Heterogeneous Beliefs, Risk Amplification, and Asset Returns*

Goutham Gopalakrishna† — EPFL and Swiss Finance Institute
Seung Joo Lee‡ — Oxford University
Theofanis Papamichalis§ — Cambridge University

May 16, 2023

Abstract

How do heterogeneous beliefs about growth prospects affect the macroeconomy's *endogenous* volatility and transition dynamics? We first set out a tractable framework that incorporates heterogeneous beliefs about technological growth, and assess their impact on the amplification of risks during crises. When productive experts become more optimistic about their growth prospects, trade is facilitated, raising investment, asset prices, and their leverage ratio considerably. However, when the economy transitions into crises, belief heterogeneity exacerbates aggregate risk, resulting in excess volatility and higher risk-premia. The economy ends up spending longer time in the crisis regime on average per year. Finally, our model generates the excess conditional asset return momentum which is empirically observed during crises, and suggests a disagreement-based factor that significantly raises pricing power in the cross-section, beyond the conventional factors proposed by the literature.

Keywords: Heterogeneous Beliefs, Endogenous Risk, Amplification

*We are grateful to Suleyman Basak, Markus Brunnermeier, Pedro Bordalo, Marc Dordal Carreras, Ding Dong, Yang Lu, Thomas Noe, Ilaria Piatti, Herkles Polemarchakis, Dimitrios Tsomocos, Mungo Wilson, Jenny Xu, and seminar audiences at HKUST for comments and suggestions.

†goutham.gopalakrishna@epfl.ch
‡seung.lee@sbs.ox.ac.uk
§tp323@cam.ac.uk

1 Introduction

Asset prices before the Global Financial Crisis (GFC) increased considerably and so did the capital investment and the leverage of financial market participants. Eventually, they collapsed during the crisis while the market volatility and the risk-premium spiked. This pattern of boom-bust cycles has repeated for a long period of history. The explanation is quite simple: if an investor believes that future asset prices will continue to grow, then she takes more leverage and eventually gets exposed to a higher amount of risk, dragging the economy into a crisis. During crises, asset markets feature higher risk-premium levels, thereby facilitating the recapitalization process for many market participants, and pushing the economy towards normalcy.

When markets are more turbulent, it is more likely that different market participants have different ideas about the financial market's direction. During the GFC, for example, there were those who bet against the US housing market, making considerable fortunes.[1] Therefore, a natural question is what would happen if different groups of market participants do not share the underlying economic process as common knowledge but instead have their own views about the economy. How does this lack of common knowledge about the real economy affect market behaviors, i.e., volatility, risk-premia, asset price, etc.? Do heterogeneous beliefs amplify or mitigate crises? Our paper attempts to shed light on those aforementioned questions.

We make three main contributions. First, we build a simple continuous-time economy occupied by two groups of agents: experts and households. Following Brunnermeier and Sannikov (2014), experts have higher productivity in (i) turning capital into output (i.e., production); and (ii) turning output into capital (i.e., investment). To introduce heterogeneous beliefs in a tractable way, we first embed exogenous technological growth in our model and assume that experts and households have different views about the economy's technological growth process[2]: two groups with imperfect information about the productivity process agree to disagree about their expected growth. We call the group of agents that believe in a higher (lower) technological growth rate 'optimists' ('pessimists'). We mostly focus on cases where more productive experts believe that the expected growth of productivity is higher than the other group (i.e., optimistic), but the opposite case is also analyzed. The second contribution is in showing that the macro-level (i.e., technological growth) disagreement is crucial in explaining asset return predictability patterns documented empirically. That is, the model successfully explains the excess asset return momentum during recessions found in the data. Importantly, this excess momentum is stronger when the disagreement is larger, which is confirmed by the predictions of our model. Finally, we build a macro-level disagreement factor from The Survey of Profes-

[1] For specific such cases, see e.g., Zuckerman (2010).

[2] We model that production technologies of experts and households must have the same growth rate as they are proportional to each other in the 'true' data-generating process. The problem is both experts and households are not aware of it and have their own views about the growth rates of their technologies.

sional Forecasters data and show that a factor model with both the intermediary factor and the disagreement factor has significant power in pricing the cross-section of asset returns. The disagreement factor has pricing power above and beyond the intermediary factor, particularly in explaining momentum and long-term reversal anomaly portfolio excess returns, which the intermediary factor alone fails to explain as documented by the literature, e.g., He et al. (2017) (HKM2017, henceforth).

Similar to Brunnermeier and Sannikov (2014), our model economy features two regimes: an efficient regime when the economy is at a stochastic steady state, and an inefficient regime where financial constraints bind and disagreement exacerbates the risk amplification. When experts, who are natural marginal investors for the capital due to their higher productivity, become more optimistic about their growth prospects, trade gets facilitated, raising the investment rate, the capital price, and the leverage ratio of experts considerably in the stochastic steady state[3]. When the economy is in the inefficient region, financial constraints amplify shocks leading to a spike in the endogenous price volatility and the risk-premium, both of which get further amplified due to belief heterogeneity. Compared to a benchmark rational expectations model, an economy where experts believe in better growth prospects and households believe in poorer growth prospects than the true growth features fatter left-tailed stationary distribution of experts' wealth share.

This amplification of risk under heterogeneous beliefs can be decomposed into two separate channels: (i) the *leverage* effect; and (ii) the *market illiquidity* effect. More optimistic experts hold higher leverage for funding their purchase of risky capital than under the benchmark case of homogeneous beliefs (i.e., rational expectations). Therefore, when negative shocks shift the economy towards the inefficienct region, they engage in a more intense fire-sale[4] of the capital asset, resulting in a higher degree of market turbulence and the amplification of risks. Another effect comes from the illiquidity of capital markets. Financial crises happen when less productive households hold a portion of the aggregate capital and the capital market is not liquid. When experts are more optimistic, a decrease in their 'wealth share' during any crisis leads to a steeper drop in the price of capital than under rational expectations.[5] It creates a higher degree of market turbulence so that the endogenous market risk hikes more in response. This capital market illiquidity is captured by a higher elasticity of capital price with respect to the wealth share of optimistic experts in equilibrium, contributing to the amplification of endogenous volatility as well as risk premium.

The dynamics of our model reveals that the economy undergoes a higher number of shorter-lived and

[3] We consider symmetric cases only: when expects become more optimistic, the less productive households have more pessimistic views about their own technological growth. It is worth noting that the *market selection hypothesis*, that markets favor agents with more accurate beliefs, does not hold in our framework, since markets are incomplete. For this issue, see e.g., Blume and Easley (2006).

[4] For fire-sales externalities in the presence of financial frictions, see e.g., Dávila and Korinek (2017).

[5] Since experts are marginal investors at the normal region, their optimism results in a higher price response (i.e., higher elasticity of price) to fluctuations in their wealth share during financial crises.

more severe crises and has a higher occupation time in a crisis compared to the benchmark rational expectations model. Due to their optimism, experts get recapitalized faster during crises by earning a large risk premium, pushing the economy out of inefficient regions, and reducing the time the economy lives under a crisis.[6] However, optimistic experts bear too much leverage and risk on their balance sheets in the stochastic steady state, making it more likely that small negative shocks push the economy into a crisis. On average, this second effect turns out to be stronger than the first effect and implies a higher occupation time in a crisis state, leading to fatter left-tailed stationary distribution of experts' wealth share.

Our paper proposes a channel through which belief heterogeneity amplifies systemic shocks, contributing to a deeper understanding of what drives *excess volatility* from e.g., Shiller (1980). Furthermore, our model exhibits an *adverse feedback loop* between heterogeneous beliefs about financial market returns and the amplification of endogenous risk: more optimistic experts and pessimistic households jointly generate more amplified endogenous risks (and risk-premia) during financial crises, whereas in turn, an increase in the endogenous volatility during each crisis raises the degree of disagreement about expected capital asset returns across two groups,[7] which in turn amplifies risks during crises again, *ad infinitum*. To our knowledge, the amplification channel coming from this "two-way interaction" is novel to the literature.

We show that our model with financial frictions and disagreement explains the empirical excess time-series momentum of asset returns by making the economy vulnerable to adverse shocks. At the stochastic steady state, optimistic experts take more leverage, increasing the economy's endogenous risk to the point that an adverse shock pushes the economy into a financial crisis. Once the economy rebounds out of a crisis, the expected asset returns are lower again, leading to a momentum crash. This is in contrast to the rational expectations model where the endogenous risk is low at the stochastic steady state, and an adverse shock of the same magnitude does not have any effect on the economy. While a higher level of disagreement increases the endogenous risk at the stochastic steady state in our model, it also leads to a faster rebound in the capital price once a crisis hits. Therefore, our model predicts that the excess momentum effect is stronger when the disagreement is high, consistent with the empirical evidence that we document.

As we turn our eyes toward the cross-sectional asset pricing, our macro-level disagreement factor from The Survey of Professional Forecasters (SPF), based on heterogeneous analyst forecasts of quarter-on-quarter GDP growth, turns out to be an important factor in addition to the intermediary factor of HKM2017. Our disagreement factor significantly improves pricing power beyond the intermediary factor, and is not explained

[6] In contrast, as households are pessimistic in this case, they are willing to provide higher leverage to optimistic experts than under the rational expectations (i.e., households try to save more in risk-free bonds issued by optimistic experts).

[7] We fix the degree of disagreement about 'expected productivity growth rates' across two groups (i.e., optimists and pessimists), but a higher level of endogenous risk raises the degree of disagreement between experts and households about 'the holding returns' they earn when investing in capital.

by other potential factors in the literature[8] and uniquely positioned in explaining momentum and long-term reversal anomaly portfolio excess returns, as anticipated. Therefore, we verify our model's asset pricing implications about how disagreements affect the capital asset market equilibrium through both time-series and cross-sectional empirical exercises.

The elicitation of objectively correct beliefs about the underlying technological process (i.e., rational expectations) is out of the scope of this paper. All agents are assumed to have static beliefs about their expected technological growth rate as in Geanakoplos (2010). Both optimists and pessimists are dogmatic in their own views about the expected technological growth rate, and their ex-ante beliefs are not affected by the time variation of sentiments.[9] In this formulation, disagreement is persistent and two groups do not reach an agreement. This setting is in accordance with the literature where variation about individual beliefs about expected returns are due to individual fixed effects (Giglio et al. (2019)). In contrast to the models of e.g., Harrison and Kreps (1978), Scheinkman and Xiong (2003), and Geanakoplos (2010), we consider risk-averse, not risk-neutral agents.[10] In terms of the methodological side, we normalize the economy by technology to get our economy stationary, and rely on the Kolmogorov forward equation (KFE) in characterizing the ergodic distribution of our state variable.

Related Literature Our model builds on Brunnermeier and Sannikov (2014) with additional components of exogenous growth prospects and embeds heterogeneous beliefs in the spirit of Basak (2000). It relates to a large literature on continuous-time macro-finance models with financial frictions in incomplete market setting.[11] Recent work in this area has turned to quantitative models that explain macroeconomic and asset pricing moments either using a rational expectations model, or using belief mechanism.[12] Di Tella (2017) shows that uncertainty shocks drive balance sheet recessions even in cases when contracting on macroeconomic state variables is possible. We follow this line of literature but focus on the general equilibrium interaction between the endogenous risk and macroeconomic variables in the presence of heterogeneous beliefs about technological growth.

[8]For example, our result is robust even after we include the cay measure of Lettau and Ludvigson (2001) and the capital share risk of Lettau et al. (2019).

[9]While two groups of agents (i.e., optimists and pessimists) have dogmatic views about the expected technological growth, still their disagreement about the capital return process is affected by a current level of endogenous volatility, which reflects what we observe as 'sentiments' in the real world.

[10]We consider log-preference for all agents. The literature that uses risk-neutrality usually excludes short-sales for equilibrium to exist. Linearity pushes optimal solutions to boundaries, and it results in extreme behaviors. For example, in Geanakoplos (2010), all agents exposed to the risky asset default in the bad state.

[11]Most notable are Basak and Cuoco (1998), He and Krishnamurthy (2011), and He and Krishnamurthy (2013).

[12]Maxted (2022) builds a macro-finance model with diagnostic expectations, and Krishnamurthy and Li (2020) builds a model where agents update their beliefs about tail risk rationally. In a recent work, Gopalakrishna (2022) introduces stochastic productivity and state-dependent exit of experts into a canonical rational expectations model to generate amplified (in volatility and risk-premium) but 'slow-moving' financial crises.

Our paper derives the endogenous amplification of aggregate risks in general equilibrium, and see how optimism and pessimism, in addition to the technological differences across two groups, affects the amplification process. In our model, heterogeneous beliefs play crucial roles in economic dynamics as they facilitate trades. Simsek (2013) studies cases in which optimists borrow from pessimists using loans collateralized by the asset that optimists purchase, closer to Geanakoplos (2001). Since pessimists attach lower values to the asset, this collateral arrangement puts the endogenous borrowing constraint on optimists, affecting their leverage choices and asset prices. Our framework focuses on the amplification of endogenous risks, abstracts from collateral constraints and instead imposes the solvency constraint that holds at any point in any contingencies. We closely follow the literature that focuses on financial frictions, heterogeneous beliefs, and other deviations from the rational expectations, e.g., Harrison and Kreps (1978),[13] Detemple and Murthy (1997), Basak and Croitoru (2000), Basak (2000), Croitoru and Basak (2004), Gallmeyer and Hollifield (2008), Gertler et al. (2020), Dong et al. (2022),[14] Maxted (2022),[15] and Camous and Van der Ghote (2023). While Camous and Van der Ghote (2023) find that diagnostic expectations exacerbate financial instability, thereby shifting the stationary density of intermediary wealth share to the left. We arrive at the similar result with heterogeneous beliefs instead. Khorrami and Mendo (2023) build a model where sentiment-driven shocks behave like uncertainty shocks and the self-fulfilled beliefs about endogenous price volatility drive business cycle fluctuations.

This paper also relates to the empirical literature focusing on heterogeneous beliefs across different groups of market agents. For example, Welch (2000) finds that there is a large degree of heterogeneity in forecasted risk premium levels even across financial economists, while Beutel and Weber (2022) point out that individuals are heterogeneous both at the information acquisition and the processing stage, thereby forming their own beliefs and choosing portfolios based on those beliefs.[16] Our results are based on the heterogeneous *per-*

[13]Harrison and Kreps (1978) assume agents "agree to disagree" about their beliefs, and asset prices can exceed their fundamental values in that case. However, the underlying microfoundation for disagreement about the dividend stream's probability distribution is not provided. While, in our setting, how agents disagree about the underlying technological growth is given as exogenous, their disagreement about the expected capital return is derived in an endogenous manner.

[14]In Dong et al. (2022), optimistic investors buy risky assets with leverage provided by pessimists, pushing up asset prices like in our model. Higher asset prices relax the financial frictions imposed on high productivity firms, mitigating degrees of misallocations and raising aggregate output.

[15]Under our heterogeneous beliefs framework, investors "agree to disagree" in their beliefs about expected technological growth. For this issue, Morris (1995) provides a thorough justification of the heterogeneous priors formulation, arguing it being fully consistent with rationality. For a deviation from rational expectations, more recently, Maxted (2022) incorporates diagnostic expectations into a model with financial intermediaries based on He and Krishnamurthy (2013), and finds that interactions between diagnostic expectation and financial frictions generate both amplification (in the short-run) and reversal (in the long-run).

[16]The previous literature also points out that different groups in the economy (e.g., households, firms, professional forecasters, etc) form different expectations not just about risk-premium but about macroeconomic variables including inflation: see e.g., Coibion et al. (2020), Candia et al. (2021), and Weber et al. (2022). As risk-premium depends on the business cycle (e.g., Cooper and Priestley (2009)), we expect that forcasted risk-premium levels across groups would differ. For equity premium, Rapach et al. (2012) argues that despite the failure of *individual* out-of-sample forecasts to outperform the historical average, *combinations* of individual forecasts deliver significant out-of-sample gains relative to the historical average on a consistent basis over time.

ceived equity premium levels across two groups (i.e., optimists and pessimists), providing novel implication about the interaction between belief heterogeneity and crisis dynamics in a general equilibrium framework.

Caballero and Simsek (2020) provide a continuous-time *risk-centric* representation of the New-Keynesian model, based on which they analyze interactions among equilibrium asset prices, sentiments (i.e., optimism and pessimism), financial speculation, and macroeconomic outcomes when output is determined by aggregate demand, in the presence of stochastic changes in exogenous volatility.[17] Belief disagreements matter as they induce optimists to speculate during normal times, which exacerbates crashes by reducing their wealth when the economy transitions to recessions. We focus more on the amplification of endogenous risks in the presence of different kinds of belief heterogeneity instead, and introduce a novel feedback channel between risk amplification and disagreements about asset return prospects.

Outline The remainder of the paper is organised as follows: Section 2 sets out the basic framework. Section 3 characterizes the equilibrium in an analytic way. Then, Section 4 provides simulation results and discusses our model's implications. Finally, Section 6 concludes.

2 The Model

We develop a continuous-time framework with two types of agents, based on which we study how heterogeneous beliefs about technological growth affect leverage choices, asset prices and the endogenous financial volatility, where the endogenous risk itself affects the degree of belief heterogeneity in asset returns across groups. Our setting is analytical yet tractable, and incorporates exogenous technological growth and heterogeneous beliefs in a general equilibrium sense. Our model is built on e.g., Basak (2000) and Brunnermeier and Sannikov (2014).

2.1 Model Setup

We will begin with the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ which is endowed with a standard Brownian motion Z_t. We assume that $Z_0 = 0$, almost surely. All economic activity will be assumed to take place in the horizon $[0, \infty)$. Let

$$\mathcal{F}^Z(t) \triangleq \sigma\{Z_s;\ 0 \leq s \leq t\}, \forall t \in [0, T]$$

be the filtration generated by $Z(\dot{})$ and let $\mathcal{N}$ denote the $\mathcal{P}$- null subsets of $\mathcal{F}^Z(T)$. We shall use the augmented

[17] Caballero and Simsek (2020) defined optimists as those who believe that the probability that the next recession comes is lower during the normal region.

filtration as follows:

$$\mathcal{F}(t) \triangleq \sigma\{\mathcal{F}^Z(t) \cup \mathcal{N}\}, \forall t \in [0, T].$$

One should interpret the $\sigma-$algebra $\mathcal{F}(t)$ as the information available to agents at time t in a complete information setting, in the sense that if $\omega \in \Omega$ is the true state of nature and if $A \in \mathcal{F}(t)$, then all agents will know whether $\omega \in A$.

