Convex Analysis and Optimization

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Chapter 1

Differentiation

Our study of the Lebesgue integral naturally leads one to the consideration of the relationship between differentiation and integration. Starting with the fundamental theorem of calculus, we will derive the integration by parts formula, using which we derive the Taylor approximation of differentiable functions. We work immediately with arbitrary euclidean spaces; the added generality actually facilitates our analysis and simplifies the proofs of some important results, as we will see below.

1.1 The Trace Norm on $\mathbb{R}^{m\times n}$

Because we define the derivative of an arbitrary function as an $m \times n$ matrix, it will be useful to furnish a norm on the real vector space of all real $m \times n$ matrices. Specifically, we will be using the trace norm $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}_+$ defined as

$$\|A\| = \operatorname{tr}(A'A)^{\frac{1}{2}}$$

for any $A \in \mathbb{R}^{m\times n}$. It is very easy to see that $\|A\|^2$ is simply the sum of the squares of all the entries of A, and it follows that

$$\|A\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{m}\sum_{j=1}^{n} |A_{ij}|.$$

We will now show that $\|\cdot\|$ possesses the properties that a matrix norm such as the operator norm should possess.

Recall that, for any $n \in N_+$, defining $S^{n\times n}$ as the set of all symmetric $n \times n$ matrices, $S^{n\times n}$ is a linear subspace of the real vector space $\mathbb{R}^{n\times n}$: this can be seen easily, since the zero $n \times n$ matrix is symmetric and, for any $a \in \mathbb{R}$ and $A, B \in S^{n\times n}$, $(aA+B)' = aA' + B' = aA + B$ and thus $aA + B \in S^{n\times n}$.

In addition, the operation $\langle \cdot, \cdot \rangle : S^{n\times n} \times S^{n\times n} \to \mathbb{R}$ defined as

$$\langle A, B \rangle = \mathrm{tr}(A'B)$$

for any $A, B \in S^{n\times n}$ is an inner product defined on $S^{n\times n}$:

1) **Linearity in First Argument**
   For any $a \in \mathbb{R}$ and $A, B, C \in S^{n\times n}$,

$$\langle aA + B, C \rangle = \mathrm{tr}((aA+B)'C) = \mathrm{tr}(a \cdot A'C + B'C) = a \cdot \mathrm{tr}(A'C) + \mathrm{tr}(B'C) = a \cdot \langle A, C \rangle + \langle B, C \rangle,$$

   so that $\langle \cdot, \cdot \rangle$ is linear in its first argument.

2) **Conjugate Symmetry**
   For any $A, B \in S^{n\times n}$,

$$\langle A, B \rangle = \mathrm{tr}(A'B) = \mathrm{tr}(BA') = \mathrm{tr}(B'A) = \langle B, A \rangle,$$

   where we used both the commutativity property of the trace operation and the symmetry of A and B.

3) **Positive Definiteness**
   For any $A \in S^{n\times n}$,

$$\langle A, A \rangle = \mathrm{tr}(A'A) = \mathrm{tr}(A^2).$$

   Letting $A = PDP'$ be the eigendecomposition of A (which exists because A is real and symmetric), $A = O$ if and only if all the diagonal entries of D are 0. Lettng $\mu_1, \cdots, \mu_n$ be the diagonal entries of D, since $A^2 = PD^2P'$ and $\mathrm{tr}(A^2) = \mathrm{tr}(D^2)$, we can see that

$$\langle A, A \rangle = \mathrm{tr}(D^2) = \sum_{i=1}^{n} \mu_i^2 \geq 0,$$

   where the inequality holds as an equality if and only if $\mu_1 = \cdots = \mu_n = 0$, or $D = O$. Therefore, $\langle A, A \rangle > 0$ if $A \neq O$.

We have just shown that $(S^{n\times n}, \langle \cdot, \cdot \rangle)$ is a real inner product space; denote by $\|\cdot\|_{tr}$ the norm induced by $\langle \cdot, \cdot \rangle$. Since

$$\|A\|_{tr} = (\langle A, A \rangle)^{\frac{1}{2}} = \mathrm{tr}(A'A)^{\frac{1}{2}}$$

for any $A \in S^{n\times n}$, we can see that $\|\cdot\|_{tr}$ equals the trace norm $\|\cdot\|$ on $S^{n\times n}$.

By the Cauchy-Schwarz inequality,

$$\left|\mathrm{tr}(A'B)\right| = |\langle A,B\rangle| \leq \|A\|_{tr}\|B\|_{tr}$$

for any $A,B \in S^{n\times n}$.
In particular, for any positive semidefinite $A \in S^{n\times n}$, letting $A = PDP'$ be its eigendecomposition and $\mu_1,\cdots,\mu_n$ be the diagonal entries of D (the eigenvalues of A), $\mu_1,\cdots,\mu_n \geq 0$. Therefore,

$$\|A\|_{tr} = \mathrm{tr}(A^2)^{\frac{1}{2}} = \left(\sum_{i=1}^{n}\mu_i^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n}\mu_i = \mathrm{tr}(A),$$

which tells us that the trace norm of a positive semidefinite matrix is majorized by its trace.

Returning to the general setting of the space of all real $m\times n$ matrices $\mathbb{R}^{m\times n}$, we can now see that the trace norm $\|\cdot\|$ on $\mathbb{R}^{m\times n}$ has the following properties:

1) **Positive Definiteness**
   Let $A \in \mathbb{R}^{m\times n}$. Suppose that $\|A\| = 0$. Then,

   $$0 = \mathrm{tr}(A'A) = \sum_{i=1}^{m}\sum_{j=1}^{n}A_{ij}^2,$$

   so that $A_{ij} = 0$ for any $1 \leq i \leq m$, $1 \leq j \leq n$. It follows that $A = O$. It is obvious that $\|A\| = 0$ if $A = O$.

2) **Absolute Homogeneity**
   Let $a \in \mathbb{R}$ and $A \in \mathbb{R}^{m\times n}$. Then,

   $$\|aA\| = \mathrm{tr}(a^2A'A)^{\frac{1}{2}} = |a|\cdot\mathrm{tr}(A'A)^{\frac{1}{2}} = |a|\cdot\|A\|.$$

3) **Triangle Inequality**
   Let $A,B \in \mathbb{R}^{m\times n}$;

   $$\|A+B\|^2 = \mathrm{tr}((A+B)'(A+B)) = \mathrm{tr}(A'A)+\mathrm{tr}(B'B)+\mathrm{tr}(B'A)+\mathrm{tr}(A'B).$$

   Letting the (i,j)th entry of A,B be denoted A_{ij},B_{ij} for any $1 \leq i \leq m$, $1 \leq j \leq n$, note that

   $$\mathrm{tr}(B'A) = \mathrm{tr}(A'B) = \sum_{j=1}^{n}\sum_{i=1}^{m}A_{ij}B_{ij},$$

and by the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{m} A_{ij}B_{ij} \leq \sum_{i=1}^{m} |A_{ij}B_{ij}| \leq \left(\sum_{i=1}^{m} A_{ij}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_{ij}^2\right)^{\frac{1}{2}}$$

for any $1 \leq j \leq n$, so that another application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}\sum_{j=1}^{n}\sum_{i=1}^{m} A_{ij}B_{ij} &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} A_{ij}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} B_{ij}^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=1}^{n}\sum_{i=1}^{m} A_{ij}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n}\sum_{i=1}^{m} B_{ij}^2\right)^{\frac{1}{2}} = \|A\|\|B\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|A+B\|^2 &= \mathrm{tr}(A'A) + \mathrm{tr}(B'B) + \mathrm{tr}(B'A) + \mathrm{tr}(A'B) \\ &\leq \|A\|^2 + \|B\|^2 + 2\cdot\|A\|\|B\| = (\|A\| + \|B\|)^2.\end{aligned}$$

We have now shown that $\|\cdot\|$ is a norm on $\mathbb{R}^{m\times n}$. Therefore, we can induce a metric d on $\mathbb{R}^{m\times n}$ by defining

$$d(A,B) = \|A-B\|$$

for any $A, B \in \mathbb{R}^{m\times n}$.

The following are more useful properties of the trace norm:

Theorem 8.1 (Properties of the Trace Norm)
Let $\|\cdot\|$ be the trace norm. Then, the following hold true:

i) For any $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, $\|AB\| \leq \|A\|\|B\|$.

ii) For any $x \in \mathbb{R}^n$, $|x| = \|x\|$.

iii) The set Ω^o of all invertible matrices on $\mathbb{R}^{n\times n}$ is an open subset of $\mathbb{R}^{n\times n}$ with respect to the metric induced by $\|\cdot\|$.

iv) The function $f : \Omega^o \to \Omega^o$ defined as $f(A) = A^{-1}$ for any $A \in \Omega^o$ is continuous with respect to the metric induced by $\|\cdot\|$.

Proof) i) For any $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$,

$$\begin{aligned}\|AB\|^2 = \operatorname{tr}(B'A'AB) &= \operatorname{tr}((A'A)(BB')) \\ &= \langle A'A, BB'\rangle \quad (A'A, BB' \text{ are } n\times n \text{ symmetric matrices}) \\ &\le \|A'A\|_{tr}\cdot\|BB'\|_{tr} \qquad \text{(The Cauchy-Schwarz Inequality)} \\ &\le \operatorname{tr}(A'A)\cdot\operatorname{tr}(BB') \quad (A'A, BB' \text{ are positive semidefinite}) \\ &= \|A\|^2\cdot\|B\|^2.\end{aligned}$$

ii) Let x be an n-dimensional real valued vector whose euclidean norm is $|x|$. Then, $\|x\|$ is well-defined as the norm of the $n\times 1$ matrix x. It is easy to see that

$$\|x\|^2 = \operatorname{tr}(x'x) = |x|^2.$$

By implication, for some $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$,

$$|Ax| = \|Ax\| \le \|A\|\cdot\|x\| = \|A\|\cdot|x|.$$

iii) Choose any $A \in \Omega^o$. Because $A^{-1} \neq O$, $\|A^{-1}\| > 0$. Let $B \in \mathbb{R}^{n\times n}$ be an element in the open ball $B(A, 1/\|A^{-1}\|)$ around A, that is,

$$\|A - B\| < \frac{1}{\|A^{-1}\|}.$$

Choose any $x \in \mathbb{R}^n$, and suppose that $x \neq \mathbf{0}$. Then,

$$\begin{aligned}|x| = \left|A^{-1}Ax\right| \le \left\|A^{-1}\right\|\cdot|Ax - Bx + Bx| \\ \le \left\|A^{-1}\right\|\cdot(\|A-B\||x| + |Bx|).\end{aligned}$$

Because $|x| > 0$, we have

$$\left\|A^{-1}\right\|\cdot\|A-B\||x| < |x|,$$

so that

$$|x| < |x| + |Bx|,$$

which implies $|Bx| > 0$, or $Bx \neq \mathbf{0}$. By contraposition, if $Bx = \mathbf{0}$, then $x = \mathbf{0}$. This tells us that the null space of B consists only of the zero vector $\mathbf{0}$, and as such

that B is an invertible matrix.
This holds for any $B \in B(A, 1/\|A\|)$, so $B(A, 1/\|A\|) \subset \Omega^o$. This in turn holds for any $A \in \Omega^o$, so Ω^o is open with respect to the metric induced by the trace norm.

iv) Define $f : \Omega^o \to \Omega^o$ as

$$f(A) = A^{-1} \quad \text{for any } A \in \mathbb{R}^{n\times n}.$$

Choose any $A \in \Omega^o$, and $B \in \mathbb{R}^{n\times n}$ such that $\|A - B\| < \delta$. Then, $B \in \Omega^o$ by the above result, and because $\|A^{-1}\| \cdot \|A - B\| < 1$,

$$\begin{aligned}\left\|B^{-1}\right\| = \left\|A^{-1}AB^{-1}\right\| &\leq \left\|A^{-1}\right\| \cdot \left\|(A-B)B^{-1} + I_n\right\| \\ &\leq \left\|A^{-1}\right\| \cdot \|A - B\| \cdot \left\|B^{-1}\right\| + \sqrt{n} \cdot \left\|A^{-1}\right\|\end{aligned}$$

implies

$$\left\|B^{-1}\right\| \leq \frac{\sqrt{n} \cdot \|A^{-1}\|}{1 - \|A^{-1}\| \cdot \|A - B\|}.$$

It then follows that

$$\begin{aligned}\|f(A) - f(B)\| = \left\|A^{-1}(A-B)B^{-1}\right\| &\leq \|A - B\| \cdot \left\|A^{-1}\right\| \cdot \left\|B^{-1}\right\| \\ &\leq \frac{\sqrt{n} \cdot \|A^{-1}\| \cdot \|A - B\|}{1 - \|A^{-1}\| \cdot \|A - B\|}.\end{aligned}$$

The right hand side goes to 0 as $\|A - B\| \to 0$, so it follows that $\|f(A) - f(B)\|$ also goes to 0 as $\|A - B\| \to 0$. This shows us that f is a continuous function on Ω^o.

Q.E.D.

1.2 Differentiation on Euclidean Space

Let E be an open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}^m$. The derivative of f at $x \in E$ is defined as the $m \times n$ matrix A such that

$$\lim_{h \to \mathbf{0}} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0.$$

Note that the fraction above is well-defined for h close to 0 because E is an open subset containing x, so that we can find a neighborhood around x contained in E. If such an A exists, we say that f is differentiable at x. If f is differentiable at every point in E, then we say that it is differentiable on E.

We first show that the derivative A of f at some point x is unique:

Lemma 1.2 (Uniqueness of the Derivative)
Let E be an open subset of $\mathbb{R}^n$, and suppose $A_1, A_2 \in \mathbb{R}^{m\times n}$ are derivatives of $f : E \to \mathbb{R}^m$ at some $x \in E$. Then, $A_1 = A_2$.

Proof) By definition, $A_1, A_2 \in \mathbb{R}^{m\times n}$ are two matrices satisfying

$$\lim_{h \to \mathbf{0}} \frac{|f(x+h) - f(x) - A_i h|}{|h|} = 0$$

for $i = 1, 2$, then for any non-zero $h \in \mathbb{R}^n$ that is small enough so that $x + h \in E$,

$$\begin{aligned} |(A_1 - A_2)h| &= |f(x+h) - f(x) - A_2 h - (f(x+h) - f(x)0A_1 h)| \\ &\leq |f(x+h) - f(x) - A_2 h| + |f(x+h) - f(x)0A_1 h|, \end{aligned}$$

so that

$$\frac{|(A_1 - A_2)h|}{|h|} \leq \frac{|f(x+h) - f(x) - A_1 h|}{|h|} + \frac{|f(x+h) - f(x) - A_2 h|}{|h|}.$$

Taking $h \to \mathbf{0}$ on both sides shows us that

$$\lim_{h \to \mathbf{0}} \frac{|(A_1 - A_2)h|}{|h|} = 0.$$

By definition, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|(A_1 - A_2)h| \leq \epsilon \cdot |h|$$

for any $h \in \mathbb{R}^n$ such that $|h| < \delta$. Fixing some non-zero $x \in \mathbb{R}^n$, this shows us that, for any $t > 0$ such that $t < \frac{\delta}{|x|}$, since $|t \cdot x| < \delta$, we have

$$|(A_1 - A_2)(tx)| = |t||(A_1 - A_2)x| \leq \epsilon \cdot |tx| = |t| \cdot \epsilon |x|.$$

Dividing both sides by $|t|$ yields

$$|(A_1 - A_2)x| \leq \epsilon \cdot |x|;$$

this holds for any $\epsilon > 0$, so $|(A_1 - A_2)x| = 0$, that is, $A_1 = A_2$.

Q.E.D.

The unique derivative $A \in \mathbb{R}^{m\times n}$ of f at x is denoted by $f'(x) \in \mathbb{R}^{m\times n}$. One of the most convenient implications of differentiability is that f is continuous at any point at which it is differentiable:

Lemma 1.3 (Differentiability implies Continuity)
Let E be an open subset of $\mathbb{R}^n$, and suppose $f : E \to \mathbb{R}^m$ is differentiable at some $x \in E$. Then, f is continuous at x.

Proof) Suppose f is differentiable at $x \in E$. Let $A \in \mathbb{R}^{m\times n}$ be the derivative of f, and choose some $\epsilon > 0$. Let $\eta > 0$ be chosen small enough so that $\eta^2 + \|A\| \cdot \eta < \epsilon$.

By definition, there exists a $\delta > 0$ satisfying

$$|f(x+h) - f(x) - Ah| \leq \eta \cdot |h|$$

for any $h \in \mathbb{R}^n$ such that $|h| < \delta$. Note that, for any $h \in \mathbb{R}^n$ such that $|h| < \delta$,

$$|f(x+h) - f(x)| - |Ah| \leq |f(x+h) - f(x) - Ah| \leq \eta \cdot |h|,$$

and by implication,

$$|f(x+h) - f(x)| \leq \epsilon \cdot |h| + |Ah| \leq \eta \cdot |h| + \|A\| \cdot |h|.$$

Therefore, for any $y \in \mathbb{R}^n$ such that $|x - y| < \min(\delta, \eta)$, we can now see that

$$|f(y) - f(x)| = |f(x + (y - x)) - f(x)| \leq (\eta + \|A\|)\, |x - y| < \eta^2 + \eta \cdot \|A\| < \epsilon.$$

This holds for any $\epsilon > 0$, so by definition f is continuous at x.

Q.E.D.

Consider a differentiable function $f : E \to \mathbb{R}^m$. While this ensures the continuity of f on E, it does not ensure the continuity of the mapping $f' : E \to \mathbb{R}^{m\times n}$ with respect to the trace norm on $\mathbb{R}^{m\times n}$. If this is also the case, that is, if f' is a continuous function as well, then we say that f is continuously differentiable on E, and we denote $f \in \mathcal{C}^1(E)$.

The simplest case we can study is the differentiation of $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, that is, linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$. In this case, we can write

$$T(x) = Ax$$

for any $x \in \mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}$ is the matrix representation of T with respect to the standard basis. Since

$$\frac{|T(x+h) - T(x) - Ah|}{|h|} = 0$$

for any $x, h \in \mathbb{R}^n$ by the linearity of T, we can see that the derivative of T at any x is exactly equal to the matrix representation A. In this case, the derivative is continuous everywhere on $\mathbb{R}^n$, so we can see that any linear transformation is also continuously differentiable on $\mathbb{R}^n$.

We can prove that the chain rule holds in this more general situation:

Theorem 1.4 (Chain Rule)
Let E be an open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}^m$ a function that is differentiable at some $x_0 \in E$. In addition, let V be some open subset of $\mathbb{R}^m$ containing the image $f(E)$, and $g : V \to \mathbb{R}^p$ a function differentiable at $f(x_0) \in V$. Then, defining $F = g \circ f : E \to \mathbb{R}^p$, F is differentiable at x_0 with derivative equal to

$$F'(x_0) = g'(f(x_0))f'(x_0) \in \mathbb{R}^{p \times n}.$$

Proof) Denote $y_0 = f(x_0) \in V$, $A = f'(x_0)$ and $B = g'(y_0)$. Then, defining

$$u(h) = f(x_0 + h) - f(x_0) - Ah$$
$$v(k) = g(y_0 + k) - g(y_0) - Bk$$

for any $h \in \mathbb{R}^n$ and $k \in \mathbb{R}^m$ for which the above functions are well-defined. Since E is open, there exists a neighborhood U of $\mathbf{0}$ such that $x_0 + U \in E$. Define the function $K : U \to \mathbb{R}^m$ as

$$K(h) = f(x_0 + h) - f(x_0)$$

for any $h \in U$. Note that, for any $h \in U$,

$$
\begin{aligned}
F(x_0+h)-F(x_0)-(BA)h &= g(f(x_0+h))-g(y_0)-(BA)h \\
&= g\left(y_0+(f(x_0+h)-f(x_0)\right)-g(y_0) \\
&\quad -B((f(x_0+h)-f(x_0))+B\left(f(x_0+h)-f(x_0)-Ah\right) \\
&= (g(y_0+K(h))-g(y_0)-B\cdot K(h))+B\left(f(x_0+h)-f(x_0)-Ah\right) \\
&= v(K(h))+B\cdot u(h),
\end{aligned}
$$

so that

$$|F(x_0+h)-F(x_0)-(BA)h| \leq |v(K(h))|+\|B\|\cdot|u(h)|.$$

We want to bound the right hand side above by $|h|$ times a small positive number $\epsilon>0$. To this end, choose some $\epsilon>0$, and let $\alpha>0$ be chosen small enough so that

$$\alpha^2+(\|A\|+\|B\|)\,\alpha<\epsilon.$$

By the differentiability of g at y_0, for any $\epsilon>0$, there exists a $\delta>0$ such that

$$|v(k)| \leq \alpha\cdot|k|$$

for any $k \in \mathbb{R}^m$ such that $|k|<\delta$.

Furthermore, by the differentiability of f at x_0, there exists an $\eta>0$ such that

$$|u(h)| \leq \alpha\cdot|h|$$

for any $h \in \mathbb{R}^n$ such that $|h|<\eta$.

Finally, since the differentiability of f at x_0 implies continuity of f at x_0,

$$\lim_{h\to\mathbf{0}} K(h)=\mathbf{0}$$

and we can take $\eta>0$ small enough so that

$$|K(h)|<\delta$$

also holds for any $h \in \mathbb{R}^n$ such that $|h|<\eta$.

Putting these results together, for any $h \in \mathbb{R}^n$ such that $|h| < \eta$,

$$|u(h)| \leq \alpha \cdot |h|,$$

and since we have $|K(h)| < \delta$, we also can conclude that

$$|v(K(h))| \leq \alpha \cdot |K(h)|.$$

Using the fact that

$$|K(h)| = |u(h) + Ah| \leq |u(h)| + \|A\| \cdot |h|$$

for any $h \in U$, we can see that

$$\begin{aligned} |v(K(h))| \leq \epsilon \cdot |K(h)| &\leq \alpha \left(|u(h)| + \|A\| \cdot |h|\right) \\ &\leq \alpha^2 \cdot |h| + \alpha \|A\| \cdot |h|. \end{aligned}$$

In other words, $|h| < \eta$ for any $h \in U$ implies

$$\begin{aligned} |F(x_0 + h) - F(x_0) - (BA)h| &\leq |v(K(h))| + \|B\| \cdot |u(h)| \\ &\leq \left(\alpha^2 + \alpha \cdot \|A\|\right) \cdot |h| + \alpha \|B\| \cdot |h| \\ &= \left[\alpha^2 + (\|A\| + \|B\|)\,\alpha\right] \cdot |h| \leq \epsilon \cdot |h|. \end{aligned}$$

This holds for any $\epsilon > 0$, so

$$\lim_{h \to \mathbf{0}} \frac{|F(x_0 + h) - F(x_0) - (BA)h|}{|h|} = 0,$$

and by definition $F'(x_0) = BA$.

Q.E.D.

1.2.1 Partial Differentiation

We now introduce a way to very easily characterize the derivative of a multivariate function using derivatives with respect to each coordinate. Let $E = \{e_1, \cdots, e_n\}$ be the standard bases of $\mathbb{R}^n$. Let E be an open subset of $\mathbb{R}^n$ and $f = (f_1, \cdots, f_m)$ a function from E into $\mathbb{R}^m$. We say that the ith coordinate function $f_i : E \to \mathbb{R}$ of f is partially differentiable at $x \in E$ with respect to the jth coordinate if the limit

$$\frac{\partial f_i}{\partial x_j}(x) = \lim_{t \to 0} \frac{f_i(x + t \cdot e_j) - f(x)}{t}$$

exists; we call the limit the partial derivative of f_i with respect to the jth coordinate at x. Note that $(D_j f_i)(x) := \frac{\partial f_i}{\partial x_j}(x)$ is essentially the (univariate) derivative of the mapping $t \mapsto f_i(x_1, \cdots, x_{j-1}, t, x_{j+1}, \cdots, x_n)$ at x_j. If all mn partial derivatives of f at x exist, then we can collect them into the Jacobian

$$J(x) = \begin{pmatrix} (D_1 f_1)(x) & \cdots & (D_n f_1)(x) \\ \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & \cdots & (D_n f_m)(x) \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Partial differentiability does not ensure differentiability; it does not even ensure continuity. However, the converse does hold true, that is, differentiability implies partial differentiability.

Theorem 1.5 (Differentiability implies Partial Differentiability)
Let E be an open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}^m$ a function on E. If f is differentiable at $x \in E$, then it is partially differentiable at $x \in E$ and the derivative $f'(x)$ is exactly the Jacobian of f at x, that is,

$$f'(x) = \begin{pmatrix} (D_1 f_1)(x) & \cdots & (D_n f_1)(x) \\ \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & \cdots & (D_n f_m)(x) \end{pmatrix}.$$

Proof) Let $\{e_1, \cdots, e_n\}$ and $\{u_1, \cdots, u_m\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Suppose that f is differentiable at x, and denote $A = f'(x)$. Choose any $1 \le i \le m$ and $1 \le j \le n$. By definition, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x+h) - f(x) - Ah| \le \epsilon \cdot |h|$$

for any $h \in \mathbb{R}^n$ such that $|h| < \delta$. Chooseing $t > 0$ such that $|t| < \delta$, since $|t \cdot e_j| = |t| < \delta$, it follows that

$$|f(x + t \cdot e_j) - f(x) - t \cdot Ae_j| \le \epsilon \cdot |t|.$$

The ith element of the vector Ae_j, which is equal to the jth column of A, is exactly the (i,j)th element of A. The definition of the euclidean norm on $\mathbb{R}^m$ now tells us that

$$|f_i(x + t \cdot e_j) - f_i(x) - t \cdot A(i,j)| \le |f(x + t \cdot e_j) - f(x) - t \cdot Ae_j| \le \epsilon \cdot |t|,$$

or equivalently,

$$\left| \frac{f_i(x + t \cdot e_j) - f_i(x)}{t} - A(i,j) \right| = \left| \frac{f_i(x + t \cdot e_j) - f_i(x) - t \cdot A(i,j)}{t} \right| \le \epsilon$$

for any $t > 0$ such that $|t| < \delta$. This holds for any $\epsilon > 0$, so we have

$$A(i,j) = \lim_{t\to 0} \frac{f_i(x+t\cdot e_j) - f_i(x)}{t}.$$

This holds for any $1 \le i \le m$ and $1 \le j \le n$, so by definition each $A(i,j)$ is the jth partial derivative of f_i at x, that is,

$$A(i,j) = (D_j f_i)(x).$$

Q.E.D.

Of interest is the case where f is a real-valued function. If E is an open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ is differentiable at x, the theorem above tells us that the partial derivatives of f exist and satisfy

$$f'(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix}.$$

The transpose of this $1 \times n$ row vector is called the gradient of f at x, and is denoted by $\boldsymbol{\nabla} f(x) \in \mathbb{R}^n$.

1.2.2 Directional Derivatives

Let (a,b) be an open interval on the real line, E an open subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}$ a real-valued function, and $\gamma : (a,b) \to E$ a function representing a parametric curve on the set E. Suppose γ and f are both differentiable on their domains, and define the function $g = f \circ \gamma : (a,b) \to \mathbb{R}$. The chain rule tells us that

$$g'(t) = f'(\gamma(t))\gamma'(t)$$

for any $t \in (a,b)$, where $g'(t)$ is real because $f'(\gamma(t)) \in \mathbb{R}^{1\times n}$ and $\gamma'(t) \in \mathbb{R}^{n\times 1}$. Furthermore, the preceding theorem implies

$$f'(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix} = \boldsymbol{\nabla} f(x)'$$

for any $x \in E$, so the derivative of g at t can be written as

$$g'(t) = \boldsymbol{\nabla} f(\gamma(t))'\gamma'(t) = \langle \boldsymbol{\nabla} f(\gamma(t)), \gamma'(t) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.

We are especially interested in the case where $\gamma : (-\delta, \delta) \to E$ is defined as

$$\gamma(t) = x + t \cdot u$$

for some $x \in E$ and a unit vector $u \in \mathbb{R}^n$, where $\delta > 0$ is chosen so that $\gamma(t)$ takes values in E for any $t \in (-\delta, \delta)$. Defining g as above, note that

$$\frac{g(t) - g(0)}{t} = \frac{f(x + t \cdot u) - f(x)}{t}$$

for any $0 < |t| < \delta$, while the derivative of g at 0 is given by

$$g'(0) = \boldsymbol{\nabla} f(\gamma(0))'\gamma'(0) = \boldsymbol{\nabla} f(x)'u.$$

It follows that

$$\boldsymbol{\nabla} f(x)'u = \lim_{t \to 0} \frac{f(x + t \cdot u) - f(x)}{t}$$

by the definition of the derivative; we call this quantity the directional derivative of f at x in the direction of u, and is denoted by $(D_u f)(x)$. Heuristically, $(D_u f)(x)$ represents the infinitesimal amount by which f increases from x in the direction of u.

Using the fact that the directional derivative is an inner product on $\mathbb{R}^n$,

$$(D_u f)(x) = \langle \boldsymbol{\nabla} f(x), u \rangle = |\boldsymbol{\nabla} f(x)| \cdot |u| \cdot \cos(\theta) = |\boldsymbol{\nabla} f(x)| \cdot \cos(\theta),$$

where θ is the angle between the vectors $\boldsymbol{\nabla} f(x)$ and u. Since the cosine function achieves its maximum when $\theta = 0$, that $(D_u f)(x)$ equals $|\boldsymbol{\nabla} f(x)| \cdot \cos(\theta)$ implies that f grows the fastest from x in the direction of $\boldsymbol{\nabla} f(x)$. In other words, the gradient of f at x is proportional to the direction in which f grows the fastest.

1.2.3 The Mean Value Theorem

We now introduce a class of theorems that have widespread applicability, especially when it comes to the approximation of functions by polynomials. We first focus on univariate functions, and then extend it to multivariate functions via the process above.

Let (E, τ) be a topological space. A real-valued function $f : E \to \mathbb{R}$ is said to achieve a local maximum (minimum) at $x \in E$ if there exists a neighborhood U around x such that $f(x) \geq f(y)$ $(f(x) \leq f(y))$ for any $y \in U$. f achieves a strict local maximum (minimum) at x if the preceding inequalities are strict.

Returning to the specific setting of euclidean spaces, let (a, b) be an open interval on the real line and $f : (a, b) \to \mathbb{R}$ a real valued function. The first theorem, Rolle's theorem, shows us that, if f is differentiable, then its derivative should equal 0 at any local extremum.

Lemma 1.6 (Rolle's Theorem)
Let $f : [a,b] \to \mathbb{R}$ achieve a local maximum (minimum) at $x \in (a,b)$. If f is differentiable at x, then $f'(x) = 0$.

Proof) Suppose that $x \in (a,b)$ is local maximum of f (the case for local minima follow similarly) and that f is differentiable at x. Then, there exists a $\delta > 0$ such that $f(x) \geq f(y)$ for any $y \in (x-\delta, x+\delta)$. For any $h \in \mathbb{R}$ such that $0 < h < \delta$, this tells us that

$$\frac{f(x+h)-f(x)}{h} \geq 0,$$

while if $-\delta < h < 0$, then

$$\frac{f(x+h)-f(x)}{h} \leq 0,$$

where the inequality is flipped because h is negative in this case. Let $\{h_n\}_{n \in N_+}$ be a sequence of positive numbers in $(0,\delta)$ converging to 0. Then,

$$f'(x) = \lim_{n\to\infty} \frac{f(x+h_n)-f(x)}{h_n} \geq 0.$$

Similarly, if $\{h_n\}_{n \in N_+}$ be a sequence of negative numbers in $(-\delta, 0)$ converging to 0, then

$$f'(x) = \lim_{n\to\infty} \frac{f(x+h_n)-f(x)}{h_n} \leq 0.$$

This shows us that $f'(x) = 0$.

Q.E.D.

The mean value theorem now follows easily from Rolle's theorem.

Theorem 1.7 (Mean Value Theorem)
Let $f:[a,b]\to\mathbb{R}$ be a continuous function that is differentiable on (a,b). Then, there is a point $x\in(a,b)$ such that

$$f(b)-f(a)=f'(x)(b-a).$$

Proof) Define the function $g:[a,b]\to\mathbb{R}$ as

$$g(x)=(f(b)-f(a))x-f(x)(b-a)$$

for any $x\in[a,b]$. Then, g is continuous on $[a,b]$ and differentiable on (a,b) with derivative equal to

$$g'(x)=f(b)-f(a)-f'(x)(b-a)$$

for any $x\in(a,b)$. In addition,

$$g(b)=(f(b)-f(a))b-f(b)(b-a)=f(b)a-f(a)b=(f(b)-f(a))a-f(a)(b-a)=g(a).$$

We want to find an $x\in(a,b)$ such that $g'(x)=0$; this can be done by finiding a local maximum/minimum of g on (a,b), and then applying Rolle's theorem.

If g is a constant function on $[a,b]$, then $g'(x)=0$ for any $x\in(a,b)$, so the claim holds trivially. Suppose now that there exists an $x\in(a,b)$ such that $g(x)>g(a)=g(b)$.. Since g is a continuous function on the compact interval $[a,b]$, by the extreme value theorem there exists an $x^*\in[a,b]$ such that $g(x^*)=\max_{x\in[a,b]}g(x)$. By assumption, $x^*\in(a,b)$, and thus by Rolle's theorem, we have $g'(x^*)=0$. On the ther hand, if there exists an $x\in(a,b)$ such that $g(x)<g(a)=g(b)$, we repeat the same argument with the minimum instead of the maximum.

Q.E.D.

The mean value theorem can be seen as an approximation of a differentiable function using a linear function, since for a differentiable function $f:\mathbb{R}\to\mathbb{R}$, it states that

$$f(x)\approx f(x_0)+f'(x_0)x$$

for any two points $x,x_0\in\mathbb{R}$ if x_0 and x are close to one another. In this context, the Taylor expansion of a univariate function can be seen as the approximation of a function with a polynomial of an aribtrary degree.

