

Justifying the First-Order Approach in Agency Frameworks with the Agent's Possibly Non-Concave Value Function

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November 28, 2022

Abstract

To justify the first-order approach in principal-agent problems, the previous literature has focused on making the agent's expected monetary utility, i.e., value function, obtained from this approach *concave* in the agent's effort. However, relying on such concavity is sometimes overly sufficient. We propose a new set of conditions for justifying the first-order approach which is derived not based on the concavity of the agent's expected monetary utility but directly from the original incentive compatibility constraint. Our suggested examples illustrate that this set of conditions can be applied to a wider range of principal-agent problems (including the one with normal or gamma density function) than all the existing sets of conditions.

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1 Introduction

Replacing the agent's original "argmax" incentive compatibility constraint with its first-order condition, which is called the *first-order approach*, has been typically adopted in solving the principal-agent problems. However, this approach is not always valid even in the standard setting.¹ Thus, identifying conditions under which using the first-order approach is valid has been one of the major issues in the principal-agent literature, and several sets of conditions for justifying this approach have been found in various agency frameworks.² Yet, there are many meaningful principal-agent settings in which using the first-order approach cannot be justified by those existing sets of conditions in the literature.

The first set of conditions was proposed by Mirrlees (1975) and Rogerson (1985) in the one-signal case. Those conditions are the well-known MLRP (monotone likelihood ratio property) and CDFC (convexity of the distribution function condition) for the distribution function of signals. Later, other conditions were proposed to generalize the one-signal CDFC to multi-signal cases. They include the generalized CDFC (i.e., GCDFC) by Sinclair-Desgagné (1994), the CISP condition (concave increasing set probability condition) by Conlon (2009), and the CDFCL condition (convexity of the distribution function condition for the likelihood ratio) by Jung and Kim (2015) among others.³ All these conditions contain the property of the CDFC condition.

However, the CDFC condition and its various extensions have a serious limitation in that they are hardly satisfied by most familiar density functions. For instance, consider a one-signal principal-agent problem in which the signal for the agent's hidden effort is generated by a simple functional form such that $x = a + \theta$, where $a \in [0, \infty)$ is the agent's effort level, and $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$ are the realized values of the signal variable, \tilde{x} , and the uncertainty variable, $\tilde{\theta} \sim N(0, \sigma^2)$, respectively.⁴ It is well known that the density function of the signal conditional on the agent's effort in this case satisfies none of the above CDF-type conditions.

To overcome this drawback, Jewitt (1988) proposed another set of conditions in the one-signal case which does not contain any CDF-type condition, and is thus more applicable in general.⁵ Recently, this set of conditions in Jewitt (1988) was also extended to the multi-signal case by Jung and Kim (2015).⁶ These sets of conditions, however, have a different kind of limitation, and thus cannot be used for many familiar cases including the above normal density example. Although they do not contain any of the troublesome CDF-type

¹For the standard principal-agent framework, see Spence and Zeckhauser (1971), Ross (1973), Mirrlees (1975), Harris and Raviv (1979), Holmstrom (1979), Shavell (1979), and Grossman and Hart (1983), among others.

²For a detailed review of the literature on the first-order approach, see e.g., Jung and Kim (2015).

³Jewitt (1988) also proposed two sets of conditions for the multi-signal case assuming that the multiple signals are independently distributed.

⁴We use letters with a tilde (e.g., \tilde{x}) to denote random variables and letters without it (e.g., x) to denote specific realized values of those random variables.

⁵See Theorem 1 in Jewitt (1988).

⁶See Proposition 7 in Jung and Kim (2015).

conditions, they contain another condition that the agent's indirect utility given the optimal contract must be concave in the signals' likelihood ratio, which cannot be satisfied in many cases, including those that impose the agent's limited liability constraint.⁷

To justify the first-order approach, the existing sets of conditions containing the CDF-type conditions (e.g., Rogerson (1985), Sinclair-Desgagné (1994), Conlon (2009), and Jung and Kim (2015)) were derived by putting all the requirements only on the signals' density function. Therefore, those CDF-type conditions are, in general, too restrictive to be satisfied by most familiar density functions. On the other hand, Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015), based on placing an additional requirement on the agent's utility function that the agent's indirect utility be concave in the likelihood ratio of the signals, were able to impose weaker requirements on the density function of the signals than those CDF-type conditions. However, there are still many cases in which such an additional requirement cannot be met. For many density functions of the signals (including the normal density function in the above example), their likelihood ratios are unbounded below. In this case, in order to guarantee the existence of the optimal contract, one needs to include the agent's limited liability constraint which requires that the agent's wage not be lower than a certain level under any circumstances.⁸ However, when the optimal contract is to be bounded below due to the agent's limited liability, so is the agent's indirect utility given that optimal contract, and thereby it cannot be concave in the likelihood ratio, satisfying neither the Jewitt (1988) conditions nor the Jung and Kim (2015) conditions.

One thing common to all the existing results is that, to justify the first-order approach, those conditions in the previous literature sufficiently make the agent's expected monetary utility obtained from this approach (i.e., value function) concave in the agent's effort. However, such concavity is sufficient but not necessary for justifying the first-order approach in two respects. First, since the agent's expected utility (i.e., value function) given an incentive contract is his expected monetary utility from that incentive contract subtracted by the cost of effort, the conditions derived by relying only on the concavity of the agent's expected monetary utility do not take the agent's cost of effort into consideration. Even if the agent's expected monetary utility is not concave in effort, his expected utility can still be concave if the cost of effort is sufficiently convex, and thus using the first-order approach can be justified. Second, but more importantly, even the concavity of the agent's expected utility (i.e., his expected monetary utility minus cost) in effort itself is not necessary for justifying the first-order approach. As shown in Figure 1, the first-order approach can be justified as long as the agent's expected utility from this approach has a maximum value at the target effort level the principal

⁷See e.g., Jewitt et al. (2008) for the solution's existence and uniqueness in the presence of the agent's limited liability constraint.

⁸This is the well-known Mirrlees' unpleasant theorem. For this issue, see e.g., Mirrlees (1975)

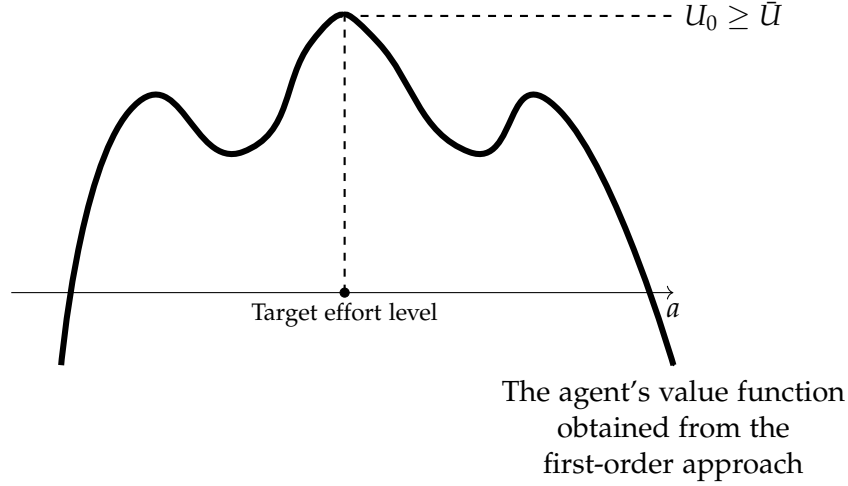


Figure 1: Possibly Non-Concave Indirect Utility of the Agent

intends to induce (i.e., the original argmax incentive constraint).

In this paper, we derive new sets of conditions that are not obtained by relying on the concavity of the agent's expected monetary utility but obtained directly from the original incentive compatibility constraint. Our new sets of conditions contain an interesting statistical condition on the signals' density functions. We illustrate that this statistical condition not only has a very useful implication but also is quite general in that it can be satisfied by the wide range of familiar density functions including the normal distributions in the above case and various other exponential families, whereas the existing conditions cannot. We also provide three alternative sets of conditions: two that can be used for the case in which that constraint is not binding at all at the optimum, and one that can be used for the case in which the agent's limited liability constraint is binding for some signal values at the optimum, respectively. Also, by using those sets of conditions, we show that the first-order approach can be validly adopted in many useful principal-agent settings (including the above normal density case) in which using the first-order approach has not been able to be justified by the literature's existing sets of conditions.

The remainder of the paper is organized as follows. In Section 2, we formulate the basic principal-agent framework, and briefly explain the issue of using the first-order approach. In Section 3, we present our main set of conditions that is derived not from the concavity of the agent's expected monetary utility but directly from the original incentive compatibility constraint. Then, in Sections 4, we propose three alternative sets of conditions that are easier to verify than our main set of conditions in both the case where it is not binding at the optimum (i.e., Section 4.1) and the case where the agent's limited liability constraint is binding for some values of the signals at the optimum (i.e., Section 4.2), respectively. In Section 5, we explain the statistical

implications of our sets of conditions, and compare them with the existing conditions. Concluding remarks are given in Section 6, and all formal proofs are relegated to the Appendix.

2 The Basic Model

We consider a one-period standard principal-agent model in which an agent works for a principal by inputting his effort $a \in [0, \infty)$. The principal cannot observe the agent's effort choice directly but can observe signals $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ that are imperfectly correlated with the agent's hidden effort, where \tilde{x}_i is a one-dimensional random variable. By taking the Mirrlees (1975) formulation, we denote $f(\mathbf{x}|a)$ as the joint density function of $\tilde{\mathbf{x}}$ conditional on the agent's effort, a . It is defined from the cumulative distribution function of $\tilde{\mathbf{x}}$ given a , i.e., $F(\mathbf{x}|a) \equiv \Pr[\tilde{\mathbf{x}} \leq \mathbf{x}|a]$, where $\mathbf{x} \in \mathbb{R}^n$ is the realized value of signal vector $\tilde{\mathbf{x}}$. We assume that the support of $f(\mathbf{x}|a)$ is independent of a , and both $F(\mathbf{x}|a)$ and $f(\mathbf{x}|a)$ are continuous and differentiable at least twice with respect to a .

When signal \mathbf{x} is realized, the principal obtains $\pi(\mathbf{x})$ as the total value of the agency relationship, and she pays to the agent his wage s which depends on \mathbf{x} , i.e., $s = s(\mathbf{x})$. The principal is risk-neutral, whereas the agent is risk-averse. It is assumed that the agent's utility function takes an additively separable form such as

$$u(s, a) = u(s) - c(a), \quad u' > 0, \quad u'' < 0, \quad c' > 0, \quad c'' > 0,$$

where $u(s)$ denotes the agent's utility from monetary payoff s and $c(a)$ denotes his disutility of exerting a .⁹ Thus, the agent's expected utility when he takes an effort a under $s(\mathbf{x})$ is given by

$$U(s(\cdot), a) \equiv \int u(s(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} - c(a). \quad (1)$$

We also assume that the agent can get \bar{U} at maximum by working for other principals, thereby \bar{U} is his reservation utility level. Furthermore, there is a limited liability constraint on the agent's side which requires that the agent's wage not be lower than \underline{s} under any circumstances, i.e., $s(\mathbf{x}) \geq \underline{s}, \forall \mathbf{x}$, where \underline{s} indicates the agent's subsistence wage level.¹⁰

So a principal-agent problem can be represented by its characteristic variables $\{\pi(\mathbf{x}), u(s), f(\mathbf{x}|a), c(a), \bar{U}, \underline{s}\}$,

⁹We use the prime and the double prime of a function to denote the first and the second derivatives of that function, respectively.

¹⁰The agent's limited liability constraint is especially needed to guarantee the optimal contract's existence in some cases. Note that the case in which there is no limited liability constraint on the agent's side is a special case where $\underline{s} = -\infty$. For this 'unpleasantness' issue, see e.g., Mirrlees (1975). Also, see e.g., Jewitt et al. (2008) for the solution's existence and uniqueness in the presence of the agent's limited liability constraint.

and the principal's optimization program given these variables is given by

$$\begin{aligned} \max_{a, s(\mathbf{x}) \geq \underline{s}} \quad & \int [\pi(\mathbf{x}) - s(\mathbf{x})] f(\mathbf{x}|a) d\mathbf{x} \\ \text{s.t.} \quad & (i) U(s(\cdot), a) \geq \bar{U}, \\ & (ii) a \in \arg \max_{a'} U(s(\cdot), a'). \end{aligned}$$

In the above, the constraints are the typical participation and incentive compatibility constraints, respectively. This optimization program indicates that the principal has to decide both the agent's wage scheme, $s(\mathbf{x}) \geq \underline{s}$, $\forall \mathbf{x}$, and the target effort level, a , simultaneously to maximize her expected payoff under the constraints that the self-interested agent actually chooses the target effort level when $s(\mathbf{x})$ is offered and that his expected utility in this case is not lower than \bar{U} .

However, the above program is generally not tractable in itself because the incentive constraint is composed of infinitely many inequality constraints. Thus, it has been typically solved by replacing the original incentive constraint with the *relaxed* constraint that the agent's expected utility given $s(\mathbf{x})$ is stationary at that effort level, a , i.e.,

$$\frac{\partial U(s(\cdot), a)}{\partial a} \equiv U_a(s(\cdot), a) = 0, \quad (2)$$

which is known as the famous first-order approach. Therefore, the principal's optimization program based on the first-order approach can be written as

$$\begin{aligned} \max_{a, s(\mathbf{x}) \geq \underline{s}} \quad & \int [\pi(\mathbf{x}) - s(\mathbf{x})] f(\mathbf{x}|a) d\mathbf{x} \\ \text{s.t.} \quad & (i) U(s(\cdot), a) \geq \bar{U}, \\ & (ii) U_a(s(\cdot), a) = \int u(s(\mathbf{x})) f_a(\mathbf{x}|a) d\mathbf{x} - c'(a) = 0, \end{aligned}$$

where $f_a(\mathbf{x}|a) \equiv \frac{\partial f(\mathbf{x}|a)}{\partial a}$ is a partial derivative of the density function $f(\mathbf{x}|a)$ with respect to action a .

Let $(s^o(\mathbf{x}), a^o > 0)$ solve the above optimization program.¹¹ By solving the Euler equation of the above program, one can derive that the optimal incentive contract, $s^o(\mathbf{x})$, should satisfy

$$\frac{1}{u'(s^o(\mathbf{x}))} = \begin{cases} \lambda + \mu \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}, & \text{if } s^o(\mathbf{x}) \geq \underline{s}, \\ \frac{1}{u'(\underline{s})}, & \text{otherwise,} \end{cases} \quad (3)$$

where λ and μ are the Lagrange multipliers of the participation and *relaxed* incentive compatibility con-

¹¹The existence of an optimal solution $(s^o(\mathbf{x}), a^o)$ is assumed. We also assume $a^o > 0$ to rule out a trivial case. For the existence of an optimal solution in the principal-agent problem, see e.g., Jewitt et al. (2008).

straints, respectively.

It is well known in the literature that the optimal contract, $s^o(\mathbf{x})$ in (3), does not always solve the original optimization program. This is because $s^o(\mathbf{x})$ in equation (3) is actually the optimal solution obtained based on replacing the original “argmax” incentive constraint with the *relaxed* one, and the principal’s opportunity set for $s(\mathbf{x})$ satisfying the relaxed incentive constraint is larger than her true opportunity set for $s(\mathbf{x})$ satisfying the original argmax incentive constraint, and thereby $s^o(\mathbf{x})$ sometimes may not be in her true opportunity set for $s(\mathbf{x})$. Thus, to guarantee that $s^o(\mathbf{x})$ in equation (3) actually solves the original program, it must be ensured that $s^o(\mathbf{x})$ satisfies the original argmax incentive constraint, that is,

$$U(s^o(\cdot), a) \leq U(s^o(\cdot), a^o), \quad \text{for all } a. \quad (4)$$

To ensure equation (4), all the literature’s existing results for justifying the first-order approach were derived to make the agent’s expected utility when he takes a under $s^o(\mathbf{x})$, $U(s^o(\cdot), a)$, concave in a .¹² Obviously, given the concavity of $U(s^o(\cdot), a)$ in a , $U_a(s^o(\cdot), a^o) = 0$ guarantees equation (4), and thus using the first-order approach is valid. However, the concavity of $U(s^o(\cdot), a)$ in a is sufficient but not necessary for ensuring equation (4). As already drawn in Figure 1, even if $U(s^o(\cdot), a)$ is not concave, the first-order approach can still be justified. The main purpose of this paper is to find a new set of conditions which ensures equation (4) without relying on the concavity of $U(s^o(\cdot), a)$, and thus is more general than the existing sets of conditions.

3 Analysis

We start with proving that μ in equation (3) is positive. The basic proof for $\mu > 0$ was already given by e.g., Mirrlees (1975) and Holmstrom (1979).¹³ But their proofs were proved with the assumption that using the first-order approach is valid. Thus, it is obvious that those proofs cannot be used for finding conditions justifying the first-order approach itself. On the other hand, Jewitt (1988) provided another proof for $\mu > 0$ without such an assumption.¹⁴ But, his approach also has a limitation in that it is valid only for the case in which the agent’s limited liability constraint is not binding at all at the optimum. However, as will be shown later, there are actually many cases in which the agent’s limited liability constraint is binding for some \mathbf{x} at the optimum, and one of our main goals is to provide the conditions under which using the first-order

¹²For example, see Mirrlees (1975), Grossman and Hart (1983), Rogerson (1985), Jewitt (1988), Sinclair-Desgagné (1994), Conlon (2009), and Jung and Kim (2015) among others.

¹³Mirrlees (1975) showed: in single-signal cases, i.e., $x \in \mathbb{R}$, $\mu > 0$ when $f(x|a)$ satisfies the monotone likelihood ratio property (MLRP), whereas Holmstrom (1979) showed it when $f(x|a)$ satisfies the first-order stochastic dominance (FOSD) condition.

¹⁴See Lemma 1 in Jewitt (1988).

approach can be justified even in those cases. Thereby, we provide a different proof for $\mu > 0$ based neither on the assumption that using the first-order approach is valid nor on the assumption that the agent's limited liability constraint is not binding at the optimum.

Lemma 1: $\mu > 0$.

Observe from equation (3) that, especially when the principal is risk-neutral, the agent's optimal contract, $s^o(\mathbf{x})$, depends on signal \mathbf{x} only through $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$. That is, $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ becomes a sufficient statistic for \mathbf{x} about a^o for designing $s(\mathbf{x})$, implying that what matters to the risk-neutral principal when she designs contracts for her agent is $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ rather than \mathbf{x} itself. In fact, $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ is the information contained in signal \mathbf{x} , indicating that how likely it is that the agent has taken a^o rather than some other nearby action when signal \mathbf{x} is realized. Based on this observation, Jung and Kim (2015) showed that analyzing principal-agent problems directly based on $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ generally has an advantage over analyzing them based on the signal vector \mathbf{x} . Thus, as in Jung and Kim (2015), we derive conditions under which the first-order approach can be justified based on $\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$.

We denote $\tilde{q} \equiv Q_{a^o}(\tilde{\mathbf{x}}) \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ as the information variable, which implies that random variable \tilde{q} and random vector $\tilde{\mathbf{x}}$ have a functional relationship such that $q = Q_{a^o}(\mathbf{x}) \equiv \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$, where q is the realized value of \tilde{q} .¹⁵ We also denote $G(q|a)$ as the cumulative distribution function of \tilde{q} given a , i.e.,

$$G(q|a) \equiv \Pr[Q_{a^o}(\tilde{\mathbf{x}}) \leq q|a],$$

and $g(q|a)$ as its probability density function.¹⁶

Based on q , define

$$w(q) \equiv u'^{-1}\left(\frac{1}{\lambda + \mu q}\right) \quad \text{and} \quad \bar{r}(q) \equiv u(w(q)). \quad (5)$$

We see from (3) that $w(q)$ denotes the optimal contract defined on the q -space when it is not constrained by the limited liability constraint, i.e., $s^o(\mathbf{x}) \geq \underline{s}$, whereas $\bar{r}(q)$ denotes the agent's indirect utility given the contract also defined on the q -space in that case. Note from equation (5) that the functional forms of $w(\cdot)$ and $\bar{r}(\cdot)$ depend only on the functional form of $u(\cdot)$, and Lemma 1 shows that both $w(q)$ and $r(q)$ are increasing in q .

Thus, the agent's indirect utility given $s^o(\mathbf{x})$ in equation (3), i.e., $u(s^o(\mathbf{x}))$, can be written as

$$u(s^o(\mathbf{x})) \equiv r(Q_{a^o}(\mathbf{x})) = r(q) = \begin{cases} \bar{r}(q), & \text{when } q \geq q_c, \\ u(\underline{s}), & \text{when } q < q_c, \end{cases} \quad (6)$$

¹⁵Note that q is a function of \mathbf{x} but it also depends on a^o . For a more detailed explanation of q , see Jung and Kim (2015).

