Recitation 3: OLS properties: Sampling distribution and fitness

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Ordinary least squares

Ordinary Least Squares: Population vs sample linear models

- Suppose that the **population linear regression model** (also known as data generating process in some books) is
$$Y_i = \beta_0 + \beta_1 X_i + u_i$$
- However, we do not know the true values of the population parameters - β_0 and β_1
- An alternative way to approach the problem is to use the **sample linear regression model** (or just model)
$$Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{u}_i$$
where $\hat{\beta}_0, \hat{\beta}_1$ are estimates of β_0, β_1

Ordinary Least Squares: Definition

- The ideal estimator minimizes the squared sum of residuals.
- Mathematically, this can be obtained by solving the following minimization problem and the first order conditions

$$\min_{\hat{\beta}_0, \hat{\beta}_1} \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2$$

$$[\hat{\beta}_0] : -2 \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$$

$$[\hat{\beta}_1] : -2 \sum_{i=1}^{n} X_i (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$$

The resulting **least squares estimators** are

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}, \quad \hat{\beta}_1 = \frac{\sum_{i=1}^{n} (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^{n} (X_i - \bar{X})^2}$$

Ordinary Least Squares: Main assumptions

- For OLS to be unbiased, consistent, efficient, and asymptotic normal, the following assumptions must be made

Assumptions

A0 Linearity: The regression is assumed to be linear in parameters.

$$\text{Okay: } Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i, \text{ Not: } Y_i = \beta_0 + \beta_1 X_i + \beta_2^2 X_i + u_i$$

A1 $E(u_i|X_i) = 0$: Conditional on letting X_i take a certain value, we are not making any systematical error in the linear regression. This is required for the OLS to be unbiased. (or $cov(X_i, u_i) = 0$)

A2 i.i.d. (random sampling): (X_i, Y_i) is assumed to be from independent, identical distribution

A3 No Outliers: Outlier has no impact on the regression results. ($E(X_i^4), E(Y_i^4) < \infty$)

A4 Homoskedasticity: $var(u_i) = \sigma_u$ (variance of u_i does not depend on X_i). $\leftrightarrow$ *heteroskedasticity*

A5 No Autocorrelation (Serial Correlation): For $i \neq j$, $cov(u_i, u_j) = 0$. Error at the previous period does not have any impact on the current period. This is usually broken in time series settings

Ordinary Least Squares: Useful alternative expression for $\hat{\beta}_1$

- OLS estimate that we are getting is a random variable - getting different estimates depending on sample we work with.
- $\hat{\beta}_1$: Recall that we can write

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

Now, replace Y_i an $\bar{Y}$ with

$$Y_i = \beta_0 + \beta X_i + u_i, \;\; \bar{Y} = \beta_0 + \beta_1 \bar{X} + \bar{u},$$

which allows us to write

$$(Y_i - \bar{Y}) = (\beta_1(X_i - \bar{X}) + (u_i - \bar{u}))$$

and get

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{n}(X_i - \bar{X})(u_i - \bar{u})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

Ordinary Least Squares: Unbiasedness of $\hat{\beta}_1$

- $E[\hat{\beta}_1]$: It can be written as

$$\begin{aligned} E[\hat{\beta}_1] &= E\left[\beta_1 + \frac{\sum_{i=1}^n (X_i - \bar{X})(u_i - \bar{u})}{\sum_{i=1}^n (X_i - \bar{X})^2}\right] \\ &= \beta_1 + E\left[\frac{\sum_{i=1}^n (X_i - \bar{X})(u_i - \bar{u})}{\sum_{i=1}^n (X_i - \bar{X})^2}\right] \end{aligned}$$

$\sum_{i=1}^n (X_i - \bar{X})(u_i - \bar{u})$ can be written to something simpler.

$$\sum_{i=1}^n (X_i - \bar{X})(u_i - \bar{u}) = \sum_{i=1}^n X_i u_i - \bar{u}\sum_{i=1}^n X_i - \bar{X}\sum_{i=1}^n u_i + n\bar{X}\bar{u} = \sum_{i=1}^n (X_i - \bar{X})u_i$$

$\rightarrow$ Since $\bar{X}$ is a sample mean of X, $\sum_{i=1}^n X_i = n\bar{X}$.

$\rightarrow$ The assumption that conditional mean is zero and (X_i, u_i) are uncorrelated means that the term on the left hand side is zero.