We can formulate multivariate versions of Rolle's theorem and the mean value theorem by making use of the chain rule and gradients. These are stated below:

Theorem 1.8 (Multivariate Mean Value Theorem)
Let E be an open set in $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ a real-valued function on E. The following hold true:

i) If $x \in E$ is a local extremum of f and f is differentiable at x, then $\boldsymbol{\nabla} f(x) = \mathbf{0}$.

ii) Suppose E is convex and that f is differentiable on E. For any $x, y \in E$, there exists a $t_0 \in (0,1)$ such that

$$f(x) - f(y) = \boldsymbol{\nabla} f(t_0 \cdot x + (1-t_0) \cdot y)'(x-y).$$

Proof) i) Let d denote the euclidean metric on $\mathbb{R}^n$. Let $x \in E$ be a local maximum of f, and assume that f is differentiable at x. It follows that there exists a $\delta > 0$ such that, for any point $y \in E$ in $B_d(x,\delta)$, we have $f(y) \leq f(x)$. Choose any unit vector $u \in \mathbb{R}^n$ and define $\gamma : (-\delta,\delta) \to E$ as

$$\gamma(t) = x + t \cdot u$$

for any $t \in (-\delta,\delta)$. Let $g = f \circ \gamma : (-\delta,\delta) \to \mathbb{R}$. Note that g achieves a local maximum at 0 since for any $t \in (-\delta,\delta)$, $\gamma(t)$ is contained in $B_d(x,\delta)$ and therefore

$$g(t) = f(\gamma(t)) \leq f(x) = g(0).$$

By Rolle's theorem,

$$0 = g'(0) = \boldsymbol{\nabla} f(x)'u.$$

This holds for any unit vector u and therefore any standard basis vector in $\mathbb{R}^n$, so we can see that $\boldsymbol{\nabla} f(x) = \mathbf{0}$.

ii) Let f be differentiable on E, and choose distinct $x, y \in E$. Define $\gamma : [0,1] \to E$ as

$$\gamma(t) = t \cdot x + (1-t) \cdot y$$

for any $t \in [0,1]$, and define $g = f \circ \gamma : [0,1] \to \mathbb{R}$. Note that γ takes values in E because E is assumed to be convex. Since f is continuous on E by differentiability and γ is continuous on $[0,1]$, g is continuous on $[0,1]$, and for any $t \in (0,1)$,

$$g'(t) = \boldsymbol{\nabla} f(t \cdot x + (1-t) \cdot y)'(x-y),$$

so that g is differentiable on $(0,1)$. By the mean value theorem, there exists a

$t^* \in (0,1)$ such that

$$g(1) - g(0) = g'(t^*).$$

Using the definition of g, we can see that

$$f(x) - f(y) = f(\gamma(1)) - f(\gamma(0)) = \boldsymbol{\nabla} f(\gamma(t^*))'(x-y) = \boldsymbol{\nabla} f(t^* \cdot x + (1-t^*) \cdot y)'(x-y).$$

Q.E.D.

Although the mean value theorem does not hold for vector-valued functions, a weaker version of the theorem in which the equality is given as an inequality remains true:

Theorem 1.9 (Mean Value Inequality for Vector-valued Functions)
Let E be a convex open set in $\mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ a differentiable function on E such that there exists an $M > 0$ such that

$$\|f'(x)\| \leq M$$

for any $x \in E$. Then

$$|f(x) - f(y)| \leq M|x-y|$$

for any $x, y \in E$.

Proof) Choose any $x, y \in E$. If $f(x) = f(y)$, then the result is trivial, so we assume $f(x) \neq f(y)$. Define $\gamma : [0,1] \to E$ as

$$\gamma(t) = t \cdot x + (1-t) \cdot y$$

for any $t \in [0,1]$, where γ once again takes values in E thanks to the convexity of E. Defining $g = f \circ \gamma : [0,1] \to \mathbb{R}^m$, note that g is continuous on $[0,1]$ due to the continuity of f and γ, and that

$$g'(t) = f'(\gamma(t)) \cdot \gamma'(t) = f'(y + t(x-y)) \cdot (x-y)$$

for any $t \in (0,1)$. It follows that

$$|g'(t)| \leq \|f'(y + t(x-y))\| \cdot |x-y| \leq M \cdot |x-y|$$

for any $t \in (0,1)$.

Now define $\varphi : [0,1] \to \mathbb{R}$ as

$$\varphi(t) = (g(1) - g(0))' g(t)$$

for any $t \in [0,1]$. Then, φ is continuous on $[0,1]$ and differentiable on $(0,1)$ with derivative

$$\varphi'(t) = (g(1) - g(0))' g'(t)$$

for any $t \in (0,1)$. Thus, by the mean value theorem, there exists some $t^* \in (0,1)$ such that

$$\begin{aligned} \varphi(1) - \varphi(0) &= |g(1) - g(0)|^2 \\ &= \varphi'(t^*) = (g(1) - g(0))' g'(t^*). \end{aligned}$$

It follows that

$$|g(1) - g(0)|^2 = \left|(g(1) - g(0))' g'(t^*)\right| \leq |g(1) - g(0)| \cdot |g'(t^*)|.$$

$g(1) - g(0) = f(x) - f(y) \neq \mathbf{0}$, so dividing both sides by $|g(1) - g(0)|$ yields

$$|f(x) - f(y)| \leq |g'(t^*)| \leq M|x - y|.$$

Q.E.D.

The mean value theorem also proves crucial to show that a close relationship holds between differentiability and partial differentiability. Specifically, if all partial derivatives exist and are continuous, then a function is continuously differentiable.

Theorem 1.10 (Characterization of Continuous Differentiability)
Let E be an open set in $\mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ a function on E. The partial derivatives $(D_j f_i)(x)$ exist for any $x \in E$, $1 \leq i \leq m$ and $1 \leq j \leq n$, and the mappings $x \mapsto D_j f_i$ are continuous, if and only if f is continuously differentiable on E.

Proof) Suppose that f is continuously differentiable. Then, by theorem 1.5, for any $x \in E$ the partial derivative $(D_j f_i)(x)$ exists and equals the (i,j)th element of the matrix $f'(x)$. Therefore,

$$\|f'(x) - f'(y)\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |(D_j f_i)(x) - (D_j f_i)(y)|^2 \right)^{\frac{1}{2}}$$

for any $x, y \in E$. The continuity of f now implies the continuity of each partial derivatives. $D_j f_i$ on E.

Conversely, suppose that the partial derivatives $D_j f_i$ exist and are continuous on E. Fix any $x \in E$ and denote

$$A = \begin{pmatrix} (D_1 f_1)(x) & \cdots & (D_n f_1)(x) \\ \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & \cdots & (D_n f_m)(x) \end{pmatrix}.$$

By the openness of E and the continuity of the functions $D_j f_i : E \to \mathbb{R}$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $B_d(x, \delta) \subset E$ and

$$|(D_j f_i)(x) - (D_j f_i)(x)| < \frac{\epsilon}{nm}$$

for any $y \in B_d(x, \delta)$ and $1 \leq i \leq m$, $1 \leq j \leq n$. Choose any $h \in \mathbb{R}^n$ such that $|h| < \delta$; then, letting $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$, define

$$v_k = (h_1, \cdots, h_k, 0, \cdots, 0)$$

for $0 \leq k \leq n$; $|v_k| \leq |h| < \delta$, so each $x + v_k$ is contained in the open ball $B_d(x, \delta)$. Furthermore, the ball $B_d(x, \delta)$ is convex, so for any $1 \leq k \leq n$, the convex combination of $x + v_k$ and $x + v_{k-1}$ lie inside $B_d(x, \delta) \subset E$.

Fix $1 \leq i \leq m$. For any $1 \leq k \leq n$, suppose $h_k > 0$ and define the mapping $g : [0, h_k] \to \mathbb{R}$ as

$$g(t) = f_i(x + v_{k-1} + t \cdot e_k)$$

for any $t \in [0, 1]$. By definition,

$$\begin{aligned} g'(t) &= \lim_{s \to 0} \frac{f_i((x + v_{k-1} + t \cdot e_k) + s \cdot e_k) - f_i(x + v_{k-1} + t \cdot e_k)}{s} \\ &= (D_k f_i)(x + v_{k-1} + t \cdot e_k) \end{aligned}$$

for any $t \in [0, 1]$, so g is differentiable on $[0, 1]$ and thus continuous on $[0, 1]$. By the mean value theorem, there then exists some $\theta_k \in (0, h_k)$ such that

$$f_i(x + v_k) - f_i(x + v_{k-1}) = g(h_k) - g(0) = (D_k f_i)(x + v_{k-1} + \theta_k \cdot e_k) \cdot h_k.$$

If $h_k < 0$, we can construct $g : [h_k, 0] \to \mathbb{R}$ and find $\theta_k \in (h_k, 0)$ satisfying the above equation in the same manner. Finally, if $h_k = 0$, then we can put $\theta_k = 0$ and the above equation will still be satisfied. Since

$$x + v_{k-1} + \theta_k \cdot e_k \in B_d(x, \delta),$$

from what we derived earlier

$$|(D_k f_i)(x+v_{k-1}+\theta_k \cdot e_k) - (D_k f_i)(x)| < \frac{\epsilon}{nm}.$$

This holds for any $1 \le k \le n$, so based on the telescoping sum

$$f_i(x+h) - f_i(x) = \sum_{k=1}^{n} \left(f_i(x+v_k) - f_i(x+v_{k-1})\right),$$

we have

$$\begin{aligned}\left|f_i(x+h) - f_i(x) - \sum_{k=1}^{n}(D_k f_i)(x) \cdot h_k\right| &\le \sum_{k=1}^{n} |f_i(x+v_k) - f_i(x+v_{k-1}) - (D_k f_i)(x) \cdot h_k| \\ &\le \sum_{i=1}^{n} |(D_k f_i)(x+v_{k-1}+\theta_k \cdot e_k) - (D_k f_i(x)| \cdot |h_k| \\ &\le \frac{\epsilon}{m} \cdot |h|.\end{aligned}$$

This in turn holds for any $1 \le i \le m$, so we have

$$\begin{aligned}|f(x+h) - f(x) - Ah| &\le \sum_{i=1}^{m} \left|f_i(x+h) - f_i(x) - \sum_{k=1}^{n}(D_k f_i)(x) \cdot h_k\right| \\ &\le \epsilon \cdot |h|.\end{aligned}$$

In other words, for any non-zero $h \in \mathbb{R}^n$ with $|h| < \delta$,

$$\frac{|f(x+h) - f(x) - Ah|}{|h|} \le \epsilon.$$

Such a $\delta > 0$ exists for any $\epsilon > 0$, so by definition f is differentiable at x with derivative equal to A, the Jacobian of f at x. Since each entry of the mapping $f' : E \to \mathbb{R}^{m \times n}$, being a partial derivative, is continuous on E, f' is itself continuous with respect to the trace norm. Therefore, $f \in \mathcal{C}^1(E)$.

Q.E.D.

1.2.4 Higher Order Derivatives

Let E be an open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}$ a real valued function on E. In what came before we introduced the concepts of the partial derivatives of f, denoted $D_1 f, \cdots, D_n f : E \to \mathbb{R}$. The preceding theorem showed, using the mean value theorem, that the function f is continuously differentiable if and only if these partial derivatives exist and are continuous. Using this characterization of continuous differentiability, we can define a new class of twice continuously differentiable functions as functions $f : E \to \mathbb{R}$ such that each partial derivative $D_j f : E \to \mathbb{R}$ is partially differentiable with continuous partial derivatives. These class of functions is denoted $\mathcal{C}^2(E)$, and the partial derivatives of $D_j f$ are denoted

$$D_{ij} f = \frac{\partial}{\partial x_i}(D_j f) = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

for any $1 \leq i, j \leq n$. We usually collect these second order partial derivatives into the Hessian $H : E \to \mathbb{R}^{n \times n}$ defined as

$$H(x) = \begin{pmatrix} (D_{11} f)(x) & \cdots & (D_{1n} f)(x) \\ \vdots & \ddots & \vdots \\ (D_{n1} f)(x) & \cdots & (D_{nn} f)(x) \end{pmatrix}$$

for any $x \in E$.

Of course, there is no reason to stop at twice continuous differentiability. In general, for any $k \geq 2$, we say that $f : E \to \mathbb{R}$ is continuously differentiable k times, or in $\mathcal{C}^k(E)$, if its partial derivatives $D_1 f, \cdots, D_k f : E \to \mathbb{R}$ are continuously differentiable $k-1$ times, or $D_1 f, \cdots, D_k f \in \mathcal{C}^{k-1}(E)$. In this case, there exist continuous functions $D_{i_1, \cdots, i_k} f : E \to \mathbb{R}$ defined as

$$(D_{i_1, \cdots, i_k} f)(x) = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} f(x) = \frac{\partial^k f(x)}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

for any $x \in E$ and $1 \leq i_1, \cdots, i_k \leq n$.

Usually, we cannot interchange the order of partial differentiation. That is, it is generally not the case that

$$D_{ij} f = D_{ji} f$$

for any $1 \leq i \neq j \leq n$, given that the partial derivatives exist for some $f : E \to \mathbb{R}$. However, this does hold given that the function f is continuously differentiable, or that the partial derivatives above are continuous. We prove this result below:

Theorem 1.11 (Young's Theorem)
Let E be an open set in $\mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ a function on E. Suppose that f is twice continuously differentiable, that is, $f \in \mathcal{C}^2(E)$. Then, for any $1 \leq i, j \leq n$ and $x \in E$,

$$(D_{ij} f)(x) = (D_{ji} f)(x).$$

Proof) Choose any $1 \leq i \neq j \leq n$ and $x \in E$. By assumption, $D_{ji}f$ is continuous at x, so that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|(D_{ji}f)(x) - (D_{ji}f)(y)| < \epsilon$$

for any $y \in \mathbb{R}^n$ such that $|x - y| < \delta$. Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and choose some $h, k \neq 0$ such that $|h|, |k| < \frac{\delta}{2}$, and define

$$\Delta(h,k) = f(x + h \cdot e_j + k \cdot e_i) - f(x + h \cdot e_j) - f(x + k \cdot e_i) + f(x).$$

Then, by applying the mean value theorem in a similar manner to what we did in theorem 1.10, there exists a θ between 0 and k such that

$$\begin{aligned}&[f(x + h \cdot e_j + k \cdot e_i) - f(x + h \cdot e_j)] - [f(x + k \cdot e_i) + f(x)] \\ &\qquad = k \cdot [(D_i f)(x + h \cdot e_j + \theta \cdot e_i) - (D_i f)(x + \theta \cdot e_i)],\end{aligned}$$

where we first see the expression on the left hand side as a univariate function with respect to the coefficient of e_i. Subsequently, the mean value theorem tells us once again that there exists some t between 0 and h such that

$$(D_i f)(x + h \cdot e_j + \theta \cdot e_i) - (D_i f)(x + \theta \cdot e_i) = h \cdot (D_{ji} f)(x + t \cdot e_j + \theta \cdot e_i),$$

this time viewing the expression on the left as a univariate function with respect to the coefficient of e_j. Putting these results together, we can see that

$$\Delta(h,k) = hk \cdot (D_{ji} f)(x + t \cdot e_j + \theta \cdot e_i)$$

for some t between 0 and h and θ between 0 and k. Since

$$|(x + t \cdot e_j + \theta \cdot e_i) - x| \leq |t| + |\theta| \leq |h| + |k| < \delta,$$

by our initial continuity result we have

$$\left| \frac{\Delta(h,k)}{hk} - (D_{ji} f)(x) \right| = |(D_{ji} f)(x) - (D_{ji} f)(x + t \cdot e_j + \theta \cdot e_i)| < \epsilon.$$

Since this holds for any non-zero h such that $|h| < \frac{\delta}{2}$, and

$$\begin{aligned}\lim_{h \to 0} \frac{\Delta(h,k)}{hk} &= \frac{1}{k} \left[\lim_{h \to 0} \frac{f(x + h \cdot e_j + k \cdot e_i) - f(x + k \cdot e_i)}{h} - \lim_{h \to 0} \frac{f(x + h \cdot e_j) - f(x)}{h} \right] \\ &= \frac{(D_j f)(x + k \cdot e_i) - (D_j f)(x)}{k},\end{aligned}$$

taking $h \to 0$ on both sides of the above inequality shows us that

$$\left| \frac{(D_j f)(x + k \cdot e_i) - (D_j f)(x)}{k} - (D_{ji} f)(x) \right| \le \epsilon.$$

This in turn holds for any $k \neq 0$ such that $|k| < \frac{\delta}{2}$, so taking $k \to 0$ on both sides shows us that

$$|(D_{ij} f)(x) - (D_{ji} f)(x)| \le \epsilon.$$

Finally, our choice of $\epsilon > 0$ was arbitrary, so it must be the case that $(D_{ij} f)(x) = (D_{ji} f)(x)$.

Q.E.D.

In light of Young's theorem, we can see that, given any $f \in \mathcal{C}^2(E)$, the Hessian $H : E \to \mathbb{R}^{n \times n}$ is symmetric matrix valued. This means that each $H(x)$ can be orthogonally diagonalized, among other useful properties. Furthermore, the results above for second order partial derivatives can be extended to partial derivatives of any order, since they can always be viewed as functions obtained by repeatedly taking second order partial derivatives.

Continuous differentiation us especially useful in the case of univariate functions. Let E be an open subset of the real line, and $f : E \to \mathbb{R}$ a real-valued function. By definition, for any $k \ge 2$, f is kth order continuously differentiable if $f' : E \to \mathbb{R}$ is $k-1$th order continuously differentiable; the kth order derivative of f is denoted

$$f^{(k)} = \frac{d}{dx} f^{(k-1)},$$

where we adopt the convention that $f^{(0)} = f$. Since (total) derivatives and partial derivatives coincide for univariate functions, $f \in \mathcal{C}^k(E)$ if and only if $f^{(1)}, \cdots, f^{(k)}$ all exist and are continuous on E.

Higher order derivatives of univariate functions appear most often when dealing with Taylor's theorem, a higher order generalization of the mean value theorem. It states that any kth order continuously differentiable function on the real line can be approximated by a kth order polynomial, where the remainder converges to 0 exponentially fast. We prove the theorem once we introduce the fundamental theorem of calculus, which allows us to move flexibly between differentiation and integration.

1.3 Differentiation and Integration

In chapter 4, we introduced and studied some properties of the Lebesgue integral on euclidean space. However, we did not study how to actually evaluate these integrals aside from the method of approximation via simple functions, as in the definition of the abstract integral. The fundamental theorem of calculus, which relates Lebesgue integration on the real line to differentiation of functions on the real line, furnishes a simple and straightforward way to evaluate integrals using derivatives.

1.3.1 The Fundamental Theorem of Calculus

There are two version of the FTC, both of which we present below. They can actually be viewed as applications of the mean value theorem, which once again testifies to its importance.

Theorem 1.12 (Fundamental Theorem of Calculus)
Let $\mathcal{L}$ be the collection of all Lebesgue measurable subsets of $\mathbb{R}$, and λ the Lebesgue measure on the real line. Then, the following hold true:

i) **(First FTC)**
Let $f : (a,b) \to \mathbb{R}$ be a Lebesgue measurable function that is integrable with respect to the Lebesgue measure, where (a,b) is allowed to be the entire real line. Define the antiderivative $F : (a,b) \to \mathbb{R}$ of f as

$$F(x) = \int_a^x f(t)dt := \int_{\mathbb{R}} \left(f \cdot I_{(a,x)}\right) d\lambda$$

for any $x \in (a,b)$. If f is continuous at some $x \in (a,b)$, then F is differentiable at x with derivative equal to $f(x)$.

In particular, if f is continuous on (a,b), then F is continuously differentiable on (a,b).

ii) **(Second FTC)**
Let $F : (a,b) \to \mathbb{R}$ be a function that is continuously differentiable on the interval (a,b), which is allowed to be the entire real line, with derivative $f : (a,b) \to \mathbb{R}$. Then, for any distinct $x, y \in (a,b)$,

$$F(y) - F(x) = \int_x^y f(t)dt.$$

Proof) i) Let f and F be defined as in the claim of the theorem. Suppose f is continuous at $x \in (a,b)$. Then, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $(x-\delta, x+\delta) \subset (a,b)$

and

$$|f(x) - f(y)| < \epsilon$$

for any $y \in \mathbb{R}$ such that $|x - y| < \delta$. Now choose any $h \in \mathbb{R}$ such that $|h| < \delta$. If $h > 0$, then

$$\begin{aligned}
|F(x+h) - F(x) - f(x)h| &= \left| \int_{\mathbb{R}} \left(f \cdot I_{(a,x+h)} \right) d\lambda - \int_{\mathbb{R}} \left(f \cdot I_{(a,x)} \right) d\lambda - \int_{\mathbb{R}} \left(f(x) \cdot I_{[x,x+h)} \right) d\lambda \right| \\
&= \left| \int_x^{x+h} (f(t) - f(x))\, dt \right| \\
&\leq \int_x^{x+h} |f(t) - f(x)| dt \\
&\leq \lambda([x, x+h)) \cdot \epsilon = \epsilon \cdot h,
\end{aligned}$$

where the last inequality follows because

$$|f(x) - f(t)| < \epsilon$$

for any $t \in (x, x+h) \subset (x, x+\delta)$.

Likewise, if $h < 0$, then

$$\begin{aligned}
|F(x+h) - F(x) - f(x)h| &= \left| \int_{\mathbb{R}} \left(f \cdot I_{(a,x+h)} \right) d\lambda - \int_{\mathbb{R}} \left(f \cdot I_{(a,x)} \right) d\lambda + \int_{\mathbb{R}} \left(f(x) \cdot I_{[x+h,x)} \right) d\lambda \right| \\
&= \left| \int_{x+h}^{x} (f(x) - f(t))\, dt \right| \\
&\leq \int_{x+h}^{x} |f(t) - f(x)| dt \\
&\leq \lambda([x+h, x)) \cdot \epsilon = \epsilon \cdot |h|,
\end{aligned}$$

where the last inequality follows for the same reason as above.

Thus, in any case,

$$\frac{|F(x+h) - F(x) - f(x)h|}{|h|} \leq \epsilon$$

for any $0 < |h| < \delta$. Such a $\delta > 0$ exists for any $\epsilon > 0$, so by definition F is differentiable at x with derivative equal to $f(x)$.

ii) Let $F : (a,b) \to \mathbb{R}$ be a continuously differentiable function on (a,b) with derivative $f : (a,b) \to \mathbb{R}$. f is a continuous function and thus Lebesgue measurable. Choose some $x, y \in (a,b)$, and assume initially that $x < y$, so that $[x,y] \subset (a,b)$. f is continuous on the compact interval $[x,y]$, so by the extreme value theorem, it is bounded on this interval. This, together with the fact that the Lebesgue measure is finite on

$[x,y]$, implies that f is Lebesgue integrable on $[x,y]$, or equivalently, the function $f \cdot I_{[x,y]}$ is Lebesgue integrable.

We can therefore define the antiderivative $G : [x,y] \to \mathbb{R}$ as

$$G(z) = \int_x^z f(t)dt$$

for any $z \in [x,y]$. For any $z \in (x,y)$, the continuity of f at z and the first FTC imply that $G'(z) = f(z)$. It follows that $F'(z) = G'(z)$ for any $z \in [x,y]$. Furthermore, we can see that G is continuous even at the endpoints x and y because

$$G(0) = 0 = \lim_{z \downarrow x} \int_x^z f(t)dt$$

by the dominated convergence theorem (each $f \cdot I_{(x,z)}$ is dominated by the Lebesgue integrable function f on (a,b)), and similarly for $G(1)$.

Defining $H = F - G$ on $[x,y]$, H is continuous on $[x,y]$ and differentiable on (x,y). Therefore, by the mean value theorem, there exists a $z \in (x,y)$ such that

$$H(y) - H(x) = H'(z)(y-x) = \left(F'(z) - G'(z)\right)(y-x) = 0.$$

Since $H(y) = F(y) - G(y)$ and $H(x) = F(x) - G(x)$, it follows that

$$F(y) - F(x) = G(y) - G(x).$$

By the definition of G as the antiderivative of f, we now have

$$F(y) - F(x) = G(y) - G(x) = \int_a^y f(t)dt - \int_a^x f(t)dt = \int_x^y f(t)dt.$$

If, on the other hand, $y < x$, then the same process with x and y interchanged tells us that

$$F(x) - F(y) = \int_y^x f(t)dt.$$

Then, multiplying both sides by -1 yields the desired result.

Q.E.D.

Corollary to Theorem 1.12 (Integration by Parts)

For any continuously differentiable functions $f,g : (a,b) \to \mathbb{R}$ on the interval (a,b), which is allowed to be the entire real line, for any distinct $x, y \in (a,b)$, we have

$$\int_x^y f'(t)g(t)dt = f(y)g(y) - f(x)g(x) - \int_x^y f(t)g'(t)dt.$$

Proof) This follows easily from the second FTC. Define $F : (a,b) \to \mathbb{R}$ as

$$F(x) = f(x)g(x)$$

for any $x \in (a,b)$. Then, by the product rule of differentiation,

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

for any $x \in (a,b)$, where F' is a continuous function because of the continuity of f, g and their derivatives. It follows from the second FTC that, for any distinct $x, y \in (a,b)$,

$$f(y)g(y) - f(x)g(x) = F(y) - F(x) = \int_x^y F'(t)dt = \int_x^y f'(t)g(t)dt + \int_x^y f(t)g'(t)dt$$

by the linearity of integration. Note that each integrand on the right hand side is integrable due to the fact that they are continuous functions on the compact interval $[x, y]$ (or $[y, x]$) and thus bounded on this interval by the extreme value theorem, along with the fact that the Lebesgue measure is finite on compact intervals.

Q.E.D.

The assumptions of the second FTC can be weakened to allow for a non-continuously differentiable F, but we omit it here for the sake of simplicity.

1.3.2 Taylor's Theorem

The integration of parts formula above is especially important, since it allows us to prove Taylor's theorem, which we state below:

Theorem 1.13 (Taylor's Theorem: Lagrange Remainder)
Let $f : (a,b) \to \mathbb{R}$ be a function that is $k+1$th order continuously differentiable for some $k \in \mathbb{N}$. Then, for any distinct $x, y \in (a,b)$,

$$f(y) = \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!}(y-x)^i + R_k(y,x),$$

where

$$R_k(y,x) = \int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt.$$

This is called the Lagrange form of the remainder.

Proof) Note first that the integral involved with the formulation of the remainder term $R_k(y,x)$ is well-defined. This is because the mapping $t \mapsto f^{(k+1)}(t)(y-t)^k$ is continuous on the compact interval $[x,y]$ (or $[y,x]$ if $y<x$), so that it is bounded by the extreme value theorem. The Lebesgue measure is finite on $[x,y]$, so this mapping is Lebesgue integrable on $[x,y]$.

We can now proceed by induction on the order k of continuous differentiability to show that the theorem holds. Fix any distinct $x,y \in (a,b)$. If $k=0$, then f is continuously differentiable once, and the result follows immediately from the second FTC:

$$f(y) = f(x) + \underbrace{\int_x^y f(t)dt}_{R_0(y,x)}.$$

Now suppose that the theorem holds for some $k \geq 0$, and choose some $f \in \mathcal{C}^{(k+2)}(E)$. Since any $k+2$th order continuously differentiable function is also $k+1$th order continuously differentiable, by the inductive hypothesis we have

$$f(y) = \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!}(y-x)^i + \int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt.$$

The remainder $R_{k+1}(y,x)$ is defined as

$$\begin{aligned}
R_{k+1}(y,x) &= f(y) - \sum_{i=0}^{k+1} \frac{f^{(i)}(x)}{i!}(y-x)^i \\
&= \int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt - \frac{f^{(k+1)}(x)}{(k+1)!}(y-x)^{k+1}.
\end{aligned}$$

Define the functions $F:(a,b) \to \mathbb{R}$ and $G:(a,b) \to \mathbb{R}$ as

$$F(t) = f^{(k+1)}(t) \quad \text{and} \quad G(t) = \frac{(y-t)^{k+1}}{(k+1)!}$$

for any $t \in (a,b)$. Then, F and G are both continuously differentiable functions on (a,b), so that, by the integration by parts formula,

$$F(y)G(y) - F(x)G(x) - \int_x^y F(t)G'(t)dt = \int_x^y F'(t)G(t)dt.$$

Using the definitions of F and G, this equation basically tells us that

$$R_{k+1}(y,x) = -\frac{f^{(k+1)}(x)}{(k+1)!}(y-x)^{k+1} + \int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt$$

$$= \int_x^y \frac{f^{(k+2)}(t)}{(k+1)!}(y-t)^{k+1} dt.$$

Therefore,

$$f(y) = \sum_{i=0}^{k+1} \frac{f^{(i)}(x)}{i!}(y-x)^i + \underbrace{\int_x^y \frac{f^{(k+2)}(t)}{(k+1)!}(y-t)^{k+1} dt}_{R_{k+1}(y,x)},$$

and the general result follows by induction.

Q.E.D.

Using the basic formula for Taylor's theorem, we can derive alternate forms of the remainder $R_k(y,x)$. Particularly useful is the Peano form of the remainder, which can be used to formulate stochastic version of the theorem, among other applications.

Theorem 1.14 (Taylor's Theorem: Peano Remainder)
Let $f:(a,b) \to \mathbb{R}$ be a function that is $k+1$th order continuously differentiable for some $k \in \mathbb{N}$. Then, for any distinct $x, y \in (a,b)$,

$$f(y) = \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!}(y-x)^i + R_k(y,x),$$

where

$$R_k(y,x) = \frac{f^{(k+1)}(x_0)}{(k+1)!}(y-x)^{k+1}$$

for some convex combination x_0 of x and y. This is called the Peano form of the remainder.

Proof) For any $k+1$th order continuously differentiable function $f:(a,b) \to \mathbb{R}$, Taylor's theorem with the Lagrange remainder tells us that, for any distinct $x, y \in (a,b)$,

$$f(y) = \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!}(y-x)^i + \int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt.$$

We need only show that there exists a convex combination x_0 of x and y such that

$$\int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt = \frac{f^{(k+1)}(x_0)}{(k+1)!}(y-x)^{k+1}$$

to complete the proof. Letting $x < y$ without loss of generality, we start by noting that $f^{(k+1)}$ is a continuous function on the compact interval $[x,y]$. As such, by the extreme value theorem, there exist $-\infty < m < M < +\infty$ such that

$$m \leq f^{(k+1)} \leq M$$

on the interval $[x,y]$. It follows that

$$m\frac{(y-t)^k}{k!} \leq \frac{f^{(k+1)}(t)}{k!}(y-t)^k \leq M\frac{(y-t)^k}{k!}$$

for any $t \in [x,y]$, and integrating both sides with respect to the Lebesgue measure over $[x,y]$ shows us that

$$m \cdot \frac{(y-x)^{k+1}}{(k+1)!} \leq \int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt \leq M \cdot \frac{(y-x)^{k+1}}{(k+1)!},$$

since $\int_x^y \frac{(y-t)^k}{k!} dt = \frac{(y-x)^{k+1}}{(k+1)!}$. Now we consider two cases:

- **One of the inequalities holds as an equality**
  Suppose without loss of generality that

  $$\int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt = M \cdot \frac{(y-x)^{k+1}}{(k+1)!}.$$

  Then, since there exists an $x^* \in [x,y]$ such that $f^{(k+1)}(x^*) = M$ by the extreme value theorem, we have

  $$\int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt = f^{(k+1)}(x^*) \cdot \frac{(y-x)^{k+1}}{(k+1)!},$$

  which is our desired result.

- **Both inequalities hold strictly**
  In this case, defining

  $$c = \left(\int_x^y \frac{f^{(k+1)}(t)}{k!}(y-t)^k dt \right) \frac{(k+1)!}{(y-x)^{k+1}},$$

  we have $m < c < M$. By the extreme value theorem, there exist $x_*, x^* \in [x,y]$ such that

  $$f^{(k+1)}(x^*) = M \quad \text{and} \quad f^{(k+1)}(x_*) = m.$$

  $f^{(k+1)}$ is continuous on the compact interval with endpoints equal to x^* and x_*, and

$$f^{(k+1)}(x_*) = m < c < M = f^{(k+1)}(x^*),$$

so by the intermediate value theorem, there exists an x_0 between x_* and x^*, and therefore between x and y, such that

$$f^{(k+1)}(x_0) = c = \left(\int_x^y \frac{f^{(k+1)}(t)}{k!} (y-t)^k dt \right) \frac{(k+1)!}{(y-x)^{k+1}},$$

which is our desired result.

Q.E.D.

The Peano form of the remainder reveals that the remainder $R_k(y,x)$ converges to 0 exponentially fast, since

$$|R_k(y,x)| \leq \frac{\left| \max_{t \in [x,y]} f^{(k+1)}(t) \right|}{(k+1)!} |y-x|^{k+1}.$$

In other words, we can denote

$$R_k(y,x) = o(|y-x|^k)$$

as $y - x \to 0$, or

$$R_k(y,x) = O(|y-x|^{k+1})$$

in little and big O notation.

Finally, we can make use of the chain rule and easily prove the multivariate analogue of Taylor's theorem.