We consider an economy with two agents, optimists and pessimists. Both types of agents can own capital, but the former are able to use capital in a more productive way.[18]

2.1.1 Technology

The aggregate amount of capital in the economy is denoted by K_t and capital owned by an individual agent i by k_t^i , where $t \in [0, \infty)$ indicates time. Physical capital k_t^O held by optimists produces output at rate:

$$y_t^O \triangleq \gamma_t^O k_t^O, \quad \forall t \in [0, \infty) \tag{1}$$

per unit of time, where γ_t^O is an exogenous productivity parameter, which evolves according to:[19]

$$\frac{d\gamma_t^O}{\gamma_t^O} \triangleq \alpha dt + \sigma dZ_t, \quad \forall t \in [0, \infty), \tag{2}$$

where dZ_t are exogenous aggregate standard Brownian shocks defined above. In a world without fiat money, output is modeled as a numeraire, and therefore, its price is normalized to one. Capital owned by individual optimists, with state space $\mathscr{F}^k \subseteq \mathbb{R}$, satisfies the following Ito's process:

$$\frac{dk_t^O}{k_t^O} \triangleq \left(\Lambda^O(\iota_t^O) - \delta^O\right) dt, \quad \forall t \in [0, \infty), \tag{3}$$

where ι_t^O is the portion of the generated output (i.e., $y_t^O = \gamma_t^O k_t^O$) used in creating new capital (i.e., the total amount of investment during the infinitesimal period $(t, t + dt)$ is given by $\iota_t^O \gamma_t^O{}_t k_t^O dt$). Function $\Lambda^O(\cdot)$, which satisfies $\Lambda^O(0) = 0$, $\Lambda^{O\prime}(0) = 1$, $\Lambda^{O\prime}(\cdot) > 0$, and $\Lambda^{O\prime\prime}(\cdot) < 0$, represents a standard investment technology with adjustment costs. In case there is no investment, capital managed by the optimist depreciates at rate δ^O. The concavity of $\Lambda^O(\cdot)$ represents investment formation friction, that can be interpreted as adjustment costs of converting output to new capital and vice versa.[20]

[18]In Section 4, we consider the opposite case where pessimists use the capital in a more productive way.

[19]Therefore, unlike Brunnermeier and Sannikov (2014), we have stochastic growth which we model as exogenous (i.e., α and σ in (2) are exogenous). Later we normalize our economy by γ_t^O to make it stationary.

[20]In contrast to the literature (e.g., Brunnermeier and Sannikov (2014) and Gopalakrishna (2022)), we abstract from the capital risk usually assumed. Its inclusion does not change our results qualitatively.

Pessimists are less productive.[21] Capital managed by her, with state space $\mathscr{F}^k \subseteq \mathbb{R}$, produces the following output:

$$y_t^P \triangleq \gamma_t^P k_t^P, \quad \forall t \in [0, \infty), \tag{4}$$

with γ_t^P is given by $\gamma_t^P = l \cdot \gamma_t^O \leq \gamma^O$, where $l \leq 1$, and evolves according to:

$$\frac{d\gamma_t^P}{\gamma_t^P} = \frac{d\gamma_t^O}{\gamma_t^O}, \quad \forall t \in [0, \infty). \tag{5}$$

In other words, optimists have proportionally higher productivity than pessimists, where the proportionality is given by $l \leq 1$, and productivity of two groups of agents grows at the same rate at every instant. Finally, capital owned by pessimists, which we denote by k_t^P follows:

$$\frac{dk_t^P}{k_t^P} \triangleq \left(\Lambda^P(\iota_t^P) - \delta^P\right) dt, \quad \forall t \in [0, \infty), \tag{6}$$

where in the same way as above, ι_t^P is a capital-owning pessimist's investment rate per unit of output. The state space $\mathscr{F}^k$ satisfies the same conditions as for optimists and $\delta^P \geq 0$ is the depreciation rate when capital is managed by pessimists. We assume $\Lambda^P(\iota_t^P) = l \cdot \Lambda^O(\iota_t^P)$ with $l < 1$ for simplicity.[22]

2.1.2 Preferences

Optimists and pessimists have preferences that are generally characterized by the instantaneous utility function $u^i(c_t^i) : \mathbb{R}_+ \to \mathbb{R}$, where $i \in \{O, P\}$, and each group i has a constant discount factor ρ^i. The consumption space defined above must also be square-integrable:

$$\int_0^\infty \left|c_t^i\right|^2 dt < \infty.$$

With $c^i \equiv \{c_t^i\}_{t=0}^\infty$, agents of type $i \in \{O, P\}$ want to maximize their expected lifetime utility function which is given by:

$$U(c^i) \triangleq \int_0^\infty e^{-\rho^i t} u^i(c_t^i) dt, \quad \forall t \in [0, \infty). \tag{7}$$

The utility function obeys the standard assumption summarized below.

[21]Therefore, à la Brunnermeier and Sannikov (2014), optimists are financial experts while pessimists are households. In Section 4, we revisit this assumption and studies the opposite case where pessimists act as experts in our economy.

[22]Therefore, we effectively assume that both (i) pessimists' productivity in turning capital into output; and (ii) their productivity in turning output into capital are both lower than of optimists with the same proportionality $l \leq 1$. This is for tractability and even if we assume different proportionality for the two productivity measures, most results in this paper do not change qualitatively. Thus, we interchangeably call optimists 'experts' and pessimists 'households' similarly to Brunnermeier and Sannikov (2014).

Assumption 1 *The utility function $u^i : \mathbb{R}_+ \rightarrow \mathbb{R}$, is concave and continuously diffentiable. Also.*

$$u^{i'}(\cdot) \triangleq \frac{\partial u^i}{\partial c_t^i}(\cdot) > 0 \;\; for \;\; i \in \{O, P\}. \tag{8}$$

2.1.3 Market for Capital

Both optimists and pessimists have the opportunity to trade physical capital in a competitive market. We denote the equilibrium market price of capital in terms of output by p_t with state space $\mathscr{F}^p \subseteq \mathbb{R}$, and assume that it satisfies the following endogenous Ito's process:

$$\frac{dp_t}{p_t} \triangleq \mu_t^p dt + \sigma_t^p dZ_t, \tag{9}$$

where μ_t^p and σ_t^p are drift and volatility of the capital price process (9), respectively. Based on the definition, capital k_t^O costs $p_t k_t^O$ for optimists. Note that, in equilibrium, p_t, μ_t^p and σ_t^p are determined endogenously.

2.1.4 Return to Capital

A straightforward application of Ito's lemma in (3) and (9) reveals that when an optimists hold k_t^O units of capital at price p_t, the total value of this capital (i.e., $p_t k_t^O$) evolves according to:

$$\frac{d(p_t k_t^O)}{p_t k_t^O} = \left(\Lambda^O(\iota_t) - \delta^O + \mu_t^p\right) dt + \sigma_t^p dZ_t. \tag{10}$$

Hence, the total return that experts earn from capital (per unit of wealth invested) is given by:

$$dr_t^{Ok} = \frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} dt + \left(\Lambda^O(\iota_t^O) - \delta^O + \mu_t^p\right) dt + \sigma_t^p dZ_t. \tag{11}$$

Similarly, a pessimist earns the return of

$$dr_t^{Pk} = \frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} dt + \left(\Lambda^P(\iota_t^P) - \delta^P + \mu_t^p\right) dt + \sigma_t^p dZ_t. \tag{12}$$

2.1.5 Beliefs

Agents of the two groups commonly observe their productivity at any instant, but have incomplete information on its exact dynamics. With equation (2) and equation (5), we know that γ_t^O and γ_t^P follow:

$$\frac{d\gamma_t^P}{\gamma_t^P} = \frac{d\gamma_t^O}{\gamma_t^O} = \alpha dt + \sigma dZ_t, \quad \forall t \in [0, \infty). \tag{13}$$

Optimists and pessimists observe the realization of their productivity, γ_t^O and γ_t^P, respectively. However, they do not know their exact processes: agents have incomplete information about the productivity process, represented by filtration $\mathcal{F}^i(t)$, with $i = O, P$, which is possibly different from the actual filtration $\mathcal{F}(t)$. We assume that they can both calculate σ from the quadratic variation of γ_t^O and γ_t^P, but they cannot elicit the exact value of α from the time-series of their productivity. Therefore, although optimists and pessimists have equivalent probability measures $\mathcal{P}^i$, $i = O, P$, respectively, also equivalent to the actual measure $\mathcal{P}$, they disagree about the expected productivity growth. We model a simple type of disagreement here: both types of agents are *dogmatic* and have a fixed bias in their beliefs about the drift of their productivity processes, in a similar manner to Yan (2008). Thus, optimists believe that their productivity γ_t^O follows:

$$\frac{d\gamma_t^O}{\gamma_t^O} = \alpha^O dt + \sigma dZ_t^O, \quad \forall t \in [0, \infty), \tag{14}$$

where α^O is possibly different from the true α, and Z_t^O is a Brownian motion related to their filtration $\mathcal{F}^O(t)$. Similarly, pessimists believe that their productivity γ_t^P evolves according to

$$\frac{d\gamma_t^P}{\gamma_t^P} = \alpha^P dt + \sigma dZ_t^P, \quad \forall t \in [0, \infty). \tag{15}$$

where α^P is possibly different from the true α, and Z_t^P is a Brownian motion related to their filtration $\mathcal{F}^P(t)$. From (13), and (14), we obtain the following consistency condition for optimists:

$$Z_t^O = Z_t - \frac{\alpha^O - \alpha}{\sigma} t. \tag{16}$$

In other words, optimists regard Z_t^O, not Z_t, as the Brownian motion driving the business cycle, while Z_t^O is in fact not a Brownian motion under the true model. Likewise for pessimists, we get

$$Z_t^P = Z_t - \frac{\alpha^P - \alpha}{\sigma} t. \tag{17}$$

From (16) and (17), we obtain the following consistency condition:

$$Z_t^O = Z_t^P + \frac{\alpha^P - \alpha^O}{\sigma} t. \tag{18}$$

We focus mostly on cases where $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$[23] and call (18) the disagreement process. From (11) and (16), optimists think that the total return that they earn from capital (per unit of wealth invested) would

[23] In Section 4, we also study cases where $\alpha^O < \alpha$ and $\alpha^P > \alpha$: i.e., when experts are pessimists and less productive households are optimists. Our analytical results do not rely upon the relative sizes of α^O or α^P though.

be given by

$$dr_t^{Ok} = \frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} dt + \left(\Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^O - \alpha}{\sigma} \sigma_t^p \right) dt + \sigma_t^p dZ_t^O. \tag{19}$$

Likewise, pessimists believe that the total return that they earn from capital (per unit of wealth invested) is given by

$$dr_t^{Pk} = \frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} dt + \left(\Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^p \right) dt + \sigma_t^p dZ_t^P. \tag{20}$$

Two points regarding equation (19) and equation (20) are important to note: (i) with $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$, optimists (pessimists) believe that the expected capital gain they earn when investing in physical capital is higher (lower) than implied under the rational expectations; (ii) the degree of disagreement (i.e., $\frac{\alpha^i - \alpha}{\sigma} \sigma_t^p$ for $i \in \{O, P\}$) between two groups in terms of the 'expected' return becomes proportional to the endogenous capital price risk σ_t^p: thereby, a higher endogenous volatility σ_t^p raises the degree of belief heterogeneity in asset returns. We believe that it is a reasonable specification: the more turbulent the market becomes, given sample points it is harder to elicit the exact process for the business cycle and financial markets, thereby it becomes more likely that many investors disagree about the expected capital gain that they would receive if invest in capital.[24,25]

2.2 Optimist's Consumption-Portfolio Problem

The non-monetary net worth w_t^O of an optimist who invests fraction x_t of her wealth in capital and consumes with the rate c_t^O evolves according to:

$$dw_t^O = x_t w_t^O dr_t^{Ok} + (1 - x_t) r_t w_t^O dt - c_t^O dt, \tag{21}$$

where r_t is the risk-free interest rate prevailing in the economy. Note that r_t is an equilibrium object to be determined endogenously. x_t represents the share of wealth optimists invest in capital. Later, it will turn out that in most cases optimists use greater-than-0 leverage (i.e., $x_t > 1$). For convenience, we call x_t 'leverage' of optimists.

Formally, each optimist solves

[24] This is a one side of the two-way interaction between belief heterogeneity and amplified volatility. In Section 4, we illustrate that our model generates the other direction: i.e., our heterogeneous belief specification amplifies the endogenous volatility σ_t^p during the crisis regime.

[25] Compared with Brunnermeier and Sannikov (2014) where more productive experts have higher dividend yields given the same investment amount, our framework features an additional channel where they become more optimistic about future returns through higher capital gain in the case of $\alpha^O \geq \alpha$.

$$\max_{\iota_t^O \geq 0, x_t \geq 0, c_{t,}^O \geq 0} \mathbb{E}_0^O \left[\int_0^\infty e^{-\rho^O t} u^O \left(c_t^O \right) dt \right], \tag{22}$$

subject to the solvency constraint $w_t^O \geq 0$ and the dynamic budget constraint (21). In optimization (22), the expectation operator $\mathbb{E}^O$ means that optimists believe dZ_t^O, not dZ_t, is the true Brownian motion driving their own filtration $\mathcal{F}^O(t)$. Therefore in characterizing (21), they use (19) for the capital return process dr_t^{Ok} instead of (11) with the true dZ_t. Note that we assume away short-sale for capital, thereby $x_t \geq 0$.

2.3 Pessimist's Consumption-Portfolio Problem

In a similar way to optimists' problem, the non-monetary net worth w_t^P of pessimists who invest fraction $\underline{x}_t$ of their wealth in capital and consume with the rate c_t^P would follow

$$dw_t^P = \underline{x}_t w_t^P dr_t^{Pk} + w_t^P (1 - \underline{x}_t) r_t dt - c_t^P dt. \tag{23}$$

Formally, each pessimist solves

$$\max_{\iota_t^P \geq 0, \underline{x}_{t,} \geq 0, c_{t,}^P \geq 0} \mathbb{E}_0^P \left[\int_0^\infty e^{-\rho^P t} u \left(c_t^P \right) dt \right], \tag{24}$$

subject to the solvency constraint $w_t^P \geq 0$ and the dynamic budget constraints (23). Likewise, in optimization (24), the expectation operator $\mathbb{E}^P$ means that pessimists believe dZ_t^P, not dZ_t, is the true Brownian motion driving their own filtration $\mathcal{F}^P(t)$. Therefore in characterizing (23), they use (20) for the capital return process dr_t^{Pk} instead of (12) with the true dZ_t. Note that we assume away short-sale for capital, thereby $\underline{x}_t \geq 0$.

2.4 Equilibrium

Intuitively, an equilibrium with full information is characterized by a map from shock histories $\{Z_S, s \in [0, t]\}$, to prices p_t and asset allocations such that, given prices, agents maximize their expected utilities[26]and markets clear.[27] To define our equilibrium more formally, we denote the set of optimists to be an interval $I = [0, 1]$ and index any optimist (i.e., expert if $\alpha^O > \alpha$) by $i \in I = [0, 1]$. Similarly, we denote the set of pessimists by $J = (1, 2]$ with index $j \in J = (1, 2]$. Since two groups (i.e., optimists and pessimists) both deviate from the benchmark model with the perceived shocks Z_t^i, $i \in \{O, P\}$, each group of agents has their own expectation operator $\mathbb{E}^i$, $i \in \{O, P\}$ as already shown in (22) and (24). Note that in equilibrium, every agent in the same

[26]Optimists solve optimization (22) subject to their dynamic budget constraint (21) and the solvency $w_t^O \geq 0$ while pessimists solve optimization (24) subject to their dynamic budget constraint (23) and the solvency $w_t^P \geq 0$.

[27]The physical capital, output, and debt markets must clear in equilibrium.

group (i.e., optimists or pessimists) chooses the same consumption and portfolio decisions.

We now proceed by stating the market clearing conditions and formally defining the equilibrium.

2.5 Market Clearing

The three markets that must clear in equilibrium at any instant are the capital, commodity and debt markets.

2.5.1 Capital Market

The total amount of capital demanded by optimists and pessimists should be equal to the aggregate supply of capital in the economy: i.e.,

$$\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj = K_t, \ \ \forall t \in [0, \infty), \tag{25}$$

where K_t is the total supply of capital. The total supply of capital in the model is not fixed as both optimists and pessimists invest in the new capital through their investments. The following equation (26) describes the evolution of the total supply of capital:

$$dK_t \triangleq \left(\int_0^1 \left(\Lambda^O \left(\iota_t^i \right) - \delta^O \right) k_t^i di + \int_1^2 \left(\Lambda^P \left(\underline{\iota}_t^j \right) - \delta^P \right) \underline{k}_t^j dj \right) dt, \ \ \forall t \in [0, \infty). \tag{26}$$

2.5.2 Good Market

Whatever is produced and not invested, has to be consumed. That is,[28]

$$\int_0^1 k_t^i \left(\gamma_t^O - \iota_t^i \gamma_t^O \right) di + \int_1^2 k_t^j \left(\gamma_t^P - \iota_t^j \gamma_t^P \right) dj = \int_0^1 c_t^i di + \int_1^2 c_t^j dj, \ \ \forall t \in [0, \infty). \tag{27}$$

2.5.3 Debt Market

The debt market clearing condition implies that the value of the debt that optimists receive should be equal to the value of loans that the pessimists extend, namely,

$$\int_0^1 \left(w_t^i - p_t k_t^i \right) di + \int_1^2 \left(w_t^j - p_t k_t^j \right) dj = 0. \tag{28}$$

By defining all three markets, we are in a position to define the economy's equilibrium.

Definition 1 *The economy's equilibrium consists of stochastic processes of (i) the price of capital p_t; (ii) interest rate r_t; (iii) investment and consumption, i.e.* $\left\{ \left(k_t^i, \iota_t^i, c_t^i \right), t \geq 0, i \in [0,1] \right\}$ *for optimists and* $\left\{ \left(k_t^j, \iota_t^j, c_t^j \right), t \geq 0, j \in (1,2] \right\}$ *for pessimists. These processes should satisfy the following three conditions:*

[28] In equation (27), we use the fact that the investment rate at time t is given by $i_t^i \gamma_t^i k_t^i$ for $i \in [0, 2]$.