¹⁶For a more detailed treatment of distributions $G(q|a)$ and $g(q|a)$, see Jung and Kim (2015). a can be different from the given a^o .

where $q_c = Q_{a^0}(\mathbf{x}_c) \equiv \frac{f_a(\mathbf{x}_c|a^0)}{f(\mathbf{x}_c|a^0)}$ solves $\frac{1}{u'(\underline{s})} = \lambda + \mu q_c$. Also, the agent's expected indirect utility when he takes an effort a under $s^o(\mathbf{x})$ can be written as

$$\begin{aligned} U(s^o(\cdot), a) &= \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a) \\ &= \int r(q)g(q|a)dq - c(a). \end{aligned}$$

It is widely known that the agent's limited liability constraint may or may not be binding at the optimum depending on the characteristic variables, $\{\pi(\mathbf{x}), u(s), f(\mathbf{x}|a), c(a), \bar{U}, \underline{s}\}$, especially on whether the information variable, $\tilde{q} = \frac{f_a(\tilde{\mathbf{x}}|a^0)}{f(\tilde{\mathbf{x}}|a^0)}$, is bounded below or not.¹⁷ If it is the case where the agent's limited liability constraint is binding for some \mathbf{x} with the optimal contract, the participation constraint may not be binding (i.e., $\lambda = 0$), and the agent may enjoy some positive rent at the optimum. Thus, to be more general, we denote the agent's expected utility when he takes a^o given $s^o(\mathbf{x})$ as

$$\begin{aligned} U(s^o(\cdot), a^o) &= \int u(s^o(\mathbf{x}))f(\mathbf{x}|a^o)d\mathbf{x} - c(a^o) \\ &= \int r(q)g(q|a^o)dq - c(a^o) = U^o \geq \bar{U}, \end{aligned} \tag{7}$$

where U^o is determined by $\{\pi(\mathbf{x}), u(s), f(\mathbf{x}|a), c(a), \bar{U}, \underline{s}\}$. Of course, $U^o = \bar{U}$ if the limited liability constraint is not binding for any \mathbf{x} at the optimum (i.e., $\lambda > 0$).

The following lemma plays the basic role in deriving our main results.

Lemma 2: *For any given a^o , if*

(L1) *$f(\mathbf{x}|a)$ satisfies that $\frac{g(q|a)}{g(q|a^o)}$ is convex in q for all a , and $\xi(q)$ satisfies that:*

(L2) *$\int \xi(q)g(q|a^o)dq = 0$,*

(L3) *$\int \xi(q) \cdot q \cdot g(q|a^o)dq = 0$, and*

(L4) *$\xi(q)$ changes sign twice from negative to positive and then to negative as q increases,*

then, we have

$$\int \xi(q)g(q|a)dq \leq 0, \quad \forall a.$$

Let S_f be the set of contracts that give the agent the same expected utility as U^o in equation (7) when he

¹⁷This is associated with what is called the Mirrlees' unpleasant theorem. See e.g., Mirrlees (1975) for this issue.

takes a^o , and satisfy the relaxed incentive constraint at that effort level. That is,

$$S_f \equiv \{s(\mathbf{x}) \mid s(\mathbf{x}) \text{ satisfies } U(s(\cdot), a^o) = U^o \text{ and } U_a(s(\cdot), a^o) = 0\}.^{18} \quad (8)$$

Then, using Lemma 2, we obtain the following Lemma 3, one of our key results.

Lemma 3: *For any given a^o , if*

((1a) = (L1)) $f(\mathbf{x}|a)$ satisfies that $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a

(2a) (double crossing property) there exists a contract $\hat{s}(\mathbf{x}) \in S_f$ such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ crosses $r(q) \equiv u(s^o(\mathbf{x}))$ twice starting from above, and

(3a) $E[\hat{r}(q)|a]$ is concave in $c(a)$,

then using the first-order approach is justified.

What Lemma 3 indicates is the following: Note that, since $s^o(\mathbf{x})$ is the optimal contract obtained from the first-order approach, $s^o(\mathbf{x})$ must be in S_f . Now, consider another contract $\hat{s}(\mathbf{x}) \in S_f$ such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ crosses $r(q) \equiv u(s^o(\mathbf{x}))$ twice starting from above as drawn in Figure 2a.¹⁹ Then, if one thinks of $r(q) - \hat{r}(q)$ as $\zeta(q)$ in Lemma 2, it can be easily seen that $\zeta(q) = r(q) - \hat{r}(q)$ satisfies the condition (L4) in Lemma 2. Furthermore, as shown in the proof of Lemma 3, the fact that both $s^o(\mathbf{x})$ and $\hat{s}(\mathbf{x})$ are in S_f implies

$$\int [r(q) - \hat{r}(q)] g(q|a^o) dq = 0, \quad (9)$$

indicating that condition (L2) in Lemma 2 is satisfied, and

$$\int [r(q) - \hat{r}(q)] \underbrace{q \cdot g(q|a^o)}_{=g_a(q|a^o)} dq = 0, \quad (10)$$

indicating that condition (L3) in Lemma 2 is satisfied. Thus, based on Lemma 2, we derive that, if condition (1a=L1) is satisfied, the agent's expected utility under $s^o(\mathbf{x})$ is lower than that under $\hat{s}(\mathbf{x})$ for all a except for a^o as drawn in Figure 2b, i.e.,

$$U(s^o(\cdot), a) - U(\hat{s}(\cdot), a) = \int [r(q) - \hat{r}(q)] g(q|a) dq \leq 0, \quad \forall a.^{20} \quad (11)$$

¹⁸Not that, since S_f is defined based on given levels of a^o , it varies as a^o changes.

¹⁹The fact that $\hat{r}(q)$ crosses $r(q)$ twice starting from above (double-crossing) is equivalent to that $\hat{s}(\mathbf{x})$ crosses $s^o(\mathbf{x})$ twice starting from above if $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP.

²⁰This result can be understood as follows: we know $U(s^o(\cdot), a^o) - U(\hat{s}(\cdot), a^o) = 0$. If $a \neq a^o$, due to the double-crossing in Figure 2a, it is more likely that the distribution $g(q|a)$ puts more weights on the negative values of $r(q) - \hat{r}(q)$, making its average value negative. Our statistical condition (1a) guarantees that this claim holds.

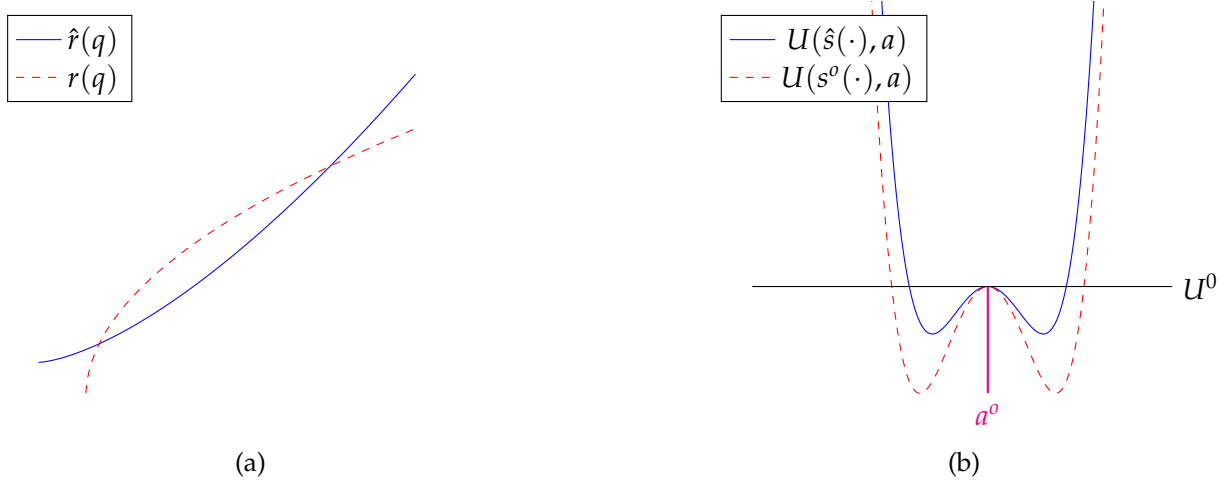


Figure 2: Double Crossing Property

Figure 2b is drawn in a way that, although $U(s^o(\cdot), a)$ and $U(\hat{s}(\cdot), a)$ have a stationary point at a^o , they do not have a global maximum point at a^o . This may happen because especially $\hat{s}(\mathbf{x})$ is assumed to be in S_f . Actually, the fact that $s^o(\mathbf{x})$ and $\hat{s}(\mathbf{x})$ are in S_f only ensures that

$$U(s^o(\cdot), a^o) = U(\hat{s}(\cdot), a^o) = U^o, \quad (12)$$

and

$$U_a(s^o(\cdot), a^o) = U_a(\hat{s}(\cdot), a^o) = 0.$$

However, condition (3a) in Lemma 3 guarantees that $U(\hat{s}(\cdot), a)$ actually has a global maximum point at a^o , i.e.,

$$U(\hat{s}(\cdot), a) \leq U(\hat{s}(\cdot), a^o) = U^o, \quad (13)$$

as drawn in Figure 3. Thus, by combining equation (11), equation (12), and equation (13), we have

$$U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a^o) = U^o, \quad \forall a,$$

which justifies the first-order approach.

Condition (1a=L1) in Lemma 3 is our main statistical condition which is different from the typical statistical conditions in the existing literature such as CDFC (convexity of the distribution function condition) by Mirrlees (1975) and Rogerson (1985), CISP condition (concave increasing set probability condition) by Conlon (2009), and CDFCL (convexity of the distribution function condition for the likelihood ratio) by Jung and Kim (2015). In Section 5, we will explain the statistical implication of (1a) as well as the difference between

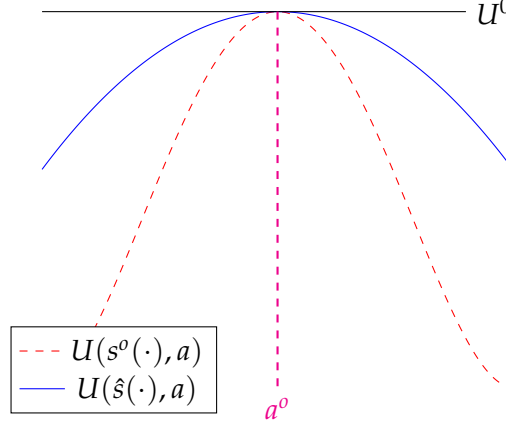


Figure 3: First-Order Approach Justified

(1a) and the above existing conditions more precisely. However, it is worth to note that (1a) is much easier to verify than the existing conditions in the previous literature and also quite general in that many familiar density functions satisfy it.

For example, in the one-signal case, consider a density function in the exponential family such as

$$f(x|a) = A(a)B(x)e^{\alpha(a)\beta(x)}, \quad x \in \mathbb{R},$$

with $\alpha(a)$ and $\beta(x)$ increasing. Then,

$$\frac{f_a(x|a)}{f(x|a)} = \alpha'(a)\beta(x) + \frac{A'(a)}{A(a)}.$$

Thus, for any given a^o ,

$$Q_{a^o}(\tilde{x}) \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \alpha'(a^o)\beta(\tilde{x}) + \frac{A'(a^o)}{A(a^o)}.$$

Since $\alpha(a)$ and $\beta(x)$ are increasing, $f(x|a)$ satisfies MLRP. Therefore,

$$G(q|a) = \Pr[Q_{a^o}(\tilde{x}) \leq q|a] = \Pr[\tilde{x} \leq x|a] = F(x|a),$$

where x solves $\frac{f_a(x|a^o)}{f(x|a^o)} = q$,²¹ and

$$g(q|a) = f(x|a) \frac{dx}{dq}.$$

²¹Thus, x which solves $\frac{f_a(x|a^o)}{f(x|a^o)} = q$ is a function of q given a^o , i.e., $x(q; a^o)$.

By using $\beta(x) = \left(q - \frac{A'(a^0)}{A(a^0)}\right) \frac{1}{\alpha'(a^0)}$ and $\frac{dx}{dq} = \frac{1}{\alpha'(a^0)\beta'(x)}$, we have

$$g(q|a) = \frac{A(a)}{\alpha'(a^0)} \delta(q) \exp \left(\frac{\alpha(a)}{\alpha'(a^0)} \left(q - \frac{A'(a^0)}{A(a^0)} \right) \right),$$

where $\delta(q) \equiv \frac{B(x)}{\beta'(x)}$.²² As a result,

$$\frac{g(q|a)}{g(q|a^0)} = \frac{A(a)}{A(a^0)} \exp \left(\frac{\alpha(a) - \alpha(a^0)}{\alpha'(a^0)} \left(q - \frac{A'(a^0)}{A(a^0)} \right) \right),$$

which is convex in q for any given a^0 and for all a , satisfying condition (1a).

More generally, even in the multi-signal case (i.e., $\mathbf{x} \in \mathbb{R}^n$), condition (1a) holds if $f(\mathbf{x}|a)$ generates, for any given a^0 ,

$$\frac{g_a(q|a)}{g(q|a)} = A(a)q + D(a),$$

which is the case for most exponential families of density functions (including normal, gamma, etc.) Note that

$$\frac{g(q|a)}{g(q|a^0)} = \exp \left(\int_{a^0}^a \frac{g_a(q|t)}{g(q|t)} dt \right).$$

Thus,

$$\frac{g(q|a)}{g(q|a^0)} = \exp (\hat{A}(a)q + \hat{D}(a)),$$

where $\hat{A}(a) \equiv \int_{a^0}^a A(t)dt$ and $\hat{D}(a) \equiv \int_{a^0}^a D(t)dt$. Therefore, one can easily see that condition (1a) is satisfied.

On the other hand, conditions (2a) and (3a) in Lemma 3 need to be elaborated more. In general, directly verifying whether conditions (2a) and (3a) are satisfied or not is not easy. In other words, for a given principal-agent problem, finding a contract $\hat{s}(\mathbf{x})$ which satisfies the double crossing property (i.e., condition (2a)) and condition (3a), if any, is not straightforward. Therefore, in what follows, we investigate the conditions which sufficiently guarantee conditions (2a) and (3a), and are easier to verify.

4 Verifying the Double Crossing Property

To verify the double crossing property between $r(q)$ and $\hat{r}(q)$ (i.e., condition (2a) in Lemma 3), we will take the following steps. First, we introduce an arbitrary contract $\hat{s}(\mathbf{x})$ under which the agent will chooses a^0 (i.e., the original ‘‘argmax’’ incentive constraint is satisfied at a^0), and his expected utility in this case is equal

²²Since $\beta(x)$ is increasing, we have $x = \beta^{-1} \left(\frac{q - \frac{A'(a^0)}{A(a^0)}}{\alpha'(a^0)} \right) \equiv \Omega(q)$. Then, $\delta(q) \equiv \frac{B(\Omega(q))}{\beta'(\Omega(q))}$.

to U^o (i.e., the participation constraint is satisfied at a^o). That is,

$$\begin{aligned} U(\hat{s}(\cdot), a) &\equiv \int u(\hat{s}(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a) \\ &\leq U(\hat{s}(\cdot), a^o) \equiv \int u(\hat{s}(\mathbf{x}))f(\mathbf{x}|a^o)d\mathbf{x} - c(a^o) = U^o, \quad \forall a. \end{aligned} \quad (14)$$

Then following Lemma 3, we derive the conditions under which $r(q) \equiv u(s^o(\mathbf{x}))$ crosses $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ instead of $r(q)$ twice starting from below. Then, we would have:

$$\begin{aligned} U(s^o(\cdot), a) &\equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a) \\ &= \int r(q)g(q|a)dq - c(a) \\ &\leq \int \hat{r}(q)g(q|a)dq - c(a) \\ &= \int u(\hat{s}(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a) \equiv U(\hat{s}(\cdot), a), \quad \forall a. \end{aligned} \quad (15)$$

In (15), the inequality comes from Lemma 2 because both $r(q)$ and $\hat{r}(q)$ satisfy the participation and the relaxed incentive constraints, and $r(q)$ crosses $\hat{r}(q)$ twice starting from below. Consequently, by combining (7), (14), and (15), we derive that

$$U(s^o(\cdot), a) \leq U(\hat{s}(\cdot), a) \leq U(\hat{s}(\cdot), a^o) = U(s^o(\cdot), a^o) = U^o, \quad \forall a,$$

which justifies the first-order approach as drawn in Figure 3.

In Figure 3, $U(\hat{s}(\cdot), a)$ is drawn in a way that the agent will choose a^o given $\hat{s}(\mathbf{x})$ and $U(\hat{s}(\cdot), a^o) = U^o$ (i.e., equation (14)), and $U(s^o(\cdot), a)$ is drawn in a way that it lies below $U(\hat{s}(\cdot), a)$ for all a except for a^o since $U(s^o(\cdot), a^o) = U^o$ (i.e., equation (7) and equation (15)).

Thus, the key point is to find an appropriate proxy contract, $\hat{s}(\mathbf{x})$, with which the double crossing property between $r(q)$ and $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ can be easily verified. To do this, however, we have to distinguish the case in which the agent's limited liability constraint is binding for some \mathbf{x} at the optimum from the case in which that constraint is not binding for all \mathbf{x} at the optimum. This is because $r(q) \equiv u(s^o(\mathbf{x}))$ in equation (6) has a different functional form depending on whether the agent's limited liability constraint is binding for some \mathbf{x} at the optimum or not. As shown in equation (6), the agent's indirect utility given $s^o(\mathbf{x})$, i.e., $u(s^o(\mathbf{x})) \equiv r(q)$, must be bounded below by $u(\underline{s})$ for some low values of q (i.e., $q < q_c$) if his limited liability constraint is binding for some \mathbf{x} at the optimum, whereas it is not bounded below by $u(\underline{s})$ (i.e., $r(q) = \bar{r}(q)$, $\forall q$) if that constraint is not binding for all \mathbf{x} at the optimum. This requires us to introduce a different $\hat{s}(\mathbf{x})$ to guarantee

the double crossing property between $r(q)$ and $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ in each case.

Associated with the above distinction, it is worth noting that the existing results for the validity of the first-order approach can also be divided into two groups, the results which can be applied only to the case in which the agent's limited liability constraint is not binding for all \mathbf{x} at the optimum, and the results which can be applied even to the case in which the agent's limited liability constraint is binding for some \mathbf{x} at the optimum.

As mentioned earlier, all the existing results were derived to make the agent's expected monetary utility given $s^o(\mathbf{x})$, i.e., $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$, concave in a . More precisely, the Mirrlees-Rogerson conditions in the one-signal case (i.e., $f(x|a)$, $x \in \mathbb{R}$, should satisfy the MLRP and the CDFC) were derived based on the fact that $R(a) \equiv \int u(s^o(x))f(x|a)dx$ is concave in a for any "increasing" function $u(s^o(x))$ if $f(x|a)$ satisfies the CDFC. Later, other conditions were found to generalize the CDFC to the multi-signal case. They include Sinclair-Desgagné's GCDFC (generalized convexity of the distribution function condition), Conlon's CISP condition (the concave increasing set probability condition), and Jung and Kim's CDFCL (convexity of the distribution function condition for the likelihood ratio). All these conditions contain the property of the CDFC because they are just extended versions of the one-signal CDFC to the multi-signal case. Since the results that contain those CDF-type conditions can be used for any "increasing" $u(s^o(\mathbf{x}))$ (or $r(q)$), they can be well applied to the case in which the agent's limited liability constraint is binding for some \mathbf{x} at the optimum because $u(s^o(\mathbf{x}))$ is generally increasing even in this case. However, they can be applied only to a limited set of cases in that most familiar density functions of the signals, $f(\mathbf{x}|a)$, hardly satisfy such CDF-type conditions.

To overcome this drawback, Jewitt (1988) proposed another set of conditions in the one-signal case which is not related to CDF-type conditions, and his conditions were generalized to multi-signal cases by Jung and Kim (2015). Both the condition on $f(\mathbf{x}|a)$ in the Jewitt (1988) (i.e., Theorem 1 in Jewitt (1988)) and that in the Jung and Kim (2015) (i.e., Proposition 7 in Jung and Kim (2015)) are weaker than the above CDF-type conditions, and thus can be satisfied by some familiar density functions.²³ This is because they were derived to make $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} = \int r(q)g(q|a)dq$ concave in a not for any "increasing" function but for any "increasing concave" function of $u(s^o(\mathbf{x}))$ (or $r(q)$). However, these conditions, although useful, can also be applied only to another limited set of cases in which the agent's limited liability constraint is not binding at the optimum. When the agent's limited liability constraint is binding for some \mathbf{x} at the optimum, $u(s^o(\mathbf{x}))$ (or $r(q)$) should be bounded below for those \mathbf{x} , and thus cannot be concave. We will elaborate on this issue in the next Section 4.2.

²³For the examples of such density functions, see Jewitt (1988, p. 1183)

4.1 When the Information Variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, Is Bounded Below

We first start from the case in which the agent's limited liability constraint is not binding for any \mathbf{x} at the optimum, and provide a set of conditions easier to verify than ((1a),(2a)) in Lemma 3. The agent's limited liability constraint will not be binding for all \mathbf{x} at the optimum if the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)} \in [\underline{q}, \bar{q}]$, has a lower bound, i.e., $\underline{q} > -\infty$, and \underline{s} is low enough such that $\bar{r}(\underline{q})$ in equation (5) is greater than $u(\underline{s})$.²⁴ Therefore, to guarantee that the agent's limited liability constraint is not binding for any \mathbf{x} at the optimum as long as the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, we will assume that that \underline{s} is low enough.

