it as the benchmark for economic and welfare analysis. We begin by revisiting the Fisherian identity, incorporating an inflation premium linked to wealth volatility into the relation. Lemma A.1 summarizes the modified identity.

Lemma A.1 (Inflation Premium) *The real interest rate of the economy is given by:*

$$r_t = i_t - \pi_t + \overbrace{\sigma_t^p \underbrace{\left(\sigma + \sigma_t^p + \sigma_t^q\right)}_{\textit{Wealth volatility}}}^{\textit{Inflation Premium}} \, . \tag{A.1}$$

Proof of Lemma A.1. The financial wealth of capitalists is equal to the value of the stock market index, $a_t = p_t A_t Q_t$, which follows from bonds being in zero net supply and capitalists being symmetric and identical in equilibrium. We start by stating capitalist's nominal state-price density ξ_t^N, which satisfies the following condition:

$$\frac{d\xi_t^N}{\xi_t^N} = -i_t dt - (\sigma + \sigma_t^q) dZ_t \ ,$$

and the real state price density ξ_t^r, which is given by

$$\xi_t^r = e^{-\rho t} \frac{1}{C_t} = p_t \xi_t^N \ . \tag{A.2}$$

Utilizing equations (2) and (3), and considering that $\theta_t = 1$ in equilibrium, the application of Ito's Lemma to equation (A.2) yields the following expression:

$$\frac{d\xi_t^r}{\xi_t^r} = \left(\underbrace{\pi_t - i_t - \sigma_t^p \left(\sigma + \sigma_t^q + \sigma_t^p\right)}_{=-r_t} \right) dt - (\sigma + \sigma_t^q) dZ_t \ ,$$

resulting in the modified Fisherian identity detailed in equation (A.1).

■

Definition A.1 *Let* $\chi^{-1} \equiv \dfrac{1-\varphi}{\chi_0+\varphi}$ *represent the effective labor supply elasticity of workers, conditional on their optimal consumption decision.*

Proposition A.1 summarizes the dynamics of the real wage, asset price, natural interest rate r_t^n, and the consumption process of capitalists within the flexible price equilibrium.

Proposition A.1 (Flexible Price Equilibrium) *In the flexible price equilibrium,*[1] *the following results are obtained:*

1. *The real wage is proportional to aggregate technology* A_t*, and given by*

$$\frac{w_t^n}{p_t} = \frac{(\epsilon-1)(1-\alpha)}{\epsilon} A_t \ .$$

2. *The equilibrium asset price* Q_t^n *is constant and given by*

$$Q_t^n = \frac{1}{\rho}\left(\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \ , \quad \textit{and} \quad \mu_t^{q,n} = \sigma_t^{q,n} = 0 \ .$$

3. *The natural interest rate* r_t^n *is constant and defined as* $r_t^n \equiv r^n = \rho + g - \sigma^2$*. The consumption of capitalists evolves according to the following equation:*

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \underbrace{\left(r^n - \rho + \sigma^2\right)}_{\equiv \mu_t^{c,n}} dt + \underbrace{\sigma}_{\equiv \sigma_t^{c.n}} dZ_t \ .$$

Proof of Proposition A.1. Starting with the optimization problem of intermediate firms, the presence of an externality à la Baxter and King (1991) imposes extra steps on the aggregation process of individual decisions across firms. Utilizing the production function, the employed labor of firm i can be expressed as

$$n_t(i) = \left(\frac{y_t(i)}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \ ,$$

[1] Variables in the flexible price (i.e., natural) equilibrium are denoted with the superscript n.

where we defined $E_t \equiv (N_{W,t})^{\alpha}$. At any given time t, each intermediate firm i determines the optimal price $p_t(i)$ to maximize its profits,

$$\max_{p_t(i)} p_t(i) \left(\frac{p_t(i)}{p_t}\right)^{-\epsilon} y_t - w_t \left(\frac{y_t}{A_t E_t}\right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t}\right)^{-\frac{\epsilon}{1-\alpha}} , \tag{A.3}$$

taking the aggregate demand of the economy y_t as given. In the flexible price equilibrium, all firms charge the same price, $p_t(i) = p_t$ for all i, and hire the same amount of labor, $n_t(i) = N_{w,t}$ for all i. From the first-order condition (A.3), we obtain the real wage as

$$\frac{w_t^n}{p_t} = \frac{\epsilon - 1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} N_{W,t}{}^{\frac{\alpha}{1-\alpha}} = \frac{\epsilon - 1}{\epsilon}(1-\alpha) y_t^{\frac{-\alpha}{1-\alpha}} (A_t)^{\frac{1}{1-\alpha}} \left(\frac{w_t^n}{p_t}\right)^{\frac{\alpha}{\chi(1-\alpha)}} A_t^{\frac{-\alpha}{\chi(1-\alpha)}} ,$$

which can be further simplified to the following expression:

$$\frac{w_t^n}{p_t} = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{\chi(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\chi\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{\chi-\alpha}{\chi(1-\alpha)-\alpha}} .$$

Aggregate production in the flexible price equilibrium is linear, $y_t = A_t N_{W,t}$. We obtain:

$$y_t = A_t \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{(1-\alpha)}{\chi(1-\alpha)-\alpha}} y_t^{\frac{-\alpha}{\chi(1-\alpha)-\alpha}} A_t^{\frac{1-\frac{\alpha}{\chi}}{\chi(1-\alpha)-\alpha}} A_t^{-\frac{1}{\chi}} .$$

The previous expression allows us to write the natural level of output y_t^n and the natural real wage $\frac{w_t^n}{p_t}$ as

$$y_t^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \text{ and } \frac{w_t^n}{p_t} = \frac{\epsilon - 1}{\epsilon}(1-\alpha) A_t ,$$

from which we obtain

$$N_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} \text{ and } C_{W,t}^n = \left(\frac{\epsilon - 1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t . \tag{A.4}$$

In equilibrium, the combined consumption of capitalists and workers equates to the total final output, as detailed in equation (7). Following from equation (A.4), we obtain:

$$\rho A_t Q_t^n + \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{1+\frac{1}{\chi}} A_t = \left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}} A_t \,.$$

where we defined Q_t^n to be the natural stock price. Therefore, we obtain an expression for Q_t^n as

$$Q_t^n = \frac{1}{\rho}\left(\frac{\epsilon-1}{\epsilon}(1-\alpha)\right)^{\frac{1}{\chi}}\left(1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right) \,,$$

and $C_t^n = \rho A_t Q_t^n$. Since Q_t^n is constant in equilibrium, its process in a flexible price economy exhibits neither drift nor volatility, which implies $\mu_t^{q,n} = \sigma_t^{q,n} = 0$. To determine the natural interest rate r_t^n, we start from the capital gain component outlined in equation (8). The application of Ito's lemma yields:

$$\mathbb{E}_t \frac{d\left(p_t A_t Q_t\right)}{p_t A_t Q_t}\frac{1}{dt} = \pi_t + \underbrace{\mu_t^q}_{=0} + g + \underbrace{\sigma_t^q}_{=0}\sigma_t^p + \sigma\left(\sigma_t^p + \underbrace{\sigma_t^q}_{=0}\right) \,.$$

Given a constant dividend yield equal to ρ, applying expectations to both sides of equation (8) and combining this expression with the equilibrium condition presented in equation (6) results in:

$$i_t^m = \rho + \pi_t + g + \sigma\sigma_t^p = i_t + \left(\sigma + \sigma_t^p\right)^2 \,.$$

Inserting the previous expression into the Fisherian identity in equation (A.1), we express the natural rate of interest r_t^n as

$$r_t^n = i_t - \pi_t + \sigma_t^p\left(\sigma + \underbrace{\sigma_t^{q,n}}_{=0} + \sigma_t^p\right) = \rho + g - \sigma^2 \,, \tag{A.5}$$

which is a function of structural parameters, including σ, thereby proving the final point of

Proposition A.1. As the consumption of capitalists C_t^n is directly proportional to the level of technology A_t, it follows that:

$$\frac{dC_t^n}{C_t^n} = gdt + \sigma dZ_t = \left(r_t^n - \rho + \sigma^2\right) dt + \sigma dZ_t \ , \tag{A.6}$$

where the last equality is derived using equation equation (A.5).

■

B Co-movements between gap variables

The following Lemma B.2 demonstrates that Assumption B.1 serves as a sufficient condition for the model to exhibit the empirical regularities of positive co-movements between asset prices and various business cycle variables, such as real wage and consumption (of capitalists and workers), as observed in data.[2]

Assumption B.1 (Labor Supply Elasticity) *The effective labor supply elasticity of workers satisfies:* $\chi^{-1} > \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}$.

Lemma B.2 (Positive comovement) *Under Assumption B.1, the consumption gaps of capitalists C_t and workers $C_{W,t}$, employment $N_{W,t}$, and real wage $\frac{w_t}{p_t}$ exhibit joint positive comovement. This relationship is approximated up to a first-order as follows:*

$$\hat{Q}_t = \hat{C}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0} \frac{\widehat{w_t}}{p_t} = \frac{1}{1+\chi^{-1}}\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)\widehat{C_{W,t}} \ ,$$

and is related to the output gap of the economy by:

$$\hat{Y}_t = \zeta \hat{Q}_t \ , \quad \text{where} \ \ \zeta \equiv \chi^{-1}\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)^{-1} > 0 \ . \tag{B.1}$$

[2]See Table I.1 in the Appendix for a plausible calibration of the model parameters.

Proof of Lemma B.2. From $C_t = \rho A_t Q_t$, we obtain $\hat{C}_t = \hat{Q}_t$. We start from the flexible price economy's good market equilibrium condition, which can be written as

$$A_t \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} = \rho A_t Q_t^n + \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}} \frac{1}{A_t^{\frac{1}{\chi}}} \,, \tag{B.2}$$

where $\frac{w_t^n}{p_t^n}$ is the real wage in the flexible price economy. We subtract equation (B.2) from the analogous good market condition in the sticky price economy, and divide by $y_t^n \equiv A_t^{1-\frac{1}{\chi}}(\frac{w_t^n}{p_t^n})^{\frac{1}{\chi}}$, which yields the following result:

$$\underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}{\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{1}{\chi}\frac{\widehat{w_t}}{p_t}} = \underbrace{\frac{C_t^n}{A_t^{1-\frac{1}{\chi}}\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}} \hat{C}_t + \underbrace{\frac{\left(\frac{w_t}{p_t}\right)^{1+\frac{1}{\chi}} - \left(\frac{w_t^n}{p_t^n}\right)^{1+\frac{1}{\chi}}}{A_t\left(\frac{w_t^n}{p_t^n}\right)^{\frac{1}{\chi}}}}_{=\frac{(\epsilon-1)(1-\alpha)}{\epsilon}\left(1+\frac{1}{\chi}\right)\frac{\widehat{w_t}}{p_t}} \,,$$

which can be written as

$$\frac{1}{\chi}\frac{\widehat{w_t}}{p_t} = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{C}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\underbrace{\left(1+\frac{1}{\chi}\right)\frac{\widehat{w_t}}{p_t}}_{=\hat{C}_{W,t}} \,,$$

which, together with $\hat{C}_t = \hat{Q}_t$, leads to

$$\hat{Q}_t = \underbrace{\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\frac{\widehat{w_t}}{p_t} = \underbrace{\frac{1}{1+\chi^{-1}}\left(\chi^{-1} - \frac{\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}{1-\frac{(\epsilon-1)(1-\alpha)}{\epsilon}}\right)}_{>0}\widehat{C_{W,t}} \,.$$

Finally, equation (B.1) follows by combining the previous expression with the market clearing condition $Y_t = C_t + C_{W,t}$, from which we obtain

$$\hat{Y}_t = \left(1 - \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\right)\hat{Q}_t + \frac{(\epsilon-1)(1-\alpha)}{\epsilon}\hat{C}_{W,t} = \zeta\hat{Q}_t \,.$$

■

C Deriving the IS equation (10)

Proof of Proposition 1. With equations (2) with $\theta_t = 1$ and (6), capitalists' consumption C_t follows

$$\frac{dC_t}{C_t} = \left(i_t + (\sigma + \sigma_t^q)^2 - \rho\right) dt + (\sigma_t + \sigma_t^q) dZ_t. \tag{C.1}$$

where we use $i_t^m = i_t + (\sigma + \sigma_t^q)^2$. Thus, with equations (A.6), we obtain

$$\begin{aligned} d\hat{Q}_t = d\hat{C}_t &= \left(i_t - \underbrace{\left(r_t^n - \frac{(\sigma + \sigma_t^q)^2}{2} + \frac{\sigma^2}{2} \right)}_{\equiv r_t^T} \right) dt + \sigma_t^q dZ_t \\ &= \left(i_t - r_t^T \right) dt + \sigma_t^q dZ_t. \end{aligned} \tag{C.2}$$

Since we have risk-premium levels $\mathrm{rp}_t = (\sigma_t + \sigma_t^q)^2$ in the sticky price economy and $\mathrm{rp}_t^n = \sigma^2$ in the flexible price economy, we can express our risk-adjusted natural rate r_t^T as

$$r_t^T = r_t^n - \frac{1}{2}\left(\mathrm{rp}_t - \mathrm{rp}_t^n\right) = r_t^n - \frac{1}{2}\hat{rp}_t, \tag{C.3}$$