Theorem 1.15 (Multivariate version of Taylor's Theorem)
Let E be a convex open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}$ an $m+1$ times continuously differentiable function for some $m \in \mathbb{N}$. Then,

$$f(x+h) = \sum_{k=0}^{m} \frac{1}{k!} \left(\sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \frac{\partial^k f(x)}{\partial x_{j_1} \cdots \partial x_{j_k}} h_{j_1} \cdots h_{j_k} \right) + R_m(x,h),$$

where

$$R_m(x,h) = \frac{1}{(m+1)!}\sum_{j_1=1}^{n}\cdots\sum_{j_{m+1}=1}^{n}\frac{\partial^{m+1} f(x+t_0\cdot h)}{\partial x_{j_1}\cdots\partial x_{j_k}}h_{j_1}\cdots h_{j_{m+1}} \qquad \text{(Peano Form)}$$

$$= \frac{1}{m!}\sum_{j_1=1}^{n}\cdots\sum_{j_{m+1}=1}^{n}\int_0^1\left(\frac{\partial^{m+1} f(x+t\cdot h)}{\partial x_{j_1}\cdots\partial x_{j_k}}h_{j_1}\cdots h_{j_{m+1}}\right)dt \qquad \text{(Lagrange Form)}$$

for some $t_0 \in [0,1]$. In addition, $R_m(x,h) = o(|h|^m) = O(|h|^{m+1})$ as $h \to \mathbf{0}$.

Proof) Choose any $x \in E$, and $h \in \mathbb{R}^n$ such that $x+h \in E$. By the openness of E, there exists a $\delta > 0$ such that $x - \delta\cdot h, x+h+\delta\cdot h \in E$. Define $\gamma : (-\delta, 1+\delta) \to E$ as

$$\gamma(t) = x + t\cdot h$$

for any $t \in (-\delta, 1+\delta)$, where γ takes values in E because E is convex. Define $g = f\circ\gamma : (-\delta, 1+\delta) \to \mathbb{R}$, and recall that, by the chain rule,

$$g'(t) = \boldsymbol{\nabla} f(\gamma(t))'h = \sum_{i=1}^{n}(D_i f)(\gamma(t))h_i$$

for any $t \in (-\delta, 1+\delta)$. Suppose that, for some $1 \le k < m+1$, g is kth order continuously differentiable and

$$g^{(k)}(t) = \sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n}(D_{j_1\cdots j_k}f)(\gamma(t))\left(\prod_{i=1}^{k}h_{j_i}\right)$$

for any $t \in (-\delta, 1+\delta)$. Then, since f is $k+1$ times continuously differentiable, for any $1 \le j_1, \cdots, j_k \le n$,

$$\frac{\partial}{\partial x_i}(D_{j_1\cdots j_k}f) = D_{ij_1\cdots j_k}f$$

exists and is continuous for any $1 \le i \le n$. It follows from the chain rule again that

$$\frac{d}{dt}(D_{j_1\cdots j_k}f)(\gamma(t)) = \sum_{i=1}^{n}(D_{ij_1\cdots j_k}f)(\gamma(t))\cdot h_i,$$

so

$$g^{(k+1)}(t) = \frac{d}{dt}g^{(k)}(t) = \sum_{i=1}^{n}\sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n}(D_{ij_1\cdots j_k}f)(\gamma(t))\left(h_i \times h_{j_1}\times\cdots\times h_{j_k}\right).$$

By induction, g is $m+1$th order continuously differentiable and, for any $1 \le k \le m+1$,

$$g^{(k)}(t) = \sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n}(D_{j_1\cdots j_k}f)(\gamma(t))\left(\prod_{i=1}^{k}h_{j_i}\right)$$

for any $t \in (-\delta, 1+\delta)$.

Now, by Taylor's theorem for univariate functions, since $0, 1 \in (-\delta, 1+\delta)$,

$$g(1) = \sum_{k=0}^{m} \frac{g^{(k)}(0)}{k!} + R_m(1,0),$$

where

$$R_m(1,0) = \frac{g^{(m+1)}(t_0)}{(m+1)!} = \int_0^1 \frac{g^{(m+1)}(t)}{m!} dt$$

for some $t_0 \in [0,1]$. Substituting the values of g and its derivatives that we found above, we can now see that

$$f(x+h) = \sum_{k=0}^{m} \frac{1}{k!} \left(\sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} (D_{j_1 \cdots j_k} f)(x) \left(\prod_{i=1}^{k} h_{j_i} \right) \right) + R_m(x,h),$$

where

$$\begin{aligned} R_m(x,h) &= \frac{1}{(m+1)!} \sum_{j_1=1}^{n} \cdots \sum_{j_{m+1}=1}^{n} (D_{j_1 \cdots j_{m+1}} f)(x + t_0 \cdot h) \left(\prod_{i=1}^{m+1} h_{j_i} \right) \\ &= \frac{1}{m!} \sum_{j_1=1}^{n} \cdots \sum_{j_{m+1}=1}^{n} \int_0^1 (D_{j_1 \cdots j_{m+1}} f)(x + t \cdot h) \left(\prod_{i=1}^{m+1} h_{j_i} \right) dt. \end{aligned}$$

This completes the proof.

Q.E.D.

A special case of interest is when f is twice continuously differentiable. Then, the multivariate version of Taylor's theorem with the Peano remainder can be written as

$$f(x+h) = f(x) + \boldsymbol{\nabla} f(x)' h + \frac{1}{2} h' (\boldsymbol{\nabla}^2 f)(x + t_0 \cdot h) h,$$

for some $t_0 \in [0,1]$, where $\boldsymbol{\nabla}^2 f : E \to \mathbb{R}^{n \times n}$ is the Hessian of f. Furthermore, if f is thrice continuously differentiable, then

$$f(x+h) = f(x) + \boldsymbol{\nabla} f(x)' h + \frac{1}{2} h' (\boldsymbol{\nabla}^2 f)(x) h + R_2(x,h),$$

where $R_2(x,h) = o(|h|^2)$ as $h \to \mathbf{0}$, although we do not have as neat a form for the remainder as we did above.

1.3.3 Interchanging the Order of Differentiation and Integration

There is also another closely related result that allows us to interchange the order of integration and partial differentiation. This result, known as the Leibniz integral rule, can be proven through the use of the dominated convergence theorem, as we show below.

Theorem 1.16 (Leibniz Integral Rule)
Let $\mathcal{L}$ be the collection of all Lebesgue measurable subsets of $\mathbb{R}$, λ the Lebesgue measure on the real line, and $J \in \mathcal{L}$ an open subset of the real line. Let $(E, \mathcal{E}, \mu)$ be a measure space, and suppose that $f : E \times J \to \mathbb{R}$ is a function such that:

i) For any $t \in J$, the section $f_t : E \to \mathbb{R}$ is a μ-integrable function.

ii) For any $x \in E$, the section $f_x : J \to \mathbb{R}$ is differentiable with the derivative at $t \in J$ denoted $\frac{df(x,t)}{dt}$.

iii) There exists a μ-integrable non-negative function $\theta \in \mathcal{E}_+$ such that

$$\left|\frac{df(x,t)}{dt}\right| \leq \theta(x)$$

for any $t \in J$ and $x \in E$.

Then, the mapping $x \mapsto \frac{df(x,t)}{dt}$ is μ-integrable for each $t \in J$, while the mapping $t \mapsto \int_E f(x,t)d\mu(x)$ is differentiable on J. Furthermore, the derivative at each $t \in J$ is given as

$$\frac{d}{dt}\int_E f(x,t)d\mu(x) = \int_E \frac{df(x,t)}{dt}d\mu(x).$$

Proof) For any $t \in J$, define the function $g_t : E \to \mathbb{R}$ as

$$g_t(x) = \frac{df(x,t)}{dt}$$

for any $x \in E$. g_t is then a $\mathcal{E}$-measurable function because it is the limit of the sequence $\{g_{n,t}\}_{n \in N_+}$ of $\mathcal{E}$-measurable functions defined as

$$g_{n,t}(x) = \frac{f(x, t+1/n) - f(x,t)}{1/n}$$

for any $x \in E$ and $n \in N_+$. Furthermore, it is μ-integrable because

$$\int_E |g_t| d\mu \leq \int_E \theta d\mu < +\infty$$

by the monotinicty of integration and the fact that g_t is dominated by the μ-integrable non-negative function θ.

Now define the function $G : J \to \mathbb{R}$ as

$$G(t) = \int_E f(x,t)d\mu(x)$$

for any $t \in J$. The proof will be completed if we can show that G is differentiable for any $t \in J$ with derivative equal to $\int_E g_t d\mu$. To this end, fix some $t \in J$.

For any non-zero $h \in \mathbb{R}$ such that $t+h \in J$ and any convex combination of t and $t+h$ is contained in J (such a h exists because J is open),

$$\frac{G(t+h)-G(t)}{h} = \int_E \frac{f(x,t+h)-f(x,t)}{h} d\mu(x)$$

by the linearity of integration. Fixing $x \in E$, because the section f_x of f is differentiable on the closed interval with endpoints $t, t+h$ and thus continuous on that interval, by the mean value theorem there exists a t_0 between t and $t+h$ such that

$$f(x,t+h)-f(x,t) = \frac{df(x,t_0)}{dt} \cdot h = g_{t_0}(x) \cdot h.$$

g_{t_0} is dominated by θ, so by implication

$$\left| \frac{f(x,t+h)-f(x,t)}{h} \right| \leq |g_{t_0}(x)| \leq \theta(x)$$

This holds for any $x \in E$, and

$$\lim_{h \to 0} \frac{f(x,t+h)-f(x,t)}{h} = \frac{df(x,t)}{dt} = g_t(x)$$

for any $x \in E$ as well. Therefore, by the dominated convergence theorem,

$$\begin{aligned} \lim_{h \to 0} \frac{G(t+h)-G(t)}{h} &= \lim_{h \to 0} \int_E \frac{f(x,t+h)-f(x,t)}{h} d\mu(x) \\ &= \int_E \left(\lim_{h \to 0} \frac{f(x,t+h)-f(x,t)}{h} \right) d\mu(x) = \int_E g_t d\mu. \end{aligned}$$

This holds for any $t \in J$, so the proof is complete.

Q.E.D.

1.4 The Banach Fixed Point Theorem

Here we take a brief detour to prove and study some applications of the Banach fixed point theorem. This theorem will be instrumental in the proofs of the inverse and implicit function theorems, which will be proven in the next section, but it is also of independent interest, with applications ranging from dynamic optimization to differential equations.

Banach's fixed point theorem, or the contraction mapping principle, is about finding a fixed point of a certain class of functions called contraction mappings. Let (E,d) be a metric space. Then, a function $f : E \to E$ is said to be a contraction or contraction mapping if there exists a $0 \leq \beta < 1$ such that

$$d(f(x), f(y)) \leq \beta \cdot d(x,y)$$

for any $x, y \in E$. Note that any contraction is Lipschitz continuous with Lipschitz constant β. Geometrically, a contraction contracts as it goes through a sequence of points in E. This fact can be exploited to algorithmically find a unique fixed point of f, that is, a point $x \in E$ satisfying $f(x) = x$. This is the content of Banach's fixed point theorem.

Theorem 1.17 (Banach Fixed Point Theorem)
Let (E,d) be a metric space, and $\varphi : E \to E$ a contraction mapping. Then, φ admits at most one fixed point. If, in addition, (E,d) is complete, then φ has a unique fixed point.

Proof) Let $0 \leq \beta < 1$ be the Lipschitz constant associated with the contraction φ. To see uniqueness, suppose $x_1, x_2 \in E$ are fixed points of φ. Then,

$$d(x_1, x_2) = d(\varphi(x_1), \varphi(x_2)) \leq \beta \cdot d(x_1, x_2).$$

It follows that

$$(1-\beta) \cdot d(x_1, x_2) \leq 0,$$

and because $1 - \beta > 0$, this implies that $d(x_1, x_2) = 0$, or $x_1 = x_2$. Thus, φ admits at most one fixed point on E.

Now suppose (E,d) is a complete metric space. We can then construct an algorithm to find x^* starting from an arbitrary point on E. Choose any $x_0 \in E$. Then, define the sequence $\{x_n\}_{n \in N_+}$ as

$$x_n = \varphi(x_{n-1})$$

for any $n \in N_+$. It follows that, for any $n \in N_+$,

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq \beta \cdot d(x_n, x_{n-1}).$$

Continuing on with this recursion, we have

$$d(x_{n+1}, x_n) \leq \beta^n \cdot d(x_1, x_0).$$

Therefore, for any $m, n \in N_+$ such that $n < m$,

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1} x_i) \leq \left(\sum_{i=n}^{m-1} \beta^i \right) d(x_1, x_0).$$

The series $\sum_{k=1}^{\infty} \beta^k$ converges to $\frac{\beta}{1-\beta}$ because $|\beta| < 1$, so it follows from the Cauchy criterion for the convergence of series that

$$\lim_{m,n \to \infty} d(x_m, x_n) = 0.$$

This shows us that $\{x_n\}_{n \in N_+}$ is a Cauchy sequence in (E, d), and by the completeness of this metric space, there exists an $x^* \in E$ such that

$$\lim_{n \to \infty} x_n = x^*$$

in the metric d. It turns out that this limit x^* is the fixed point; by the continuity of φ,

$$\varphi(x^*) = \lim_{n \to \infty} \varphi(x_n) = \lim_{n \to \infty} x_{n+1} = x^*.$$

Note that we reached this fixed point x^* by starting from an arbitrary point in E and then recursively obtaining function values under φ. This is what is meant by a contraction "contracting" to a single point.

Q.E.D.

In practice, we often want to find contraction mappings on some complete metric space to apply the fixed point theorem. The following theorem furnishes some simple sufficient conditions a functional defined on bounded function spaces can satisfy to become a contraction mapping:

Theorem 1.18 (Blackwell Sufficient Conditions for a Contraction)
Let (E,d) be a metric space, $B(E,\mathbb{R})$ the set of all bounded functions taking values in $\mathbb{R}$, and $\|\cdot\|_{\mathcal{C}}$ the supremum norm on $B(E,\mathbb{R})$. Consider a mapping $T : B(E,\mathbb{R}) \to B(E,\mathbb{R})$. Suppose

i) **(Monotonicity)**
$T[f] \leq T[g]$ for any $f,g \in B(E,\mathbb{R})$ and $f \leq g$.

ii) **(Discounting)**
There exists a $\beta \in (0,1)$ such that

$$T[f+c] \leq T[f] + \beta \cdot c$$

for any $f \in B(E,\mathbb{R})$ and $c \geq 0$.

Then, T is a contraction mapping on $B(E,\mathbb{R})$.

Proof) Let $f,g \in B(E,\mathbb{R})$. Then,

$$f(x) = g(x) + (f(x) - g(x)) \leq g(x) + \|f-g\|_{\mathcal{C}}$$

for any $x \in E$, so we have $f \leq g + \|f-g\|_{\mathcal{C}}$. Similarly, we can see that $g \leq f + \|f-g\|_{\mathcal{C}}$. We can now see that

$$\begin{aligned} T[f] &\leq T[g + \|f-g\|_{\mathcal{C}}] && \text{(Monotonicity)} \\ &\leq T[g] + \beta \cdot \|f-g\|_{\mathcal{C}}, && \text{(Discounting)} \end{aligned}$$

and likewise,

$$T[g] \leq T[f] + \beta \cdot \|f-g\|_{\mathcal{C}}.$$

Together, these tell us that

$$|T[f] - T[g]| \leq \beta \cdot \|f-g\|_{\mathcal{C}},$$

and since $\beta \in (0,1)$, T is a contraction mapping on $B(E,\mathbb{R})$.

Q.E.D.

1.4.1 Application to Differential Equations

One major application of Banach's fixed point theorem is to the theory of ordinary differential equations. Let $\mathcal{U}\times\mathcal{I}$ be an open rectangle in the set $\mathbb{R}^n\times\mathbb{R}$, and $f:\mathcal{U}\times\mathcal{I}\to\mathbb{R}$ a function. We are interested in the first-order ODE

$$\begin{aligned} y_1'(t) &= f(y_1(t),\cdots,y_n(t),t)\\ &\vdots\\ y_n'(t) &= f(y_1(t),\cdots,y_n(t),t), \end{aligned}$$

written compactly as $y'(t)=f(y(t),t)$, where $y:\mathcal{I}\to\mathbb{R}^n$ is a differentiable function.

We are interested in the Initial Value Problem (IVP), which is the problem of finding, for any $(y_0,t_0)\in\mathcal{U}\times\mathcal{I}$, a $\delta>0$ and a differentiable function $y:(t_0-\delta,t_0+\delta)\to\mathbb{R}^n$ asuch that

- $y(t_0)=y_0$,
- $(y(t),t)\in\mathcal{U}\times\mathcal{I}$ for any $t\in(t_0-\delta,t_0+\delta)$, and
- $y'(t)=f(y(t),t)$ for any $t\in(t_0-\delta,t_0+\delta)$.

The next theorem furnishes sufficient conditions for the existence of such a (local) solution to the IVP:

Theorem 1.19 (Picard-Lindelof Theorem)
Let $\mathcal{U}\times\mathcal{I}$ be an open rectangle on $\mathbb{R}^n\times\mathbb{R}$, and $f:\mathcal{U}\times\mathcal{I}\to\mathbb{R}^n$ a function such that:

i) f is continuous on $\mathcal{U}\times\mathcal{I}$, and

ii) f is Lipschitz with respect to the first n arguments with Lipschitz constant independent of the last argument, that is, there exists an $L>0$ such that

$$|f(y,t)-f(z,t)|\le L|y-z|$$

for any $(y,t),(z,t)\in\mathcal{U}\times\mathcal{I}$.

Then, for any set of initial values $(y_0,t_0)\in\mathcal{U}\times\mathcal{I}$, there exists a $\delta>0$ and a unique solution $y:(t_0-\delta,t_0+\delta)\to\mathbb{R}^n$ to the IVP $y'=f(y,t)$.

Proof) Fix $(y_0,t_0)\in\mathcal{U}\times\mathcal{I}$. Since $\mathcal{U}$ and $\mathcal{I}$ are open subsets of $\mathbb{R}^n$ and $\mathbb{R}$, there exists an $r\in(0,1)$ such that the closed balls $\overline{B}_{d^n}(y_0,r)$ and $\overline{B}_d(t_0,r)$ are contained in $\mathcal{U}$ and $\mathcal{I}$. In particular,

$$R=\overline{B}_{d^n}(y_0,r)\times\overline{B}_d(t_0,r)$$

is a compact subset of $\mathbb{R}^{n+1}$, so that the continuity of f implies, together with the extreme value theorem, that there exists an $M > 0$ such that

$$|f(y,t)| \leq M$$

for any $(y,t) \in R$. Define

$$\delta = \min\left(\frac{r}{2L}, \frac{r}{M}, r\right) > 0.$$

We claim that this is exactly the window δ over t_0 that we are looking for.

Define $I_\delta = (t_0 - \delta, t_0 + \delta) \subset \mathcal{I}$, and let

$$E = \{y : I_\delta \to \mathbb{R}^n \mid y \text{ is continuous, } y(t) \in \overline{B}_d(t_0, r)\ \forall t \in I_\delta\}.$$

Note that E is a subset of the set $C_b(I_\delta, \mathbb{R}^n)$ of bounded and continuous functions from I_δ to $\mathbb{R}^n$. Letting $\|\cdot\|_{\mathcal{C}}$ be the supremum norm on $C_b(I_\delta, \mathbb{R}^n)$ and $d_{\mathcal{C}}$ the metric induced by this norm, recall that $(C_b(I_\delta, \mathbb{R}^n), \|\cdot\|_{\mathcal{C}})$ is a Banach space, that is, a complete normed vector space (for a proof, refer to chapter 6 of the probability theory text). Since E is a closed suset of $C_b(I_\delta, \mathbb{R}^n)$, the restriction of this function space to E is also a Banach space.

So far, we have seen that $(E, d_{\mathcal{C}})$ is a complete metric space. To apply Banach's fixed point theorem, we now must construct a contraction mapping on E. To this end, define $T : E \to C_b(I_\delta, \mathbb{R}^n)$ as

$$T[y](t) = y_0 + \int_{t_0}^{t} f(y(s), s)ds$$

for any $t \in I_\delta$ and $y \in E$. Note that the function $T[y]$ is well-defined because the continuity of f and y imply the continuity and therefore Lebesgue measurability of the mapping $s \mapsto f(y(s), s)$, f is bounded on R by the extreme value theorem, and the Lebesgue measure is finite on the interval I_δ. In addition, by the second fundamental theorem of calculus, $T[y]$ is continuously differentiable on I_δ with derivative equal to $f(y(t), t)$ for any $t \in I_\delta$. Thus, $T[y]$ is also continuous on I_δ. Finally, for any $t \in I_\delta$, since $y \in E$ and $(y(s), s) \in R$ for any $s \in I_\delta$, we have

$$|T[y](t) - y_0| \leq \left|\int_{t_0}^{t} f(y(s), s)ds\right| \leq M \cdot |t - t_0| < M \cdot \delta \leq r,$$

so that $T[y](t) \in \overline{B}_{d^n}(y_0, r)$. This shows us that $T[y] \in E$ for any $y \in E$, so that T is a mapping form E into E.

To show that T is a contraction mapping on E, choose any $\varphi, y \in E$ and note that, for any $t \in I_\delta$ such that $t \geq t_0$,

$$\begin{aligned}
|T[y](t) - T[\varphi](t)| &= \left| \int_{t_0}^{t} f(\varphi(s), s) ds - \int_{t_0}^{t} f(y(s), s) ds \right| \\
&\leq \int_{t_0}^{t} |f(\varphi(s), s) - f(y(s), s)| ds \\
&\leq L \cdot \int_{t_0}^{t} |\varphi(s) - y(s)| ds \\
&\leq L(t - t_0) \cdot \|\varphi - y\|_{\mathcal{C}} \\
&\leq L\delta \cdot \|\varphi - y\|_{\mathcal{C}} \leq \frac{1}{2} \cdot \|\varphi - y\|_{\mathcal{C}}.
\end{aligned}$$

For $t < t_0$, we can reach the same conclusion by switching t and t_0, so it follows that

$$\|T[y] - T[\varphi]\|_{\mathcal{C}} = \sup_{t \in I_\delta} |T[y](t) - T[\varphi](t)| \leq \frac{1}{2} \cdot \|\varphi - y\|_{\mathcal{C}}.$$

This holds for any $y, \varphi \in E$, so T is a contraction mapping on E.

Now, the Banach fixed point theorem tells us that there exists a unique $y^* \in E$ such that

$$y^* = T[y^*].$$

Notice how we constructed T. Differentiating both sides of the definition of T for some $y \in E$ by t shows that

$$y'(t) = f(y(t), t) \quad \text{for any } t \in I_\delta, \quad \text{and} \quad y(t_0) = y_0,$$

$$\text{if and only if} \quad T[y] = y \quad \text{on } I_\delta.$$

In other words, $y \in E$ is a solution to the IVP if and only if it is a fixed point for T. Therefore, the preceding result shows that we have found a unique solution $y^* \in E$ to our IVP.

Q.E.D.

1.5 The Inverse and Implicit Function Theorems

In this section we study two theorems of central importance to optimization theory. The inverse function theorem gives sufficient conditions for a restriction of a function to be invertible with a continuously differentiable inverse. The implicit function theorem, meanwhile, shows us how to restrict an implicit function so that each argument of the function can be expressed as a continuous differentiable function of the other arguments.

Theorem 1.20 (Inverse Function Theorem)
Let E be an open subset of $\mathbb{R}^n$ and $f: E \to \mathbb{R}^n$ a function that is $\mathcal{C}^1(E)$. Suppose $f(a) = b$ for some $a \in E$ and $b \in \mathbb{R}^n$, and that the derivative $f'(a)$ of f at a is invertible. Then, the following hold true:

i) There exist open sets $U \subset E$ and $V \subset \mathbb{R}^n$ such that $a \in U$, $b \in V$, and the restriction of f to U is a bijection onto V.

ii) There exists a continuously differentiable function $g: V \to U$ such that $f(g(y)) = y$ and $g(f(x)) = x$ for any $y \in V$ and $x \in U$, with

$$g'(y) = f'(g(y))^{-1}$$

for any $y \in V$.

Proof) We proceed in steps.

Step 1: Finding U

Denote $A = f'(a)$, and let $\epsilon = \frac{1}{2\|A^{-1}\|}$. By continuous differentiability, there exists an open ball U around a such that

$$\|f'(x) - A\| < \epsilon$$

for any $x \in U$. We define $f|_U$ as the restriction of f to U.

We first show that $f|_U$ is an injective mapping. For any $y \in \mathbb{R}^n$, define the function $\varphi_y : U \to \mathbb{R}^n$ as

$$\varphi_y(x) = x + A^{-1}(y - f(x))$$

for any $x \in U$. Note that φ_y is differentiable on U with derivative

$$\varphi_y'(x) = I_n - A^{-1}f'(x) = A^{-1}\left(A - f'(x)\right)$$

for any $x \in U$, so we can see that

$$\left|\varphi_y'(x)\right| \le \left\|A^{-1}\right\| \cdot \|A - f'(x)\| < \frac{1}{2}$$

for any $x \in U$ by what we established above. Since U is convex and open, by the mean value inequality for vector valued functions,

$$|\varphi_y(x_1) - \varphi_y(x_2)| \le \frac{1}{2}|x_1 - x_2|$$

for any $x_1, x_2 \in U$.

Step 2: Constructing the Function g

Suppose that $y = f|_U(x_1) = f|_U(x_2)$ for some $x_1, x_2 \in U$. Then,

$$|x_1 - x_2| = |\varphi_y(x_1) - \varphi_y(x_2)| \le \frac{1}{2}|x_1 - x_2|,$$

which implies $x_1 = x_2$. This shows us that f is injective on U.

Now define $V = f(U)$. Then, $f|_U : U \to V$ is both injective and surjective, so that it admits an inverse $g : V \to U$, which by definition satisfies $g(f(x)) = x$ and $f(g(y)) = y$ for any $x \in U$ and $y \in V$.

Step 3: Showing the Openness of V

To show that V is open, we can show that, for any point in V, there exists a neighborhood around that point that is contained in V. Choose some $y_0 \in V$. Since $y_0 \in f(U)$, there exists an $x_0 \in U$ such that $f(x_0) = y_0$. Since U is an open set, there exists an $r > 0$ such that the closed ball $\bar{B}(x_0, r)$ is contained in U.

Suppose $|y - y_0| < \epsilon \cdot r$. Then,

$$\begin{aligned}|\varphi_y(x_0) - x_0| &= \left|x_0 + A^{-1}(y - f(x_0)) - x_0\right| \\ &\le \left\|A^{-1}\right\| \cdot |y - f(x_0)| = \left\|A^{-1}\right\| \cdot |y - y_0| < \frac{1}{2}r.\end{aligned}$$

On the other hand, if $x \in \bar{B}(x_0, r) \subset U$, then

$$\begin{aligned}|\varphi_y(x) - x_0| &\le |\varphi_y(x) - \varphi_y(x_0)| + |\varphi_y(x_0) - x_0| \\ &< \frac{1}{2}|x - x_0| + \frac{1}{2}r \le r,\end{aligned}$$

since φ_y is a contraction on U, so it follows that $\varphi_y(x) \in \bar{B}(x_0, r)$. Thus, the restriction of φ_y to the closed set $\bar{B}(x_0, r)$ also maps into $\bar{B}(x_0, r)$, so that it becomes a contraction mapping. Since a closed subset of the metric space $\mathbb{R}^n$, which is complete under the euclidean metric, is also complete under the euclidean metric, by the Banach fixed point theorem there exists a unique $x^* \in \bar{B}(x_0, r) \subset U$ such that

$$\varphi_y(x^*) = x^*,$$

or equivalently, $y = f(x^*)$. What we have done is shown that, for any y in the open ball $B(y_0, \epsilon \cdot r)$, there exists an $x \in U$ such that $y = f(x) \subset f(U) = V$. Therefore, $B(y_0, \epsilon \cdot r) \subset V$, and since our choice of $y_0 \in V$ was arbitrary, this shows that V is an open subset of $\mathbb{R}^n$.

Step 4: Proving the Differentiability of g

It remains to show that g is continuously differentiable on V with derivative equal to the inverse of the corresponding derivative of f. Choose some $y \in V$ and non-zero $k \in \mathbb{R}^n$ such that $y + k \in V$. Defining $x = g(y)$ and $z(k) = g(y+k)$, by the definition of g we have $y = f(x)$ and $y + k = f(z(k))$. Letting $h(k) = z(k) - x \in \mathbb{R}^n$, we can write

$$y = f(x), \quad y + k = f(x + h(k)).$$

Note that $h(k) \neq \mathbf{0}$ if $k \neq \mathbf{0}$, since $h(k) = \mathbf{0}$ implies $z(k) = x$ and therefore that $y = f(x) = f(z(k)) = y + k$, a contradiction.

Now we have

$$\begin{aligned} |\varphi_y(x + h(k)) - \varphi_y(x)| = \left| h + A^{-1}(y - f(x + h(k))) \right| = \left| h(k) - A^{-1}k \right| \\ \leq \frac{1}{2}|h(k)| \end{aligned}$$

by the fact that φ_y is a contraction on U. In other words,

$$|h(k)| - \left\| A^{-1} \right\| \cdot |k| \leq |h(k)| - \left\| A^{-1}k \right\| \leq \left| h(k) - A^{-1}k \right| \leq \frac{1}{2}|h(k)|,$$

or, rearranging terms,

$$|h(k)| \leq 2\left\| A^{-1} \right\| \cdot |k| = \frac{1}{\epsilon} \cdot |k|.$$

It follows that

$$\lim_{k \to \mathbf{0}} h(k) = \mathbf{0}.$$

Recall that, if B is a matrix such that $\|A-B\| < \frac{1}{\|A^{-1}\|}$, then B is also invertible. Since

$$\|f'(x)-A\| < \epsilon = \frac{1}{2\|A^{-1}\|} < \frac{1}{\|A^{-1}\|},$$

we can see that $f'(x) = f'(g(y))$ is invertible; denote $B = f'(g(y))^{-1}$. Then,

$$g(y+k)-g(y)-Bk = h(k)-Bk,$$

and since $k = f(z(k))-y = f(x+h(k))-f(x)$, we can see that

$$\begin{aligned} g(y+k)-g(y)-Bk &= (BB^{-1})h(k) - B\left[f(x+h(k))-f(x)\right] \\ &= -B\left[f(x+h(k))-f(x)-B^{-1}h(k)\right], \end{aligned}$$

where $B^{-1} = f'(x)$. As such,

$$\begin{aligned} \frac{|g(y+k)-g(y)-Bk|}{|k|} &\leq \|B\| \cdot \frac{\left|f(x+h(k))-f(x)-B^{-1}h(k)\right|}{|k|} \\ &\leq \frac{\|B\|}{\epsilon}\left(\frac{\left|f(x+h(k))-f(x)-B^{-1}h(k)\right|}{|h(k)|}\right). \end{aligned}$$

Taking $k \to \mathbf{0}$ on both sides of the equation yields

$$\lim_{k\to\mathbf{0}} \frac{|g(y+k)-g(y)-Bk|}{|k|} = 0,$$

and as such

$$g'(y) = B = f'(g(y))^{-1}.$$

This holds for any $y \in V$, so g is differentiable on V.

Step 5: Proving Continuous Differentiability of g

To show that g is actually continuously differentiable, we need only note that $g' = G \circ f' \circ g : V \to \mathbb{R}^{n\times n}$, where $G : \Omega^o \to \Omega^o$ is the inverse matrix mapping. We saw that one of the properties of the trace norm is that G is continuous, f' is continuous by the assumption that $f \in \mathcal{C}^1(E)$, and g is continuous on V because it is differentiable at every point of V. Continuity is preserved across compositions, so it follows that g' is also a continuous mapping. This proves that $g \in \mathcal{C}^1(V)$.

Q.E.D.

The primary use of the inverse function theorem is to prove the implicit function theorem. Before moving onto the implicit function theorem, we first introduce some new notation.

We denote any point in $\mathbb{R}^{n+m}$ by the pair (x,y), where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$; we eschew the usual column vector representation for notational simplicity. Any matrix $A \in \mathbb{R}^{m\times(n+m)}$ can be partitioned as $A = (A_x, A_y)$, where $A_x \in \mathbb{R}^{m\times n}$ and $A_y \in \mathbb{R}^{m\times m}$. It follows that, for any $(x,y) \in \mathbb{R}^{n+m}$, we can write

$$A(x,y) = A_x \cdot x + A_y \cdot y \in \mathbb{R}^{m\times 1}.$$

Below, we state and prove the implicit function theorem:

Theorem 1.21 (Implicit Function Theorem)
Let E be an open subset of $\mathbb{R}^{n+m}$ and $f : E \to \mathbb{R}^m$ a function that is $\mathcal{C}^1(E)$. Suppose $f(a,b) = \mathbf{0}$ for some $(a,b) \in E$ and, letting

$$f'(a,b) = \begin{pmatrix} (D_1 f_1)(a,b) & \cdots & (D_n f_1)(a,b) & (D_{n+1} f_1)(a,b) & \cdots & (D_{n+m} f_1)(a,b) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (D_1 f_m)(a,b) & \cdots & (D_n f_m)(a,b) & (D_{n+1} f_m)(a,b) & \cdots & (D_{n+m} f_m)(a,b) \end{pmatrix}$$

with

$$f'(a,b)_x = \begin{pmatrix} (D_1 f_1)(a,b) & \cdots & (D_n f_1)(a,b) \\ \vdots & \ddots & \vdots \\ (D_1 f_m)(a,b) & \cdots & (D_n f_m)(a,b) \end{pmatrix}$$

$$f'(a,b)_y = \begin{pmatrix} (D_{n+1} f_1)(a,b) & \cdots & (D_{n+m} f_1)(a,b) \\ \vdots & \ddots & \vdots \\ (D_{n+1} f_m)(a,b) & \cdots & (D_{n+m} f_m)(a,b) \end{pmatrix},$$

assume that $f'(a,b)_y$ is invertible. Then, there exist open sets $U \subset \mathbb{R}^{n+m}$ containing (a,b), an open set $W \subset \mathbb{R}^n$ containing a, and a unique function $g : W \to \mathbb{R}^m$ such that

$$(x, g(x)) \in U \quad \text{and} \quad f(x, g(x)) = \mathbf{0}$$

for any $x \in W$. In addition, g is continuously differentiable on W and

$$\begin{aligned} g'(a) &= -\left(f'(a,b)_y\right)^{-1} f'(a,b)_x \\ &= -\begin{pmatrix} (D_{n+1} f_1)(a,b) & \cdots & (D_{n+m} f_1)(a,b) \\ \vdots & \ddots & \vdots \\ (D_{n+1} f_m)(a,b) & \cdots & (D_{n+m} f_m)(a,b) \end{pmatrix}^{-1} \begin{pmatrix} (D_1 f_1)(a,b) & \cdots & (D_n f_1)(a,b) \\ \vdots & \ddots & \vdots \\ (D_1 f_m)(a,b) & \cdots & (D_n f_m)(a,b) \end{pmatrix}. \end{aligned}$$

Proof) Define $A = f'(a,b)$, $A_x = f'(a,b)_x$ and $A_y = f'(a,b)_y$. The theorem follows almost immediately from the inverse function theorem. We once again proceed in steps.

