Introduction to Econometrics 2: Recitation 9

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Quantile Regression

Motivation

- The usual linear regression, which has the form

$$y = X\beta + u$$

  where $E(Xu) = 0$ (or more restrictively, $E(u|X) = 0$)
- This setup is aimed at capturing the conditional mean of y given X.
- However, there is no reason to restrict our attention to just a conditional mean. We might want a conditional median etc.
- **Quantile regression** aims to capture different values of β depending on the location of the conditional distribution.

Quantile Regression

Setup

- The quantile regression seeks to estimate the conditional quantile

  $$q_\tau(y|X) = X\beta_\tau$$

  where $\tau \in [0, 1]$ is the percentile of our choice satisfying $F_{y|X}(X\beta_\tau|X) = \tau$.
- With

  $$F_{y|X}(X\beta_\tau|X) = \Pr(y \leq X\beta_\tau|X) = \tau$$

  we can write

  $$\tau - \Pr(y \leq X\beta_\tau|X) = 0$$

  Since $\Pr(y \leq X\beta_\tau|X)$ is equal to $E[1(y - X\beta_\tau \leq 0)] = E[1(u \leq 0)]$, we can obtain the condition

  $$E[\tau - 1(y - X\beta_\tau \leq 0)|X] = 0$$

  this also implies $E[(\tau - 1(y - X\beta_\tau \leq 0))X] = 0$

Quantile Regression

Setup: Check Function

- The check function can be defined as

$$\rho_\tau(u) = u(\tau - 1(u \leq 0))$$

  - Median: Let $\tau = 1/2$. Then the check function becomes

$$\rho_{1/2}(u) = \begin{cases} -\frac{1}{2}u & (u \leq 0) \\ \frac{1}{2}u & (u > 0) \end{cases} = \frac{1}{2}|u| = \frac{1}{2}|y - X\beta_{1/2}|$$

    This becomes equivalent to solving the least absolute deviation problem.
  - $\tau = 1/3$: Then the check function becomes

$$\rho_{1/3}(u) = \begin{cases} -\frac{2}{3}u & (u \leq 0) \\ \frac{1}{3}u & (u > 0) \end{cases}$$

    which has a kink at $u = 0$ and is asymmetric.

Quantile Regression

Finding the Quantile Estimator

- The minimization problem solved in quantile regression is

$$\min_{\beta} E[\rho_\tau(y - X\beta_\tau)|X]$$

which can be written as

$$E[\rho_\tau(y - X_i\beta_\tau)|X] = (\tau - 1)\int_{-\infty}^{a}(y-a)f_{Y|X}(y|x)dy + \tau\int_{a}^{\infty}(y-a)f_{Y|X}(y|x)dy$$

where $a = X\beta_\tau$

- Take the first order condition w.r.t. a to get

$$-(\tau - 1)\int_{-\infty}^{a} f_{Y|X}(y|x)dy - \tau\int_{a}^{\infty} f_{Y|X}(y|x)dy = 0$$
$$\iff -\tau + F(a|X) = 0$$

Quantile Regression

Finding the Quantile Estimator

- Thus, the β_τ that solves

$$X\beta_\tau = a = F^{-1}_{Y|X}(\tau|X)$$

  is the β_τ that we are looking for
- We can also solve for

$$\min_\beta E[\rho_\tau(y - X\beta_\tau)X]$$

  or in a sample analogue

$$\min_\beta \frac{1}{n}\sum_{i=1}^{n}\rho_\tau(y_i - X_i\beta_\tau)X_i$$

- For suitable conditions, the resulting estimator is CAN.

