A Proxy Contract Based Approach to the First-Order Approach in Agency Models

Jin Yong Jung Kangnam University Son Ku Kim Seoul National University Seung Joo Lee Oxford University

July 24, 2024

The agency problem: Holmström (1979)

Principal's canonical problem (x is the multi-dimensional signal):

$$\max_{a,s(\cdot)} \int (\pi(\mathbf{x}) - s(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} \quad \text{s.t.}$$
(i) (PC) $U(s(\cdot), a) > \overline{U}$

(ii) (IC)
$$a \in \underset{a'}{\operatorname{arg\,max}} \ U(s(\cdot), a') = \int u(s(\mathbf{x})) f(\mathbf{x}|a') d\mathbf{x} - a'$$

$$(iii)$$
 (LL) $s(x) \geq \underline{s}$

Note: the limited-liability (LL) $s(x) \ge \underline{s}$ for the solution existence (e.g., Mirrlees (1975)): especially when

First-Order Approach

Principal's canonical problem (x is the multi-dimensional signal):

$$\max_{a,s(\cdot)} \int (\pi(\mathbf{x}) - s(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x} \quad \text{s.t.}$$

$$(i) (PC) \quad U(s(\cdot), a) \ge \overline{U}$$

$$(ii)'$$
 (IC)-relaxed $U_a(s(\cdot),a)=\int u(s(\mathbf{x}))f_a(\mathbf{x}|a)d\mathbf{x}-1=0$

(iii) (LL)
$$s(x) \ge \underline{s}$$

Note: the limited-liability (LL) $s(x) \ge \underline{s}$ for the solution existence (e.g., Mirrlees (1975)): especially when

$$\frac{f_a}{f}(\mathbf{x}|a) \to -\infty$$
, when $x \to \underline{x}$ (2)

Optimal contract $(s^{\circ}(x), a^{\circ})$ based on the first-order approach:

$$\frac{1}{u'(s^{o}(\mathbf{x}))} = \begin{cases} \lambda + \mu \frac{f_{a}(\mathbf{x}|a^{o})}{f(\mathbf{x}|a^{o})}, & \text{if } s^{o}(\mathbf{x}) \geq \underline{s}, \\ \frac{1}{u'(\underline{s})}, & \text{otherwise,} \end{cases}$$

with $\lambda \geq 0$ and $\mu > 0$

• Existence and uniqueness: Jewitt, Kadan, and Swinkels (2008)

If the agent's value function $U(s^{\circ}(\cdot), a)$,

$$U(s^{\circ}(\cdot),a) = \int u(s^{\circ}(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - a$$

is 'concave' in a, then the first-order approach is valid (e.g., Mirrlees (1975))

The previous literature since Mirrlees (1975): 'sufficient' conditions for $U(s^o(\cdot), a)$ to be 'concave' in a

The previous literature

Question (Focus of the literature)

How can we make $U(s^{\circ}(\cdot), a)$ concave in a?

Strategy 1: put conditions on $f(\mathbf{x}|a)$, the technology, only:

 One-signal (i.e., x is scalar): Mirrlees (1975) and Rogerson (1985): MLRP (monotone likelihood ratio property) and CDFC (convexity of the distribution function condition)

Multi-signal extension of CDFC: Sinclair-Desgagné (1994, GCDFC: generalized CDFC), Conlon (2009, CISP: concave increasing set property), and Jung and Kim (2015, CDFCL: convexity of the distribution function condition for the likelihood ratio)

3 Too restricted (e.g., normal, gamma distributions excluded)

Question (Focus of the literature)

How can we make $U(s^{\circ}(\cdot), a)$ concave in a?