Step 1: Transforming f into $F: E \to \mathbb{R}^{n+m}$

Define $F: E \to \mathbb{R}^{n+m}$ as

$$F(x,y) = (x, f(x,y))$$

for any $(x,y) \in E$. Then,

$$F'(x,y) = \begin{pmatrix} I_n & O_{n\times m} \\ \underbrace{f'(x,y)_x}_{m\times n} & \underbrace{f'(x,y)_y}_{m\times m} \end{pmatrix}$$

at each point on E. As such,

$$\|F'(x_1,y_1) - F'(x_2,y_2)\| = \left\| \begin{pmatrix} O_{n\times(n+m)} \\ f'(x_1,y_1) - f'(x_2,y_2) \end{pmatrix} \right\| = \|f'(x_1,y_1) - f'(x_2,y_2)\|$$

for any $(x_1,y_1),(x_2,y_2) \in E$; since f' is continuous by assumption, so is F', making $F \in \mathcal{C}^1(E)$. Finally,

$$\det\big(F'(a,b)\big) = \det \begin{pmatrix} I_n & O_{n\times m} \\ A_x & A_y \end{pmatrix} = \det(A_y) \neq 0$$

with $F(a,b) = (a, f(a,b)) = (a, \mathbf{0})$.

Step 2: Applying the Inverse Function Theorem to construct U and W

Applying the inverse function theorem to F, we can see that:

i) There exist open sets $U \subset E$ and $V \subset \mathbb{R}^{n+m}$ such that $(a,b) \in U$, $(a,\mathbf{0}) \in V$, and the restriction of F to U is a bijection onto V.

ii) There exists a continuously differentiable function $G: V \to U$ that is the inverse of F restricted to U, and

$$G'(x,y) = F'(G(x,y))^{-1}$$

for any $(x,y) \in V$.

We define W as

$$W = \{x \in \mathbb{R}^n \mid (x, \mathbf{0}) \in V\}.$$

For any $x \in \mathbb{R}^n$, $(x, \mathbf{0}) \in V$ and, by the openness of V, there exists an $\epsilon > 0$ such that $B_{n+m}((x, \mathbf{0}), \epsilon) \subset V$. Now consider the ball $B_n(x, \epsilon)$ around x. For any $z \in B_n(x, \epsilon)$,

$$|(x, \mathbf{0}) - (z, \mathbf{0})| = |x - z| < \epsilon,$$

so $(z, \mathbf{0}) \in B_{n+m}((x, \mathbf{0}), \epsilon) \subset V$. It follows that $B_n(x, \epsilon) \subset W$, and since we can find such an $\epsilon > 0$ for any $x \in W$, W is an open subset of $\mathbb{R}^n$.

Step 3: Finding the function g

Choose any $x \in W$. Then, $(x, \mathbf{0}) \in V$, and since F restricted to U is surjective onto V, there exists an $(z, y) \in U$ such that

$$F(z, y) = (z, f(z, y)) = (x, \mathbf{0}),$$

so that $z = x$ and $f(x, y) = \mathbf{0}$. In addition, F restricted to U is injective as well, so this $y \in \mathbb{R}^m$ is the unique vector such that $(x, y) \in U$ and $f(x, y) = \mathbf{0}$; denote $y = g_x$.

The preceding holds for any $x \in W$, so we can define the function $g : W \to \mathbb{R}^m$ as $g(x) = g_x$ for any $x \in W$. Then, from what we saw just now, g is the unique function such that, for any $x \in W$,

$$(x, g(x)) \in U \quad \text{and} \quad f(x, g(x)) = \mathbf{0}.$$

Step 4: Continuous Differentiability of g

It remains to prove the continuous differentiability of g. For any $x \in W$, since $F(x, g(x)) = (x, \mathbf{0})$, by design

$$G(x, \mathbf{0}) = (x, g(x)).$$

Let G_2 collect the lower m functions of G; then, for any $1 \leq i \leq m$, this tells us that

$$G_{n+i}(x, \mathbf{0}) = g_i(x),$$

so that, for any $t \neq 0$ and $1 \leq j \leq n$,

$$\frac{g_i(x+t\cdot e_j)-g(x)}{t} = \frac{G_{n+i}(x+t\cdot e_j,\mathbf{0})-G_{n+i}(x,\mathbf{0})}{t}.$$

Taking $t \to 0$ on both sides shows us that

$$(D_j g_i)(x) = (D_j G_{n+i})(x,\mathbf{0})$$

for any $x \in W$; the partial derivatives of g_i exist. Since G is continuously differentiable on V by the inverse function theorem, it has continuous partial derivatives, and as such $D_j g_i$ is continuous on W as well. This holds for any $1 \leq j \leq n$ and $1 \leq i \leq m$, so by the characterization of continuous differentiability, $g \in \mathcal{C}^1(W)$.

Step 5: Obtaining $g'(a)$

We can now easily obtain the derivative of g at a. The inverse function theorem tells us that

$$\begin{aligned} G'(a,\mathbf{0}) = F'(G(a,\mathbf{0}\ ))^{-1} &= F'(a,g(a))^{-1} \\ &= \begin{pmatrix} I_n & O_{n\times m} \\ A_x & A_y \end{pmatrix}^{-1} = \begin{pmatrix} I_n & O_{n\times m} \\ -A_y^{-1}A_x & I_m \end{pmatrix}. \end{aligned}$$

By the differentiabiltiy of g, the derivative of g at a is given as exactly the matrix

$$g'(a) = \begin{pmatrix} (D_1 g_1)(a) & \cdots & (D_n g_1)(a) \\ \vdots & \ddots & \vdots \\ (D_1 g_m)(a) & \cdots & (D_n g_m)(a) \end{pmatrix} = \begin{pmatrix} (D_1 G_{n+1})(a,\mathbf{0}) & \cdots & (D_n G_{n+1})(a,\mathbf{0}) \\ \vdots & \ddots & \vdots \\ (D_1 G_m)(a,\mathbf{0}) & \cdots & (D_n G_m)(a,\mathbf{0}) \end{pmatrix}.$$

This exactly the lower left $m \times n$ block of $G'(a,\mathbf{0})$, so we have

$$g'(a) = -A_y^{-1}A_x,$$

which is exactly what we claim.

Q.E.D.

Of special interest is the case where $m = 1$. Suppose E is a subset of $\mathbb{R}^{n+1}$, and let $f : E \to \mathbb{R}$ be continuously differentiable on E. Consider the level curve $L = \{(x,y) \in E \mid f(x,y) = 0\}$. For any point $(a,b) \in \mathbb{R}^{n+1}$ on the level curve L such that $\frac{\partial f}{\partial y}(a,b) \neq 0$, the implicit function theorem tells us that there exist an open set $U \subset \mathbb{R}^{n+1}$ around (a,b), an open set $W \subset \mathbb{R}^n$ around a, and a unique function $g : W \to \mathbb{R}$ such that

$$(x, g(x)) \in U \quad \text{and} \quad f(x, g(x)) = 0,$$

or more succinctly, $(x, g(x)) \in L \cap U$. Furthermore, it tells us that this g is continuously differentiable on W, and that

$$g'(a) = -\frac{1}{\frac{\partial f}{\partial y}(a,b)} \begin{pmatrix} \frac{\partial f}{\partial x_1}(a,b) & \cdots & \frac{\partial f}{\partial x_n}(a,b). \end{pmatrix}.$$

In other words, the gradient of g at a is given as

$$\boldsymbol{\nabla} g(a) = \begin{pmatrix} -\left(\frac{\partial f}{\partial x_1}(a,b)\right) / \left(\frac{\partial f}{\partial y}(a,b)\right) \\ \vdots \\ -\left(\frac{\partial f}{\partial x_n}(a,b)\right) / \left(\frac{\partial f}{\partial y}(a,b)\right) \end{pmatrix}.$$

Chapter 2

Convex Analysis

2.1 Separating Hyperplane Theorems

These class of theorems, used in proofs of duality in microeconomics, furnish sufficient conditions for two convex sets to be (strictly) separated.

First consider a finite-dimensional euclidean space $\mathbb{R}^n$. A hyperplane on $\mathbb{R}^n$ is a vector space of dimensional $n-1$; the following result characterizes hyperplanes on finite dimensional euclidean spaces:

Lemma 2.1 (Characterization of Hyperplanes)
A subset H of $\mathbb{R}^n$ is a hyperplane on H if and only if there exists a non-zero $v \in \mathbb{R}^n$ such that

$$H = \{x \in \mathbb{R}^n \mid v'x = 0\}.$$

Proof) Suppose H is a hyperplane on $\mathbb{R}^n$. Then, it has dimension $n-1$, so there exists a basis $\{v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^n$ of H that is orthogonal with respect to the standard inner product on $\mathbb{R}^n$. We can extend this basis to an orthogonal basis $\{v_1, \cdots, v_{n-1}, v\}$ of $\mathbb{R}^n$. Define

$$H' = \{x \in \mathbb{R}^n \mid v'x = 0\}.$$

Then, for any $x \in H$, because $x = \sum_{i=1}^{n-1} a_i v_i$ for some $a_1, \cdots, a_n \in \mathbb{R}$,

$$v'x = \sum_{i=1}^{n-1} a_i \left(v'v_i\right) = 0,$$

since v is orthogonal to $v_1, \cdots, v_{n-1}$. Thus, $x \in H'$ and we have $H \subset H'$. Likewise, if $x \in H'$, then $v'x = 0$, and because $\{v_1, \cdots, v_{n-1}, v\}$ is a basis of $\mathbb{R}^n$, there exist

$a_1, \cdots, a_{n-1}, a \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n-1} a_i v_i + av.$$

It follows that

$$v'x = a \cdot (v'v) = 0,$$

so it must be the case that $a = 0$ and thus $x \in H$. Therefore,

$$H = H' = \{x \in \mathbb{R}^n \mid v'x = 0\}.$$

Conversely, suppose that

$$H = \{x \in \mathbb{R}^n \mid v'x = 0\}$$

for some non-zero $v \in \mathbb{R}^n$. Then, defining $T : \mathbb{R}^n \to \mathbb{R}$ as

$$T(x) = v'x$$

for any $x \in \mathbb{R}^n$, T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$, and H is the null space of T. Since the range of T is $\mathbb{R}$ itself (v is non-zero), by the dimension theorem it follows that $rank(T) = 1$ and

$$rank(T) + nullity(T) = dim(\mathbb{R}^n) = n.$$

Thus, the nullity of T is $n-1$, and because the nullity of T is exactly the dimension of H, H is a hyperplane on $\mathbb{R}^n$.
Q.E.D.

For any non-zero $v \in \mathbb{R}^n$, define the hyperplane

$$H = \{x \in \mathbb{R}^n \mid v'x = 0\}.$$

For any $a \in \mathbb{R}$, there exists an $x_0 \in \mathbb{R}^n$ such that $v'x_0 = a$, and as such, the space

$$H' = \{x \in \mathbb{R}^n \mid v'x = a\}$$

is a translation $H + x_0$ of H. We call spaces like H', which are translations of hyperplanes, affine hyperplanes on $\mathbb{R}^n$.

We will be focusing on the separation of convex sets in euclidean spaces, so it will be useful to give below some properties of convex sets in $\mathbb{R}^n$.

Lemma 2.2 (Properties of Convex Sets in $\mathbb{R}^n$)
Let A, B be non-empty convex sets in $\mathbb{R}^n$. Then, the following hold true:

i) For any $a \in \mathbb{R}$, the linear combination of A and B defined as

$$C = aA + B = \{a \cdot x + y \in \mathbb{R}^n \mid x \in A, y \in B\}$$

is a non-empty convex subset of $\mathbb{R}^n$.

ii) The closure $\overline{A}$ of A is a convex subset of $\mathbb{R}^n$.

Proof) i) Choose any $\lambda \in [0,1]$ and $a \cdot x_1 + y_1, a \cdot x_2 + y_2 \in C$. Then,

$$\begin{aligned}\lambda \cdot (a \cdot x_1 + y_1) + (1-\lambda) \cdot (a \cdot x_2 + y_2) \\ = a\left(\lambda \cdot x_1 + (1-\lambda) \cdot x_2\right) + \left(\lambda \cdot y_1 + (1-\lambda) \cdot y_2\right).\end{aligned}$$

Since $\lambda \cdot x_1 + (1-\lambda) \cdot x_2 \in A$ and $\lambda \cdot y_1 + (1-\lambda) \cdot y_2 \in B$ by the convexity of A and B, we can see that

$$\lambda \cdot (a \cdot x_1 + y_1) + (1-\lambda) \cdot (a \cdot x_2 + y_2) \in C$$

as well. This shows that C is convex.

ii) Let $x, y \in \overline{A}$ and choose some $\lambda \in [0,1]$. There then exist sequences $\{x_k\}_{k \in N_+}$ and $\{y_k\}_{k \in N_+}$ of points in A that converge to x and y. For any $k \in N_+$, $\lambda \cdot x_k + (1-\lambda) \cdot y_k \in A$ by the convexity of A. Thus,

$$\lambda \cdot x + (1-\lambda) \cdot y = \lim_{k \to \infty} \left(\lambda \cdot x_k + (1-\lambda) \cdot y_k\right) \in \overline{A},$$

and $\overline{A}$ is a convex set.

Q.E.D.

2.1.1 Basic Separation on Hilbert Spaces

Separating hyperplanes tell us that, given two convex subsets of a finite-dimensional euclidean space, there exists an affine hyperplane that separates them. In what follows, we present a more generalized version for closed convex sets in Hilbert space.

We first state the basic separation theorem:

Theorem 2.3 (Basic Separation Theorem)
Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over the real field, and $\|\cdot\|$ the norm on H induced by the inner product $\langle \cdot, \cdot \rangle$. Let W be a nonempty closed convex subset of H, and $x \in H$ a point not contained in W.

Then, there exists a non-zero $v \in H$ and $c \in \mathbb{R}$ such that

$$\langle v, x \rangle > c > \langle v, y \rangle$$

for any $y \in A$.

Proof) By the Hilbert projection theorem, there exists a $y^* \in W$ such that

$$\|x - y*\| = \inf_{y \in W} \|x - y\|.$$

If $\|x - y^*\| = 0$, then $x = y^*$, which contradicts the fact that $x \notin W$, so it must be the case that $\|x - y^*\| > 0$.

Define $v = x - y^*$ and $c = \frac{\langle v, x \rangle - \langle v, y^* \rangle}{2} \in \mathbb{R}$. Since $\|v\| > 0$, v is non-zero, and we can immediately see that

$$0 < \|v\|^2 = \langle v, x - y^* \rangle = \langle v, x \rangle - \langle v, y^* \rangle,$$

so that we have $\langle v, x \rangle > c > \langle v, y^* \rangle$. It remains to show that the inequality holds for any $y \in A$.

For any $y \in A$, by the definition of y^*, we have

$$0 < \|x - y^*\| = \inf_{z \in W} \|x - z\| \leq \|x - y\|.$$

For any $k \in N_+$, define

$$z_k = \frac{k-1}{k} y^* + \frac{1}{k} y.$$

Since W is a convex set and $y^*, y \in W$, it follows that $z_k \in W$ as well. We now find that

$$\begin{aligned}\|v\|^2 = \|x - y^*\|^2 \le \|x - z_k\|^2 &= \langle x - z_k, x - z_k \rangle \\ &= \langle v + \frac{1}{k}(y^* - y), v + \frac{1}{k}(y^* - y) \rangle \\ &= \|v\|^2 + 2 \cdot \frac{1}{k} \langle v, y^* - y \rangle + \frac{1}{k^2} \|y^* - y\|^2.\end{aligned}$$

Therefore,

$$2 \cdot \frac{1}{k} \langle v, y^* - y \rangle + \frac{1}{k^2} \|y^* - y\|^2 \ge 0,$$

and multiplying both sides by k yields

$$2 \cdot \langle v, y^* - y \rangle + \frac{1}{k} \|y^* - y\|^2 \ge 0.$$

Taking $k \to \infty$ on both sides, we finally obtain the inequality

$$2 \cdot \langle v, y^* - y \rangle \ge 0,$$

or equivalently, $\langle v, y^* \rangle \ge \langle v, y \rangle$. Therefore, for any $y \in W$,

$$\langle v, x \rangle > c > \langle v, y^* \rangle \ge \langle v, y \rangle.$$

Q.E.D.

The basic separation theorem can be used to give a characterization of any closed convex set in a Hilbert space. Let $(H, \langle \cdot, \cdot \rangle)$ be the real Hilbert space given above, and W a closed convex subset of H. The support function $\mu_W : H \to (-\infty, +\infty]$ of W is defined as

$$\mu_W(x) = \sup_{y \in W} \langle x, y \rangle$$

for any $x \in H$; μ_W does not take $-\infty$ as a value because the set $\{\langle x, y \rangle \mid y \in W\}$ is a non-empty subset of $\mathbb{R}$ for any $x \in H$.

Define the set

$$W' = \{y \in H \mid \langle x, y \rangle \le \mu_W(x) \ \forall x \in H\}.$$

Then, $W \subset W'$, since if $y \in W$, then

$$\langle x, y \rangle \le \sup_{z \in W} \langle x, z \rangle = \mu_W(x)$$

for any $x \in H$ by definition. Conversely, suppose that $y \in W^c$. Then, by the basic separation theorem, there exists a non-zero $v \in H$ and a $c \in \mathbb{R}$ such that

$$\langle v, y\rangle > c > \langle v, z\rangle$$

for any $z \in W$. It follows that

$$\langle v, y\rangle > c \geq \sup_{z \in W} \langle v, z\rangle = \mu_W(v),$$

so that $y \notin W'$. It follows that $W^c \subset (W')^c$, and therefore

$$W = W' = \{y \in H \mid \langle x, y\rangle \leq \mu_W(x)\ \forall x \in H\}.$$

This is the dual representation of W. It can be seen from this representation that the support function μ_W contains all the information on W; given μ_W, we are able to recover W, and vice versa.

If $H = \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, then the support function for W would be defined as

$$\mu_W(x) = \sup_{y \in W} x'y,$$

and the dual representation of W would be

$$W = \{y \in \mathbb{R}^n \mid x'y \leq \mu_W(x)\ \forall x \in \mathbb{R}^n\}.$$

2.1.2 Farkas' Lemma

Another application of the basic separation theorem is in Farkas' lemma, which gives sufficient and necessary conditions for the solvability of a system of linear inequalities. The statement and proof are given below:

Theorem 2.4 (Farkas' Lemma)
Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then,

1. **Either**: $Ax = b$ has a solution $x \in \mathbb{R}^n_+$
2. **Or**: There exists a $p \in \mathbb{R}^m$ such that $A'p \geq \mathbf{0}$ and $p'b < 0$.

Proof) Necessity is simple. Suppose that there exists some $x \in \mathbb{R}^n_+$ such that $Ax = b$, and assume that there exists a $p \in \mathbb{R}^m$ such that $A'p \geq \mathbf{0}$ and $p'b < 0$. Then, pre-multiplication by p' yields

$$0 \leq p'Ax = p'b < 0,$$

where $p'Ax \geq 0$ because all the elements of $A'p$ and x are non-negative. This is a contradiction, so there cannot exist such a $p \in \mathbb{R}^m$.

Now suppose that the system $Ax = b$ has no solution on $\mathbb{R}^m_+$. Letting $v_1, \cdots, v_n \in \mathbb{R}^m$ be the columns of A, define the set

$$C = \{Ax \mid x \in \mathbb{R}^n_+\} = \Big\{\sum_{i=1}^{n} a_i \cdot v_i \mid a_1, \cdots, a_n \geq 0\Big\}.$$

In the terminology used in convex analysis, C is the conic hull of the vectors $v_1, \cdots, v_n$, that is, the set of all conic combinations of these vectors. It is clearly convex; for any $\lambda \in [0,1]$ and $Ax, Ay \in C$,

$$\lambda \cdot Ax + (1-\lambda) \cdot Ay = A(\lambda cdot x + (1-\lambda) \cdot y) \in C,$$

since $\mathbb{R}^n_+$ is convex. C is also closed; that a conic hull of a finite set is closed is a well-known result that is tedious to prove, to we simply assume it here. As such, C is a closed, convex and non-empty (it contains the zero vector) subset of $\mathbb{R}^m$ such that $b \notin C$.

$\mathbb{R}^m$ can be viewed as a Hilbert space under the standard inner product on $\mathbb{R}^m$, so the basic separation theorem tells us that there exists a non-zero $p \in \mathbb{R}^m$ and $c \in \mathbb{R}$ such that

$$p'b < c < p'Ax$$

for any $x \in \mathbb{R}^n_+$. We will now show that p satisfies the conditions of the theorem. Putting

$x = \mathbf{0}$ shows us that $p'b < c < 0$, which proves the second condition. For the first condition, define $y = A'p \in \mathbb{R}^n$ and suppose $y_i < 0$ for some $1 \leq i \leq n$. Then, letting $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$, there exists an $M > 0$ such that $M \cdot y_i < c < 0$. Therefore, putting $x = M \cdot e_i \in \mathbb{R}^n_+$ in the above inequality reveals that

$$c < p'Ax = y'x = M \cdot y'e_i = M \cdot y_i < c,$$

a contradiction. It follows that $y = A'p \in \mathbb{R}^n_+$, or equivalently, $A'p \geq \mathbf{0}$.

Q.E.D.

Farkas' lemma has many alternative formulations, some arguably more convenient to use than others; we state some of them below as a corollary.

Corlloray to Theorem 2.4 (Alternative Formluations of Farkas' Lemma)
Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following hold true:

i) Either $Ax \leq b$ admits a solution $x \in \mathbb{R}^n_+$, or there exists a $p \in \mathbb{R}^m_+$ such that $A'p \geq \mathbf{0}$ and $p'b < 0$.

ii) Either $Ax \leq b$ admits a solution $x \in \mathbb{R}^n$, or there exists a $p \in \mathbb{R}^m_+$ such that $A'p = \mathbf{0}$ and $p'b < 0$.

Proof) i) The statement that $Ax \leq b$ admits a solution $x \in \mathbb{R}^n_+$ is equivalent to claiming that there exists a solution $(x, z) \in \mathbb{R}^{n+m}_+$ to the equation

$$\begin{pmatrix} A & I_m \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = b.$$

By Farkas' lemma, there either exists a solution $x \in \mathbb{R}^n_+$ to $Ax \leq b$, or there exists a $p \in \mathbb{R}^m$ such that

$$\begin{pmatrix} A'p \\ p \end{pmatrix} \geq \mathbf{0}, \quad \text{and} \quad p'b < 0.$$

This shows us that the existence of a $p \in \mathbb{R}^m_+$ satisfying $A'p \geq \mathbf{0}$ and $p'b < 0$.

ii) The statement that $Ax \leq b$ admits a solution $x \in \mathbb{R}^n$ is equivalent to claiming that there exists a solution $(u, v) \in \mathbb{R}^{2n}_+$ to the equation

$$\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \leq b.$$

By i), there either exists a solution $x \in \mathbb{R}^n$ to $Ax \leq b$, or there exists a $p \in \mathbb{R}^m_+$ such that

$$\begin{pmatrix} A'p \\ -A'p \end{pmatrix} \geq \mathbf{0}, \quad \text{and} \quad p'b < 0.$$

This shows us that the existence of a $p \in \mathbb{R}^m_+$ such that $A'p = \mathbf{0}$ and $p'b < 0$.

Q.E.D.

Farkas' lemma can also be used to show prove the Fredholm alternative, which provides two mutually exclusive and exhaustive cases when solving a sysetem of linear equations.

Corlloray to Theorem 2.4 (Fredholm Alternative)
Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then,

1. **Either**: $Ax = b$ has a solution $x \in \mathbb{R}^n$
2. **Or**: There exists a $p \in \mathbb{R}^m$ such that $A'p = \mathbf{0}$ and $p'b \neq 0$.

Proof) The statement that $Ax = b$ has a solution $x \in \mathbb{R}^n$ is equivalent to claiming that there exists a solution $(u, v) \in \mathbb{R}^{2n}_+$ to the system

$$\begin{pmatrix} A & -A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = b.$$

Thus, by Farkas' lemma, either there exists a solution $x \in \mathbb{R}^n$ to $Ax = b$, or there exists a $p \in \mathbb{R}^m$ such that

$$\begin{pmatrix} A'p \\ -A'p \end{pmatrix} \geq \mathbf{0} \quad \text{and} \quad p'b < 0.$$

This implies the existence of a $p \in \mathbb{R}^m$ such that $A'p = \mathbf{0}$ and $p'b \neq 0$.

Q.E.D.

2.1.3 Separation Results on Euclidean Spaces

We can apply the basic separation theorem to derive separation results on euclidean space. We start with the strict separation theorem:

Theorem 2.5 (Strict Separation Theorem)
Let A, B be disjoint nonempty closed convex subsets of $\mathbb{R}^n$, and suppose at least one of them is compact. Then, there exists a non-zero $v \in \mathbb{R}^n$ and a $c \in \mathbb{R}$ such that

$$v'x > c > v'y$$

for any $x \in A$ and $y \in B$.

Proof) Suppose, without loss of generality, that A is compact. Define the function $f : A \to [0, +\infty)$ as

$$f(x) = d(x, B)$$

for any $x \in A$. Since f is a continuous function on the compact set A, there exists an $x^* \in A$ such that

$$f(x^*) = \min_{x \in A} f(x).$$

Since A and B are disjoint, and $x^* \in A$, it follows that $x^* \notin B$. By the basic separation theorem, there exists a $y^* \in A$ such that

$$f(x^*) = d(x^*, B) = |x^* - y^*|,$$

and for the non-zero vector $v = x^* - y^* \in \mathbb{R}^n$ and $c = \frac{v'x^* - v'y^*}{2} \in \mathbb{R}$, we have

$$v'x^* > c > v'y$$

for any $y \in B$. It remains to be seen that $v'x > c$ for any $x \in A$.

For any $x \in A$ and $k \in N_+$, define

$$z_k = \frac{k-1}{k} x^* + \frac{1}{k} x;$$

by the convexity of A, $z_k \in A$. Furthermore, by our choice of x^*,

$$|z_k - y^*|^2 \geq d(z_k, B)^2 = f(z_k)^2 \geq f(x^*)^2 = |x^* - y^*|^2 = |v|^2 > 0.$$

Since

$$
\begin{aligned}
|v|^2 \leq |z_k - y^*|^2 &= (z_k - x^* + v)'(z_k - x^* + v) \\
&= |z_k - x^*|^2 + 2v'(z_k - x^*) + |v|^2,
\end{aligned}
$$

we have the inequality

$$0 \leq |z_k - x^*|^2 + 2v'(z_k - x^*).$$

Note that, by construction,

$$z_k - x^* = \frac{1}{k}(x - x^*);$$

therefore,

$$0 \leq \frac{1}{k^2}|x - x^*|^2 + 2\frac{1}{k}v'(x - x^*),$$

and multiplying both sides by k yields

$$0 \leq \frac{1}{k}|x - x^*|^2 + 2v'(x - x^*).$$

Taking $k \to \infty$ finally reveals that $v'x \geq v'x^* > c$, and because this holds for any $x \in A$, we have

$$v'x > c > v'y$$

for any $x \in A$ and $y \in B$.
Q.E.D.

The next result formulates a result analogous to the basic separation theorem for boundary points.

Theorem 2.6 (Supporting Hyperplane Theorem for Closed Sets)
Let A be a nonempty closed convex subset of $\mathbb{R}^n$, and $x \in A$ a point on the boundary of A. Then, there exists a non-zero vector $v \in \mathbb{R}^n$ such that $v'x \geq v'y$ for any $y \in A$.

Proof) Choose any $x \in \partial A$. For any $k \in N_+$, the $\frac{1}{k}$-ball $B(x, 1/k)$ around x contains a point that is not in A; let this point be $x_k \in \mathbb{R}^n$. Since x_k is a point that is not contained in the closed and convex set A, by the basic separation theorem there exists a non-zero $v_k \in \mathbb{R}^n$ such that $v_k' x_k \geq v_k' y$ for any $y \in A$. Suppose that v_k is normalized to a unit vector (simply divide both sides of the inequality by $|v_k|$).

It follows that $\{x_k\}_{k \in N_+}$ is a sequence converging to x and $\{v_k\}_{k \in N_+}$ a sequence of vectors in $\mathbb{R}^n$ taking values on the unit circle T in $\mathbb{R}^n$. Since T is compact, there exists a convergent subsequence of $\{v_k\}_{k \in N_+}$ with limit $v \in T$; for notational brevity, assume that $\{v_k\}_{k \in N_+}$ itself converges to v. Note that v is non-zero because it is a unit vector. For any $y \in A$,

$$v_k' x_k \leq v_k' y$$

for any $k \in N_+$. Taking $k \to \infty$ on both sides yields

$$v'x \leq v'y.$$

This holds for any $y \in A$, so the proof is complete.
Q.E.D.

Putting together the supporting hyperplane theorem for closed sets and the basic separation theorem, we can obtain the following result:

Theorem 2.7 (Separation of Interior and Boundary)
Let A be a closed convex subset of $\mathbb{R}^n$ with nonempty interior, and $x \in A$ a point not in the interior A^o of A. Then, there exists a non-zero vector $v \in \mathbb{R}^n$ such that $v'x > v'y$ for any $y \in A^o$.

Proof) Since $x \notin A^o$, either $x \in \partial A$ or $x \notin A$. In either case, since A is a nonempty closed convex subset of $\mathbb{R}^n$, there exists a non-zero vector $v \in \mathbb{R}^n$ such that $v'x \geq v'y$ for any $y \in A$.

Choose any $y \in A^o$ and suppose, for the sake of contradiction, that $v'x = v'y$. Intuitively, this results in a contradiction because we can take an open ball around y contained in A, and there must exist some points in this ball that lie on either side of the hyperplane $H = \{w \in \mathbb{R}^n \mid v'w = v'x\}$. We now formalize this intuition.

By the definition of an interior point, there exists an $\epsilon > 0$ such that $B(y, \epsilon) \subset A$. Define

$$z = y + \left(\frac{v}{|v|}\right) \cdot \frac{\epsilon}{2} \in \mathbb{R}^n.$$

Then,

$$|z-y| = \frac{\epsilon}{2} < \epsilon,$$

and

$$v'z = v'y + |v| \cdot \frac{\epsilon}{2} = v'x + |v| \cdot \frac{\epsilon}{2} > v'x.$$

The first inequality tells us that $z \in B(y,\epsilon) \subset A$, while the second inequality contradicts the fact that $v'x \geq v'w$ for any $w \in A$. Therefore, it must be the case that $v'x > v'y$.

Q.E.D.

The supporting hyperplane theorem for closed sets can be extended to arbitrary convex subsets of $\mathbb{R}^n$ by utilizing a useful property of convex sets. Note that, for any $A \subset \mathbb{R}^n$,

$$\partial\overline{A} = \overline{A} \setminus \overline{A}^o \subset \overline{A} \setminus A^o = \partial A,$$

since the interior $\overline{A}^o$ of the closure $\overline{A}$ contains the interior A^o of A. In general, the reverse inclusion does not hold, but for convex sets with non-empty interior, it does; this is the content of the lemma below.

Lemma 2.8 If A is a convex subset of $\mathbb{R}^n$, then $A^o = \overline{A}^o$, that is, A and $\overline{A}$ have the same interior.

Proof) A^o is clearly contained in $\overline{A}^o$ because $A \subset \overline{A}$. If $\overline{A}^o$ is empty, then so is A^o, and the claim holds trivially. Suppose that $\overline{A}^o$ is non-empty. We must prove that $\overline{A}^o$ is contained in A^o. To do so, we make use of convex hulls and the basic separation theorem; for more on convex hulls, consult the section on convex functions.

The intuition of the proof is to first construct a box Ω around a point x in the interior of $\overline{A}$ that is contained in $\overline{A}$. Since each vertex in that box is a point in the closure of A, we can slightly perturb each of them to obtain a new box Ω' containing x with vertices in A. The convexity of A shows us that this box Ω' is contained in A. Finally, we show that some open ball around x is contained in Ω' and therefore in A, from which we can conclude that x is also in the interior of A. To do so, we assume that some point y in this ball is not conatined in Ω'; since Ω' is the convex hull generated by the perturbed vertices of Ω, it is a non-empty closed convex set, so we can use the basic separation theorem to find a hyperplane separating y and Ω'. However, since the radius of this ball is taken to be the shortest distance between x and each of the perturbed distances, at least one vertex of Ω' must necessarily lie on each side of the hyperplane separating the ball and Ω', which results in a contradiction.