■

D Stochastic Stabilization in Section 4.3

Proof of Proposition 3. We derive the equilibrium when there is a Poisson (with ν as its parameter) probability that the economy returns to full stabilization after $\hat{T}^{\text{HOFG}}$. $\nu \in [0, +\infty)$, where $\nu = 0$ means no return to stabilization (as in Proposition 2). Central bank solves:

$$
\begin{aligned}
&\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_t^2 dt + \mathbb{E}_0 \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\cdot e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)}\cdot \hat{Q}_t^2 dt,\\
&\text{s.t.} \begin{cases}
d\hat{Q}_t = -(\underbrace{r_1^T(\sigma_1^{q,L})}_{<0})dt + (\sigma_1^{q,L})dZ_t, & \text{for } t < T,\\
d\hat{Q}_t = -(\underbrace{r_2^T(\sigma_2^{q,L})}_{>0})dt + (\sigma_2^{q,L})dZ_t, & \text{for } T \le t < \hat{T}^{\text{HOFG}},\\
d\hat{Q}_t = 0, & \text{for } t \ge \hat{T}^{\text{HOFG}},
\end{cases} ,\\
&\text{with } \hat{Q}_0 = r_1^T(\sigma_1^{q,L})T + r_2^T(\sigma_2^{q,L})(\hat{T}^{\text{HOFG}} - T).
\end{aligned}
\tag{D.1}
$$

where the discounting becomes $\rho+\nu > \rho$ after $\hat{T}^{\text{HOFG}}$, which is itself endogenous. The loss function in (D.1) can be written then as

$$
\begin{aligned}
&\min_{\sigma_1^{q,L},\sigma_2^{q,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_t^2 dt + \mathbb{E}_0 \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t}\cdot e^{-\nu\left(t-\hat{T}^{\text{HOFG}}\right)}\cdot \hat{Q}_t^2 dt\\
=& \min_{\hat{T},\sigma_1^{q,L},\sigma_2^{q,L}} \underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_{\text{d}}(t;\hat{T}^{\text{HOFG}})^2 dt}_{\text{From deterministic fluctuation}} + \underbrace{\left(\sigma_1^{q,L}\right)^2\left[\frac{1-e^{-\rho T}}{\rho^2} - \frac{Te^{-\rho\hat{T}^{\text{HOFG}}}}{\rho}\left(\frac{\nu}{\rho+\nu}\right)\right]}_{\text{From stochastic fluctuation}}\\
&+ \underbrace{\left(\sigma_2^{q,L}\right)^2\left[\left(\frac{e^{-\rho T}-e^{-\rho\hat{T}^{\text{HOFG}}}}{\rho^2}\right) - \left(\hat{T}^{\text{HOFG}}-T\right)e^{-\rho\hat{T}^{\text{HOFG}}}\frac{\nu}{\rho(\rho+\nu)}\right]}_{\text{From stochastic fluctuation}}
\end{aligned}
\tag{D.2}
$$

where $\hat{Q}_d(t;\hat{T}^{\text{HOFG}})$ is defined in (II.2): we observe new terms appear compared with the baseline case of $\nu = 0$. Now, notice that if the central bank is allowed to maximize with respect to ν, then we obtain a corner solution with $\nu \to +\infty$. This means that the most efficient would be to immediately return to perfect stabilization, with a very small probability of no adjustment.

The central bank has control over $\sigma_1^{q,L}$, $\sigma_2^{q,L}$, and $\hat{T}^{\text{HOFG}}$, in addition to its conventional monetary policy tool $\{i_t\}$. We derive the first-order condition for $\hat{T}^{\text{HOFG}}$ as follows:

$$
\begin{aligned}
&2 \cdot \underbrace{r_2^T(\sigma_1^{q,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt + \underbrace{\left(\sigma_1^{q,L}\right)^2 e^{-\rho \hat{T}^{\text{HOFG}}} \left(\frac{\nu}{\rho+\nu}\right) T}_{>0} \\
&+ \underbrace{\left(\sigma_2^{q,L}\right)^2 \left[\frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho+\nu} + \left(\hat{T}^{\text{HOFG}} - T\right) e^{-\rho \hat{T}^{\text{HOFG}}} \left(\frac{\nu}{\rho+\nu}\right)\right]}_{>0} = 0
\end{aligned}
\tag{D.3}
$$

from which we obtain

$$
\int_0^{\infty} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt = \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) dt < 0 \; . \tag{D.4}
$$

The first-order condition for $\hat{T}^{\text{HOFG}}$ indicates that, at the optimum, the central bank reduces the value of $\hat{T}^{\text{HOFG}}$ compared to $\hat{T}^{\text{TFG}}$ (traditional forward guidance). This is because when the central bank utilizes traditional forward guidance and achieves perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$, the expression above becomes

$$
\int_0^{\hat{T}^{\text{TFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T} \| \sigma_1^{q,L} = \sigma_1^{q,n} = 0, \sigma_2^{q,L} = \sigma_2^{q,n} = 0) dt = 0 \; , \tag{D.5}
$$

which is derived by plugging $\sigma_1^{q,L} = 0$ and $\sigma_2^{q,L} = 0$ into equation (D.3).

Given that at the optimum, $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ (which we will demonstrate),

$$
\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} = 0, \sigma_2^{q,L} = 0) < \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}} \| \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) \; .
$$

Therefore, we deduce from equation (D.1) that at the optimum, $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$, as evidenced by comparing (D.4) with (D.5).

To characterize the optimal values of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, a **variational argument** is required. This is because $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ influence the levels of $r_1^T(\sigma_1^{q,L})$, $r_2^T(\sigma_2^{q,L})$, and $\hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}})$.

Specifically, we can derive:

$$\frac{\partial r_1^T(\sigma_1^{q,L})}{\partial \sigma_1^{q,L}} = -\left(\bar{\sigma} + \sigma_1^{q,L}\right) < 0, \;\; \frac{\partial r_2^T(\sigma_2^{q,L})}{\partial \sigma_2^{q,L}} = -\left(\underline{\sigma} + \sigma_2^{q,L}\right) < 0 \; .$$

Determining $\sigma_1^{q,L}$ An increase in $\sigma_1^{q,L}$ leads to a decrease in $r_1^T(\sigma_1^{q,L})$, which alters the trajectory of $\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})$. This change is illustrated in Figure D.1, as depicted by the transition from the thick blue line to the dashed red line.

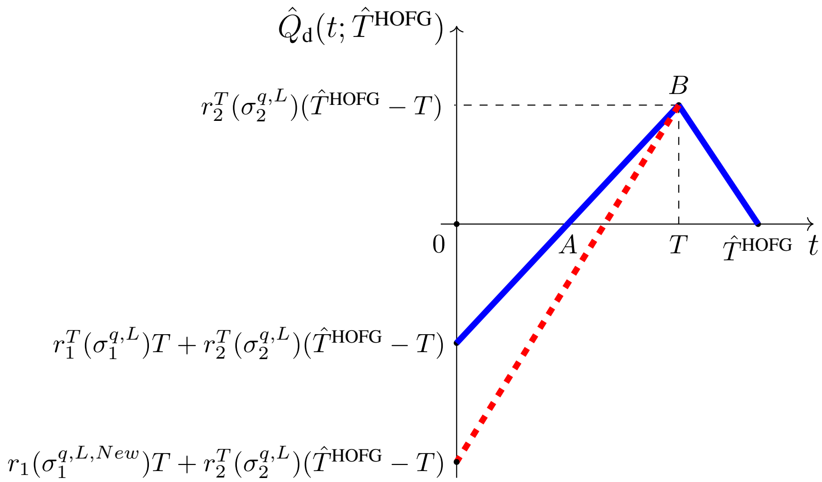

Figure D.1: Variation along $\sigma_1^{q,L}$. Increase to $\sigma_1^{q,L,New} > \sigma_1^{q,L}$.

Differentiating $\hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG}) = \int_t^{\hat{T}^{\rm HOFG}} r_s^T ds$ with respect to $\sigma_1^{q,L}$, we obtain:

$$\frac{\partial \hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})}{\partial \sigma_1^{q,L}} = \int_t^T -\left(\bar{\sigma} + \sigma_1^{q,L}\right) ds = -\left(\bar{\sigma} + \sigma_1^{q,L}\right)(T-t), \;\; \forall t \leq T \; .$$

To find optimal $\sigma_1^{q,L}$, we differentiate the objective function in (D.2) by $\sigma_1^{q,L}$ and obtain the following condition:

$$\left(\bar{\sigma} + \sigma_1^{q,L}\right) \int_0^T e^{-\rho t} \hat{Q}_{\rm d}(t;\hat{T}^{\rm HOFG})(T-t)dt = \left(\sigma_1^{q,L}\right) \left\{ \frac{1-e^{-\rho T}}{\rho^2} - \frac{e^{-\rho \hat{T}^{\rm HOFG}}}{\rho}\left[1 - \frac{\rho}{\rho+\nu}\right] \cdot T \right\} . \tag{D.6}$$

First, we obtain

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})(T-t)dt = \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds\cdot(T-t)\Big|_0^T}_{=0} + \int_0^T \underbrace{\int_0^t e^{-\rho s}\hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds}_{<0}\, dt < 0\,,$$

where $\int_0^t e^{-\rho s}\hat{Q}_{\mathrm{d}}(s;\hat{T}^{\mathrm{HOFG}})ds < 0$ for $t \leq T$, as derived in equation (D.4). Also, as we know

$$\begin{aligned}\frac{1-e^{-\rho T}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]T &\geq \frac{1-e^{-\rho T}}{\rho^2} - \frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}T \\ &= \underbrace{\int_0^T te^{-\rho t}dt}_{>0} + \underbrace{\frac{T}{\rho}e^{-\rho T} - \frac{T}{\rho}e^{-\rho\hat{T}^{\mathrm{HOFG}}}}_{\geq 0} > 0,\end{aligned} \tag{D.7}$$

from (D.6), we obtain that $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ at optimum.[3]

Determining $\sigma_2^{q,L}$ An increase in $\sigma_2^{q,L}$ leads to a decrease in $r_2^T(\sigma_2^{q,L})$, which alters the shape of $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$. This effect is illustrated in Figure D.2 by the transition from the thick blue line to the dashed red line. To further analyze this, we differentiate $\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})$ with respect to $\sigma_2^{q,L}$ and obtain:

$$\frac{\partial\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\mathrm{HOFG}})}{\partial\sigma_2^{q,L}} = \begin{cases}\int_T^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\mathrm{HOFG}}-T)\,, & t<T\,, \\ \int_t^{\hat{T}^{\mathrm{HOFG}}} -\left(\underline{\sigma}+\sigma_2^{q,L}\right)ds = -\left(\underline{\sigma}+\sigma_2^{q,L}\right)(\hat{T}^{\mathrm{HOFG}}-t)\,, & T\leq t\leq\hat{T}^{\mathrm{HOFG}}\,.\end{cases}$$

[3]Note that in (D.6), due to the additional term

$$\frac{e^{-\rho\hat{T}^{\mathrm{HOFG}}}}{\rho}\left[1-\frac{\rho}{\rho+\nu}\right]\cdot T,$$

$\sigma_1^{q,L}$ becomes more negative at optimum taking $\hat{T}^{\mathrm{HOFG}}$ and $\sigma_2^{q,L}$ as given, compared with our benchmark case in which $\nu = 0$.

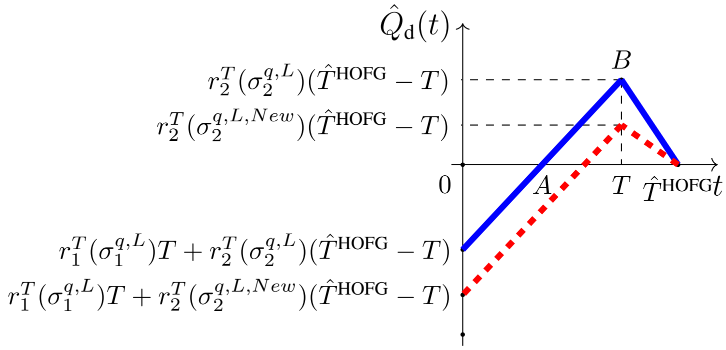

Figure D.2: Variation along $\sigma_2^{q,L}$. Increase to $\sigma_2^{q,L,New} > \sigma_2^{q,L}$.