1. Given their perceived Brownian motion Z_t^O and Z_t^P, respectively, optimists $i \in [0,1]$ solve optimization (22) subject to their dynamic budget constraint (21) and the solvency $w_t^i \geq 0$ while pessimists $j \in (1,2]$ solve optimization (24) subject to their dynamic budget constraint (23) and the solvency $w_t^j \geq 0$.

2. Capital (i.e., (25) and (26)), consumption (i.e., (27)), and debt (i.e., (28)) markets clear.

3. Consistency condition holds. In other words, the disagreement process (18) must hold.

3 Characterization for Equilibrium

In this section, we will discuss how to find the equilibrium price p_t, both optimists and pessimists' consumption decisions, the risk-free interest rates r_t as well as the endogenous drift μ_t^p and volatility σ_t^p of the capital price p_t's process, given the history of perceived shock processes $\{Z_s^O, Z_s^P, 0 \leq s \leq t\}$.

We first start with some definitions.

Definition 2 *The aggregate wealth of both optimists and pessimists is given by summing up their individual wealth respectively, that is,*[29]

$$W_t = \int_0^1 w_t^i di \qquad \underline{W}_t = \int_1^2 w_t^j dj, \quad \forall t \in [0, \infty).$$

Observe that the capital market clearing condition (25) and the debt market clearing condition (28) become

$$\int_0^1 x_t^i w_t^i di + \int_1^2 \underline{x}_t^j w_t^j dj = p_t K_t, \quad \forall t \in [0, \infty), \tag{29}$$

where K_t is the supply of aggregate capital that follows the process (26) and

$$W_t + \underline{W}_t = p_t K_t \tag{30}$$

holds from the debt market equilibrium condition (28). And the good market equilibrium condition (27) can be written as

$$\int_0^1 \frac{x_t^i w_t^i}{p_t} \left(\gamma_t^O - \iota_t^O \gamma_t^O \right) di + \int_1^2 \frac{\underline{x}_t^j w_t^j}{p_t} \left(\gamma_t^P - \underline{\iota}_t^P \gamma_t^P \right) dj = \int_0^1 c_t^i di + \int_1^2 c_t^j dj. \tag{31}$$

We now provide the definition of the economy's state variable, the wealth share that optimists possess. By representing each variable in the model in terms of optimists' wealth share which is bounded between 0 and 1, we can fully characterize the equilibrium price and quantity variables in a similar manner to Brunnermeier and Sannikov (2014).

[29] Actually, $w_t^i = w_t^O$ in equation (21) for $i \in [0,1]$. Likewise, $w_t^j = w_t^P$ in equation (23) for $j \in (1,2]$.

The proportion of wealth that optimists possesses which we denote by η_t is given by,

$$\eta_t = \frac{W_t}{W_t + \underline{W}_t}. \tag{32}$$

We further postulate that the dynamics of η_t, which is endogenous, evolve with the process:

$$\frac{d\eta_t}{\eta_t} = \mu_t^{\eta}(\eta_t)dt + \sigma_t^{\eta}(\eta_t)dZ_t. \tag{33}$$

Based on (30) and (32), we observe that η_t can be written as

$$\eta_t = \frac{W_t}{p_t K_t}. \tag{34}$$

The economy's entire dynamicswill be driven by η_t in our Markov equilibrium in a similar way to Brunnermeier and Sannikov (2014). Since every agent of the same group solves the same portfolio problem, let $x_t = x_t^i$ for $i \in [0, 1]$ for optimists in equilibrium. Evidently, in equilibrium we have,

$$x_t \leq \frac{1}{\eta_t}, \tag{35}$$

which translates to the fact that the maximum 'leverage' that optimists can obtain is bounded above by $\frac{1}{\eta_t}$. In this paper we identify two regions: the first one is when (35) binds and the second one is when leverage is strictly less than $\frac{1}{\eta_t}$, i.e., when (35) does not bind. We call the first region '*normal*' and the second region by '*crisis*'. In other words, in the *normal* region, all the physical capital will be owned by optimists, while in the *crisis* regime, some of the capital must be purchased by pessimists.

3.1 Internal Investment

Hereafter, we proceed by defining the investment functions $\Lambda^O(\cdot)$ that optimists use as follows:

$$\Lambda^O(\iota_t^O) = \frac{1}{k}\left(\sqrt{1 + 2k\iota_t^O} - 1\right), \ \ \forall t \in [0, \infty), \tag{36}$$

which satisfies all standard assumptions discussed in Section 2.1.1. We do not allow for disinvestment, thus $i_t^O \geq 0$ for $i \in [0, 1]$. We use the mathematical form in (36) of $\Lambda^O(i_t^i)$ for simplicity and acknowledge that our results do not change qualitatively even if we use different forms for $\Lambda^O(\cdot)$ that satisfy conditions in Section 2.1.1. Similarly as we already explained in Section 2.1.1, we define the internal investment function $\Lambda^P(i_t^P)$ that pessimists use as

$$\Lambda^P(\iota_t^P) = l \cdot \Lambda^O(\iota_t^P), \ \ \forall \iota_t^P. \tag{37}$$

From now on, we express our equilibrium with the following normalized asset price:

Definition 3 *The normalized asset price q_t is defined as $q_t \equiv \dfrac{p_t}{\gamma_t^O}$.*

The normalized asset price q_t can be interpreted also as the price-earnings ratio of physical capital. It turns out that when we write our model in terms of q_t instead of p_t, we can characterize the stationary equilibrium, as we have exogenous growth in our economy.

3.2 Solving an Optimist's Consumption-Portfolio Problem

An optimist $i \in [0,1]$ solves the optimization (22) subject to her wealth dynamics (21) and the solvency $w_t^i \geq 0$. We now fully characterize her optimal consumption c_t^{O*}, her optimal investment i_t^{O*}, and the equilibrium risk free interest rate r_t^*. We also focus on the case where all the optimists have the same logarithmic utility function, i.e., $u^O(c_t^O) = \log c_t^O$, for mathematical tractability.

Proposition 1 *Assume that all optimists have the same logarithmic utility, i.e., $u^O(c_t^O) = \log c_t^O$. Then:*
(i) The optimal consumption c_t^{O*} is given by:

$$c_t^{O*} = \rho^O w_t^O.$$

(ii) The equilibrium interest rate r_t^* is given by:

$$r_t^* = \left(\frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma} \sigma_t^p \right) - x_t \left(\sigma_t^p\right)^2,$$

where x_t is her optimal portfolio choice (i.e., leverage) as defined in (21).
(iii) The optimal investment rate ι_t^{O*} is given by:

$$\iota_t^{O*}(q_t) = \frac{q_t^2 - 1}{2k}.$$

Proof. Since optimists believe that dZ_t^O, not dZ_t, is the true Brownian motion driving the business cycle, they believe the wealth share η_t follows:

$$\frac{d\eta_t}{\eta_t} = \left(\mu_t^\eta(\eta_t) + \frac{\alpha^O - \alpha}{\sigma} \sigma_t^\eta(\eta_t) \right) dt + \sigma_t^\eta(\eta_t) dZ_t^O, \tag{38}$$

which is consistent with the true η_t process in (33). Based on Merton (1971), we conjecture her value function $V(\cdot)$ will depend on her own wealth w_t^O and the aggregate wealth share of optimists η_t with the following form:

$$V\left(w_t^O, \eta_t\right) = \frac{\log w_t^O}{\rho^O} + f\left(\eta_t\right). \tag{39}$$

Based on (21) and (38), the Hamiltonian-Jacobi-Bellman (HJB) equation for an individual optimist's problem can be written as[30,31]

$$\rho^O V(\cdot) = \max_{x_t, c_t^O, i_t^O} \log c_t^O + \left[w_t^O \left(x_t \left(\frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma} \sigma_t^p \right) + (1 - x_t) r_t \right) - c_t^O \right] \frac{dV_t}{dw_t^O}(\cdot)$$
$$+ \frac{\left(x_t w_t^O \sigma_t^p\right)^2}{2} \frac{d^2 V_t}{d\left(w_t^O\right)^2}(\cdot) + \left(\eta_t \left(\mu_t^\eta(\eta_t) + \frac{\alpha^O - \alpha}{\sigma} \sigma_t^\eta(\eta_t) \right) \right) \frac{dV}{d\eta_t}(\cdot) + \frac{\left(\eta_t \sigma_t^\eta(\eta_t)\right)^2}{2} \frac{d^2 V}{d\eta_t^2}(\cdot).$$

The first order condition with respect to c_t^O are given by:

$$\frac{1}{c_t^{O*}} = \frac{dV_t}{dw_t^O}\left(w_t^O, \eta_t\right) = \frac{1}{\rho^O w_t^O}, \tag{40}$$

which gives $c_t^{O*} = \rho^O w_t^O$ at optimum. As every aggregate variable including q_t, i_t^O, μ_t^p, σ_t^p, x_t, and r_t will be expressed as functions of the aggregate wealth share η_t, Merton (1971) justifies our choice of value function in (39). The first-order condition with respect to x_t gives the optimal portfolio choice given in (ii).

To prove (iii), we observe that investment appears only in a intra-temporal manner. Thus the corresponding optimization problem is static and satisfies the first-order condition: $\Lambda^{O\prime}(i_t) = q_t^{-1}$ (i.e., marginal Tobin's q). With the form of $\Lambda^O(i_t)$ from Section 3.1, we finally obtain:

$$i_t^{O*}(q_t) = \frac{q_t^2 - 1}{2k}. \tag{41}$$

■

Note from (ii) that given i_t^O, μ_t^p, σ_t^p, and x_t, the equilibrium interest rate r_t^* increases due to the distorted belief of optimists when $\alpha^O > \alpha$. If optimists believe that the expected capital gain they earn when investing in capital will be higher, they will try to get more loans from pessimists, raising the equilibrium interest rate r_t^*.

3.3 Solving a Pessimist's Consumption-Portfolio Problem

Each pessimist $j \in (1, 2]$ optimizes her lifetime utility (24) subject to the evolution of his wealth (23) and the solvency $w_t^j \geq 0$. The following Proposition 2 solves her optimal choices for consumption c_t^{P*}, leverage $\underline{x}_t$,

[30] We use the following relation from equation (19):

$$\mathbb{E}_t^O\left(dr^{Ok}\right) = \left(\frac{\gamma_t^O - \iota_t^O \gamma_t^O}{p_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma}\sigma_t^p \right) dt = \left(\frac{1 - \iota_t^O}{q_t} + \Lambda^O(\iota_t^O) - \delta^O + \mu_t^p + \frac{\alpha^0 - \alpha}{\sigma}\sigma_t^p \right) dt.$$

[31] Due to the assumed value function form in (39), we ignore the cross-derivative term of $V(\cdot)$ with respect to w_t^O and η_t.

and investment ι_t^{P*}. Note that $\underline{x}_t \geq 0$ must hold since the short-sale of capital is not allowed.

Proposition 2 *Assume that all pessimists have the same logarithmic utility, i.e., $u^P(c_t^P) = \log c_t^P$. Then:*

(i) The optimal consumption c_t^{P*} is given by:

$$c_t^{P*} = \rho^P w_t^P.$$

(ii) The optimal portfolio choice (i.e., leverage) $\underline{x}_t$ as defined in (23) is given by:[32]

$$\underline{x}_t = \max\left\{ \frac{\left(\dfrac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \dfrac{\alpha^P - \alpha}{\sigma}\sigma_t^p\right) - r_t^*}{\left(\sigma_t^q\right)^2}, 0 \right\}$$

(iii) The optimal investment rate ι_t^{P*} is given by:

$$i_t^{P*}(q_t) = \frac{q_t^2 - 1}{2k},$$

which is the same as i_t^{O*}. Therefore, optimists and pessimists have the same rate of investment, i.e., $i_t^{O*}(\cdot) = i_t^{P*}(\cdot)$.

Proof. Since pessimists believe that dZ_t^P, not dZ_t, is the true Brownian motion driving the business cycle, they believe the wealth share η_t follows:

$$\frac{d\eta_t}{\eta_t} = \left(\mu_t^\eta(\eta_t) + \frac{\alpha^P - \alpha}{\sigma}\sigma_t^\eta(\eta_t)\right)dt + \sigma_t^\eta(\eta_t)dZ_t^P, \tag{42}$$

which is consistent with the true η_t process in (33). Based on Merton (1971), we conjecture her value function $\underline{V}(\cdot)$ will depend on her own wealth w_t^P and the aggregate wealth share of optimists η_t as the state variable:

$$\underline{V}\left(w_t^P, \eta_t\right) = \frac{\log w_t^P}{\rho^P} + \underline{f}\left(\eta_t\right). \tag{43}$$

[32]Therefore, if pessimists' *perceived* risk-premium levels are below 0, they only invest in risk-free loans issued by optimists.

Based on (23) and (42), the Hamiltonian-Jacobi-Bellman (HJB) equation for an individual optimist's problem can be written as[33]

$$\rho^P \underline{V}(\cdot) = \max_{\underline{x}_t \geq 0, c_t^P, i_t^P} \log c_t^P + \left[w_t^P \left(\underline{x}_t \left(\frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^p \right) + (1 - \underline{x}_t) r_t \right) - c_t^P \right] \frac{d\underline{V}_t}{dw_t^P}(\cdot)$$
$$+ \frac{\left(\underline{x}_t w_t^P \sigma_t^p\right)^2}{2} \frac{d^2 \underline{V}_t}{d\left(w_t^P\right)^2}(\cdot) + \left(\eta_t \left(\mu_t^\eta(\eta_t) + \frac{\alpha^P - \alpha}{\sigma} \sigma_t^\eta(\eta_t) \right) \right) \frac{d\underline{V}}{d\eta_t}(\cdot) + \frac{\left(\eta_t \sigma_t^\eta(\eta_t)\right)^2}{2} \frac{d^2 \underline{V}}{d\eta_t^2}(\cdot).$$

The first order condition with respect to c_t^P are given by:

$$\frac{1}{c_t^{P*}} = \frac{d\underline{V}_t}{dw_t^P}\left(w_t^P, \eta_t\right) = \frac{1}{\rho^P w_t^P}, \tag{44}$$

which gives $c_t^{P*} = \rho^P w_t^P$ at optimum. As every aggregate variable including q_t, i_t^P, μ_t^p, σ_t^p, $\underline{x}_t$, and r_t will be expressed as functions of the aggregate wealth share η_t, Merton (1971) justifies our choice of value function in (43). The first-order condition with respect to x_t gives the optimal portfolio choice given in (ii).

To prove (iii), we observe that investment appears only in a intra-temporal manner. Thus the corresponding optimization problem is static and satisfies the first-order condition: $\Lambda^{P\prime}(i_t) = l \cdot q_t^{-1}$. With equation (37) from Section 3.1 that $\Lambda^P(\cdot) = l \cdot \Lambda^O(\cdot)$, we finally obtain:

$$i_t^{P*}(q_t) = \frac{q_t^2 - 1}{2k}, \tag{45}$$

which is the same as $i_t^{O*}(q_t)$ in (41).

■

Finally, in order to conclude with the characterization of equilibrium, we need to derive the evolution of the state variable η_t. First, we define the fraction of physical capital held by optimists and pessimists by

$$\psi_t \equiv \frac{\int_0^1 k_t^i di}{K_t} = \frac{k_t^O}{K_t}, \quad 1 - \psi_t = \frac{\int_1^2 k_t^j dj}{K_t} = \frac{k_t^P}{K_t}, \tag{46}$$

where the aggregate capital K_t follows the process (26). Then leverages of two groups of agents, i.e., x_t and $\underline{x}_t$, respectively, can be also characterized by the following proposition.

[33] We use the following relation from equation (20):

$$\mathbb{E}_t^P\left(dr^{Pk}\right) = \left(\frac{\gamma_t^P - \iota_t^P \gamma_t^P}{p_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma}\sigma_t^p\right) dt = \left(l \cdot \frac{1 - \iota_t^P}{q_t} + \Lambda^P(\iota_t^P) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma}\sigma_t^p\right) dt.$$

Proposition 3 *In equilibrium, leverages of two groups of agents, i.e.,* x_t *and* $\underline{x}_t$ *are given by*

$$x_t = \frac{\psi_t}{\eta_t} \qquad \underline{x}_t = \frac{1-\psi_t}{1-\eta_t} \tag{47}$$

Proof. It follows immediately from the definitions of wealth share and capital share, i.e., (34) and (46).

■

Thus the evolution of the proportion of wealth held by optimists is established in the following Proposition 4. Note that in Proposition 4, we express the aggregate dynamics in the true Brownian motion dZ_t.

Proposition 4 *The evolution of optimists' wealth relative to the entire economy is given by*

$$\frac{d\eta_t}{\eta_t} = \mu^{\eta}(\eta_t)dt + \sigma^{\eta}(\eta_t)dZ_t \tag{48}$$

where

$$\mu^{\eta}(\eta_t) = \left(\frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p\right)^2 - \frac{\psi_t - \eta_t}{\eta_t}\frac{\alpha^O - \alpha}{\sigma}\sigma_t^p + \frac{1-\iota_t^O}{q_t} + (1-\psi_t)\left(\delta^P - \delta^O\right) + (1-l)\left(1-\psi_t\right)\Lambda^O(i_t^O) - \rho^O,$$

and

$$\sigma^{\eta}(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p.$$

Proof. Optimists' aggregate wealth W_t defined in Definition 2 would evolve with the process

$$dW_t = r_t W_t dt + \psi_t p_t K_t \left(dr_t^{Ok} - r_t dt\right) - c_t^O dt, \tag{49}$$

where we use $c_t^i = c_t^O$ for $i \in [0,1]$ for all optimists.[34] By applying Ito's quotient rule on (34),[35] we have

$$\frac{d\eta_t}{\eta_t} = \frac{dW_t}{W_t} - \frac{d\left(p_t K_t\right)}{p_t K_t} + \left(\frac{d\left(p_t K_t\right)}{p_t K_t}\right)^2 - \frac{dW_t}{W_t}\frac{d\left(p_t K_t\right)}{p_t K_t}. \tag{50}$$

In addition, from the process (26) of the aggregate capital K_t, we obtain

$$\begin{aligned} \frac{1}{K_t}\frac{dK_t}{dt} &= \left(\Lambda^O\left(i_t^O\right) - \delta^O\right)\psi_t + \left(\Lambda^P\left(i_t^P\right) - \delta^P\right)(1-\psi_t) \\ &= \left(\Lambda^O\left(i_t^O\right) - \delta^O\right) - \left(\delta^P - \delta^O\right)(1-\psi_t) - (1-\psi_t)(1-l)\Lambda^O\left(i_t^O\right), \end{aligned} \tag{51}$$

[34] We know $c_t^O = \rho^O W_t$ holds at optimum from equation (40).

