Recitation 9: IV estimation

Seung-hun Lee

Columbia University
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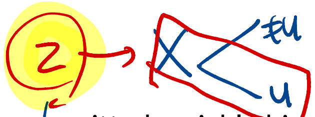
Instrumental variables

Why use IV? Deal with $\boxed{E[u_i|X_i] \neq 0}$



- Recall that OLS estimates can be biased when we have omitted variable bias, measurement error, and simultaneity bias
- Instrumental variable: Allows us to eliminate bias
  - When you have an independent variable X, there are parts of this variable that are correlated with u and the other parts that are independent of u.
  - If we are able to find Z that is correlated with X but not with u, using this Z variable would allow you to sort variable X into what is correlate with u and what is not.
  - Then, using the part that is not correlated with u, variable Z allows us to get unbiased estimates.
- Uses of IV
  - Endogenous variable: X determined as choice, not as given
  - Simultaneity bias: Y can lead to changes in X
  - Omitted variable bias: Unincorporated determinant of Y
  - Measurement error in X

Y: democracy in Africa

X: agri share of income

Z: rainfall

Association vs causation

- Association refers to any type of statistical dependence
  - It mostly refers to correlation, which is

$$\text{corr}(x, y) = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x)}\sqrt{\text{var}(y)}}$$

- Running an OLS is practically equivalent to finding the correlation
  - From the population regression,

$$\beta_1 = \frac{E[X - E[X]]E[Y - E[Y]]}{(E[X - E[X]])^2} = \text{corr}(X, Y)\sqrt{\frac{\text{var}(Y)}{\text{var}(X)}}$$

    implying that β_1 is obtained by multiplying $corr(X, Y)$ with some constant
  - Same correlation direction (variances are nonnegative)
  - Not good for finding causal effect of X on Y: Switching the two would give same results

What conditions should IV satisfy?

first stage

- Must satisfy
  - **Relevance:** Variable Z satisfies relevancy condition if $cov(X, Z) \neq 0$
  - **Exogeneity:** Variable Z satisfies exogeneity condition if $cov(Z, u) = 0$
- In words,
  - Variable Z should be somewhat correlated with the variable X Ideally strongly
  - Variable Z should not be correlated with u
  - (For exogeneity): Variable Z should affect Y only through X, or when X is controlled for, Z alone should not affect Y **(exclusion)**
- More on exclusion (on model $\boxed{Y = \beta_0 + \beta_1 X + u}$)

$$
\begin{aligned}
cov(Z, u) &= cov(Z, Y - \beta_0 - \beta_1 X) = 0 \\
&= \boxed{cov(Z, Y)} - \cancel{cov(Z, \beta_0)} - \boxed{cov(Z, \beta_1 X)} = 0 \\
&\Longrightarrow \boxed{cov(Z, Y) = cov(Z, \beta_1 X)}
\end{aligned}
$$

This condition means that Z is correlated with Y *only through* X

2SLS estimation

$Y_i = \beta_0 + \beta_1 \underset{\hat{\delta}_1 Z}{X} + u$

1st

- **Regress with X as dependent, Z as independent variable.** Regress

$$X = \delta_0 + \boxed{\delta_1 Z} + v$$

$\hat{\delta}_0 + \hat{\delta}_1 Z = \hat{X}$

From this regression, obtain the predicted values of X, denoted as $\hat{X} = \hat{\delta}_0 + \hat{\delta}_1 Z$. This $\hat{X}$ is the part that is related with Z but is uncorrelated with u. $(\because \text{Cor}(Z,u)=0)$

2nd

- **Regress with Y as dependent, $\hat{X}$ as independent variable.** Regressing with this $\hat{X}$ will satisfy the $E[u|\hat{X}] = 0$ condition, as the $\hat{X}$ is uncorrelated with u. Thus, your regression equation looks like this:

$$Y = \beta_0 + \beta_1 \hat{X} + u$$

$\hat{\beta}_0, \hat{\beta}_1$

Then you run a OLS regression on the above equation and get the 2SLS estimator $\hat{\beta}_{\text{TSLS}}$.

ivregress 2sls in STATA

Estimating using a covariance method

$$Cov(Y,Z) = Cov(\beta_1 X, Z)$$

$$= \beta_1 Cov(X, Z)$$

- Note that (from exclusion condition)

$$cov(Z, Y) = cov(Z, \beta_1 X) \implies cov(Z, Y) = \beta_1 cov(Z, X)$$

- From this, we can get

$$\beta_1 = \left(\frac{cov(Z, Y)}{cov(Z, X)} \right)$$

  where division is possible because we require a relevancy condition ($cov(Z, X) \neq 0$)

