Introduction to Econometrics 2: Recitation 12

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April 29th, 2020

Regression Discontinuity

Introduction

- **Regression discontinuity** was brought into the mainstream of applied economics very recently.
- Whenever you learn applied microeconomics class in your second year, you are bound to read or present about a paper that uses this research design.
- This approach exploits the discontinuities in the treatment assignment.
  - So the key difference between regression discontinuity and previous frameworks is that RDD does not require the overlap condition in the sense that $p(X_i) \in (0, 1)$.
  - In fact, one of the design has $p(X_i) = 1$ for $X_i \geq c$ and 0 otherwise.

Regression Discontinuity

Introduction

- Another thing that RDD should satisfy is that for every other covariates, the distribution should remain continuous even through the discontinuity.
- This is usually presented visually through graphs - one with the covariate that determines assignment to the treatment, usually called the forcing (running) variable W_i and the treatment status D_i, the other with the W_i and all other covariates X_i.
- If we are in the public finance context and study the effect of a particular welfare program with a means-tested benefit assignment mechanism, we could use the means-tested criteria as our W_i.
- If we are to study the effect of a specific educational program offered to those below a particular test score, that score could be the W_i.

Regression Discontinuity

Lee (2007, JOEconometrics)

- This paper focuses on the elections won/lost by a very narrow margin.
- The running variable is the Democrat vote share − Republican vote share, or a margin of victory/loss.
- One of the findings is that Democrats who barely win an election are much more likely to run for office and succeed in the next election, compared to Democrats who barely lose. (Carries over to other electoral outcomes)

Ost, Pan, Webber (2018, JLE)

- This paper estimates the return to college using admin data on college enrollment and earnings from Ohio.
- The running variable here is a GPA
- Dismissal leads to a short-run increase in earnings and tuition savings wiped out by gains from college graduates 8 years after. They also carry out the test that the result is not driven by manipulation.

Regression Discontinuity

Howell (2017, AER)

- This paper assess the impact of R&D subsidies from the government to new ventures.
- The rank on the application to the R&D is used as a running variable,
- The paper finds that an early-stage award approximately doubles the probability that a firm receives subsequent venture capital and has large, positive impacts on patenting and revenue.

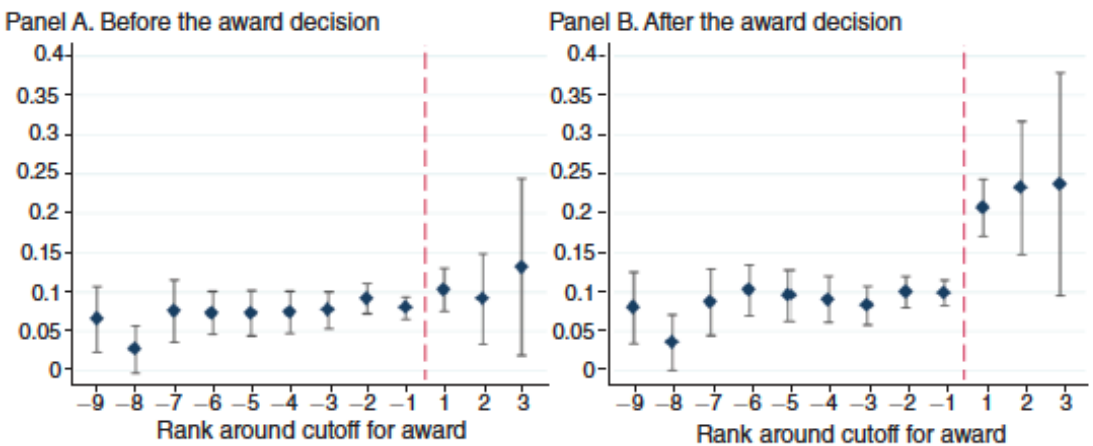


FIGURE 3. PROBABILITY OF VENTURE CAPITAL BEFORE AND AFTER GRANT BY RANK

Regression Discontinuity

Framework

- We will start with a **sharp RDD**, where the assignment into the treatment is determined by

$$D_i = 1(W_i \geq c(X_i))$$

  where $c(X_i)$ can be a known function of X_i or just a constant.
- We also keep the potential outcome framework from before, but we will slightly change this to

$$\begin{aligned} Y_i &= D_i Y_i(1) + (1 - D_i) Y_i(0) \\ &= Y_i(0) + D_i(Y_i(1) - Y_i(0)) \\ &= Y_i(0) + 1(W_i \geq c(X_i)) \cdot (Y_i(1) - Y_i(0)) \end{aligned}$$