[35] Ito's quotient rule states that:

$$\frac{d\left(XY^{-1}\right)}{XY^{-1}} = \frac{dX}{X} - \frac{dY}{Y} + \left(\frac{dY}{Y}\right)^2 - \frac{dX}{X}\frac{dY}{Y}.$$

where we used the property that $i_t^O = i_t^P$ in equilibrium as seen in (41) and (45). Applying Ito's product rule to the price process (9) and the capital process (51), and comparing with (19), we obtain

$$\frac{d\left(p_t K_t\right)}{p_t K_t} = dr_t^{Ok} - \frac{1-\iota_t^O}{q_t}dt - \left(\delta^P - \delta^O\right)\left(1-\psi_t\right)dt - (1-\psi_t)(1-l)\Lambda^O\left(i_t^O\right)dt. \tag{52}$$

Since $\mathbb{E}_t^O\left(dr_t^{Ok}\right) - r_t = x_t\left(\sigma_t^p\right)^2$ from the optimists' optimal portfolio decision à la Merton (1971), and from the fact that

$$\mathbb{E}_t^O\left(dr_t^{Ok}\right) = \mathbb{E}_t\left(dr_t^{Ok}\right) + \frac{\alpha^O - \alpha}{\sigma}\sigma_t^p dt, \tag{53}$$

where $\mathbb{E}_t$ is the expectation operator corresponding to the true model, we can finally plug in (49), (52), and (53) into (50) and obtain

$$\begin{aligned}\frac{d\eta_t}{\eta_t} = &\left(\frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p\right)^2 dt - \frac{\psi_t - \eta_t}{\eta_t}\frac{\alpha^O - \alpha}{\sigma}\sigma_t^p dt + \frac{1-\iota_t^O}{q_t}dt + \left(1-\psi_t\right)\left(\underline{\delta}^P - \delta^O\right)dt \\ &+ \left(1-l\right)\left(1-\psi_t\right)\Lambda^O\left(i_t^O\right)dt - \rho^O dt + \frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p dZ_t,\end{aligned} \tag{54}$$

where we used $c_t^O = \rho^O W_t$ in the equilibrium. From (54), we finally obtain

$$\mu^\eta(\eta_t) = \left(\frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p\right)^2 - \frac{\psi_t - \eta_t}{\eta_t}\frac{\alpha^O - \alpha}{\sigma}\sigma_t^p + \frac{1-\iota_t^O}{q_t} + \left(1-\psi_t\right)\left(\delta^P - \delta^O\right) + \left(1-l\right)\left(1-\psi_t\right)\Lambda^O(i_t^O) - \rho^O, \tag{55}$$

and

$$\sigma^\eta(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\sigma_t^p. \tag{56}$$

■

Now by knowing the drift μ_t^η and the volatility σ_t^η of our state variable η, we calculate the the drift μ_t^p and the volatility σ_t^p of the price of capital p_t.

Proposition 5 *In the Markov equilibrium where q_t is a function of our state variable η_t with $q_t = q(\eta_t)$, the drift μ_t^p of the price of capital p_t is given by*

$$\mu_t^p = \alpha + \frac{q'\left(\eta_t\right)}{q_t}\mu^\eta(\eta_t)\eta_t + \frac{1}{2}\sigma^\eta(\eta_t)\eta_t\frac{q''(\eta_t)}{q(\eta_t)} + \sigma\sigma^\eta(\eta_t)\eta_t\frac{q'\left(\eta_t\right)}{q(\eta_t)}, \tag{57}$$

where $\sigma_t^\eta\left(\eta_t\right)$ is given by equation (56) and the volatility of the price of capital, σ_t^p, is given by

$$\sigma_t^p = \frac{\sigma}{1-\left(x_t\eta_t - \eta_t\right)\dfrac{q'\left(\eta_t\right)}{q(\eta_t)}}. \tag{58}$$

Proof. Applying Ito's lemma to the Markov relation $q_t = q(\eta_t)$, we obtain

$$dq_t = q'(\eta_t)\, d\eta_t + \frac{1}{2} q''(\eta_t)\,(d\eta_t)^2. \tag{59}$$

Plugging in equation (48) to (59) and using $(d\eta_t)^2 = \eta_t^2 \sigma^\eta(\eta_t)^2 dt$, we have

$$\frac{dq_t}{q_t} = \left(\eta_t \mu^\eta(\eta_t)\frac{q'(\eta_t)}{q(\eta_t)} + \frac{\eta_t^2 \sigma^\eta(\eta_t)^2}{2}\frac{q''(\eta_t)}{q(\eta_t)}\right) dt + \eta_t \sigma^\eta(\eta_t)\frac{q'(\eta_t)}{q(\eta_t)} dZ_t. \tag{60}$$

Now, with the definition of normalized asset price (i.e., price-earnings ratio) $q_t = \frac{p_t}{\gamma_t^O}$, we also have:[36]

$$\frac{dq_t}{q_t} = \left(\mu_t^p - \alpha + \sigma^2 - \sigma_t^p \sigma\right) dt + \left(\sigma_t^p - \sigma\right) dZ_t. \tag{61}$$

Comparing (60) and (61) yields:

$$\sigma_t^p = \sigma + \eta_t \sigma^\eta(\eta_t)\frac{q'(\eta_t)}{q(\eta_t)}, \tag{62}$$

and

$$\mu_t^p = \alpha + \eta_t \mu^\eta(\eta_t)\frac{q'(\eta_t)}{q(\eta_t)} + \frac{\eta_t^2 \sigma^\eta(\eta_t)^2}{2}\frac{q''(\eta_t)}{q(\eta_t)} + \sigma\sigma_t^\eta \eta_t \frac{q'(\eta_t)}{q(\eta_t)}. \tag{63}$$

From (56) in Proposition 4, we have that

$$\sigma^\eta(\eta_t) = \frac{x_t\eta_t - \eta_t}{\eta_t}\sigma_t^p, \tag{64}$$

where we used $x_t \eta_t = \psi_t$ from (47). Therefore, by (62) and (64) we have

$$\sigma_t^p = \frac{\sigma}{1 - (x_t\eta_t - \eta_t)\dfrac{q'(\eta_t)}{q(\eta_t)}}. \tag{65}$$

■

3.4 Price of Capital and Optimal Leverages

We derive a closed formed solution for leverage and a first-order differential equation for the price of capital. The following theorems summarise the main results of the paper.

Proposition 6 *The equilibrium domain consists of sub-intervals $[0, \eta^\psi)$, where $\psi(\eta) < 1$, and $[\eta^\psi, 1]$ where $\psi(\eta) = 1$.*

[36] Note that in equation (61), we use the 'true' process for γ_t^O in (13).

The capital price function $q(\eta)$ in our Markov equilibrium satisfies:

$$q(0) = l \cdot \frac{1 - i(q(0))}{\rho^P} \tag{66}$$

and

$$q(\eta) = \frac{1 - i(q(\eta))}{\rho^P(1-\eta) + \rho^O \eta} \quad on \quad [\eta^\psi, 1] \tag{67}$$

The following procedures can be used to compute $\psi(\eta)$ and $q'(\eta)$ from $(\eta, q(\eta))$ on $(\eta^\psi, 0)$:

(i) Find ψ that satisfies

$$\left(\rho^P(1-\eta) + \rho^O \eta\right) q(\eta) = \psi + (1-\psi)l - i(q)\left(\psi + (1-\psi)l\right) \tag{68}$$

(ii) Compute $q(\eta)$ where $q(\eta)$ is given by the solution of the equation:

$$\frac{(1-l)(1-i(q(\eta)))}{q(\eta)} + (1-l)\Lambda^O\left(i(q(\eta))\right) + \delta^P - \delta^O + \frac{\alpha^O - \alpha^P}{\sigma}\sigma^p(\eta) = \left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right)\sigma^p(\eta)^2, \tag{69}$$

where $\sigma^p(\eta)$ is given by (65). From (69), $\sigma^p(\eta)$ can also be expressed as:

$$\sigma^p(\eta) = \frac{\frac{\alpha^O - \alpha^P}{\sigma} + \sqrt{\left(\frac{\alpha^O - \alpha^P}{\sigma}\right)^2 + 4\left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right)\left(\frac{(1-l)(1-i(q))}{q(\eta)} + (1-l)\Lambda^O(i(q)) + \delta^P - \delta^O\right)}}{2\left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right)}, \tag{70}$$

where $q = q(\eta)$.

Proof. From the good market equilibrium condition (27), we have

$$k_t^O\left(\gamma_t^O - \iota_t^O \gamma_t^O\right) + k_t^P\left(\gamma^P - \iota_t^P \gamma^P\right) = c_t^O + c_t^P, \ \ \forall t \in [0, \infty). \tag{71}$$

Observing that $c_t^O = \rho^O w_t^O$ and $c_t^P = \rho^P w_t^P$ in equilibrium, we can divide (71) by $p_t K_t$ and obtain

$$\left(\rho^P(1-\eta) + \rho^O \eta\right) q = \psi + (1-\psi)l - i(q)\left(\psi + (1-\psi)l\right). \tag{72}$$

Since $\mathbb{E}_t^O\left(dr_t^{Ok}\right) - r_t = x_t\left(\sigma_t^p\right)^2$ from the optimists' optimal portfolio decision, we obtain

$$\frac{1 - i(q(\eta))}{q(\eta)} + \Lambda^O(i(q(\eta))) - \delta^O + \mu_t^p + \frac{\alpha^O - \alpha}{\sigma}\sigma^p(\eta) - r_t = \left(\frac{\psi}{\eta}\right)\sigma^p(\eta)^2. \tag{73}$$

For pessimists, it must be the case where

$$l \cdot \frac{1 - i(q(\eta))}{q(\eta)} + l \cdot \Lambda^O(i(q(\eta))) - \delta^P + \mu_t^p + \frac{\alpha^P - \alpha}{\sigma}\sigma^p(\eta) - r_t \leq \left(\frac{1-\psi}{1-\eta}\right)\sigma^p(\eta)^2. \tag{74}$$

with equality when $\psi < 1$. Finally, by subtracting (74) from (73), we get

$$\frac{(1-l)(1-i(q(\eta)))}{q(\eta)} + (1-l)\Lambda^O\left(i(q(\eta))\right) + \delta^P - \delta^O + \frac{\alpha^O - \alpha^P}{\sigma}\sigma^p(\eta) = \left(\frac{\psi}{\eta} - \frac{1-\psi}{1-\eta}\right)\sigma^p(\eta)^2, \tag{75}$$

when $\psi_t < 1$, i.e., pessimists hold some physical capital.

■

4 Analysis

We solve our model numerically in a similar way to Brunnermeier and Sannikov (2014). For our numerical solutions, we study two cases: (i) when more productive experts are optimists and less productive households are pessimists, i.e., $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$; (ii) the opposite case where experts are pessimists and households are optimists, i.e., $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$.[37,38] We use the following parametrization:[39]

	l	δ^O	δ^P	ρ^O	ρ^P	χ	σ	k	α	α^O set	α^P set
Values	0.4	0	0	0.07	0.065	1	0.08	18	0.05	[0.05, 0.06, 0.07]	[0.05, 0.04, 0.03]

Table 1: Parameterization for $\alpha^O \geq \alpha \geq \alpha^P$

4.1 Results

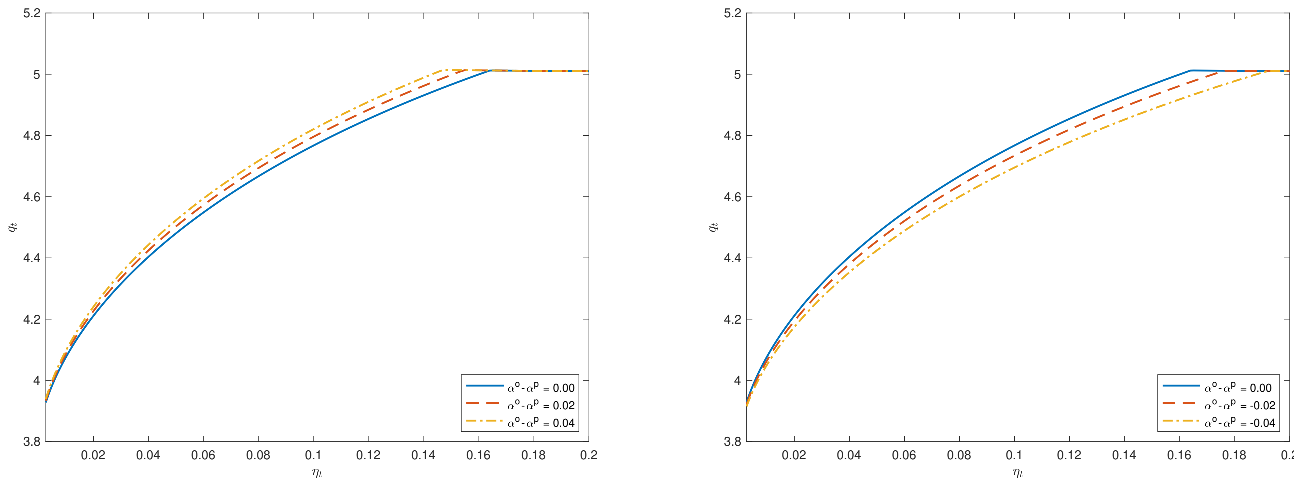
Figure 1 shows the normalized capital price (i.e., price-earnings ratio) q_t as a function of our state variable η_t. Figure 1a corresponds to the original case where more productive experts are optimists (i.e., $\alpha^O \geq \alpha$), while figure 1b represents the opposite case where less productive households are optimists (i.e., $\alpha^P \geq \alpha$). As we already discussed in Section 3.4, two regions are identified. The first region, is when more productive experts manage the entire aggregate capital. We call this region (i.e., when $\psi_t = 1$) the *efficient* region. When experts' wealth share η_t holds reach a level η^ψ defined in Proposition 6, so that the economy is at the efficient

[37] Until now, we have called those who have higher productivity in both turning capital into output and output into new investment 'optimists'. This nomenclature is partially true as they are optimistic only when $\alpha^O \geq \alpha$. In Section 4, we interchangeably call them 'experts' as we also consider the other case where $\alpha^O < \alpha$.

[38] When we vary α^O or α^P, we always consider symmetric cases where $|\alpha^O - \alpha| = |\alpha - \alpha^P|$ in both cases.

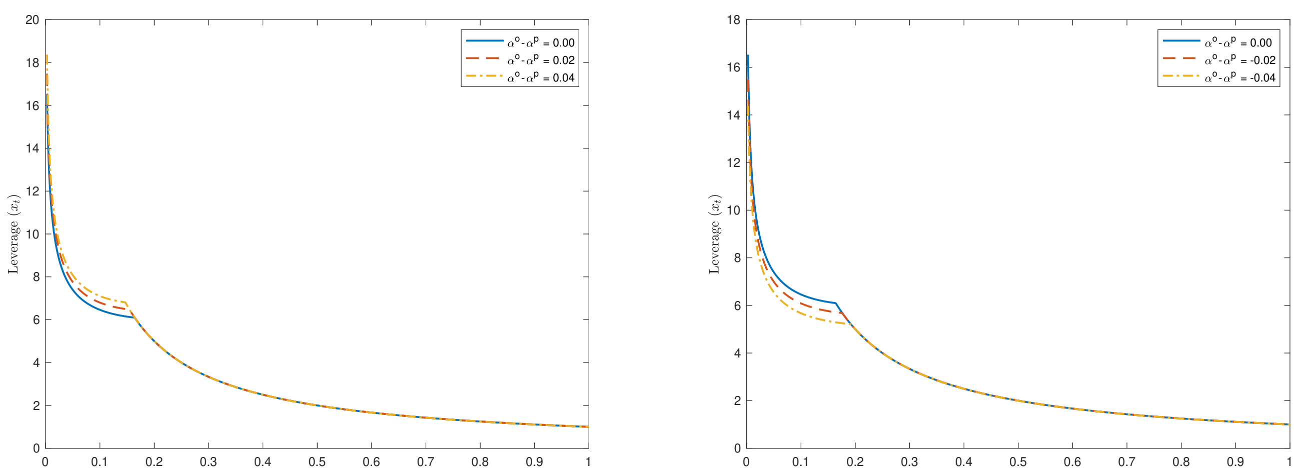
[39] Parameters with $\{\alpha, \alpha^O, \alpha^P\} = \{0.05, 0.07, 0.03\}$ are chosen to target a probability of crisis of 5%.

region, the price of capital q_t reaches its peak regardless of (α^O, α^P). For $\eta > \eta^\psi$, q_t starts declining slowly as η_t rises: when η rises, the wealth share of households goes down, thereby the leverage ratio experts can attain from households falls as seen in Figure 2, lowering their demand for physical capital. When η_t is lower than the threshold η^ψ, the economy is in the *inefficient* (i.e., crisis) region. In this region, the less productive households start participating in the capital market as marginal investors and will hold a proportion of capital, lowering the equilibrium capital price as η_t falls.



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$ (b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 1: Price-earnings ratio q_t as a function of η_t



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$ (b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 2: Leverage x_t as a function of η_t

As we observe, belief heterogeneity does not affect the price of capital when the economy is at the efficient region. This result can be expected since the entire capital is held by experts (i.e., $\psi_t = 1$).[40] In this region, the 'risk-adjusted' return that the households obtain from holding capital is less than the prevailing risk-free rate and thus households prefer to invest in risk-free bonds issued by experts.