Step 1: Constructing the Box Ω

Choose any $x \in \overline{A}^o$. By the definition of an interior point, there exists an $\epsilon > 0$ such that the open ball $B(x,\epsilon)$ is contained in $\overline{A}$. Letting $\{e_1,\cdots,e_n\}$ be the standard basis of $\mathbb{R}^n$, let E collects the 2^n vertices of the box with edges of length $\frac{2\cdot\epsilon}{n}$ and centered at x, that is, the box

$$\Omega = \prod_{i=1}^{n}\left[x - \frac{\epsilon}{n}, x + \frac{\epsilon}{n}\right].$$

Each $v \in E$ has the form

$$v = x + \frac{\epsilon}{n}\sum_{i=1}^{n} s_i \cdot e_i$$

for some $s_1,\cdots,s_n \in \{-1,1\}$; we collect these signs in the n-dimensional vector $s(v)$. Note that Ω is actually the convex hull generated by $E = \{v^{(1)},\cdots,v^{(2^n)}\}$, that is,

$$\Omega = \Big\{\sum_{i=1}^{2^n}\lambda_i \cdot v^{(i)} \mid \lambda_1,\cdots,\lambda_{2^n} \in \mathbb{R}_+,\ \sum_{i=1}^{2^n}\lambda_i = 1\Big\}.$$

For any $y \in \Omega$, there exist $\lambda_1,\cdots,\lambda_{2^n} \in \mathbb{R}_+$ such that $\sum_{i=1}^{2^n}\lambda_i = 1$ and $y = \sum_{i=1}^{2^n}\lambda_i \cdot v^{(i)}$. For any $1 \le i \le n$,

$$\left|x - v^{(i)}\right| < \epsilon,$$

so it follows that

$$|x-y| \le \sum_{i=1}^{2^n}\lambda_i \cdot \left|x - v^{(i)}\right| < \epsilon,$$

so that $\Omega \subset B(x,\epsilon)$.

Step 2: Constructing the Box Ω' and an Open Ball around x

Since each point in $B(x,\epsilon)$, including the vertices collected in E, are points in $\overline{A}$, the ϵ/n-neighborhood of these vertices contains a point in A; that is, we can find $u^{(1)},\cdots,u^{(2^n)} \in A$ such that

$$\left|u^{(i)} - v^{(i)}\right| < \frac{\epsilon}{n}$$

for $1 \leq i \leq 2^n$. Since

$$v^{(i)} = x + \frac{\epsilon}{n}\left(I_n \cdot s(v^{(i)})\right),$$

for any $1 \leq j \leq n$,

$$u_j^{(i)} \begin{cases} > x_j & \text{if } s(v^{(i)})_j = 1 \\ < x_j & \text{if } s(v^{(i)})_j = -1 \end{cases}.$$

Let Ω' be the convex hull generated by $E' = \{u^{(1)}, \cdots, u^{(2^n)}\}$. Since E' is a subset of the convex set A, the convex hull Ω' that it generates is also a subset of A. Define

$$\delta = \min_{1 \leq i \leq 2^n, 1 \leq j \leq n} \left| u_j^{(i)} - x_j \right| > 0,$$

and choose some $y \in B(x, \delta)$; we will show that $y \in \Omega'$, which will demonstrate that $B(x, \delta) \subset \Omega' \subset A$ and thus that x is an interior point of A.

Step 3: Using the Basic Separation Theorem

Suppose that $y \notin \Omega'$. Ω' is a non-empty, closed and convex set, so by the basic separation theorem, there exists a non-zero vector $v \in \mathbb{R}^n$ such that

$$\langle v, y \rangle > \langle v, z \rangle$$

for any $z \in \Omega'$. By implication, $\langle v, y - u^{(i)} \rangle > 0$ for any $1 \leq i \leq 2^n$. Choose $1 \leq i \leq 2^n$ such that

$$s(v^{(i)})_j = 1 \quad \text{if } v_j \geq 0$$
$$s(v^{(i)})_j = -1 \quad \text{if } v_j < 0.$$

Then, note that, if $v_j \geq 0$, then $s(v^{(i)}) = 1$ and thus $u_j^{(i)} > x_j$. Since

$$y_j - x_j \leq |y_j - x_j| < \delta \leq u_j^{(i)} - x_j,$$

we can see that $y_j - u_j^{(i)} < 0$, so that $v_j(y_j - u_j^{(i)}) \leq 0$. On the other hand, if $v_j < 0$, then $s(v^{(i)}) = -1$ and $u_j^{(i)} < x_j$. It follows that

$$x_j - y_j \leq |y_j - x_j| < \delta \leq x_j - u_j^{(i)},$$

so that $y_j - u_j^{(i)} > 0$ and $v_j(y_j - u_j^{(i)}) < 0$. Therefore,

$$\langle v, y - u^{(i)} \rangle = \sum_{j=1}^{n} v_i(y_j - u_j^{(i)}) \leq 0,$$

which contradicts the fact that $\langle v, y - u^{(i)} \rangle > 0$. In other words, $y \in \Omega'$, which completes the proof.

Q.E.D.

We can now present a version of the supporting hyperplane theorem that does not require the closedness assumption.

Theorem 2.9 (Supporting Hyperplane Theorem)
Let A be a non-empty convex subset of $\mathbb{R}^n$, and $x \in A$ a point on the boundary of A. Then, there exists a non-zero vector $v \in \mathbb{R}^n$ such that $v'x \geq v'y$ for any $y \in A$.

Proof) This follows easily from the supporting hyperplane theorem for closed sets. Letting $\overline{A}$ be the closure of A, since A is convex $\partial\overline{A} = \partial A$ by the preceding lemma. Therefore, $x \in \partial A = \partial\overline{A}$; x is a point on the boundary of the non-empty, closed and convex set $\overline{A}$. By the supporting hyperplane theorem for closed sets, there exists a non-zero $v \in \mathbb{R}^n$ such that

$$v'x \geq v'y$$

for any $y \in \overline{A}$, and therefore for any $y \in A$.

Q.E.D.

The general supporting hyperplane theorem now allows us to prove a version of the separating hyperplane theorem for general disjoint convex sets.

Theorem 2.10 (Separating Hyperplane Theorem)
Let A, B be non-empty disjoint convex subsets of $\mathbb{R}^n$. Then, there exists a non-zero vector $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that

$$v'x \leq c \leq v'y$$

for any $x \in A$ and $y \in B$.

Proof) Define the set $C = A - B$; since A and B are non-empty convex sets, so is C. Note that $\mathbf{0} \notin C$, since if it were an element of C, then there would exist $x \in A$ and $y \in B$ such that $x = y$, which contradicts the assumption that A and B are disjoint. We consider two cases.

Initially, suppose that $\mathbf{0} \notin \overline{C}$. Since $\overline{C}$ is a non-empty closed and convex subset, and $\mathbf{0}$ is some point outside $\overline{C}$, the basic separation theorem tells us that there exists a non-zero $v \in \mathbb{R}^n$ such that $v'z > 0$ for any $z \in \overline{C}$. Since $C \subset \overline{C}$, it follows that $v'x \geq v'y$ for any $x \in A$ and $y \in B$.

On the other hand, suppose $\mathbf{0} \in \overline{C}$. In this case, $\mathbf{0} \in \partial C$, so that, by the supporting hyperplane theorem, there exists a non-zero $v \in \mathbb{R}^n$ such that $v'z \geq 0$ for any $z \in C$, that is, $v'x \geq v'y$ for any $x \in A$ and $y \in B$.

In any case, we have seen that there must exist a non-zero $v \in \mathbb{R}^n$ such that

$$v'x \geq v'y \quad \text{for any } x \in A, y \in B.$$

Define

$$c_A = \inf_{x \in A} v'x, \quad \text{and} \quad c_B = \sup_{y \in B} v'y.$$

These values exist in $\mathbb{R}$ because A, B are non-empty and the mappings $x \mapsto v'x$ on A and $y \mapsto v'y$ on B are bounded below and above, respectively. Choose any $y \in B$. Then, since $v'x \geq v'y$ for any $x \in A$, we have

$$c_A = \inf_{x \in A} v'x \geq v'y.$$

This in turn holds for an $y \in B$, so we have

$$c_A \geq \sup_{y \in B} v'y = c_B.$$

Therefore, taking $c = \frac{c_A + c_B}{2}$, for any $x \in A$ and $y \in B$ we have

$$v'x \geq c \geq v'y.$$

Q.E.D.

2.2 Convex Functions

In this section we derive some results concerning convex functions. Let E be a non-empty convex subset of the euclidean space $\mathbb{R}^n$, and $f : E \to \mathbb{R}$ a real-valued function. We say that f is a convex function if, for any $\lambda \in [0,1]$ and $x, y \in E$,

$$f(\lambda \cdot x + (1-\lambda) \cdot y) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(y).$$

We say f is strictly convex if the inequality above holds as a strict inequality for $\lambda \in (0,1)$ and distinct $x, y \in E$. The defining property above can be extended in a natural manner.

Lemma 2.11 Let E be a convex subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}$ a convex function. Then, for any $x_1, \cdots, x_k \in E$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{R}_+$ such that $\sum_{i=1}^{k} \lambda_i = 1$,

$$\sum_{i=1}^{k} \lambda_i \cdot x_i \in E$$

and

$$f\left(\sum_{i=1}^{k} \lambda_i \cdot x_i\right) \leq \sum_{i=1}^{k} \lambda_i \cdot f(x_i).$$

Proof) We proceed by induction. We know that the claim holds for $k = 2$ by the definition of a convex set and a convex function.

Now suppose that the claim holds for some $k \geq 2$. Choose any $x_1, \cdots, x_{k+1} \in E$ and $\lambda_1, \cdots, \lambda_{k+1} \in \mathbb{R}_+$ such that $\sum_{i=1}^{k+1} \lambda_i = 1$. Note that there must exist a $1 \leq i \leq k+1$ such that $\lambda_i < 1$; assume without loss of generality that $\lambda_{k+1} < 1$. Then, defining $\gamma_i = \frac{\lambda_i}{1-\lambda_{k+1}}$ for any $1 \leq i \leq k$, each $\gamma_i \geq 0$ with sum

$$\sum_{i=1}^{k} \gamma_i = \frac{\sum_{i=1}^{k} \lambda_i}{1-\lambda_{k+1}} = 1.$$

By the inductive hypothesis,

$$y = \sum_{i=1}^{k} \gamma_i \cdot x_i \in E$$

and

$$f(y) \leq \sum_{i=1}^{k} \gamma_i \cdot f(x_i).$$

Since

$$z := \sum_{i=1}^{k+1} \lambda_i \cdot x_i = (1 - \lambda_{k+1}) \cdot y + \lambda_{k+1} \cdot x_{k+1},$$

where $y, x_{k+1} \in E$, by the definition of a convex set $z \in E$. Furthermore,

$$\begin{aligned} f\left(\sum_{i=1}^{k+1} \lambda_i \cdot x_i\right) &= f((1 - \lambda_{k+1}) \cdot y + \lambda_{k+1} \cdot x_{k+1}) \\ &\leq (1 - \lambda_{k+1}) \cdot f(y) + \lambda_{k+1} \cdot f(x_{k+1}) \leq \sum_{i=1}^{k+1} \lambda_i \cdot f(x_i). \end{aligned}$$

The claim now follows by induction.

Q.E.D.

The set

$$C = \left\{ \sum_{i=1}^{k} \lambda_i \cdot x_i \mid \lambda_1, \cdots, \lambda_k \in \mathbb{R}, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

is called the convex hull generated the set $\{x_1, \cdots, x_k\}$; we have just shown that any convex set E contains the convex hull generated by finite sets of vectors in E.

2.2.1 Continuity of Convex Functions

It is useful to work with convex functions because they possess certain desirable properties. One of these properties is continuity. We already showed that convex functions defined on an open interval in the real line is continuous. Here we extend that result to show convex functions defined on any arbitrary convex set of euclidean space is also continuous. We start by proving an auxiliary lemma.

Lemma 2.12 Let E be a convex open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}$ a convex function. Suppose f is locally bounded, that is, for any $x \in E$ there exists a neighborhood U of x and some $M > 0$ such that $|f(y)| \leq M$ for any $y \in U$. Then, f is continuous on E.

Proof) We actually show a stronger result that f is locally Lipschitz at any point on E, that is, for any $x_0 \in E$ there exists a neighborhood U of x_0 and an $L \geq 0$ such that $|f(x) - f(y)| < L \cdot |x - y|$ for any $x, y \in U$. Suppose that f is not locally Lipschitz at some point $x_0 \in E$. Since f is locally bounded at x_0, there exists an $\epsilon > 0$ and $M > 0$ such that the open ball $B(x_0, \epsilon)$ is contained in E (by the openness of E) and $|f(x)| \leq M$ for any $x \in B(x_0, \epsilon)$. f is not locally Lipschitz at x_0, so $\frac{4M}{\epsilon} > 0$ cannot be a Lipschitz constant for f on any neighborhood of x_0; therefore, there exist $x, y \in B(x_0, \epsilon/2)$ such that

$$f(y)-f(x)=|f(x)-f(y)|\geq\frac{4M}{\epsilon}\cdot|x-y|;$$

where we assume $f(x)\leq f(y)$ without loss of generality.

Define the point $z=y+\frac{\epsilon}{2|x-y|}\cdot(y-x)$, so that $|z-y|=\frac{\epsilon}{2}$. Choosing $\delta=\frac{\epsilon}{2|x-y|}$, define $\gamma:[-\delta,\delta]\to\mathbb{R}^n$ as

$$\gamma(t)=t\cdot x+(1-t)\cdot y=y+t(x-y)$$

for any $t\in[-\delta,\delta]$; for any $t\in[-\delta,\delta]$, $\gamma(t)\in B(x_0,\epsilon)$, since

$$|\gamma(t)-x_0|\leq|x_0-y|+|t||x-y|\leq|x_0-y|+\frac{\epsilon}{2}<\epsilon.$$

This means that γ takes values in E, so that we can define the function $g=f\circ\gamma:[-\delta,\delta]\to\mathbb{R}$. Note that g is a convex function on $[-\delta,\delta]$; for any $\lambda\in[0,1]$ and $t_1,t_2\in[-\delta,\delta]$,

$$\begin{aligned}g(\lambda\cdot t_1+(1-\lambda)\cdot t_2)&=f\left(y+(\lambda\cdot t_1+(1-\lambda)\cdot t_2)\cdot(x-y)\right)\\&=f\left(\lambda\cdot(y+t_1(x-y))+(1-\lambda)\cdot(y+t_2(x-y))\right)\\&\leq\lambda\cdot f(y+t_1(x-y))+(1-\lambda)\cdot f(y+t_2(x-y))\\&=\lambda\cdot g(t_1)+(1-\lambda)\cdot g(t_2).\end{aligned}$$

Since

$$\begin{aligned}g(1)&=f(\gamma(1))=f(x),\\g(0)&=f(\gamma(0))=f(y),\\g(-\delta)&=f(\gamma(-\delta))=f(z),\end{aligned}$$

the property of convex functions on the real line shows us that

$$f(x)-f(y)\geq\frac{f(y)-f(z)}{\delta}=\frac{f(z)-f(y)}{\frac{\epsilon}{2|x-y|}}.$$

Multiplying $-\frac{\epsilon}{2|x-y|}$ on both sides now reveals

$$f(z)-f(y)\geq\frac{f(y)-f(x)}{|x-y|}\cdot\frac{\epsilon}{2}>\frac{4M}{\epsilon}\cdot\frac{\epsilon}{2}=2M.$$

However, since $z,y\in B(x_0,\epsilon)$,

$$|f(z)-f(y)|\leq|f(z)|+|f(y)|\leq 2M$$

by the local boundedness condition. This is a contradiction, so f should be locally Lipschitz at x_0. This holds for any $x_0 \in E$, so it follows that f is continuous on E.

Q.E.D.

The continuity of convex functions will be established as soon as we show that any convex function is locally bounded at each point in its domain. This is exactly what we establish below.

Theorem 2.13 (Local Boundedness and Continuity of Convex Functions)
Let E be a convex open subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}$ a convex function. Then, f is locally bounded at each point $x_0 \in E$, and therefore continuous on E.

Proof) Choose any $x_0 \in E$. We will construct an open set U containing x_0 on which f is bounded. Since E is open, there exists an $\epsilon > 0$ such that the ball $B(x_0, 2\epsilon) \subset E$. Letting $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{R}^n$, we define

$$V = \Big\{ \sum_{i=1}^{n} [\lambda_i \cdot (x_0 + \epsilon \cdot e_i) + \gamma_i \cdot (x_0 - \epsilon \cdot e_i)] \mid \lambda_1, \cdots, \lambda_n, \gamma_1, \cdots, \gamma_n \in \mathbb{R}_+, \ \sum_{i=1}^{n} (\lambda_i + \gamma_i) = 1 \Big\}.$$

Heuristically, V is the convex hull generated by the vertices of the rotated box with side of length 2ϵ and center at x_0. Since x_0 is the center of U (this can be seen by choosing $\lambda_i = \gamma_i = \frac{1}{2n}$ for any $1 \leq i \leq n$), there exists a $\delta > 0$ such that $B(x_0, \delta) \subset V$. We now define

$$U = B(x_0, \delta).$$

We will show that f is bounded on U.

First we show that f is upper bounded on U. For any $x \in U$, there exist strictly positive values $\lambda_1, \cdots, \lambda_n, \gamma_1, \cdots, \gamma_n$ satisfying

$$\sum_{i=1}^{n} (\lambda_i + \gamma_i) = 1$$

such that

$$x = \sum_{i=1}^{n} [\lambda_i \cdot (x_0 + \epsilon \cdot e_i) + \gamma_i \cdot (x_0 - \epsilon \cdot e_i)],$$

so by the convexity of f,

$$
\begin{aligned}
f(x) &\leq \sum_{i=1}^{n} [\lambda_i \cdot f(x_0 + \epsilon \cdot e_i) + \gamma_i \cdot f(x_0 - \epsilon \cdot e_i)] \\
&\leq \max_{1 \leq i \leq n} |f(x_0 \pm \epsilon \cdot e_i)| = M < +\infty,
\end{aligned}
$$

where M is finite because it is the maximum of a finite number of positive elements.

To show that f is lower bounded on U, choose some $x \in U$ and note that

$$|(2x_0 - x) - x_0| = |x_0 - x| < \delta,$$

so that $2x_0 - x \in U$ as well. Since $x_0 = \frac{1}{2}x + \frac{1}{2}(2x_0 - x)$, by convexity we have

$$f(x_0) \leq \frac{1}{2} f(x) + \frac{1}{2} f(2x_0 - x),$$

which through rearrangement becomes

$$f(x) \geq 2 \cdot f(x_0) - f(2 \cdot x_0 - x).$$

Since $2 \cdot x_0 - x \in U$, the above result shows us that $f(2 \cdot x_0 - x)$ is bounded above by M, which in turn implies

$$f(x) \geq 2 \cdot f(x_0) - M.$$

We have shown that there exists an $M > 0$ such that, for any $x \in U$,

$$2f(x_0) - M \leq f(x) < M.$$

Thus, f is bounded on the open neighborhood U.

We can construct such a U for any $x_0 \in E$, so it follows that f is locally bounded on each point of E, and in light of the preeding lemma, f is continuous on E.

Q.E.D.

2.2.2 Subdifferentials and Directional Derivatives

Consider a convex open subset E of $\mathbb{R}^n$. The subdifferential of f at $x \in E$ is defined as

$$\partial f(x) = \{v \in \mathbb{R}^n \mid f(z) \geq f(x) + v'(z-x) \quad \forall z \in E\},$$

and elements of $\partial f(x)$ are called subgradients of f at x.

Our goal is to show that f is differentiable at some $x \in E$ if and only if $\partial f(x)$ is a singleton, and that this single element equals the gradient of f at x. We first show that $\partial f(x)$ is non-empty for any $x \in E$.

Theorem 2.14 (Non-emptiness of Subdifferential)
Let E be a non-empty open and convex subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ some convex function on E. Then, the subgradient $\partial f(x)$ of f is non-empty for any $x \in E^o$.

Proof) Define the epigraph of f as

$$\mathrm{epi}(f) = \{(x,q) \in E \times \mathbb{R} \mid f(x) \leq q\},$$

and denote by $\overline{\mathrm{epi}(f)}$ the closure of the epigraph.

$\mathrm{epi}(f)$ is a convex set; for any $(x_1,q_1),(x_2,q_2) \in \mathrm{epi}(f)$ and $\alpha \in [0,1]$, $\alpha x_1 + (1-\alpha)x_2 \in E$ by the convexity of E, and

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha \cdot f(x_1) + (1-\alpha) \cdot f(x_2) \leq \alpha q_1 + (1-\alpha)q_2,$$

so that $(\alpha x_1 + (1-\alpha)x_2, \alpha q_1 + (1-\alpha)q_2) \in \mathrm{epi}(f)$.

For any $x \in E$, $(x, f(x))$ is contained in the boundary $\partial\mathrm{epi}(f)$ of the epigraph. Clearly,

$$(x, f(x)) \in \mathrm{epi}(f) \subset \overline{\mathrm{epi}(f)}.$$

Suppose that $(x, f(x))$ is contained in the interior $\mathrm{epi}(f)^o$ of the epigraph. Then, there exists an $\epsilon > 0$ such that $(y,q) \in \mathrm{epi}(f)$ for any $(y,q) \in B((x,f(x)),\epsilon)$. Choosing

$$q = f(x) - \frac{\epsilon}{2} < f(x),$$

since $(x,q) \in B((x,f(x)),\epsilon)$, we must have $(x,q) \in \mathrm{epi}(f)$; but this is a contradiction, since $q < f(x)$. Thus, $(x,f(x)) \notin \mathrm{epi}(f)^o$, and we can conclude that

$$(x, f(x)) \in \overline{\mathrm{epi}(f)} \setminus \partial\mathrm{epi}(f) = \partial\mathrm{epi}(f).$$

Now choose any $x \in E$. We have seen that $(x, f(x))$ is a boundary point of the non-empty convex set epi(f). Therefore, by the supporting hyperplane theorem for convex (not necessarily closed) sets, there exists a non-zero vector $(v,a) \in \mathbb{R}^{n+1}$ such that

$$v'x + a \cdot f(x) \leq v'z + a \cdot f(z)$$

for any $z \in E$, which holds since $(z, f(z)) \in \text{epi}(f)$ for each $z \in E$. Rearringing terms shows us that $a \cdot f(z) \geq a \cdot f(x) + v'(x-z)$ for any $z \in E$. We now show that $a \neq 0$.

Suppose $a = 0$. Then, v would have to be non-zero, since $(v,a) \in \mathbb{R}^{n+1}$ is non-zero, and we have

$$v'(x-z) \leq 0$$

for any $z \in E$. Because $x \in E$ and E is open, there exists an $\epsilon > 0$ such that $B(x,\epsilon) \subset E$. Assuming $v_i > 0$ for some $1 \leq i \leq n$ without loss of generality, define

$$z = x - \epsilon \cdot e_i,$$

where e_i is the ith standard basis vector in $\mathbb{R}^n$. Then, $z \in B(x,\epsilon)$, so that $z \in E$, and we have $v'z = v'x - \epsilon \cdot v_i < v'x$, which contradicts our assumption that $v'x \leq v'z$ for any $z \in E$. Therefore, it must be the case that $a \neq 0$, and defining $\hat{v} = -\frac{v}{a} \in \mathbb{R}^n$,

$$f(z) \geq f(x) + \hat{v}'(z-x)$$

for any $z \in E$. By definition, $\hat{v} \in \partial f(x)$.

Q.E.D.

We are immediately able to obtain a convenient characterization of convex functions. Key to this characterizations are affine functions on $\mathbb{R}^n$. We say that a function $h : \mathbb{R}^n \to \mathbb{R}^m$ is affine if there exist $A \in \mathbb{R}^{m \times n}$ and $a \in \mathbb{R}^m$ such that

$$h(x) = Ax + a$$

for any $x \in \mathbb{R}^n$. Note that, if $a = \mathbf{0}$, then h is simply linear; the presence of the intercept a that shifts this linear transformation is some direction is what makes this function affine. We can actually study affinity in a more abstract setting, like we did with linearity, by utilizing the concept of affine combinations. In that setting, it turns out that any affine transformation can be expressed as the sum of a linear transformation plus an intercept.

Theorem 2.15 (Characterization of Convex Functions)
Let E be a non-empty open convex subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ some convex function on E. Let

W_f be the collection of all real-valued affine functions h on $\mathbb{R}^n$ such that $h(x) \leq f(x)$ for any $x \in E$. Then, for any $x \in E$,

$$f(x) = \sup_{h \in W_f} h(x),$$

where the supremum is pointwise.

Proof) Choose any $x \in E$. By definition, for any $h \in W_f$, we have $f(x) \geq h(x)$, so that

$$f(x) \geq \sup_{h \in W_f} h(x).$$

On the other hand, let $v \in \partial f(x)$, where $\partial f(x)$ is non-empty by the preceding theorem. Then, by definition

$$f(z) \geq f(x) + v'(z - x)$$

for any $z \in E$. Defining the affine function $\phi : \mathbb{R}^n \to \mathbb{R}$ as

$$\phi(z) = v'z + (f(x) - v'x)$$

for any $z \in \mathbb{R}^n$, we can easily see that

$$f(x) = \phi(x) \quad \text{and} \quad f(z) \geq \phi(z)$$

for any $z \in E$ such that $z \neq x$. This tells us that $\phi \in W_f$, and that

$$f(x) = \phi(x) \leq \sup_{h \in W_f} h(x).$$

Therefore, we may conclude that

$$f(x) = \sup_{h \in W_f} h(x).$$

Q.E.D.

The functions collected in W_f are referred to as the affine minorants of f.

Given a non-empty open convex subset E of $\mathbb{R}^n$ and a convex function $f : E \to \mathbb{R}$, the right directional derivative of f at $x \in E$ in the direction $y \in \mathbb{R}^n$ is defined as

$$f'(x; y) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda y) - f(x)}{\lambda},$$

granted the limit exists. Note that the value $f(x+\lambda y)$ is well-defined for small λ because E is open. Fortunately, the following lemma shows that this limit always exists:

Theorem 2.16 (Existence of Right Directional Derivatives)
Let E be a non-empty open convex subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ some convex function on E. Then, for any $x \in E$, $y \in \mathbb{R}^n$ and $0 < \lambda_1 < \lambda_2$ such that $x+\lambda_2 y \in E$,

$$\frac{f(x+\lambda_1 y)-f(x)}{\lambda_1} \leq \frac{f(x+\lambda_2 y)-f(x)}{\lambda_2}.$$

By implication, the mapping

$$\lambda \mapsto \frac{f(x+\lambda y)-f(x)}{\lambda}$$

is increasing, so that the right directional derivative of f at x in the direction y exists in $[-\infty,+\infty)$. In addition, the mapping $y \mapsto f'(x;\cdot)$ is real-valued and convex on $\mathbb{R}^n$.

Proof) Let $x \in E$, $y \in \mathbb{R}^n$ and $0 < \lambda_1 < \lambda_2$ be chosen as above. Defining $t = \frac{\lambda_1}{\lambda_2} \in (0,1)$, the fact that $x+\lambda_2 y$ and x are contained in E implies that

$$f\left((1-t)x+t(x+\lambda_2 y)\right) \leq (1-t)f(x)+t\cdot f(x+\lambda_2 y)$$

by the convexity of f. Since $(1-t)x+t(x+\lambda_2 y) = x+\lambda_1 y$, we can see that

$$f(x+\lambda_1 y)-f(x) \leq \lambda_1 \cdot \frac{f(x+\lambda_2 y)-f(x)}{\lambda_2},$$

so that we have the desired inequality.

Define $g : \mathbb{R}^n \to [-\infty,+\infty)$ as

$$g(y) = f'(x;y)$$

for any $y \in \mathbb{R}^n$. From the previous theorem, we can see that there exists a subgradient $v \in \partial f(x)$ of f, since $x \in E^o$; by definition,

$$f(z) \geq f(x)+v'(z-x)$$

for any $z \in E$. Choose any $y \in \mathbb{R}^n$, and note that, for small enough $\lambda > 0$, $x+\lambda y \in E$ and therefore

$$f(x+\lambda y) \geq f(x)+\lambda\left(v'y\right).$$

Rearranging terms yields

$$\frac{f(x+\lambda y)-f(x)}{\lambda} \geq v'y.$$

Now sending $\lambda \downarrow 0$ on both sides yields

$$f'(x;y) = g(y) \geq v'y > -\infty,$$

so that $g(y) \in \mathbb{R}$. It can also be seen that this holds for any $v \in \partial f(x)$, so

$$g(y) \geq \sup_{v \in \partial f(x)} v'y.$$

It remains to be seen that g is convex. Choose any $y_1, y_2 \in \mathbb{R}^n$ and $\alpha \in [0,1]$. For any $\lambda > 0$,

$$\begin{aligned}
\frac{f(x+\lambda(\alpha y_1 + (1-\alpha)y_2)) - f(x)}{\lambda} &= \frac{f(\alpha(x+\lambda y_1) + (1-\alpha)(x+\lambda y_2)) - f(x)}{\lambda} \\
&\leq \frac{\alpha f(x+\lambda y_1) + (1-\alpha) f(x+\lambda y_2) - f(x)}{\lambda} \\
&= \alpha \cdot \frac{f(x+\lambda y_1) - f(x)}{\lambda} + (1-\alpha)\frac{f(x+\lambda y_2) - f(x)}{\lambda}
\end{aligned}$$

by the convexity of f, so taking $\lambda \downarrow 0$ on both sides yields

$$f'(x; \alpha y_1 + (1-\alpha)y_2) \leq \alpha f'(x;y_1) + (1-\alpha) f'(x;y_2).$$

This shows us that g is convex.

Q.E.D.

Right directional derivatives are useful because of their peculiar relationship to the subgradients of a convex function:

Theorem 2.17 (Characterization of RDDs with Subgradients)
Let E be a non-empty open convex subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ some convex function on E. Then, for any $x \in E$ and $y \in \mathbb{R}^n$,

$$f'(x;y) = \sup_{v \in \partial f(x)} v'y.$$

Proof) Define $g : \mathbb{R}^n \to \mathbb{R}$ as $g(y) = f'(x;y)$ for any $y \in \mathbb{R}^n$ and let W_g be the set of affine minorants of the convex function g. We first show that every $h \in W_g$ is a linear function.

Choose any $h \in W_g$, and let there exist $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that

$$h(y) = v'y + c$$

for all $y \in \mathbb{R}^n$. By definition,

$$g(y) \geq h(y) = v'y + c$$

for any $y \in \mathbb{R}^n$. Since $g(\mathbf{0}) = 0$, we immediately have $c \leq 0$.

Note that, for any $y \in \mathbb{R}^n$, $t > 0$ and $\lambda > 0$,

$$\frac{f(x + \lambda(ty)) - f(x)}{\lambda} = t \cdot \frac{f(x + (t\lambda)y) - f(x)}{t\lambda},$$

so that, taking $\lambda \downarrow 0$, we have the equality

$$f'(x; ty) = t \cdot f'(x; y).$$

It follows that

$$t(v'y) + c = v'(ty) + c \leq g(ty) = t \cdot g(y),$$

and dividing both sides by t yields

$$v'y + \frac{c}{t} \leq g(y).$$

Taking $t \to \infty$ on both sides, we are left with the expression

$$v'y \leq g(y),$$

and because this holds for any $y \in \mathbb{R}^n$,

$$h(y) = v'y - c \leq v'y \leq g(y)$$

for any $y \in \mathbb{R}^n$. Therefore, for any $h \in W_g$, there exists a linear function ϕ such that $h \leq \phi \leq g$ on $\mathbb{R}^n$. Since

$$g(y) = \sup_{h \in W_g} h(y)$$

for any $y \in \mathbb{R}^n$, defining W_g^L as the collection of all linear minorants of g on $\mathbb{R}^n$, we must have

$$g(y) = \sup_{\phi \in W_g^L} \phi(y).$$

for any $y \in \mathbb{R}^n$.

Choose any $\phi \in W_g^L$; there exists a $v \in \mathbb{R}^n$ such that $\phi(y) = v'y$ and $\phi(y) \leq g(y)$ for any $y \in \mathbb{R}^n$. For any $z \in E$, defining $y = z - x \in \mathbb{R}^n$,

$$f(z) - f(x) = f(x+y) - f(x) \geq f'(x;y) = g(y) \geq \phi(y) = v'y = v'(z-x),$$

where the first inequality holds because $f'(x;y)$ is the infimum of

$$\frac{f(x+\lambda y) - f(x)}{\lambda}$$

with respect to $\lambda > 0$. Therefore, $v \in \partial f(x)$ by definition, and this holds for any $\phi \in W_g^L$, meaning that

$$f'(x;y) = g(y) = \sup_{\phi \in W_g^L} \phi(y) = \sup_{v \in \partial f(x)} v'y,$$

as desired.

Q.E.D.

2.2.3 Convex Functions and Differentiability

We are ready to prove the main result of this section:

Theorem 2.18 (Differentiability of Convex Functions)
Let E be a non-empty open convex subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ some convex function on E. f is differentiable at some $x \in E$ if and only if $\partial f(x)$ is a singleton. In this case, the gradient $\boldsymbol{\nabla} f(x)$ of f at x is is the only element in $\partial f(x)$.