- We also keep the regressional framework for the potential outcomes, $Y_i(d) = \mu(X_i, W_i, d) + \epsilon_i(d)$. So we have

$$E[Y_i(d)|X_i, W_i] = \mu(X_i.W_i.d)$$

Regression Discontinuity

Framework

- One assumption that we carry in this section is the continuity of the conditional expectation on potential outcomes at $W_i = c(X_i)$

Continuity of $\mu(X_i, W_i, d)$ at $W_i = c$

If W_i has a known cutoff at c, we assume that $E[Y_i(1)|X_i, W_i]$ and $E[Y_i(0)|X_i, W_i]$ are continuous around $W_i = c$. Put if differently,

$$E[Y_i(d)|X_i, W_i = c^+] = E[Y_i(d)|X_i, W_i = c^-] \text{ for each } d \in \{0, 1\}$$

- The above condition also implies that $(\epsilon_i(1), \epsilon_i(0))$ also has a continuous distribution around $W_i = c(X_i)$.
- If this condition is not satisfied, this implies that there could be more than the W_i variable that jumps around the cutoff

Regression Discontinuity

Fuzzy RDD Framework

- There is another version of the RDD which is more general than the sharp RDD.
- A **fuzzy RDD** setup exploits the jump in the probability of being treated at $W_i = c$.
- This jump is not necessarily from 0 to 1.
- Formally, we say we look for discontinuities of

$$\Pr(D_i|X_i, W_i)$$

at $W_i = c(X_i)$.

Regression Discontinuity

Identification: Sharp RDD

- Assume that $c(X_i)$ is a constant c. We can write

$$E[Y_i|X_i, W_i] = E[Y_i(0)|X_i, W_i] + E[(Y_i(1) - Y_i(0))1(W_i \geq c(X_i))|X_i, W_i]$$

- And we can separate the above into two. One with $W_i = c^+$ and the other with $W_i = c^-$.

$$E[Y_i|X_i, W_i = c^+] = E[Y_i(0)|X_i, W_i = c^+] + E[(Y_i(1) - Y_i(0))|X_i, W_i = c^+]$$
$$E[Y_i|X_i, W_i = c^-] = E[Y_i(0)|X_i, W_i = c^-]$$

- Therefore,

$$\begin{aligned} E[Y_i|X_i, W_i = c^+] - E[Y_i|X_i, W_i = c^-] &= E[Y_i(0)|X_i, c^+] - E[Y_i(0)|X_i, c^-] \\ &\quad + E[(Y_i(1) - Y_i(0))|X_i, c^+] \\ &= E[(Y_i(1) - Y_i(0))|X_i, c^+] \end{aligned}$$

where the first line on the right hand side vanishes due to the continuity assumptions.

Regression Discontinuity

Identification: Sharp RDD

- Reusing the continuity argument, we can write

$$E[(Y_i(1) - Y_i(0))|X_i, W_i = c^+] = E[(Y_i(1) - Y_i(0))|X_i, W_i = c]$$

- Therefore, we have backed out

$$E[(Y_i(1) - Y_i(0))|X_i, W_i = c] = E[Y_i|X_i, W_i = c^+] - E[Y_i|X_i, W_i = c^-]$$

  which is the average treatment effect at $W_i = c$ and a given X_i
- This is similar to the LATE in the sense if we let $Z_i = 1(W_i \geq c)$, we can get back what is effectively equivalent to the LATE estimator (with the difference in the probability of treatment exactly at 1).
- Also, we are looking at the treatment effect on the compliers, albeit a very narrow group of people.