However, in the inefficient region where less productive households hold a portion of capital, the degree and direction of the belief heterogeneity start to matter. As experts become more optimistic, i.e., α^O rises from α and households become more pessimistic, i.e., α^P decreases from α, the threshold for the efficient region η^ψ falls: as experts, who are marginal investors, become more optimistic about the productivity growth, they still demand capital and do not engage in fire-sale even when their wealth share η_t is not high enough (i.e., around the threshold η^ψ under rational expectations). It can be seen in Figure 2a where the threshold where experts start deleveraging[41] falls as they become more optimistic. When experts become more pessimistic and less productive households become optimistic in contrast, i.e., $\alpha^O < \alpha$ and $\alpha^P > \alpha$, the optimists' wealth share threshold η^ψ increases in the degree of belief heterogeneity instead. Also comparing Figure 1a and 1b, we observe that the effect of heterogeneous beliefs in this case turns out to be stronger than the former case in which experts are optimists. It is because experts, who are already marginal investors, become pessimistic and thereby have lower demands for capital now in general, allowing inefficient households to participate in the market as marginal investors and raising the crisis threshold η^ψ strongly.[42]

Figure 3 represents the endogenous volatility σ_t^p as a function of the experts' wealth share η_t. We observe that as more productive experts become more optimistic, i.e., α^O rises from α, and less productive households are more pessimistic, i.e., α^P falls from α, we have more amplification of the endogenous risk in the inefficient (crisis) region.[43] It can be understood with regard to equation (58): we can rewrite (58) as

$$\sigma_t^p \left(1 - (x_t - 1) \frac{\frac{dq(\eta_t)}{q(\eta_t)}}{\frac{d\eta_t}{\eta_t}} \right) \equiv \sigma_t^p \left(1 - (x_t - 1)\, \varepsilon_{q,\eta}\right) = \sigma, \tag{76}$$

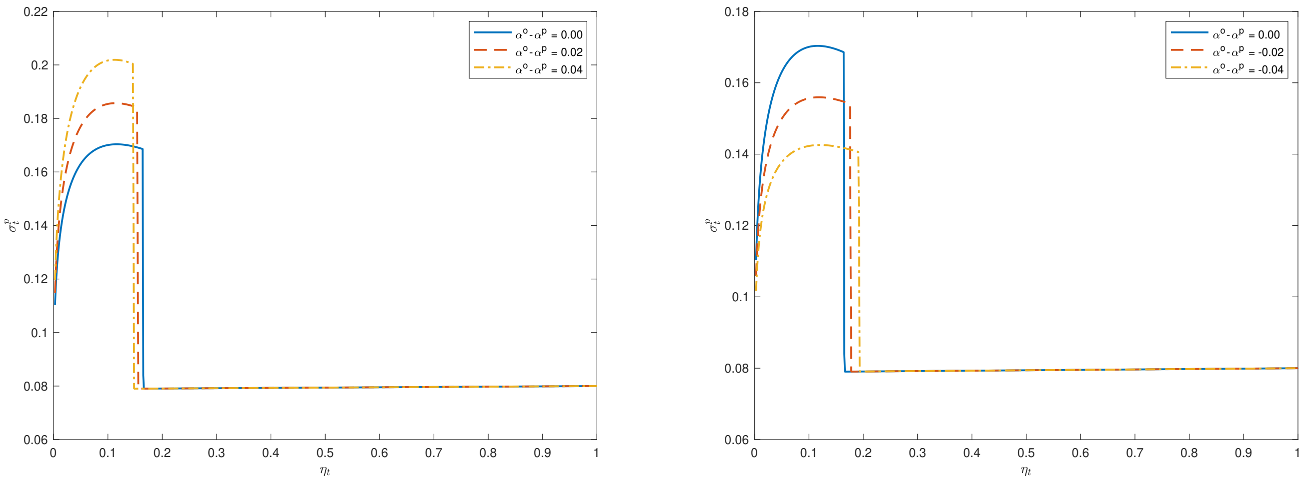
where $\varepsilon_{q,\eta}$ is defined as the elasticity of the price-earnings ratio (i.e., normalized capital price) with respect to the experts' wealth share η_t. When experts become more optimistic, their leverage ratio x_t becomes higher

[40] This can be seen in Figure A2 in Appendix.

[41] In Figure 2, the points where the slope of the graphs becomes discontinuous are the thresholds where experts start to deleverage. Even as experts fire-sell their capital assets and try to lower their leverage, we observe the leverage x_t rises at the inefficient (crisis) region in equilibrium. It is because the capital price q_t drops too much as experts engage in fire-sale of their capital assets, leading to huge increases in 'marked-to-market' leverage. For the issue of procyclicality of leverage, see e.g., Adrian and Shin (2014).

[42] As investment rate ι_t is an increasing function the capital price q_t from (41), it is expected to have the same shape and properties as Figure 1. The full dynamics of investment per unit of capital can be seen in Figure A1.

[43] As we already discussed in Figure 1 and 2, we still observe that the threshold η^ψ decreases in this case. More optimistic experts still demand capital strongly even when their wealth share is low enough, so the economy still is in the efficient region.



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 3: Endogenous Volatility σ_t^p as a function of η_t

around the new threshold η^ψ compared with the benchmark case as seen in Figure 2a, as the threshold point η^ψ where they start fire-selling their capital assets falls due to higher demands coming from their optimism. With experts' leverage ratio increasing with their optimism, we have more intense fire-sales of their capital assets when negative shocks to their wealth share are realized, yielding amplification of the endogenous risk at the inefficient region. Therefore, this *leverage* effect raises the equilibrium endogenous risk σ_t^p from (76).

Another effect comes from the elasticity term $\varepsilon_{q,\eta}$ in (76). This elasticity of the price of capital with respect to the wealth share of 'natural' marginal investors (i.e., experts) can be interpreted as a measure of market illiquidity (or the inverse of market depth). As experts become more optimistic, a % increase in their wealth share obviously leads to higher % increases in the price of capital in the inefficient region as seen in Figure 1a: the inefficient (crisis) region is where the households must hold a portion of capital and when the market risk-premium is the highest. Under experts' stronger optimism, an increase in their wealth share affects the equilibrium capital price more strongly, leading to a higher elasticity $\varepsilon_{q,\eta}$. This *market illiquidity* effect raises the equilibrium endogenous risk σ_t^p from (76) too. Adding *leverage* effect and *market illuqidity* effect yields more amplified endogenous risks during crises.

Figure 3b represents the case where experts become pessimistic and households are optimistic in contrast, i.e., $\alpha^O < \alpha$ and $\alpha^P > \alpha$. We have less amplification of endogenous risks and the higher thresholds η^ψ. Also comparing Figure 3a and 3b, we observe the effect of belief heterogeneity in this case turns out to be stronger than the former case as before.

Belief-Driven Adverse Loop Therefore, we have *two-way* interactions between belief heterogeneity about growth, (α^O, α^P), and the amplification of endogenous financial volatility, σ_t^p: (i) more optimistic experts and pessimistic households jointly yield more amplified endogenous risks during crises: i.e., as α^O increases and α^P decreases from the true α, we have higher σ_t^p at the inefficient region as seen in Figure 3a; (ii) In turn, a higher endogenous volatility σ_t^p raises the degree of disagreement about the expected capital return, as seen in (19) and (20) given (α^O, α^P), which in turn amplifies the risk during crises even further.

Interest Rates and Perceived Risk-Premium To study the behavior of the equilibrium risk-free interest rate r_t, we focus on regions around the threshold points η^ψ where the crisis regime starts. Figure 4a focuses on cases where experts are more optimistic, i.e., $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$, and Figure 4b look at the opposite case as before. In Figure 4a, we observe: (i) the risk-free rate drops a lot (i.e., undefined) around the crisis threshold η^ψ;[44] (ii) as experts become more optimistic, the interest rate becomes higher at the normal region, while it becomes lower during crises, implying that crises are more severe. More optimistic experts increases their leverage demand, leading to on average higher interest rates during normal times.[45,46] When the economy enters the crisis region, i.e., when $\eta \leq \eta^\psi$, in contrast, more optimistic experts generate a more severe crisis with amplified endogenous risks as seen in Figure 3, creating a stronger precautionary savings demand and pushing down the risk-free rate more from their incentive not to hit the solvency constraint, i.e., $w_t^O \geq 0$. As we have more amplified endogenous risk σ_t^q during crises when experts are more optimistic, both the experts' *perceived* risk-premium and *true* risk-premium rise, while the former becomes higher, as seen in Figure 5a. From Figure A3a, we can see that pessimists' *perceived* risk-premium becomes non-negative only at the crisis region, allowing them to hold some capital, i.e., $\psi_t < 1$.[47]

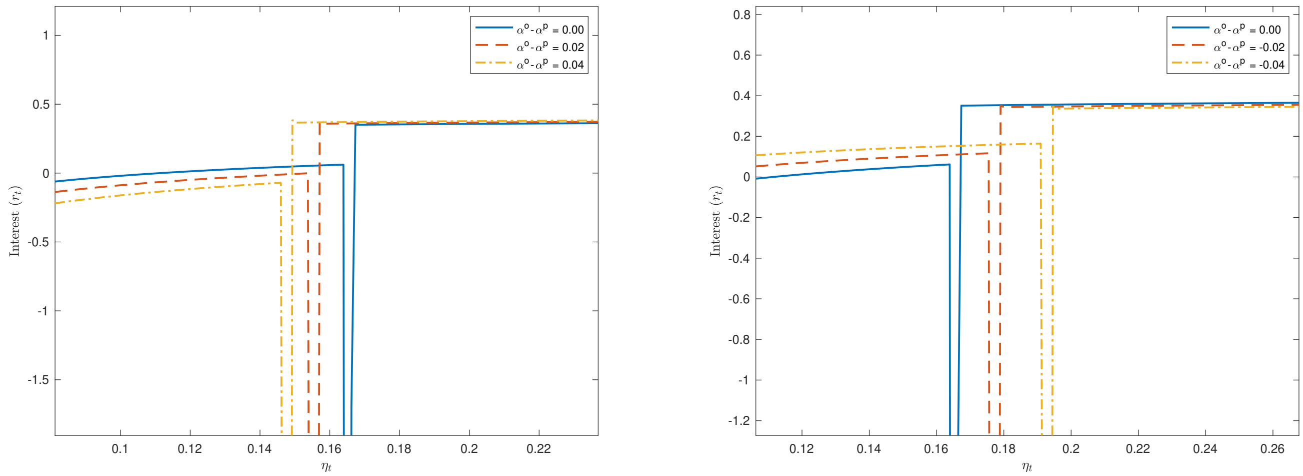
Figures 4b and 5b illustrate the opposite case where experts become more pessimistic. As we see in Figure 3b, we have less amplification of endogenous risks and the higher thresholds η^ψ, mitigating the precautionary savings motive and raising the risk-free rate at the crisis regime. However, during the normal times, we have

[44] Brunnermeier and Sannikov (2014) notice that due to this kink in the interest rate r_t, we can write the same risk-free rate in the form of dr_t, which results in the same results of our model. Our exercise here is to see how two different regimes (i.e., normal and crisis) feature different levels of the risk-free rate generally.

[45] As less productive households become more pessimistic, this leads to their higher demands for risk-free loans issued by experts, pushing down the risk-free rate. In our framework, as experts are more productive and participate in the capital market as *natural* marginal investors, this effect from pessimists turns out to be weaker than the effect that the higher demands of experts for leverage have on the interest rate.

[46] When experts (optimists in this case) start deleveraging and fire-sell their capital assets, the interest rate drops a lot in an almost discontinuous manner. It is expected: as experts suddenly fire-sell their capital and reduce their leverage demand, the interest rate must drop to clear the bond market. After the economy enters the inefficient region, i.e., η_t is lower than η^ψ, the interest rate level becomes stabilized from this discontinuity.

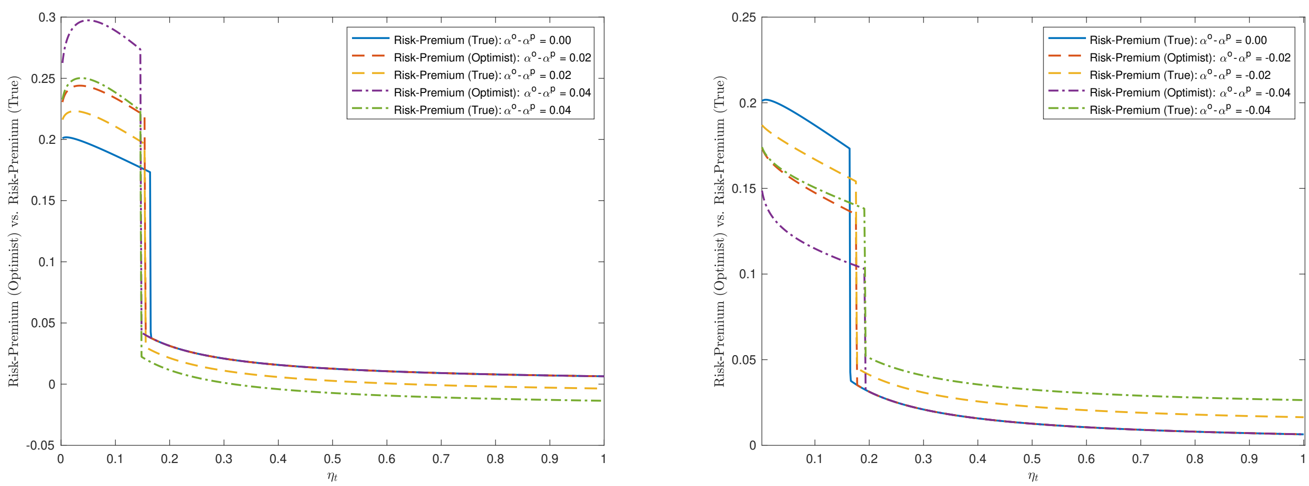
[47] Figures 5 and A3 together confirm the finding in the literature that different types of agents (e.g., households, firms, professional forecasters, etc) form their own beliefs about markets, based on which they choose their portfolio demand. For the recent treatment of this issue, see Beutel and Weber (2022).



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$ (b) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

Figure 4: Interest Rate r_t as a function of η_t: $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

lower risk-free rates as experts' leverage demand is weaker. Both optimists' perceived risk-premium and the true risk-premium decline during crises.



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$ (b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure 5: Risk-Premium (Optimists' and True Value) as a Function of η_t

Other relevant figures of our numerical simulation are provided in Appendix.

4.2 Ergodic Distribution

Now, we consider the stationary (ergodic) distribution of η_t in the long run: the objective here is to analyze how the optimism and pessimism of experts and households affect the *average* time the economy lives under the inefficient (i.e., crisis) regime.

Starting form equation (48), the dynamics of η_t, we can employ the Kolmogorov forward equation (KFE) in a similar way to Brunnermeier and Sannikov (2014). If we let $d(\eta)$ as the stationary density of η_t, it should satisfy

$$0 = -\frac{\partial}{\partial\eta}\left(\mu^{\eta}(\eta)\eta\cdot d(\eta)\right) + \frac{1}{2}\frac{\partial^2}{\partial\eta^2}\left(\left(\sigma^{\eta}(\eta)\eta\right)^2 d(\eta)\right). \tag{77}$$

With the transformation $D(\eta) \equiv \left(\sigma^{\eta}(\eta)\eta\right)^2 \cdot d(\eta)$, equation (77) can be written as

$$\frac{D'(\eta)}{D(\eta)} = 2\frac{\mu^{\eta}(\eta)\eta}{\left(\sigma^{\eta}(\eta)\eta\right)^2}, \tag{78}$$

which can be solved easily by integrating both sides of (78).

Figure 6 draws the stationary distribution of our state variable η_t in the presence of belief heterogeneity: we observe that the time the economy spends the most is around its stochastic steady states.[48] When negative shocks shift the economy toward the inefficient region however, experts' higher degree of optimism (thereby households' higher degree of pessimism) on average makes the economy more vulnerable to financial crises: the economy spends more time in its inefficient region.[49]

This can be understood as follows: as we see in Figure 3, experts' more optimism (i.e., higher α^O) amplifies the endogenous volatility σ_t^p during crises. In that case, experts receive higher risk-premium[50] when investing in physical capital during the crises, getting recapitalized faster and moving the economy out of the inefficient region. This feature can be checked in Figure A4 where the drift of the wealth share process $\mu^{\eta}(\eta)\eta$ is higher in the inefficient region when experts get more optimistic. This channel reduces the time the economy spends in crises.

However, when experts become more optimistic, they bear too much risk in their balance sheet with the higher leverage ratio x_t around the new threshold η^{ψ}, which itself falls with the degree of optimism. It raises the probability that the economy is pushed from the efficient to inefficient region when negative shocks are realized. This feature is checked in Figure A4 in which the drift of the wealth share process $\mu^{\eta}(\eta)\eta$ becomes more negative in the efficient region as experts become more optimistic. This channel increases the time the

[48] There are two stochastic steady states where $\mu^{\eta}(\eta) = 0$, one of which is $\eta = 1$, definitely

[49] For example, the case of $\{\alpha^O, \alpha^P\} = \{0.07, 0.03\}$ targets a probability 5% that the economy is in financial crisis.

[50] From Proposition 1, we know that the optimists' *perceived* risk-premium is given by $x_t(\sigma_t^p)^2$ in our log-utility case.

economy spends in the inefficient region (i.e., crises). It turns out that the second channel (i.e., from efficient to inefficient) is stronger than the first channel (i.e., from inefficient to efficient): so our economy undergoes a higher number of shorter-lived and more severe (with higher levels of volatility) financial crises than under the benchmark model with rational expectations, as experts who are natural marginal investors become more optimistic.