When the agent's limited liability constraint is not binding for all \mathbf{x} at the optimum, equation (3) reduces to

$$\frac{1}{u'(s^o(\mathbf{x}))} = \lambda + \mu \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} = \lambda + \mu q, \quad \text{for all } q, \quad (16)$$

and equation (6) reduces to

$$u(s^o(\mathbf{x})) \equiv r(Q_{a^o}(\mathbf{x})) = r(q) = \bar{r}(q), \quad \text{for all } q. \quad (17)$$

Furthermore, since the agent's participation constraint must be binding (i.e., $\lambda > 0$) in this case, equation (7) also reduces to

$$U(s^o(\cdot), a^o) = \int u(s^o(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) = \int r(q) g(q|a^o) dq - c(a^o) = \bar{U}. \quad (18)$$

Then, based on equations equation (16), equation (17) and equation (18), we have the following proposition.

Proposition 1 *Given that the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, if, for any given a^o ,*

- (1a) $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a ,
- (2c) $c(a)$ is convex in $m(a) \equiv \int q g(q|a) dq$, and
- (3c) $\bar{r}(q)$ is concave in q ,

then the first-order approach is justified.

The conditions in Proposition 1 sufficiently guarantee conditions ((1a),(2a)) in Lemma 3 in the case where the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, is bounded below, and thereby the agent's limited liability constraint is not binding at all at the optimum. Especially, conditions (2c) and (3c) are provided as sufficient conditions for condition (2a) in Lemma 3 (i.e., the double crossing property between $r(q)$ and $\hat{r}(q)$) in this case.

²⁴This is the case to which Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015) can be applied.

To guarantee the double crossing property between $r(q)$ and $\hat{r}(q)$ in this case, we can require that $r(q)$ in equation (6) be concave in q (i.e., condition (3c) in Proposition 1). This is because $r(q) = \bar{r}(q)$, $\forall q$ in this case and we set a linear proxy contract $\hat{r}(q)$ as seen in Figure 4.

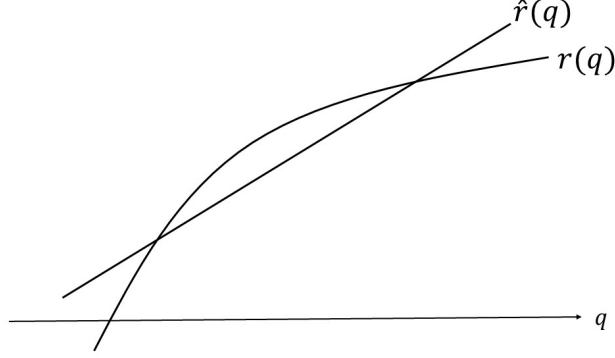


Figure 4: When the Limited Liability Constraint Does Not Bind

We introduce an arbitrary proxy contract, $\hat{s}(\mathbf{x})$, with which the agent's indirect utility is linear in q , i.e., $u(\hat{s}(\mathbf{x})) \equiv \hat{r}(q) = Aq + B$, where A and B are to be set to satisfy both the participation and the relaxed incentive constraints at a^o . Condition (2c) is given to ensure that the agent will actually choose a^o given $\hat{s}(\mathbf{x})$, satisfying equation (14) with $U^o = \bar{U}$. Also, condition (3c) is given to guarantee the double crossing property between $r(q)$ and $\hat{r}(q)$, satisfying equation (15). Thus, one can easily see that

$$U(s^o(\cdot), a) \leq U(s^o(\cdot), a^o) = \bar{U}, \quad \forall a,$$

which justifies the first-order approach.

The following Example 1 explains how the conditions in Proposition 1 can be applied to principal-agent problems where the agent's limited liability constraint is not binding for any \mathbf{x} at the optimum.

Example 1 Consider a one-signal principal-agent problem, $\{\pi(x), u(s), f(x|a), c(a), \bar{U}, \underline{s}\}$, where $\pi(x) = x$, $u(s) = \frac{1}{r}s^r$, $r \leq \frac{1}{2}$, and $c(a)$ is increasing and convex in a . We assume that the signal generating function has a simple multiplicative form, $\tilde{x} = h(a)\tilde{\theta}$, where $h(0) = 0$, $h(a)$ is increasing, and $\tilde{\theta}$ is exponentially distributed with mean 1, i.e., the density function of $\tilde{\theta}$ is $p(\theta) = e^{-\theta}$, $\theta \in [0, \infty)$. We also assume that \underline{s} is low enough. Then,

$$f(x|a) = \frac{1}{h(a)} e^{-\frac{x}{h(a)}}, \quad (19)$$

where $E[x|a] = h(a)$. Since, given a^o ,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \frac{h'(a^o)}{[h(a^o)]^2} [\tilde{x} - h(a^o)],$$

$f(x|a)$ satisfies MLRP. Since q is bounded below, i.e., $q \geq -\frac{h'(a^o)}{h(a^o)}$, and since \underline{s} is assumed to be low enough, the agent's limited liability constraint is not binding at all at the optimum. Using $g(q|a)dq = f(x|a)dx$, we derive

$$g(q|a) = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp \left(-\frac{1}{h(a)} \left(\frac{[h(a^o)]^2}{h'(a^o)} q + h(a^o) \right) \right). \quad (20)$$

Thus, it is easily seen that condition (1a) holds. Furthermore, from (16) and (17), $r(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$ is concave in q since $r \leq \frac{1}{2}$, which implies that condition (3c) is satisfied. Now, since

$$m(a) \equiv \int qg(q|a)dq = \frac{h'(a^o)}{[h(a^o)]^2} [h(a) - h(a^o)],$$

condition (2c) will hold if $c(a)$ is convex in $h(a)$, i.e., $c(a) = \phi(h(a))$, $\phi'' \geq 0$. As a result, the only condition that is needed to justify the first-order approach in this case is that $c(a)$ is convex in $h(a)$. Of course when $h(a)$ is concave in a , this condition trivially holds.

Using the first-order approach for the principal-agent problem in the above example cannot be justified by any of the existing sets of conditions if $h(a)$ is not concave in a . First, note that neither $f(x|a)$ in equation (19) nor $g(q|a)$ in equation (20) satisfies any of the CDF-type conditions. This indicates that the Mirrlees-Rogerson conditions and any existing conditions of those conditions (i.e., GCDFC by Sinclair-Desgagné (1994), the CISP condition by Conlon (2009), and the CDFCL by Jung and Kim (2015)) cannot be used for justifying the first-order approach in this case. On the other hand, Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015), which do not contain any CDF-type condition, cannot be satisfied if $h(a)$ is not concave in a . This is because the concavity of $E[x|a]$ is necessary for Theorem 1 in Jewitt (1988), whereas the concavity of $m(a)$ is necessary for Proposition 7 in Jung and Kim (2015). Therefore, those conditions cannot be also used for justifying the first-order approach in this case if $h(a)$ is not concave in a . Of course, if $h(a)$ is concave in a , condition (2c) will also be satisfied because $c(a)$ is increasing and convex in a . Thus, what Example 1 shows is that, even if $h(a)$ is not concave in a , using the first-order approach in this case can still be justified by the conditions in Proposition 1 as long as $c(a)$ is convex in $h(a)$.

One shortcoming of Proposition 1 is that it still has condition (3c), following Jewitt (1988) and Jung and Kim (2015). For example, it cannot be used when $r > \frac{1}{2}$ in Example 1. However, even if the agent's monetary utility in q -space, $\bar{r}(q)$, is convex, the next Proposition 2 justifies the first-order approach if some conditions

(i.e., (2c') and (3c') in Proposition 2) are satisfied. First, we define $M(a; t)$ as the moment generating function of $g(q|a)$, i.e.,

$$M(a; t) \equiv \int e^{tq} g(q|a) dq.$$

which we assume exists for the distribution $g(q|a)$ for $t > 0$.

In cases where $\bar{r}(q)$ is possibly convex, in order to verify the double crossing property between $r(q)$ and a possible proxy contract $\hat{r}(q)$, we introduce an arbitrary contract $\hat{s}_t(\mathbf{x})$, $t > 0$, with which the agent's indirect utility has an exponential form such as $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = A \cdot e^{tq}$, where A and $t > 0$ are to be set to satisfy both the participation and the relaxed incentive constraints as well as the agent's limited liability constraint, i.e., $\hat{s}_t(\mathbf{x}) \geq \underline{s}$, $\forall \mathbf{x}$. Since \underline{s} is assumed to be small enough, we always make sure $\hat{s}_t(\mathbf{x}) \geq \underline{s}$ for such $t > 0$.

Then, we have the following proposition.

Proposition 2 *Given that $u(s) > 0$ for $\forall s$, and the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^0)}{f(\tilde{\mathbf{x}}|a^0)}$, is bounded below, if, for any given a^0 ,*

(1a) $\frac{g(q|a)}{g(q|a^0)}$ *is convex in $q = \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)}$ for all a ,*

(2c') (i) *there exists $t > 0$ such that $\frac{c'(a^0)}{M'(a^0; t)} M(a^0; t) - c(a^0) = \bar{U}$, and (ii) $c(a)$ is convex in $M(a; t) \equiv \int e^{tq} g(q|a) dq$ for such $t > 0$, and*

(3c') $\ln \bar{r}(q)$ *is concave in q ,*

then the first-order approach is justified.

In contrast to Proposition 1, where we relied on the double-crossing property between $r(q)$ and the linear $\hat{r}_t(q)$ as shown in Figure 4, we now construct an *exponential* $\hat{r}_t(q)$ that double-crosses $\bar{r}(q)$ from above even if $\bar{r}(q)$ is convex, as in Figure 5a. This is possible when $\ln \bar{r}(q)$, instead of $\bar{r}(q)$, is concave as we can construct the linear $\ln \hat{r}_t(q)$ that double-crosses $\ln \bar{r}(q)$ from above, as shown in Figure 5b. Condition (2c') guarantees that under the proxy exponential contract $\hat{r}_t(q)$, the agent would voluntarily participate and choose $a = a^0$, and Lemma 3 with the double-crossing between $\hat{r}_t(q)$ and $\bar{r}(q)$ justifies the first-order approach in this case. Since we rely on double-crossing of log-utilities, we assume $u(s) > 0$ for $\forall s$.

Note that if $\bar{r}(q)$ is concave (i.e., (3c) in Proposition 1), then $\ln \bar{r}(q)$ becomes concave (i.e., (3c') in Proposition 2). Thus, (3c') is weaker than (3c). In turn, (2c') in Proposition 2 that $c(a)$ is convex in $M(a; t)$ is usually stronger than (2c) in Proposition 1 that $c(a)$ is convex in $m(a)$. Based on this trade-off between Proposition 1 and 2, it turns out that in Example 1, Proposition 2 allows us to justify the first-order approach even in the case of $\frac{1}{2} < r < 1$ when $h(a)$ is concave enough,²⁵ as shown in the next Example 2.

²⁵As we relax the concavity of $\bar{r}(q)$ condition (i.e., (2c) in Proposition 1) to include cases where $\bar{r}(q)$ is convex, we impose stronger concavity condition on $h(a)$ that stems from (2c') in Proposition 2.

Example 2 Consider the same one-signal principal-agent problem as Example 1, except here we have $\frac{1}{2} < r < 1$ so $\bar{r}(q)$ is convex. From

$$g(q|a) = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp \left(-\frac{1}{h(a)} \left(\frac{[h(a^o)]^2}{h'(a^o)} q + h(a^o) \right) \right), \quad (21)$$

its moment generating function is given by

$$M(a; t) = \frac{h(a^o)^2}{h(a^o)^2 - h'(a^o)h(a)t} \exp \left(-t \frac{h'(a^o)}{h(a^o)} \right), \quad (22)$$

which is defined when $t < \frac{h(a^o)^2}{h(a)h'(a^o)}$. If $c(a)$ becomes convex in $M(a; t)$ in (22) for all feasible $t > 0$, then we satisfy (ii) of condition (2c'). For example, $M(a; t)$ becomes concave in a when $h(a)$ is concave enough, i.e.,

$$h''(a) < -\frac{2th'(a^o)h'(a)^2}{h(a^o)^2 - h'(a^o)h(a)t}, \quad (23)$$

and in this case $c(a)$ is convex in $M(a; t)$. With $\bar{r}(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$, $\ln \bar{r}(q) = \frac{r}{1-r} \ln(\lambda + \mu q) - \ln r$ becomes concave in q , satisfying condition (3c'). For (i) of condition (2c'), we can set t as

$$\frac{h(a^o)}{h'(a^o)} > t = \frac{h(a^o)^2 c'(a^o)}{[\bar{U} + c(a^o)]h'(a^o)^2 + h(a^o)h'(a^o)c'(a^o)} > 0$$

that satisfies $\frac{c'(a^o)}{M'(a^o; t)} M(a^o; t) - c(a^o) = \bar{U}$. Therefore, we conclude that we can justify the first-order approach in Example 1 even when $\bar{r}(q)$ is convex, if $h(a)$ satisfies the condition in (23). Examples 1 and 2 illustrate that our approach is completely different from the previous literature.²⁶

4.2 When the Information Variable, $\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)}$, Is Unbounded Below

Our approach in Propositions 1 and 2 can be similarly applied even in cases where the agent's limited liability constraint binds. Whether the agent's limited liability constraint is binding for some x at the optimum mainly depends on technology $f(x|a)$ and the subsistence wage level \underline{s} . Especially when the density function of the signals has its information variable, $\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)}$ which is not bounded below, i.e., $\tilde{q} \in (-\infty, \bar{q})$,²⁷ we need to have \underline{s} bounded below, i.e., $\underline{s} > -\infty$, to guarantee the existence of the optimal contract. Then, as shown in equation (3), the agent's limited liability must be binding for some x under the optimal contract.

²⁶The detailed derivation of Example 2 is provided in Appendix.

²⁷This is the case for many familiar density functions of the signals (e.g., normal distribution, gamma distribution, Chi-square distribution, etc.).

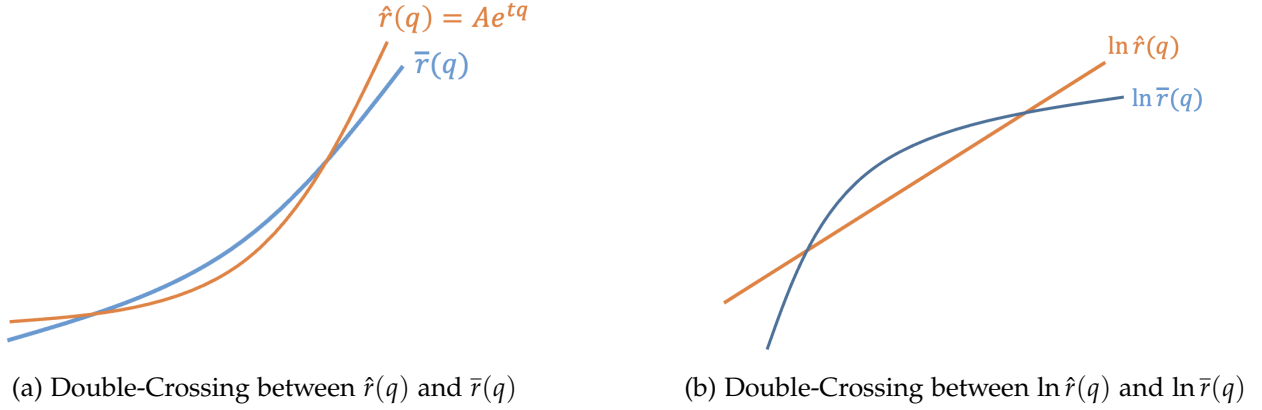


Figure 5: When $\bar{r}(q)$ is Convex while $\ln \bar{r}(q)$ is Concave

In this case, in order to verify the double crossing property between $r(q)$ and a possible proxy contract $\hat{r}(q)$, we introduce an arbitrary contract $\hat{s}_t(\mathbf{x}), t > 0$, with which the agent's indirect utility has an exponential form such as $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = A \cdot e^{tq} + B$, where A, B , and $t > 0$ are to be set to satisfy both the participation and the relaxed incentive constraints as well as the agent's limited liability constraint, i.e., $\hat{s}_t(\mathbf{x}) \geq \underline{s}, \forall \mathbf{x}$.

Then, we have the following proposition.

Proposition 3 *Given that the information variable, $\tilde{q} \equiv \frac{f_a(\bar{\mathbf{x}}|a^0)}{f(\bar{\mathbf{x}}|a^0)}$, is unbounded below, if, for any given a^0 ,*

(1a) $\frac{g(q|a)}{g(q|a^0)}$ *is convex in $q = \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)}$ for all a ,*

(2b) (i) *there exists $t > 0$ such that $\frac{c'(a^0)}{M'(a^0;t)}M(a^0;t) - c(a^0) \leq \bar{U} - u(\underline{s})$, and (ii) $c(a)$ is convex in $M(a;t)$ for such t , and*

(3b) $\ln(\bar{r}(q) - u(\underline{s}))$ *is concave in q for $\forall q \geq q_c$,*

then the first-order approach is justified.

The conditions in Proposition 3 sufficiently guarantee conditions ((1a),(2a)) in Lemma 3 in the case where the information variable, $\tilde{q} \equiv \frac{f_a(\bar{\mathbf{x}}|a^0)}{f(\bar{\mathbf{x}}|a^0)}$, is unbounded below, and thus the agent's limited liability constraint is binding for some \mathbf{x} at the optimum. Especially, conditions (2b) and (3b) are given as sufficient conditions for condition (2a) in Lemma 3 (i.e., the double crossing property between $r(q)$ and $\hat{r}(q)$) in this case.

In Section 4.1, to guarantee the double crossing property between $r(q)$ and $\hat{r}(q)$ in cases where the agent's limited liability constraint is not binding at all at the optimum, Proposition 1 requires $r(q)$ in (6) be concave in q (i.e., (3c) in Proposition 1) and Proposition 2 requires $\ln r(q)$ be concave in q instead (i.e., (3c') in Proposition 2). In Propositions 1 and 2, conditions on $r(q)$ are equivalent to those on $\bar{r}(q)$ as $r(q) = \bar{r}(q)$ for $\forall q$, or $q_c = -\infty$ in those cases.²⁸ However, in cases where the limited liability constraint binds for $q < q_c$ as in (6), $r(q)$ cannot

²⁸Proposition 1 and 2 differ in which forms of proxy contracts to be used (e.g., Proposition 1 uses a linear $\hat{r}(q)$ while Proposition 2

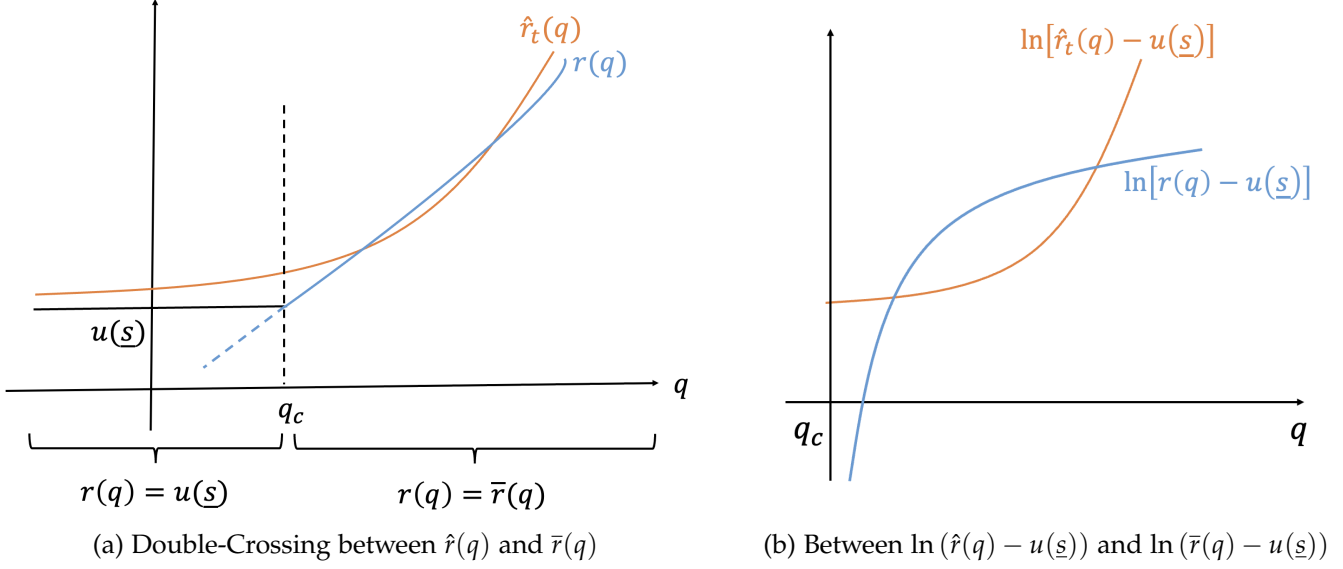


Figure 6: When the limited liability binds at $q \leq q_c$

be concave around q_c , as seen in Figure 6. Figure 6a illustrates that regardless of whether $\bar{r}(q)$ is convex or concave for $q \geq q_c$, $r(q)$ cannot be concave around q_c , where the limited liability starts to bind.