Strategy 2: put conditions on both u(s) and f(x|a):

Theorem 1 in Jewitt (1988):

$$w(z) \equiv u\left(u'^{-1}\left(\frac{1}{z}\right)\right)$$
 is concave in $z > 0$ (3)

or Proposition 7 in Jung and Kim (2015):

$$U(s^{\circ}(\mathbf{x}), a^{\circ}) \equiv r(q)$$
 is concave in $q \equiv \frac{f_a}{f}(\mathbf{x}|a^{\circ})$ (4)

→ (3) and (4) are equivalent

② Problem: Cannot be used when the agent's limited liability $s(x) \ge \underline{s}$ binds:

$$U(s^o(\mathbf{x}), a^o)$$
 becomes convex in $q \equiv \frac{f_a}{f}(\mathbf{x}|a^o)$

around x where $s(x) \ge \underline{s}$ binds



Examples show the existing conditions are not enough

Example (Poisson distribution: (LL) not binding)

- **1** The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r > \frac{1}{2}$
- $oldsymbol{0}$ The single-dimensional signal x, which is non-negative integer, follows

$$f(x|a) = \frac{[h(a)]^{x}}{\Gamma(x+1)} e^{-h(a)}.$$
 (5)

which is the Poisson distribution with mean h(a) that is increasing in a

Issue with $r > \frac{1}{2}$:

- $U(s^o(\mathbf{x}), a^o) \equiv r(q)$ becomes convex in q, not satisfying Jewitt (1988) and Jung and Kim (2015)
- Our Proposition 2 justifies the first-order approach in this case if h(a) becomes concave 'enough'
- Jewitt (1988) and Jung and Kim (2015) imposes h(a) is concave

Trade-off: h(a) should be more strictly concave, but $r > \frac{1}{2}$ allowed

Examples show the existing conditions are not enough

Example (Exponential distribution: (LL) not binding)

- **1** The agent's utility is $u(s) = \frac{1}{r}s^r$, with r < 0
- $oldsymbol{0}$ The single-dimensional signal x, which is non-negative, follows

$$f(x|a) = \frac{1}{h(a)}e^{-\frac{x}{h(a)}},\tag{6}$$

which is the exponential distribution with mean h(a) that is increasing in a

 \bullet h(a) is convex to some degree (not too much) in a

Issue with h(a):

- Jewitt (1988) and Jung and Kim (2015) imposes $h(\cdot)$ is concave
- Our Proposition 3 justifies the first-order approach in this case if $U(s^o(\mathbf{x}), a^o) \equiv r(q)$ becomes concave 'enough'

Trade-off: r(q) should be more strictly concave, but convex h(a) allowed

Examples show the existing conditions are not enough

Example (Normal distribution: (LL) binding)

- **1** The agent's utility is $u(s) = \frac{1}{r}s^r$, with r < 1
- ② The single-dimensional signal $x \sim N(h(a), \sigma^2)$, i.e.,

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-h(a))^2}{2\sigma^2}}$$

which is a normal distribution with mean h(a) that is increasing in a

1 In this case, (LL) $s(x) \ge \underline{s}$ must be imposed

Issue with normal distribution:

- The previous literature cannot justify the first-order approach in this simple example
- $U(s^o(\mathbf{x}), a^o) \equiv r(q)$ becomes convex at points where (LL) binds, not satisfying Jewitt (1988) and Jung and Kim (2015): even if $r < \frac{1}{2}$



Example (Normal distribution: (LL) binding)

- **1** The agent's utility is $u(s) = \frac{1}{r}s^r$, with r < 1
- **②** The single-dimensional signal $x \sim N(h(a), \sigma^2)$, i.e.,

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-h(a))^2}{2\sigma^2}}$$

which is a normal distribution with mean h(a) that is increasing in a

3 In this case, (LL) $s(x) \ge \underline{s}$ must be imposed

Issue with normal distribution:

- Our Proposition 4 justifies the first-order approach in this case if h(a) becomes concave 'enough', regardless of r < 1
- Jewitt (1988) and Jung and Kim (2015) imposes h(a) is concave

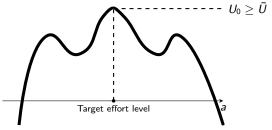
Trade-off: h(a) should be more strictly concave, but (LL) allowed



Our paper: different approach

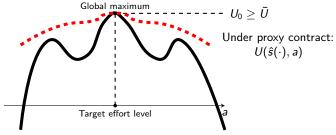
Big Question (Possibly Non-Concave Expected Monetary Utility of the Agent)

Why should the agent's expected monetary utility $U(s^{\circ}(\cdot), a)$ be concave in a?