To find the optimal $\sigma_2^{q,L}$, we differentiate the objective function in (D.2) by $\sigma_2^{q,L}$ and obtain

$$\left(\underline{\sigma} + \sigma_2^{q,L}\right)\left(\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - t)dt\right)$$
$$= (\sigma_2^{q,L})\left\{\frac{e^{-\rho T} - e^{-\rho \hat{T}}}{\rho^2} - \frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho}\left[1 - \frac{\rho}{\rho+\nu}\right]\left(\hat{T}^{\text{HOFG}} - T\right)\right\}, \tag{D.8}$$

from which we can demonstrate that at the optimum, $\sigma_2^{q,L} < 0$ must be satisfied, given that

$$\int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - t)dt$$
$$< \int_0^T e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})(\hat{T}^{\text{HOFG}} - T)dt + \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\underbrace{\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})}_{>0}(\hat{T}^{\text{HOFG}} - T)dt$$
$$= (\hat{T}^{\text{HOFG}} - T)\underbrace{\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t}\hat{Q}_{\mathrm{d}}(t;\hat{T}^{\text{HOFG}})dt}_{<0} < 0\ ,$$

where the final inequality is derived from equation (D.4), and

$$\frac{e^{-\rho T} - e^{-\rho \hat{T}}}{\rho^2} - \frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho}\left[1 - \frac{\rho}{\rho+\nu}\right]\left(\hat{T}^{\text{HOFG}} - T\right) \geq \frac{e^{-\rho T} - e^{-\rho \hat{T}}}{\rho^2} - \frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho}\left(\hat{T}^{\text{HOFG}} - T\right)$$
$$= \int_T^{\hat{T}^{\text{HOFG}}} e^{-\rho t}(t - T)dt > 0. \tag{D.9}$$

Equation (D.8) proves that $\sigma_2^{q,L} < 0$ at optimum.[4] Therefore, we have proven that during periods of high TFP volatility (i.e., $t < T$) and low TFP volatility with forward guidance (i.e., $T \leq t \leq \hat{T}^{\text{HOFG}}$), a central bank aims to target financial volatility levels below those in a flexible price economy: $\sigma_1^{q,L} < \sigma_1^{q,n} = 0$ and $\sigma_2^{q,L} < \sigma_2^{q,n} = 0$. Such intervention reduces the required risk premium and raises the asset price level $\hat{Q}_t$, thereby increasing output.

■

Proof of Corollary 1. Note that $\nu = \infty$ implies that full stabilization immediately follows after $\hat{T}^{\text{HOFG}}$ when the zero policy rate regime is over. It corresponds to the traditional forward guidance case of Section 4.1, so when $\nu = \infty$, the only feasible $\left(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}\right)$ would be $(0, 0, \hat{T})$ in this case. Since for every ν, $\left(\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}^{\text{HOFG}}\right) = (0, 0, \hat{T}^{\text{TFG}})$ is feasible, we obtain

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right) \leq \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu = \infty\right) .$$

To obtain the strict inequality between the two sides, we compare the first-order conditions for $\hat{T}^{\text{HOFG}}$ when $\nu = \infty$ and $\nu \to \infty$. When $\nu = \infty$, the optimality is given by (15), which can be written as

$$\int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt = 0 , \tag{D.10}$$

where $\hat{Q}_{\text{d}}$ is defined in (II.2). In contrast, when $\nu \to \infty$, the first-order condition of $\hat{T}^{\text{HOFG}}$ in (D.3) becomes

$$2 \cdot \underbrace{r_2^T(\sigma_1^{q,L})}_{>0} \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \hat{Q}_{\text{d}}(t; \hat{T}^{\text{HOFG}}) dt + \underbrace{\left(\sigma_1^{q,L}\right)^2 e^{-\rho \hat{T}^{\text{HOFG}}} T}_{>0} + \underbrace{\left(\sigma_2^{q,L}\right)^2 \left[\left(\hat{T}^{\text{HOFG}} - T\right) e^{-\rho \hat{T}^{\text{HOFG}}}\right]}_{>0} = 0$$

[4] Note that in (D.8), due to the additional term

$$\frac{e^{-\rho \hat{T}^{\text{HOFG}}}}{\rho} \left[1 - \frac{\rho}{\rho + \nu}\right] \cdot (\hat{T}^{\text{HOFG}} - T),$$

$\sigma_2^{q,L}$ becomes more negative at optimum taking $\hat{T}^{\text{HOFG}}$ and $\sigma_1^{q,L}$ as given, compared with our benchmark case in which $\nu = 0$.

which is different from the above (D.10). Therefore, we obtain

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Q,*}\left(\{\hat{Q}_t\}_{t\geq 0}, \nu = \infty\right) .$$

■

E Stochastic T in Section 3 and Section 4.1

Here we prove the result of Section 3 and Section 4.1 that $\sigma_t^q = \sigma_t^{q,n} \equiv 0$ still holds even when the mandatory ZLB duration T is stochastic. First, we do not consider the traditional forward guidance policy.

For illustration purposes, we assume that T follows a discrete distribution: T_1, T_2, and T_3 with probabilities p_1, p_2, and p_3 with $p_1 + p_2 + p_3 = 1$. The same logic can be applied to more general cases where T has a continuous distribution. We keep assuming that after T is realized, i.e., the ZLB ends, the monetary authority achieves perfect stabilization based on a rule in (11). We similarly rely on the backward induction. First, we know certainly that after T_3, the economy is fully stabilized, implying $\sigma_t^q = 0$ for $t \geq T_3$. For $t \in [T_2, T_3)$,

1. If the ZLB already ended at T_1 or T_2, then $\sigma_t^q = 0$.

2. The ZLB has not ended: then it is certain that $T = T_3$ and $\hat{Q}_t = 0$ for $t \geq T_3$, which means that $\sigma_t^q = 0$ for $t \in (T_2, T_3)$. In that case, $\hat{Q}_{T_2} = \underline{r}(T_3 - T_2) < 0$ is determined.

For $t \in [T_1, T_2)$, we know that

1. If the ZLB already ended at T_1, then $\sigma_t^q = 0$.

2. The ZLB has not ended: then it is for sure that $T = T_2$ or $T = T_3$. At $t = T_2 - dt$ for small $dt > 0$, $\hat{Q}_{T_2 - dt}$ is determined by a conditional probability-weighted linear combination of 0 (when $T = T_2$) and $\underline{r}(T_3 - T_2)$ (when $T = T_3$), so that

$$\hat{Q}_{T_2 - dt} = \underline{r}dt + \frac{p_2}{1 - p_1} \cdot 0 + \frac{p_3}{1 - p_1} \cdot \underline{r}(T_3 - T_2).$$

Since $\hat{Q}_{T_2-dt}$ is determined, $\sigma_t^q = 0$ for $t \in [T_1, T_2)$.

For $t < T_1$, we know that

1. $T = T_1$ or T_2 or T_3. At $t = T_1 - dt$ for small $dt > 0$, $\hat{Q}_{T_1-dt}$ is determined by a conditional probability-weighted linear combination of 0 (when $T = T_1$), $\underline{r}(T_2 - T_1)$ (when $T = T_2$) and $\underline{r}(T_3 - T_1)$ (when $T = T_3$), so that

$$\hat{Q}_{T_1-dt} = \underline{r}dt + p_1 \cdot 0 + p_2 \cdot \underline{r}(T_2 - T_1) + p_3 \cdot \underline{r}(T_3 - T_1).$$

Since $\hat{Q}_{T_1-dt}$ is determined, $\sigma_t^q = 0$ for $t < T_1$.

Therefore, $\sigma_t^q = \sigma_t^{q,n} = 0$ for all t even if ZLB duration T is stochastic.

Traditional forward guidance When T is stochastic, the zero rate duration under traditional forward guidance, i.e., $\hat{T}$ in Section 4.1, becomes stochastic as well and dependent on T. The above logic can be applied in this case, and we can similarly prove that if the monetary authority commits to perfectly stabilizing the economy after any realized $\hat{T}$, then $\sigma_t^q = \sigma_t^{q,n} = 0$ for $t \leq \hat{T}$.

F Macroprudential Policies

This section examines two types of macroprudential policies designed to stimulate the economy at the ZLB. Firstly, we consider a fiscal subsidy aimed at encouraging capitalists to undertake higher levels of risk, thereby boosting asset prices and other real economic activities. Secondly, we explore the impact of direct fiscal transfers from capitalists to hand-to-mouth workers, who typically exhibit a higher marginal propensity to consume. This policy is shown to increase overall stock market dividends, and consequently, asset prices $\hat{Q}_t$ and consumption. To assess the impact of macroprudential policies on the business cycle, forward guidance is excluded from our analysis in this section. We maintain the same scenario as outlined in Section 3, and assume that monetary policy reverts to the perfect stabilization rule specified in equation (11) for $t \geq T$.

F.1 Fiscal Subsidy on Stock Market Investment

In the period up to T, where $r_t^n = \underline{r} < 0$ and monetary policy is constrained by the ZLB, the risk-premium level $\mathrm{rp}_1^n = \bar{\sigma}^2$ required by capitalists leads to a reduction in asset prices, $\hat{Q}_t$. To counteract this, we propose a subsidy policy aimed at incentivizing capitalists' holdings of the risky stock market index. This intervention is expected to increase $\hat{Q}_t$, thereby addressing the aggregate demand externalities responsible for dragging the economy into a ZLB recession.[5]

We begin by examining a government subsidy for the purchase of (risky) stock market index shares.[6] Specifically, instead of the usual expected return i_t^m, a capitalist earns an expected return of $(1+\tau)i_t^m$ for every dollar invested in the stock market, where $\tau \geq 0$ is the stock subsidy. To fund this intervention, the government imposes a 'lump-sum' tax L_t on capitalists. Consequently, a capitalist solves the optimization problem with a modified flow budget constraint given by:

$$\begin{aligned} &\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dt \\ &\text{s.t. } da_t = \left(a_t \left(i_t + \theta_t((1+\tau)i_t^m - i_t)\right) - \bar{p}C_t - L_t\right) dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \, . \end{aligned} \tag{F.1}$$

In equilibrium, capitalists finance the stock market subsidy by paying taxes L_t equal to $\tau \bar{p} A_t Q_t i_t^m$. Setting $\theta_t = 1$ in equilibrium, we can express the stock market's expected

[5] Numerous studies have examined the link between externalities (e.g., pecuniary or aggregate-demand) and macroprudential policies. Notable references include Caballero and Krishnamurthy (2001), Lorenzoni (2008), Farhi et al. (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010), Stein (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), Dàvila and Korinek (2018), among others.

[6] In our model, a subsidy for stock investments functions similarly to a tax break on capital income, a policy commonly implemented *in practice* by governments. We opt for the subsidy model for simplicity in notation.

return as follows:

$$i_t^m = \frac{i_t + (\bar{\sigma} + \sigma_t^q)^2}{1+\tau} = \underbrace{\rho}_{\substack{\text{Dividend}\\\text{yield}}} + \underbrace{g + \mu_t^q + \sigma_t \sigma_t^q}_{\text{Capital gain}} \ . \tag{F.2}$$

As detailed in Section 3, given that σ_t^q and i_t equal zero for $t \leq T$, equation (F.2) simplifies to

$$i_t^m = \frac{\bar{\sigma}^2}{1+\tau} \ ,$$

which is lower than $\bar{\sigma}^2$ and inversely proportional to τ. Thus, a positive subsidy rate $\tau > 0$ increases $\hat{Q}_t$ along the path up to time T, when the economy achieves perfect stabilization with $\hat{Q}_T = 0$. Proposition F.2 summarizes this result.

Proposition F.2 (Fiscal Subsidy on Stock Market Expected Returns) *Under the ZLB environment of Section 3, where a fiscal subsidy $\tau \geq 0$ is applied to the expected return of stock markets, the dynamics of $\hat{Q}_t$ during the period $t < T$ are given by:*

$$d\hat{Q}_t = -\left(\underbrace{\underline{r}}_{\equiv r^n(\bar{\sigma})<0} + \underbrace{\frac{\tau}{1+\tau}\bar{\sigma}^2}_{>0} \right) dt \ , \tag{F.3}$$

for $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 < 0$ and $\hat{Q}_T = 0$. When $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2 > 0$, the subsidy $\tau > 0$ lifts the economy out of the ZLB and immediate stabilization becomes possible by adhering to the policy rule outlined in equation (11).

Proof. See Appendix G.

■

In equation (F.3), a positive subsidy $\tau > 0$ increases the effective natural rate from $\underline{r}$ to $\underline{r} + \frac{\tau}{1+\tau}\bar{\sigma}^2$. This rise narrows the gap between the ZLB and the 'effective' natural rate, consequently raising $\hat{Q}_t$ relative to the scenario described in Section 3. It is important to note that as τ approaches infinity, the expression $\underline{r}+\frac{\tau}{1+\tau}\bar{\sigma}^2$ converges to $\underline{r}+\bar{\sigma}^2 = \rho+g > 0$.

In this situation, the economy moves away from the ZLB and the monetary authority can achieve perfect stabilization by adhering to the policy rule outlined in equation (11).