Regression Discontinuity

Identification: Fuzzy RDD

- For the fuzzy RDD, we can write as follows

$$E[Y_i|X_i, W_i] = E[Y_i(0)|X_i, W_i] + E[(Y_i(1) - Y_i(0))\Pr(D_i = 1|X_i, W_i)|X_i, W_i]$$

  So we can do the similar approach as in sharp RDD and get

$$E[Y_i|x, c^+] = E[Y_i(0)|x, c^+] + \Pr(D_i = 1|x, c^+)E[Y_i(1) - Y_i(0)|x, c^+]$$
$$E[Y_i|x, c^-] = E[Y_i(0)|x, c^-] + \Pr(D_i = 1|x, c^-)E[Y_i(1) - Y_i(0)|x, c^-]$$

- So using the continuity assumption, the difference in the LHS is

$$E[Y_i|x, c^+] - E[Y_i|x, c^-] = [\Pr(D_i = 1|x, c^+) - \Pr(D_i = 1|x, c^-)]E[Y_i(1) - Y_i(0)|x, c]$$

- Thus, we also identify an average treatment effect at $W_i = c$ and $X_i = x$, which is now divided by something that is not necessarily 1.

$$E[Y_i(1) - Y_i(0)|X_i, W_i = c] = \frac{E[Y_i|X_i, W_i = c^+] - E[Y_i|X_i, W_i = c^-]}{\Pr(D_i = 1|x, W_i = c^+) - \Pr(D_i = 1|x, W_i = c^-)}$$

Regression Discontinuity

Implementation: Nonparametrics

- One way to implement RDD in practice is to make use of nonparametric regression.
- The bandwidth, however, should be bounded at c.
  - If you are doing a nonparametric regression from the left of the bandwidth, you should apply this on $[c - \epsilon, c)$
  - For the right hand side, $[c, c + \epsilon]$.
  - If the domain reaches beyond c from either side, there is a huge risk of misfitting the data.
- Ideally, carry out a local linear (or polynomial) regression
- We also run into curse of dimensionality and minimizing AMISE in determining optimal bandwidth (bias-variance tradeoff).
- Calonico, Cattaneo, and Titunik (2014) suggests the use of local quadratic estimator with the optimal bandwidth selected by the minimization of AMISE, which is typically known as IK bandwidth (Imbens, Kalyanaraman, 2012).

Regression Discontinuity

Implementation: Parametrics

- One way is to run a separate regression for both left and right of the cutoff. That is, run

$$Y_i = X_i\beta^- + (c - w_i)X_i\gamma^- + \epsilon_i^- \quad \text{for } w_i < c$$
$$Y_i = X_i\beta^+ + (w_i - c)X_i\gamma^+ + \epsilon_i^+ \quad \text{for } w_i \geq c$$

  allowing us to fit a different trend on the left & right of the cutoff c.

- There is also a way in which we make use of both sides of the cutoff. Specifically, we run

$$Y_i = X_i\beta + X_i \cdot 1(W_i \geq c)\gamma + X_i \cdot (c - W_i)\delta + X_i \cdot (c - W_i) \cdot 1(W_i \geq c)\mu + \epsilon_i$$

  Here, what happens is that
  - $W_i \geq c$: $Y_i = X_i(\beta + \gamma) + X_i \cdot (c - W_i)(\delta + \mu)$
  - $W_i < c$: $Y_i = X_i(\beta) + X_i \cdot (c - W_i)(\delta)$

- In an (Y_i, W_i) plane, γ captures the jump, δ, μ determines slope

Regression Discontinuity

Implementation: Diagnostics

- First, we plot Y_i as a function of W_i for a given X_i and see if there is a mean shift.
- We plot the propensity score $\Pr(D_i = 1|X_i, W_i)$ and look for a shift.
- One thing that should not shift is the distribution of X_i's. So plot X_i's as function of W_i and confirm that there is no shift.
- Another thing to worry regarding W_i is the possibility of manipulating or gaming. Therefore, use a McCrary test to confirm that the distribution of W_i do not change at $W_i = c$. (No bunching!)
- What if we do not know the exact cutoff? We can figure out what they might be based on the distribution of W_i (Chay, McEwan, and Urquiola, 2005)
- We lose a lot of data in selecting the optimal bandwidth around the cutoff, hurting external validity.