Proof) **Necessity**

Suppose that f is differentiable at x. Then,

$$\boldsymbol{\nabla} f(x)'y = f'(x;y) = \sup_{v \in \partial f(x)} v'y$$

for any $y \in \mathbb{R}^n$, where the first equality follows from the chain rule and the second from the prevous theorem. For any $v \in \partial f(x)$,

$$f(z) - f(x) \geq v'(z-x)$$

for any $z \in E$ by definition, so fixing $z \in E$, we have

$$f(z) - f(x) \geq \sup_{v \in \partial f(x)} v'(z-x) = \boldsymbol{\nabla} f(x)'(z-x).$$

It follows that $\boldsymbol{\nabla} f(x) \in \partial f(x)$ as well, so that, for any $y \in \mathbb{R}^n$,

$$\boldsymbol{\nabla} f(x)'y = \max_{v \in \partial f(x)} v'y.$$

Suppose $v \in \partial f(x)$ is a subgradient of f at x that is distinct form $\boldsymbol{\nabla} f(x)$. Then,

$$\boldsymbol{\nabla} f(x)'y \geq v'y$$

for any $y \in \mathbb{R}^n$, which implies that $\boldsymbol{\nabla} f(x) = v$. Therefore,

$$\partial f(x) = \{\boldsymbol{\nabla} f(x)\},$$

that is, the subdifferential of f at x consists only of the gradient of f at x.

Sufficiency

Conversely, suppose that the subdifferential of f at x consists of a single vector $x^* \in \mathbb{R}^n$.

The previous theorem immediately tells us that

$$f'(x;y) = \sup_{v \in \partial f(x)} v'y = x^{*\prime}y$$

for any $y \in \mathbb{R}^n$. Defining $g : \mathbb{R}^n \to \mathbb{R}$ as

$$g(y) = f(x+y) - f(x) - x^{*\prime}y$$

for any $y \in \mathbb{R}^n$, note that $g(\mathbf{0}) = 0$ and

$$\frac{g(\lambda y)}{\lambda} = \frac{f(x+\lambda y) - f(x)}{\lambda} - x^{*\prime}y$$

for any $\lambda > 0$. Taking $\lambda \downarrow 0$ on both sides, by definition

$$g'(\mathbf{0};y) = f'(x;y) - x^{*\prime}y = 0;$$

this holds for any $y \in \mathbb{R}^n$.

Now fix $\lambda > 0$, and define the mapping $h_\lambda : \mathbb{R}^n \to \mathbb{R}$ as

$$h_\lambda(y) = \frac{g(\lambda y)}{\lambda}$$

for any $y \in \mathbb{R}^n$. h_λ has the following properties:

- **Positivity**
  For any $y \in \mathbb{R}^n$,

$$h_\lambda(y) = \frac{f(x+\lambda y) - f(x)}{\lambda} - x^{*\prime}y \geq f'(x;y) - x^{*\prime}y = 0,$$

  where the inequality follows because $\frac{f(x+\lambda y)-f(x)}{\lambda}$ is decreasing in λ.

- **Convergence to 0**
  For any $y \in \mathbb{R}^n$, $h_\lambda(y) \to 0$ as $\lambda \downarrow 0$.

- **Convexity**
  h_λ is convex; for any $y_1, y_2 \in \mathbb{R}^n$ and $\alpha \in [0,1]$,

$$\begin{aligned}
h_\lambda(\alpha y_1 + (1-\alpha)y_2) &= \frac{g(\alpha(\lambda y_1) + (1-\alpha)(\lambda y_2))}{\lambda} \\
&= \frac{f(\alpha(x+\lambda y_1) + (1-\alpha)(x+\lambda y_2)) - f(x)}{\lambda} - x^{*\prime}(\alpha y_1 + (1-\alpha)y_2) \\
&\leq \alpha\left(\frac{f(x+\lambda y_1) - f(x)}{\lambda} - x^{*\prime}y_1\right) + (1-\alpha)\left(\frac{f(x+\lambda y_2) - f(x)}{\lambda} - x^{*\prime}y_2\right) \\
&= \alpha h_\lambda(y_1) + (1-\alpha)h_\lambda(y_2).
\end{aligned}$$

Choose any $\epsilon > 0$. Let B be the unit ball in $\mathbb{R}^n$ centered at the origin, and Δ a convex hull that contains B generated by the finite set $\{a_1, \cdots, a_{n+1}\} \subset \mathbb{R}^n$. Then, for any $u \in B$, there exist $\gamma_i, \cdots, \gamma_{n+1} \in [0,1]$ such that

$$u = \sum_{i=1}^{n+1} \gamma_i \cdot a_i \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

It follows then that

$$\begin{aligned} 0 \leq \frac{g(\lambda u)}{\lambda} = h_\lambda(u) &\leq \sum_{i=1}^{n+1} \gamma_i \cdot h_\lambda(a_i) \\ &\leq \max_{1 \leq i \leq n+1} h_\lambda(a_i). \end{aligned}$$

This holds for any $u \in B$, so we have

$$0 \leq \sup_{u \in B} \frac{g(\lambda u)}{\lambda} \leq \max_{1 \leq i \leq n+1} h_\lambda(a_i).$$

Each $h_\lambda(a_i)$ goes to 0 as $\lambda \downarrow 0$, so

$$\lim_{h \downarrow 0} \left(\max_{1 \leq i \leq n+1} h_\lambda(a_i) \right) = 0$$

as well. By implication, there exists a $\delta > 0$ such that, for any $0 < \lambda < \delta$,

$$0 \leq \sup_{u \in B} \frac{g(\lambda u)}{\lambda} \leq \max_{1 \leq i \leq n+1} h_\lambda(a_i) < \epsilon$$

for any $u \in B$.

Therefore, for any non-zero $y \in \mathbb{R}^n$ such that $0 < |y| < \delta$, we have

$$\left| \frac{g(y)}{|y|} \right| = \frac{g(y)}{|y|} = \frac{g(|y| \cdot u)}{|y|} < \epsilon,$$

since $u = \frac{y}{|y|} \in B$. Here, δ depends only on ϵ, so by definition,

$$\lim_{|y| \to 0} \left| \frac{g(y)}{|y|} \right| = 0.$$

In other words,

$$\lim_{|y| \to 0} \frac{|f(x+y) - f(x) - x^{*\prime} y|}{|y|} = 0,$$

and as such, f is differentiable at x with gradient

$$\boldsymbol{\nabla} f(x) = x^*.$$

Q.E.D.

Differentiable convex functions can be characterized in terms of their derivatives as follows:

Theorem 2.19 (Properties of Differentiable Convex Functions)
Let E be a non-empty open convex subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ a function on E. Then, the following hold true:

i) **(First Order Characterization)**
Let f be continuously differentiable. Then, f is convex if and only if, for any distinct $x, y \in E$,

$$f(y) - f(x) \geq \boldsymbol{\nabla} f(x)'(y-x).$$

Likewise, f is strictly convex if and only if, for any distinct $x, y \in E$, the above inequality holds strictly.

ii) **(Second Order Characterization)**
Let f be twice continuously differentiable. Then, f is convex if and only if the Hessian matrix $\boldsymbol{\nabla}^2 f(x)$ is postive semidefinite (definite) for any $x \in E$.

If $\boldsymbol{\nabla}^2 f(x)$ is positive definite for any $x \in E$, then f is strictly convex, but the converse does not hold.

Proof) **First Order Characterization**

Suppose that f is convex. For any $x \in E$, since f is differentiable, the previous theorem shows us that the subdifferential $\partial f(x)$ is a singleton consisting only of the gradient $\boldsymbol{\nabla} f(x)$. By definition of the subdifferential, for any $y \in E$ we have

$$f(y) \geq f(x) + \boldsymbol{\nabla} f(x)'(y-x).$$

Now let f be strictly convex, and choose any distinct $x, y \in E$. As we saw in the section on direcional derivatives,

$$\lim_{\lambda \downarrow 0} \frac{f(x+\lambda(y-x)) - f(x)}{\lambda} = \boldsymbol{\nabla} f(x)'(y-x).$$

and since the mapping $\lambda \mapsto \frac{f(x+\lambda(y-x))-f(x)}{\lambda}$ is increasing,

$$\lim_{\lambda \downarrow 0} \frac{f(x+\lambda(y-x)) - f(x)}{\lambda} = \inf_{\lambda \in (0,1)} \frac{f(x+\lambda(y-x)) - f(x)}{\lambda}.$$

Putting the two together, we can see that

$$\begin{aligned}
\boldsymbol{\nabla} f(x)'(y-x) &= \inf_{\lambda\in(0,1)} \frac{f(x+\lambda(y-x))-f(x)}{\lambda} \\
&\le \frac{f(x+0.5(y-x))-f(x)}{0.5} \\
&= 2\cdot\left(f\left(\frac{1}{2}y+\frac{1}{2}x\right)-f(x)\right) \\
&< 2\cdot\left(\frac{1}{2}f(y)-\frac{1}{2}f(x)\right) = f(y)-f(x),
\end{aligned}$$

where the strict inequality follows by strict convexity. It follows that

$$f(y) > f(x) + \boldsymbol{\nabla} f(x)'(y-x).$$

Conversely, suppose that

$$f(y)-f(x) \ge \boldsymbol{\nabla} f(x)'(y-x)$$

for any distinct $x, y \in E$. Then, for any distinct $x, y \in E$ and $\lambda \in (0,1)$,

$$\begin{aligned}
f(y)-f(\lambda\cdot y+(1-\lambda)\cdot x) &\ge (1-\lambda)\cdot\boldsymbol{\nabla} f(\lambda\cdot y+(1-\lambda)\cdot x)'(y-x) \\
f(x)-f(\lambda\cdot y+(1-\lambda)\cdot x) &\ge -\lambda\cdot\boldsymbol{\nabla} f(\lambda\cdot y+(1-\lambda)\cdot x)'(y-x).
\end{aligned}$$

Multiplying λ to the first inequality and $(1-\lambda)$ to the second inequality and then summing up yields

$$\begin{aligned}
&\lambda\cdot f(y)+(1-\lambda)\cdot f(x)-f(\lambda\cdot y+(1-\lambda)\cdot x) \\
&\quad\ge \lambda(1-\lambda)\cdot\boldsymbol{\nabla} f(x+\lambda\cdot(y-x))'(y-x)-\lambda(1-\lambda)\cdot\boldsymbol{\nabla} f(x+\lambda\cdot(y-x))'(y-x) = 0,
\end{aligned}$$

so that

$$f(\lambda\cdot y+(1-\lambda)\cdot x) \le \lambda\cdot f(y)+(1-\lambda)\cdot f(x).$$

This holds for any distinct $x, y \in E$ and $\lambda \in (0,1)$, so by definition f is convex. For strict convexity, just replace all the above inequalities with strict inequalities.

Second Order Characterization

If f is convex, then by the first order characterization, it satisfies

$$f(y) \geq f(x) + \boldsymbol{\nabla} f(x)'(y-x)$$

for any distinct $x, y \in E$. We will show that $\boldsymbol{\nabla}^2 f(x)$ is positive semidefinite for any $x \in E$ in this case.

Choose any $x \in E$ and non-zero vector $u \in \mathbb{R}^n$. By the openness of E, $x + t \cdot u \in E$ for t in a small enough neighborhood U of 0. Define the function $\gamma : U \to \mathbb{R}$ as

$$\gamma(t) = \boldsymbol{\nabla} f(x + t \cdot u)'u = \sum_{j=1}^{n} (D_j f)(x + t \cdot u) u_j.$$

Then, for any $t \in U$,

$$\begin{aligned}
\gamma'(t) &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} (D_{ij} f)(x + t \cdot u) u_i \right) u_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ij} f)(x + t \cdot u) u_i u_j = u' \left[\boldsymbol{\nabla}^2 f(x + t \cdot u) \right] u.
\end{aligned}$$

It follows that $\gamma'(0) = u' \left[\boldsymbol{\nabla}^2 f(x) \right] u$.

We now show that γ is increasing on U. For any $t_1, t_2 \in U$, note that

$$f(x + t_2 \cdot u) \geq f(x + t_1 \cdot u) + (t_2 - t_1) \cdot \gamma(t_1)$$

$$f(x + t_1 \cdot u) \geq f(x + t_2 \cdot u) - (t_2 - t_1) \cdot \gamma(t_2),$$

so that

$$\gamma(t_1) \leq \frac{f(x + t_2 \cdot u) - f(x + t_1 \cdot u)}{t_2 - t_1} \leq \gamma(t_2).$$

Therefore,

$$\frac{\gamma(t) - \gamma(0)}{t} \geq 0$$

for any non-zero $t \in U$. It follows that

$$u' \left[\boldsymbol{\nabla}^2 f(x) \right] u = \gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} \geq 0.$$

This shows us that $\boldsymbol{\nabla}^2 f(x)$ is positive semidefinite.

Conversely, suppose that $\boldsymbol{\nabla}^2 f(x)$ is positive semidefinite for any $x \in E$. Then, by the multivariate version of Taylor's theorem, for any distinct $x, y \in E$, there exists a $t \in [0,1]$

such that

$$f(y) = f(x) + \nabla f(x)'(y-x) + \frac{1}{2}(y-x)' \left[\nabla^2 f(t \cdot x + (1-t) \cdot y) \right] (y-x).$$

By positive semidefiniteness,

$$f(y) - f(x) - \nabla f(x)'(y-x) \geq 0.$$

This holds for any distinct $x, y \in E$, so by the first order characterization, f is convex. If $\nabla^2 f(x)$ is positive definite for any $x \in E$, then we can infer that f is strictly convex by replacing the inequality above with a strict inequality.

Q.E.D.

We can show that the converse of the second order characterization for strictly convex functions does not hold via a simple counterexample. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^4$ for any $x \in \mathbb{R}$. Then, f is strictly convex on $\mathbb{R}$, since it is the composition of two strictly convex functions (both equal to the quadratic function), but its second derivative at 0 is $f''(0) = 0$.

Chapter 3

Static Optimization

3.1 Unconstrained Optimization

Here we deal with problems of the form

$$\max_{x \in E} \quad f(x)$$

for some open subset E of $\mathbb{R}^n$ and function $f : E \to \mathbb{R}$. We saw with Rolle's theorem and the multivariate mean value theorem that, if f is differentiable at some local extremum x^*, one of the necessary conditions is that the gradient $\boldsymbol{\nabla} f(x^*)$ of f at x^* must be equal to 0. This leads one to naturally wonder if there exists a set of sufficient conditions involving this gradient condition that guarantees some point x^* is a local extremum of f on E. It turns out that Taylor's theorem furnishes us with a simple set of sufficient conditions based only on the first two derivatives of a function.

We first deal with the problem of finding global optimizers, since this turns out to be easier than finiding sufficient conditions for local optimizers. The formal statement and proof are given below:

Theorem 3.1 (Necessary and Sufficient Conditions for Unconstrained Global Maximization)
Let E be a convex open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ a twice continuously differentiable function on E. Consider the unconstrained optimization problem

$$\max_{x \in E} \quad f(x).$$

Then, the following hold true:

i) **(First Order Necessary Conditions for Global Maximization)**
If $x^* \in E$ is a global maximum of f on E, then $\boldsymbol{\nabla} f(x^*) = \mathbf{0}$.

ii) **(Second Order Sufficient Conditions for Global Maximization)**
Let $\boldsymbol{\nabla}^2 f(x)$ be the Hessian matrix of f at $x \in E$. If $\boldsymbol{\nabla} f(x^*) = \mathbf{0}$ and the Hessian $\boldsymbol{\nabla}^2 f(x)$

is negative definite for any $x \in E$, then x^* is a strict global maximum of f on E.

Proof) Again, the necessary condition follows from theorem 1.8 because any global maximum is also a local maximum. As such, we once again focus on the second order conditions.

Suppose that $\boldsymbol{\nabla} f(x^*) = \mathbf{0}$ and let $\boldsymbol{\nabla}^2 f(x)$ be negative definite for any $x \in E$. Choose any $x \in E$ such that $x \neq x^*$. Then, by the multivariate version of Taylor's theorem, there exists a $t_0 \in [0,1]$ such that

$$\begin{aligned} f(x) &= f(x^*) + \boldsymbol{\nabla} f(x^*)'(x - x^*) + \frac{1}{2}(x - x^*)' \left[\boldsymbol{\nabla}^2 f(t_0 \cdot x + (1 - t_0) \cdot x^*)\right](x - x^*) \\ &= f(x^*) + \frac{1}{2}(x - x^*)' \left[\boldsymbol{\nabla}^2 f(t_0 \cdot x + (1 - t_0) \cdot x^*)\right](x - x^*). \end{aligned}$$

By the negative definiteness of $\boldsymbol{\nabla}^2 f(t_0 \cdot x + (1 - t_0) \cdot x^*)$ and the fact that $x - x^* \neq \mathbf{0}$,

$$f(x) - f(x^*) = \frac{1}{2}(x - x^*)' \left[\boldsymbol{\nabla}^2 f(t_0 \cdot x + (1 - t_0) \cdot x^*)\right](x - x^*) < 0,$$

so that $f(x) < f(x^*)$. This holds for any $x \in E$ such that $x \neq x^*$, so by definition x^* is a global maximizer of f.

Q.E.D.

Corollary to Theorem 3.1 (Necessary and Sufficient Conditions for Unconstrained Global Minimization)
Let E be a convex open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ a twice continuously differentiable function on E. Consider the unconstrained optimization problem

$$\min_{x \in E} \quad f(x).$$

Then, the following hold true:

i) **(First Order Necessary Conditions for Global Minimization)**
If $x^* \in E$ is a global minimum of f on E, then $\boldsymbol{\nabla} f(x^*) = \mathbf{0}$.

ii) **(Second Order Sufficient Conditions for Global Minimization)**
Let $\boldsymbol{\nabla}^2 f(x)$ be the Hessian matrix of f at $x \in E$. If $\boldsymbol{\nabla} f(x^*) = \mathbf{0}$ and the Hessian $\boldsymbol{\nabla}^2 f(x)$ is positive definite for any $x \in E$, then x^* is a strict global minimum of f on E.

Proof) The problem of minimizing f over E is equivalent to maximizing $-f$ over E. Thus, the first order necessary conditions hold without modification, while

$$\nabla^2 f(x) = -\nabla^2(-f)(x)$$

for any $x \in E$, so that the negative definiteness of the Hessian must become positive definiteness.

Q.E.D.

Similar conditions can be furnished for unconstrained local optimization. This time, we require only the negative/positive definiteness of the Hessian matrix at the local extremum, and we can drop the convexity requirement for the domain.

Theorem 3.2 (Necessary and Sufficient Conditions for Unconstrained Local Maximization)
Let E be an open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ a twice continuously differentiable function on E. Consider the unconstrained optimization problem

$$\max_{x \in E} \quad f(x).$$

Then, the following hold true:

i) **(First Order Necessary Conditions for Local Maximization)**
If $x^* \in E$ is a local maximum of f on E, then $\nabla f(x^*) = \mathbf{0}$.

ii) **(Second Order Sufficient Conditions for Local Maximization)**
Let $\nabla^2 f(x)$ be the Hessian matrix of f at $x \in E$. If $\nabla f(x^*) = \mathbf{0}$ and the Hessian $\nabla^2 f(x^*)$ is negative definite for some $x^* \in E$, then x^* is a strict local maximum of f on E.

Proof) The first order necessary conditions follow from theorem 1.8, the multivariate mean value theorem, so it remains to prove that the second order conditions are actually sufficient for a strict local maximum.

Let $x^* \in E$ satisfy $\nabla f(x^*) = \mathbf{0}$ and assume $\nabla^2 f(x^*)$ is negative definite. Since E is open, there exists an $\eta > 0$ such that the open ball $B(x^*, \eta)$ is contained in E; note that this open ball is convex.

By the multivariate version of Taylor's theorem, for any $x \in B(x^*, \eta)$ there exists a $t_0 \in [0,1]$ such that

$$f(x) = f(x^*) + \nabla f(x^*)'(x - x^*) + \frac{1}{2}(x - x^*)' \left[\nabla^2 f(t_0 \cdot x + (1 - t_0) \cdot x^*)\right](x - x^*),$$

so that, in light of the fact that $\boldsymbol{\nabla} f(x^*) = \mathbf{0}$, we have

$$f(x) - f(x^*) = \frac{1}{2}(x - x^*)' \left[\boldsymbol{\nabla}^2 f(t_0 \cdot x + (1 - t_0) \cdot x^*)\right](x - x^*).$$

The proof here is now complicated by the fact that the Hessian is only known to be negative definite at x^*, not $t_0 \cdot x + (1 - t_0) \cdot x^*$. We overcome this wrinkle by using the fact that, for values of x close to x^*, the Hessian at x is also close to the Hessian at x^* and thus negative definite.

Formally, we proceed via a proof by contradiction. Suppose that x^* is not a strict local maximizer of f. Then, there exists a point in every neighborhood of x^* distinct from x^* whose value under f is at least as large as $f(x^*)$. In particular, any $k \in N_+$, there exists an x_k in the open ball $B(x^*, \eta/k)$ not equal to x^* such that $f(x_k) \geq f(x^*)$. As such, there exists a corresponding sequence $\{t_k\}_{k \in N_+}$ in $[0,1]$ such that

$$f(x_k) - f(x^*) = \frac{1}{2}(x_k - x^*)' \left[\boldsymbol{\nabla}^2 f(t_k \cdot x_k + (1 - t_k) \cdot x^*)\right](x_k - x^*) \geq 0.$$

Since

$$|x^* - x_k| < \frac{\eta}{k}$$

for any $k \in N_+$, the sequence $\{x_k\}_{k \in N_+}$ converges to x^*. Defining $\tilde{x}_k = t_k \cdot x_k + (1 - t_k) \cdot x^* = x^* + t_k(x_k - x^*)$ for any $k \in N_+$, it follows that $\tilde{x}_k \to x^*$ as $k \to \infty$ as well. Because the twice continuous differentiability of f implies that the Hessian $\boldsymbol{\nabla}^2 f$ is continuous on E, this implies that

$$\lim_{k \to \infty} \left\| \boldsymbol{\nabla}^2 f(\tilde{x}_k) - \boldsymbol{\nabla}^2 f(x^*) \right\| = 0.$$

Let $s_k = \frac{x_k - x^*}{|x_k - x^*|}$ for any $k \in N_+$. Then, the above inequality can be rewritten as

$$\frac{|x_k - x^*|^2}{2} \left(s_k' \left[\boldsymbol{\nabla}^2 f(\tilde{x}_k)\right] s_k \right) \geq 0$$

for any $k \in N_+$. Since $\{s_k\}_{k \in N_+}$ is a sequence taking values in the unit disk $S = \{z \in \mathbb{R}^n \mid |z| = 1\}$, which is compact and thus bounded by the Heine-Borel theorem, by the Bolzano-Weierstrass theorem we can conclude that $\{s_k\}_{k \in N_+}$ admits a convergent sequence with limit $s \in S$. For notational simplicity, assume that $\{s_k\}_{k \in N_+}$ is itself this subsequence. Therefore,

$$\lim_{k \to \infty} s_k' \left[\boldsymbol{\nabla}^2 f(\tilde{x}_k)\right] s_k = s' \left[\boldsymbol{\nabla}^2 f(x^*)\right] s < 0,$$

where the last equality holds by the negative definiteness of $\boldsymbol{\nabla}^2 f(x^*)$ and the fact that $s \neq \mathbf{0}$. It follows from the definition of convergence that there exists an $N \in N_+$ such that

$$\left| s_k' \left[\nabla^2 f(\tilde{x}_k) \right] s_k - s' \left[\nabla^2 f(x^*) \right] s \right| < -s' \left[\nabla^2 f(x^*) \right] s,$$

and in particular

$$s_k' \left[\nabla^2 f(\tilde{x}_k) \right] s_k < 0$$

for any $k \geq N$. Since $|x_N - x^*| > 0$ by the distinctness of x_k and x^*, this implies that

$$\frac{|x_N - x^*|^2}{2} \left(s_N' \left[\nabla^2 f(\tilde{x}_N) \right] s_N \right) < 0,$$

which contradicts our initial assumption. Thus, x^* is a strict local maximizer of f.

Q.E.D.

Corollary to Theorem 3.2 (Necessary and Sufficient Conditions for Unconstrained Local Minimization)
Let E be an open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ a twice continuously differentiable function on E. Consider the unconstrained optimization problem

$$\min_{x \in E} \quad f(x).$$

Then, the following hold true:

i) **(First Order Necessary Conditions for Local Minimization)**
If $x^* \in E$ is a local minimum of f on E, then $\nabla f(x^*) = \mathbf{0}$.

ii) **(Second Order Sufficient Conditions for Local Minimization)**
Let $\nabla^2 f(x)$ be the Hessian matrix of f at $x \in E$. If $\nabla f(x^*) = \mathbf{0}$ and the Hessian $\nabla^2 f(x^*)$ is postive definite for some $x^* \in E$, then x^* is a strict local minimum of f on E.

Proof) Again, we change the problem to one of maximizing $-f$. Then, the first order necessary conditions are unchanged, and we require a positive definite $\nabla^2 f(x^*)$ instead of a negative definite Hessian at x^*.

Q.E.D.

3.2 Constrained Optimization

Now we move onto optimization problems of the following form:

$$\begin{aligned} \max_{x\in E} \quad & f(x) \\ \text{subject to} \quad & g_1(x)\geq 0,\cdots,g_k(x)\geq 0, \end{aligned}$$

where $f, g_1, \cdots, g_k$ are real valued functions defined on an open subset E of $\mathbb{R}^n$. We say that $x^* \in E$ is a local maximizer (minimizer) of f subject to the constraints $g_1(x) \geq 0, \cdots, g_k(x) \geq 0$ if there exists a $\delta > 0$ such that, for any $x \in B(x^*, \delta)$ such that $g_1(x) \geq 0, \cdots, g_k(x) \geq 0$, $f(x) \leq f(x^*)$ ($f(x) \geq f(x^*)$). We first present the Karusch-Kuhn-Tucker (KKT) theorem, which provides first order necessary conditions for a point on E to be a local maximum to the constrained optimization problems with inequality constraints.

Theorem 3.3 (KKT First Order Conditions for Constrained Maximization)
Let E be an open subset of $\mathbb{R}^n$ and $f, g_1, \cdots, g_k : E \to \mathbb{R}^n$ continuously differentiable functions on E. Suppose $x^* \in E$ is a local maximum of the constrained optimization problem

$$\begin{aligned} \max_{x\in E} \quad & f(x) \\ \text{subject to} \quad & g_1(x)\geq 0,\cdots,g_k(x)\geq 0. \end{aligned}$$

Suppose that the first $m \leq k$ constraints are binding at x^*, that is, $g_1(x^*) = 0, \cdots, g_m(x^*) = 0, g_{m+1}(x^*) > 0, \cdots, g_k(x^*) > 0$, and that the gradients $\boldsymbol{\nabla} g_1(x^*), \cdots, \boldsymbol{\nabla} g_m(x^*) \in \mathbb{R}^n$ are linearly independent. Then, the following first order conditions hold:

i) **(Stationarity)**
There exist $\lambda_1^*, \cdots, \lambda_k^* \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(x^*) + \sum_{i=1}^{k} \lambda_i^* \boldsymbol{\nabla} g(x^*) = \mathbf{0}.$$

ii) **(Primal Feasibility)**
For any $1 \leq i \leq k$, $g_i(x^*) \geq 0$.

iii) **(Dual Feasibility)**
For any $1 \leq i \leq k$, $\lambda_i^* \geq 0$.

iv) **(Complementary Slackness)**

For any $1 \leq i \leq k$,

$$\lambda_i^* \cdot g_i(x^*) = 0.$$

Proof) The proof is long, complicated and oftentimes not very intuitive, so we present here some intuition for the proof before delving into the details. Focusing on the simplest case of only one biniding inequality constraint, what we want to do is to show that the gradients of the objective function f and the constraint g at the maximum is inversely proportional to each other, so that they go in opposite directions.

To this end, we think of vectors "tangent" to the constraint set at x^*. What we mean by these are the directions of entry of curves that enter the constraint set through x^*, where $g(x^*) = 0$. Let v be one such vector. Then, it should either be orthogonal to the gradient $\boldsymbol{\nabla} g(x^*)$, in which case v are tangent in the original sense of the word, so that the curve move across the surface of the constraint set, or it should point roughly in the direction of $\boldsymbol{\nabla} g(x^*)$, so that the value of g increases from 0 and the constraint is satisfied, that is, so that the curve enters into the interior of the constraint set. This condition can be written as $\boldsymbol{\nabla} g(x^*)'v \geq 0$. On the other hand, the gradient $\boldsymbol{\nabla} f(x^*)$, which represents the direction in which f grows the fastest, should point away from any vector v tangent to the constraint set. This is because moving roughly in the direction of v means returning to the constraint set, on which f cannot be increased beyond $f(x^*)$. This condition is written as $\boldsymbol{\nabla} f(x^*)'v \leq 0$.

In short, the gradient $\boldsymbol{\nabla} f(x^*)$ should not point in the same direction as any vector v entering the constraint set through x^*, and that v should instead point in the direction of the constraint set, or $\boldsymbol{\nabla} g(x^*)$. Thus, any vector whose angle with $\boldsymbol{\nabla} g(x^*)$ is acute should form an obtuse angle with $\boldsymbol{\nabla} f(x^*)$ and vice versa, which reveals that $\boldsymbol{\nabla} g(x^*)$ and $\boldsymbol{\nabla} f(x^*)$ should ultimately point in the opposite direction.

In the proof sketch above there are three main steps. First, we need to show that any tangent vector v actually does point roughly in the direction of $\boldsymbol{\nabla} g(x^*)$ in the sense that $\boldsymbol{\nabla} g(x^*)'v \geq 0$. Similarly, we need to show that any tangent vector v should point away from $\boldsymbol{\nabla} f(x^*)$ in the sense that $\boldsymbol{\nabla} f(x^*)'v \leq 0$. Finally, we must prove that these conditions imply the stationarity condition, that is, the inverse proportionality of $\boldsymbol{\nabla} g(x^*)$ and $\boldsymbol{\nabla} f(x^*)$. The proof below is accordingly split into parts corresponding to these three steps.

Step 0: Notations and Preliminary Definitions

Since x^* is a local maximum for the constrained maximization problem above, by definition there exists a $\delta > 0$ such that $f(x) \le f(x^*)$ for any $x \in B(x^*, \delta)$ that satisfies the inequality constraints. By assumption, only the first m constraints are binding at x^*. This implies that we can view x^* as a local maximum of the problem

$$\begin{aligned} \max_{x \in E} \quad & f(x) \\ \text{subject to} \quad & g_1(x) \ge 0, \cdots, g_m(x) \ge 0, \end{aligned}$$

that is, we can ignore the non-binding constraints. To see this, note that, because $g_{m+1}, \cdots, g_k$ are continuous functions that are all positive at x^*, there exists an $0 < \eta < \delta$ such that, on $B(x^*, \eta)$, $g_{m+1}(x) > 0, \cdots, g_k(x) > 0$. As such, for any $x \in B(x^*, \eta)$ such that $g_1(x) \ge 0, \cdots, g_m(x) \ge 0$, because x is contained in $B(x^*, \delta)$, we must have $f(x) \le f(x^*)$.

Given the above modification of the problem, define the function $g : E \to \mathbb{R}^m$ as $g = (g_1, \cdots, g_m)$. By the continuous differentiability of $g_1, \cdots, g_m$, g is continuously differentiable. The set of all points in the neighborhood $B(x^*, \eta)$ that satisfy the constraints is denoted by

$$L = \{x \in B(x^*, \eta) \mid g(x) \ge \mathbf{0}\};$$

notice how L becomes the level surface of all points satisfying $g(x) = \mathbf{0}$ if the constraints are equalities. Now we can say that $f(x) \le f(x^*)$ for any $x \in L$. In addition, L is non-empty because $x^* \in L$ by assumption.

Define the set T as the set of all $v \in \mathbb{R}^n$ for which there exist a neighborhood U of 0 and a continuously differentiable function $\gamma : U \to \mathbb{R}^n$ such that:

- $\gamma(t) \in B(x^*, \eta)$ for any $t \in U$
- $\gamma(t) \in L$ for any $t \ge 0$
- $\gamma(0) = x^*$
- $\gamma'(0) = v$.

T is our set of all directions of entry of continuously differentiable curves that enter L through x^*.

Define the matrix

$$A = g'(x^*) = \begin{pmatrix} \boldsymbol{\nabla} g_1(x^*)' \\ \vdots \\ \boldsymbol{\nabla} g_m(x^*)' \end{pmatrix} \in \mathbb{R}^{m \times n},$$

which by assumption has linearly independent rows. By implication, $m \leq n$, and we define

$$N = \{x \in \mathbb{R}^n \mid Ax \geq \mathbf{0}\}.$$

N is our set of all vectors that point roughly in the same direction as the gradient of g at x^*.

Step 1: Tangent vectors point in the direction of $g'(x^*)$

Our goal in this section is to show that $T = N$.

$T \subset N$ follows easily. Suppose that $v \in T$, so that there exists some neighborhood U of 0 and continuously differentiable curve $\gamma : U \to B(x^*, \eta)$ on L for any $t \geq 0$ such that $\gamma(0) = x^*$ and $\gamma'(0) = v$. Since $g(\gamma(t)) \geq \mathbf{0}$ for any $t \geq 0$, note that, for any $t > 0$ and $1 \leq i \leq m$,

$$\frac{g_i(\gamma(t)) - g_i(\gamma(0))}{t} = \frac{g_i(\gamma(t)) - g_i(x^*)}{t} = \frac{g_i(\gamma(t))}{t} \geq 0$$

because x^* satisfies the first m constraints as equalities. Therefore, taking $t \downarrow 0$ shows us, by the chain rule, that

$$0 \leq \lim_{t \downarrow 0} \frac{g_i(\gamma(t))}{t} = \boldsymbol{\nabla} g_i(x^*)'\gamma'(0) = \boldsymbol{\nabla} g_i(x^*)'v.$$

This holds for any $1 \leq i \leq m$, so $Av \geq \mathbf{0}$ and $v \in N$.