The agent's expected monetary utility obtained from the first-order approach

Figure: Possibly Non-Concave Expected Monetary Utility of the Agent



The agent's expected monetary utility: $U(s^{\circ}(\cdot), a)$

Our approach:

- Finding a proxy function $\hat{s}(\mathbf{x})$ where the proxy value $U(\hat{s}(\cdot), a)$ is maximized at $a = a^{\circ}$, the same target action level
- ② Proving $U(s^{\circ}(\cdot), a) \leq U(\hat{s}(\cdot), a)$, $\forall a$, justifying the first-order approach

Key idea: double-crossing property between $s^{\circ}(\cdot)$ and $\hat{s}(\cdot)$ in q-space

Fundamental Lemma

Change of variables to q-space

À la Jung and Kim (2015), define the likelihood ratio

$$\tilde{q} \equiv Q_{a^o}(\tilde{\mathbf{x}}) \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$$

The optimal contract $s^{\circ}(x)$ in q-space becomes:

$$s^{\circ}(x) \equiv w(q) \equiv (u')^{-1} \left(\frac{1}{\lambda + \mu q}\right)$$

The agent's indirect utility given $s^{o}(\cdot)$

$$u(s^{\circ}(\mathbf{x})) \equiv r(q) = \left\{ egin{array}{ll} u(w(q)) \equiv \overline{r}(q), & ext{when } q \geq q_c \\ u(\underline{s}), & ext{when } q < q_c \end{array} \right.$$

- Threshold q_c solves $u'(\underline{s})^{-1} = \lambda + \mu q_c > 0$:
- $q \leq q_c$: the limited liability binds

Distribution function for q given a (possbly different from a°)

$$G(q|a) \equiv Pr[Q_{a^{\circ}}(\tilde{\mathbf{x}}) \leq q|a], \quad dG(q|a) = g(q|a)dq$$

Double-crossing: constructing a proxy contract

Define $U^o \geq \overline{U}$ at the optimum:

$$U^{\circ} = U(s^{\circ}(\mathbf{x}), a^{\circ}) = \int u(s^{\circ}(\mathbf{x})) f(\mathbf{x}|a^{\circ}) d\mathbf{x} - a^{\circ}$$
 (7)

Lemma (How to construct a proxy contract $\hat{s}(\cdot)$)

- (1a) $f(\mathbf{x}|a)$ satisfies that $\frac{g(q|a)}{g(q|a^o)}$ is convex in $q = \frac{f_a(\mathbf{x}|a^o)}{f(\mathbf{x}|a^o)}$ for all a
- (2a) (Double-crossing) \exists a contract $\hat{s}(x)$ satisfying

(i) Same (PC)
$$U(\hat{s}(\cdot), \mathbf{a}^{\circ}) = \int u(\hat{s}(\mathbf{x}))f(\mathbf{x}|\mathbf{a}^{\circ})d\mathbf{x} - \mathbf{a}^{\circ} = U^{\circ}$$
 (8)

(ii) Same (IC)
$$a^o \in \underset{a'}{\operatorname{arg max}} \int u(\hat{s}(\mathbf{x})) f(\mathbf{x}|a') d\mathbf{x} - a'$$
 (9)

such that $\hat{r}(q) \equiv u(\hat{s}(\mathbf{x}))$ double-crosses $r(q) \equiv u(s^o(\mathbf{x}))$ from above in q-space

then using the first-order approach is justified



Intuition

(1a) and (2a) jointly imply:

$$U(s^{\circ}(\cdot),a)-U(\hat{s}(\cdot),a)=\int (r(q)-\hat{r}(q))\,g(q|a)dq\leq 0,\quad \forall a\in \mathcal{S}$$

Why?: we know that $U(s^{\circ}(\cdot), a^{\circ}) = U(\hat{s}(\cdot), a^{\circ})$ when $a = a^{\circ}$

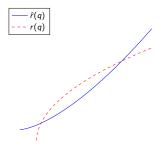


Figure: r(q) and $\hat{r}(q)$: double-crossing

For example, when a^{\uparrow} from a° :

- G(q|a) shifts toward higher q, where $r(q) \hat{r}(q)$ becomes more negative
- (1a) condition operationalizes this intuition



(1a) and (2a) jointly imply:

$$U(s^{\circ}(\cdot),a)-U(\hat{s}(\cdot),a)=\int \left(r(q)-\hat{r}(q)\right)g(q|a)dq\leq 0, \quad orall a$$

But: it might be the following case

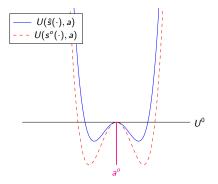


Figure: First-order approach not justified?