Therefore, we require that the agent's indirect utility given $s^o(\mathbf{x})$ before constrained by the limited liability constraint, i.e., $\bar{r}(q)$ in equation (5), to satisfy $\ln(\bar{r}(q) - u(\underline{s}))$ is concave in q for $q \geq q_c$ (i.e., condition (3b)), and introduce an arbitrary proxy contract $\hat{s}_t(\mathbf{x})$, $t > 0$ such that $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = Ae^{tq} + B$, which is convex. In fact, condition (i) of (2b) ensures that $B \geq u(\underline{s})$,²⁹ thereby given our exponential-affine proxy contract $\hat{r}_t(q)$, $\ln(\hat{r}_t(q) - u(\underline{s}))$ becomes convex for $q \geq q_c$, allowing itself to double-cross the concave $\ln(\bar{r}(q) - u(\underline{s}))$ from above. In turn, our proxy contract $\hat{r}_t(q)$ must double-cross $r(q)$ from above as in Figure 6b, satisfying (2a) of Lemma 3. Since $\ln(\bar{r}(q) - u(\underline{s}))$ becomes concave if $\bar{r}(q)$ is concave, (3b) of Proposition 3 is weaker than (3c) of Proposition 1.³⁰

Condition (ii) of (2b) (i.e., the convexity of $c(a)$ in $M(a; t)$, $t > 0$) guarantees that the agent will choose a^o given $\hat{s}_t(\mathbf{x})$, that is,

$$U(\hat{s}_t(\cdot), a) \equiv \int u(\hat{s}_t(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - c(a) \leq U(\hat{s}_t(\cdot), a^o) \equiv \int u(\hat{s}_t(\mathbf{x}))f(\mathbf{x}|a^o)d\mathbf{x} - c(a^o) = U^o, \quad \forall a. \quad (24)$$

uses an exponential $\hat{r}(q)$). Therefore, Proposition 2 relaxes the condition on concavity of $\bar{r}(q)$ (i.e., (3c') in Proposition 2 is weaker than (3c) in Proposition 1) while imposing the stronger condition on $c(a)$ (i.e., (2c') in Proposition 2 is stronger than (2c) in Proposition 1).

²⁹ $B \geq u(\underline{s})$ also guarantees that our proxy contract $\hat{r}_t(q)$ (or $\hat{s}_t(x)$) satisfies the limited liability constraint: $\hat{r}_t(q) \geq u(\underline{s})$ or $\hat{s}_t(x) \geq \underline{s}$ for all q or x .

³⁰Jewitt (1988) (i.e., Theorem 1) and Jung and Kim (2015) (i.e., Proposition 7) also assume that $\bar{r}(q)$ is concave in their justification of the first-order approach. Our (3b) in Proposition 3 is a weaker condition.

Therefore, Lemma 3 implies:

$$U(s^o(\cdot), a) \equiv \int r(q)g(q|a)dq - c(a) \leq \int \hat{r}_t(q)g(q|a)dq - c(a) = U(\hat{s}_t(\cdot), a), \quad \forall a. \quad (25)$$

The inequality in (25) comes from that both $r(q)$ and $\hat{r}_t(q)$ satisfy the participation and the relaxed incentive constraints, and $r(q)$ crosses $\hat{r}_t(q)$ twice starting from below. By combining equation (7), equation (24), and equation (25), we finally have

$$U(s^o(\cdot), a) \leq U(s^o(\cdot), a^o) = U^o, \quad \forall a,$$

which justifies the first-order approach.

One thing to note is that, even though condition (2b) looks a bit complicated, it can be reduced to a much simple form especially when there exists $t > 0$ such that the agent's cost function $c(a)$ is linear in $M(a; t)$, i.e., $c(a) = \alpha M(a; t) + \beta$. In this case, (2b) holds if $\bar{U} - u(\underline{s}) \geq -\beta$, which always holds for $\beta > 0$. In Appendix, we provide Proposition 5 that is similar to Proposition 3 in nature with a different condition: (2b') in Proposition 5 is easier to check than (i) and (ii) of (2b) in Proposition 3.³¹

The following two examples illustrate how the conditions in Proposition 3 can be applied to the canonical agency problems in which the agent's limited liability constraint is binding for some x at the optimum (i.e., $\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)}$ is unbounded below).

Example 3 Consider a one-signal principal-agent problem, $\{\pi(x), u(s), f(x|a), c(a), \bar{U}, \underline{s}\}$, where $\pi(x) = x$ and $u(s) = \frac{1}{r}s^r$, $r < 1$. The agent's cost function is given by $c(a) = D(e^{ka} - 1)$, $D > 0$, $k > 0$, and assume that the signal generating function has a simple additive form such as $\tilde{x} = a + \tilde{\theta}$, $\tilde{\theta} \sim N(0, \sigma^2)$. Then,

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}. \quad (26)$$

Since, given a^o ,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \frac{\tilde{x} - a^o}{\sigma^2},$$

we have

$$\tilde{q}|a \sim N\left(\frac{a - a^o}{\sigma^2}, \frac{1}{\sigma^2}\right).$$

Therefore,

$$g(q|a) = \frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{2} \left(q - \frac{a - a^o}{\sigma^2}\right)^2\right). \quad (27)$$

³¹The reason we keep Proposition 3 instead of 5 in Appendix is that (2b) in Proposition 3 looks similar to (2c') in Proposition 2 so that we can keep consistency of our Propositions 1, 2, and 3.

Since $\tilde{x} \in (-\infty, \infty)$, \tilde{q} would be unbounded below, and the agent's limited liability constraint must be binding for some low values of \tilde{q} (i.e., for some low values of \tilde{x}).

From equation (27), we have

$$\frac{g(q|a)}{g(q|a^0)} = \exp \left((a - a^0)q - \frac{(a - a^0)^2}{2\sigma^2} \right).$$

Therefore, one easily sees that condition (1a) is satisfied. Also, from (5), $\bar{r}(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$ becomes concave in q when $r \leq \frac{1}{2}$, which satisfies condition (3b) in Proposition 3. Even for $r > \frac{1}{2}$, we know that $\ln(\bar{r}(q) - u(\underline{s}))$ becomes concave in $q \geq q_c$ because³²

$$\frac{d^2}{dq^2} \ln(\bar{r}(q) - u(\underline{s})) = \frac{\bar{r}'(q)}{\bar{r}(q) - u(\underline{s})} \left(\frac{\bar{r}''(q)}{\bar{r}'(q)} - \frac{\bar{r}'(q)}{\bar{r}(q) - u(\underline{s})} \right), \quad (28)$$

where (i) $\bar{r}'(q) > 0$ and (ii) $\bar{r}(q) \geq u(\underline{s}) \geq 0$ for $\forall q \geq q_c$ ³³ lead to

$$\begin{aligned} \frac{\bar{r}''(q)}{\bar{r}'(q)} - \frac{\bar{r}'(q)}{\bar{r}(q) - u(\underline{s})} &= \frac{\bar{r}''(q)}{\bar{r}'(q)} - \frac{\bar{r}'(q)}{\bar{r}(q)} \left(1 - \frac{u(\underline{s})}{\bar{r}(q)} \right)^{-1} \\ &\leq \frac{\bar{r}''(q)}{\bar{r}'(q)} - \frac{\bar{r}'(q)}{\bar{r}(q)} = \frac{2r - 1}{(1 - r)(\lambda + \mu q)} - \frac{r}{(1 - r)(\lambda + \mu q)} = -\frac{1}{\lambda + \mu q} < 0. \end{aligned} \quad (29)$$

Therefore, regardless of the value of $\forall r < 1$, (3b) in Proposition 3 always holds. Furthermore, since $c(a) = D(e^{ka} - 1)$, $D > 0$, $k > 0$, by picking up $t = k\sigma^2$, we have

$$M(a; t = k\sigma^2) = \int e^{k\sigma^2 q} g(q|a) dq = \exp \left(k(a - a^0) + \frac{k^2 \sigma^2}{2} \right) = K \cdot e^{ka},$$

where $K \equiv \exp \left(\frac{k^2 \sigma^2}{2} - ka^0 \right)$. Therefore, the agent's cost function can be represented as

$$c(a) = \frac{D}{K} M(a; t = k\sigma^2) - D,$$

which implies that condition (2b) holds if $D \leq \bar{U} - u(\underline{s})$.

This example clearly shows the advantage of the set of conditions in Proposition 3 over the existing sets of conditions. Using the first-order approach in the above example cannot be justified by any of the existing sets of conditions. First, it is easy to see that neither $f(x|a)$ in equation (26) nor $g(q|a)$ in equation (27) satisfies any

³²We can see trivially from (28) that if $\bar{r}(q)$ is concave (e.g., when $r < \frac{1}{2}$), then $\ln(\bar{r}(q) - u(\underline{s}))$ becomes concave.

³³Since the optimal contract s satisfies $u'(s)^{-1} = s^{1-r} = \lambda + \mu q$ for $q \geq q_c$ and we assume the marginal utility of agent is positive (i.e., $u'(s) > 0$ for $\forall s$), $\lambda + \mu q > 0$ for $q \geq q_c$. Thus, $\lambda + \mu q_c \geq 0$ holds and $\underline{s}^{1-r} \geq 0$, implying $u(\underline{s}) \geq 0$ when $\frac{1}{2} < r < 1$. Otherwise, the optimal contract cannot exist.

of the the CDF-type conditions. Thus, the Mirrless-Rogerson conditions or any extension of those conditions (e.g., the GCDFC by Sinclair-Desgagné (1994), the CISP condition by Conlon (2009), and the CDFCL by Jung and Kim (2015)) cannot be used for justifying the first-order approach in this case. Furthermore, the Jewitt (1988) conditions and the extension by Jung and Kim (2015) cannot be also used for justifying the first-order approach in this case: both Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015) (i) focus on cases where the agent's limited liability is not binding at the optimum; (ii) require that $\bar{r}(q)$ be concave in q .³⁴ In contrast, using the first-order approach in this case can be actually justified by the conditions in Proposition 3 as long as $\bar{U} - U(\underline{s}) \geq D$, (i) even if the agent's limited liability constraint is binding for $q \leq q_c$; (ii) regardless of whether $\bar{r}(q)$ is concave or convex for $q \geq q_c$.

Actually, the simple signal generating function with a linear form such as $\tilde{x} = a + \tilde{\theta}$, where $\tilde{\theta}$ is normally distributed, has not been able to be used in a wide range of principal-agent problems, because the first-order approach was not justified under this simple signal generating function by the previous literature, which has been a big obstacle in applying the agency theory to various economic problems. What Example 3 illustrates is that the first-order approach can be validly adopted in the principal-agent problem in which the signals for the agent's effort are normally distributed if the agent's cost of effort has a somewhat convex functional form such as an exponential form.

Example 4 Consider a one-signal principal-agent problem, $\{\pi(x), u(s), f(x|a), c(a), \bar{U}, \underline{s}\}$, where $\pi(x) = x$, $u(s) = \frac{1}{r}s^r$, $r < 1$, $c(a) = ka^\eta$, $k > 0$, $\eta \geq 1$, and $\tilde{x} \in (0, \infty)$ has the gamma distribution with shape parameter a , i.e.,

$$f(x|a) = \frac{x^{a-1}\beta^{-a}}{\Gamma(a)}e^{-\frac{x}{\beta}}. \quad (30)$$

Since, given a^0 ,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^0)}{f(\tilde{x}|a^0)} = \log \tilde{x} - K,$$

where $K \equiv \log \beta + \frac{\Gamma'(a^0)}{\Gamma(a^0)}$, \tilde{q} is unbounded below. Thus, the agent's limited liability constraint must be binding for some low values of \tilde{q} (i.e., for some low value of \tilde{x}). We have

$$g(q|a) = \frac{\beta^{-a}}{\Gamma(a)} \exp \left(a(q + K) - \frac{1}{\beta}e^{q+K} \right). \quad (31)$$

Thus, we can see from (31) that condition (1a) in Proposition 3 is satisfied. Also, from Example 3, $\ln(\bar{r}(q) - u(\underline{s}))$ becomes concave with $\bar{r}(q) = \frac{1}{r}(\lambda + \mu q)^{\frac{r}{1-r}}$ for any $r < 1$, satisfying condition (3b) in Proposition 3.

³⁴In those cases where the agent's limited liability constraint is not binding at the optimum, $r(q) = \bar{r}(q)$ for $\forall q$ as in Section 4.1.

Since the agent's cost function is given by $c(a) = ka^\eta$, $k > 0$, by picking up $t = 1$, we have

$$M(a; t = 1) = \int e^{1 \cdot q} g(q|a) dq = e^{-K} \int x f(x|a) dx = \beta e^{-K} a.$$

Therefore, the agent's cost function can be represented as

$$c(a) = \frac{ke^{\eta K}}{\beta^\eta} M(a; t = 1)^\eta,$$

which implies that condition (2b) holds when $(\eta - 1)c(a^0) < \bar{U} - u(\underline{s})$. In the case of $\eta = 1$, this condition must hold as $\bar{U} > u(\underline{s})$.

What Example 4 shows is even more striking. Unlike Example 3, justifying the first-order approach does not even need a convex cost function of effort in the case where $f(x|a)$ is a gamma density function with shape parameter a . Like Example 3, the first-order approach for the principal-agent problem in the above example cannot be justified by any of the existing sets of conditions. One can see that neither $f(x|a)$ in equation (30) nor $g(q|a)$ in equation (31) satisfies any of the CDF-type conditions. Thus, the Mirrlees-Rogerson conditions and any extensions of those conditions (e.g., GCDFC by Sinclair-Desgagné (1994), CISP condition by Conlon (2009), and CDFCL by Jung and Kim (2015)) cannot be used for justifying the first-order approach in this case. Furthermore, since, for any given a^0 , $\tilde{q} \equiv \frac{f_a(\tilde{x}|a^0)}{f(\tilde{x}|a^0)} = \log \tilde{x} - K \in (-\infty, \infty)$, which is unbounded below, the agent's limited liability constraint must be binding for low values of \tilde{x} at the optimum. Thus, by the same reason as in Example 3, Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015) cannot be also used for justifying the first-order approach in this case. However, as shown, using the first-order approach in this case can be actually justified by our new conditions in proposition 3.

5 The Comparison with the Existing Conditions

As explained, our conditions in Propositions 1, 2, and 3 are different from the previous literature's conditions in that they are directly derived to satisfy equation (4), whereas the existing conditions were derived to make the agent's expected monetary utility given $s^0(\mathbf{x})$ (i.e., $R(a) \equiv \int u(s^0(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$) concave in a .

As previously explained, the existing results can be categorized into two groups, the results that contain the CDF-type conditions (i.e., Mirrlees-Rogerson's CDFC, Sinclair-Desgagné's GCDFC, Conlon's CISP condition, and Jung and Kim's CDFCL), and the results that do not contain any CDF-type condition but instead contain a restriction on the agent's utility function (i.e., Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015)). The CDF-type conditions on $f(\mathbf{x}|a)$ in the first group, which were given to make

$R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ concave in a for any “increasing” $u(s^o(\mathbf{x}))$, can be applied even to the case in which the agent’s limited liability constraint is binding for some \mathbf{x} at the optimum because $u(s^o(\mathbf{x}))$ is increasing even in those cases. However, these conditions have a limitation in that they are hardly satisfied by most familiar density functions.

On the other hand, the conditions on $f(\mathbf{x}|a)$ that appear in the results of the second group were given to make $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ concave in a for any “increasing concave” $u(s^o(\mathbf{x}))$. These conditions must be weaker than the above CDF-type conditions because they require that the concavity of $R(a)$ be satisfied for a smaller set of $u(s)$.³⁵ However, the results in the second group have other limitations in that they cannot be used for cases in which the signals’ density function has its information variable, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$, unbounded below, and thus the agent’s limited liability constraint is binding for some \mathbf{x} at the optimum. Unfortunately, many familiar probability density functions (e.g., normal, gamma, etc.) belong to this case. In fact, to match with for any “increasing concave” $u(s^o(\mathbf{x}))$, the results in the second group contain another restriction on the agent’s utility function $u(s)$ such that $u(s^o(\mathbf{x})) \equiv r(q)$ is concave in q . However, this restriction cannot be satisfied in this case due to the agent’s binding limited liability constraint.

As a result, it is clear that our conditions in Proposition 3, which can be used even for the case in which the agent’s limited liability constraint is binding for some \mathbf{x} at the optimum,³⁶ have the advantage over the existing results. When the agent’s limited liability constraint is binding for some \mathbf{x} at the optimum, among the existing results, only the results that contain the CDF-type conditions can be used. However, the CDF-type conditions are too restrictive to be satisfied by most familiar density functions of the signals. In contrast, our conditions in Proposition 3 (especially condition (1a)) can be satisfied by the wide range of density functions including the normal and the gamma density functions, as shown in Examples 3 and 4.

On the other hand, the advantage of our conditions in Propositions 1 and 2, which can be used only for the case in which the agent’s limited liability constraint is not binding at all, over the existing results needs to be explained more carefully. This is because the results which do not contain the CDF-type conditions (i.e., Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015)) can still be used for this case.³⁷ Thus, it will be interesting to compare our conditions in Propositions 1 and 2 with the conditions in Proposition 7 in Jung and Kim (2015) which are up to now the most general conditions in the second group.³⁸ In Section 5, we first investigate the statistical implications of condition (1a) and then, based on these statistical implications,

³⁵These conditions are satisfied by some familiar density functions (e.g., Chi-square, Poisson etc.).

³⁶Note that another advantage of Proposition 3 over the previous literature is that we can have a possibly convex $\bar{r}(q)$ for $q \geq q_c$.

³⁷We already discussed that (3c’) in Proposition 2 is weaker than (3c) in Proposition 1 (which is equivalent to Theorem 1 in Jewitt (1988) and Proposition 7 in Jung and Kim (2015)’s condition on the agent’s indirect monetary utility $\bar{r}(q)$).

³⁸Jung and Kim (2015) show that their conditions in Proposition 7 are more general than the conditions in Theorem 1 in Jewitt (1988).

compare our new conditions in Propositions 1 and 2 with Proposition 7 in Jung and Kim (2015). In addition, we illustrate the fundamental differences between our conditions directly derived to satisfy equation (4) and the existing conditions derived to make $R(a) \equiv \int u(s^o(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ concave in a .

Proposition 7 in Jung and Kim (2015) states that, if, for any given a^o ,

- (1J-1) $\int^z G(q|a)dq$ is convex in a for all z ,
- (1J-2) $m(a) \equiv \int qg(q|a)dq$ is concave in a ,³⁹ and
- (2J) $r(q)(= \bar{r}(q))$ is concave in q ,

then the first-order approach is justified.

Notice that (2J) above is identical to (3c) in Proposition 1. Thus, to compare our conditions in Proposition 1 with the conditions in Proposition 7 in Jung and Kim (2015), comparing conditions ((1a),(2c)) with conditions ((1J-1),(1J-2)), given that (2J) (or equivalently (3c)) is satisfied, will be enough. To put forth the conclusion first, the direct comparison between ((1a),(2c)) and ((1J-1),(1J-2)) is not possible because one does not imply the other. This is basically because condition ((1a),(2c)) are derived to directly satisfy equation (4), whereas conditions ((1J-1),(1J-2)) are derived to make $R(a)$ concave in a . However, an indirect comparison between ((1a),(2c)) and ((1J-1),(1J-2)) is possible. Conditions ((1J-1),(1J-2)) are, in general, hard to verify. So our strategy here is as follows:

Step 1: First, we introduce another set of conditions called TP_3 -based conditions.

Step 2: We show that $(TP_3, (1J-2))$ conditions are sufficient for ((1a),(2c)) and easier to verify.

Step 3: We show that $(TP_3, (1J-2))$ conditions are sufficient for ((1J-1),(1J-2)) and easier to verify.

And we explain ((1a),(2c)) and ((1J-1),(1J-2)) do not imply each other. The TP_3 -based conditions are, for any given a^o , given by $(TP_3, (1J-2))$ as follows:

- (TP_3) $g(q|a)$ is totally positive of degree 3 (i.e., TP_3), and
- (1J-2) $m(a) \equiv \int qg(q|a)dq$ is concave in a .⁴⁰

Definition 1 (Total Positivity): A function $f(x, a)$, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, is totally positive of degree n (i.e., TP_n) if, for

³⁹Condition (1J-2) is generally implied by (1J-1) when z goes to infinity. But the reason we list (1J-2) as a separate condition is because $\int G(q|a)dq$ sometimes may not exist.