Proving the reverse inclusion is trickier, and requires the use of the implicit function theorem. The rows of A comprise a linearly independent subset of $\mathbb{R}^n$. There exist vectors $z_1, \cdots, z_{n-m}$ in $\mathbb{R}^n$ such that $\{\boldsymbol{\nabla} g_1(x^*), \cdots, \boldsymbol{\nabla} g_m(x^*), z_1, \cdots, z_{n-m}\}$ becomes a basis of $\mathbb{R}^n$; let the matrix

$$B = \begin{pmatrix} \underbrace{A}_{m \times n} \\ \underbrace{Z}_{(n-m) \times n} \end{pmatrix}$$

be the non-singular $n \times n$ matrix constructed by putting its rows equal to the above basis.

Now choose some $v \in N$, so that $Av \geq \mathbf{0}$. We define the function $G : \mathbb{R} \times B(x^*, \eta) \to \mathbb{R}$

as follows:

$$G(t,x) = \begin{pmatrix} g(x) - t \cdot Av \\ Z(x - x^* - t \cdot v) \end{pmatrix}$$

for any $(t,x) \in \mathbb{R} \times B(x^*,\eta)$. Being the Cartesian product of open sets, $\mathbb{R} \times B(x^*,\eta)$ is an open subset of $\mathbb{R}^{1+n}$, and G is continuously differentiable on $\mathbb{R} \times B(x^*,\eta)$ with derivative

$$G'(t,x) = \begin{pmatrix} -Av & g'(x) \\ -Zv & Z \end{pmatrix}$$

for any $(t,x) \in \mathbb{R} \times B(x^*,\eta)$. In addition, $G(0,x^*) = \mathbf{0}$, and the rightmost $n \times n$ block of the derivative

$$G'(0,x^*) = \begin{pmatrix} -Av & A \\ -Zv & Z \end{pmatrix}$$

is simply B and therefore invertible. By the implicit function theorem, there then exist an open set $U \subset \mathbb{R}^{1+n}$ containing $(0,x^*)$, an open set $W \subset \mathbb{R}$ containing 0, and a continuously differentiable mapping $\gamma : W \to \mathbb{R}^n$ satisfying

$$(t,\gamma(t)) \in U \subset \mathbb{R} \times B(x^*,\eta) \quad \text{and} \quad G(t,\gamma(t)) = \mathbf{0}$$

for any $t \in W$, with $\gamma(0) = x^*$. Since this means that

$$G(t,\gamma(t)) = \begin{pmatrix} g(\gamma(t)) - t \cdot Av \\ Z\left(\gamma(t) - x^* - t \cdot v\right) \end{pmatrix} = \mathbf{0},$$

we have $g(\gamma(t)) = t \cdot Av \geq \mathbf{0}$ for any $t \in W$ such that $t \geq 0$, so that $\gamma(t) \in L$ for any non-negative $t \in W$. Finally,

$$\gamma'(0) = -B^{-1} \begin{pmatrix} -Av \\ -Zv \end{pmatrix} = v.$$

Therefore, we have found a neighborhood W around 0 and a continuously differentiable function $\gamma : W \to \mathbb{R}^n$ such that $\gamma(t) \in L$ for any $t \geq 0$, $\gamma(0) = x^*$ and $\gamma'(0) = v$. By definition, $v \in T$, and this proves that $N \subset T$.

Step 2: Tangent vectors point away from $\nabla f(x^*)$

Here we formulate the result for vectors in N, because the previous step revealed that $N = T$. Let $v \in N$, and let $\gamma : U \to B(x^*,\eta)$ be the associated continuously differentiable curves such that $\gamma(t) \in L$ for any $t \geq 0$, $\gamma(0) = x^*$ and $\gamma'(0) = v$. Define $k : U \to \mathbb{R}$ as

$$k(t) = f(\gamma(t))$$

for any $t \in U$. Then, by the chain rule, k is continuously differentiable on U with derivative

$$k'(t) = \boldsymbol{\nabla} f(\gamma(t))'\gamma'(t)$$

for any $t \in U$. In addition, for any $t \geq 0$, since $\gamma(t)$ satisfies the constraints,

$$f(\gamma(t)) \leq f(x^*) = f(\gamma(0)).$$

It follows that

$$\frac{k(t) - k(0)}{t} = \frac{f(\gamma(t)) - f(\gamma(0))}{t} \leq 0$$

for any $t > 0$, so that

$$0 \geq \lim_{t \downarrow 0} \frac{k(t) - k(0)}{t} = k'(0) = \boldsymbol{\nabla} f(x^*)'v.$$

Therefore, we have established that $\boldsymbol{\nabla} f(x^*)'v \leq 0$ for any $v \in N$.

Step 3: Completion of the Proof

Proving the remainder of the KKT theorem reduces to a simple application of Farkas' lemma. Farkas' lemma tells us that:

1. **Either:** The system $A'\lambda = -\boldsymbol{\nabla} f(x^*)$ has a solution $\lambda \in \mathbb{R}^m_+$, or
2. **Or:** There exists some $v \in \mathbb{R}^n$ such that $Av \geq \mathbf{0}$ and $-\boldsymbol{\nabla} f(x^*)'v < 0$.

Suppose there exists a $v \in \mathbb{R}^n$ such that $Av \geq \mathbf{0}$ and $-\boldsymbol{\nabla} f(x^*)'v < 0$. Then, $v \in T$, which implies $v \in N$. By the result we showed earlier, we must have $-\boldsymbol{\nabla} f(x^*)'v \geq 0$, a contradiction. Therefore, there must exist some $\lambda^* = (\lambda_1^*, \cdots, \lambda_m^*) \in \mathbb{R}^m_+$ such that

$$-\boldsymbol{\nabla} f(x^*) = A'\lambda^* = \sum_{i=1}^{m} \lambda_i^* \cdot \boldsymbol{\nabla} g_i(x^*),$$

or, by rearranging terms,

$$\boldsymbol{\nabla} f(x^*) + \sum_{i=1}^{m} \lambda_i^* \cdot \boldsymbol{\nabla} g_i(x^*) = \mathbf{0}.$$

We have therefore proved stationarity and dual feasbility. The complementary slackness condition follows by putting $\lambda_i^* = 0$ for $m+1 \leq i \leq k$. Then,

$$\lambda_i^* \cdot g_i(x^*) = 0$$

for any $1 \leq i \leq k$, and

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i^* \cdot \nabla g_i(x^*) = \mathbf{0}.$$

Q.E.D.

Corollary to Theorem 3.3 (KKT First Order Conditions for Constrained Minimization)
Let E be an open subset of $\mathbb{R}^n$ and $f, g_1, \cdots, g_k : E \to \mathbb{R}^n$ continuously differentiable functions on E. Suppose $x^* \in E$ is a local maximum of the constrained optimization problem

$$\begin{aligned} \min_{x \in E} \quad & f(x) \\ \text{subject to} \quad & g_1(x) \leq 0, \cdots, g_k(x) \leq 0. \end{aligned}$$

Suppose that the first $m \leq k$ constraints are binding at x^*, that is, $g_1(x^*) = 0, \cdots, g_m(x^*) = 0, g_{m+1}(x^*) < 0, \cdots, g_k(x^*) < 0$, and that the gradients $\nabla g_1(x^*), \cdots, \nabla g_m(x^*) \in \mathbb{R}^n$ are linearly independent. Then, the following first order conditions hold:

i) **(Stationarity)**
There exist $\lambda_1^*, \cdots, \lambda_k^* \in \mathbb{R}$ such that

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i^* \nabla g(x^*) = \mathbf{0}.$$

ii) **(Primal Feasibility)**
For any $1 \leq i \leq k$, $g_i(x^*) \leq 0$.

iii) **(Dual Feasibility)**
For any $1 \leq i \leq k$, $\lambda_i^* \geq 0$.

iv) **(Complementary Slackness)**

For any $1 \leq i \leq k$,

$$\lambda_i^* \cdot g_i(x^*) = 0.$$

Proof) Once again we re-formulate the problem as one in which we maximize $-f$ subject to $-g_1(x) \geq 0, \cdots, -g_k(x) \geq 0$. Then, by the preceding theorem, if x^* is a local maximizer of this problem (thus a local minimizer of the original problem), then there exist $\lambda_1^*, \cdots, \lambda_k^* \geq 0$ such that

$$\lambda_i^* g_i(x^*) = 0$$

for $1 \leq i \leq k$ and

$$-\nabla f(x^*) - \sum_{i=1}^{k} \lambda_i^* \cdot \nabla g_(x^*) = \mathbf{0}.$$

The last equation can be expressed as

$$\nabla f(x^*) + \sum_{i=1}^{k} \lambda_i^* \cdot \nabla g_(x^*) = \mathbf{0},$$

so we can see that the KKT conditions above are satisfied. Note how the direction of the inequalities in the inequality constraints must be reversed for us to obtain the same KKT conditions as in the maximization case.

Q.E.D.

The values $\lambda_1^*, \cdots, \lambda_k^* \geq 0$ derived above are called the Lagrange multipliers for the associated constrained optimization problem, and the requirement that the gradients of the binding constraints at x^* must be linearly independent is called the Non-degenerate Constraint Qualification (NDCQ). There are also other kinds of constraint qualifications that allow the above conditions to be derived as necessary conditions of local optimization.

What the KKT theorem tells us is that, given any constrained maximization problem

$$\begin{aligned} \max_{x \in E} \quad & f(x) \\ \text{subject to} \quad & g_1(x) \geq 0, \cdots, g_k(x) \geq 0, \end{aligned}$$

we can transform it into one in which we "maximize" the Lagrangian function defined as

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^{k} \lambda_i \cdot g_i(x)$$

for any $x \in E$ and $\lambda \in \mathbb{R}^k_+$. This is because, given a local maximizer $x^* \in E$ and associated Lagrange multipliers $\lambda^* \in \mathbb{R}^k_+$, the first order necessary conditions for the unconstrained maximization of the Lagrangian are

$$\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial x} = \boldsymbol{\nabla} f(x^*) + \sum_{i=1}^{k} \lambda_i^* \cdot \boldsymbol{\nabla} g_i(x^*) = \mathbf{0},$$

which are exactly the KKT stationarity conditions derived above.

As with unconstrained optimization, we can provide second order sufficient conditions for local maximization in terms of the Hessian of the objective function. We state and prove the formal statement below:

Theorem 3.4 (KKT Second Order Conditions for Constrained Local Maximization)
Let E be an open subset of $\mathbb{R}^n$ and $f, g_1, \cdots, g_k : E \to \mathbb{R}^n$ twice continuously differentiable functions on E. Suppose $x^* \in E$ satisfies the following conditions:

i) There exist $\lambda_1^*, \cdots, \lambda_k^* \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(x^*) + \sum_{i=1}^{k} \lambda_i^* \boldsymbol{\nabla} g(x^*) = \mathbf{0}.$$

ii) $g_1(x^*) = 0, \cdots, g_l(x^*) = 0, g_{l+1}(x^*) > 0, \cdots, g_k(x^*) > 0.$

iii) For any $1 \leq i \leq k$, $\lambda_i^* \geq 0$ and $\lambda_i^* \cdot g_i(x^*) = 0$.

iv) Defining the set $N \subset \mathbb{R}^n$ as

$$N = \{v \in \mathbb{R}^n \mid \forall 1 \leq i \leq l,\ \boldsymbol{\nabla} g_i(x^*)' v \geq 0\},$$

for any non-zero $v \in N$

$$v' \left[\boldsymbol{\nabla}^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \cdot \left(\boldsymbol{\nabla}^2 g_i(x^*) \right) \right] v < 0.$$

Then, x^* is a strict local maximizer of the problem

$$\begin{aligned} \max_{x \in E} \quad & f(x) \\ \text{subject to} \quad & g_1(x) \geq 0, \cdots, g_k(x) \geq 0. \end{aligned}$$

Proof) The proof proceeds in a very similar fashion to the proof of the second order sufficient conditions for unconstrained maximization.

Define the function $F : E \to \mathbb{R}$ as

$$F(x) = f(x) + \sum_{i=1}^{k} \lambda_i^* \cdot g_i(x)$$

for any $x \in E$. Then, note that the sufficient conditions above require that $\boldsymbol{\nabla} F(x^*) = \mathbf{0}$ and

$$v' \left[\boldsymbol{\nabla}^2 F(x^*) \right] v < 0$$

for any $v \in N$. These are exactly the conditions for x^* to be a strict local maximum of the unconstrained maximization problem with objective function F, except for the weaker requirement that $\boldsymbol{\nabla}^2 F(x^*)$ only has to be negative definite for vectors $v \in N$. Thus, we can proceed in much the same way.

Since E is open, there exists an open ball $B(x^*, \eta)$ centered in x^* such that $B(x^*, \eta) \subset E$. Note that this ball is both open and convex, so that for any $x \in B(x^*, \eta)$, the multivariate version of Taylor's theorem applied to F tells us that

$$\begin{aligned} F(x) &= F(x^*) + \boldsymbol{\nabla} F(x^*)'(x - x^*) + \frac{1}{2}(x - x^*)' \left[\boldsymbol{\nabla}^2 F(\tilde{x}) \right] (x - x^*) \\ &= F(x^*) + \frac{1}{2}(x - x^*)' \left[\boldsymbol{\nabla}^2 F(\tilde{x}) \right] (x - x^*) \end{aligned}$$

for some convex combination $\tilde{x}$ of x and x^*, where we used the fact that $\boldsymbol{\nabla} F(x^*) = \mathbf{0}$. Using the fact that $\lambda_i^* \cdot g_1(x^*) = 0$ for any $1 \le i \le k$, we can write the above as

$$f(x) - f(x^*) = -\sum_{i=1}^{k} \lambda_i^* \cdot g_i(x) + \frac{1}{2}(x - x^*)' \left[\boldsymbol{\nabla}^2 F(\tilde{x}) \right] (x - x^*).$$

As we did earlier, suppose that x^* is not a strict local maximum of the constrained optimization problem above. Then, for any $m \in N_+$ there exists an $x_m \in B(x^*, \eta/m)$ distinct from x^* such that $g_1(x_m) \ge 0, \cdots, g_k(x_m) \ge 0$ and $f(x_m) \ge f(x^*)$. Let $\tilde{x}_m$ be the convex combination of x_m and x^* that satisfies the equation

$$f(x_m) - f(x^*) = -\sum_{i=1}^{k} \lambda_i^* \cdot g_i(x_m) + \frac{1}{2}(x_m - x^*)' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m) \right] (x_m - x^*)$$

for any $m \in N_+$. Since the left hand side is non-negative,

$$\frac{1}{2}(x_m - x^*)' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m) \right] (x_m - x^*) \ge \sum_{i=1}^{k} \lambda_i^* \cdot g_i(x_m),$$

and since $\lambda_i^* \geq 0$ and $g_i(x_m) \geq 0$ for $1 \leq i \leq k$, we have

$$\frac{1}{2}(x_m - x^*)' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m)\right](x_m - x^*) \geq 0.$$

It remains to show that this results in a contradiction.

By design, $x_m \to x^*$ as $m \to \infty$, so $\tilde{x}_m \to x^*$ as $m \to \infty$ as well. The continuous differentiability of f and each of the constraint functions $g_1, \cdots, g_k$ now implies that

$$\lim_{m\to\infty} \left\| \boldsymbol{\nabla}^2 F(\tilde{x}_m) - \boldsymbol{\nabla}^2 F(x^*) \right\| = 0$$

as well. For any $m \in N_+$, define $v_m = \frac{x_m - x^*}{|x_m - x^*|}$; we can now write

$$\frac{|x_m - x^*|^2}{2} v_m' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m)\right] v_m \geq 0.$$

Since $\{v_m\}_{m\in N_+}$ is a sequence of vectors in the unit disk on $\mathbb{R}^n$, which is compact, by the Heine-Borel and Bolzano-Weierstrass theorems there exists a convergent subseuqnece of $\{v_m\}_{m\in N_+}$; for notational simplicity, take this subsequence to be $\{v_m\}_{m\in N_+}$ itself. Denote the limit of this subsequence as $v \in \mathbb{R}^n$, and by the closedness of the unit disk, $|v| = 1$.

We will now show that $\boldsymbol{\nabla} g_i(x^*)'v \geq 0$ for $1 \leq i \leq l$. Note that, for any $m \in N_+$, we can write

$$g_i(x_m) = g_i(x^* + (x_m - x^*)) = g_i(x^* + |x_m - x^*| \cdot v_m).$$

By the multivariate mean value theorem, there exists a $t_m \in [0,1]$ such that

$$g_i(x_m) - g_i(x^*) = |x_m - x^*| \cdot \left[\boldsymbol{\nabla} g_i(x^* + t_m \cdot |x_m - x^*| v_m)' v_m\right],$$

or

$$\frac{g_i(x_m) - g_i(x^*)}{|x_m - x^*|} = \boldsymbol{\nabla} g_i(x^* + t_m \cdot |x_m - x^*| v_m)' v_m.$$

Since $\{t_m\}_{m\in N_+}$ takes values in the compact set $[0,1]$, by the Heine-Borel and Bolzano-Weierstrass theorems again, there exists a subsequence of $\{t_m\}_{m\in N_+}$ that converges to some point $t \in [0,1]$; once again, we let this subsequence be $\{t_m\}_{m\in N_+}$ itself. It is important not to let the notational simplicity fool us; the sequences we discuss and the limits we take going forward run along two nested subsequences of N_+.

Taking $m \to \infty$ on both sides, we can now see, by the continuity of the gradient of g_i, that

$$\lim_{m\to\infty} \frac{g_i(x_m) - g_i(x^*)}{|x_m - x^*|} = \boldsymbol{\nabla} g_i(x^*)'v.$$

On the other hand, for any $m \in N_+$, we know

$$\frac{g_i(x_m) - g_i(x^*)}{|x_m - x^*|} = \frac{g_i(x_m)}{|x_m - x^*|} \geq 0$$

because x_m satisfies the constraints and $g_i(x*) = 0$ for the first l binding constraints. Putting these two observations together, we obtain the inequality

$$\boldsymbol{\nabla} g_i(x^*)'v = \lim_{m \to \infty} \frac{g_i(x_m) - g_i(x^*)}{|x_m - x^*|} \geq 0.$$

This shows us that $v \in N$, and by assumption,

$$v' \left[\boldsymbol{\nabla}^2 F(x^*)\right] v < 0.$$

Since $v_m' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m)\right] v_m \to v' \left[\boldsymbol{\nabla}^2 F(x^*)\right] v$ as $m \to \infty$, there exists an $N_0 \in N_+$ such that

$$\left| v_m' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m)\right] v_m - v' \left[\boldsymbol{\nabla}^2 F(x^*)\right] v \right| < -v' \left[\boldsymbol{\nabla}^2 F(x^*)\right] v,$$

and in particular,

$$v_m' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m)\right] v_m < 0,$$

for any $m \geq N_0$. Since $|x_m - x^*| > 0$ for any $m \in N_+$, we now have

$$\frac{|x_m - x^*|^2}{2} v_m' \left[\boldsymbol{\nabla}^2 F(\tilde{x}_m)\right] v_m < 0$$

for any $m \geq N_0$, which contradicts the result derived above. Therefore, x^* is a strict local maximizer of the constrained optimization problem at hand.

Q.E.D.

Corollary to Theorem 3.4 (KKT Second Order Conditions for Constrained Local Minimization)

Let E be an open subset of $\mathbb{R}^n$ and $f, g_1, \cdots, g_k : E \to \mathbb{R}^n$ twice continuously differentiable functions on E. Suppose $x^* \in E$ satisfies the following conditions:

i) There exist $\lambda_1^*, \cdots, \lambda_k^* \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(x^*) + \sum_{i=1}^{k} \lambda_i^* \boldsymbol{\nabla} g(x^*) = \mathbf{0}.$$

ii) $g_1(x^*) = 0, \cdots, g_l(x^*) = 0, g_{l+1}(x^*) < 0, \cdots, g_k(x^*) < 0.$

iii) For any $1 \leq i \leq k$, $\lambda_i^* \geq 0$ and $\lambda_i^* \cdot g_i(x^*) = 0$.

iv) Defining the set $N \subset \mathbb{R}^n$ as

$$N = \{v \in \mathbb{R}^n \mid \forall 1 \leq i \leq l,\ \boldsymbol{\nabla} g_i(x^*)'v \leq 0\},$$

for any non-zero $v \in N$

$$v'\left[\boldsymbol{\nabla}^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \cdot \left(\boldsymbol{\nabla}^2 g_i(x^*)\right)\right] v < 0.$$

Then, x^* is a strict local minimizer of the problem

$$\begin{aligned} \max_{x \in E} \quad & f(x) \\ \text{subject to} \quad & g_1(x) \leq 0, \cdots, g_k(x) \leq 0. \end{aligned}$$

Proof) Suppose the above conditions hold. Reformulating the conditions in terms of $-f$ and $-g_1, \cdots, -g_k$, what we know is that there exist $\lambda_1^*, \cdots, \lambda_k^* \geq 0$ such that

$$-\boldsymbol{\nabla} f(x^*) - \sum_{i=1}^{k} \lambda_i^* \boldsymbol{\nabla} g(x^*) = \mathbf{0}$$

and

$$v'\left[-\boldsymbol{\nabla}^2 f(x^*) - \sum_{i=1}^{m} \lambda_i^* \cdot \left(\boldsymbol{\nabla}^2 g_i(x^*)\right)\right] v < 0$$

for any $v \in \mathbb{R}^n$ such that $-\boldsymbol{\nabla} g_i(x^*)'v \geq 0$ for $1 \leq i \leq l$, and that x^* also satisfies $-g_i(x^*) \geq 0$ for $1 \leq i \leq k$, where the first l constraints are binding and the rest non-binding. By the preceding theorem x^* is a strict local maximizer of the problem

$$\begin{aligned} \max_{x \in E} \quad & -f(x) \\ \text{subject to} \quad & -g_1(x) \geq 0, \cdots, -g_k(x) \geq 0. \end{aligned}$$

This essentially means that x^* is a strict local minimizer of the original minimization problem.

Q.E.D.

3.3 The Envelope Theorem

Often the functions involved in a constrained optimization problem involve parameters of interest. Think, for example, of income and commodity prices in a utility maximization problem. In this case, we want to find out how our objective function responds to changes in these parameters. The envelope theorem provides a useful simplification of this problem that reduces total derivatives to partial derivatives.

Theorem 3.5 (The Envelope Theorem)
Let E be an open subset of $\mathbb{R}^n$, Θ an open subset of $\mathbb{R}^m$, and $f, g_1, \cdots, g_k : E \times \Theta \to \mathbb{R}^n$ that are continuously differentiable on $E \times \Theta$. Let the function $x^* : \Theta \to E$ be defined so that $x^*(\theta)$ is a global solution to the problem

$$\begin{aligned} \max_{x \in E} \quad & f(x,\theta) \\ \text{subject to} \quad & g_1(x,\theta) \geq 0, \cdots, g_k(x,\theta) \geq 0 \end{aligned}$$

for any $\theta \in \Theta$. Assume that the NDCQs are satisfied at each $x^*(\theta)$, and let $\lambda^* : \Theta \to \mathbb{R}^k$ be defined so that, for any $\theta \in \Theta$, $\lambda^*(\theta)$ are the set of Lagrange multipliers corresponding to the maximizer $x^*(\theta)$. Suppose x^* and λ^* are both differentiable at some $\theta_0 \in \Theta$.

Then, the value function $V : \Theta \to \mathbb{R}$ defined as

$$V(\theta) = f(x^*(\theta),\theta)$$

for any $\theta \in \Theta$ is differentiable at θ_0 with gradient

$$\boldsymbol{\nabla} V(\theta_0) = \boldsymbol{\nabla}_\theta f(x^*(\theta_0),\theta_0) + \sum_{i=1}^{k} \lambda_i^*(\theta_0) \cdot \boldsymbol{\nabla}_\theta g(x^*(\theta_0),\theta_0).$$

Proof) The notations $\boldsymbol{\nabla}_x f(x,\theta)$ and $\boldsymbol{\nabla}_\theta f(x,\theta)$ be defined as

$$\boldsymbol{\nabla}_x f(x,\theta) = \begin{pmatrix} \frac{\partial f(x,\theta))}{\partial x_1} \\ \vdots \\ \frac{\partial f(x,\theta))}{\partial x_n} \end{pmatrix}, \quad \boldsymbol{\nabla}_\theta f(x,\theta) = \begin{pmatrix} \frac{\partial f(x,\theta))}{\partial \theta_1} \\ \vdots \\ \frac{\partial f(x,\theta))}{\partial \theta_n} \end{pmatrix}$$

for any $(x,\theta) \in E \times \Theta$. We define $\boldsymbol{\nabla}_x g_i(x,\theta)$ and $\boldsymbol{\nabla}_\theta g_i(x,\theta)$ in the same manner for $1 \leq i \leq k$.

By definition, $V(\theta) = f(x^*(\theta),\theta)$ for any $\theta \in \Theta$. By the differentiability of f on $E \times \Theta$

and x^* at θ_0, V is differentiable at θ_0 with derivative given as

$$\begin{aligned} V'(\theta_0) &= \left(\boldsymbol{\nabla}_x f(x^*(\theta_0), \theta_0)' \quad \boldsymbol{\nabla}_\theta f(x^*(\theta_0), \theta_0)' \right) \begin{pmatrix} Dx^*(\theta_0) \\ I_m \end{pmatrix} \\ &= \boldsymbol{\nabla}_x f(x^*(\theta_0), \theta_0)' Dx^*(\theta_0) + \boldsymbol{\nabla}_\theta f(x^*(\theta_0), \theta_0)', \end{aligned}$$

where

$$Dx^*(\theta_0) = \begin{pmatrix} (D_1 x_1^*)(\theta_0) & \cdots & (D_m x_1^*)(\theta_0) \\ \vdots & \ddots & \vdots \\ (D_1 x_n^*)(\theta_0) & \cdots & (D_m x_n^*)(\theta_0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla} x_1^*(\theta_0)' \\ \vdots \\ \boldsymbol{\nabla} x_n^*(\theta_0)' \end{pmatrix}.$$

It follows that

$$\boldsymbol{\nabla} V(\theta_0) = \sum_{j=1}^n \frac{\partial f(x^*(\theta_0), \theta_0)}{\partial x_j} \boldsymbol{\nabla} x_j^*(\theta_0) + \boldsymbol{\nabla}_\theta f(x^*(\theta_0), \theta_0).$$

For any $\theta \in \Theta$, by the stationarity condition,

$$\boldsymbol{\nabla}_x f(x^*(\theta), \theta) + \sum_{i=1}^k \lambda_i^*(\theta) \cdot \boldsymbol{\nabla}_x g_i(x^*(\theta), \theta) = \mathbf{0},$$

by the primal feasibility condition, $g_i(x^*(\theta), \theta) \geq 0$ for $1 \leq i \leq k$, and by the complementary slackness condition,

$$\lambda_i^*(\theta) \cdot g_i(x^*(\theta), \theta) = 0$$

for $1 \leq i \leq k$. Since

$$\sum_{i=1}^k \lambda_i^*(\theta) \cdot g_i(x^*(\theta), \theta) = 0$$

on Θ, differentiating both sides by θ at θ_0 yields

$$\begin{aligned} \mathbf{0} = & \sum_{i=1}^k \boldsymbol{\nabla} \lambda_i^*(\theta_0) \cdot g_i(x^*(\theta_0), \theta_0) \\ & + \sum_{i=1}^k \lambda_i^*(\theta_0) \left[\sum_{j=1}^n \frac{\partial g_i(x^*(\theta_0), \theta_0)}{\partial x_j} \boldsymbol{\nabla} x_j^*(\theta_0) + \boldsymbol{\nabla}_\theta g_i(x^*(\theta_0), \theta_0) \right]. \end{aligned}$$

For any $1 \leq i \leq k$, if the ith constraint is binding under θ_0, then $g_i(x^*(\theta_0), \theta_0) = 0$ and therefore

$$\boldsymbol{\nabla} \lambda_i^*(\theta_0) \cdot g_i(x^*(\theta_0), \theta_0) = \mathbf{0}.$$

On the ohter hand, if the ith constraint is not binding under θ_0, then $g_i(x^*(\theta_0),\theta_0) > 0$. The differentiability of x^* at θ and the differentiability of g_i on $E \times \Theta$ indicates that the mapping $\theta \mapsto g_i(x^*(\theta),\theta)$ is continuous at θ_0. By implication, there exists a $\delta > 0$ such that

$$|g_i(x^*(\theta),\theta) - g_i(x^*(\theta_0),\theta_0)| < g_i(x^*(\theta_0),\theta_0),$$

and in particular

$$g_i(x^*(\theta),\theta) > 0,$$

for any $\theta \in \Theta$ such that $|\theta_0 - \theta| < \delta$. Thus, $g_i(x^*(\theta),\theta) > 0$ in the neighborhood $B(\theta_0,\delta)$ around θ_0, and by complementary slackness, $\lambda_i^*(\theta) = 0$ on this neighborhood as well. It follows that $\boldsymbol{\nabla}\lambda_i^*(\theta_0) = \mathbf{0}$, from which we have

$$\boldsymbol{\nabla}\lambda_i^*(\theta_0) \cdot g_i(x^*(\theta_0),\theta_0) = \mathbf{0}.$$

We have just shown that

$$\sum_{i=1}^{k} \boldsymbol{\nabla}\lambda_i^*(\theta_0) \cdot g_i(x^*(\theta_0),\theta_0) = \mathbf{0},$$

so it follows that

$$\begin{aligned}
\sum_{i=1}^{k} \lambda_i^*(\theta_0) \cdot \boldsymbol{\nabla}_\theta g_i(x^*(\theta_0),\theta_0) &= -\sum_{i=1}^{k} \lambda_i^*(\theta_0) \left[\sum_{j=1}^{n} \frac{\partial g_i(x^*(\theta_0),\theta_0)}{\partial x_j} \boldsymbol{\nabla} x_j^*(\theta_0) \right] \\
&= -\sum_{j=1}^{n} \left[\sum_{i=1}^{k} \lambda_i^*(\theta_0) \cdot \frac{\partial g_i(x^*(\theta_0),\theta_0)}{\partial x_j} \right] \boldsymbol{\nabla} x_j^*(\theta_0).
\end{aligned}$$

By the stationarity condition, for any $1 \leq j \leq n$

$$\frac{\partial f(x^*(\theta_0),\theta_0)}{\partial x_j} + \sum_{i=1}^{k} \lambda_i^*(\theta_0) \cdot \frac{\partial g_i(x^*(\theta_0),\theta_0)}{\partial x_j} = 0,$$

so we have

$$\sum_{i=1}^{k} \lambda_i^*(\theta_0) \cdot \boldsymbol{\nabla}_\theta g_i(x^*(\theta_0),\theta_0) = \sum_{j=1}^{n} \frac{\partial f(x^*(\theta_0),\theta_0)}{\partial x_j} \cdot \boldsymbol{\nabla} x_j^*(\theta_0).$$

Finally, substituting this result into the original formula for $\boldsymbol{\nabla} V(\theta_0)$ yields

$$\boldsymbol{\nabla} V(\theta_0) = \boldsymbol{\nabla}_\theta f(x^*(\theta_0),\theta_0) + \sum_{i=1}^{k} \lambda_i^*(\theta_0) \cdot \boldsymbol{\nabla}_\theta g(x^*(\theta_0),\theta_0).$$

Q.E.D.

Chapter 4

Correspondences

Let E and F be arbitrary sets. A correspondence $\Gamma : E \to F$ from E into F is a function from E into the power set 2^F of F. In this sense, correspondences can be thought of as set-valued functions; note the contrast with measures, which are defined on collections of subsets but take real or complex values. The graph $Gr(\Gamma)$ of Γ is a set in $E \times F$ defined as

$$Gr(\Gamma) = \{(x,y) \in E \times F \mid x \in E, y \in \Gamma(x)\}.$$

The image of a set $U \subset X$ under Γ is defined in the usual way, as the union of the sets $\Gamma(x)$ for $x \in U$. Formally, we write $\Gamma(U) = \bigcup_{x \in U} \Gamma(x)$ for the image of U under Γ. Note that we need not impose any topological or measure-theoretical structure on the sets E and F in order for a correspondence to be well-defined. This greatly contributes to the generality of correspondences.

Like functions, correspondences can also be continuous. There are two related concepts of continuity: upper and lower hemicontinuity. The intuition for the two concepts of continuity is simple. Consider a function $f : E \to F$. f is continuous at some point x if, for any open set V in F containing $f(x)$, there exists a neighborhood U around x contained in the inverse image $f^{-1}(V)$. With a correspondence $\Gamma : E \to F$, the starting point is the same; we want to say that Γ is continuous at x if, for any open set V containing $\Gamma(x)$, there exists a neighborhood U around x such that the image of U under Γ is contained in V.

It is here that we encounter a slight complication; what does it mean for $\Gamma(x)$, or the image of U under Γ, to be contained in V? We can think of two distinct notions of containment. First, we can require the set $\Gamma(x) \subset V$ to be a subset of V, in accordance with the usual notion of the containment of sets. On the other hand, we may require $\Gamma(x)$ to only intersect V at some point in F; these two notions of containment coincide when Γ is singleton valued, that is, when Γ is basically a function. If we define containment as set containment, then Γ is continuous at x if, for any open set V such that $\Gamma(x) \subset V$, there exists a neighborhood U of x such that $\Gamma(U) = \bigcup_{y \in U} \Gamma(y) \subset V$. If we require only intersection for containment, then Γ is continuous at x if, for any open set V such that $\Gamma(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $\Gamma(y) \cap V \neq \emptyset$ for any $y \in U$. The former is the concept of upper hemicontinuity, and the latter defines lower hemicontinuity.