Quantile Regression

Examples

- Levin (2001)
  - In educational production function literature, the effect of class size on various outcomes for the student is controversial
  - Uses quantile regression in estimating educational production
  - Also controls for peer effects - turns out that this effect matters more especially for those in the low achievement group
  - Class size does not have as significant effect
- Autor, Houseman, Kerr (2017)
  - Studies effect of direct hire assistance and temporary-help job placement programs in Detroit on distribution of participant's earnings over a 7-quarter period
  - For low-tail, none are effective. For high-tail, direct hire raises earnings but temporary-help does not
  - Autor, Houseman (2010) study the same program without QR and finds on avg that earnings increased

Nonparametric Regression

Motivation

- Consider a setting where we observe the data (y_i, x_i) which is i.i.d. and drawn from a DGP $P_0(y|x)$.
- We are interested in backing out the whole or part of the data generating process $P_0(y|x)$ **without any modeling assumptions**
- This approach is called a **nonparametric** approach
- We normally use nonparametric approach to
  - Conduct a diagnostic checking of an estimated parametric model,
  - To conveniently display key features of the dataset
  - Conduct an inference under very weak assumptions.

Nonparametric Regression

Idea

- For discrete-valued Y and X, this is relatively easy. Just back out the data generating process of interest

$$\hat{P}(y \in A|x \in B) = \frac{n^{-1}\sum_{i=1}^{n} 1(y_i \in A, x_i \in B)}{n^{-1}\sum_{i=1}^{n} 1(x_i \in B)}$$

  Provided that $P_0(x \in B) \neq 0$, then the above estimator converges to the true $P_0(y|x)$ as $n \to \infty$.
- For continuous cases, $A \equiv (-\infty, y], B \equiv [x - \epsilon, x + \epsilon]$ and use $\lim_{\epsilon \to 0} \hat{P}(A|B)$ to back out the DGP
- This can be problematic when we have too many dimensions of X
- Therefore, the method to conduct nonparametric estimation in this context is kernel density function approach.

Nonparametric Regression

Kernels

- Consider the problem of estimating the probability density function $f(x)$ of a random scalar variable X at $X = x$
- Let $\{X_1, ..., X_n\}$ be a random sample of X. If f is a smooth function, we can approximate f by

$$f(x) \simeq \frac{\int_{x-h}^{x+h} f(u)du}{2h}$$

  for a small h

- The numerator on the right hand side can be estimated using a sample analogue of the following form

$$\frac{1}{n}\sum_{i=1}^{n} 1\{x - h \leq X_i \leq x + h\}$$

Nonparametric Regression

Kernels

- Combining the two expressions, we can approximate $f(x)$ with

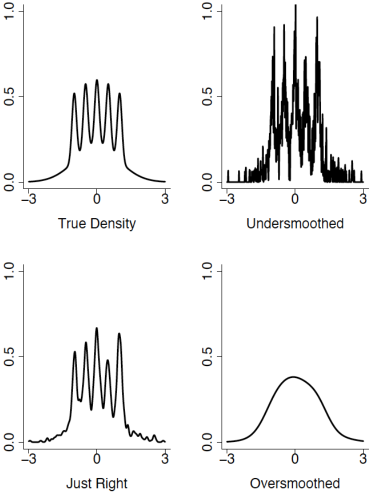
$$\begin{aligned}
\hat{f}(x) &= \frac{n^{-1}\sum_{i=1}^{n} 1[x-h \le X_i \le x+h]}{2h} \\
&= \frac{1}{2nh}\sum_{i=1}^{n} 1[x-h \le X_i \le x+h] \\
&= \frac{1}{nh}\sum_{i=1}^{n} \frac{1}{2} 1\left[\left|\frac{x-X_i}{h}\right| \le 1\right] \\
&= \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)
\end{aligned}$$

  where $K(u) = \frac{1}{2}1(|u| \le 1)$ - a uniform kernel.

- $K(\cdot)$ is a **kernel density estimator**. It should be a real-to-real function, positive and smooth and integrates to 1 on its support
- Generally type of kernels rarely matter

Bandwidth

- However, bandwidths do matter, as they are involved in the the trade-off between bias and variance