⁴⁰Since TP_3 -based conditions are given to sufficiently guarantee ((1J-1),(1J-2)), the first-order approach can be justified if conditions (TP_3) , (1J-2), and (2J) are satisfied.

every $x_1 < x_2 < \dots < x_n$ and $a_1 < a_2 < \dots < a_n$,

$$T(f, k) \equiv \begin{vmatrix} f(x_1, a_1) & \dots & f(x_1, a_k) \\ \vdots & \vdots & \vdots \\ f(x_k, a_1) & \dots & f(x_k, a_k) \end{vmatrix} \geq 0, \quad \text{for all } k = 1, 2, \dots, n.^{41}$$

To show that conditions ((1a),(2c)) are more general than the above TP_3 -based conditions, we first consider the case in which there is a single signal, i.e., $x \in \mathbb{R}$, and $f(x|a)$ satisfies the MLRP, and thus there is a 1:1 relation between \tilde{q} and \tilde{x} . Then, we consider the case in which either there are multiple signals, i.e., $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, or $f(x|a)$, $x \in \mathbb{R}$, does not satisfy the MLRP, and thus there is no 1:1 relation between \tilde{q} and \tilde{x} .

5.1 When $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP

When there is a single signal, i.e., $x \in \mathbb{R}$, and its density function, $f(x|a)$, satisfies the MLRP, there is a 1:1 relation between \tilde{q} and \tilde{x} for any given a^0 . Then, as previously shown,

$$G(q|a) = F(x|a) \quad \text{for all } a, \quad (32)$$

where x solves $\frac{f_a(x|a^0)}{f(x|a^0)} = q$. We thus have $g(q|a)Q'_{a^0}(x) = f(x|a)$, where $Q'_{a^0}(x) \equiv \frac{dQ_{a^0}(x)}{dx}$ is independent of a . Thus, we have

$$\frac{g(q|a)}{g(q|a^0)} = \frac{f(x|a)}{f(x|a^0)} \quad \text{and} \quad \frac{g_a(q|a)}{g(q|a)} = \frac{f_a(x|a)}{f(x|a)} \quad \text{for all } a. \quad (33)$$

From (33), we obtain an interesting result that, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, (1a) reduces to

$$(1d) \quad \frac{f(x|a)}{f(x|a^0)} \text{ is convex in } q = \frac{f_a(x|a^0)}{f(x|a^0)} = \frac{g_a(q|a^0)}{g(q|a^0)} \text{ for any given } a^0 \text{ and for all } a.$$

In other words, if $f(x|a)$ satisfies MLRP for $x \in \mathbb{R}$, condition (1a) can be replaced by condition (1d), which is much easier to verify because it does not require us to explicitly calculate $g(q|a)$ from $f(x|a)$. For instance in Example 3, note that, as $\tilde{x} = a + \tilde{\theta}$, $\tilde{\theta} \sim N(0, \sigma^2)$, $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP. Thus, condition (1a) can be easily verified by using condition (1d) without even calculating $g(q|a)$ from $f(x|a)$. From equation (26), one can derive that

$$q = \frac{f_a(x|a^0)}{f(x|a^0)} = \frac{x - a^0}{\sigma^2},$$

and

$$\frac{f(x|a)}{f(x|a^0)} = \exp \left(-\frac{1}{2\sigma^2} (2(a^0 - a)x + a^2 - (a^0)^2) \right).$$

⁴¹For a detailed explanation of “total positivity”, see Karlin (1968)

Since q is linear in x , and $\frac{f(x|a)}{f(x|a^0)}$ is convex in x for any given a^0 , it is easy to confirm that condition (1a) is satisfied.

Lemma 4: Given two functions, $\phi(x)$ and $\psi(x)$, $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$, where $\phi(x)$ is increasing in x ,

$$\begin{vmatrix} 1 & \phi(x_1) & \psi(x_1) \\ 1 & \phi(x_2) & \psi(x_2) \\ 1 & \phi(x_3) & \psi(x_3) \end{vmatrix} \begin{matrix} \geq \\ \leq \end{matrix} 0, \text{ for every } x_1 < x_2 < x_3,$$

if and only if $\psi(x)$ is convex (concave) in $\phi(x)$.

Using Lemma 4, we derive the following lemma and corollary.

Lemma 5: Given $x \in \mathbb{R}$, $f(x|a)$ is TP_3 if and only if

- (i) $f(x|a)$ satisfies the MLRP, and
- (ii) $\frac{f(x|a)}{f(x|a^0)}$ is convex in $q = \frac{f_a(x|a^0)}{f(x|a^0)}$ for any given a^0 and for all a (i.e., condition (1d)).

Corollary 1: Given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^0 .

One thing to note is that Corollary 1 should not be read as “Given that $x \in \mathbb{R}$, $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^0 ”. The statement, “Given that $x \in \mathbb{R}$, $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^0 ”, is true only when $f(x|a)$ satisfies the MLRP. This is because, although, as can be seen from equation (33), the MLRP for $f(x|a)$ (i.e., $T(f, 2) \geq 0$ for every $x_1 < x_2$ and $a_1 < a_2$) implies the MLRP for $g(q|a)$ (i.e., $T(g, 2) \geq 0$ for every $q_1 < q_2$ and $a_1 < a_2$ for any given a^0), the converse is not always true.

For instance, consider a class of density functions which is a convex mixture of two probability density functions $p_H(x)$ and $p_L(x)$ such that

$$f(x|a) = \alpha(a)p_H(x) + (1 - \alpha(a))p_L(x),$$

where $p_H(x) = -6x^2 + 6x$ and $p_L(x) = 1$ with $x \in [0, 1]$, and $\alpha(a) \in [0, 1]$ is increasing in a .⁴² Then,

$$f(x|a) = \alpha(a)(-6x^2 + 6x - 1) + 1,$$

⁴²This example is from Jung and Kim (2015).

and

$$\frac{f_a(x|a)}{f(x|a)} = \frac{(-6x^2 + 6x - 1)\alpha'(a)}{1 + (-6x^2 + 6x - 1)\alpha(a)}.$$

Thus, it is easy to see that $f(x|a)$ does not satisfy the MLRP.

Define, for any given a^o ,

$$\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)} = \frac{(-6\tilde{x}^2 + 6\tilde{x} - 1)\alpha'(a^o)}{1 + (-6\tilde{x}^2 + 6\tilde{x} - 1)\alpha(a^o)}.$$

Then, \tilde{q} is distributed with support $[\frac{\alpha'(a^o)}{\alpha(a^o)-1}, \frac{\alpha'(a^o)}{\alpha(a^o)+2}]$. Thus,

$$G(q|a) = \Pr[\tilde{q} \leq q|a] = \Pr\left[-6\tilde{x}^2 + 6\tilde{x} - 1 \leq \frac{q}{\alpha'(a^o) - \alpha(a^o)q} \middle| a\right].$$

If $x_1(q)$ and $x_2(q)$ be two roots of $-6x^2 + 6x - 1 = \frac{q}{\alpha'(a^o) - \alpha(a^o)q}$, where $x_1(q) \leq x_2(q)$, then $x_2(q) = 1 - x_1(q)$, where $x_1(q) \in [0, \frac{1}{2}]$ is increasing in q , whereas $x_2(q) \in [\frac{1}{2}, 1]$ is decreasing in q . Therefore,

$$G(q|a) = \Pr[\tilde{x} \leq x_1(q)|a] + \Pr[\tilde{x} \geq x_2(q)|a].$$

Since $f(x|a)$ is symmetric around $x = \frac{1}{2}$,

$$\Pr[\tilde{x} \leq x_1(q)|a] = \Pr[\tilde{x} \geq x_2(q)|a].$$

Thus,

$$G(q|a) = 2\Pr[\tilde{x} \leq x_1(q)|a] = 2F(x_1(q)|a),$$

and

$$g(q|a) = 2f(x_1(q)|a)x_1'(q).$$

Consequently, we have

$$\frac{g_a(q|a)}{g(q|a)} = \frac{f_a(x_1(q)|a)}{f(x_1(q)|a)}.$$

Since $\frac{f_a(x|a)}{f(x|a)}$ is increasing in $x \in [0, \frac{1}{2}]$, we finally have $\frac{g_a(q|a)}{g(q|a)}$ is increasing in q , indicating that $g(q|a)$ satisfies the MLRP.

Based on Lemma 5 and Corollary 1, we see that, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, the following three statements are equivalent.

(1a) For any given a^o , $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)} = \frac{g_a(q|a^o)}{g(q|a^o)}$ for all a .

(1d) For any given a^o , $\frac{f(x|a)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$ for all a
(i.e., $f(x|a)$ is TP_3 given the MLRP for $f(x|a)$).

(1e) For any given a^o , $\frac{g(q|a)}{g(q|a_t)}$ is convex in $\frac{g_a(q|a_t)}{g(q|a_t)}$ for all a, a_t
(i.e., $g(q|a)$ is TP_3 given the MLRP for $g(q|a)$).⁴³

Thus, given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, condition (1a) indicates that $g(q|a)$ is TP_3 for any given a^o .

Therefore, we have the following proposition.

Proposition 4 *Given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, conditions $((TP_3), (1J-2))$ imply conditions $((1a), (2c))$, but the converse is not true.*

As explained above, (1a) is equivalent to condition (TP_3) provided that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP. However, condition (2c) is obviously more general than condition (1J-2) because $c(a)$ is increasing and convex in a . In other words, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, our conditions $((1a), (2c))$ require that $g(q|a)$ be TP_3 for any given a^o (or equivalently $f(x|a)$ be TP_3) but do not necessarily require that $m(a) \equiv \int qg(q|a)dq$ be concave in a (i.e., (1J-2)).

To understand why conditions $((1a), (2c))$ are more general than conditions $((TP_3), (1J-2))$ intuitively, the following characteristics of a density function with TP_3 will be useful.

Lemma 6: *If a density function $f(x|a)$ is TP_3 , then, for any increasing concave function $u(x)$,*

$$u^*(a) \equiv \int u(x)f(x|a)dx \text{ is increasing concave in } \mu(a) \equiv \int xf(x|a)dx.$$

Lemma 6 shows one of the most interesting characteristics of a density function, $f(x|a)$, with TP_3 . When the density function satisfying TP_3 is combined with any increasing concave function, the expected value of that function is increasing concave in the density's mean value. As explained in Lemma 5 and Corollary 1, given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, condition (1a) is equivalent to that $g(q|a)$ is TP_3 for any given a^o . Thus, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP, another interpretation for the conditions in Proposition 1 is possible based on Lemma 6 as follows.

Denote

$$U(s^o(\cdot), a) \equiv \int r(q)g(q|a)dq - c(a) = R(a) - c(a) \equiv \xi(m(a)) - \phi(m(a)),$$

⁴³It is worth noting that (1a) is weaker than (1e) in general. This is because (1a) requires that (1e) hold only for $a_t = a^o$ but not for all a_t . However, (1a) and (1e) become equivalent when $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP.

where $m(a) \equiv \int qg(q|a)dq$, i.e., the mean value of the distribution $g(q|a)$. Then, (1a) and (3c)⁴⁴ in Proposition 1 imply that $R(a)$ is concave in $m(a)$ (i.e., $\xi'' \leq 0$) by Lemma 6. Furthermore, since $c(a)$ is convex in $m(a)$ (i.e., $\phi'' \geq 0$) by the condition (2c), conditions ((1a),(2c),(3c)) in Proposition 1 ensure that, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, $U(s^o(\cdot), a)$ is concave in $m(a)$, which sufficiently guarantees equation (4) since $m(a)$ is increasing in a . To justify the first-order approach, all the existing results were derived to make $U(s^o(\cdot), a)$ concave in a (to be more precise, $R(a)$ concave in a). However, the first-order approach can be more generally justified by showing that there exists an increasing function of a , such as $m(a)$, in which $U(s^o(\cdot), a)$ is concave. This is actually what the conditions in Proposition 1 entails.

We now explain why there is no direct comparison available between conditions ((1J-1),(1J-2)) and conditions ((1a),(2c)). First, note that our conditions ((1a),(2c)) do not imply conditions ((1J-1),(1J-2)). For instance, consider the case in Example 1 where $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP. Assume that $h(a)$ is not concave in a . Then, since $m(a) \equiv \int qg(q|a)dq = \frac{h'(a^o)}{[h(a^o)]^2}(h(a) - h(a^o))$, neither (1J-1) nor (1J-2) is satisfied. The concavity of $m(a) \equiv \int qg(q|a)dq$ is essential not only for (1J-2) itself but also for (1J-1). However, conditions ((1a),(2c)) will be satisfied as long as $c(a)$ is convex in $h(a)$.

Conditions ((1J-1),(1J-2)) do not imply conditions ((1a),(2c)), either. To see this, it is worth to note that conditions ((TP₃),(1J-2)) are sufficient but not necessary for conditions ((1J-1),(1J-2)). As shown in Jewitt (1988), conditions ((1J-1),(1J-2)) are necessary and sufficient for $R(a) \equiv \int r(q)g(q|a)dq$ to be increasing and concave in a for any increasing concave function $r(q)$. On the other hand, conditions ((TP₃),(1J-2)) are sufficient but not necessary for $R(a) \equiv \int r(q)g(q|a)dq$ to be increasing and concave in a for any increasing concave function $r(q)$. The sufficient part comes from that $R(a) \equiv \int r(q)g(q|a)dq \equiv \xi(m(a))$ is increasing concave in $m(a)$ by (TP₃), and $m(a)$ is concave in a by (1J-2). However, even if $R(a) \equiv \int r(q)g(q|a)dq \equiv \xi(m(a))$ is increasing concave in a for any increasing concave function $r(q)$ given that $m(a) \equiv \int qg(q|a)dq$ is concave in a (i.e., (1J-1) and (1J-2)), it does not necessarily mean that $R(a)$ is increasing concave in $m(a)$, i.e., $\xi'' \leq 0$. Furthermore, even if $R(a)$ is increasing concave in $m(a)$ for any increasing concave function $r(q)$, $g(q|a)$ may not be always TP₃. Therefore, condition (1J-1) is more general than condition (TP₃) given that condition (1J-2) is satisfied.⁴⁵ This tells that conditions ((1J-1),(1J-2)) do not imply conditions ((1a),(2c)) because (1a) is equivalent to (TP₃) provided that $x \in \mathbb{R}$ and $f(x|a)$ satisfies the MLRP. Consequently, there is no relation of inclusion between conditions ((1J-1),(1J-2)) and conditions((1a),(2c)).

In fact, Figure 7 below demonstrates the relation among ((1a),(2c)), ((1J-1),(1J-2)), and ((TP₃),(1J-2)) more clearly. Given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, ((TP₃),(1J-2)) is equivalent to ((1a),(1J-2)) due to Lemma

⁴⁴Since the agent's limited liability is not binding at the optimum, $r(q) = \bar{r}(q)$ for $\forall q$.

⁴⁵Without condition (1J-2) being satisfied, condition (TP₃) does not always imply condition (1J-1).

5. In turn, $((1a), (1J-2))$ implies $((1a), (2c))$ because the concave $m(a)$ implies that our $c(a)$ is convex in $m(a)$. Also $((TP_3), (1J-2))$ implies $((1J-1), (1J-2))$ from Lemma 6. Still $((1J-1), (1J-2))$ and $((1a), (2c))$ do not include each other.

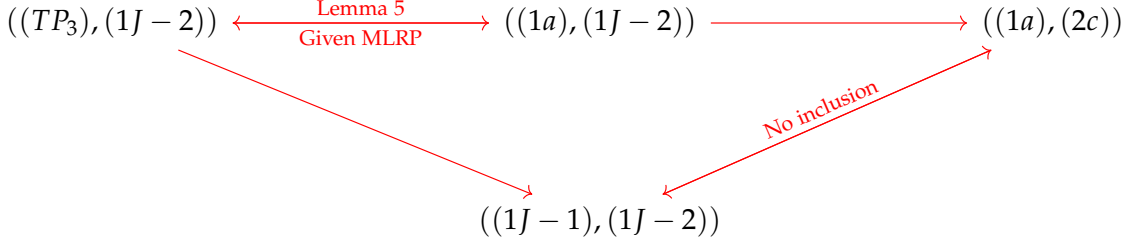


Figure 7: Relation Diagram between Conditions

As a result, direct comparison between our conditions $((1a), (2c))$ and conditions $((1J-1), (1J-2))$ is not possible. However, we clearly observe our conditions $((1a), (2c))$ are at least more general than TP_3 -based conditions, $((TP_3), (1J-2))$, when $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP.

5.2 When $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, or $f(x|a)$, $x \in \mathbb{R}$, does not satisfy MLRP

When there are multiple signals, i.e., $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, or density function $f(x|a)$ does not satisfy the MLRP even if $x \in \mathbb{R}$, there is no 1:1 relation between \tilde{q} and $\tilde{\mathbf{x}}$. Thus, some of the results that are derived when there is a 1:1 relation between \tilde{q} and \tilde{x} in the previous subsection may not hold. However, the main result in the previous subsection still holds even in this case. That is, although there is no relation of inclusion between our conditions $((1a), (2c))$ in Proposition 1 and conditions $((1J-1), (1J-2))$ of Jung and Kim (2015), the conditions $((1a), (2c))$ are more general than the TP_3 -based conditions, $((TP_3), (1J-2))$. Nevertheless, it is worth noting that there are two non-trivial differences in this case compared with the previous case.

Consider a multi-signal case where there is a random vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$, $n \geq 2$, with density $f(\mathbf{x}|a)$. Although there might be multiple \mathbf{x} satisfying $\frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)} = q$, for any given q , one can calculate $g(q|a)$ from $f(\mathbf{x}|a)$ by using the transformation method of random variables.⁴⁶ In order to use the transformation method, we introduce another random vector $\tilde{\mathbf{y}} = (\tilde{y}_1, \dots, \tilde{y}_n)$ with a density function $\hat{f}(\mathbf{y}|a)$ such that $\tilde{y}_j = \tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^0)}{f(\tilde{\mathbf{x}}|a^0)}$ and $\tilde{y}_i = \tilde{x}_i$ for all $i = 1, \dots, n$, $i \neq j$. If there exists a coordinate x_j in which $\frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)}$ is increasing for any \mathbf{x}_{-j} , the density function of $\tilde{\mathbf{y}}$ can be expressed as

$$\hat{f}(\mathbf{y}|a) = f(\mathbf{x}|a) \cdot |J| = f(x_j(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \times \left| \frac{\partial x_j(q, \mathbf{x}_{-j})}{\partial q} \right|,$$

⁴⁶In the one-signal case in which $f(x|a)$, $x \in \mathbb{R}$, does not satisfy MLRP, one easily obtains $g(q|a) = \sum_k f(x_k(q)|a) \left| \frac{dx_k(q)}{dq} \right|$ where $x_k(q) \in X(q) \equiv \left\{ x \mid \frac{f_a(x|a^0)}{f(x|a^0)} = q \right\}$.

where J is the transformation's Jacobian and $x_j(q, \mathbf{x}_{-j})$ solves $\frac{f_a(x_j, \mathbf{x}_{-j}|a^0)}{f(x_j, \mathbf{x}_{-j}|a^0)} = q$ for given \mathbf{x}_{-j} .⁴⁷ Note that, in this case, $|J| = \left| \frac{\partial x_j(q, \mathbf{x}_{-j})}{\partial q} \right|$, which is independent of a because $x_j(q, \mathbf{x}_{-j})$ is independent of a . Then, we have:

$$g(q|a) = \int \hat{f}(q, \mathbf{y}_{-j}|a) d\mathbf{y}_{-j} = \int f(x_j(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \cdot |J| d\mathbf{x}_{-j} = \int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j},$$

where $X(q) \equiv \left\{ \mathbf{x} \mid \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)} = q \right\}$. Thus, although

$$\frac{g_a(q|a^0)}{g(q|a^0)} = \frac{\int_{\mathbf{x} \in X(q)} f_a(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}} = \frac{\int_{\mathbf{x} \in X(q)} q f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}} = q = \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)} \quad (34)$$

for any given a^0 even in this case, it is generally true that

$$\frac{g(q|a)}{g(q|a^0)} = \frac{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}} \neq \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^0)}, \text{ and } \frac{g_a(q|a)}{g(q|a)} = \frac{\int_{\mathbf{x} \in X(q)} f_a(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}} \neq \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)}, \quad (35)$$

where $q = \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)}$.