$\rightarrow$ Therefore, UNDER CLASSICAL ASSUMPTIONS, $E[\hat{\beta}_1] = \beta_1$.

Ordinary Least Squares: Unbiasedness of $\hat{\beta}_0$

- $\hat{\beta}_0$: The formula for $\hat{\beta}_0$ is $\bar{Y} - \hat{\beta}_1 \bar{X}$. By changing $\bar{Y}$, we can get

$$\begin{aligned} \hat{\beta}_0 &= (\beta_0 + \beta_1 \bar{X} + \bar{u}) - \hat{\beta}_1 \bar{X} \\ &= \beta_0 + (\beta_1 - \hat{\beta}_1)\bar{X} + \bar{u} \end{aligned}$$

  Then we can say the following about the sampling distribution

- $E[\hat{\beta}_0]$: We can write

$$E[\hat{\beta}_0] = \beta_0 + E[(\beta_1 - \hat{\beta}_1)\bar{X}] + E[\bar{u}] = \beta_0$$

  since $\hat{\beta}_1$ is unbiased and conditional expectation of u_i is zero.

$\rightarrow$ Thus, under our current assumptions, $\hat{\beta}_0$ is unbiased.

Ordinary Least Squares: Variances of $\hat{\beta}_0$ and $\hat{\beta}_1$

- Might take bit of a work, but when you follow the notes, you get

$$var(\hat{\beta}_0) = \frac{\sigma_u^2}{n}\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n (X_i - \bar{X})^2}, \; var(\hat{\beta}_1) = \frac{\sigma_u^2}{\sum_{i=1}^n (X_i - \bar{X})^2}$$

- At the end of the day, we can say

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma_u^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right)$$
$$\hat{\beta}_0 \sim N\left(\beta_0, \frac{\sigma_u^2}{n}\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n (X_i - \bar{X})^2}\right)$$

- The importance of this is that now we can conduct a hypothesis test and create a test statistic based on this distribution

Ordinary Least Squares: How well does the model capture the data?

Measure of fitness

- These numbers tell us how informative the sample linear regression we used is in telling us about the population data
- R^2: It is defined as a fraction of total variation which is explained by the model. Mathematically, this is

$$Y_i = \underbrace{\hat{\beta}_0 + \hat{\beta}_1 X_i}_{\hat{Y}_i} + u_i,\ \ \bar{Y} = \underbrace{\hat{\beta}_0 + \hat{\beta}_1 \bar{X}}_{\bar{\hat{Y}}} + \bar{u},$$

$$\implies Y_i - \bar{Y} = (\hat{Y}_i - \bar{\hat{Y}}) - (u_i - \bar{u})$$

$$\implies \sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(\hat{Y}_i - \bar{\hat{Y}})^2 + \sum_{i=1}^{n}(u_i - \bar{u})^2 - 2\sum_{i=1}^{n}(\hat{Y}_i - \bar{\hat{Y}})(u_i - \bar{u})$$

Ordinary Least Squares: Getting to R^2

- Note that
$$\sum_{i=1}^{n}(\hat{Y}_i - \bar{\hat{Y}})(u_i - \bar{u}) = \sum_{i=1}^{n}\hat{Y}_i u_i - \bar{\hat{Y}}\sum_{i=1}^{n}u_i - \bar{u}\sum_{i=1}^{n}\hat{Y}_i + n\bar{u}\bar{\hat{Y}}$$
- Since $\sum_{i=1}^{n} u_i = n\bar{u}$, $\sum_{i=1}^{n}\hat{Y}_i = n\bar{\hat{Y}}$ and $\sum_{i=1}^{n}\hat{Y}_i u_i = n\bar{u}\bar{\hat{Y}}$, all terms cancel each other out.
- So we are left with

$$\underbrace{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}_{TSS} = \underbrace{\sum_{i=1}^{n}(\hat{Y}_i - \bar{\hat{Y}})^2}_{ESS} + \underbrace{\sum_{i=1}^{n}(u_i - \bar{u})^2}_{RSS}$$
$$\implies 1 = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{\hat{Y}})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2} + \frac{\sum_{i=1}^{n}(u_i - \bar{u})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

Ordinary Least Squares: Getting to R^2

- Thus, the R^2 can be found as

$$R^2 = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{\hat{Y}})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2} = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS}$$

- Intuitively, higher R^2 implies that the model explains more of the total variance, which implies that the regression fits the data well.