Nonparametric Regression

Bandwidth: Characterizing Bias

$$
\begin{aligned}
E[\hat{f}(x)] &= \frac{1}{h}\int_{-\infty}^{\infty} K\left(\frac{x-s}{h}\right) f(s)ds \\
&= -\frac{1}{h}\int_{-\infty}^{\infty} K(t)f(x-ht)(-hdt) \ (\because \frac{x-s}{h} = t \text{ transformation}) \\
&= \int_{-\infty}^{\infty} K(t)f(x-ht)dt \\
&= \int_{-\infty}^{\infty} K(t)\left[f(x) - f'(x)ht + \frac{f''(x)h^2t^2}{2} + o(h^2)\right] dt \\
&= f(x) - 0 + \frac{1}{2}\int_{-\infty}^{\infty} K(t)h^2t^2f''(x)dt + o(h^2)
\end{aligned}
$$

Where $\int_{-\infty}^{\infty} K(t)dt = 1$ justifies $f(x)$ and $\int_{-\infty}^{\infty} tK(t)dt = 0$ (since kernel is symmetric around 0) justifies second term in the last line being 0

- Thus, bias is

$$E[\hat{f}(x)] - f(x) = \frac{1}{2}\int_{-\infty}^{\infty} K(t)h^2t^2f''(x)dt$$

Nonparametric Regression

Bandwidth: Characterizing Variance

$$
\begin{aligned}
Var[\hat{f}(x)] &= E\left[\frac{1}{n^2h^2}\left(\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)\right)^2\right] - (E[\hat{f}(x)])^2 \\
&= E\left[\frac{1}{n^2h^2}\left(\sum_{i=1}^{n} K^2\left(\frac{x-X_i}{h}\right) + 2\sum_{i<j} K\left(\frac{x-X_i}{h}\right) K\left(\frac{x-X_j}{h}\right)\right)\right] - (E[\hat{f}(x)])^2 \\
&= \frac{1}{nh^2}\int_{-\infty}^{\infty} K^2\left(\frac{x-s}{h}\right) f(s)ds + \frac{n(n-1)}{n^2h^2}\left(\int_{-\infty}^{\infty} K\left(\frac{x-s}{h}\right) f(s)ds\right)^2 \\
&- \frac{1}{h^2}\left(\int_{-\infty}^{\infty} K\left(\frac{x-s}{h}\right) f(s)ds\right)^2
\end{aligned}
$$

Then, use variable transformation and Taylor approximation.

- Thus, the leading term of bias is

$$\frac{1}{nh}\int_{-\infty}^{\infty} K^2(t)f(x)dt \simeq O\left(\frac{1}{nh}\right)$$

- So low h drops bias at the expense of rising variance

Nonparametric Regression

Optimal Bandwidth?

- Optimal h minimizes the loss function, or asymptotic mean integrated squared error (AMISE), defined as

$$\int E(\hat{f}(x) - f(x))^2 dx$$

- We can show that $MSE = \text{Variance}+\text{Bias}^2$
- From the discussion about the variance and bias,

$$\text{Variance} + \text{Bias}^2 = \frac{1}{nh}\int_{-\infty}^{\infty} K^2(t)f(x)dt + \frac{h^4}{4}(f''(x))^2\left(\int_{-\infty}^{\infty} K(t)t^2dt\right)^2$$

- Thus, the final version of AMISE can be written as

$$\int(\text{Variance}+\text{Bias}^2)dx = \frac{1}{nh}\int_{-\infty}^{\infty} K^2(t)dt+\frac{h^4}{4}\int_{-\infty}^{\infty}(f''(x))^2\left(\int_{-\infty}^{\infty} K(t)t^2dt\right)^2 dx$$

Nonparametric Regression

Optimal Bandwidth?

- If $A = \frac{1}{4}\int_{-\infty}^{\infty}(f''(x))^2\left(\int_{-\infty}^{\infty}K(t)t^2dt\right)^2dx$, $B = \int_{-\infty}^{\infty}K^2(t)dt$. Then

$$AMISE = Ah^4 + \frac{B}{nh}$$

- Then, the minimization problem becomes $\min_h AMISE$. Therefore, we find h satisfying

$$4Ah^3 - Bn^{-1}h^{-2} = 0 \iff h^5 = \frac{B}{4An} \iff h = \left(\frac{B}{4An}\right)^{1/5}$$

  - In this framework, the bias and standard errors are both in $n^{-2/5}$ and AMISE will be in $n^{-4/5}$. Therefore, we may not have a CAN estimator at $n^{-1/2}$
  - Even bigger problem arises from $\int f''(x)dx$ term in A, as we are not sure of $f(x)$ to begin with, we do not know what $f''(x)$ would be.