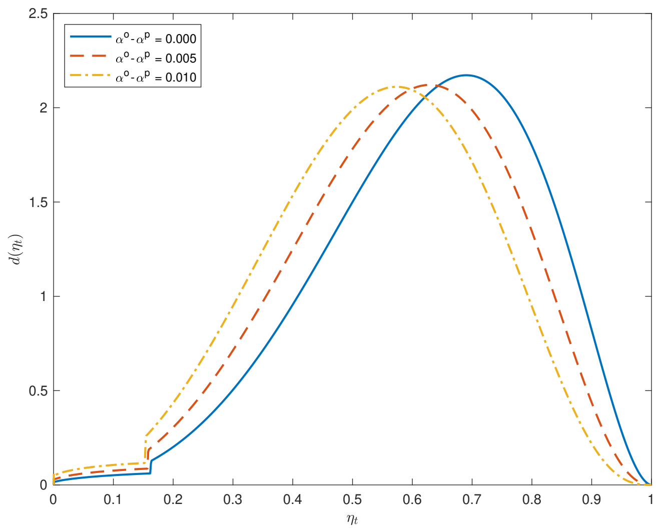


Figure 6: Ergodic Distribution of η_t with $\alpha^O \geq \alpha \geq \alpha^P$[51]

5 Quantitative Analysis

5.1 Time-series return predictability

In this section, we study the time-series return predictability patterns of stock returns. To that end, we regress the excess return on the S&P500 on its lagged excess return controlling for recession periods.

$$r^e_{t+h} = \alpha(h) + \beta_1(h) r^e_t + \beta_2(h) * r^e_t * 1_{Rec} + \epsilon_{t+h}$$

The recession indicator 1_{Rec} takes a value 1 during the NBER recessionary months, and 0 otherwise. The coefficient $\beta_2(h)$ measures the excess conditional momentum. Figure (7) displays the empirical auto-correlation coefficients. Empirically, the aggregate stock return exhibits a strong momentum in the shorter horizons, and reversal over horizon 5 months. This effect is stronger in the crisis period which can be seen in the right

[51]Under the current calibration in the absence of belief heterogeneity, i.e., $\alpha = \alpha^O = \alpha^P = 0.05$, we observe that the experts take over the economy in the long run, resulting in the degenerate distribution as seen in Figure 6 (i.e., Dirac mass at $\eta = 1$).

panel, corroborating with Cujean and Hasler (2017) who find that the return predictability is concentrated during recessionary periods. To compute the model implied correlation coefficients, we first construct the excess return over a period of length Δ based on the following definition.[52]

$$R_t^e = \int_{t-\Delta}^{t} \left(\frac{d(q_u K_u) + (\hat{A}_u - \iota_u) K_u dt}{q_u K_u} - r_{f,u} du \right)$$

We then simulate the model 1000 times for 5000 years and compute the average slope cofficients from the following regression[53]

$$R_{t+h}^e = \alpha(h) + \beta_{1,model}(h) R_t^e + \beta_{2,model}(h) * R_t^e * 1_{crisis} + \epsilon_{t+h} \tag{79}$$

If the sign of coefficients is positive (negative), the return exhibits momentum (reversal). The bottom two graphs in the Figure (7) display the model-implied correlation coefficients. The model captures the 'excess' short-term momentum and long-term momentum crash quite well.[54] After a series of negative shocks, the economy enters a crisis with depressed wealth share of optimists and high expected returns. An increase in the expected returns during the recapitalization phase leads to conditional momentum in asset returns. The optimists recapitalize until the economy transitions back to the efficient region where the expected returns are low, leading to momentum crash. While the empirical conditional momentum crashes after a few lags, the model implied conditional momentum persists for several periods. We next turn to analyse the role of disagreement in the excess conditional momentum pattern.

5.1.1 Impact of disagreement

To study the impact of disagreement on conditional return predictabilitiy patterns empirically, we run the following regression:

$$r_{t+h}^e = \alpha(h) + \beta_1(h) r_t^e + \beta_2(h) * r_t^e * 1_{Rec} + \beta_3(h) * r_t^e * 1_{Rec} * 1_d + \epsilon_{t+h} \tag{80}$$

where 1_d is a dummy variable that takes a value of 1 if the disagreement D_t that is defined in (83) in Section 5.2 is above its median. The coefficient $\beta_3(h)$ measures the excess conditional momentum when disagreement is high. We run a similar regression in the model with disagreement proxied by the 'net leverage' defined in

[52]The scaled aggregate productivity $\hat{A}_t := \frac{A_t}{\gamma_O}$ is given by $\hat{A}_t = \psi_t + l * (1 - \psi_t)$.

[53]The parameters in the model are chosen to target a probability of crisis of 5%.

[54]We define the 'excess' momentum/reversal to be the momentum/reversal in crisis period relative to the normal period.

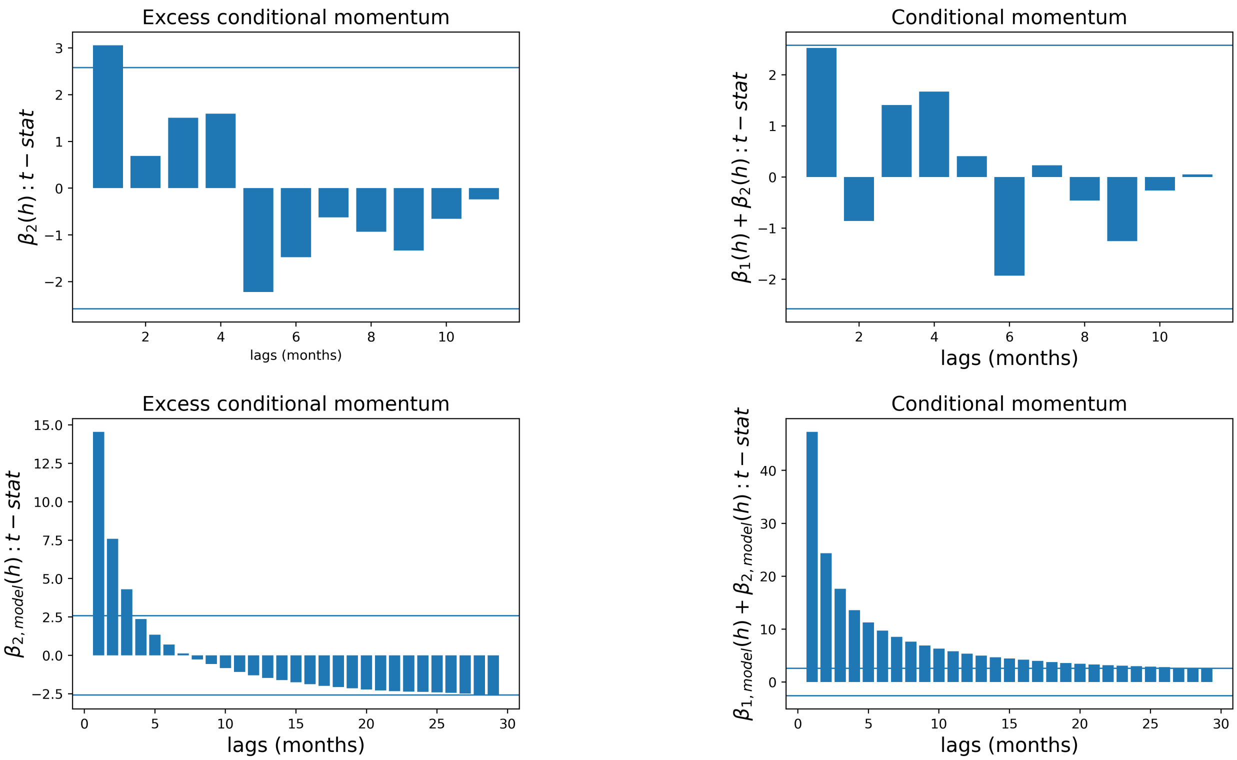


Figure 7: Time series return predictability. Top two panels present the empirical autocorrelation coefficients from regressing the excess return on the S&P500 on its lagged excess return, as shown in the equation (79). The data is at monthly frequency from 1945 till 2022. The bottwom two panels present the model implied autocorrelation coefficients from the regression (79). The model is simulated 1000 times for 5000 years at a monthly frequency. Each correlation coefficient represents the average value across simulations.

(81). The net leverage captures the component of the optimists' total leverage attributed to disagreement.

$$x_t^{net} = x_t - x_t^{REE} \tag{81}$$

where x_t^{REE} is the leverage of experts under the rational expectation equilibrium. While two groups' disagreement about the expected technological growth is constant in the model, it triggers a feedback loop between the belief heterogeneity and the wealth share of optimists, impacting their leverage. This implies that the *net* leverage, the component corresponding to the disagreement, is time-varying. In the model regression, the dummy variable 1_d takes a value 1 if the net conditional leverage of optimists is larger than its median value in each simulated path.[55] Figure (8) presents the results from the regression. The model successfully captures the excess conditional momentum during high disagreement periods which is evident in the data. Higher disagreement increases the vulnerability of the economy in stochastic steady state. Thus, an adverse

[55]The reason we use conditional leverage is because in our model, disagreement affects equilibrium quantities only during crisis.

shock leads to a crisis with high expected return on capital. With higher disagreement, the optimists recapitalize faster since they earn a larger risk premium. The actual conditional risk premium is larger for higher disagreement levels, leading to excess conditional momentum. After a few months, the momentum crashes since the economy transitions into the efficient region, where the actual risk premium falls due to a healthier economic environment. Next, we turn to cross-sectional implications of macro-level disagreement.

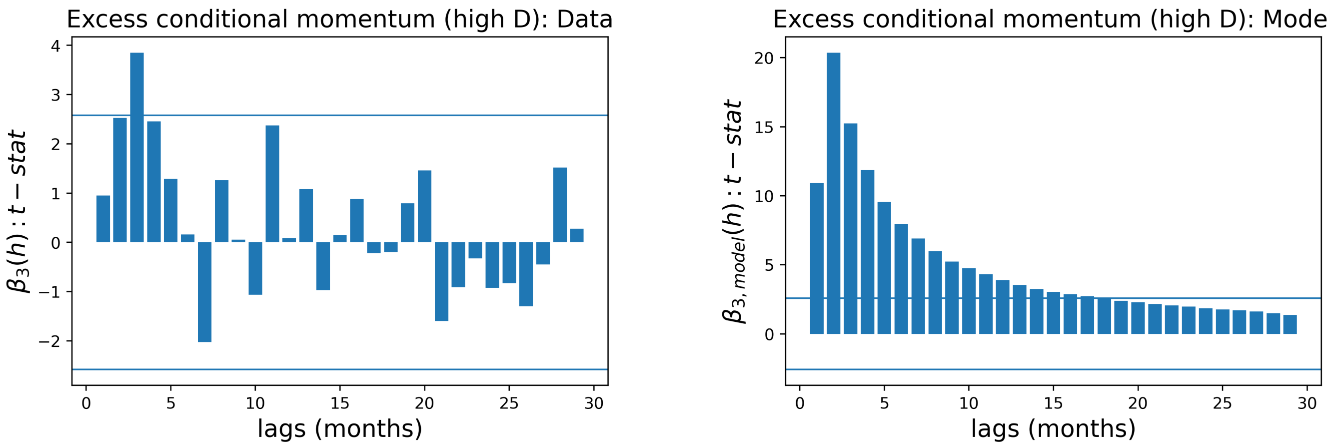


Figure 8: Time series return predictability. The left panel presents empirical autocorrelation coefficients from regressing the excess return on S&P500 on its lagged excess return, as shown in equation (80). The data is at monthly frequency from 1945 till 2022. Disagreement data is computed from the definition (83). The disagreement data is available at a quarterly frequency and hence interpolated to get monthly values. The left panel presents the conditional t-stats when the disagreement is high ($\beta_3(h)$). The right panels presents the model-implied conditional t-stats when the disagreement is high ($\beta_{3,model}(h)$).

5.2 Factor model

We propose a three factor model with a role for disagreement along with the aggregate wealth and the intermediary wealth. The intermediary wealth is proxied by the intermediary capital ratio from HKM2017. That is, the capital ratio, denoted by η_t is the fraction of total assets held in equity by the bank holding companies in the US.

$$\eta_t = \frac{\text{Equity}_t}{\text{Assets}_t} \tag{82}$$

The aggregate wealth W_t is proxied by the traditional market factor. The third novel factor disagreement is empirically measured as

$$D_t = \frac{f_{75} - f_{25}}{|f_{50}|} \tag{83}$$

where f_i denotes the i^{th} percentile analyst forecast of quarter-on-quarter GDP growth rate for the $(T+2)^{th}$ quarter ahead at date T. Figure (9) plots the disagreement measure at a quarterly frequency from 1970Q1 till

2022Q4. The disagreement spiked during the GFC 2008, as well as during the COVID crisis of 2020. For the asset pricing tests, we construct and use a disagreement growth measure d_t which is shown in Figure (9).[56]

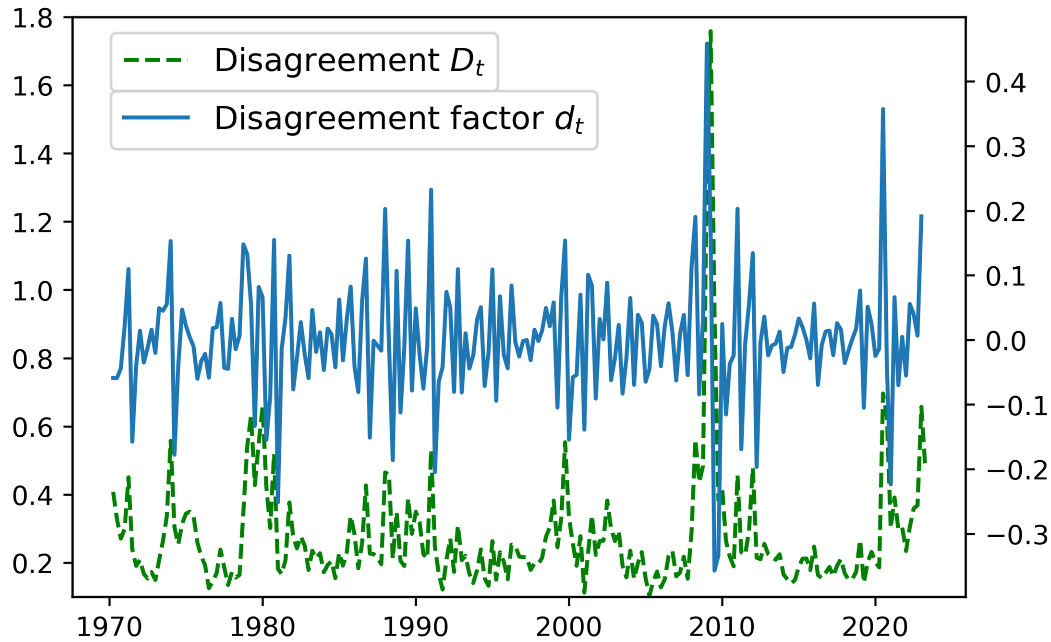


Figure 9: Disagreement D_t is computed as the interquartile dispersion of the 2nd quarter ahead GDP QoQ projection scaled by median growth projection. It corresponds to the left axis. The data is taken from The Survey of Professional Forecasters. The disagreement factor d_t, corresponding to the right axis, is computed as the change in $\log(1 + D_t)$. The shaded areas represent NBER recessionary periods.

We run a two-stage Fama-McBeth regression with the three factors $f_t := [M_t, \eta_t, d_t]'$, where M_t is the market excess return. First, we regress the time series excess return of each asset i on the three factors to estimate corresponding betas $\hat{\beta}_{i,t}$ using the following regression equation.

$$R^e_{i,t} = a_i + \beta'_{i,f} f_t + v_{i,t} \tag{84}$$

In the second stage regression, the estimated betas are used to obtain the risk prices for each of the factors by the following regression

$$E[R^e_{i,t}] = \alpha_i + \hat{\beta}'_{i,f} \lambda_f + \varepsilon_i \tag{85}$$

where $E[R^e_{i,t}]$ is the excess return on the asset i, and λ_f is the risk price of the corresponding factor $f \in f_t$. We adjust the standard errors of the estimated λ_f in order to account for the fact that regressors $\hat{\beta}_{i,f}$ are

[56]The growth in disagreement is computed as $d_t := log((1 + D_t)/(1 + D_{t-1}))$.

estimated quantities.

Test assets: To test the ability of disagreement in pricing the cross-section of asset returns above and beyond the intermediary factor and the market factor, we use the following test assets from the period 1970Q1 till 2022Q4: 25 size and book-to-market portfolios, 25 size and momentum sorted portfolios, 10 long-term reversal portfolios, 25 profitability and investment portfolios, and 10 maturity sorted US treasury bond portfolios from CRSP Fama bond dataset with maturities in six month intervals up to five years. Using a large number of test assets passes the criticism by Lewellen et al. (2010). In addition to the equity and bond portfolios used above, we extend the analysis to include other asset classes with the following test assets from 1970Q1 till 2012Q4 (due to data availability)- 18 option portfolios, 20 CDS portfolios, and 12 FX portfolios used in HKM2017.

Table (2) presents the results from the first-stage regression using the equity and bond portfolios as test assets.[57] The table displays the following statistics for the two asset classes (equities, and equities and bonds)- the average excess return, standard deviation of the excess return in each asset class, mean and standard deviation of the factor exposures of the excess returns to the factors f_t. We consider two models for each asset classes- a two-factor model with market and intermediary factor, and a three-factor model with market, intermediary, and disagreement factor. The regression controls for the price-dividend ratio and cyclically adjusted price-earnings ratio (CAPE) obtained from Robert Shiller's website. These two variables are commonly used to predict asset returns.[58] Consistent with HKM2017, there is a large variation in the intermediary factor exposure in both asset classes. In addition, the exposure to disagreement factor also shows a considerable variability in both asset classes. That is, the standard deviation of β_d is around 10 times its mean, indicating that the excess returns of test assets have sufficient variation in its exposure to the disagreement factor.

5.3 Cross-sectional asset pricing test

Table (3) presents the risk price estimates for the two-factor and the three-factor model along with Shanken-corrected t-stats. The risk price of disagreement factor in the three-factor model is positive and statistically significant in both equity and bond portfolios. The inclusion of disagreement factor increases the adjusted R-squared significantly by 18% in the case of equity portfolios. The intermediary factor, although has a

[57] We focus on the equity and bond portfolios first since data is available for extended period from 1970Q1 till 2022Q4. Later, we focus on expanded test assets including multiple asset classes from 1970Q1 till 2012Q4 due to limited data availability.

[58] Later, we also control for the cay measure by Lettau and Ludvigson (2001), and capital share ratio by Lettau et al. (2019). Since capital share data is only available until 2013Q4, we drop this from our main regression. However, as shown in the robustness section in appendix (B), the results are unchanged by inclusion of these control variables.

