Recitation 4

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Not everything in world is linearly related

- The effect of X may grow larger/smaller as X increases (think of wage and age)
- For correlations like this, nonlinear regressors are necessary
- When incorporating such regressors, the interpretation of each coefficient becomes trickier.
- Quadratic relations: Think about wage and age - wages increase with age, but (usually) at a decreasing pace

$$W = \beta_0 + \beta_1 X + \beta_2 X^2 + u$$

- The marginal effect of X on W can be written as

$$\frac{\partial W}{\partial X} = \beta_1 + 2\beta_2 X$$

Back to linear models: So how is the marginal effect different?

- In a linear regressor only format, the marginal effect is

$$W = \beta_0 + \beta_1 X + u$$

$$\frac{\partial W}{\partial X} = \beta_1$$

- The difference is that with quadratic terms, we can express cases where *marginal* changes to W with respect to X is not a constant, but depends on some value of X
  - In the above case, if $\beta_2 > 0$, marginal increase in W increases with X (and vice versa)

We are interested in percent changes (because measurement units)...

- We might be interested in how the changes in the X in terms of *percentages* affect changes in the dependent variable Y.
- (Natural) Log regressors captures the idea of percentage changes.
- Recall from calculus the first order approximation: For any differentiable function f, and a very small change in x, the following relationship holds.

$$f(x + \Delta x) \simeq f(x) + f'(x)[(x + \Delta x) - x]$$

- Define $y = f(x) = \ln x$. Then $f(x + \Delta x) = y + \Delta y$ and $f'(x) = \frac{1}{x}$. We then get

$$\Delta y = \frac{\Delta x}{x} \implies \ln(x + \Delta x) - \ln x \simeq \frac{\Delta x}{x} \implies \ln\left(1 + \frac{\Delta x}{x}\right) \simeq \frac{\Delta x}{x}$$

- Therefore, the changes in log values allows us to capture changes in percentages, at least for very small amount of change.

How much *level* changes in Y due to *percentage* changes in X?

- **lin-**log: Consider the model $Y = \beta_0 + \beta_1 \log X + u$.
- I take a before and after approach by changing X by Δx.
- Then the total amount of Y would be $Y + \Delta y$. Formally,

$$Y + \Delta y = \beta_0 + \beta_1 \log(X + \Delta x) + u$$

Subtract $Y = \beta_0 + \beta_1 \log X + u$ from this equation to get

$$\Delta y = \beta_1 \log(X + \Delta x) - \beta_1 \log X = \beta_1 \log\left(1 + \frac{\Delta x}{X}\right) = \beta_1 \frac{\Delta x}{X}$$

- Therefore,

$$\beta_1 = \frac{\Delta y}{(\Delta x / X)}$$

Note that the percentage change in X is $\frac{\Delta x}{X} \times 100$. In words, change in X by 1 percent, raises Y by $\beta_1 \times 0.01$.

How much *percentage* changes in Y due to *level* changes in X?

- log-**lin**: Consider the model $\log Y = \beta_0 + \beta_1 X + u$.
- Conduct a similar before and after analysis as we did before to get:

$$\log(Y + \Delta y) = \beta_0 + \beta_1 (X + \Delta x) + u$$

- Then, subtract $\log Y = \beta_0 + \beta_1 X + u$. This gets us

$$\log\left(1 + \frac{\Delta y}{Y}\right) \simeq \frac{\Delta y}{Y} = \beta_1 \Delta x$$

- Then, β_1 can be backed out as

$$\beta_1 = \frac{(\Delta y / Y)}{\Delta x}$$

Again, using the fact that percentage change in Y can be represented as $\frac{\Delta y}{Y} \times 100$. This implies that a 1 unit change in X raises Y by $(100 \times \beta_1)\%$.

How much *percentage* changes in Y due to *percentage* changes in X?

- log-log: Consider the equation $\log Y = \beta_0 + \beta_1 \log X + u$.
- Similar approach allows us to write

$$\log(Y + \Delta y) = \beta_0 + \beta_1 \log(X + \Delta x) + u$$

- Subtract the original equation to obtain

$$\log\left(1 + \frac{\Delta y}{Y}\right) = \beta_1 \log\left(1 + \frac{\Delta x}{X}\right) \implies \frac{\Delta y}{Y} = \beta_1 \frac{\Delta x}{X}$$

- which implies

$$\beta_1 = \frac{(\Delta y / Y)}{(\Delta x / X)}$$

This implies that 1% change in X leads to β_1% change in Y. This is the *elasticity* interpretation

IRL: Marginal effect of X_1 on Y maybe a function of some other variables!

- Suppose that we are interested in the relationship between test scores (Y) and class size(X_1).
- However, one might guess that the effect of class size may differ depending on some other variables.
  - e.g. schools in districts where there are more funding (X_2) are more likely to enjoy the benefits of small school classroom
- In math, the marginal effect of X_1 on Y may depend on X_2.
- To capture this idea in a model, we can incorporate an **interaction term** involving X_1 and X_2, which can be written as $X_1 \times X_2$

Binary $\times$ Binary

- Suppose that there are two binary variables, D_1, D_2.
- Notice the regression equation below

$$Y = \beta_0 + \beta_1 D_1 + \beta_2 D_2 + \beta_3 (D_1 \times D_2) + u$$

- By now, you know that β_1 captures the group difference in average for those in group D_1 and β_2 captures the same for those in group D_2.
- Note that $D_1 \times D_2$ becomes 1 only individual is in both groups. The average for these individuals are captured by β_3 coefficient.
- To sum up:
  - $E[Y|D_1 = 1] = \beta_0 + \beta_1 + \beta_2 \times D_2 + \beta_3 \times D_2$
  - $E[Y|D_1 = 0] = \beta_0 + \beta_2 \times D_2$
  - $E[Y|D_1 = 1] - E[Y|D_1 = 0] = \beta_1 + \beta_3 \times D_2$
- So the effect of D_1 differs depending on D_2 as well.
- This setup allows us to analyze the difference in effect of binary variable depending on another binary factor.

Binary $\times$ Continuous

- Instead of a second binary variable, we include a continuous variable X_1.
- Now we write
$$Y = \beta_0 + \beta_1 D_1 + \beta_2 X_1 + \beta_3 (D_1 \times X_1) + u$$
- In earlier example, classroom size would be a continuous variable, so that will be our X_1.
- Now, define $D_i = 1$ if a school district receives funding and 0 if otherwise.
- The effect of classroom size would now be
$$\frac{\partial Y}{\partial X_1} = \beta_2 + \beta_3 D_1$$
- Now the effect of X_1 depends on D_1 - whether the district receives funding or not

Continuous $\times$ Continuous

Interaction terms

- Consider this regression, where both X_1 and X_2 are continuous variables.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \times X_2) + u$$

- In this equation, the effect of X_1 on Y, and that for X_2 on Y are

$$\frac{\partial Y}{\partial X_1} = \beta_1 + \beta_3 X_2 \quad \frac{\partial Y}{\partial X_2} = \beta_2 + \beta_3 X_1$$

- Now you see that the marginal impact of X_1 on Y is dependent of X_2.