Ordinary Least Squares: Setting up the hypothesis test

- From the sample distribution of $\hat{\beta}_1$, we can break down into two cases
- **Know** σ_u: Since the $\hat{\beta}_1$ takes a normal distribution, we can “standardize" it to get the test statistic and the distribution for it

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{var(\hat{\beta}_1)}} \sim N(0,1)$$

and compare against the critical values (depending on significance level, two vs one-sided test)

Ordinary Least Squares: Hypothesis test methods

- **Don't know** σ_u; need to have an estimate for $var(\hat{\beta}_1)$ due to not knowing σ_u. The test statistics and its distribution is

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\widehat{var(\hat{\beta}_1)}}} \sim t_{n-2}$$

  where $\widehat{var(\hat{\beta}_1)}$ is the estimate for the variance and t_{n-2} is a t-distribution with $n-2$ degrees of freedom.
- The d.f. is determined by the number of observations, where 2 is subtracted because we are estimating β_0 and β_1 in the process.
- When n is large, t-distribution becomes similar to the normal distribution

Ordinary Least Squares: Confidence interval

- **Confidence interval**: A 95% confidence interval is a range of numbers that form a random interval that has a 95% chance of including a (nonrandom) true value of a parameter.
- This can be obtained by inverting the rejection region that we have used in the critical value approach.

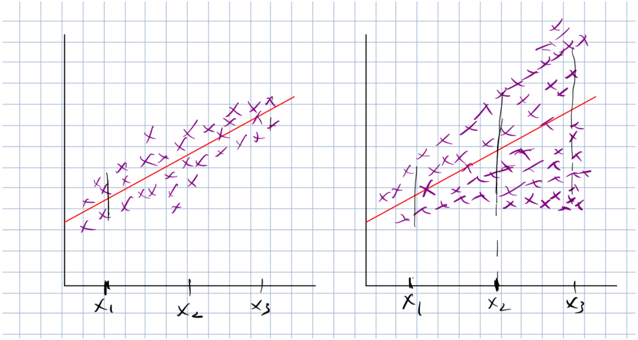
$$\Pr\left(-1.96 \leq \frac{\hat{\beta}_1 - \beta_1}{\sqrt{var(\hat{\beta}_1)}} \leq 1.96\right) = 0.95$$

$$\implies \Pr\left(\hat{\beta}_1 - 1.96 \times \sqrt{var(\hat{\beta}_1)} \leq \beta_1 \leq \hat{\beta}_1 + 1.96 \times \sqrt{var(\hat{\beta}_1)}\right) = 0.95$$

- If they encompass the null test value, then we cannot reject the null hypothesis. Otherwise, we can reject the null.

Errors may have different distribution across observations

- The assumption that $var(u_i)$ is constant may not hold. Thus, be open for heteroskedasticity
- If we stick to homoskedasticity in this case, the standard errors are incorrectly estimated (usually underestimated)



- In such case, standard errors of our estimators must take this into account.

...but what does heteroskedasticity change?

```
. regress testscr str

      Source |       SS           df       MS      Number of obs   =       420
-------------+----------------------------------   F(1, 418)       =     22.58
       Model |  7794.11919         1  7794.11919   Prob > F        =    0.0000
    Residual |  144315.475       418  345.252333   R-squared       =    0.0512
-------------+----------------------------------   Adj R-squared   =    0.0490
       Total |  152109.594       419  363.030058   Root MSE        =    18.581

------------------------------------------------------------------------------
     testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |   -2.27981   .4798255    -4.75   0.000    -3.222981   -1.336638
       _cons |    698.933   9.467491    73.82   0.000     680.3232    717.5428
------------------------------------------------------------------------------
```

```
. regress testscr str, vce(robust)

Linear regression                               Number of obs     =        420
                                                F(1, 418)         =      19.26
                                                Prob > F          =     0.0000
                                                R-squared         =     0.0512
                                                Root MSE          =     18.581

------------------------------------------------------------------------------
             |               Robust
     testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |   -2.27981   .5194894    -4.39   0.000    -3.300947   -1.258672
       _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

- The variance rises (usually) in the heteroskedastic regression, so we may make a wrong hypothesis test
- The coefficients are unchanged, since estimation of OLS estimates did not rely on homoskedasticity