	Equities		Equities and Bonds	
	Two-factor	Three-factor	Two-factor	Three-factor
Mean excess return	2.06	2.06	1.88	1.88
Std. excess return	0.69	0.69	0.84	0.84
Mean β_M	1.0	1.0	0.9	0.9
Std β_M	0.23	0.23	0.37	0.37
Mean β_η	0.09	0.09	0.08	0.08
Std β_η	0.11	0.11	0.11	0.11
Mean β_d	-	0.004	-	0.004
Std β_d	-	0.04	-	0.04
Assets	85	85	95	95
Quarters	211	211	211	211
Controls	Yes	Yes	Yes	Yes

Table 2: Expected returns and risk exposures. Equity assets include 25 size and book-to-market portfolios, 25 size and momeutum portfolios, 10 long-term reversal portfolios, and 25 profitability and investment portfolios. Bond portfolios include 10 maturity sorted portfolios from CRSP Fama bond portfolio dataset. Mean and std. of excess return is the difference in mean return and risk free rate of the corresponding test assets. The frequency is quarterly and time period is from 1970Q1 till 2022Q4. The mean and std. of betas ($\beta_M, \beta_\eta, \beta_d$) measure the average and standard deviation of the exposure of excess return to the market factor, the intermediary capital ratio, and the disagreement factor respectively.

positive price of risk estimate, is not statistically significant.[59]

Figure (10) displays the actual average excess returns against the predicted average excess returns for the two-factor model. The predicted excess returns is from the two-factor model with the market and the intermediary factor. Figure (11) displays the same with predicted excess returns from the three-factor model with disagreement. The predicted returns from the three-factor model aligns better along the 45-degree line compared to the predicted returns from the two-factor model. This observation is confirmed with Table (3) that shows that the mean absolute pricing error from the three-factor model is 14 basis points lower than the error from the two-factor model.

5.4 More asset classes

While the previous subsection used equities and bonds as test assets, we now turn to testing the performance of the three factor model on a wider range of asset classes. We use 10 US corporate bond portfolios sorted on yield spreads from Nozawa (2017), 18 option portfolios, 20 CDS portfolios, and 12 Foreign exchange portfolios from HKM2017. The data is at quarterly frequency from 1970Q1 till 2012Q4 due to limited data availability. While the timespan of the test assets is limited, using a wider range of asset classes provide a

[59]HKM2017 documents that the intermediary risk factor is not significant with 25 size and book-to-market portfolios and 10 momentum sorted portfolios.

	Equities		Equities and Bonds	
	Two-factor	Three-factor	Two-factor	Three-factor
Market	-0.01	-0.01	0.01	0.01
t-Stat Shanken	(-0.71)	(-0.5)	(1.17)	(1.05)
Intermediary	-0.01	0.0	0.02	0.02
t-Stat Shanken	(-0.29)	(0.02)	(1.08)	(1.09)
Disagreement	-	0.07	-	0.07
t-Stat Shanken	-	(2.91)	-	(2.81)
MAPE %	2.0	1.79	2.22	2.08
Adj. R2	0.00	0.18	0.23	0.35
Assets	85	85	95	95
Quarters	211	211	211	211

Table 3: Risk price estimates for equities and US government bond portfolios. Equity test assets include 25 size and book-to-market portfolios, 25 size and momentum portfolios, 10 long-term reversal portfolios, and 25 profitability and investment sorted portfolios. The 'Equity and bonds' portfolio include all of the above assets, plus 10 maturity sorted US government bond portfolios taken from the CRSP Fama bond portfolio dataset. The data is at quarterly frequency from 1970Q1 till 2022Q4. The factors are market, intermediary capital ratio, and disagreement. The disagreement factor d_t is computed as growth rate in the inter-quartile dispersion of GDP forecast scaled by median forecast by analysts. The forecast data is taken from The Survey of Professional Forecasters. The standard errors of risk price estimates are adjusted for generated regressor problem using Shanken correction. MAPE denotes the mean absolute pricing error in annualized terms.

stringent test for the three factor model. In particular, it is of interest to see if the disagreement factor prices the cross-section of asset returns above and beyond what is captured by the intermediary and market factor. Table (4) shows that the exposure of excess returns to the extended test assets shows sufficient variability. The standard deviation of β_d is around 20 times the mean exposure. The mean and standard deviation of the intermediary factor exposure is unchanged between the two-factor and the three-factor model.

Table (5) presents the risk price estimates from the two-factor and three-factor model with HKM test assets. The test assets in the last two columns includes 25 size and momentum, and 10 long-term reversal portfolios to the HKM test assets. The risk price of disagreement is positive and statistically significant in both cases. The adjusted R-squared increases by 14% and the mean absolute pricing error decreases by 38 basis points when momentum portfolio is added to the HKM test assets. Figure (12) displays the average realized excess returns vs the average predicted excess returns for HKM+Momemtum test assets. The predicted returns in the three-factor model line up towards the 45-degree line better than the two-factor model, which is consistent with the reduction in pricing error shown in the table (5). Overall, the evidence for the disagreement factor in pricing the cross-section of asset returns above and beyond the intermediary and the market factor is strong for equity, bond, and other asset classes.

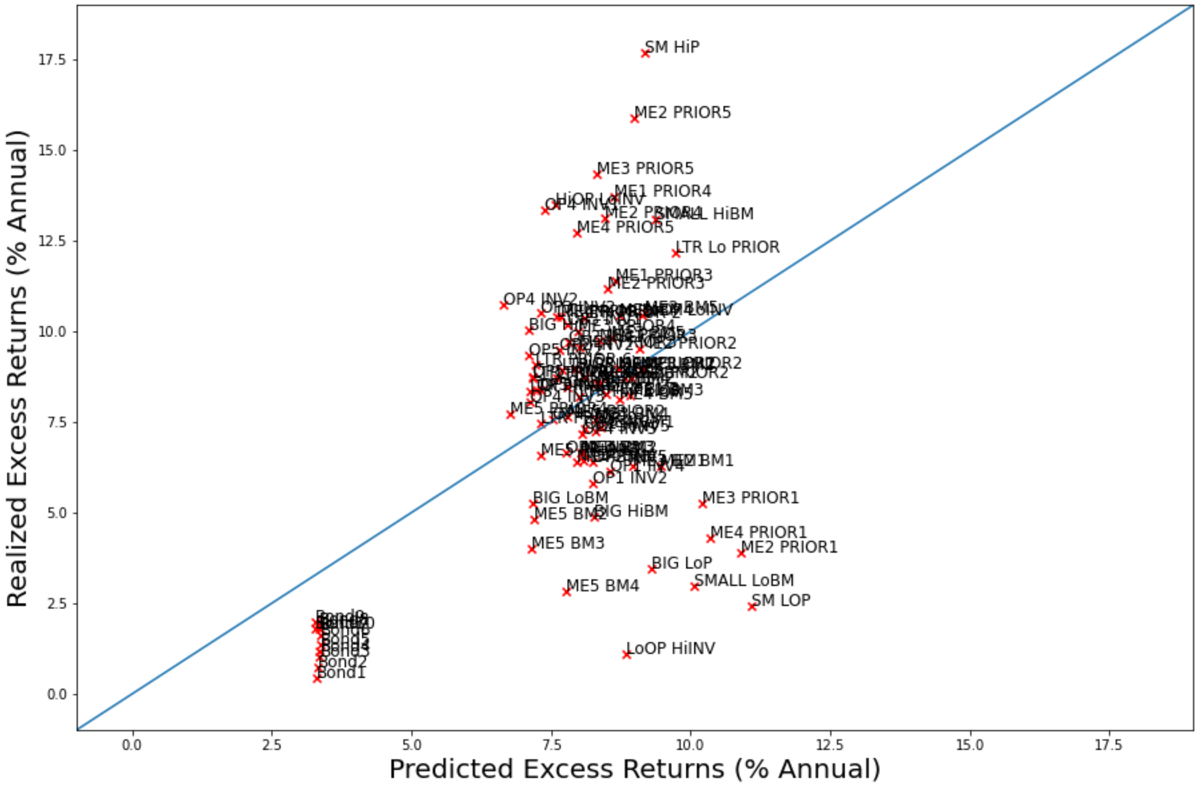


Figure 10: Pricing errors on equity and bond portfolios. Realized excess returns versus predicted excess returns using the two-factor model with market and intermediary factors. The data is at quarterly frequency and from 1970Q1 till 2022Q4.

5.5 Two-way sorted portfolios

To further verify the positive risk price of disagreement, we independently double sort the test assets based on their exposures to the intermediary factor and the disagreement factor, controlling for the market factor, price-dividend ratio, cyclically adjusted earnings ratio (CAPE), and cay variable. A rolling-window of 60 periods is used to estimate the betas and for sorting the assets. Table (6) presents the average excess returns for three-by-three double-sorted portfolios. The high-minus-low return of the portfolios sorted with respect to the disagreement beta is consistently positive in all three quintiles as seen in the last column of table (6). Except for the lowest quintile, excess returns increase monotonically with the disagreement beta. The economic magnitude of the high-minus-low returns is also large with upto 2.32% in annualized terms. Thus, the disagreement factor has pricing power beyond the market and the intermediary factor.

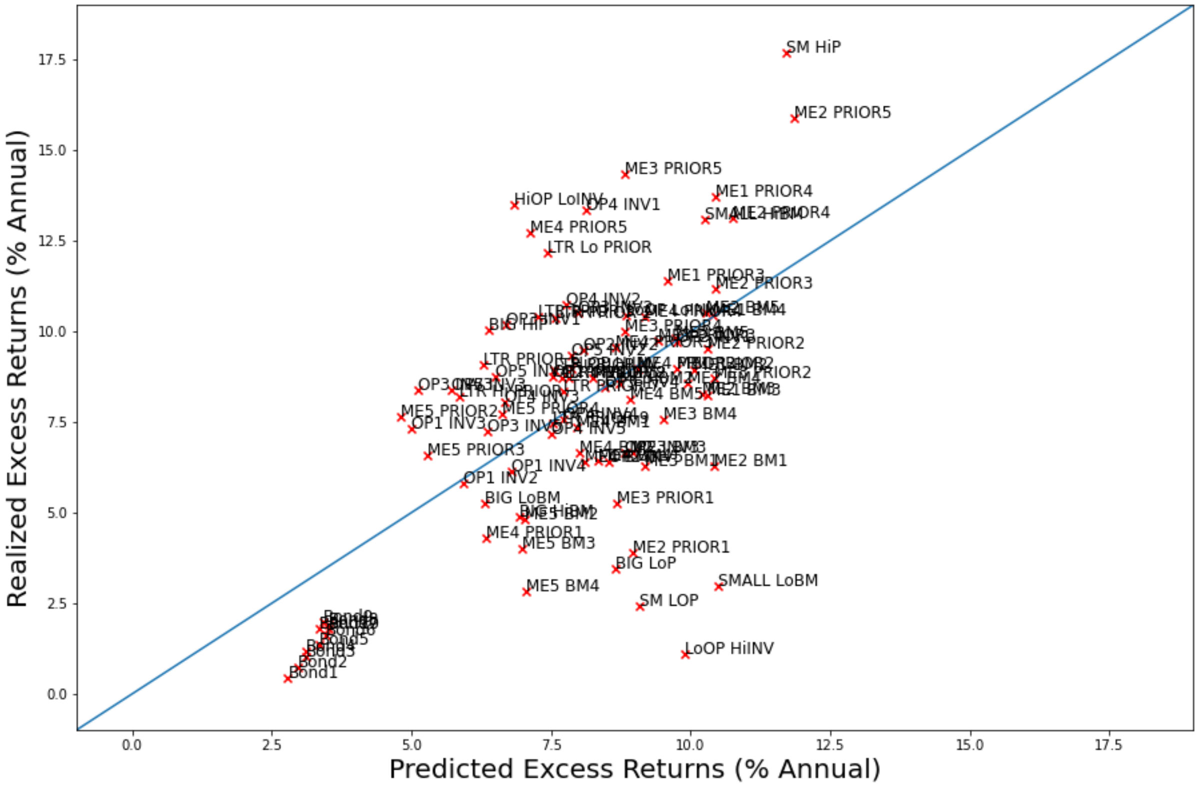


Figure 11: Pricing errors on equity and bond portfolios. Realized excess returns versus predicted excess returns using the three-factor model with market, intermediary, and disagreement factors. The data is at quarterly frequency and from 1970Q1 till 2022Q4.

6 Conclusion

In an economy with two groups of agents who have different productivity in (i) turning capital into outputs; (ii) turning output into capital (i.e., investment), we introduce heterogeneous beliefs in a tractable way: two groups, optimists and pessimists, with imperfect information about the process of their productivity growth, agree to disagree about the expected growth of their productivity measures. We mostly focus on cases where more productive experts believe that the expected growth of their productivity is higher than the other group (i.e., optimistic), but the opposite case is also analyzed.

The analysis reveals that when the experts are optimistic, they take too much leverage at the stochastic steady state making the economy vulnerable to adverse shocks. At the steady state, the capital price and the investment rate is high, and risk premium is low. The adverse shocks get amplified not only due to the

	HKM		HKM+Momentum	
	Two-factor	Three-factor	Two-factor	Three-factor
Mean excess return	0.85	0.85	1.18	1.18
Std. excess return	1.31	1.31	1.32	1.32
Mean β_M	0.46	0.46	0.61	0.61
Std. β_M	0.45	0.45	0.47	0.47
Mean β_η	0.03	0.03	0.05	0.04
Std. β_η	0.09	0.09	0.1	0.1
Mean β_d	-	0.002	-	0.002
Std. β_d	-	0.03	-	0.04
Assets	94	94	129	129
Quarters	171	171	171	171
Controls	Yes	Yes	Yes	Yes

Table 4: Expected returns and risk exposures. HKM assets include 25 size and book-to-market portfolios, 10 U.S. Govt. bond portfolios, 10 U.S. corporate bond portfolios, 18 option portfolios, 20 CDS portfolios, and 12 foreign exchange portfolios taken from HKM2017. HKM+Momentum includes 25 size and momentum portfolios and 10 long-term reversal portfolios in addition to the HKM assets. Mean and std. of excess return is the difference in mean return and risk free rate of the corresponding test assets. The frequency is quarterly and time period is from 1970Q1 till 2012Q4. The mean and std. of betas ($\beta_M, \beta_\eta, \beta_d$) measure the average and standard deviation of exposure of the excess return to market factor, intermediary capital ratio, and disagreement measure respectively.

financial frictions, but also due to the difference in beliefs, through an adverse feedback loop between the expert wealth and the belief channel. The amplified crisis state has high risk premium, and low output, where the amplification is starker for high disagreement levels. Compared to a rational expectations benchmark, our economy with optimistic (pessimistic) experts has a higher occupation time in a crisis, and features higher (lower) instability out of the crisis. The model can explain the excess time series momentum during crises, as well as the empirical fact that the excess momentum effect is stronger for higher disagreement levels. Lastly, we show that a factor pricing model with both the intermediary and disagreement factor significantly improves the pricing power in the cross-section of asset returns.

	HKM		HKM+Momentum	
	Two-factor	Three-factor	Two-factor	Three-factor
Market	0.02	0.01	0.02	0.01
t-stat Shanken	(1.46)	(0.83)	(1.59)	(0.97)
Intermediary	0.09	0.10	0.06	0.07
t-stat Shanken	(4.19)	(3.09)	(2.86)	(2.14)
Disagreement	-	0.1	-	0.12
t-stat Shanken	-	(1.93)	-	(2.93)
MAPE %	1.66	1.34	2.35	1.97
Adj. R2	0.83	0.89	0.59	0.73
Assets	94	94	129	129
Quarters	171	171	171	171

Table 5: Risk price estimates for HKM and HKM+Momentum portfolios. HKM assets include 25 size and book-to-market portfolios, 10 U.S. Govt. bond portfolios, 10 U.S. corporate bond portfolios, 18 option portfolios, 20 CDS portfolios, and 12 foreign exchange portfolios taken from HKM2017. HKM+Momentum includes 25 size and momentum portfolios and 10 long-term reversal portfolios in addition to the HKM assets. The data is at quarterly frequency from 1970Q1 till 2012Q4. The factors are market, intermediary capital ratio, and disagreement. The disagreement factor d_t is computed as growth rate in the inter-quartile dispersion of GDP forecast scaled by median forecast by analysts. The forecast data is taken from The Survey of Professional Forecasters. The standard errors of risk price estimates are adjusted for generated regressor problem using Shanken correction. MAPE denotes the mean absolute pricing error in annualized terms.

		Disagreement			
		(1)	(2)	(3)	(3)-(1)
	(1)	6.47	4.71	6.83	0.36
Intermediary	(2)	7.37	7.67	9.69	2.32
	(3)	7.26	9.09	9.26	2.00
	(3)-(1)	0.79	4.38	2.43	-

Table 6: Average excess returns. The table reports the annualized mean excess return on equity and bond portfolios double-sorted on their exposures to the intermediary factor and the disagreement factor using the three-factor model. The data is at quarterly frequency from 1970Q1 till 2022Q4. The intermediary factor is from HKM2017, and disagreement factor is computed from the growth rate of inter-quartile dispersion in GDP projection scaled by the median projection.