Tax on whom? We now consider an alternative funding scheme for the stock market subsidy τ by imposing a lump-sum tax L_t on hand-to-mouth workers. Under this policy, the budget constraint of the workers (1) becomes

$$\frac{w_t}{\bar{p}} N_{W,t} = C_{W,t} + \frac{L_t}{\bar{p}} \,. \tag{F.4}$$

Hand-to-mouth workers, characterized by a marginal propensity to consume of one, experience a proportional reduction in their consumption due to taxation. This fall in workers' consumption adversely impacts stock dividends and prices, $\hat{Q}_t$. In this context, the formula for the stock market's expected return i_t^m is as follows:

$$i_t^m = \underbrace{\frac{y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t}}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right] = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \mathbb{E}_t \left[\frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \right] , \tag{F.5}$$

where we used an equilibrium tax equal to $\tau i_t^m \bar{p} A_t Q_t$ to obtain the last equality. Proposition F.3 summarizes our findings, highlighting the crucial role of tax scheme design in determining the effectiveness of the macroprudential policy.

Proposition F.3 (Fiscal Subsidy and Tax on Workers) *The positive impact of a subsidy τ on asset prices is precisely offset by the reduced consumption of hand-to-mouth workers due to taxation L_t. Consequently, this results in no net effect on the dynamics of $\{\hat{Q}_t\}$ during a ZLB episode, apart from a redistribution of wealth from workers to capitalists. The trajectory of asset prices under this taxation scheme corresponds with the benchmark scenario, which lacks forward guidance and macroprudential interventions, as depicted in Figure 1.*

Proof. See Appendix G.

■

F.2 Fiscal Redistribution

Lastly, we consider a redistribution policy in the form of a fiscal transfer $L_t > 0$ from capitalists to hand-to-mouth workers during a ZLB episode.[7] This policy increases aggregate demand due to the high marginal propensity to consume of workers and, in turn, the total dividends paid by the stock market index. The expected return on the stock market i_t^m then becomes:

$$i_t^m = \frac{y_t - \frac{w_t}{\bar{p}} N_{W,t}}{A_t Q_t} + \mathbb{E}_t \left[\frac{d(\cancel{\bar{p}} A_t Q_t)}{\cancel{\bar{p}} A_t Q_t} \frac{1}{dt} \right] = \rho + \underbrace{\frac{L_t}{\bar{p} A_t Q_t}}_{>0} + \mathbb{E}_t \left[\frac{d(\cancel{\bar{p}} A_t Q_t)}{\cancel{\bar{p}} A_t Q_t} \frac{1}{dt} \right] .$$

Assuming capitalists finance this transfer L_t by paying a portion φ of their wealth a_t, the dividend yield increases to $\rho + \varphi$ from a baseline yield (before transfers) of ρ. This adjustment raises the effective natural rate of interest from $\underline{r}$ to $\underline{r} + \varphi$, resulting in an increase in asset prices $\hat{Q}_t$ and a narrower output gap during a ZLB episode. Proposition F.4 summarizes this result.

Proposition F.4 (Fiscal Redistribution) *In the ZLB environment presented in Section 3, and under a redistribution scheme where a $\varphi \geq 0$ portion of capitalists' wealth is transferred to hand-to-mouth workers, the dynamic IS equation for $\hat{Q}_t$ becomes:*

$$d\hat{Q}_t = -(\underbrace{\underline{r}}_{<0} + \varphi)\, dt \;, \tag{F.6}$$

for $\underline{r} + \varphi < 0$. After time T, the central bank perfectly stabilizes the economy and eliminates the volatility in asset prices, $\sigma_t^q = 0$, for all $t \geq T$. When $\underline{r} + \varphi > 0$, fiscal transfers lift the

[7] A policy subsidizing firms' payroll, financed through a lump-sum tax L_t on capitalists, produces identical results. When firms incur net payroll costs of $w_t N_{W,t} - L_t$, the consequent rise in employment effectively creates a transfer of income equivalent to L_t to the workers. We opt for the direct transfer formulation for simplicity in notation.

economy out of the ZLB and immediate stabilization is possible by adhering to the policy rule outlined in equation (11)*, with* $\underline{r} + \varphi$ *as the effective natural rate.*

Proof. See Appendix G.

■

From the capitalists' perspective, this policy effectively reduces their expected wealth growth by φ, taking the expected stock market return i_t^m as given. At the ZLB, i_t^m does not react to fiscal transfers due to the the binding constraint on the policy rate i_t.[8] As a result, the equilibrium growth rates of capitalists' wealth and the stock price index fall by φ, due to a less significant initial decline in asset prices $\hat{Q}_0$ at the start of the ZLB episode. Therefore, fiscal transfers to workers with a high marginal propensity to consume not only enhance aggregate demand but also create additional wealth effects which manifest through increases in dividend yields and asset prices, $\hat{Q}_t$.

G Macroprudential Policies - Derivations

Proof of Proposition F.2. We begin by solving the capitalist's problem presented in equation (F.1), considering a subsidy rate τ on stock market investments for $t \leq T$:

$$\begin{aligned} &\max_{C_t,\theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \\ &\text{s.t. } da_t = (a_t(i_t + \theta_t((1+\tau)\, i_t^m - i_t)) - \bar{p}C_t - L_t)dt + \theta_t a_t \left(\bar{\sigma} + \sigma_t^q\right) dZ_t \ . \end{aligned} \tag{G.1}$$

Since the subsidy τ is financed through a lump-sum tax on capitalists, the dividend process in equation (4) and the stock market valuation equation (5) remain unchanged. As a result, $\bar{p}C_t = \rho a_t$ and $C_t = \rho A_t Q_t$. Equilibrium taxes L_t equal to $\tau i_t^m a_t$, and the budget constraint

[8] Note from the capitalists' optimization that risk-premium rp_t is given by $\bar{\sigma}^2$ during the ZLB, and $i_t^m = i_t + \text{rp}_t$.

in equation (G.1) becomes

$$\begin{aligned}\frac{dC_t}{C_t} = \frac{da_t}{a_t} &= ((1+\cancel{\tau})i_t^m - \rho - \cancel{\tau i_t^m})dt + (\bar{\sigma} + \sigma_t^q)dZ_t \\ &= (i_t^m - \rho)dt + (\bar{\sigma} + \sigma_t^q)dZ_t \ ,\end{aligned} \tag{G.2}$$

where we used equilibrium condition $\theta_t = 1$. Since $\xi_t^N = e^{-\rho t}\frac{1}{\bar{p}C_t}$, we obtain:

$$\begin{aligned}\frac{d\xi_t^N}{\xi_t^N}(i_t^m, \sigma_t^q) &= -\rho dt - \frac{dC_t}{C_t} + \left(\frac{dC_t}{C_t}\right)^2 \\ &= -\cancel{\rho} dt - [(i_t^m - \cancel{\rho})dt + (\bar{\sigma} + \sigma_t^q)dZ_t] + (\bar{\sigma} + \sigma_t^q)^2 dt \\ &= -\left[i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] dt - (\bar{\sigma} + \sigma_t^q)dZ_t \ .\end{aligned} \tag{G.3}$$

The subsidy τ on the expected return i_t^m alters the original Euler equation $\mathbb{E}_t \frac{d\xi_t^N}{\xi_t^N} = -i_t dt$. Consequently, the revised expression with a subsidy τ must be :

$$\mathbb{E}_t\left[\frac{d\xi_t^N}{\xi_t^N}((1+\tau)i_t^m, \sigma_t^q)\right] = -\left[(1+\tau)i_t^m - (\bar{\sigma} + \sigma_t^q)^2\right] = -i_t dt \ ,$$

from which we obtain equation (F.2):

$$i_t^m = \frac{i_t + (\bar{\sigma} + \sigma_t^q)^2}{1+\tau} = \frac{\bar{\sigma}^2}{1+\tau} \ ,$$

where the final equality results from substituting $i_t = 0$ and $\sigma_t^q = 0$ into the equation. From equation (G.2),it follows that:

$$\frac{dC_t}{C_t} = (i_t^m - \rho)dt + \bar{\sigma} dZ_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho\right) dt + \bar{\sigma} dZ_t \ , \tag{G.4}$$

with which we obtain

$$d\ln C_t = \left(\frac{\bar{\sigma}^2}{1+\tau} - \rho - \frac{\bar{\sigma}^2}{2}\right) dt + \bar{\sigma} dZ_t \ .$$

Finally, by using equation (A.6) from Online Appendix A, we derive the natural counterpart to the above expression:

$$d \ln C_t^n = \left(\underbrace{\bar{r}}_{<0} - \rho + \frac{\bar{\sigma}^2}{2} \right) + \bar{\sigma} dZ_t \ . \tag{G.5}$$

Combining both expressions, we obtain the dynamic IS equation in (F.3).

■

Proof of Proposition F.3. By equation (F.5), the condition that characterizes the equilibrium stock market return i_t^m is given by:

$$i_t^m = \frac{y_t - \overbrace{\frac{w_t}{\bar{p}}}^{=C_{W,t}+\frac{L_t}{\bar{p}}} N_{W,t}}{A_t Q_t} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} = \underbrace{\rho - \tau i_t^m}_{\text{Dividend yield}} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \ ,$$

from which we obtain $(1+\tau) i_t^m = \rho + g + \mu_t^q$ using $\sigma_t^q = 0$. Since $(1+\tau) i_t^m = \bar{\sigma}^2$ by equation (F.2), we infer that μ_t^q remains constant in comparison to the scenario without subsidy, conditional on $i_t = 0$ and $\sigma_t^q = 0$. Therefore, the subsidy policy does not alter the $\{\hat{Q}_t\}$ process. To align this intuition with the mathematical representation, we begin by examining the process for C_t, which is different from that in equation (G.4), as capitalists are now exempt from paying taxes L_t:

$$\begin{aligned} \frac{dC_t}{C_t} &= ((1+\tau) i_t^m - \rho) dt + \bar{\sigma} dZ_t \\ &= (\bar{\sigma}^2 - \rho) dt + \bar{\sigma} dZ_t \ . \end{aligned}$$

Given that the previous expression remains unchanged in the presence of subsidy τ, it can be inferred that a policy subsidizing the expected return of the stock market and financed by a lump-sum tax on workers does not impact the $\{\hat{Q}_t\}$ process. Consequently, the dynamics of $\{\hat{Q}_t\}$ are identical to those in an economy without this policy.

■

Proof of Proposition F.4. A fiscal transfer $L_t > 0$ from capitalists to hand-to-mouth workers increases the aggregate dividends in the financial market. This results in a reduced need for expected future capital gains, which translates into higher asset prices $\hat{Q}_t$ at the ZLB. The expected stock market return i_t^m under these circumstances is given by:

$$
\begin{aligned}
i_t^m &= \frac{A_t N_{W,t} - \overbrace{\frac{w_t}{\bar{p}} N_{W,t}}^{=C_{W,t} - \frac{L_t}{\bar{p}}}}{A_t Q_t} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} = \rho + \underbrace{\frac{L_t}{\bar{p} A_t Q_t}}_{>0} + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} \\
&= \rho + \varphi + \frac{d(\bar{p} A_t Q_t)}{\bar{p} A_t Q_t} \frac{1}{dt} ,
\end{aligned}
$$

where the last equality follows from L_t being equal to $\varphi \bar{p} A_t Q_t$ in equilibrium.

To derive equation (F.6), we start from the capitalists' optimization problem:

$$
\begin{aligned}
&\max_{C_t, \theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \\
&\text{s.t. } da_t = (a_t(i_t + \theta_t(i_t^m - i_t)) - \bar{p} C_t - L_t) dt + \theta_t a_t (\bar{\sigma} + \sigma_t^q) dZ_t ,
\end{aligned}
$$

which features equilibrium conditions for C_t and θ_t identical to those described in equations (5) and (6), together with $\sigma_t^q = 0$. As a result, $C_t = \rho \bar{p} A_t Q_t$ and $i_t^m = i_t + (\bar{\sigma} + \sigma_t^q)^2$ follows. In an equilibrium where $\sigma_t^q = 0$ and i_t is constrained by the ZLB, the wealth process for capitalists is given by:

$$
\frac{dC_t}{C_t} = \frac{da_t}{a_t} = (i_t^m - \rho - \varphi)\, dt + \bar{\sigma}_t dZ_t = (\bar{\sigma}^2 - \varphi - \rho) dt + \bar{\sigma}_t dZ_t ,
$$

from which we derive

$$
d \ln C_t = \left(\frac{\bar{\sigma}^2}{2} - \varphi - \rho \right) dt + \bar{\sigma}_t dZ_t .
$$

Subtracting the process for C_t^n in equation (G.5) yields the dynamic IS equation in (F.6).

■

H Standard New Keynesian Models and the Higher-Order Forward Guidance

We now illustrate that the higher-order forward guidance policy of Section 4.2 can be implemented in a standard non-linear New Keynesian model,[9] instead of the Two-Agent New Keynesian (TANK) model of Section 2.

H.1 Setting

The representative household owns the entire firms and receives their profits through lump-sum transfers. As in Section 2, we assume a perfectly rigid price that allows an analytical tractability: $p_t = \bar{p}$, $\forall t$. The household solves

$$\max_{\{B_t, C_t, L_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[\log C_t - \frac{L_s^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right] dt \quad , \quad \text{s.t. } \dot{B}_t = i_t B_t - \bar{p} C_t + w_t L_t + D_t, \tag{H.1}$$

where C_t and L_t are her consumption and labor supply, respectively, η is the Frisch elasticity of labor supply, B_t is her nominal holding of bonds, and D_t are the entire firms' profits and fiscal transfers from the government. w_t is the wage level, and i_t is the policy rate set by the central bank. The bond market is in zero net supply in equilibrium, i.e., in equilibrium $B_t = 0$. Finally, ρ is the time discount rate.