For instance, consider multi-signal cases where $\tilde{\mathbf{x}} \sim N(\mu(a), \Sigma)$ where $\mu(a) = [\mu_1(a), \dots, \mu_n(a)]'$ and Σ is a covariance matrix. Then,

$$f(\mathbf{x}|a) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left(-\frac{1}{2} [\mathbf{x} - \mu(a)]' \Sigma^{-1} [\mathbf{x} - \mu(a)] \right),$$

and

$$\frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)} = [\mu'(a)]' \Sigma^{-1} [\mathbf{x} - \mu(a)].$$

Thus, we have

$$\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|a^0)}{f(\tilde{\mathbf{x}}|a^0)} \sim N \left(m(a), \sigma_q^2 \right),$$

⁴⁷Even in the multi-signal cases where there is no x_j in which $\frac{f_a(x_j, \mathbf{x}_{-j}|a^0)}{f(x_j, \mathbf{x}_{-j}|a^0)}$ is increasing, we observe equations (34) and (35) still hold. For example, if the support $\{\mathbf{x} | f(\mathbf{x}|a) > 0\}$ can be decomposed into subsets X_1, \dots, X_m such that $\tilde{\mathbf{y}}$ is a 1:1 transformation of X_k onto one subset of the support $\{\mathbf{y} | \hat{f}(\mathbf{y}|a) > 0\}$, the density function of $\tilde{\mathbf{y}}$ can be expressed by $\hat{f}(\mathbf{y}|a) = \sum_k f(x_j^k(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \cdot |J_k|$, where $x_j^k(q, \mathbf{x}_{-j})$ solves $\frac{f_a(x_j, \mathbf{x}_{-j}|a^0)}{f(x_j, \mathbf{x}_{-j}|a^0)} = q$ on X_k for given \mathbf{x}_{-j} and J_k is the Jacobian of the transformation on X_k , from which one can easily obtain $g(q|a) = \int \sum_k f(x_j^k(q, \mathbf{x}_{-j}), \mathbf{x}_{-j}|a) \cdot |J_k| d\mathbf{x}_{-j}$.

where $m(a) \equiv [\mu'(a^o)]' \Sigma^{-1} [\mu(a) - \mu(a^o)]$ (thereby $m(a^o) = 0$) and $\sigma_q^2 \equiv [\mu'(a^o)]' \Sigma^{-1} \mu'(a^o)$, and

$$\frac{g_a(q|a)}{g(q|a)} = \frac{q - [\mu'(a^o)]' \Sigma^{-1} [\mu(a) - \mu(a^o)]}{[\mu'(a^o)]' \Sigma^{-1} \mu'(a^o)} [\mu'(a^o)]' \Sigma^{-1} \mu'(a).$$

Therefore, one easily sees: $\frac{g(q|a)}{g(q|a^o)} \neq \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^o)}$ and $\frac{g_a(q|a)}{g(q|a)} \neq \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)}$ for $\forall a \neq a^o$ where $q = [\mu'(a^o)]' \Sigma^{-1} [\mathbf{x} - \mu(a^o)]$. This shows that, when there is no 1:1 relation between $\tilde{\mathbf{x}}$ and \tilde{q} , equation (35) is generally true.

From (35), one observes two non-trivial differences between this case and the previous case (i.e., Section 5.1) where $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP. First, (1a) cannot be reduced to (1d) as in Section 5.1. Therefore, to verify condition (1a) in this case, one should explicitly calculate $g(q|a)$ from $f(\mathbf{x}|a)$. Second, condition (1a) is not equivalent to that $g(q|a)$ is TP_3 for any given a^o . In fact, the condition that $g(q|a)$ is TP_3 for any given a^o is stronger than condition (1a) in this case. In order for $g(q|a)$ to be TP_3 for any given a^o , it is needed that, for any given a^o ,

- (1) $g(q|a)$ satisfies the MLRP, and
- (2)=(1e) $\frac{g(q|a)}{g(q|a_t)}$ is convex in $\frac{g_a(q|a_t)}{g(q|a_t)}$ for all a, a_t .

However, condition (1a) requires neither that $g(q|a)$ satisfy the MLRP nor that condition (1e) hold for all a_t .⁴⁸ Thus, in contrast with the previous case, when there is no 1:1 relation between \tilde{q} and $\tilde{\mathbf{x}}$, condition (1a) is more general than (TP_3). As a result, our conditions ((1a),(2c)) are even more general than conditions ((TP_3),(1J-2)) in this case.

However, there is a meaningful exception even in this case. Consider a density function $f(\mathbf{x}|a)$, $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, which generates, for any given a^o ,

$$\frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)} = \alpha(a) \cdot q + \beta(a), \quad (36)$$

where $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$, $\alpha(a^o) = 1$, $\beta(a^o) = 0$, and $\alpha(a) \geq 0$. Note that most exponential-family density functions with an appropriate parameterization satisfy equation (36).

⁴⁸When there is no 1:1 relation between \tilde{q} and $\tilde{\mathbf{x}}$, the MLRP for $f(\mathbf{x}|a)$ does not always imply MLRP for $g(q|a)$ for a given a^o . For example, consider a case of discrete signals with $f(x_1, x_2|a) = e^{2\sqrt{a}x_1 + ax_2 - K(a)}$, $x_i \in \{0, 1\}$, $i = 1, 2$, $a > 0$, where $K(a) = \log[(1 + e^{2\sqrt{a}})(1 + e^a)]$. $\frac{f_a(x_1, x_2|a)}{f(x_1, x_2|a)} = \frac{x_1}{\sqrt{a}} + x_2 - K'(a)$ implies that $f(x_1, x_2|a)$ satisfies MLRP. Define $\tilde{q} = \frac{\tilde{x}_1}{\sqrt{a^o}} + \tilde{x}_2 - K'(a^o)$ and let $a^o < 1$. Then, $g(q_1|a) = f(0, 0|a) = e^{-K(a)}$, $g(q_2|a) = f(0, 1|a) = e^{-K(a)}$, $g(q_3|a) = f(1, 0|a) = e^{2\sqrt{a} - K(a)}$ and $g(q_4|a) = f(1, 1|a) = e^{2\sqrt{a} + a - K(a)}$, where $q_1 = -K'(a^o)$, $q_2 = 1 - K'(a^o)$, $q_3 = \frac{1}{\sqrt{a^o}} - K'(a^o)$, and $q_4 = 1 + \frac{1}{\sqrt{a^o}} - K'(a^o)$ so $q_1 < q_2 < q_3 < q_4$. Thus, $\frac{g_a(q_1|a)}{g(q_1|a)} = -K'(a)$, $\frac{g_a(q_2|a)}{g(q_2|a)} = 1 - K'(a)$, $\frac{g_a(q_3|a)}{g(q_3|a)} = \frac{1}{\sqrt{a}} - K'(a)$, and $\frac{g_a(q_4|a)}{g(q_4|a)} = \frac{1}{\sqrt{a}} + 1 - K'(a)$. Since, when $a > 1$, $\frac{g_a(q_2|a)}{g(q_2|a)} = 1 - K'(a) > \frac{1}{\sqrt{a}} - K'(a) = \frac{g_a(q_3|a)}{g(q_3|a)}$, we observe $g(q|a)$ does not satisfy MLRP in general.

Since

$$\frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^0)} = \exp \left(\int_{a^0}^a \frac{f_a(\mathbf{x}|t)}{f(\mathbf{x}|t)} dt \right) = \exp ((A(a) - A(a^0))q + B(a) - B(a^0)),$$

where $A(a) \equiv \int_0^a \alpha(t)dt$ and $B(a) \equiv \int_0^a \beta(t)dt$, one can see that, for any given q , $\frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^0)}$ has the same value for all $\mathbf{x} \in X(q) \equiv \left\{ \mathbf{x} \mid \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)} = q \right\}$. Thus,

$$\begin{aligned} \frac{g(q|a)}{g(q|a^0)} &= \frac{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}} = \frac{\int_{\mathbf{x} \in X(q)} \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^0)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}} \\ &= \frac{\exp ((A(a) - A(a^0))q + B(a) - B(a^0)) \int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a^0) \cdot |J| d\mathbf{x}_{-j}} = \frac{f(\mathbf{x}|a)}{f(\mathbf{x}|a^0)}. \end{aligned}$$

Furthermore, from equation (36), we also have

$$\begin{aligned} \frac{g_a(q|a)}{g(q|a)} &= \frac{\int_{\mathbf{x} \in X(q)} f_a(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}} = \frac{\int_{\mathbf{x} \in X(q)} \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}} \\ &= \frac{(\alpha(a)q + \beta(a)) \int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}}{\int_{\mathbf{x} \in X(q)} f(\mathbf{x}|a) \cdot |J| d\mathbf{x}_{-j}} = \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)}. \end{aligned} \tag{37}$$

Therefore, even if there is no 1:1 relation between \tilde{q} and $\tilde{\mathbf{x}}$, all the results in Section 5.1 equally hold in this case. That is, condition (1a) reduces to the condition (1d) and thus verifying (1a) can be replaced by verifying (1d) which does not require to calculate $g(q|a)$ from $f(\mathbf{x}|a)$ directly. Furthermore, because $\alpha(a) \geq 0$, $g(q|a)$ satisfies MLRP from equation (37). Therefore, based on equation (36) and equation (37), one can also see that condition (1a) is equivalent to that $g(q|a)$ is TP_3 for any given a^0 .

6 Conclusion

All the existing results for justifying the first-order approach in the principal-agent problems have been derived to make the agent's expected monetary utility obtained from that approach "concave" in the agent's effort. However, to justify the first-order approach, relying on such concavity is sometimes overly sufficient. We have proposed a new set of conditions which is derived not from the concavity of the agent's expected monetary utility but directly from the original "argmax" incentive constraint, and shown that our approach

can be applied to a wider range of principal-agent problems than the previous literature. The examples we suggested (i.e., Examples 1, 2, 3, and 4) illustrate interesting cases where the previous literature cannot justify their use of the first-order approach while our new sets of conditions, in contrast, can.

This set of conditions contains of a statistical condition on the density function of the signals, which is quite general and easy to verify. We also have provided a few alternative sets of conditions (i.e., Propositions 1, 2, and 3) that are derived to be applied specifically to cases in which the agent's limited liability constraint is not binding at the optimum and to the case in which it is binding for some values of the signal vector at the optimum, respectively. Then, statistical implications of these sets of conditions have been explored, and the comparison between these conditions and the existing conditions has been provided.

When the agent's limited liability is binding for some values of the signal vector at the optimum, among the existing results in the previous literature, only the sets of conditions containing the CDF-type conditions can be applied. While the CDF-type conditions are hardly satisfied by most familiar probability density functions, our corresponding set of conditions (i.e., the conditions in Proposition 3) can be used for many useful density functions including the normal density function and gamma density function. On the other hand, when the agent's limited liability constraint is not binding at all at the optimum, Jewitt (1988) conditions (i.e., Theorem 1 in Jewitt (1988)) and Jung and Kim (2015) conditions (i.e., Proposition 7 in Jung and Kim (2015)) which do not contain any CDF-type conditions can still be applied. We have shown that there is no direct comparison available between our corresponding set of conditions (i.e., the conditions in Proposition 1) and the conditions in Proposition 7 in Jung and Kim (2015) in the sense that one does not imply the other. This is because Jung and Kim (2015) conditions, like other existing sets of conditions, were derived to make the agent's expected monetary utility concave in effort, whereas our conditions are derived to directly satisfy the original "argmax" incentive constraint without relying on such concavity. Nevertheless, we have shown that our conditions are more general than the TP_3 -based conditions which were proposed to verify Jung and Kim (2015) conditions.

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7 Appendix

7.1 Proofs for Lemmas and Propositions

Proof of Lemma 1. Using equation (2), the relaxed incentive constraint at a^o is

$$\begin{aligned}
c'(a^o) &= \int u(s^o(\mathbf{x})) f_a(\mathbf{x}|a^o) d\mathbf{x} \\
&= \int u(s^o(\mathbf{x})) \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} f(\mathbf{x}|a^o) d\mathbf{x} \\
&= \int (u(s^o(\mathbf{x})) - E[u(s^o(\mathbf{x}))]) \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} f(\mathbf{x}|a^o) d\mathbf{x} \\
&= \text{Cov} \left(u(s^o(\mathbf{x})), \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} \right).
\end{aligned}$$

In the above equation, the third equality comes from the fact that $\int E[u(s^o(\mathbf{x}))] f_a(\mathbf{x}|a^o) d\mathbf{x} = 0$ since $E[u(s^o(\mathbf{x}))]$ is constant, and the last equality comes from the fact that $E \left[\frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} | a^o \right] = \int f_a(\mathbf{x}|a^o) d\mathbf{x} = 0$. Suppose to the contrary that $\mu \leq 0$. Then, $\text{Cov} \left(u(s^o(\mathbf{x})), \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} \right) \leq 0$ from equation (3), which contradicts $c'(a^o) > 0$. Therefore, μ must be positive.

■

Proof of Lemma 2. Let $\psi(c) \equiv \xi(c)g(c|a^o)$, and define $\psi^{(1)}(q) \equiv \int^q \xi(c)g(c|a^o)dc = \int^q \psi(c)dc$ and $\psi^{(2)}(q) \equiv \int^q \int^c \xi(t)g(t|a^o)dt dc = \int^q \psi^{(1)}(c)dc$. Then, we have $\psi^{(1)}(\bar{q}) = 0$ from (L2). Furthermore, since

$$\begin{aligned}
\int \xi(q)q \cdot g(q|a^o)dq &= q \int^q \xi(c)g(c|a^o)dc \Big|_{\underline{q}}^{\bar{q}} - \int \int^q \xi(c)g(c|a^o)dc dq \\
&= - \int \int^q \xi(c)g(c|a^o)dc dq = 0,
\end{aligned}$$

where the second equality comes from (L2), and the last equality is from (L3), we have $\psi^{(2)}(\bar{q}) = 0$. Note from (L4) that $\psi(c)$ changes sign twice from negative to positive and to negative as c increases. Thus, $\psi^{(1)}(q)$ changes sign once from negative to positive as q increases since $\psi^{(1)}(\bar{q}) = 0$. Since $\psi^{(1)}(q)$ changes sign once from negative to positive as q increases and since $\psi^{(2)}(\bar{q}) = \int \psi^{(1)}(q)dq = 0$, we have

$$\psi^{(2)}(q) = \int^q \int^c \xi(t)g(t|a^o)dt dc \leq 0, \quad \forall q. \tag{A.1}$$

Denote

$$\int \xi(q)g(q|a)dq = \int \xi(q) \frac{g(q|a)}{g(q|a^o)} g(q|a^o)dq = \int \xi(q)\Gamma(q,a)g(q|a^o)dq, \tag{A.2}$$

where $\Gamma(q, a) \equiv \frac{g(q|a)}{g(q|a^o)}$. Then, by taking integration by parts twice, we have

$$\begin{aligned}
\int \xi(q) \Gamma(q, a) g(q|a^o) dq &= \Gamma(q, a) \int^q \xi(c) g(c|a^o) dc \Big|_{\underline{q}}^{\bar{q}} - \int \left(\int^q \xi(c) g(c|a^o) dc \right) \Gamma_q(q, a) dq \\
&= -\Gamma_q(q, a) \int^q \int^c \xi(t) g(t|a^o) dt dc \Big|_{\underline{q}}^{\bar{q}} + \int \left(\int^q \int^c \xi(t) g(t|a^o) dt dc \right) \Gamma_{qq}(q, a) dq \quad (\text{A.3}) \\
&= \int \left(\int^q \int^c \xi(T) g(t|a^o) dt dc \right) \Gamma_{qq}(q, a) dq,
\end{aligned}$$

where $\Gamma_q(q, a) \equiv \frac{\partial}{\partial q} \Gamma(q, a)$ and $\Gamma_{qq}(q, a) \equiv \frac{\partial^2}{\partial q^2} \Gamma(q, a)$. In equation (A.3), the second equality comes from the fact that $\psi^{(1)}(\bar{q}) = 0$, and the last equality is from the fact that $\psi^{(2)}(\bar{q}) = 0$. Thus, by using equation (A.1), equation (A.2), equation (A.3), and the fact that $\Gamma_{qq}(q, a) \geq 0$ (i.e., $\frac{g(q|a)}{g(q|a^o)}$ is convex in q), we finally have

$$\int \xi(q) g(q|a) dq = \int \left(\int^q \int^c \xi(t) g(t|a^o) dt dc \right) \Gamma_{qq}(q, a) dq \leq 0, \quad \forall a.$$

■

Proof of Lemma 3. Since $s^o(\mathbf{x})$ is the optimal contract obtained under the first-order approach, we know $s^o(\mathbf{x}) \in S_f$. Thus, from, equation (8),

$$\begin{aligned}
U(s^o(\cdot), a^o) &= \int u(s^o(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) \\
&= \int r(q) g(q|a^o) dq - c(a^o) = U^o,
\end{aligned} \quad (\text{A.4})$$

and

$$\begin{aligned}
U_a(s^o(\cdot), a^o) &= \int u(s^o(\mathbf{x})) f_a(\mathbf{x}|a^o) d\mathbf{x} - c'(a^o) \\
&= \int u(s^o(\mathbf{x})) \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)} f(\mathbf{x}|a^o) d\mathbf{x} - c'(a^o) = \int r(q) \cdot q \cdot g(q|a^o) dq - c'(a^o) = 0.
\end{aligned} \quad (\text{A.5})$$

Also, since $\hat{s}(\mathbf{x}) \in S_f$,

$$U(\hat{s}(\cdot), a^o) = \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a^o) d\mathbf{x} - c(a^o) = \int \hat{r}(q) g(q|a^o) dq - c(a^o) = U^o, \quad (\text{A.6})$$

and

$$U_a(\hat{s}(\cdot), a^o) = \int u(\hat{s}(\mathbf{x})) f_a(\mathbf{x}|a^o) d\mathbf{x} - c'(a^o) = \int \hat{r}(q) \cdot q \cdot g(q|a^o) dq - c'(a^o) = 0. \quad (\text{A.7})$$

Thus, from equation (A.4) and equation (A.6),

$$\int (r(q) - \hat{q}(q))g(q|a^0)dq = 0, \quad (\text{A.8})$$

and, from equation (A.5) and equation (A.7),

$$\int (r(q) - \hat{r}(q))q \cdot g(q|a^0)dq = 0. \quad (\text{A.9})$$

By substituting $r(q) - \hat{r}(q)$ for $\xi(q)$ in Lemma 2, one can easily see that $\xi(q) = r(q) - \hat{r}(q)$ satisfies conditions (L2) and (L3) in Lemma 2. Furthermore, as $\hat{r}(q)$ crosses $r(q)$ twice starting from above, $\xi(q) = r(q) - \hat{r}(q)$ satisfies (L4) in Lemma 2. Therefore, given that (L1) is satisfied, we derive from Lemma 2 that

$$\int (r(q) - \hat{r}(q))g(q|a)dq \leq 0, \quad \forall a, \quad (\text{A.10})$$

implying that

$$U(s^0(\cdot), a) \leq U(\hat{s}(\cdot), a), \quad \forall a. \quad (\text{A.11})$$

Let $E[\hat{r}(q)|a] \equiv \phi(c(a))$. Then,

$$\begin{aligned} U(\hat{s}(\cdot), a) &= \int \hat{r}(q)g(q|a)dq - c(a) \\ &= E[\hat{r}(q)|a] - c(a) = \phi(c(a)) - c(a). \end{aligned}$$

Since $\phi'(c(a^0)) = 1$ from equation (A.7) and since $\phi'' < 0$, $U(\hat{s}(\cdot), a)$ has a global maximum point at a^0 because

$$\phi'(c(a)) \geq 1, \quad \forall a \leq a^0, \quad \text{and} \quad \phi'(c(a)) < 1, \quad \forall a > a^0.$$

Therefore, we have

$$U(\hat{s}(\cdot), a) \leq U(\hat{s}(\cdot), a^0) = U^0, \quad \forall a. \quad (\text{A.12})$$

As a result, by combining equation (A.11) and equation (A.12), we have

$$U(s^0(\cdot), a) \leq U(s^0(\cdot), a^0) = U^0, \quad \forall a,$$

which justifies the first-order approach.

■

Proof of Proposition 1. Given a^0 , consider an arbitrary contract $\hat{s}(\mathbf{x})$ such that $u(\hat{s}(\mathbf{x})) \equiv \hat{r}(q) = Aq + B$ which satisfies both the participation and the “relaxed” incentive constraints at a^0 . Thus, since

$$\hat{R}(a) \equiv \int \hat{r}(q)g(q|a)dq = A \cdot m(a) + B,$$

from the participation constraint in equation (18), $\hat{r}(q)$ should satisfy

$$\hat{R}(a^0) - c(a^0) = A \cdot m(a^0) + B - c(a^0) = \bar{U}. \quad (\text{A.13})$$

Also, from the relaxed incentive constraint, $\hat{r}(q)$ should satisfy

$$\hat{R}'(a^0) - c'(a^0) = A \cdot m'(a^0) - c'(a^0) = 0. \quad (\text{A.14})$$

Note that

$$m(a^0) = \int qg(q|a^0)dq = \int \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)} f(\mathbf{x}|a^0) d\mathbf{x} = \int f_a(\mathbf{x}|a^0) d\mathbf{x} = 0,$$

which indicates that the expected value of information is always zero (i.e., no information ex-ante).⁴⁹ Thus, by solving equation (A.13) and equation (A.14), we obtain

$$A = \frac{c'(a^0)}{m'(a^0)}, \text{ and } B = \bar{U} + c(a^0). \quad (\text{A.15})$$

Then, using equation (A.15), we have

$$\hat{R}(a) - c(a) = \frac{c'(a^0)}{m'(a^0)} m(a) + \bar{U} + c(a^0) - c(a). \quad (\text{A.16})$$

Thus, $\hat{R}(a) - c(a)$ has a maximum value at a^0 if

$$\begin{aligned} \hat{R}'(a) - c'(a) \geq 0 &\iff \frac{m'(a)}{m'(a^0)} \geq \frac{c'(a)}{c'(a^0)}, \quad \forall a \leq a^0, \\ \hat{R}'(a) - c'(a) \leq 0 &\iff \frac{m'(a)}{m'(a^0)} \leq \frac{c'(a)}{c'(a^0)}, \quad \forall a > a^0. \end{aligned} \quad (\text{A.17})$$

Note that, since $g(q|a)$ exhibits the FOSD, $m'(a) \geq 0$ for all a . Thus, by defining $c(a) \equiv \phi(m(a))$, one can

⁴⁹Also, note that

$$m'(a^0) = \int qg_a(q|a^0)dq = \int \frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)} f_a(\mathbf{x}|a^0) d\mathbf{x} = \int \left(\frac{f_a(\mathbf{x}|a^0)}{f(\mathbf{x}|a^0)} \right)^2 f(\mathbf{x}|a^0) d\mathbf{x} = \text{Var}(q|a^0).$$

easily see that equation (A.17) holds if (2c) is satisfied (i.e., $\phi'' \geq 0$). Therefore,

$$\hat{R}(a) - c(a) \leq \hat{R}(a^0) - c(a^0) = \bar{U}, \quad \text{for any given } a^0 \text{ and for all } a, \quad (\text{A.18})$$

implying that the arbitrary contract, $\hat{s}(\mathbf{x})$, also satisfies the original “argmax” incentive constraint.