(2a) makes sure that $U(\hat{s}(\cdot), a)$ is maximized at $a = a^{\circ}$, therefore:

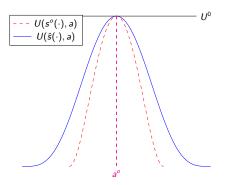


Figure: First-Order Approach Justified

So $U(s^{\circ}(\cdot), a)$ must be maximized at $a = a^{\circ}$

• The first-order approach (FOA) justified

18 / 33

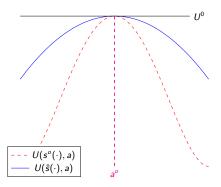


Figure: First-Order Approach Justified

So $U(s^{\circ}(\cdot), a)$ must be maximized at $a = a^{\circ}$

• The first-order approach (FOA) justified

19 / 33

When the Limited Liability (LL) Not Binds

Simplest case: linear contract in *q*-space

Proposition (Proposition 1)

Given that the likelihood ratio, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|\mathbf{a}^o)}{f(\tilde{\mathbf{x}}|\mathbf{a}^o)}$, is bounded below, a given \mathbf{a}^o ,

$$(1a) \ \frac{g(q|a)}{g(q|a^o)} \ \text{is convex in} \ q \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)} \ \text{for all} \ a$$

(2b)
$$m(a) \equiv \int qg(q|a)dq$$
 is concave in a

(3b)
$$r(q)$$
 is concave in q

then the first-order approach is justified

Note: Now $\overline{r}(q) = r(q)$ due to the nonbinding (LL)

- (2b) and (3b) are from Jewitt (1988) and Jung and Kim (2015)
- Find $\hat{s}(\mathbf{x})$ such that $u(\hat{s}(\mathbf{x})) \equiv \hat{r}(q)$ becomes linear in q



^aWe assume \underline{s} is small enough, so (LL) does not bind at optimum

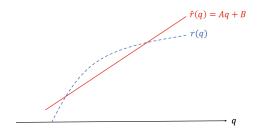


Figure: When the Agent's Limited Liability Constraint Does Not Bind

Simplest case: our proxy contract $\hat{r}(q)$ is linear in q

- (2b) makes sure under $\hat{r}(q)$, the agent will choose $a = a^{\circ}$
- With (1a) and (3b), we apply the lemma above (double-crossing)

Violating (3b): what if $\overline{r}(q)$ becomes convex in q?

Define the moment generating function (MGF) of g(q|a):

$$M(a;t) \equiv \int e^{tq} g(q|a) dq$$

Proposition (Proposition 2)

Given that u(s) > 0 for all s and $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|\mathbf{a}^o)}{f(\tilde{\mathbf{x}}|\mathbf{a}^o)}$, is bounded below, given \mathbf{a}^o ,

$$(\mathbf{1a}) \ \frac{g(q|\mathbf{a})}{g(q|\mathbf{a}^o)} \ \text{is convex in} \ q \equiv \frac{f_{\mathbf{a}}(\widetilde{\mathbf{x}}|\mathbf{a}^o)}{f(\widetilde{\mathbf{x}}|\mathbf{a}^o)} \ \text{for all} \ \mathbf{a}$$

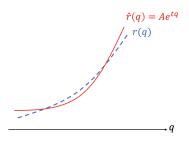
(2b')
$$\phi(a; t, \overline{U}) \equiv \frac{M_a(a; t)}{M(a; t)} \times [\overline{U} + a]$$
 is decreasing in a for any given $t > 0$

(3b') In r(q) is concave in q

then the first-order approach is justified

(3b'): $\ln \overline{r}(q)$, not $\overline{r}(q)$, is concave so $\overline{r}(q)$ can be convex (\rightarrow weaker)