As we prove in Lee and Dordal i Carreras (2024), we obtain

$$-i_t dt = \mathbb{E}_t \left(\frac{d\xi_t^N}{\xi_t^N} \right), \quad \text{where } \xi_t^N = e^{-\rho t} \frac{1}{\bar{p}} \frac{1}{C_t}, \tag{H.2}$$

as the intertemporal optimality condition of problem (H.1), where $\frac{d\xi_t^N}{\xi_t^N}$ is the instantaneous (nominal) stochastic discount factor, and its expected value equals the (minus) nominal risk-free rate $-i_t dt$. Due to the rigid price assumption, the real and nominal risk-free rates

[9] For a treatment of non-linearity in a standard New Keynesian model, see our companion paper, i.e., Lee and Dordal i Carreras (2024).

of the economy are equal, i.e., $r_t = i_t$, where r_t is the real interest rate.

We can rewrite equation (H.2) as

$$\mathbb{E}_t\left(\frac{dC_t}{C_t}\right) = (i_t - \rho)dt + \underbrace{\mathrm{Var}_t\left(\frac{dC_t}{C_t}\right)}_{\substack{\text{Endogenous} \\ \text{precautionary savings}}}, \tag{H.3}$$

where the last term $\mathrm{Var}_t(\frac{dC_t}{C_t})$ arises from the endogenous volatility of the aggregate consumption process $\{C_t\}$. Note that this term is a second-order term and is typically dropped out in log-linearized models. In contrast, equation (H.3) properly accounts for consumption volatility and allows it to affect the drift of the aggregate consumption process, where the volatility as well as the drift is an endogenous object. This additional term reflects the usual *precautionary savings channel*, in which a more volatile business cycle leads to an increased demand for riskless savings, which in turn leads to a drop in current consumption and a higher expected growth for the consumption process, so $\mathbb{E}_t\left(\frac{dC_t}{C_t}\right)$ is increasing in $\mathrm{Var}_t(\frac{dC_t}{C_t})$.

Firms We assume the usual Dixit-Stiglitz monopolistic competition among firms, where the demand each firm i faces is given by

$$D_t(p_t^i, p_t) = \left(\frac{p_t^i}{p_t}\right)^{-\varepsilon} Y_t,$$

with

$$p_t = \left(\int_0^1 \left(p_t^i\right)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}},$$

where p_t^i is an individual firm i's price, p_t is the price aggregator, and Y_t is the aggregate output. With rigid prices, firms never change their prices so $p_t^i = p_t = \bar{p}$ and $D_t(p_t^i, p_t) = D_t(\bar{p}, \bar{p}) = Y_t$ for all $i \in [0, 1]$ and $\forall t$. Therefore, each firm i equally produces to meet the aggregate demand Y_t.

A firm i produces with the production function: $Y_t^i = A_t L_t^i$, where L_t^i is firm i's labor hiring, and A_t is the total factor productivity (TFP) assumed to be exogenous and follow a geometric Brownian motion with drift:

$$\frac{dA_t}{A_t} = gdt + \sigma dZ_t, \tag{H.4}$$

where g is its expected growth rate and σ is what we call 'fundamental' volatility, assumed to be constant over time.[10] It follows that firms' profits to be rebated can be written as $D_t = \bar{p}Y_t - w_t L_t$. We assume that all the aggregate variables are adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ generated by the process in (H.4) in a given *filtered* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$.

Flexible price equilibrium as benchmark With the assumed Dixit-Stiglitz monopolistic competition among firms, we can characterize the counterfactual flexible price equilibrium where firms can freely choose their prices. The flexible price equilibrium outcomes are called 'natural' as central banks in the presence of price rigidities target these outcomes with monetary tools. As proven in Lee and Dordal i Carreras (2024), the natural output Y_t^n follows

$$\frac{dY_t^n}{Y_t^n} = \left(\underbrace{r^n}_{\text{Natural rate}} - \rho + \sigma^2\right) dt + \underbrace{\sigma}_{\text{Natural volatility}} dZ_t, \tag{H.5}$$

where $r^n = \rho + g - \sigma^2$ is defined as the natural interest rate. Note that the natural rate r^n here equals its level in our Two-Agent New Keynesian model of Section 2. From the monetary authority's perspective, the process in (H.5) is an exogenous process that monetary policy cannot affect nor control. Note that natural output Y_t^n follows a geometric Brownian motion with the volatility σ, which equals the volatility of A_t process in (H.4).

Rigid price equilibrium and the 'gap' economy Going back to the 'rigid' price economy, we introduce σ_t^s as the *excess* volatility of the growth rate of the output process $\{Y_t\}$,

[10] As in Section 4, we assume in Appendix H.2 that σ jumps up to bring the economy into a ZLB recession.

compared with the benchmark flexible price economy output in (H.5). Then:

$$\mathrm{Var}_t\left(\frac{dY_t}{Y_t}\right) = (\sigma + \sigma_t^s)^2 dt \tag{H.6}$$

holds by definition. Note that σ_t^s is an *endogenous* volatility to be determined in equilibrium. It will play a similar role to asset price volatility σ_t^q of Section 2. By plugging (H.6) into the nonlinear Euler equation (H.3), we obtain

$$\frac{dY_t}{Y_t} = \left(i_t - \rho + (\sigma + \sigma_t^s)^2\right) dt + (\sigma + \sigma_t^s) dZ_t. \tag{H.7}$$

With the usual definition of output gap $\hat{Y}_t = \ln\left(\frac{Y_t}{Y_t^n}\right)$, we obtain

$$d\hat{Y}_t = \left(i_t - \left(r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2\right)\right) dt + \sigma_t^s dZ_t, \tag{H.8}$$

which has an isomorphic mathematical form to equation (10) of Section 2, with the following *risk-adjusted* natural rate defined as

$$r_t^T = r^n - \frac{1}{2}(\sigma + \sigma_t^s)^2 + \frac{1}{2}\sigma^2. \tag{H.9}$$

The above equation (H.7) features a similarly interesting feedback effect that is omitted in log-linearized models:[11] given the policy rate i_t, a rise in the endogenous volatility σ_t^s pushes up the drift of (H.8) and lowers output gap $\hat{Y}_t$. The intuition follows from the households' precautionary behavior we see in (H.3): households respond to a higher economic volatility with increased savings and lower consumption, thereby inducing a recession. In a similar manner to Section 2, we define precautionary premium $\mathrm{pp}_t \equiv (\sigma + \sigma_t^s)^2$ and its gap $\hat{pp}_t \equiv \mathrm{pp}_t - \mathrm{pp}_t^n = (\sigma + \sigma_t^s)^2 - \sigma^2$, so that (H.9) becomes

$$r_t^T \equiv r^n - \frac{1}{2}\hat{pp}_t. \tag{H.10}$$

[11] For illustrative purposes, compare (H.8) with the conventional IS equation given by $d\hat{Y}_t = (i_t - r^n)\, dt + \sigma_t^s dZ_t$ where the endogenous aggregate volatility σ_t^s has no first-order effect on the drift.

This precautionary premium $\hat{pp}_t$ will serve a similar role to risk-premium in Section 2. With (H.9), equation (H.8) can be written as

$$d\hat{Y}_t = \left(i_t - r_t^T\right) dt + \sigma_t^s dZ_t. \tag{H.11}$$

Due to the isomorphic mathematical form of $\hat{Y}_t$ process in equation (H.11) to equation (10), we know immediately that the policy rule following

$$i_t = r_t^T + \phi_y \hat{Y}_t \tag{H.12}$$

with $\phi_y > 0$ will achieve perfect stabilization, i.e., $\hat{Y}_t = 0$, as a unique equilibrium.

H.2 The Zero Lower Bound and Traditional Forward Guidance

ZLB Recession Following Section 3 and given that the natural rate r^n is of the same form as in Section 2, we consider a scenario where $\sigma_t = \bar{\sigma}$ for $0 \leq t \leq T$ and $\sigma_t = \underline{\sigma} < \bar{\sigma}$ for $t \geq T$. More specifically, we assume that TFP volatilites during these periods are such that the natural rate r_t^n satisfies: $\underline{r} \equiv r^n(\bar{\sigma}) = \rho + g - \bar{\sigma}^2 < 0$ and $\bar{r} \equiv r^n(\underline{\sigma}) = \rho + g - \underline{\sigma}^2 > 0$, resulting in the ZLB binding in the first period.

Recovery Without Guidance First, as in Section 3, after period T, we assume that the monetary authority follows the Taylor rule presented in (H.12), achieving perfect economic stabilization defined by $\hat{Y}_t = 0$ for $t \geq T$. We infer by backward induction from equation (H.8) that perfect stabilization with certainty at T necessarily implies the absence of volatility in output gap $\hat{Y}_t$ process in the preceding periods, $t < T$.[12] Therefore, it follows that $\sigma_t^s = 0$ and $r_t^T = \underline{r} < 0$ for $t < T$ whenever the monetary authority can credibly commit to follow the policy rule in equation (H.12) for $t \geq T$. In this scenario, the dynamics

[12]For instance, at $T - \Delta$, where Δ is an infinitesimally small time interval, $\sigma_{T-\Delta}^s = 0$ is the only rational solution to equation (H.8) consistent with $\hat{Y}_T = 0$ for any possible realization of the stochastic component of the TFP process, $dZ_{T-\Delta}$. This result deterministically pins down the output gap of the preceding period, $\hat{Y}_{T-\Delta}$, leading by backward induction to $\sigma_t^s = 0$ for $t \leq T$.

of $\hat{Y}_t$ according to (H.8) simplify to:

$$d\hat{Y}_t = -\underline{r}\, dt\;, \qquad \text{for } t < T\;, \tag{H.13}$$

with associated boundary condition $\hat{Y}_T = 0$ and initial output gap given by $Y_0 = \underline{r}\,T$. The trajectory of $\{\hat{Y}_t\}$ following equation (H.13) is illustrated in Figure H.3.

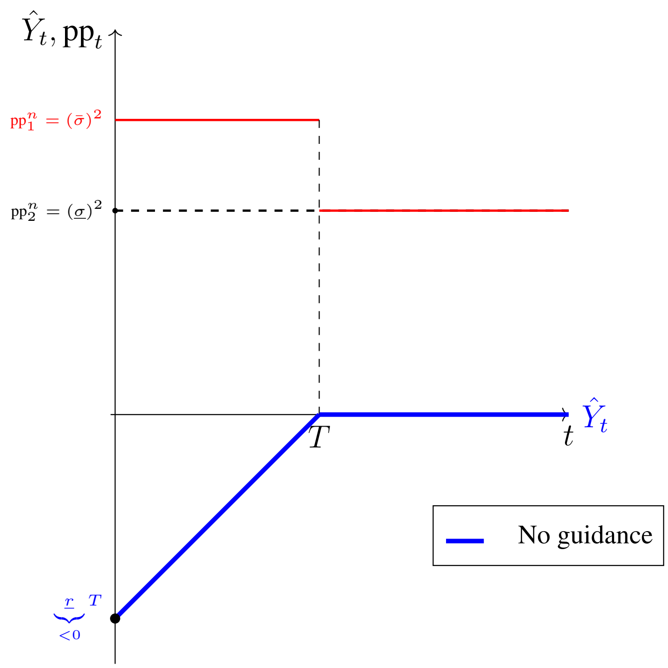

Figure H.3: ZLB dynamics, economic recovery without guidance (Benchmark).

The initial increase in σ_t from $\underline{\sigma}$ to $\bar{\sigma}$ raises the precautionary premium defined in Appendix H.1 from $\text{pp}_2^n = (\underline{\sigma})^2$ to $\text{pp}_1^n = \bar{\sigma}^2$. This leads to a decline in output $\hat{Y}_t$ because the ZLB prevents the risk-free rate from falling into negative territory, as would be necessary for complete stabilization. As a result, the heightened degree of precautionary savings, in conjunction with the ZLB, leads to a reduction in consumption demand. The equilibrium output gap path is again consistent with the dynamics described in Werning (2012) and Cochrane (2017), despite our model featuring a distinct IS equation (H.8) with endogenous excess volatility σ_t^s influencing the drift in the $\hat{Y}_t$ process, a departure from traditional New-Keynesian models. This result arises because ensuring future stabilization for $t \geq T$

effectively eliminates any excess endogenous volatility σ_t^s during a ZLB episode.