Since $r(q) \equiv u(s^0(\mathbf{x}))$ is concave in q by (3c), and since it also satisfies both the participation and the relaxed incentive constraints, $r(q)$ must cross $\hat{r}(q)$ twice starting from below as drawn in Figure 4. Actually, $r(q)$ and $\hat{r}(q)$ must cross because they both satisfy the same participation constraint, and they must cross twice because $r(q)$ is (increasing and) concave in q whereas $\hat{r}(q)$ is (increasing and) linear in q .⁵⁰

The fact that both $r(q)$ and $\hat{r}(q)$ satisfy the participation constraint at a^0 gives

$$\int (r(q) - \hat{r}(q))g(q|a^0)dq = 0, \quad (\text{A.19})$$

and the fact that they also satisfy the relaxed incentive constraint at a^0 gives

$$\int (r(q) - \hat{r}(q))qg(q|a^0)dq = 0. \quad (\text{A.20})$$

Thus, by combining (1a) with equation (A.19), equation (A.20), and the double crossing property between $r(q)$ and $\hat{r}(q)$, we have from Lemma 2 that

$$\int (r(q) - \hat{r}(q))g(q|a)dq \leq 0, \quad \forall a. \quad (\text{A.21})$$

Therefore, from equation (A.18) and equation (A.21), we finally derive

$$\begin{aligned} U(s^0(\cdot), a) &= R(a) - c(a) \leq \hat{R}(a) - c(a) \\ &\leq \hat{R}(a^0) - c(a^0) = \bar{U} = U(s^0(\cdot), a^0), \quad \text{for any given } a^0 \text{ and for all } a, \end{aligned}$$

which justifies the first-order approach.

■

Proof of Proposition 2. As we already explained, let $\hat{r}_t(q) = Ae^{tq}$ for some $t > 0$. Then $\ln \hat{r}_t(q) = tq + \ln A$ becomes linear in q . For $\hat{r}_t(q)$ to satisfy the agent’s participation constraint, we need to have

$$\hat{R}(a^0) = AM(a^0; t) = \bar{U} + c(a^0).$$

⁵⁰ $\hat{r}(q)$ must double-cross $r(q)$ since it induces the agent to participate and voluntarily choose $a = a^0$, as explained in Innes (1990).

For $\hat{r}_t(q)$ to satisfy the agent's incentive compatibility constraint, we need to have

$$\hat{R}'(a^0) = AM'(a^0; t) = c'(a^0), \quad A = \frac{c'(a^0)}{M'(a^0; t)}.$$

Therefore, t must be chosen such that

$$M(a^0; t) \frac{c'(a^0)}{M'(a^0; t)} = \bar{U} + c(a^0).$$

to satisfy the participation constraint of the agent. Under $\hat{r}_t(q)$, the agent's indirect utility function (i.e., value function) becomes $\hat{R}(a) - c(a) = AM(a; t) - c(a)$, where $\hat{R}_t(a) - c(a)$ has a maximum value at a^0 if

$$\begin{aligned} \hat{R}'_t(a) - c'(a) \geq 0 &\iff \frac{M'(a; t)}{M'(a^0; t)} \geq \frac{c'(a)}{c'(a^0)}, \quad \forall a \leq a^0, \\ \hat{R}'_t(a) - c'(a) \leq 0 &\iff \frac{M'(a; t)}{M'(a^0; t)} \leq \frac{c'(a)}{c'(a^0)}, \quad \forall a > a^0. \end{aligned} \tag{A.22}$$

Note that, since $g(q|a)$ exhibits the FOSD, $M'(a; t) \geq 0$ for all a . Thus, by defining $c(a) \equiv \phi(M(a; t))$, one can easily see that equation (A.22) holds if (2c') is satisfied (i.e., $\phi'' \geq 0$). Therefore,

$$\hat{R}_t(a) - c(a) \leq \hat{R}_t(a^0) - c(a^0) = U^0, \quad \text{for any given } a^0 \text{ and for all } a, \tag{A.23}$$

implying that $\hat{s}_t(\mathbf{x})$ actually satisfies the original “argmax” incentive constraint. As the linear $\ln \hat{r}_t(q)$ double-crosses the concave $\ln r(q)$, $\hat{r}_t(q)$ must double-cross $r(q)$ since it induces the agent to participate and voluntarily choose $a = a^0$, as explained in Innes (1990). Following Lemma 3, the first-order approach is justified.

■

Proof of Proposition 3. For any given a^0 , consider an arbitrary contract $\hat{s}_t(\mathbf{x})$ such that $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = Ae^{tq} + B$, $t > 0$, which satisfies both the participation constraint and the “relaxed” incentive constraint at a^0 . Therefore, since

$$\hat{R}_t(a) \equiv \int \hat{r}_t(q) g(q|a) dq = A \cdot M(a; t) + B, \tag{A.24}$$

from the participation constraint in equation (7) and the relaxed incentive constraint, $\hat{r}_t(q)$ should satisfy

$$\hat{R}_t(a^0) - c(a^0) = A \cdot M(a^0; t) + B - c(a^0) = U^0 \geq \bar{U}. \tag{A.25}$$

and

$$\hat{R}'_t(a^0) - c'(a^0) = A \cdot M'(a^0; t) - c'(a^0) = 0. \tag{A.26}$$

Then, solving equation (A.43) and equation (A.44) gives

$$A = \frac{c'(a^0)}{M'(a^0; t)}, \quad (\text{A.27})$$

and

$$B = U^0 + c(a^0) - c'(a^0) \frac{M(a^0; t)}{M'(a^0; t)}. \quad (\text{A.28})$$

Furthermore, since

$$\hat{r}_t(-\infty) = A \cdot e^{t(-\infty)} + B = B = U^0 + c(a^0) - c'(a^0) \frac{M(a^0; t)}{M'(a^0; t)},$$

there exists $t > 0$ such that

$$\hat{r}_t(-\infty) = B = \geq \bar{U} + c(a^0) - c'(a^0) \frac{M(a^0; t)}{M'(a^0; t)} \geq u(\underline{s}),$$

where the first inequality comes from that $U^0 \geq \bar{U}$ and the second inequality comes from (2b). This indicates that $\hat{s}_t(\mathbf{x})$ also satisfies the agent's limited liability constraint. Now, using equation (A.24), equation (A.45), and equation (A.46), we have

$$\hat{R}_t(a) - c(a) = \frac{c'(a^0)}{M'(a^0; t)} [M(a; t) - M(a^0; t)] + U^0 + c(a^0) - c(a).$$

Thus, $\hat{R}_t(a) - c(a)$ has a maximum value at a^0 if

$$\begin{aligned} \hat{R}'_t(a) - c'(a) \geq 0 &\iff \frac{M'(a; t)}{M'(a^0; t)} \geq \frac{c'(a)}{c'(a^0)}, \quad \forall a \leq a^0, \\ \hat{R}'_t(a) - c'(a) \leq 0 &\iff \frac{M'(a; t)}{M'(a^0; t)} \leq \frac{c'(a)}{c'(a^0)}, \quad \forall a > a^0. \end{aligned} \quad (\text{A.29})$$

Note that, since $g(q|a)$ exhibits the FOSD, $M'(a; t) \geq 0$ for all a . Thus, by defining $c(a) \equiv \phi(M(a; t))$, one can easily see that equation (A.29) holds if (2b) is satisfied (i.e., $\phi'' \geq 0$). Therefore,

$$\hat{R}_t(a) - c(a) \leq \hat{R}_t(a^0) - c(a^0) = U^0, \quad \text{for any given } a^0 \text{ and for all } a, \quad (\text{A.30})$$

implying that $\hat{s}_t(\mathbf{x})$ actually satisfies the original “argmax” incentive constraint. With $B \geq u(\underline{s})$, we observe $\ln(\hat{r}_t(q) - u(\underline{s}))$ becomes convex in $q \geq q_c$ as:

$$\frac{d^2}{dq^2} \ln(Ae^{tq} + B - u(\underline{s})) = \frac{(B - u(\underline{s})) At^2 e^{tq}}{(Ae^{tq} + B - u(\underline{s}))^2} > 0.$$

Since $\ln(\bar{r}(q) - u(\underline{s}))$ is concave in $q \geq q_c$ by (3b), and since $r(q) \equiv u(s^o(\mathbf{x}))$ also satisfies both the participation constraint and the “relaxed” incentive constraint, $r(q)$ must cross $\hat{r}_t(q)$ twice starting from below as drawn in Figure 6b. In fact, $r(q)$ and $\hat{r}_t(q)$ must cross because they both satisfy the same participation constraint, and they must cross twice because $\ln(\bar{r}(q) - u(\underline{s}))$ is concave in q whereas $\ln(\hat{r}_t(q) - u(\underline{s}))$ is convex in q with both $\bar{r}(q)$ and $\hat{r}_t(q)$ ’s minimum value higher than $u(\underline{s})$.⁵¹

The fact that both $r(q)$ and $\hat{r}_t(q)$ satisfy the same participation constraint at a^o gives

$$\int (r(q) - \hat{r}_t(q))g(q|a^o)dq = 0, \quad (\text{A.31})$$

and the fact that they also satisfy the relaxed incentive constraint at a^o gives

$$\int (r(q) - \hat{r}_t(q))q \cdot g(q|a^o)dq = 0. \quad (\text{A.32})$$

Thus, by combining (1a) with (A.31), (A.32), and the double crossing property between $r(q)$ and $\hat{r}_t(q)$, we have from Lemma 2 that

$$\int (r(q) - \hat{r}_t(q))g(q|a)dq \leq 0, \quad \forall a. \quad (\text{A.33})$$

Therefore, from equation (A.30) and equation (A.33), we finally have

$$\begin{aligned} U(s^o(\cdot), a) &= R(a) - c(a) \leq \hat{R}_t(a) - c(a) \\ &\leq \hat{R}_t(a^o) - c(a^o) = U^o = U(s^o(\cdot), a^o), \quad \text{for any given } a^o \text{ and for all } a, \end{aligned}$$

which justifies the first-order approach.

■

Proof of Lemma 4. Note that

$$\begin{vmatrix} 1 & \phi(x_1) & \psi(x_1) \\ 1 & \phi(x_2) & \psi(x_2) \\ 1 & \phi(x_3) & \psi(x_3) \end{vmatrix} = (\phi(x_3) - \phi(x_2))(\phi(x_2) - \phi(x_1)) \left(\frac{\psi(x_3) - \psi(x_2)}{\phi(x_3) - \phi(x_2)} - \frac{\psi(x_2) - \psi(x_1)}{\phi(x_2) - \phi(x_1)} \right).$$

Since $\phi(x)$ is increasing in x , $\phi(x_1) \leq \phi(x_2) \leq \phi(x_3)$ for every $x_1 < x_2 < x_3$. Therefore, we have, for every $x_1 < x_2 < x_3$,

⁵¹If $r(q)$ crosses $\hat{r}_t(q)$ only once, then equation (A.32) below is not possible. For detailed discussion for this, see Lemma 1 in Innes (1990).

$$\begin{vmatrix} 1 & \phi(x_1) & \psi(x_1) \\ 1 & \phi(x_2) & \psi(x_2) \\ 1 & \phi(x_3) & \psi(x_3) \end{vmatrix} \begin{matrix} \geq \\ \leq \end{matrix} 0 \iff \frac{\psi(x_3) - \psi(x_2)}{\phi(x_3) - \phi(x_2)} - \frac{\psi(x_2) - \psi(x_1)}{\phi(x_2) - \phi(x_1)} \begin{matrix} \geq \\ \leq \end{matrix} 0,$$

which indicates that $\psi(x)$ is convex (concave) in $\phi(x)$.

■

Proof of Lemma 5. Assume that $x_1 < x_2 < x_3$ and $a_1 < a_2 < a_3$, and, without loss of generality, let $a^0 = a_2$.

(i) the “if” part: Since

$$\begin{aligned} T(f, 2) &\equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) \\ f(x_2|a_1) & f(x_2|a_2) \end{vmatrix} = f(x_1|a_1)f(x_2|a_1) \times \begin{vmatrix} 1 & \frac{f(x_1|a_2)}{f(x_1|a_1)} \\ 1 & \frac{f(x_2|a_2)}{f(x_2|a_1)} \end{vmatrix} \\ &= f(x_1|a_1)f(x_2|a_1) \left(\frac{f(x_2|a_2)}{f(x_2|a_1)} - \frac{f(x_1|a_2)}{f(x_1|a_1)} \right), \end{aligned} \quad (\text{A.34})$$

the MLRP for $f(x|a)$ implies $T(f, 2) \geq 0$. Also, given the MLRP for $f(x|a)$, condition (ii) in Lemma 5 implies that $\frac{f(x|a_1)}{f(x|a^0)}$ is convex in $q = \frac{f_a(x|a^0)}{f(x|a^0)}$, $\forall a_1 < a^0$ as well as that $\frac{f(x|a_3)}{f(x|a^0)}$ is convex in $q = \frac{f_a(x|a^0)}{f(x|a^0)}$, $\forall a_3 > a^0$. Therefore, we have⁵²

$$\frac{\frac{f(x_2|a_1)}{f(x_2|a^0)} - \frac{f(x_1|a_1)}{f(x_1|a^0)}}{q_2 - q_1} \leq \frac{\frac{f(x_3|a_1)}{f(x_3|a^0)} - \frac{f(x_2|a_1)}{f(x_2|a^0)}}{q_3 - q_2} \leq 0 \quad (\text{A.35})$$

and

$$0 \leq \frac{\frac{f(x_2|a_3)}{f(x_2|a^0)} - \frac{f(x_1|a_3)}{f(x_1|a^0)}}{q_2 - q_1} \leq \frac{\frac{f(x_3|a_3)}{f(x_3|a^0)} - \frac{f(x_2|a_3)}{f(x_2|a^0)}}{q_3 - q_2}, \quad (\text{A.36})$$

where $q_i = \frac{f_a(x_i|a^0)}{f(x_i|a^0)}$. By combining equation (A.35) and equation (A.36), we derive

$$\begin{aligned} \frac{\frac{f(x_2|a_1)}{f(x_2|a^0)} - \frac{f(x_1|a_1)}{f(x_1|a^0)}}{\frac{f(x_2|a_3)}{f(x_2|a^0)} - \frac{f(x_1|a_3)}{f(x_1|a^0)}} &\leq \frac{\frac{f(x_3|a_1)}{f(x_3|a^0)} - \frac{f(x_2|a_1)}{f(x_2|a^0)}}{\frac{f(x_3|a_3)}{f(x_3|a^0)} - \frac{f(x_2|a_3)}{f(x_2|a^0)}} \leq 0. \end{aligned} \quad (\text{A.37})$$

By definition we can rewrite $T(f, 3)$ as

$$T(f, 3) \equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a^0) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a^0) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a^0) & f(x_3|a_3) \end{vmatrix} = \left(\prod_{i=1}^3 f(x_i|a^0) \right) \times \begin{vmatrix} \frac{f(x_1|a_1)}{f(x_1|a^0)} & 1 & \frac{f(x_1|a_3)}{f(x_1|a^0)} \\ \frac{f(x_2|a_1)}{f(x_2|a^0)} & 1 & \frac{f(x_2|a_3)}{f(x_2|a^0)} \\ \frac{f(x_3|a_1)}{f(x_3|a^0)} & 1 & \frac{f(x_3|a_3)}{f(x_3|a^0)} \end{vmatrix}$$

⁵²Due to MLRP, for $a_1 < a^0$, $\frac{f(x|a_1)}{f(x|a^0)}$ becomes decreasing in $\forall x$.

$$= \left(\prod_{i=1}^3 f(x_i|a^0) \right) \times \begin{vmatrix} 1 & \frac{f(x_1|a_3)}{f(x_1|a^0)} & \frac{f(x_1|a_1)}{f(x_1|a^0)} \\ 1 & \frac{f(x_2|a_3)}{f(x_2|a^0)} & \frac{f(x_2|a_1)}{f(x_2|a^0)} \\ 1 & \frac{f(x_3|a_3)}{f(x_3|a^0)} & \frac{f(x_3|a_1)}{f(x_3|a^0)} \end{vmatrix},$$

one can easily check that equation (A.37) implies $T(f, 3) \geq 0$.

(ii) the “only if” part: If $f(x|a)$ is TP_3 , then, by definition, $T(f, 2) \geq 0$ and $T(f, 3) \geq 0$. First, from equation (A.34), it is obvious that $T(f, 2) \geq 0$ implies the MLRP for $f(x|a)$. Second, notice that

$$\begin{aligned} T(f, 3) &\equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a^0) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a^0) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a^0) & f(x_3|a_3) \end{vmatrix} \\ &= (a^0 - a_1) \times \begin{vmatrix} f(x_1|a_1) & \frac{f(x_1|a^0) - f(x_1|a_1)}{a^0 - a_1} & f(x_1|a_3) \\ f(x_2|a_1) & \frac{f(x_2|a^0) - f(x_2|a_1)}{a^0 - a_1} & f(x_2|a_3) \\ f(x_3|a_1) & \frac{f(x_3|a^0) - f(x_3|a_1)}{a^0 - a_1} & f(x_3|a_3) \end{vmatrix} \\ &= (a^0 - a_1) \times \left\{ \prod_{i=1}^3 f(x_i|a_1) \right\} \times \underbrace{\begin{vmatrix} 1 & \frac{f(x_1|a^0) - f(x_1|a_1)}{(a^0 - a_1)f(x_1|a_1)} & \frac{f(x_1|a_3)}{f(x_1|a_1)} \\ 1 & \frac{f(x_2|a^0) - f(x_2|a_1)}{(a^0 - a_1)f(x_2|a_1)} & \frac{f(x_2|a_3)}{f(x_2|a_1)} \\ 1 & \frac{f(x_3|a^0) - f(x_3|a_1)}{(a^0 - a_1)f(x_3|a_1)} & \frac{f(x_3|a_3)}{f(x_3|a_1)} \end{vmatrix}}_{\equiv A}. \end{aligned}$$

Since $a^0 > a_1$ and since

$$\lim_{a_1 \rightarrow a^0} A = \begin{vmatrix} 1 & \frac{f_a(x_1|a^0)}{f(x_1|a^0)} & \frac{f(x_1|a_3)}{f(x_1|a^0)} \\ 1 & \frac{f_a(x_2|a^0)}{f(x_2|a^0)} & \frac{f(x_2|a_3)}{f(x_2|a^0)} \\ 1 & \frac{f_a(x_3|a^0)}{f(x_3|a^0)} & \frac{f(x_3|a_3)}{f(x_3|a^0)} \end{vmatrix},$$

$T(f, 3) \geq 0$ implies that $\frac{f(x|a_3)}{f(x|a^0)}$ is convex in $q = \frac{f_a(x|a^0)}{f(x|a^0)}$, $\forall a_3 > a^0$ by Lemma 4.