- We prove (2b') is a bit stronger than (implies) (2b) instead
- In this case, our proxy contract $\hat{r}(q)$ is exponential in q



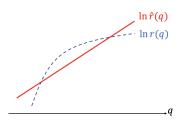


Figure: Double-Crossing: $\hat{r}(q)$ and $\bar{r}(q)$

Figure: $\ln \hat{r}(q)$ and $\ln \bar{r}(q)$

Double-crossing: proxy contract $\hat{r}(q)$ is exponential in q so $\ln \hat{r}(q)$ is linear

- (2b') makes sure under $\hat{r}(q)$, the agent will choose $a = a^{\circ}$
- (1a) and (3b') allow us to apply the lemma above (double-crossing)

Going back to an example

Example (Poisson distribution: (LL) not binding)

- **1** The agent's utility is $u(s) = \frac{1}{r}s^r$, with $r > \frac{1}{2}$

$$f(x|a) = \frac{[h(a)]^{x}}{\Gamma(x+1)} e^{-h(a)}.$$
 (10)

which is the Poisson distribution with mean h(a) that is increasing in a

Even with $r > \frac{1}{2}$:

Our Proposition 2 justifies the first-order approach in this case if

$$\frac{h''(a)}{h'(a)} \le -\frac{1}{\overline{U}+a} < 0. \tag{11}$$

• Jewitt (1988) and Jung and Kim (2015) imposes h(a) is concave

Trade-off: h(a) should be more strictly concave, but $r > \frac{1}{2}$ allowed

Violating (2b): what if m(a) becomes convex in a?

Proposition (Proposition 3)

Given that u(s) < 0 for all s and $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|\mathbf{a}^o)}{f(\tilde{\mathbf{x}}|\mathbf{a}^o)}$, is bounded below, \tilde{a} given \mathbf{a}^o ,

(1a)
$$\frac{g(q|a)}{g(q|a^o)}$$
 is convex in $q \equiv \frac{f_a(\tilde{\mathbf{x}}|a^o)}{f(\tilde{\mathbf{x}}|a^o)}$ for all a

$$(2\mathbf{b}'') \ \phi\left(\mathbf{a}; t, \overline{U}\right) \equiv \frac{M_{\mathbf{a}}(\mathbf{a}; t)}{M(\mathbf{a}; t)} \times [\overline{U} + \mathbf{a}] \text{ is decreasing in } \mathbf{a} \text{ for any given } t < 0$$

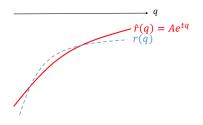
$$(3b'') - \ln[-r(q)]$$
 is concave in q

then the first-order approach is justified

(3b"): $-\ln[-\overline{r}(q)]$, not $\overline{r}(q)$, is concave so $\overline{r}(q)$ is more concave (\rightarrow stronger)

- We prove (2b") is a bit weaker than (implied by) (2b) instead
- In this case, our proxy contract $\hat{r}(q)$ is negative exponential (concave) in q

^aWe assume <u>s</u> is small enough, so (LL) does not bind at optimum



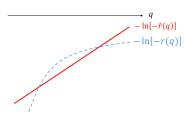


Figure: Double-Crossing: $\hat{r}(q)$ and $\bar{r}(q)$

Figure: $\ln \hat{r}(q)$ and $\ln \overline{r}(q)$

Double-crossing: proxy contract $\hat{r}(q)$ is more concave in q so $\ln \hat{r}(q)$ is linear

- (2b") makes sure under $\hat{r}(q)$, the agent will choose $a=a^o$
- (1a) and (3b") allow us to apply the lemma above (double-crossing)

Going back to an example

Example (Exponential distribution: (LL) not binding)