Remarks Again, central banks can prevent the emergence of endogenous excess volatility σ_t^s at the ZLB through a 'credible' commitment to stabilize the business cycle by a predetermined future date $T < +\infty$. Even if the monetary authority is constrained by the ZLB and thus unable to adhere to the policy rule in (H.12), which directly targets aggregate volatility, the additional stability costs resulting from policy inaction can be effectively mitigated by pledging to stabilize upon exiting the ZLB. One implication of this result is that the impact of the ZLB could vary significantly between countries: those with monetary authorities committed to stabilization after the ZLB period may only face the demand-driven recession from the level effect: the ZLB is higher than the natural rate $r_t^n = \underline{r} < 0$. In contrast, countries lacking the capacity or willingness to stabilize in the future might incur additional costs due to potential increases in σ_t^s during a ZLB episode.

Traditional Forward Guidance We define traditional forward guidance as the communication strategy where the central bank credibly commits to maintaining a zero policy rate for a duration of time $\hat{T}^{\text{TFG}} > T$ exceeding the initial period of high fundamental volatility. We further assume that the central bank reverts to the policy rule defined in equation (H.12) after the forward guidance period ends, resulting in a perfect stabilization for $t \geq \hat{T}^{\text{TFG}}$. Following from the same backward induction rationale presented above, stabilization with certainty after $\hat{T}^{\text{TFG}}$ results in the absence of endogenous excess volatility, i.e., $\sigma_t^s = 0$, for $t < \hat{T}^{\text{TFG}}$. The dynamics of $\hat{Y}_t$ is thus described by

$$d\hat{Y}_t = \begin{cases} -\underline{r}\ dt\ , & \text{for } t < T\ , \\ -\bar{r}\ dt\ , & \text{for } T \leq t < \hat{T}^{\text{TFG}}\ , \end{cases} \tag{H.14}$$

with associated boundary condition $\hat{Y}_{\hat{T}^{\text{TFG}}} = 0$, resulting in an initial output gap of $\hat{Y}_0 = \underline{r}\,T + \bar{r}\,(\hat{T}^{\text{TFG}} - T)$.

The dynamics of $\{\hat{Y}_t\}$ governed by equation (H.14) are depicted in Figure H.4. Traditional forward guidance induces an artificial economic boom between T and $\hat{T}^{\text{TFG}}$, thereby alleviating recessionary pressures within the interval $0 \leq t < T$. Specifically, traditional forward guidance increases output gap between T and $\hat{T}^{\text{TFG}}$, which results in a narrower initial output gap $\hat{Y}_0$ due to the forward-looking nature of consumption dynamics.

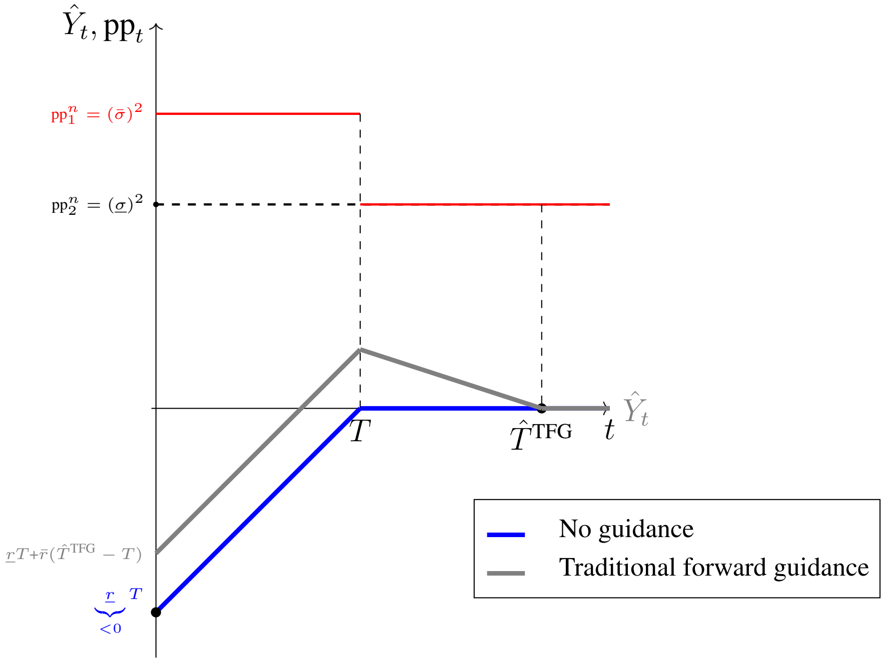

Figure H.4: ZLB dynamics under traditional forward guidance.

Optimal Traditional Forward Guidance As in Section 4, we can determine the optimal forward guidance duration $\hat{T}^{\text{TFG}}$ by minimizing the quadratic loss function represented by:[13]

$$\mathbb{L}^Y\left(\{\hat{Y}_t\}_{t\geq 0}\right) = \mathbb{E}_0 \int_0^\infty e^{-\rho t}\left(\hat{Y}_t\right)^2 dt \,, \tag{H.15}$$

subject to the dynamics outlined in equation (13). The first-order condition with respect to $\hat{T}^{\text{TFG}}$ results in:

$$\int_0^\infty e^{-\rho t}\left(\hat{Y}_t\right) dt = 0 \,. \tag{H.16}$$

[13] Deriving the quadratic loss function in equation (14) is standard: see e.g., Woodford (2003).

H.3 Higher-Order Forward Guidance

The principal cause of ZLB recessions in the standard model of Appendix H is an excessively high precautionary premium pp_t that leads to a higher precautionary savings demand and depressed consumption demand, driven by increased fundamental volatility σ_t. As a result, central banks might alternatively consider focusing on mitigating aggregate volatility by steering agents' actions toward a favorable trajectory for the excess volatility $\{\sigma_t^s\}$ during the ZLB period, aiming to support consumption demand.[14]

Context Due to the isomorphic structure of dynamics between our two-agent New Keynesian (TANK) model of Section 2 and the standard New Keynesian model of Appendix H, we can implement a similar higher-order forward guidance provided in Section 4.2. In the traditional forward guidance policy previously discussed, the central bank's commitment to perfect stabilization (with certainty) at $\hat{T}^{\mathrm{TFG}}$ facilitates a smoother transition toward economic recovery. However, this approach prevents any deviation of σ_t^s from zero, its natural level, during the ZLB period, as depicted in Figure H.5. This suggests that to sustain alternative equilibria where σ_t^s deviates from zero, the central bank must refrain from promising perfect stabilization upon exiting the ZLB at $\hat{T}^{\mathrm{TFG}}$, as illustrated in Figure H.6.

1. Central bank achieves perfect stabilization with certainty after $\hat{T}^{\mathrm{TFG}}$ (i.e., $\hat{Y}_t = 0$, for $t \geq \hat{T}^{\mathrm{TFG}}$)

$\downarrow$

2. $\hat{Y}_{\hat{T}^{\mathrm{TFG}}} = 0$ guarantees $\sigma_t^s = 0$, $\mathrm{pp}_t = \mathrm{pp}_t^n$ for $t < \hat{T}^{\mathrm{TFG}}$

Figure H.5: Mechanism under traditional forward guidance.

[14]The precautionary premium, pp_t, is given by $\mathrm{pp}_t = (\bar{\sigma} + \sigma_t^s)^2$ for $t < T$ and $\mathrm{pp}_t = (\underline{\sigma} + \sigma_t^s)^2$ for $T \leq t < \hat{T}^{\mathrm{TFG}}$. Therefore, a negative σ_t^s can reduce the precautionary premium pp_t below its natural level, thereby improving aggregate demand at the ZLB.

$\boxed{\neg 2.\ \sigma_t^s < 0,\ \mathrm{pp}_t < \mathrm{pp}_t^n \text{ for } t < \hat{T}^{\mathrm{TFG}}}$

$$\downarrow$$

$\boxed{\neg 1.\ \hat{Y}_{\hat{T}^{\mathrm{TFG}}} \neq 0\text{: central bank commits not to perfectly stabilize the economy after } \hat{T}^{\mathrm{TFG}}}$

Figure H.6: Mechanism under higher-order forward guidance.

Implementation We define $\hat{T}^{\mathrm{HOFG}}$ as the duration of zero policy rate under our 'higher-order' policy. We model the commitment constraint described in Figure H.6 by assuming that after the forward guidance regime with i_t equal to zero ends at $\hat{T}^{\mathrm{HOFG}}$, the monetary authority implements a *passive policy rule* with i_t fixed at $\bar{r} > 0$, which allows for the existence of multiple equilibria. The central bank then coordinates the economy's agents into an optimal path within the admissible solutions set, subject to the constraints: $\sigma_t^s = 0$ for $t \geq \hat{T}^{\mathrm{HOFG}}$ and $\mathbb{E}_0 \hat{Y}_\infty = 0$. The latter is necessary to meet the economy's transversality condition, while the former simplifies the optimization problem by assuming the central bank ends its influence on the excess business cycle volatility σ_t^s at the conclusion of the forward guidance period. Together with the dynamic IS equation in (H.11), these constraints indicate that the output gap is initially expected to close, $\mathbb{E}_0 \hat{Y}_{\hat{T}^{\mathrm{HOFG}}} = 0$, by the end of the forward guidance period at $\hat{T}^{\mathrm{HOFG}}$. In Section H.4, we will additionally assume that the central bank permanently reverts to the active Taylor rule in equation (H.12) with a constant probability less than one after $\hat{T}^{\mathrm{HOFG}}$.

Formalism We denote the natural precautionary premiums as $\mathrm{pp}_1^n \equiv \bar{\sigma}^2$ for $t < T$ (high fundamental volatility region), $\mathrm{pp}_2^n \equiv \underline{\sigma}^2$ for $T \leq t < \hat{T}^{\mathrm{HOFG}}$ (low fundamental volatility region), and $\mathrm{pp}_3^n \equiv \underline{\sigma}^2$ for $t \geq \hat{T}^{\mathrm{HOFG}}$ (low fundamental volatility region post-forward guidance period).[15]

We can simplify the optimization problem by assuming that the central bank maintains consistent excess volatility and precautionary premium levels within each regime. Specif-

[15]The precautionary premium is defined as $\mathrm{pp}_t = (\sigma_t + \sigma_t^s)^2$, and the expression for the natural level pp_t^n stems from no excess volatility in a flexible price economy, i.e., our benchmark economy.

ically, the excess volatility σ_t^s is set to be $\sigma_1^{s,L}$ for $t < T$, $\sigma_2^{s,L}$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and zero for $t \geq \hat{T}^{\text{HOFG}}$. The precautionary premia associated with each period are $\text{pp}_1 \equiv (\bar{\sigma} + \sigma_1^{s,L})^2 < \text{pp}_1^n$ for $t < T$, $\text{pp}_2 \equiv (\underline{\sigma} + \sigma_2^{s,L})^2 < \text{pp}_2^n$ for $T \leq t < \hat{T}^{\text{HOFG}}$, and $\text{pp}_3 \equiv (\underline{\sigma})^2$ for $t \geq \hat{T}^{\text{HOFG}}$.[16] This simplified problem is represented in Figure H.7. Finally, the risk-adjusted natural rate in (H.10) is expressed as r_1^T for $t < T$ and r_2^T for $T \leq t < \hat{T}^{\text{HOFG}}$, each being a function of $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, respectively. This is represented by:

$$\begin{aligned} r_1^T\left(\sigma_1^{s,L}\right) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{\left(\bar{\sigma} + \sigma_1^{s,L}\right)^2}{2} > \underline{r} \equiv r_1^T(0) \ \text{ when } \ \sigma_1^{s,L} < 0 \,, \\ r_2^T\left(\sigma_2^{s,L}\right) &\equiv \rho + g - \frac{\underline{\sigma}^2}{2} - \frac{\left(\underline{\sigma} + \sigma_2^{s,L}\right)^2}{2} > \bar{r} \equiv r_2^T(0) \ \text{ when } \ \sigma_2^{s,L} < 0 \,. \end{aligned} \tag{H.17}$$

From equation (H.17), we observe that lower precautionary premia during the forward guidance period up to $\hat{T}^{\text{HOFG}}$ lead to increased risk-adjusted rates and, consequently, higher values of output gap $\{\hat{Y}_t\}$ along the expected equilibrium path (in comparison to a traditional forward guidance policy of the same duration). This results in reduction of the expected quadratic loss function in (H.15). However, as indicated by our IS equation (H.11), a σ_t^s different from zero introduces stochastic fluctuations in the trajectory of $\hat{Y}_t$, resulting in potential additional stabilization costs in the future. The green line in Figure H.8 illustrates the expected trajectory (or deterministic component) of $\{\hat{Y}_t\}$ under a higher-order forward guidance policy as detailed in this section. The dashed lines alongside the expected path depict two possible sample paths that stem from stochastic variations in $\{\hat{Y}_t\}$, which are

$\neg 2.$ $\sigma_t^s = \sigma_1^{s,L} < 0$ for $t < T$; $\sigma_2^{s,L} < 0$ for $T \leq t \leq \hat{T}^{\text{HOFG}}$; $\sigma_t^s = 0$ for $t > \hat{T}^{\text{HOFG}}$

$\neg 1.$ $\hat{Y}_{\hat{T}^{\text{HOFG}}} \neq 0$: central bank pegs its policy rate $i_t = \bar{r}$ after $\hat{T}^{\text{HOFG}}$

Figure H.7: Simplified higher-order forward guidance.