- Taking the sample counterparts, we get our IV estimator

$$\hat{\beta}_{TSLS} = \frac{s_{ZY}}{s_{ZX}}$$

Reduced form method

- Denote

$$X = \pi_0 + \pi_1 Z + v \text{ (where } cov(Z, v) = 0\text{)}$$
$$Y = \gamma_0 + \gamma_1 Z + w \text{ (where } cov(Z, w) = 0\text{)}$$

- Rewrite the first equation in terms of Z and get

$$Z = \frac{X}{\pi_1} - \frac{\pi_0}{\pi_1} - \frac{v}{\pi_1}$$

- Then plug this into the second equation. Reorganizing this equation, you should get

$$Y = \left(\gamma_0 - \frac{\pi_0\gamma_1}{\pi_1}\right) + \left(\frac{\gamma_1}{\pi_1}\right) X + \left(w - \frac{\gamma_1}{\pi_1} u\right)$$

- As a result, β_1 from the equation with X as independent variable is $\beta_1 = \frac{\gamma_1}{\pi_1}$.

IV estimate is consistent!

$$\beta_1 + \frac{\frac{1}{n}\sum_i (z_i-\bar z)u_i}{\frac{1}{n}\sum_i (z_i-\bar z)(x_i)}$$

$\to 0$

exogeneity

- Consistency: 2SLS can be written as

$$\hat\beta_{1,TSLS} = \frac{s_{zy}}{s_{zx}} = \boxed{\frac{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar Z)(Y_i-\bar Y)}{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar Z)(X_i-\bar X)}}$$

$=0$ (relevance)

$\simeq 0$: weak IV

As $n \to \infty$, we can show that $\hat\beta_{1,TSLS} \to \beta_1$

- $\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar Z)(Y_i-\bar Y) \xrightarrow{p} cov(Z,Y)$
- $\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar Z)(X_i-\bar X) \xrightarrow{p} cov(Z,X)$
- Note that $cov(Z,Y) = cov(Z,\beta_0+\beta_1 X+u) = \beta_1 cov(Z,X) + cov(Z,u)$
- If Z is a valid IV, $cov(Z,u) = 0$
- So $\hat\beta_{1,TSLS} \to \frac{\beta_1 cov(Z,X)}{cov(Z,X)} = \beta_1$ QED

IV estimators are normally distributed (but only in large samples)

$\sqrt{n}(\hat{\theta}-\theta)$

- Break down the 2SLS estimators into

$$\hat{\beta}_{TSLS} = \frac{s_{zy}}{s_{zx}} = \frac{\sum_{i=1}^{n}(Z_i-\bar{Z})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(Z_i-\bar{Z})(X_i-\bar{X})} = \frac{\sum_{i=1}^{n}(Z_i-\bar{Z})Y_i}{\sum_{i=1}^{n}(Z_i-\bar{Z})X_i}$$

$$= \frac{\sum_{i=1}^{n}(Z_i-\bar{Z})(\beta_0+\beta_1 X_i+u_i)}{\sum_{i=1}^{n}(Z_i-\bar{Z})X_i} = \beta_1 + \frac{\sum_{i=1}^{n}(Z_i-\bar{Z})u_i}{\sum_{i=1}^{n}(Z_i-\bar{Z})X_i} = \beta_1 + \frac{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z})u_i}{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z})X_i}$$

- After subtracting both sdies by β_1 and multiplying both sides by $\sqrt{n}$, we get

$$\sqrt{n}(\hat{\beta}_{TSLS}-\beta_1) = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Z_i-\bar{Z})u_i}{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z})X_i}$$

CLT $N(E(Zu)=0, \text{var})$

$\xrightarrow{p} cov(Z,X)$

- Denominator $\xrightarrow{p} cov(Z,X)$ by (weak) law of large numbers
- Numerators $\sim N(0, var[(Z-\mu_Z)u])$ by central limit theorem.
- Thus, $\hat{\beta}_{TSLS}$ has a normal distribution.

→ So Hypothesis Test continues as usual

Multivariate case and Identification issues

Note: in first stage, all the W's should be included

- Suppose that we have

$$Y_i = \beta_0 + \beta_1 X_{1i} + \ldots + \beta_k X_{ki} + \delta_1 W_{1i} + \ldots + \delta_l W_{li} + u_i$$

$Z_1 \cdots \cdots Z_k \cdots Z_m$

  where X variables are endogenous and W variables are exogenous.