- 1 The agent's utility is $u(s) = \frac{1}{r}s^r$, with r < 0
- $oldsymbol{2}$ The single-dimensional signal x, which is non-negative, follows

$$f(x|a) = \frac{1}{h(a)}e^{-\frac{x}{h(a)}},$$
 (12)

which is the exponential distribution with mean h(a) that is increasing in a

- \bullet h(a) is convex to some degree (not too much) in a
- Our Proposition 3 justifies the first-order approach in this case if

$$\frac{h''(a)}{h'(a)} \le \underbrace{-\frac{1}{\overline{U}+a}}_{>0}, \quad \forall a \in (0, \overline{a}], \tag{13}$$

if
$$h(a) + h'(a)[\overline{U} + a] < 0$$
 for $\forall a \in (0, \overline{a}]$

Trade-off: r(q) should be more strictly concave, but convex h(a) allowed

When the Limited Liability (LL) Binds

Finding a proxy contract when (LL) binds for $q \leq q_c$

Define the moment generating function (MGF) of g(q|a):

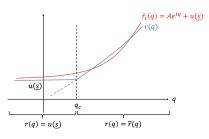
$$M(a;t) \equiv \int e^{tq} g(q|a) dq$$

Proposition (Proposition 4)

Given that the likelihood ratio, $\tilde{q} \equiv \frac{f_a(\tilde{\mathbf{x}}|\mathbf{a}^o)}{f(\tilde{\mathbf{x}}|\mathbf{a}^o)}$, is unbounded below, given \mathbf{a}^o ,

- $\textbf{(1a)} \ \frac{g(q|\textbf{a})}{g(q|\textbf{a}^{\circ})} \ \text{is convex in} \ q = \frac{f_{\textbf{a}}(\textbf{x}|\textbf{a}^{\circ})}{f(\textbf{x}|\textbf{a}^{\circ})} \ \text{for all} \ \textbf{a}$
- (2c) $\phi(a; t, \overline{U} u(\underline{s})) \equiv \frac{M_a(a; t)}{M(a; t)} \times [\overline{U} u(\underline{s}) + a]$ is decreasing in a for any given t > 0
- (3c) $\ln[\overline{r}(q) u(\underline{s})]$ is concave in q for all $q > q_c$, where q_c solves $\overline{r}(q_c) = u(\underline{s})$ then the first-order approach is justified
- (3c): $\ln[\overline{r}(q) u(\underline{s})]$, not $\overline{r}(q)$, is concave so $\overline{r}(q)$ can be convex $(\rightarrow$ weaker)
 - (2c): (2b') with $\bar{U} u(\underline{s})$





 $\frac{\ln[\hat{r}_t(q) - u(\underline{s})]}{qc}$

Figure: Double-Crossing: $\hat{r}(q)$ and $\bar{r}(q)$

Figure: $\ln \hat{r}(q)$ and $\ln \bar{r}(q)$

Double-crossing: proxy contract $\hat{r}(q)$ is affine-exponential in q

- (2c) makes sure under $\hat{r}(q)$, the agent will choose $a=a^{\circ}$
- (1a) and (3c) allow us to apply the lemma above (double-crossing)

Going back to an example

Example (Normal distribution: (LL) binding)

- ① The agent's utility is $u(s) = \frac{1}{r}s^r$, with r < 1
- ② The single-dimensional signal $x \sim N(h(a), \sigma^2)$, i.e.,

$$f(x|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-h(a))^2}{2\sigma^2}}$$

which is a normal distribution with mean h(a) that is increasing in a

- **3** In this case, (LL) $s(x) \ge \underline{s}$ must be imposed
 - Our Proposition 4 justifies the first-order approach if

$$\frac{h''(a)}{h'(a)} \le -\frac{1}{\overline{U} - u(\underline{s}) + a} < 0, \quad \forall a, \tag{14}$$



Comparison with the earlier literature

To compare with Jung and Kim (2015)'s conditions (1J-1) and (1J-2):

- We introduce the total positivity of degree 3 (TP₃) (Karlin (1968))
- Our (1a) condition is equivalent to (TP₃) under MLRP
- Thus, Proposition 1 implies Jung and Kim (2015)

$$((\mathsf{TP}_3),(1\mathsf{J}-2)) \xleftarrow{\mathsf{Lemma\ 5}}_{\mathsf{Given\ MLRP}} ((1\mathsf{a}),(2\mathsf{b})) \xrightarrow{\mathsf{Implies}} ((1\mathsf{J}-1),(1\mathsf{J}-2))$$

Figure: Relation Diagram between Conditions

Still, Our Propositions 2, 3, 4 extend the first-order approach's applicability