[16] Proposition H.5 later proves that $\sigma_1^{s,L} < 0$ and $\sigma_2^{s,L} < 0$ at the optimum. For illustration purposes, we assume these conditions are satisfied in the rest of the argument of Appendix H.

caused by σ_t^s different from zero until $\hat{T}^{\text{HOFG}}$.

In summary, central banks operating under our higher-order guidance with commitment face a trade-off between achieving lower precautionary premiums and higher output levels prior to $\hat{T}^{\text{HOFG}}$, and the subsequent costs of de-stabilization caused by $\sigma_t^s \neq 0$. This balancing act involves a careful choice of $\sigma_1^{s,L}$, $\sigma_2^{s,L}$, and $\hat{T}^{\text{HOFG}}$, as we discuss next. It will turn out that due to the additional stabilization effects coming from negative $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, the duration of zero policy rate $\hat{T}^{\text{HOFG}}$ falls from $\hat{T}^{\text{TFG}}$ that satisfies equation (H.16).

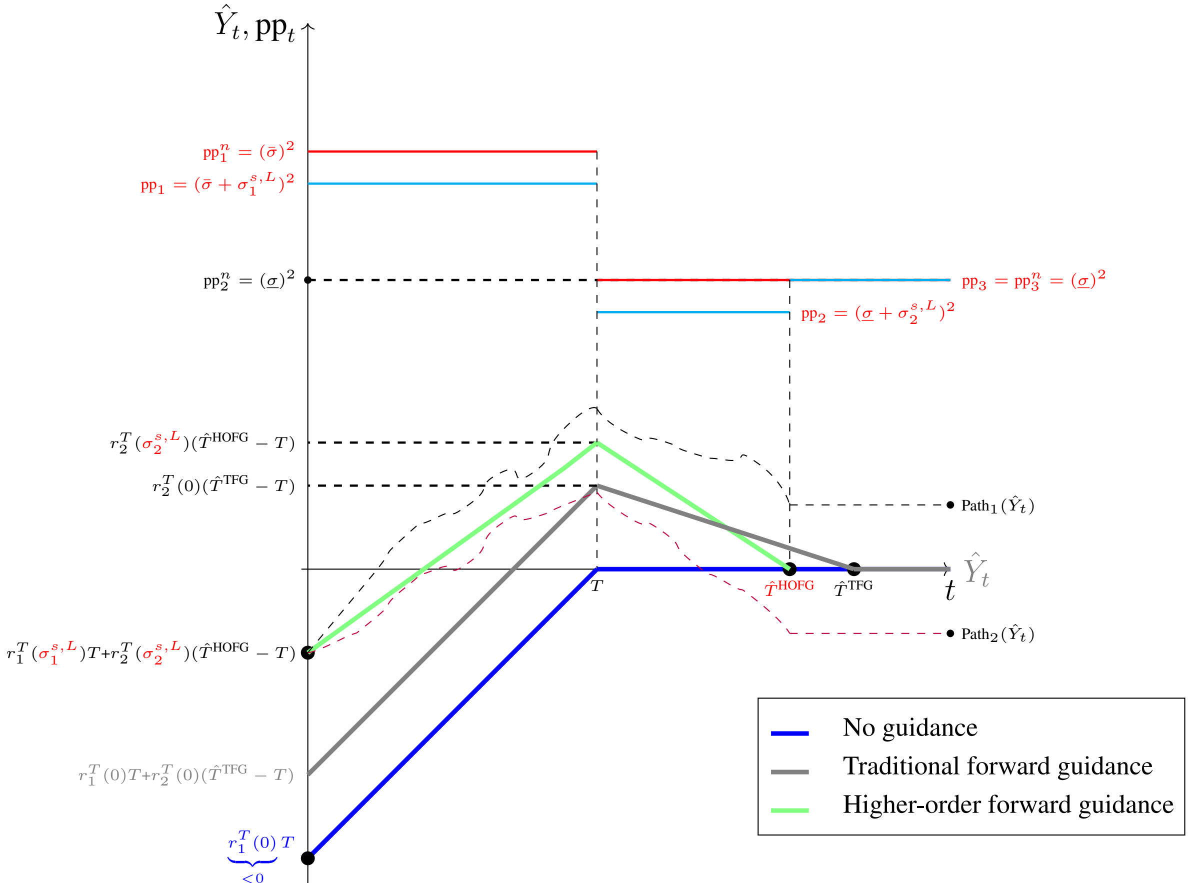

Figure H.8: Intervention dynamics of $\{\hat{Y}_t\}$ with $\sigma_1^{s,L} < 0$, $\sigma_2^{s,L} < 0$, and $\hat{T}^{\text{HOFG}} < \hat{T}^{\text{TFG}}$.

Optimal Higher-Order Forward Guidance The initial output gap $\hat{Y}_0$ is determined by the condition $\mathbb{E}_0 \hat{Y}_{\hat{T}^{\text{HOFG}}} = 0$ previously discussed and the dynamic IS equation in (H.11) as follows:

$$\hat{Y}_0 = r_1^T(\sigma_1^{s,L})\,T + r_2^T(\sigma_2^{s,L})\,(\hat{T}^{\text{HOFG}} - T)\ . \tag{H.18}$$

The central bank minimizes the loss function given by (H.15) by selecting the optimal values for $\sigma_1^{s,L}$, $\sigma_2^{s,L}$, and $\hat{T}^{\text{HOFG}}$. The formulation of the optimization problem is:

$$\min_{\sigma_1^{s,L},\sigma_2^{s,L},\hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^\infty e^{-\rho t}\left(\hat{Y}_t\right)^2\, dt,\ \text{s.t. } d\hat{Y}_t = \begin{cases} -r_1^T(\sigma_1^{s,L})dt + \sigma_1^{s,L}dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{s,L})dt + \sigma_2^{s,L}dZ_t, & \text{for } T \le t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \ge \hat{T}^{\text{HOFG}}, \end{cases} \tag{H.19}$$

with initial output gap $\hat{Y}_0$ determined by equation (H.18). The following Proposition H.5, which is the same as Proposition 2 in Section 4.2, summarizes the resulting optimal commitment path for the central bank under higher-order forward guidance.

Proposition H.5 (Optimal Commitment Path) *The solution to the central bank's higher-order forward guidance optimization problem in (H.19) results in an optimal commitment path characterized by $\sigma_1^{s,L} < 0$, $\sigma_2^{s,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. In addition, optimal higher-order forward guidance always results in an equal or lower expected quadratic loss than the traditional forward guidance discussed in Appendix H.2.*

Proof. Identical to Proposition 2 in Section 4.2.

■

H.4 Higher Order Forward Guidance *with* Stochastic Stabilization

In the previous section, we assumed that following the end of the forward guidance regime at $\hat{T}^{\text{HOFG}}$, the monetary authority would passively peg the policy rate i_t to the natural rate $\bar{r}$ and set σ_t^s to zero indefinitely. This setup allows for σ_t^s to deviate from zero during the ZLB period, as illustrated in Figure H.8. We now relax these assumptions while maintaining the support for the existence of multiple equilibria provided by the earlier framework. In specific, we assume that after forward guidance ends, the central bank not only follows the outlined passive rule but also commits to a stochastic return to the perfect stabilization rule in equation (H.12). This commitment is represented as a constant probability outcome determined by a Poisson process. Accordingly, the output gap $\hat{Y}_t$ after $\hat{T}^{\text{HOFG}}$ follows the process given by:

$$d\hat{Y}_t = -\hat{Y}_t d\Pi_t \, , \quad \text{s.t. } \; d\Pi_t = \begin{cases} 1 \, , & \text{with probability } \nu dt \, , \\ 0 \, , & \text{with probability } 1 - \nu dt \, , \end{cases}$$

where $d\Pi_t$ is a Poisson random variable, with rate parameter $\nu \geq 0$.

The central bank's optimization problem can thus be expressed as:

$$\begin{aligned} &\min_{\sigma_1^{s,L}, \sigma_2^{s,L}, \hat{T}^{\text{HOFG}}} \mathbb{E}_0 \int_0^{\hat{T}^{\text{HOFG}}} e^{-\rho t} \left(\hat{Y}_t\right)^2 \, dt + \int_{\hat{T}^{\text{HOFG}}}^{\infty} e^{-\rho t} \cdot e^{-\nu\left(t - \hat{T}^{\text{HOFG}}\right)} \cdot \left(\hat{Y}_t\right)^2 \, dt \, , \\ &\text{s.t. } \; d\hat{Y}_t = \begin{cases} -r_1^T(\sigma_1^{s,L})dt + \sigma_1^{s,L} dZ_t, & \text{for } t < T, \\ -r_2^T(\sigma_2^{s,L})dt + \sigma_2^{s,L} dZ_t, & \text{for } T \leq t < \hat{T}^{\text{HOFG}}, \\ 0, & \text{for } t \geq \hat{T}^{\text{HOFG}}, \end{cases} \end{aligned} \tag{H.20}$$

with $\hat{Y}_0$ determined by equation (H.18). Proposition H.6 outlines the optimal commitment path for the central bank under higher-order forward guidance with stochastic stabilization.

Proposition H.6 (Optimal Commitment Path with Stochastic Stabilization) *The solution to the central bank's forward guidance optimization problem in (H.20) results in an optimal commitment path characterized by $\sigma_1^{s,L} < 0$, $\sigma_2^{s,L} < 0$, and $\hat{T}^{HOFG} < \hat{T}^{TFG}$. In addition, optimal higher-order forward guidance with a stochastic stabilization probability always results in an equal or lower expected quadratic loss than the traditional forward guidance discussed in Appendix H.2.*

Furthermore, an increased probability of stabilization, indicated by higher values of

ν, leads to a reduction in the optimal values of $\sigma_1^{s,L}$ and $\sigma_2^{s,L}$, resulting in a decrease in precautionary premia at the ZLB.

Proof. Identical to Proposition 3 in Section 4.3.

■

Finally, Corollary H.3 asserts that introducing a minimal degree of uncertainty about the timing of future stabilization in its communications is always optimal for the central bank. This approach facilitates the application of higher-order forward guidance, resulting in equilibrium paths that are strictly superior from a quadratic loss perspective, compared to those under traditional forward guidance.

Corollary H.3 (Discontinuity at the Limit) *The limit case where stabilization parameter ν equals $+\infty$ corresponds to the traditional forward guidance problem described in Appendix H.2. As ν approaches $+\infty$ from the left, the central bank's expected quadratic loss function exhibits a discontinuity. Specifically, the expected quadratic loss is always lower when there's a minimal probability of stabilization. Formally:*

$$\lim_{\nu \to +\infty^-} \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu\right) < \mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu = \infty\right) ,$$

where $\mathbb{L}^{Y,*}\left(\{\hat{Y}_t\}_{t\geq 0}, \nu\right)$ represents the quadratic loss function defined in equation (H.15), evaluated at its optimum for an economy characterized by a Poisson rate ν.

Proof. Identical to Corollary 1 in Section 4.3.

■

I Welfare Derivation

In this section, we derive the quadratic welfare function in equation (14), in a similar way to Woodford (2003) with a key difference: as there are two types of agents in the economy, we need to consider some welfare weights attached to each type.