Bandwidth Choice in Practice

- Silverman: Use $1.06\sigma n^{-1/5}$ for a normal kernel
- Robust version: $h = 0.9 \min\{s, IQ/1.34\} n^{-1/5}$
- Cross Validation: Use a leave-one-out estimator defined as

$$\hat{f}_{-i}(x) = \frac{1}{nh} \sum_{j \neq i} K\left(\frac{x - x_j}{h}\right)$$

- Local bandwidth: Make h larger in a low density area (decrease variance). In a wiggly region, it is better to take h smaller. (minimize biases).

Nonparametric Regression

Curse of Dimensionality

- Now assume that x is not necessarily a scalar, but of dimension d.
- Then we can use a d-dimensional kernel K and estimate $f(x)$ with

$$\hat{f}(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

  where K can be a d-product of a uni-dimensional kernels
- Not necessarily confined to using a same bandwith for all d kernels.
- We may want to sphericize if the kernel estimators are correlated.
- A bigger concern has to do with the computation cost from using a large d - a **curse of dimensionality**

Nonparametric Regression

Curse of Dimensionality

- Return to calculating $E[\hat{f}(x)^2]$ term in AMISE,

$$
\begin{aligned}
E[\hat{f}(x)^2] &= E\left[\frac{1}{n^2h^{2d}}\left(\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)\right)^2\right] \\
&= E\left[\frac{1}{n^2h^{2d}}\left(\sum_{i=1}^{n} K^2\left(\frac{x-X_i}{h}\right) + 2\sum_{i<j} K\left(\frac{x-X_i}{h}\right)K\left(\frac{x-X_j}{h}\right)\right)\right] \\
&= \frac{1}{nh^{2d}}\int_{-\infty}^{\infty} K^2\left(\frac{x-s}{h}\right)f(s)ds + \frac{n(n-1)}{n^2h^{2d}}\left(\int_{-\infty}^{\infty} K\left(\frac{x-s}{h}\right)f(s)ds\right)^2
\end{aligned}
$$

- The leading term is

$$
\frac{1}{nh^{2d}}\int_{-\infty}^{\infty} K^2\left(\frac{x-s}{h}\right)f(s)ds \simeq \frac{1}{nh^d}\int K^2(t)f(x-ht)dt = O\left(\frac{1}{nh^d}\right)
$$

- The optimal h will be in $n^{-\frac{1}{4+d}}$ and convergence occurs in $n^{-\frac{2}{4+d}}$ - at an even slower rate. So more n required to guarantee precision.

Nonparametric Regression

Nadaraya-Watson Estimation

- Given the data (y_i, x_i), we are attempting to capture $E[g(y,x)|x] = m(x)$ for some $g(y,x)$.
- For instance, we can attempt to estimate the conditional expectation by letting $g(y,x) = y$.
- Note that the conditional expected value can be written as

$$
\begin{aligned}
E[y|x] &= \int y f_{Y|X}(y|x) dy \\
&= \int y \frac{f_{Y,X}(y,x)}{f_X(x)} dy = \frac{\int y f_{Y,X}(y,x) dy}{\int f_{Y,X}(y,x) dy}
\end{aligned}
$$

- We can obtain the nonparametric estimator for the conditional expected value by replacing $f_{Y,X}$ with its kernel estimator

Nadaraya-Watson Estimation

- The numerator becomes

$$\begin{aligned}\int y\hat{f}(y,x)dy &= \int y \frac{1}{nh^2}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) K\left(\frac{y-Y_i}{h}\right) dy \\ &= \frac{1}{nh^2}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) \int yK\left(\frac{y-Y_i}{h}\right) dy \\ &= \frac{1}{nh^2}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) \int (Y_i+sh)K(s)(hds) \; (\because s = \frac{y-Y_i}{h}) \\ &= \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) Y_i \; (\because \int K(s)ds = 1, \int sK(s)ds = 0)\end{aligned}$$