Similarly, notice that

$$T(f, 3) = (a_3 - a^0) \times \begin{vmatrix} f(x_1|a_1) & f(x_1|a^0) & \frac{f(x_1|a_3) - f(x_1|a^0)}{a_3 - a^0} \\ f(x_2|a_1) & f(x_2|a^0) & \frac{f(x_2|a_3) - f(x_2|a^0)}{a_3 - a^0} \\ f(x_3|a_1) & f(x_3|a^0) & \frac{f(x_3|a_3) - f(x_3|a^0)}{a_3 - a^0} \end{vmatrix}$$

$$\begin{aligned}
&= (a_3 - a^o) \times \left(\prod_{i=1}^3 f(x_i|a^o) \right) \times \begin{vmatrix} \frac{f(x_1|a_1)}{f(x_1|a^o)} & 1 & \frac{f(x_1|a_3)-f(x_1|a^o)}{(a_3-a^o)f(x_1|a^o)} \\ \frac{f(x_2|a_1)}{f(x_2|a^o)} & 1 & \frac{f(x_2|a_3)-f(x_2|a^o)}{(a_3-a^o)f(x_2|a^o)} \\ \frac{f(x_3|a_1)}{f(x_3|a^o)} & 1 & \frac{f(x_3|a_3)-f(x_3|a^o)}{(a_3-a^o)f(x_3|a^o)} \end{vmatrix} \\
&= (a_3 - a^o) \times \left(\prod_{i=1}^3 f(x_i|a^o) \right) \times \underbrace{\begin{vmatrix} 1 & \frac{f(x_1|a_3)-f(x_1|a^o)}{(a_3-a^o)f(x_1|a^o)} & \frac{f(x_1|a_1)}{f(x_1|a^o)} \\ 1 & \frac{f(x_2|a_3)-f(x_2|a^o)}{(a_3-a^o)f(x_2|a^o)} & \frac{f(x_2|a_1)}{f(x_2|a^o)} \\ 1 & \frac{f(x_3|a_3)-f(x_3|a^o)}{(a_3-a^o)f(x_3|a^o)} & \frac{f(x_3|a_1)}{f(x_3|a^o)} \end{vmatrix}}_{\equiv B}.
\end{aligned}$$

Since $a_3 > a^o$ and since

$$\lim_{a_3 \rightarrow a^o} B = \begin{vmatrix} 1 & \frac{f_a(x_1|a^o)}{f(x_1|a^o)} & \frac{f(x_1|a_1)}{f(x_1|a^o)} \\ 1 & \frac{f_a(x_2|a^o)}{f(x_2|a^o)} & \frac{f(x_2|a_1)}{f(x_2|a^o)} \\ 1 & \frac{f_a(x_3|a^o)}{f(x_3|a^o)} & \frac{f(x_3|a_1)}{f(x_3|a^o)} \end{vmatrix},$$

$T(f, 3) \geq 0$ also implies that $\frac{f(x|a_1)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$, $\forall a_1 < a^o$ by Lemma 4. Consequently, $T(f, 3) \geq 0$ implies that, for any given a^o , $\frac{f(x|a)}{f(x|a^o)}$ is convex in $q = \frac{f_a(x|a^o)}{f(x|a^o)}$ for all a .

■

Proof of Corollary 1:. We have, for any given a^o ,

$$\begin{aligned}
T(f, 2) &\equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) \\ f(x_2|a_1) & f(x_2|a_2) \end{vmatrix} = Q'_{a^o}(x_1)Q'_{a^o}(x_2) \begin{vmatrix} g(q_1|a_1) & g(q_1|a_2) \\ g(q_2|a_1) & g(q_2|a_2) \end{vmatrix} \\
&= Q'_{a^o}(x_1)Q'_{a^o}(x_2)T(g, 2),
\end{aligned}$$

where $q_i = Q_{a^o}(x_i) \equiv \frac{f_a(x_i|a^o)}{f(x_i|a^o)}$, $i = 1, 2$. Since $f(x|a)$ satisfies MLRP, we have $Q'_{a^o}(x) \geq 0$, $\forall x$. Thus, given MLRP for $f(x|a)$,

$$\begin{aligned}
&T(f, 2) \geq 0, \quad \text{for every } x_1 < x_2 \text{ and } a_1 < a_2 \\
&\iff T(g, 2) \geq 0, \quad \text{for every } q_1 < q_2 \text{ and } a_1 < a_2.
\end{aligned} \tag{A.38}$$

Likewise, we have, for any given a^o ,

$$T(f, 3) \equiv \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix}$$

$$\begin{aligned}
&= Q'_{a^o}(x_1)Q'_{a^o}(x_2)Q'_{a^o}(x_3) \begin{vmatrix} g(q_1|a_1) & g(q_1|a_2) & g(q_1|a_3) \\ g(q_2|a_1) & g(q_2|a_2) & g(q_2|a_3) \\ g(q_3|a_1) & g(q_3|a_2) & g(q_3|a_3) \end{vmatrix} \\
&= Q'_{a^o}(x_1)Q'_{a^o}(x_2)Q'_{a^o}(x_3)T(g,3),
\end{aligned}$$

where $q_i = Q_{a^o}(x_i) \equiv \frac{f_a(x_i|a^o)}{f(x_i|a^o)}$, $i = 1, 2, 3$. Therefore, by the same way, we derive that, given MLRP for $f(x|a)$,

$$\begin{aligned}
&T(f,3) \geq 0, \quad \text{for every } x_1 < x_2 < x_3 \text{ and } a_1 < a_2 < a_3 \\
&\iff T(g,3) \geq 0, \quad \text{for every } q_1 < q_2 < q_3 \text{ and } a_1 < a_2 < a_3
\end{aligned} \tag{A.39}$$

Thus, by combining (A.38) and (A.39), we have that $f(x|a)$ is TP_3 if and only if $g(q|a)$ is TP_3 for any given a^o given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP.

■

Proof of Proposition 4. From Lemma 5 and Corollary 1, it is shown that, given that $x \in \mathbb{R}$ and $f(x|a)$ satisfies MLRP, condition (TP_3) is equivalent to condition (1a). Furthermore, condition (1J-2) implies condition (2c) since $c(a)$ is increasing convex in a . However, condition (2c) does not imply condition (1J-2).

■

Proof of Lemma 6. Define, for a given density function $f(x|a)$,

$$\psi(a, k) \equiv \int \phi(x, k) f(x|a) dx,$$

where $x \in \mathbb{R}$, $a \in \mathbb{R}$, and k is the parameter that determines the functional form of $\phi(x, k)$. Then, by the “basic composition formula” by Karlin (1968),⁵³ we have

$$\begin{aligned}
&\begin{vmatrix} \psi(a_1, k_1) & \psi(a_1, k_2) & \psi(a_1, k_3) \\ \psi(a_2, k_1) & \psi(a_2, k_2) & \psi(a_2, k_3) \\ \psi(a_3, k_1) & \psi(a_3, k_2) & \psi(a_3, k_3) \end{vmatrix} = \iiint_{x_1 < x_2 < x_3} \begin{vmatrix} \phi(x_1, k_1) & \phi(x_1, k_2) & \phi(x_1, k_3) \\ \phi(x_2, k_1) & \phi(x_2, k_2) & \phi(x_2, k_3) \\ \phi(x_3, k_1) & \phi(x_3, k_2) & \phi(x_3, k_3) \end{vmatrix} \\
&\quad \times \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix} dx_1 dx_2 dx_3. \tag{A.40}
\end{aligned}$$

Let $\phi(x, k_1) \equiv 1$, $\phi(x, k_2) \equiv x$, and $\phi(x, k_3) \equiv u(x)$. We have $\psi(a, k_1) = 1$, $\psi(a, k_2) = \mu(a) \equiv \int x f(x|a) dx$, and

⁵³For the detailed proof of the basic composition formula, see Karlin (1968) p. 17. The equation (A.40) is a direct extension of the famous Cauchy-Binet theorem.

$\psi(a, k_3) = u^*(a) \equiv \int u(x)f(x|a)dx$. Thus, by using equation (A.40),

$$\begin{vmatrix} 1 & \mu(a_1) & u^*(a_1) \\ 1 & \mu(a_2) & u^*(a_2) \\ 1 & \mu(a_3) & u^*(a_3) \end{vmatrix} = \iiint_{x_1 < x_2 < x_3} \begin{vmatrix} 1 & x_1 & u(x_1) \\ 1 & x_2 & u(x_2) \\ 1 & x_3 & u(x_3) \end{vmatrix} \times \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix} dx_1 dx_2 dx_3. \quad (\text{A.41})$$

The fact that $f(x|a)$ is TP_3 implies

$$T(f, 2) = \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) \\ f(x_2|a_1) & f(x_2|a_2) \end{vmatrix} \geq 0, \quad \text{for every } x_1 < x_2 \text{ and } a_1 < a_2,$$

which is equivalent to MLRP for $f(x|a)$. Thus, both $\mu(a)$ and $u^*(a)$ are increasing in a . Since $u(x)$ is concave in x , we have from Lemma 4 that

$$\begin{vmatrix} 1 & x_1 & u(x_1) \\ 1 & x_2 & u(x_2) \\ 1 & x_3 & u(x_3) \end{vmatrix} \leq 0, \quad \text{for every } x_1 < x_2 < x_3.$$

Also, the fact that $f(x|a)$ is TP_3 implies

$$T(f, 3) = \begin{vmatrix} f(x_1|a_1) & f(x_1|a_2) & f(x_1|a_3) \\ f(x_2|a_1) & f(x_2|a_2) & f(x_2|a_3) \\ f(x_3|a_1) & f(x_3|a_2) & f(x_3|a_3) \end{vmatrix} \geq 0, \quad \text{for every } x_1 < x_2 < x_3 \text{ and } a_1 < a_2 < a_3.$$

Thus, from equation (A.41), we have

$$\begin{vmatrix} 1 & \mu(a_1) & u^*(a_1) \\ 1 & \mu(a_2) & u^*(a_2) \\ 1 & \mu(a_3) & u^*(a_3) \end{vmatrix} \leq 0, \quad \text{for every } a_1 < a_2 < a_3,$$

which indicates that $u^*(a)$ is increasing concave in $\mu(a) \equiv \int xf(x|a)dx$, given that $\mu(a)$ is increasing in a .

■

7.2 Additional Results

Derivation of Example 2. Given

$$g(q|a) = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp \left(-\frac{1}{h(a)} \left(\frac{[h(a^o)]^2}{h'(a^o)} q + h(a^o) \right) \right),$$

the moment generating function for arbitrary a will be given as

$$\begin{aligned} M(a; t) &= \int e^{tq} g(q|a) dq = \frac{[h(a^o)]^2}{h'(a^o)h(a)} \int_{-\frac{h'(a^o)}{h(a^o)}}^{\infty} \exp \left[-\frac{1}{h(a)} \left(\frac{h(a^o)^2}{h'(a^o)} q + h(a^o) \right) + tq \right] dq \\ &= \frac{[h(a^o)]^2}{h'(a^o)h(a)} \exp \left(-\frac{h(a^o)}{h(a)} \right) \int_{-\frac{h'(a^o)}{h(a^o)}}^{\infty} \exp \left[-\left(\frac{h(a^o)^2}{h(a)h'(a^o)} - t \right) q \right] dq \\ &= \frac{[h(a^o)]^2}{h'(a^o)h(a)} \frac{1}{\frac{h(a^o)^2}{h(a)h'(a^o)} - t} \exp \left(-t \frac{h'(a^o)}{h(a^o)} \right) = \frac{h(a^o)^2}{h(a^o)^2 - h'(a^o)h(a)t} \exp \left(-t \frac{h'(a^o)}{h(a^o)} \right), \end{aligned}$$

which is equation (22). And then we can calculate its derivative $M'(a; t)$ as:

$$M'(a; t) = \frac{-h(a^o)^2}{[h(a^o)^2 - h'(a^o)h(a)t]^2} (-h'(a^o)h'(a)t) \exp \left(-t \frac{h'(a^o)}{h(a^o)} \right),$$

thereby

$$\frac{M(a; t)}{M'(a; t)} = \frac{h(a^o)^2 - h'(a^o)h(a)t}{h'(a^o)h'(a)t}.$$

Therefore, we obtain

$$\frac{M(a^o; t)}{M'(a^o; t)} c'(a^o) - c(a^o) = \frac{h(a^o)^2 - h'(a^o)h(a^o)t}{[h'(a^o)]^2 t} c'(a^o) - c(a^o) = \bar{U},$$

which is satisfied when

$$t = \frac{h(a^o)^2 c'(a^o)}{[\bar{U} + c(a^o)] h'(a^o)^2 + h(a^o) h'(a^o) c'(a^o)} > 0.$$

It is clear that the above t is lower than $\frac{h(a^o)}{h'(a^o)}$.

■

Now, we characterize the next Proposition 5 which is similar to Proposition 3 except that (2b') in Proposition 5 is different from (2b) in Proposition 3. It turns out that (2b) and (2b') are not entirely the same conditions.

Proposition 5 Given that the information variable, $\tilde{q} \equiv \frac{f_a(\tilde{x}|a^o)}{f(\tilde{x}|a^o)}$, is unbounded below, if, for any given a^o ,

(1a) $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(\mathbf{x}|a)}{f(\mathbf{x}|a)}$ for all a ,

(2b') The following function is decreasing in action a for any given $t > 0$:

$$\frac{M'(a; t)}{M(a; t)} \times \frac{\bar{U} - u(\underline{s}) + c(a)}{c'(a)}$$

(3b) $\ln(\bar{r}(q) - u(\underline{s}))$ is concave in q for $\forall q \geq q_c$,

then the first-order approach is justified.

Proof of Proposition 5. For any given a^o , consider an arbitrary contract $\hat{s}_t(\mathbf{x})$ such that $u(\hat{s}_t(\mathbf{x})) \equiv \hat{r}_t(q) = Ae^{tq} + u(\underline{s})$,⁵⁴ which satisfies both the participation constraint and the “relaxed” incentive constraint at a^o .

Therefore, since

$$\hat{R}_t(a) \equiv \int \hat{r}_t(q) g(q|a) dq = A \cdot M(a; t) + u(\underline{s}), \quad (\text{A.42})$$

from the participation constraint in equation (7) and the relaxed incentive constraint, $\hat{r}_t(q)$ should satisfy

$$\hat{R}_t(a^o) - c(a^o) = A \cdot M(a^o; t) + u(\underline{s}) - c(a^o) = U^o \geq \bar{U}. \quad (\text{A.43})$$

and

$$\hat{R}'_t(a^o) - c'(a^o) = A \cdot M'(a^o; t) - c'(a^o) = 0. \quad (\text{A.44})$$

Then, solving equation (A.43) and equation (A.44) gives (A^o, t^o) such that

$$A^o = \frac{c'(a^o)}{M'(a^o; t^o)}, \quad (\text{A.45})$$

and

$$\frac{M'(a^o; t^o)}{M(a^o; t^o)} = \frac{c'(a^o)}{U^o - u(\underline{s}) + c(a^o)}. \quad (\text{A.46})$$

We know from Proposition 3 that condition (3b) guarantees that $\hat{r}_t(q)$ double-crosses $r(q)$ from above.⁵⁵ Thus, showing that (2b') implies that under the proxy contract $\hat{r}_t(q)$, the agent will choose $a = a^o$ voluntarily will be enough to prove Proposition 5 with the help of Lemma 3. We need to show:

$$\hat{R}_t(a) - c(a) \leq \hat{R}_t(a^o) - c(a^o) = U^o,$$

i.e., $A^o M(a; t^o) \leq U^o - u(\underline{s}) + c(a)$, and equivalently, $\ln M(a; t^o) + \ln A^o \leq \ln(U^o - u(\underline{s}) + c(a))$. If we define a new function $\phi(a; a^o) \equiv \ln M(a; t^o) + \ln A^o - \ln(U^o - u(\underline{s}) + c(a))$, due to (A.43) and (A.46), $\phi(a = a^o; a^o) = 0$

⁵⁴In proving Proposition 3, we used the proxy contract $\hat{r}_t(q) = Ae^{tq} + B$, $t > 0$, where $B \geq u(\underline{s})$. Here, we fix $B = u(\underline{s})$ and find A and $t > 0$ that satisfy the agent's participation and relaxed incentive compatibility constraints.

⁵⁵In this case, $B = u(\underline{s})$ implies $\ln(\hat{r}_t(q) - u(\underline{s}))$ becomes linear, while $\ln(\bar{r}(q) - u(\underline{s}))$ is concave in $q \geq q_c$ by condition (3b).

and $\phi'(a = a^0; a^0) = 0$ hold. Therefore, if $\phi'(a; a^0) \geq 0$ for $\forall a < a^0$ and $\phi'(a; a^0) \leq 0$ for $\forall a > a^0$, $\phi(a; a^0)$ would have a global maximum at $a = a^0$, implying that the agent chooses $a = a^0$ under $\hat{r}_t(q) = A^0 e^{t^0 q} + u(\underline{s})$. We have

$$\begin{aligned}\phi'(a; a^0) &= \frac{M'(a; t^0)}{M(a; t^0)} - \frac{c'(a)}{U^0 - u(\underline{s}) + c(a)} \\ &= \frac{c'(a)}{U^0 - u(\underline{s}) + c(a)} \left(\frac{M'(a; t^0)}{M(a; t^0)} \cdot \frac{U^0 - u(\underline{s}) + c(a)}{c'(a)} - 1 \right) \\ &= \underbrace{\frac{c'(a)}{U^0 - u(\underline{s}) + c(a)}}_{>0} \left(\underbrace{\frac{M'(a; t^0)}{M(a; t^0)} \cdot \frac{U^0 - u(\underline{s}) + c(a)}{c'(a)}}_{\equiv h(a; t^0, U^0)} - \underbrace{\frac{M'(a^0; t^0)}{M(a^0; t^0)} \cdot \frac{U^0 - u(\underline{s}) + c(a^0)}{c'(a^0)}}_{=h(a^0; t^0, U^0)=1} \right),\end{aligned}$$

where we use $h(a^0; t^0, U^0) = 1$ (i.e., equation (A.46)). Therefore, if $h(a; t^0, U^0)$ is decreasing in a , $\phi(a; a^0)$ achieves its global maximum at $a = a^0$, thereby proving Proposition 5. Note that condition (2b) is equivalent to $h(a; t, \bar{U})$ is decreasing in a for any $t > 0$, which implies

$$\frac{h_a(a; t^0, \bar{U})}{h(a; t^0, \bar{U})} = \frac{M''(a; t^0)}{M'(a; t^0)} - \frac{M'(a; t^0)}{M(a; t^0)} + \frac{\overbrace{c'(a^0)}^{>0}}{\bar{U} - u(\underline{s}) + c(a^0)} - \frac{c''(a)}{c'(a)} < 0, \quad (\text{A.47})$$

given $t = t^0$. The above (A.47) implies

$$\frac{h_a(a; t^0, U^0)}{h(a; t^0, U^0)} = \frac{M''(a; t^0)}{M'(a; t^0)} - \frac{M'(a; t^0)}{M(a; t^0)} + \frac{\overbrace{c'(a^0)}^{>0}}{U^0 - u(\underline{s}) + c(a^0)} - \frac{c''(a)}{c'(a)} < 0, \quad (\text{A.48})$$

since $U^0 \geq \bar{U}$. Finally, equation (A.48) implies $h(a; t^0, U^0)$ is decreasing in a , proving Proposition 5.

■

Example 3. We check the first-order approach's validity in Example 3 based on Proposition 5. We already verified that condition (2b) in Proposition 3 (and thereby the first-order approach) hold if $D \leq \bar{U} - u(\underline{s})$. To check (2b') in Proposition 5 with $c(a) = D(e^{ka} - 1)$, we calculate

$$\begin{aligned}h(a; t, \bar{U}) &= \frac{M'(a; t)}{M(a; t)} \cdot \frac{\bar{U} - u(\underline{s}) + c(a)}{c'(a)} = \frac{t}{\sigma^2} \times \frac{\bar{U} - u(\underline{s}) + D(e^{ka} - 1)}{Dke^{ka}} \\ &= \frac{t}{\sigma^2} \times \frac{(\bar{U} - u(\underline{s}) - D)e^{-ka} + D}{Dk},\end{aligned}$$

which is decreasing in a if and only if $\bar{U} - u(\underline{s}) \geq D$. Thus Propositions 3 and 5 give the same condition that guarantees the validity of the first-order approach.

■

Comparison between Propositions 3 and 5. Then, are two Propositions (i.e., Propositions 3 and 5) the same set of conditions? Think about the following case: $x \sim N(a, \sigma^2)$ as in Example 3 with the following agent's cost function:

$$c(a) = D \left(\left(1 - \frac{a}{\eta} \right)^{-\eta} - 1 \right), \quad a \in [0, \eta).$$

Then we have,

$$\begin{aligned} h(a; t, \bar{U}) &= \frac{M'(a; t)}{M(a; t)} \cdot \frac{\bar{U} - u(\underline{s}) + c(a)}{c'(a)} = \frac{t}{\sigma^2} \times \frac{\bar{U} - u(\underline{s}) + D \left(\left(1 - \frac{a}{\eta} \right)^{-\eta} - 1 \right)}{D \left(\left(1 - \frac{a}{\eta} \right)^{-\eta-1} \right)} \\ &= \frac{t}{\sigma^2} \times \left(\frac{\bar{U} - u(\underline{s}) - D}{D} \left(1 - \frac{a}{\eta} \right)^{\eta+1} + 1 - \frac{a}{\eta} \right), \end{aligned}$$

which is decreasing in a (i.e., (2b') in Proposition 5) if and only if $\bar{U} - u(\underline{s}) \geq D$. To satisfy (2b) in Proposition 3, there must be $t > 0$ such that

$$\frac{c'(a^0)}{M'(a^0; t)} M(a^0; t) - c(a^0) \leq \bar{U} - u(\underline{s}), \quad (\text{A.49})$$

where

$$\begin{aligned} \frac{c'(a^0)}{M'(a^0; t)} M(a^0; t) - c(a^0) &= \frac{\sigma^2}{t} D \left(1 - \frac{a^0}{\eta} \right)^{-\eta-1} - D \left(1 - \frac{a^0}{\eta} \right)^{-\eta} + D \\ &= D \left(1 - \frac{a^0}{\eta} \right)^{-\eta} \left(\frac{\sigma^2}{t} \times \frac{\eta}{\eta - a^0} - 1 \right) + D \rightarrow \infty \end{aligned}$$

as $a^0 \rightarrow \infty$, violating equation (A.49). Therefore, conditions (2b) in Proposition 3 and (2b') in Proposition 5 are not equivalent to each other in this case. ■