I.1 Efficient steady state with a production subsidy

I.1.1 First-Best Allocation

A first-best allocation must be the solution of the following optimization problem.

$$\max_{C_t, N_{W,t}, C_{W,t}} \omega_1 \log \frac{C_t}{A_t} + \omega_2 \left(\frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \right) \text{ s.t. } C_t + C_{W,t} = A_t N_{W,t}, \tag{I.1}$$

where $\omega_1 > 0$ and $\omega_2 > 0$ are welfare weights attached to capitalists and workers, respectively. For expositional purposes, we define $x_t \equiv N_{W,t}$ and $y_t \equiv \frac{C_{W,t}}{A_t}$: then the first-order conditions for (I.1) can be written as

$$y_t^{-\varphi} = x_t^{\chi_0}, \ \ \frac{\omega_1}{\omega_2} = x_t^{\chi_0}(x_t - y_t). \tag{I.2}$$

I.1.2 Optimization for workers and firms

Following Woodford (2003), we introduce a production subsidy $\tau > 0$ offered to the firms, financed through a lump-sum tax on workers. The production subsidy ensures that our flexible price equilibrium (or steady-state) allocation $\left(N_{W,t}^n, \frac{C_{W,t}^n}{A_t}, \frac{C_t^n}{A_t}\right)$ is efficient and satisfies equation (I.2). With the subsidy τ, workers solve

$$\max_{C_{W,t}, N_{W,t}} \frac{\left(\frac{C_{W,t}}{A_t}\right)^{1-\varphi}}{1-\varphi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \text{ s.t. } p_t C_{W,t} = w_t N_{W,t} - p_t T_t, \tag{I.3}$$

where $T_t = \tau y_t$ is the (real) lump-sum tax imposed on workers. Equation (I.3)'s first-order condition is written as:

$$(N_{W,t})^{\chi_0+\varphi} \left(\frac{w_t}{p_t A_t} - \tau \right)^{\varphi} = \frac{w_t}{p_t A_t}. \tag{I.4}$$

We can express $N_{W,t}$ that satisfies equation (I.4) as a function of the normalized real wage $\frac{w_t}{p_t A_t}$, i.e., $N_{W,t} \equiv f_N(\frac{w_t}{p_t A_t})$. Under the flexible price equilibrium, each firm's optimization is changed from (A.3) with the introduction of τ as follows, with $E_t = (N_{W,t})^{\alpha}$:

$$\max_{p_t(i)} (1+\tau) p_t(i) \left(\frac{p_t(i)}{p_t} \right)^{-\epsilon} y_t - w_t \left(\frac{y_t}{A_t E_t} \right)^{\frac{1}{1-\alpha}} \left(\frac{p_t(i)}{p_t} \right)^{\frac{-\epsilon}{1-\alpha}}, \tag{I.5}$$

which at the optimum leads to

$$\frac{w_t^n}{p_t^n A_t} = \frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}. \tag{I.6}$$

Based on equation (I.4) and equation (I.6), we can obtain

$$\begin{aligned} N_{W,t}^n &= f_N\left(\frac{w_t^n}{p_t^n A_t}\right) = f_N\left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\right), \\ \frac{C_{W,t}^n}{A_t} &= \left[\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau\right] f_N\left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\right). \end{aligned} \tag{I.7}$$

Since our goal is to align the allocation implied by equation (I.7) with the first-best allocation implied by equation (I.2), $N_{W,t}^n$ and $\frac{C_{W,t}^n}{A_t}$ in equation (I.7) must satisfy (I.2):

$$\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau = f_N\left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}\right)^{-\frac{\chi_0+\varphi}{\varphi}}. \tag{I.8}$$

Plugging equation (I.6) into equation (I.4), we obtain

$$\left(N_{W,t}^n\right)^{\chi_0+\varphi} \left(\frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon} - \tau\right)^{\varphi} = \frac{(1+\tau)(\epsilon-1)(1-\alpha)}{\epsilon}. \tag{I.9}$$

Solving jointly equation (I.8) and equation (I.9), we conclude the optimal τ^* must satisfies

$$\frac{(1+\tau^*)(\epsilon-1)(1-\alpha)}{\epsilon} = 1.^{17} \tag{I.10}$$

[17] As in Woodford (2003), τ^* is a function of primitive parameters ϵ and α.

This optimal τ^* in equation (I.10) eliminates the mark-up of firms and restores efficiency. With $\tau = \tau^*$, the normalized real wage $\frac{w_t^n}{p_t^n A_t}$ in (I.6) becomes 1 and we obtain the following benchmark efficient allocation from equation (I.7):

$$N_{W,t}^n \equiv \bar{x} = (1-\tau^*)^{-\frac{\varphi}{\chi_0+\varphi}}, \quad \frac{C_{W,t}^n}{A_t} \equiv \bar{y} = (1-\tau^*)^{\frac{\chi_0}{\chi_0+\varphi}}, \tag{I.11}$$

and

$$\frac{C_t^n}{A_t} = \bar{x} - \bar{y} = (1-\tau^*)^{-\frac{\varphi}{\chi_0+\varphi}} \cdot \tau^*. \tag{I.12}$$

The last step is to ensure that the welfare weights ω_1 and ω_2 in (I.1) satisfy equation (I.2).[18] By plugging equation (I.11) into the second condition of equation (I.2), we obtain

$$\frac{\omega_1}{\omega_2} = (N_{W,t}^n)^{\chi_0}\left(N_{W,t}^n - \frac{C_{W,t}^n}{A_t}\right) = (1-\tau^*)^{-\frac{(\chi_0+1)\varphi}{\chi_0+\varphi}} \cdot \tau^*. \tag{I.13}$$

Thus, with $\omega_1 > 0$ and $\omega_2 > 0$ satisfying equation (I.13), our allocation with $\tau = \tau^*$ is efficient. We now approximate a joint welfare in equation (I.1) with ω_1 and ω_2 satisfying (I.13) up to a second-order.

I.1.3 Derivation of a quadratic loss function

The steady-state values $\bar{x}$ and $\bar{y}$ (or the flexible price equilibrium values) of x_t and y_t are provided in equation (I.11). From the economy-wide resource constraint and given our assumption of perfectly rigid prices, we express

$$\frac{C_t}{A_t} = N_{W,t} - \frac{C_{W,t}}{A_t} = x_t - y_t, \tag{I.14}$$

With (I.14), we express our social welfare in (I.1) with ω_1 and ω_2 satisfying equation (I.13) as

$$U(x_t, y_t, \Delta_t) \equiv \omega_1 \log(x_t - y_t) + \omega_2 \left(\frac{y_t^{1-\varphi}}{1-\varphi} - \frac{x_t^{1+\chi_0}}{1+\chi_0}\right), \tag{I.15}$$

[18] Since ω_1 and ω_2 are chosen arbitrarily, we make sure that our allocation with a production subsidy can be on the efficient frontier, which is generated by a varying set of $\{\omega_1, \omega_2\}$.

which achieves its maximum value $\bar{U}$ when $x_t = \bar{x}$, and $y_t = \bar{y}$.[19] A second-order approximation of equation (I.15) around the efficient benchmark allocation $(\bar{x}, \bar{y})$ in equation (I.11) results in:

$$U_t - \bar{U} = \frac{1}{2}U_{xx} \cdot \bar{x}^2 \cdot (\hat{x}_t)^2 + \frac{1}{2}U_{yy} \cdot \bar{y}^2 \cdot (\hat{y}_t)^2 + U_{xy} \cdot \bar{x} \cdot \bar{y} \cdot \hat{x}_t \cdot \hat{y}_t + h.o.t, \tag{I.16}$$

where all the second-order partial derivatives (U_{xx}, U_{yy}, U_{xy}) are evaluated at the benchmark point $(\bar{x}, \bar{y})$ and given by

$$\begin{aligned} U_{xx} &= -\omega_2(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \left(\frac{1}{\tau^*} + \chi_0\right), \\ U_{yy} &= -\omega_2(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \left(\frac{1}{\tau^*} + \frac{\varphi}{1-\tau^*}\right), \\ U_{xy} &= \omega_2(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}} \frac{1}{\tau^*}, \end{aligned} \tag{I.17}$$

where we use the relation between ω_1 and ω_2 in equation (I.13) in the process of derivation. Since ω_2 can be regarded a free parameter, we set $\omega_2 \equiv 1$ from now on.

Log-linearization Log-linearizing the worker's optimization condition (I.4), with τ^* given by (I.10), results in

$$\widehat{N_{W,t}} = \frac{1 - \frac{\varphi}{1-\tau^*}}{\chi_0 + \varphi} \widehat{\left(\frac{w_t}{p_t}\right)}. \tag{I.18}$$

Log-linearizing the budget constraint of workers in (I.1) results in

$$\widehat{C_{W,t}} = \frac{1 + \frac{\chi_0}{1-\tau^*}}{\chi_0 + \varphi} \widehat{\left(\frac{w_t}{p_t}\right)}. \tag{I.19}$$

Linearizing the economy-wide resource constraint (I.14) with $\hat{Q}_t = \hat{C}_t$ and solving jointly

[19] We have $U_x = U_y = 0$ at $x_t = \bar{x}$ and $y_t = \bar{y}$ where U_x and U_y are the partial derivatives with respect to x_t and y_t, respectively and $\bar{x}$ and $\bar{y}$ are defined in (I.11).

with equations (I.18) and (I.19), we can obtain

$$\widehat{\left(\frac{w_t}{p_t}\right)} = \frac{\tau^*(\chi_0+\varphi)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\hat{Q}_t$$
$$\hat{x}_t \equiv \widehat{N_{W,t}} = \frac{\tau^*\left(1-\frac{\varphi}{1-\tau^*}\right)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\hat{Q}_t, \tag{I.20}$$
$$\hat{y}_t \equiv \widehat{C_{W,t}} = \frac{\tau^*\left(1+\frac{\chi_0}{1-\tau^*}\right)}{\tau^* - \left(\chi_0 + \frac{\varphi}{1-\tau^*}\right)}\hat{Q}_t.$$

Plugging equation (I.17) into the second-order approximation to the welfare function, i.e., equation (I.16), we obtain

$$\begin{aligned} U_t - \bar{U} =& -\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)(1-\tau^*)^{\frac{-2\varphi}{\chi_0+\varphi}}(\hat{x}_t)^2 \\ & -\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\frac{\varphi}{1-\tau^*}\right)(1-\tau^*)^{\frac{2\chi_0}{\chi_0+\varphi}}(\hat{y}_t)^2 + (1-\tau^*)^{\frac{-(\chi_0-1)\varphi}{\chi_0+\varphi}}\frac{1}{\tau^*}(1-\tau^*)^{\frac{\chi_0-\varphi}{\chi_0+\varphi}}\hat{x}_t\hat{y}_t \\ =& -\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)(\hat{x}_t)^2 - \frac{1}{2}(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\left(\frac{1-\tau^*}{\tau^*}+\varphi\right)(\hat{y}_t)^2 + (1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\frac{1}{\tau^*}\hat{x}_t\hat{y}_t. \end{aligned} \tag{I.21}$$

Finally by plugging equation (I.20) into equation (I.21), we obtain

$$U_t - \bar{U} = \tilde{\gamma}_q\left(\hat{Q}_t\right)^2 + h.o.t, \tag{I.22}$$

with

$$\begin{aligned} \tilde{\gamma}_q =& -\frac{1}{2}(1-\tau^*)^{\frac{-(\chi_0+1)\varphi}{\chi_0+\varphi}}\left(\frac{1}{\tau^*}+\chi_0\right)\left(\frac{\tau^*\left(1-\frac{\varphi}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)^2 \\ & -\frac{1}{2}(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\left(\frac{1-\tau^*}{\tau^*}+\varphi\right)\left(\frac{\tau^*\left(1+\frac{\chi_0}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)^2 \\ & +(1-\tau^*)^{\frac{\chi_0(1-\varphi)}{\chi_0+\varphi}}\frac{1}{\tau^*}\left(\frac{\tau^*\left(1-\frac{\varphi}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right)\left(\frac{\tau^*\left(1+\frac{\chi_0}{1-\tau^*}\right)}{\tau^*-\left(\chi_0+\frac{\varphi}{1-\tau^*}\right)}\right) < 0. \end{aligned} \tag{I.23}$$

Conditional loss function Equations (I.22) and (I.23) lead to our dynamic loss function in (14):

$$\mathbb{L}^{Q}\left(\left\{\hat{Q}_t\right\}_{t\geq 0}\right)=\mathbb{E}_0\int_0^{\infty}e^{-\rho t}\left(\hat{Q}_t\right)^2 dt. \tag{I.24}$$

References

Baxter, Marianne and Robert King, "Productive externalities and business cycles," *Working Paper*, 1991.

Bianchi, Javier and Enrique G. Mendoza, "Overborrowing, Financial Crises and 'Macro-Prudential' Taxes," *NBER Working Paper*, 2010.

Caballero, Ricardo J. and Arvind Krishnamurthy, "International and domestic collateral constraints in a model of emerging market crises," *Journal of Monetary Economics*, 2001, *48* (3), 513–548.

Cochrane, John, "The new-Keynesian liquidity trap," *Journal of Monetary Economics*, 2017, *92*, 47–63.

Dàvila, Eduardo and Anton Korinek, "Pecuniary Externalities in Economies with Financial Frictions," *The Review of Economic Studies*, 2018, *85* (1), 352–395.

Farhi, Emmanuel and Iván Werning, "Dealing with the Trilemma: Optimal Capital Controls with Fixed Exchange Rates," *Working Paper*, 2012.

— **and** — , "A Theory of Macroprudential Policies in the Presence of Nominal Rigidities," *Econometrica*, 2016, *84* (5), 1645–1704.

— **and** — , "Fiscal Unions," *American Economic Review*, 2017, *107* (12), 3788–3834.

— **, Mike Golosov, and Aleh Tsyvinski**, "A Theory of Liquidity and Regulation of Financial Intermediation," *Review of Economic Studies*, 2009, *76* (3), 973–992.

Jeanne, Olivier and Anton Korinek, “Excessive Volatility in Capital Flows: A Pigouvian Taxation Approach,” *American Economic Review Papers and Proceedings*, 2010, *100* (2), 403–007.

Korinek, Anton. and Alp Simsek, “Liquidity Trap and Excessive Leverage,” *American Economic Review*, 2016, *106* (3), 699–738.

Lee, Seung Joo and Marc Dordal i Carreras, “Self-fulfilling Volatility and a New Monetary Policy,” *Working Paper*, 2024.

Lorenzoni, Guido, “Inefficient Credit Booms,” *The Review of Economic Studies*, 2008, *75* (3), 809–833.

Stein, Jeremy, “Monetary Policy as Financial-Stability Regulation,” *Quarterly Journal of Economics*, 2012, *127* (1), 57–95.

Werning, Iván, “Managing a Liquidity Trap: Monetary and Fiscal Policy,” *Working Paper*, 2012.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, 2003.