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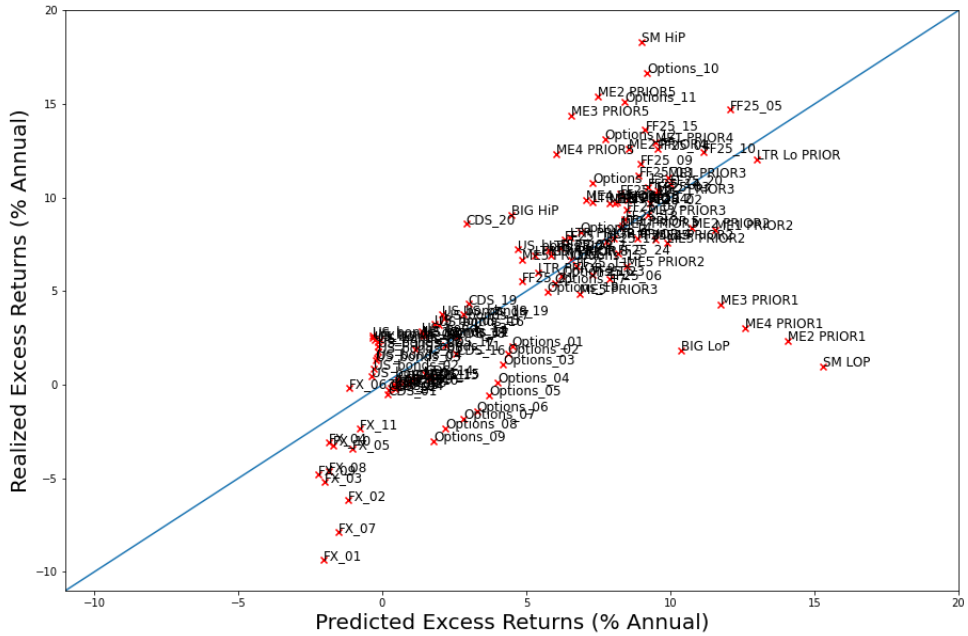
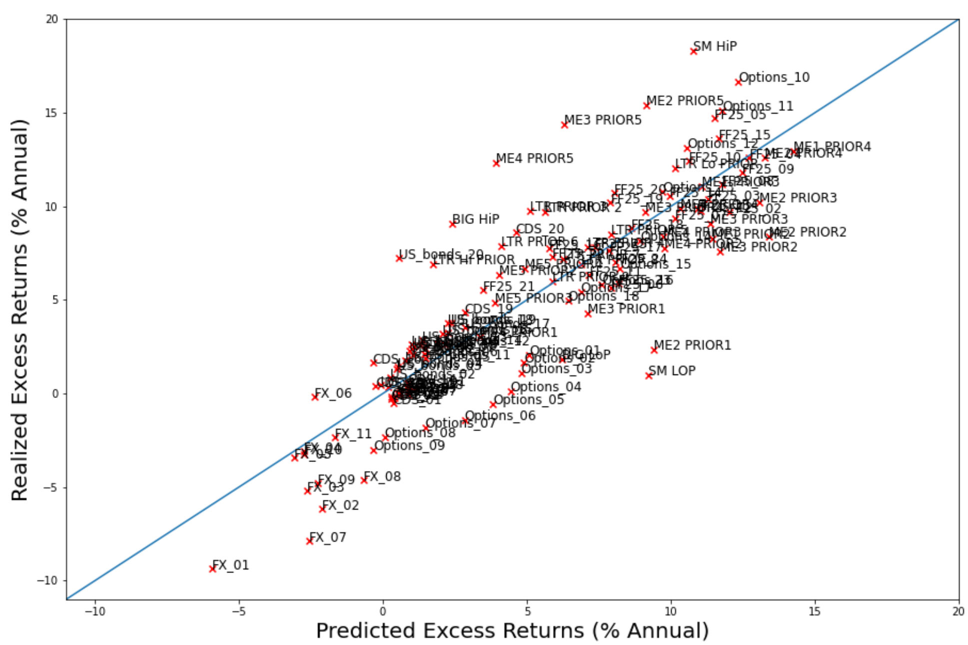


(a) Pricing error in two-factor model.

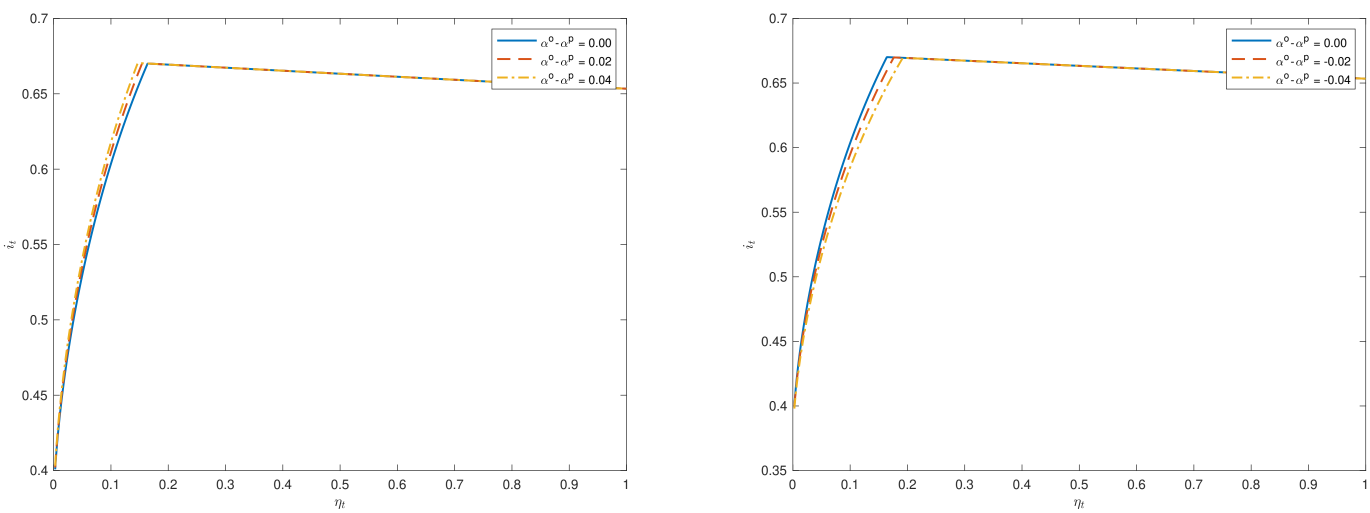


(b) Pricing error in three-factor model.

Figure 12: Pricing errors on HKM+Momentum portfolios. Realized excess returns versus predicted excess returns using the two-factor model with the market and the intermediary factor in panel (a), and the three-factor model with the market, intermediary, and disagreement factors in panel (b). The data is at quarterly frequency and from 1970Q1 till 2012Q4.

7 Appendix

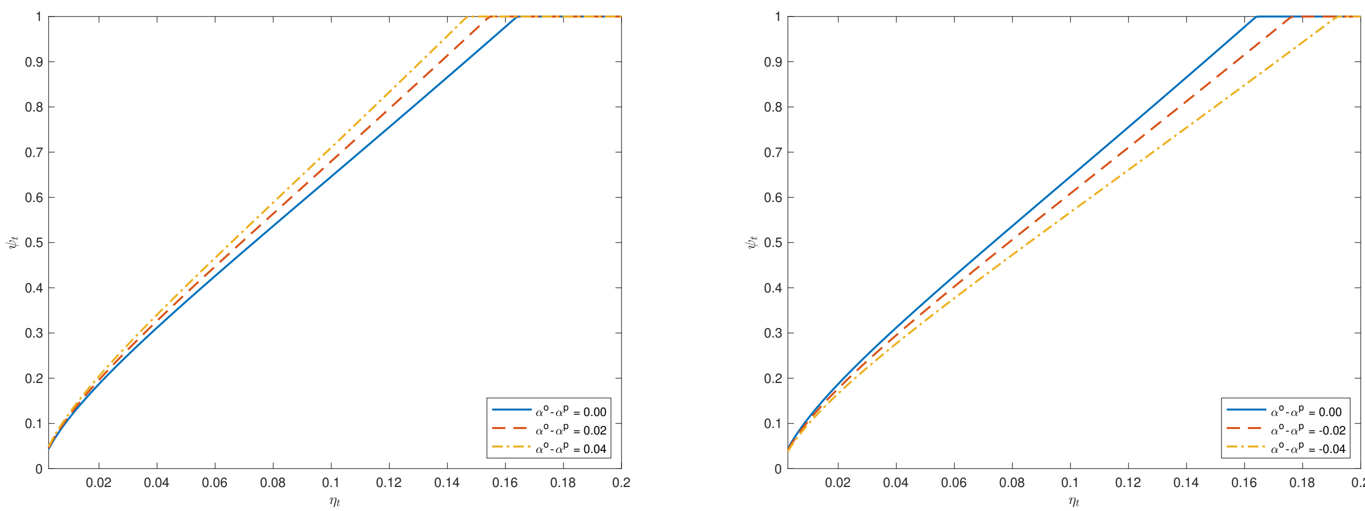
A Additional Figures



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

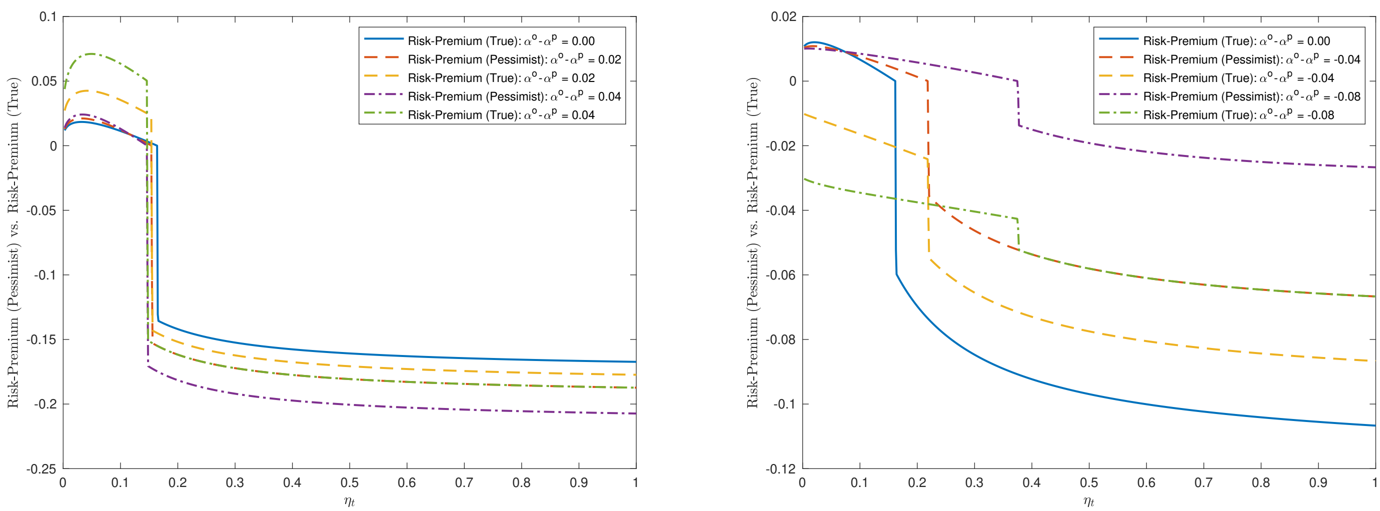
Figure A1: Investment Rate ι_t as a Function of η_t



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

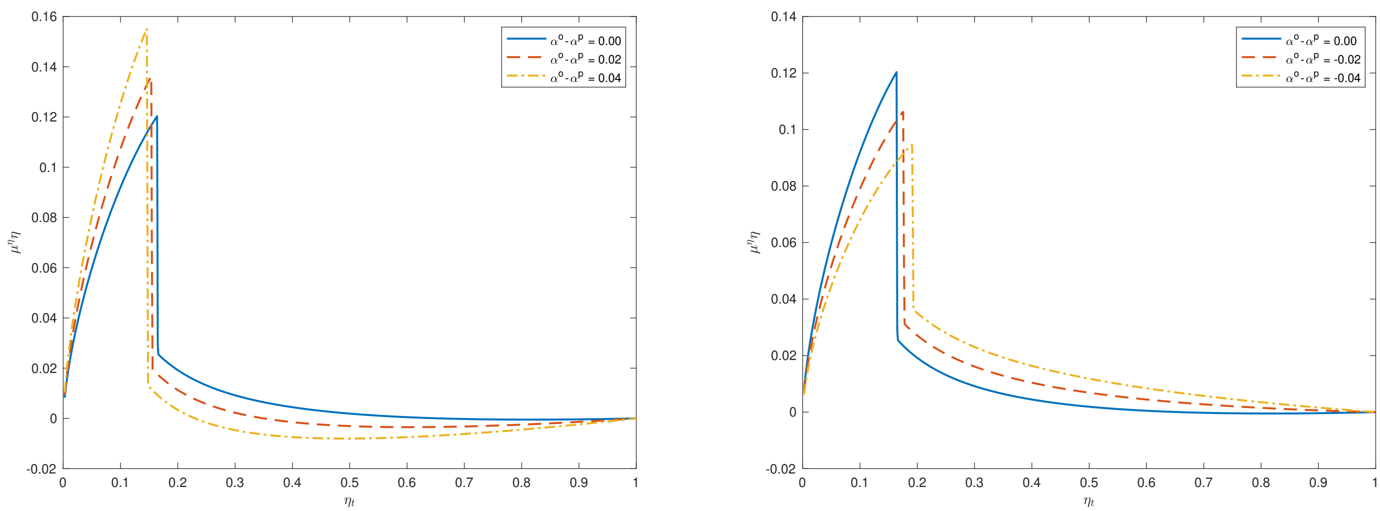
Figure A2: Capital Share ψ_t as a Function of η_t



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

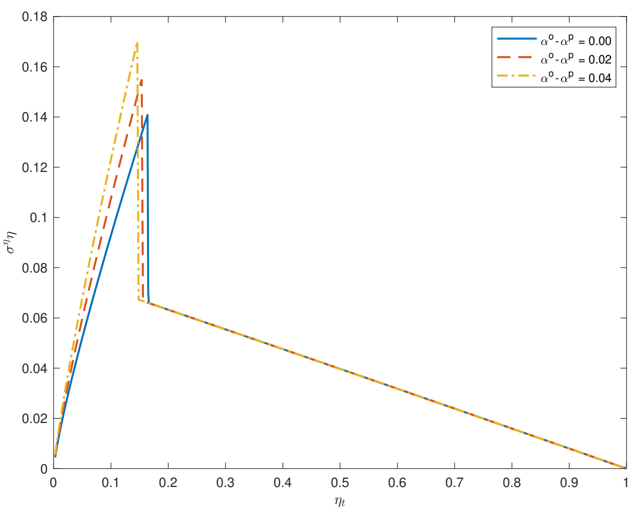
Figure A3: Risk-Premium (Pessimists' and True Value) as a Function of η_t



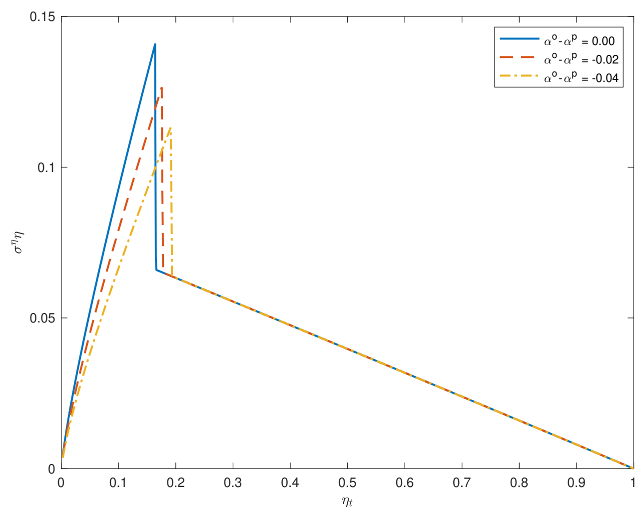
(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$

(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure A4: Wealth Share Drift $\mu_\eta(\eta_t) \cdot \eta_t$ as a Function of η_t



(a) $\alpha^O \geq \alpha$ and $\alpha^P \leq \alpha$



(b) $\alpha^O \leq \alpha$ and $\alpha^P \geq \alpha$

Figure A5: Wealth Share Volatility $\sigma^\eta(\eta_t) \cdot \eta_t$ as a Function of η_t

B Robustness check

	HKM		HKM+Momentum	
	Two-factor	Three-factor	Two-factor	Three-factor
Mean excess return	0.85	0.85	1.2	1.2
Std excess return	1.31	1.31	1.32	1.32
Mean β_M	0.46	0.46	0.62	0.62
Std β_M	0.46	0.46	0.48	0.48
Mean β_η	0.03	0.03	0.04	0.04
Std β_η	0.09	0.09	0.1	0.1
Mean β_d	-	-0.0	-	-0.001
Std β_d	-	0.03	-	0.04
Assets	94	94	129	129
Quarters	195	195	195	195
Controls	Yes	Yes	Yes	Yes

Table A1: Expected returns and risk exposures- Robustness check. HKM assets include 25 size and book-to-market portfolios, 10 U.S. Govt. bond portfolios, 10 U.S. corporate bond portfolios, 18 option portfolios, 20 CDS portfolios, and 12 foreign exchange portfolios taken from HKM2017. HKM+Momentum includes 25 size and momentum portfolios and 10 long-term reversal portfolios in addition to the HKM assets. Mean and std. of excess return is the difference in mean return and risk free rate of the corresponding test assets. The frequency is quarterly and time period is from 1970Q1 till 2012Q4. The mean and std. of betas ($\beta_W, \beta_\eta, \beta_d$) measure the average and standard deviation of exposure of the excess return to market factor, intermediary capital ratio, and disagreement measure respectively. Controls include price-dividend ratio, cyclically adjusted earnings ratio (CAPE), cay, and capital share risk.

	HKM		HKM+Momentum	
	Two-factor	Three-factor	Two-factor	Three-factor
Market	0.02	0.01	0.02	0.01
	(1.6)	(0.86)	(1.83)	(1.04)
Intermediary	0.09	0.09	0.05	0.06
	(4.48)	(3.05)	(3.01)	(2.13)
Disagreement	-	0.11	-	0.12
	-	(2.04)	-	(3.25)
MAPE %	1.7	1.34	2.36	1.95
Adj. R2	0.82	0.89	0.60	0.76
Assets	94	94	129	129
Quarters	195	195	195	195

Table A2: Risk price estimates for HKM and HKM+Momentum portfolios- Robustness check. HKM assets include 25 size and book-to-market portfolios, 10 U.S. Govt. bond portfolios, 10 U.S. corporate bond portfolios, 18 option portfolios, 20 CDS portfolios, and 12 foreign exchange portfolios taken from HKM2017. HKM+Momentum includes 25 size and momentum portfolios and 10 long-term reversal portfolios in addition to the HKM assets. The data is at quarterly frequency from 1970Q1 till 2012Q4. The factors are market, intermediary capital ratio, and disagreement. The disagreement factor d_t is computed as growth rate in the inter-quartile dispersion of GDP forecast scaled by median forecast by analysts. The forecast data is taken from The Survey of Professional Forecasters. The standard errors of risk price estimates are adjusted for generated regressor problem using Shanken correction. MAPE denotes the mean absolute pricing error in annualized terms.