Machine Learning

Introduction

- Machine learning in the context of econometrics is interested in model selection for prediction - finding the parsimonious model in the context of a very large dataset.
- It ranges from regularized models such as LASSO, Ridge regression, and elastic nets and methods that are much more involved - random forests and neural networks.
- Often, we are interested in controlling a large set of covariates
- In determining the structure, we face a trade-off: We are tempted to increase the number of covariates to fit the data better.
- However, in doing so, we risk overfitting - the out-sample prediction can perform poorly in terms of variance.
- Therefore, some method to penalize complexity is required.

Information Criterion

- We start with a set $\mathcal{M}$ of candidate models. Then for $M \in \mathcal{M}$, we define a penalty parameter $p(M)$ that increases in complexity.
  - Akaike's Information Criterion: Choose M satisfying

$$\max_M \left(\max_\theta \sum_{i=1}^{n} \log l_i(\theta; M) - p(M) \right)$$

  - Bayesian Information Criterion: Choose M satisfying

$$\max_M \left(\max_\theta \sum_{i=1}^{n} \log l_i(\theta; M) - p(M)\frac{\log n}{2} \right)$$

- BIC tends to be harsher for complexities and performs better for model selection
- For out-sample performance, AIC performs better.
- Both of them, however, can be impractical to calculate when we work with large models. If p is a number of potential covariates, and n is the sample size, we run into the case where $p >> n$.

Shrinkage

- One way out is to penalize complexities with the convex norm penalties. Define

$$\|\theta\|_m = \sqrt[m]{\sum_{k=1}^{p} |\theta_k|^m}$$

  and for $m = 0$, we have $\|\theta\|_p = p$
- Depending on the selection of m, we get different estimators that incorporates different shrinkage mechanisms
- The general idea of shrinkage is as follows: We know that OLS is BLUE. But what if we are willing to relax the 'unbiased' condition? Then can we find some estimators that have lower variance and even a lower mean squared error? James, Stein (1961) shows that it is possible and generalizable.

LASSO and Ridge

- Ridge regression minimizes

$$\sum_{i=1}^{n}(Y_i - X_i\theta_0)^2 + \lambda\sum_{k=1}^{p}\theta_k^2$$

and yields

$$\hat{\theta} = (X'X + \lambda I_p)^{-1}X'y$$

- With LASSO, the minimization is

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - X_i\theta_0)^2 + \lambda\sum_{k=1}^{p}|\theta_k|$$

and the first order condition can be written as

$$\frac{2}{n}\sum_{i} X_{ik}(y_i - X_i\theta) = \lambda s_k,$$

Where λs_k is not necessarily 0, consistent with the idea that we allow for some bias in order to find an estimator that reduces MSE.

Machine Learning

So what is the difference between LASSO and Ridge?

- Ridge regression drives your coefficients towards 0 but not equal to it.
- So when it comes to model selection in terms of ruling out irrelevant variables, LASSO does better.
- Ridge has quicker computation and works better with multicollinearity.
- So the choice of the estimation method depends on your goal of minimizing MSE as well as computation power