- Assume that we have found a total of m (not necessarily equal to k) variables that could qualify as IVs, all of them satisfying
  - **Just-identified**: When $m = k$. There are just enough instruments to identify k endogenous variables
  - **Overidentified**: When $m > k$. There are more than enough instruments.
  - **Underidentified**: When $m < k$. There are not enough instruments. The coefficients for X's will not be identified
- Need at least as much instrumental variables as the number of endogenous regressors you have

$M \geq K$

$$y = \beta_0 + \beta_1 X_1 + \delta_1 W_1 + \delta_2 W_2 + u_i$$

$$Z_1$$

~~1) $X_1 = \gamma_0 + \gamma_1 Z_1 + u_i$~~

2) $X_1 = \gamma_0 + \gamma_1 Z_1 + \gamma_2 W_1 + \gamma_3 W_2 + u_i$

IV assumptions

Not guaranteed for $E[u_i|X_{1i} \cdots X_{ki}]$

IV assumptions, multiple variable case

IV1 $\boxed{E[u_i|W_{1i}, ..., W_{li}] = 0}$ (At least for exogenous variables, this is satisfied)

IV2 $(Y_i, X_{1i}, .., X_{ki}, W_{1i}, .., W_{li}, Z_{1i}, ..., Z_{mi})$ are IID (across $i=1 \cdots n$)

IV3 The Y, X, W, Z variables all have nonzero finite 4th moments ($\because$ CLT, asymp normality)

IV4 The instruments are valid. That is $cov(Z_{ji}, u_i) = 0$ for all $j = 1, ..., m$ and relevancy conditions are satisfied for all Z's.

test (partly)

We can directly test relevance condition

- Assume we have

$$Y_i = \beta_0 + \boxed{\beta_1 X_i} + \underline{\beta_2 W_{1i} + \ldots + \beta_{1+r} W_{ri}} + u_i$$

  where only X_i is endogenous.

- Suppose we have m instruments. Then our first stage equation looks like

$$\boxed{X_i} = \pi_0 + \underline{\pi_1 Z_{1i} + \ldots + \pi_m Z_{mi}} + \underline{\pi_{(m+1)i} W_{1i} + \ldots + \pi_{(m+r)i} W_{ri}} + v_i$$

- We say our instrument is relevant if at least one of $\pi_1, \ldots, \pi_m$ is statistically nonzero.
- Run the F-test with these null and alternative hypothesis

$$H_0 : \pi_1 = \ldots = \pi_m = 0 \text{ vs. } \boxed{H_1 : \neg H_0}$$

reject H_0
(so $\exists$ relevant IV)

- By rule of thumb, we want F-statistics to be larger than 10. Otherwise, we have a weak instrument problem (and IV estimator may not be normally distributed even in large numbers)

We can get a taste of exogeneity condition with many instruments

- Consider the following case: We want to regress

$$Y_i = \beta_0 + \beta_1 X_{1i} + \ldots + \beta_k X_{ki} + \delta_1 W_{1i} + \ldots + \delta_l W_{li} + u_i$$

  and we found m possible candidates for instrumental variables, $Z_1, \ldots, Z_m$

- Overidentifying restriction test we conduct here is a J-test that uses TSLS estimators and not the hypothesized β values. The steps are as follows
  1. Regress the above equation using 2SLS.Obtain the residuals $\boxed{\hat{u}_i} = Y_i - \widehat{Y}_i$
  2. Regress residuals onto instruments and other controls, namely

$$\boxed{\hat{u}_i} = \alpha_0 + \alpha_1 Z_{1i} + \ldots + \alpha_m Z_{mi} + \alpha_{m+1} W_{1i} + \ldots + \alpha_{m+l} W_{li} + v_i$$

  3. Compute the F-statistics from $H_0 : \alpha_1 = \ldots = \alpha_m = 0$ vs. $H_1 : \neg H_0$ → Violate exog.
  4. Derive the J statistics as follows $J = m \times F$. J follows $\boxed{\chi^2_{m-k}}$ distribution
  5. If you have an endogenous IV, you will end up rejecting the null
  6. Then, you need to make a guess on which instrument is violating the exogeneity condition, drop those, and redo the above procedure. (Good luck!)

Do not violate exog

Overidentification test as a ‘distance’ between different IVs

(If time permits)

- Assume a case with one endogenous variable X and two IVs (Z_1, Z_2)
- The estimates for the coefficient for X are

$$\underline{\hat{\beta}_{Z_1} = \frac{cov(Z_1, Y)}{cov(Z_1, X)}}, \ \hat{\beta}_{Z_2} = \frac{cov(Z_2, Y)}{\underline{cov(Z_2, X)}}$$

  The numbers look different, although both $\hat{\beta}$’s are suppose to be the same coefficient estimating impact of X_1 on Y.

- The overidentification test checks whether the differences between $\hat{\beta}_{Z_1}, \hat{\beta}_{Z_2}$ are large.
  - If they converge to same item, we do not have to worry. Otherwise, one or more IV might be faulty

What if we have just enough IVs?

- To be very honest, there is not much we can do in terms of rigorous testing approach. This is usually the area of judgement call.
- One way to do this is to use the approach of exclusion restriction - you argue that the instrumental variable Z affects Y only through X.
  - If Z affects Y directly without X, then the exclusion restriction fails.
- You try to persuade others based on logic, previous practices, or intuition (usually a combination of all three).