- The denominator can be written as $\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)$. Thus, the estimator for the conditional expectation becomes

$$\frac{\sum_{i=1}^{n} Y_i K\left(\frac{x-X_i}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)}$$

Nonparametric Regression

Nadaraya-Watson Estimation

- Effectively we are putting weight $\frac{K\left(\frac{x-X_i}{h}\right)}{\sum_{i=1}^n K\left(\frac{x-X_i}{h}\right)}$ on each observation
- This is also a **local constant estimation** in the sense that when solving the following minimization problem

$$\hat{f}(x) = \arg\min_a \frac{1}{nh} \sum_{i=1}^n (Y_i - a)^2 K\left(\frac{x - X_i}{h}\right)$$

  The first order condition on a yields the following results

$$\sum_{i=1}^n Y_i K\left(\frac{x - X_i}{h}\right) = a \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right) \implies a = \frac{\sum_{i=1}^n Y_i K\left(\frac{x-X_i}{h}\right)}{\sum_{i=1}^n K\left(\frac{x-X_i}{h}\right)}$$

- As for the optimal h, it is possible to use cross-validation method and minimizing AMISE.
- This method can be unreliable if $f(x) \to 0$

Nonparametric Regression

Nadaraya-Watson Estimation

- We can also do something more general. e.g. **local linear estimation**.
- We regress $g(y,x)$ on a constant (a) and a linear term ($b(x-X_i)$).
- In mathematical expression, we solve

$$\min_{a,b} \frac{1}{nh}\sum_{i=1}^{n}(Y_i - a - b(x-X_i))^2 K\left(\frac{x-X_i}{h}\right)$$

  and obtain that $\hat{a} = \hat{g}$ and $\hat{b}$ is an estimate of $\frac{\partial g(x)}{\partial x}$.
- If it happens that the true functional form is linear, then this estimate does not produce a bias.
- In addition, local linear estimation performs better than local constant estimation in the boundaries of the support for X.
- We can do even more with **local polynomial estiation** by regressing $g(y,x)$ on a constant, $x-X_i$, $(x-X_i)^2$ and so on.

Nonparametric Regression

Semi-nonparametric Estimation

- Suppose that we are sure that $f(x)$ can be characterized by $f_{m,\sigma}$ where m, σ indexes some properties of the density function f.
- Then, by Weierstrass approximation theorem, we can choose a family of positive functions which increases in complexity $P_\theta^1, P_\theta^2, \ldots$ and maximize over the loglikelihood

$$\sum_{i=1}^{n} \log f_{m,\sigma}(X_i) P_\theta^M(X_i)$$

- Mixture of normals: Suppose that $Y|X$ is drawn from the two distributions
    - $N_1(m_1(x,\theta), \sigma_1^2(x,\theta))$ with probability $q_1(x,\theta)$
    - $N_2(m_2(x,\theta).\sigma_2^2(x,\theta))$ with probability $q_2(x,\theta)$

  Then, we apply a maximum likelihood of the following form

$$\min_\theta \sum_i \sum_k q_k(x_i,\theta)[(y_i - m_k(x_i,\theta))' \sigma_k(x_i,\theta)^{-1}(y_i - m_k(x_i,\theta)) + \log\det\sigma_k(x_i,\theta)]$$

Nonparametric Regression

Semi-nonparametric Estimation: Series Estimation

- Let $\{P_k(x_i)|k = 1, 2...\}$ be the orthonormal basis for a smooth function.
  - $\int P_k(x)^2 dx = 1, \int P_k(x)P_m(x) = 0 \ (k \neq m)$
- These could be polynomials of degree k, sine functions and so on.
- Run a linear regression that has the following form

$$y_i = \sum_{k=1}^{M} P_k(x_i)\theta_k + \epsilon_i$$

- The $\sum_{k=1}^{M} P_k(x_i)\theta_k$ part is a series approximation to $g(x)$.
- However, depending on the number of M that we choose, the curse of dimensionality can kick in.