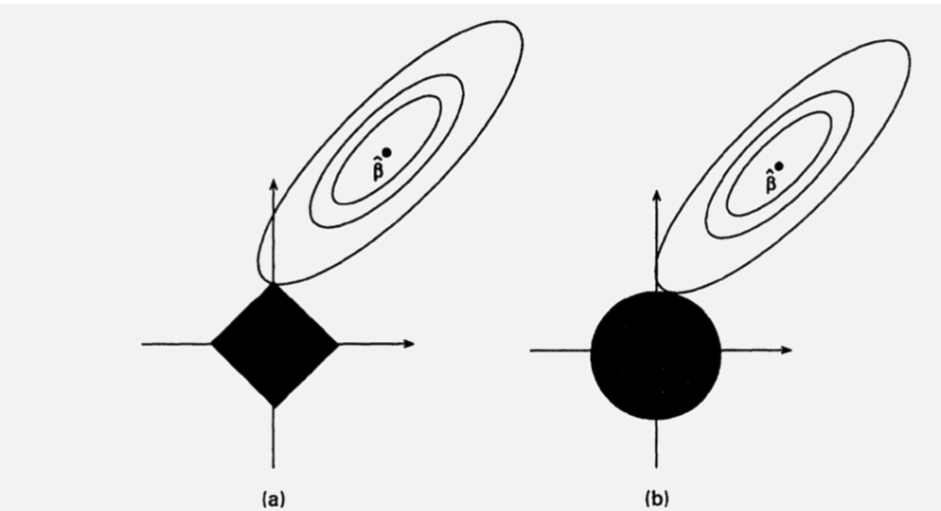


Fig. 2. Estimation picture for (a) the lasso and (b) ridge regression

Elastic Net abd LARS

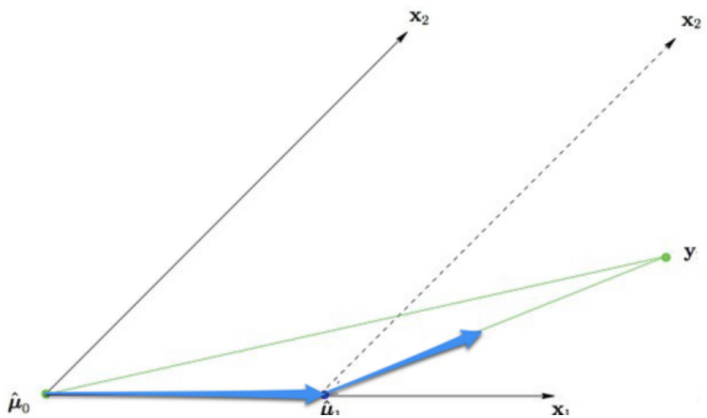
- The idea of **elastic net** is to use a penalty term that is a convex combination of both l_1 and l_2 penalty term. Specifically,

$$\sum_{i=1}^{n}(Y_i - X_i\theta_0)^2 + \lambda \sum_{k=1}^{n}\left[\alpha\|\theta\|_1^2 + (1-\alpha)\|\theta\|_2^2\right]$$

- If $\lambda = 0$ we have an OLS. For $\lambda \neq 0$, if $\alpha = 1$, we have LASSO, and if $\alpha = 0$ we get Ridge.

Elastic Net abd LARS

- For **Least Angle Regression (LARS)** the algorithm is
  1. Start with a constant model: For all non-constant covariates, coefficients are 0 and $\lambda = \infty$.
  2. Find the covariate that has the largest covariance with the residual.
  3. Change the coefficient for that variable in the same direction as its covariance with the residual until you find a different covariate that has as much covariance. λ decrease and θ is updated along the way.
  4. Change the coefficient for the two variables until you find a third variable with the same covariance with the residual.
  5. Stop when all predictors are in the model



Choosing λ

- If your goal is to minimize the prediction error, you can use the following cross-validation methods to come up with one
  - Leave-one-out: Leave one observation out at a time
  - Training vs Test sample: You split the sample into two, build a model given some λ and gain prediction error from the test sample. Then, select λ minimizing the prediction error there.
  - K-fold cross validation: For $K-1$ blocks, estimate the model given some λ and compute the error on the remaining block. Get this for K blocks and compute the average error. Select λ minimizing the error.
- In general, selecting λ depends on your goals

Consistency of LASSO

- We want to identify what the true model is in a setting with potentially an infinite number of covariates.
- To do that, we first set a sparsity assumption, where the number of nonzero coefficients is not too large.
- As a thought experiment, we increase the number of regressors as $n \to \infty$ in order to approximate the true model.
- To satisfy the assumption, we require that the number of nonzero regressors to rise at a slower rate than n.
- Specifically, $s_n = o\left(\sqrt{\frac{n}{\log p}}\right)$.

Consistency of LASSO

- If we let $\lambda_n = O\left(\sqrt{\frac{\log p}{n}}\right)$ and have $s_n = o\left(\sqrt{\frac{n}{\log p}}\right)$, then we can show that the prediction error goes to 0 and the LASSO estimator is consistent. Or

$$\frac{1}{n}\sum_i (X_i\hat{\theta}_n - X_i\hat{\theta}_{0n})^2 = O_p(\lambda_n^2 s_n)$$

and

$$\|\hat{\theta}_n - \hat{\theta}_{0n}\|_1 = o_p(\lambda_n s_n) = o_p(1)$$

- This comes at the cost of a slightly slower rate of convergence, which is tolerable as long as it is faster than $n^{-1/4}$