In the section below, we formally introduce the two hemicontinuity concepts, and study their useful sequential characterizations.

4.1 Upper and Lower Hemicontinuity

Now we formally introduce the two hemicontinuity concepts. Let (E, τ) and (F, s) be topological spaces. The correspondence $\Gamma : E \to F$ is said to be **upper hemicontinuous** at $x \in E$ if:

$$\forall V \in s \text{ s.t. } \Gamma(x) \subset V, \quad \exists \text{ a neighborhood } U \in \tau \text{ of } x \text{ s.t. } \Gamma(U) = \bigcup_{y \in U} \Gamma(y) \subset V.$$

Likewise, we say that Γ is **lower hemicontinuous** at $x \in E$ if:

$$\forall V \in s \text{ s.t. } \Gamma(x) \cap V \neq \emptyset, \quad \exists \text{ a neighborhood } U \in \tau \text{ of } x \text{ s.t. } \Gamma(y) \cap V \neq \emptyset \quad \forall y \in U.$$

Γ is continuous at $x \in E$ if and only if it is both lower and upper hemicontinuous at x. The concepts of continuity introduced so far can be extended to the entire set E if they hold for any $x \in E$. If $\Gamma(x) = \emptyset$, then Γ is trivially lower hemicontinuous at x.

The following result provides a characterization of hemicontinuity in terms of sequences and elucidates the relationship between the hemicontinuity of correspondences and the topological properties of their graphs.

Theorem 4.1 (Characterization of Hemicontinuity)
Let $(E, d), (F, \rho)$ be metric spaces and τ, s the corresponding metric topologies. Suppose $\Gamma : E \to F$ is a nonempty-valued correspondence on E The following hold true:

i) Suppose $\Gamma(x)$ is compact for some $x \in E$. Then, Γ is upper hemicontinuous at x if and only if, for any sequence $\{x_n\}_{n \in N_+}$ converging to x and a sequence $\{y_n\}_{n \in N_+}$ in F such that $y_n \in \Gamma(x_n)$ for any $n \in N_+$, there exists a convergent subsequence of $\{y_{n_k}\}_{k \in N_+}$ with limit in $\Gamma(x)$.

ii) Γ is lower hemicontinuous at $x \in E$ if and only if, for any sequence $\{x_n\}_{n \in N_+}$ converging to x and $y \in \Gamma(x)$, there exists a subsequence $\{x_{n_k}\}_{k \in N_+}$ of $\{x_n\}_{n \in N_+}$ and a sequence $\{y_k\}_{k \in N_+}$ such that $y_k \in \Gamma(x_{n_k})$ for any $k \in N_+$ and $y_k \to y$ as $k \to \infty$.

iii) **(Closed Graph Theorem)**
Suppose Γ is compact valued. If Γ is upper hemicontinuous, then its graph is closed.
If the graph of Γ is closed and, in addition, $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$ is contained in a compact set for any bounded set $A \subset E$, then Γ is upper hemicontinuous.

Proof) **Characterization of Upper Hemicontinuity**

Suppose, for some $x \in E$, that $\Gamma(x)$ is compact.

Necessity

Let Γ be upper hemicontinuous at x, $\{x_n\}_{n\in N_+}$ be some sequence in E converging to x and $\{y_n\}_{n\in N_+}$ a sequence in F such that $y_n \in \Gamma(x_n)$ for any $n \in N_+$. We want to show that $\{y_n\}_{n\in N_+}$ has a convergent subsequence with limit in $\Gamma(x)$.

First, we want to show that $\{y_n\}_{n\in N_+}$ has a convergent subsequence. Since $\Gamma(x)$ is a compact set contained in the open (and closed) set F, and any metric space is a locally compact Hausdorff space, there exists an open set V with compact closure such that

$$\Gamma(x) \subset V \subset \overline{V} \subset F.$$

By the upper hemicontinuity of Γ, there exists a neighborhood U of x such that $\Gamma(U) \subset V \subset \overline{V}$; since $\{x_n\}_{n\in N_+}$ converges to x, there exists an $N \in N_+$ such that $x_n \in U$ for any $n \geq N$. It follows that, for any $n \geq N$, $y_n \in \Gamma(x_n) \subset \overline{V}$, so that $\{y_n\}_{n\geq N}$ takes values in the compact set $\overline{V}$. Because compactness is equivalent to sequential compactness in metric spaces, it follows that $\{y_n\}_{n\geq N}$ has a subsequence that converges to some point y in $\overline{V}$. Denote this subsequence by $\{y_{n_k}\}_{k\in N_+}$.

Next, we must show that $y \in \Gamma(x)$. Since $\Gamma(x)$ is compact and thus closed, this can be achieved by simply showing that $\rho(y,\Gamma(x)) = 0$, that is, the distance between y and the closed set $\Gamma(x)$ equals 0. To this end, choose any $\epsilon > 0$ and consider the inverse image

$$V = \{z \in F \mid \rho(z,\Gamma(x)) < \epsilon\}$$

of the distance function $\rho(\cdot,\Gamma(x)) : F \to \mathbb{R}_+$. V contains $\Gamma(x)$, and since the distance function is continuous on F and $(-\infty,\epsilon)$ is an open subset of $\mathbb{R}$, V is an open subset of F. Therefore, by upper hemicontinuity, there exists a neighborhood U around x such that $\Gamma(U) \subset V$. Since $\{x_n\}_{n\in N_+}$ and thus $\{x_{n_k}\}_{k\in N_+}$ converges to x, we can once again choose an $N \in N_+$ such that

$$x_{n_k} \in U \quad \text{for any } k \geq N.$$

In other words, for any $k \geq N$, $y_{n_k} \in \Gamma(x_{n_k}) \subset \Gamma(U) \subset V$ and therefore $\rho(y_{n_k},\Gamma(x)) < \epsilon$. By the continuity of the distance function and the fact that $\{y_{n_k}\}_{k\in N_+}$ converges to y,

$$\rho(y,\Gamma(x)) = \lim_{n\to\infty} \rho(y_{n_k},\Gamma(x)) \leq \epsilon.$$

This holds for any $\epsilon > 0$, so it follows that $\rho(y,\Gamma(x)) = 0$, or in other words, $y \in \Gamma(x)$.

Sufficiency

Conversely, suppose that, for any sequence $\{x_n\}_{n\in N_+}$ converging to x and a sequence $\{y_n\}_{n\in N_+}$ such that $y_n \in \Gamma(x_n)$, $\{y_n\}_{n\in N_+}$ has a subsequence that converges to a point in $\Gamma(x)$. We want to show that Γ is upper hemicontinuous at x.

Suppose that Γ is not upper hemicontinuous at x. Then, there exists an open set V in F such that $\Gamma(x) \subset V$ and, for any neighborhood U of x, there exists a $z \in U$ such that $\Gamma(z) \cap V^c \neq \emptyset$. We construct the sequence $\{x_n\}_{n\in N_+}$ by choosing the point x_n in the $1/n$-ball $B_d(x, 1/n)$ around x such that $\Gamma(x_n) \cap V^c \neq \emptyset$. Furthermore, we can construct $\{y_n\}_{n\in N_+}$ so that $y_n \in \Gamma(x_n) \cap V^c$ for any $n \in N_+$. Since $x_n \to x$ in the metric d, by assumption there exists a subsequence $\{y_{n_k}\}_{k\in N_+}$ of $\{y_n\}_{n\in N_+}$ that converges to some point $y \in \Gamma(x)$. However, since the sequence $\{y_n\}_{n\in N_+}$ is contained in the closed set V^c, the limit y should belong to V^c. We now have the contradiction

$$y \in \Gamma(x) \subset V \quad \text{and} \quad y \notin V.$$

It follows that Γ is upper hemicontinuous at x.

Characterization of Lower Hemicontinuity

Necessity

Suppose Γ is lower hemicontinuous at $x \in E$, and choose $y \in \Gamma(x)$ and any sequence $\{x_n\}_{n\in N_+}$ converging to x. We want to find a subsequence of $\{x_{n_k}\}_{k\in N_+}$ and a sequence $\{y_k\}_{k\in N_+}$ such that $y_k \in \Gamma(x_{n_k})$ for any $k \in N_+$ that converges to y.

To this end, note that, because $y \in \Gamma(x)$, the 1-ball $B_\rho(y, 1)$ around y intersects with $\Gamma(x)$; $B_\rho(y,1) \cap \Gamma(x) \neq \emptyset$. Thus, by lower hemicontinuity, there exists a neighborhood U_1 around x such that $\Gamma(z) \cap B_\rho(y,1) \neq \emptyset$ for any $z \in U_1$. Since $\{x_n\}_{n\in N_+}$ converges to x, there exists an $n_1 \in N_+$ such that $x_{n_1} \in U_1$, and choose $y_1 \in \Gamma(x_{n_1} \cap B_\rho(y,1)$.

Now suppose, for some $k \geq 1$, that we have found $n_1 < \cdots < n_k$ and $y_1, \cdots, y_k \in F$ such that

$$y_k \in \Gamma(x_{n_k}) \cap B_\rho(y, 1/k).$$

Since $B_\rho(y, 1/(k+1))$ once again intersects $\Gamma(x)$, by lower hemicontinuity there exists a neighborhood U_{k+1} around x such that $\Gamma(z) \cap B_\rho(y, 1/(k+1)) \neq \emptyset$ for any $z \in U_{k+1}$. Since $\{x_n\}_{n\in N_+}$ converges to x, U_{k+1} contains infinitely many values in $\{x_n\}_{n\in N_+}$, so that we can choose some $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in U_{k+1}$. Finally, we can choose

$$y_{k+1} \in \Gamma(x_{n_{k+1}}) \cap B_\rho(y, 1/(k+1)).$$

Having constructed the sequence $\{y_k\}_{k \in N_+}$ and subsequence $\{x_{n_k}\}_{k \in N_+}$ in this way, we can see that

$$y_k \in \Gamma(x_{n_k}) \quad \text{for any } k \in N_+$$

and, since $\rho(y_k, y) < \frac{1}{k}$ for any $k \in N_+$, $\{y_k\}_{k \in N_+}$ converges to y.

Sufficiency

Suppose now that, for any sequence $\{x_n\}_{n \in N_+}$ converging to x and $y \in \Gamma(x)$, there exists a subsequence $\{x_{n_k}\}_{k \in N_+}$ of $\{x_n\}_{n \in N_+}$ and a sequence $\{y_k\}_{k \in N_+}$ such that $y_k \in \Gamma(x_{n_k})$ for any $k \in N_+$ and $y_k \to y$ as $k \to \infty$. We want to show that Γ is lower hemicontinuous at x.

Again, assume that Γ is not lower hemicontinuous at x. Then, there exists an open set V in F such that $V \cap \Gamma(x) \neq \emptyset$ and, for any neighborhood U of x, $\Gamma(z) \subset V^c$ for some $z \in U$. Since $V \cap \Gamma(x) \neq \emptyset$, there exists a $y \in \Gamma(x)$ such that $y \in V$ as well.

We now construct the sequence $\{x_n\}_{n \in N_+}$ as in the upper hemicontinuity case, by choosing the point in the $1/n$-ball $B_d(x, 1/n)$ such that $\Gamma(x_n) \subset V^c$. Since $d(x_n, x) < 1/n$ for any $n \in N_+$, $\{x_n\}_{n \in N_+}$ converges to x; thus, by assumption, there exists a subsequence $\{x_{n_k}\}_{k \in N_+}$ of $\{x_n\}_{n \in N_+}$ and a sequence $\{y_k\}_{k \in N_+}$ such that $y_k \in \Gamma(x_{n_k})$ for any $k \in N_+$ and $y_k \to y$ as $k \to \infty$. It follows that, for any $k \in N_+$,

$$y_k \in \Gamma(x_{n_k}) \subset V^c,$$

and because V^c is a closed set containing $\{y_k\}_{k \in N_+}$, its limit y is contained in V^c. However, this contradicts the fact that $y \in \Gamma(x) \cap V \subset V$, so it must be the case that Γ is lower hemicontinuous at x.

Proof of Closed Graph Theorem

Suppose that Γ is compact-valued.

Suppose Γ is upper hemicontinuous on E. Recall that graph of Γ is defined as

$$Gr(\Gamma) = \{(x, y) \in E \times F \mid x \in E, y \in \Gamma(x)\}.$$

To show that it is closed, let $(x, y) \in E \times F$ be a limit point of $Gr(\Gamma)$; then, there exists a sequence $\{(x_n, y_n)\}_{n \in N_+}$ in $Gr(\Gamma)$ converging to (x, y). For any $n \in N_+$, since

$(x_n, y_n) \in Gr(\Gamma)$, we have $y_n \in \Gamma(x_n)$. By the characterization of upper hemicontinuity established above, there exists a subsequence $\{y_{n_k}\}_{k \in N_+}$ of $\{y_n\}_{n \in N_+}$ that converges to a point in $\Gamma(x)$; since y is the limit of $\{y_{n_k}\}_{k \in N_+}$, it follows that $y \in \Gamma(x)$. Therefore, $(x, y) \in Gr(\Gamma)$, and $Gr(\Gamma)$ is closed.

Conversely, suppose that $Gr(\Gamma)$ is closed, and that, for any bounded set $A \subset E$, the image $\Gamma(A)$ is contained in a compact set. For any $x \in E$, let $\{x_n\}_{n \in N_+}$ be a sequence converging to x and $\{y_n\}_{n \in N_+}$ a sequence in F such that $y_n \in \Gamma(x_n)$ for any $n \in N_+$. Since $\{x_n\}_{n \in N_+}$ is convergent, it is bounded, and therefore by assumption there exists a compact set K such that

$$\bigcup_n \Gamma(x_n) \subset K.$$

Since $\{y_n\}_{n \in N_+}$ is a sequence in the compact set K, there exists a subsequence $\{y_{n_k}\}_{k \in N_+}$ of $\{y_n\}_{n \in N_+}$ that converges to some point $y \in K$. Then, the sequence $\{(x_{n_k}, y_{n_k})\}_{k \in N_+}$ converges to (x, y), and because $\{(x_{n_k}, y_{n_k})\}_{k \in N_+}$ lies in $Gr(\Gamma)$, by the closedness of $Gr(\Gamma)$, we have $(x, y) \in Gr(\Gamma)$. By definition, $y \in \Gamma(x)$, so that, by the characterization above, Γ is upper hemicontinuous at x. This in turn holds for any $x \in E$, so Γ is upper hemicontinuous on E.

Q.E.D.

Note that, if (F, ρ) is a euclidean space, then the third result follows when $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$ is bounded for any bounded $A \subset E$. This is because, by the Heine-Borel theorem, any bounded subset in a euclidean space is contained in a larger compact set.

The following are some continuous correspondences that we often encounter:

Theorem 4.2 (Examples of Continuous Correspondences)
The following are examples of how to construct continuous correspondences:

i) Let (E,τ) and (F,s) be topological spaces, and $f: E \to F$ a function. Define the correspondence $\Gamma: E \to F$ as $\Gamma(x) = \{f(x)\}$ for any $x \in E$. Then, Γ is a continuous correspondence if and only if f is a continuous function.

ii) Let (E,τ) and (F,s) be topological spaces, K a compact subset of F, and define $\Gamma: E \to F$ as $\Gamma(x) = K$ for any $x \in E$. Γ is a continuous correspondence.

iii) Let $(E,\tau),(F,s),(G,r)$ be topological spaces, $f: E \to F$ a continuous function, and $\Gamma: F \to G$ a continuous correspondence. Then, the correspondence $\Gamma \circ f: E \to G$ defined as

$$(\Gamma \circ f)(x) = \Gamma(f(x))$$

for any $x \in E$ is a continuous correspondence.

iv) Let $f: \mathbb{R}^n \to \mathbb{R}_+$ be a continuous function, and define $\Gamma: \mathbb{R}^n_+ \to \mathbb{R}_+$ as $\Gamma(x) = [0, f(x)]$ for any $x \in \mathbb{R}^n$. Γ is a continuous correspondence.

Proof) i) Suppose that Γ is continuous, and choose any $x \in E$. Then, for any open set V containing $f(x)$, by upper hemicontinuity there exists a neighborhood U of x such that $\Gamma(z) \subset V$ for any $z \in U$. In other words, $U \subset f^{-1}(V)$, and by definition f is continuous at x. This holds for any $x \in E$, so f is a continuous function.

Conversely, suppose that f is a continuous function, and choose any $x \in E$. Then, for any open set V such that $\Gamma(x) = \{f(x)\} \subset V$, there exists a neighborhood U of x such that $U \subset f^{-1}(V)$. This implies that, for any $z \in U$, $\Gamma(z) = \{f(z)\} \subset V$, so that Γ is upper hemicontinuous at x.

Now let V be an open set in F such that $V \cap \Gamma(x) \neq \emptyset$. This means that $f(x) \in V$, so by the same process as above, we can see that there exists a neighborhood U of x such that $f(z) \in V$ for any $z \in U$. In other words, $\Gamma(z) \cap V = \{f(z)\} \cap V \neq \emptyset$ for any $z \in U$, so that Γ is lower hemicontinuous at x. This holds for any $x \in E$, so Γ is a continuous function.

ii) Let Γ be defined as $\Gamma(x) = K$ for any $x \in E$, where K is a compact set. Then, for any $x \in E$, letting V be an open set containing $\Gamma(x)$, V is an open set containing K, so that $\Gamma(z) \subset V$ for any $z \in E$. Taking E as the neighborhood around x, Γ is upper hemicontinuous at x by definition.

Now let V be an open set such that $V \cap \Gamma(x) = V \cap K \neq \emptyset$. Then, $\Gamma(z) \cap V = K \cap V \neq \emptyset$ for any $z \in E$, so Γ is lower hemicontinuous at x as well. This holds for any $x \in E$, so Γ is a continuous function.

iii) Let $\Phi = \Gamma \circ f$ be defined as in the claim. For any $x \in E$ and an open set V containing $\Phi(x) = \Gamma(f(x))$, by the upper hemicontinuity of Γ there exists a neighborhood U of $f(x)$ such that $\Gamma(y) \subset V$ for any $y \in U$. The continuity of f now tells us that there exists a neighborhood W of x such that $W \subset f^{-1}(U)$. Therefore, for any $x' \in W$, $f(x') \in U$ and therefore

$$\Phi(x') = \Gamma(f(x')) \subset V.$$

By definition, Φ is upper hemicontinuous at x.

Now choose any open set V such that $\Phi(x) \cap V \neq \emptyset$. By the lower hemicontinuity of Γ, there exists a neighborhood U of $f(x)$ such that $\Gamma(y) \cap V \neq \emptyset$ for any $y \in U$. Again, the continuity of f now tells us that there exists a neighborhood W of x such that $W \subset f^{-1}(U)$. Thus, for any $x' \in W$, $f(x') \in U$ and

$$\Phi(x') \cap V = \Gamma(f(x')) \cap V \neq \emptyset.$$

By definition, Φ is lower hemicontinuous at x. This holds for any $x \in E$, so Φ is continuous on E.

iv) Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a continuous function, and define $\Gamma : \mathbb{R}^n \to \mathbb{R}_+$ as $\Gamma(x) = [0, f(x)]$ for any $x \in \mathbb{R}^n$. Clearly, Γ is a non-empty compact valued correspondence. For any $x \in \mathbb{R}^n$, let $\{x_k\}_{k \in N_+}$ be a sequence converging to x and $\{y_k\}_{k \in N_+}$ a sequence such that $y_k \in \Gamma(x_k) = [0, f(x_k)]$ for any $k \in N_+$. Since $\{f(x_k)\}_{k \in N_+}$ is a convergent sequence (by the continuity of f, it converges to $f(x)$), it is bounded, say, by $M \in (0, +\infty)$, and since $\{y_k\}_{k \in N_+}$ is a sequence contained in the compact set $[0, M]$, there exists a convergent subsequence $\{y_{k_l}\}_{l \in N_+}$ of $\{y_k\}_{k \in N_+}$. Letting y be the limit of this subsequence, since $0 \leq y_{k_l} \leq f(x_{k_l})$ for any $l \in N_+$, taking $l \to \infty$ on both sides yields $0 \leq y \leq f(x)$, or equivalently, $y \in [0, f(x)] = \Gamma(x)$. By the characterization theorem above, Γ is upper hemicontinuous at x.

To show lower hemicontinuity, for any $x \in \mathbb{R}^n$, let $\{x_k\}_{k \in N_+}$ be a sequence converging to x and $y \in \Gamma(x) = [0, f(x)]$. Defining $y_k = \min(f(x_k), y)$ for any $k \in N_+$, $y_k \in [0, f(x_k)] = \Gamma(x_k)$ for any $k \in N_+$, and, by the continuity of the minimum function,

$$\lim_{k \to \infty} y_k = \min(f(x), y) = y.$$

Therefore, Γ is lower hemicontinuous at x. This holds for any $x \in \mathbb{R}^n$, so Γ is a continuous correspondence.

Q.E.D.

4.2 Berge's Maximum Principle

The next theorem is fundamental for optimization, not only in microeconomics but across economics as a whole. It states that any maximization problem involving a continuous function, a continuous constraint and a compact domain yields a continuous value function and an upper hemicontinuous solution set.

Theorem 4.3 (Theorem of the Maximum)
Let (E,τ) and (Θ,s) be topological spaces, $\Gamma:\Theta\to E$ a non-empty compact-valued and continuous correspondence, and $f:E\times\Theta\to\mathbb{R}$ a continuous function on $E\times\Theta$. Define the value function $v:\Theta\to[-\infty,+\infty]$ as

$$v(\theta)=\sup_{x\in\Gamma(\theta)} f(x,\theta)$$

and the set of maximizers $G:\Theta\to E$ as

$$G(\theta)=\operatorname*{argmax}_{x\in\Gamma(\theta)} f(\theta,x)=\{x\in\Gamma(\theta)\mid f(x,\theta)=v(\theta)\}$$

for any $\theta\in\Theta$. Then, v is a continuous function taking values in $\mathbb{R}$, and G is a non-empty compact valued and upper hemicontinuous correspondence. By implication, if G is single-valued on Θ, it defines a continuous function.

Proof) The difficult part of the proof involves showing that v and G are continuous in the sense described above, so we first prove the results that are not related to continuity.

Step 0: Non-Continuity Results

We first show that v is real-valued and G is a non-empty compact valued correspondence. For any $\theta\in\Theta$, since $\Gamma(\theta)$ is compact and the section $f(\cdot,\theta)$ is a continuous function, by the extreme value theorem there exists an $x^*\in\Gamma(\theta)$ such that

$$f(\theta,x^*)=\sup_{x\in\Gamma(\theta)} f(\theta,x)=v(\theta)\in\mathbb{R}.$$

Therefore, $x^*\in G(\theta)$, which shows us that v is real-valued and G is non-empty valued.

To show that G is compact-valued, let us first show that it is closed. Suppose x is a limit point of $G(\theta)$. Then, since x is a limit point of the set $G(\theta)$, which is a subset of $\Gamma(\theta)$, a compact set, by the fact that any compact set is also limit point compact, we can see that $x\in\Gamma(\theta)$. Now suppose that $f(x,\theta)\neq v(\theta)$. Assuming $f(x,\theta)>v(\theta)$ without loss of generality, since $f(\cdot,\theta)$ is continuous on E and thus lower semicontinuous, there exists a neighborhood U of x such that $f(z,\theta)>v(\theta)$ for any $z\in U$. Since x is a limit point of $G(\theta)$, $G(\theta)\cap U \neq \emptyset$, so that there exists some $z \in G(\theta)\cap U$. This implies that

$$v(\theta) = f(z,\theta) > v(\theta),$$

a contradiction, so it must be the case that $f(x,\theta) = v(\theta)$. Therefore, $x \in G(\theta)$ and $G(\theta)$ is a closed set. Finally, $G(\theta)$ is a closed set of the compact set $\Gamma(\theta)$, which implies that $G(\theta)$ is also compact.

Step 1: Continuity of v

We will show that v is continuous by showing that it is both lower and upper semicontinuous. Lower and upper semicontinuity follow from the lower and upper hemicontinuity of Γ, respectively. Note that, for upper semicontinuity, it is enough to show that, for any $\theta \in \Theta$ and $\epsilon > 0$, there exists a neighborhood U_θ around θ such that $v(\theta') \leq v(\theta) + \epsilon$. To see why, suppose that this property holds, and consider the set $A = \{\theta \in \Theta \mid v(\theta) < a\}$ for some $a \in \mathbb{R}$. If A is empty, then it is trivially open, so suppose A is non-empty. In this case, choosing any $\theta \in A$ and putting $\epsilon = \frac{a - v(\theta)}{2} > 0$, it follows that there exists a neighborhood U_θ around θ such that

$$v(\theta') \leq v(\theta) + \epsilon < v(\theta) + (a - v(\theta)) = a.$$

In other words, $U_\theta \subset A$; this holds for any $\theta \in A$, so A is the union of open sets and therefore open itself. A similar result holds for lower semicontinuity.

Upper Semicontinuity of v

We employ the proof method introduced above. Choose any $\epsilon > 0$ and $\theta \in \Theta$. We want to show that $v(\theta') \leq v(\theta) + \epsilon$ for any θ' in some neighborhood around θ. To do so, it suffices to show that, for any $x \in \Gamma(\theta)$, there exist neighborhoods around x and θ such that

$$f(x',\theta') \leq f(x,\theta) + \epsilon$$

for any x' and θ' in their respective neighborhoods. Then, the compactness of $\Gamma(\theta)$ tells us that $\Gamma(\theta)$ is covered by a finite number of these neighborhoods around x. Taking the finite intersection of the corresponding neighborhoods around θ, we can see that, for any $x \in \Gamma(\theta)$ and θ' in this new neighborhood, we have $f(x,\theta') \leq v(\theta) + \epsilon$. The upper hemicontinuity of Γ tells us that this inequality holds for any $x \in \Gamma(\theta')$ as well. The desired inequality is then established by taking suprema over $x \in \Gamma(\theta')$ over both sides. Below we formalize this intuition.

First, choose any $x \in \Gamma(\theta)$. Our goal is to form an open cover of $\Gamma(\theta)$ using open balls around x. To this end, note that the upper semicontinuity of f with respect to the product topology $\tau \times s$ at (x, θ) implies that the set

$$\Omega(x) = \{(x', \theta') \in E \times \Theta \mid f(x', \theta') < f(x, \theta) + \epsilon\}$$

is open, that is, contained in $\tau \times s$. Since the set of all open rectangles is a base generating $\tau \times s$ and $(x, \theta) \in \Omega(x, \theta)$, it follows that there exist neighborhoods $V(x) \subset E$ and $U(x) \subset \Theta$ around x and θ such that

$$V(x) \times U(x) \subset \Omega(x).$$

Note that $\{V(x)\}_{x \in \Gamma(\theta)}$ now forms an open cover of $\Gamma(\theta)$; by the compactness of $\Gamma(\theta)$, there exist $x_1, \cdots, x_n \in \Gamma(\theta)$ such that

$$\Gamma(\theta) \subset \bigcup_{i=1}^{n} V(x_i).$$

The right hand side above is one big open set, so by the upper hemicontinuity of Γ there exists a neighborhood $U(\theta)$ of θ such that

$$\Gamma(U) \subset \bigcup_{i=1}^{n} V(x_i).$$

Now define the neighborohood U of θ as

$$U = U(\theta) \cap U(x_1) \cap \cdots \cap U(x_n).$$

Choose any $\theta' \in U$ and $x \in \Gamma(\theta')$. Then, because $\theta' \in U(\theta)$,

$$x \in \Gamma(\theta') \subset \bigcup_{i=1}^{n} V(x_i)$$

and $x \in V(x_i)$ for some $1 \leq i \leq n$. $\theta' \in U(x_i)$ as well, so we have

$$f(x, \theta') < f(x_i, \theta) + \epsilon \leq v(\theta) + \epsilon.$$

This holds for any $x \in \Gamma(\theta')$, so

$$v(\theta') = \sup_{x \in \Gamma(\theta')} f(x, \theta') \leq v(\theta) + \epsilon.$$

Therefore, we have found a neighborhood U of θ such that, for any $\theta' \in U$, $v(\theta') \leq v(\theta) + \epsilon$. By the remark preceding this section, this shows us that v is upper semicontinuous.

Lower Semicontinuity of v

We proceed in much the same way as the preceding section. Choose any $\theta \in \Theta$ and $\epsilon > 0$. We want to show that there exists a neighborhood of θ such that $v(\theta) - \epsilon \geq v(\theta')$ for any θ' in this neighborhood.

Let $x \in \Gamma(\theta)$ be chosen so that

$$v(\theta) - \frac{\epsilon}{2} < f(x,\theta);$$

such an x exists by the definition of the supremum. By the lower semicontinuity of f at (θ,x), the set

$$\Omega = \{(x',\theta') \in E \times \Theta \mid f(x',\theta') > f(x,\theta) - \epsilon/2\}$$

is open in the product topology $\tau \times s$. Since Ω contains (x,θ), there exist neighborhoods V and W of x and θ such that $V \times W \subset \Omega$.

Note that $x \in \Gamma(\theta) \cap V$, so that $\Gamma(\theta) \cap V \neq \emptyset$. By the lower hemicontinuity of Γ, there exists a neighborhood W' of θ such that $\Gamma(\theta') \cap V \neq \emptyset$ for any $\theta' \in W'$. Define $U = W' \cap W$. For any $\theta' \in U$, since $\Gamma(\theta') \cap V \neq \emptyset$, there exists an $x' \in \Gamma(\theta')$ and $x' \in V$. Since $(x',\theta') \in V \times W \subset \Omega$, we have

$$f(x,\theta) - \frac{\epsilon}{2} < f(x',\theta').$$

Finally, $x' \in \Gamma(\theta')$ shows us that

$$f(x,\theta) - \frac{\epsilon}{2} < f(x',\theta') \leq \sup_{y \in \Gamma(\theta')} f(y,\theta') = v(\theta').$$

By our choice of x, we can finally conclude that

$$v(\theta) - \epsilon < v(\theta').$$

Step 2: Upper Hemicontinuity of G

The final step of our proof involves proving the upper hemicontinuity of G. Choose some $\theta \in \Theta$ and an open set V in E such that $G(\theta) \subset V$. We want to find a neighborhood around θ whose image is contained in V. The idea of the construction is to first express $G(\theta)$ as the intersection of two correspondences. In other words, since $G(\theta)$ is the collection of all x in $\Gamma(\theta)$ such that $f(x,\theta)=v(\theta)$, we can write

$$G(\theta)=B(\theta)\cap\Gamma(\theta),$$

where we define the correspondence $B:\Theta\to E$ as

$$B(\theta)=\{x\in E\mid f(x,\theta)=v(\theta)\}$$

for any $\theta\in\Theta$. Given $G(\theta)\subset V$, if $\Gamma(\theta)\subset V$ the desired result follows by the upper hemicontinuity of V. On the other hand, if $\Gamma(\theta)\cap V^c\neq\emptyset$, then we cover the compact set $\Gamma(\theta)\cap V^c$ with balls around x on which (x,θ) lie outside the graph $Gr(B)$ of B. Since no point in $G(\theta)$ can lie outside the graph of B, we can then construct a neighborhood of θ whose image does not intersect V^c, and instead lies in V. We now formalize this intuition.

First, we prove that the graph of B is closed, so that we can take neighborhoods of points outside the graph. The proof proceeds almost identically to the proof of the closedness of $G(\theta)$; suppose that (x,θ) is a limit point of $Gr(B)$ that lies outside $Gr(B)$, that is, $f(x,\theta)\neq v(\theta)$. Assuming without loss of generality that $f(x,\theta)>v(\theta)$, the lower semicontinuity of the mapping $(x',\theta')\mapsto f(x',\theta')-v(\theta')$ (which follows because f and v are continuous), there exists an neighborhood Ω of (x,θ) such that

$$f(x',\theta')>v(\theta')$$

for any $(x',\theta')\in\Omega$. As such, $\Omega\subset Gr(B)^c$. However, by the definition of a limit point, $Gr(B)\cap\Omega\neq\emptyset$, which is a contradiction. Therefore, $Gr(B)$ should contain the limit point (x,θ), and $Gr(B)$ is closed.

Choose any $\theta\in\Theta$ and open set V in E such that

$$G(\theta)=B(\theta)\cap\Gamma(\theta)\subset V.$$

If $\Gamma(\theta)\subset V$, then by the upper hemicontinuity of Γ, there exists a neighborhood U of θ such that $G(U)\subset\Gamma(U)\subset V$, and the proof is complete.

Now suppose that $\Gamma(\theta)\not\subset V$, that is, $\Gamma(\theta)\cap V^c\neq\emptyset$. $\Gamma(\theta)\cap V^c$ is a closed subset of the compact set $\Gamma(\theta)$, so it is itself compact. Choose any $x\in\Gamma(\theta)\cap V^c$; $x\notin G(\theta)$, since otherwise $x\in G(\theta)\subset V$, a contradiction. Since $x\in\Gamma(\theta)$, this must mean that $f(x,\theta)\neq v(\theta)$, that is, $(x,\theta)\notin Gr(B)$. The openness of $Gr(B)$ furnishes us with a neighborhood $\Omega(x)\in\tau\times s$ around (x,θ) such that $\Omega(x)\subset Gr(B)^c$, and as earlier, this implies that there exist neighborhoods $V(x) \subset E$ and $U(x) \subset \Theta$ around x and θ such that

$$V(x) \times U(x) \subset \Omega(x) \subset Gr(B)^c.$$

Then, $\{V(x)\}_{x \in \Gamma(\theta) \cap V^c}$ is an open cover of the compact set $\Gamma(\theta) \cap V^c$; it follows that there exist $x_1, \cdots, x_n \in \Gamma(\theta) \cap V^c$ such that

$$\Gamma(\theta) \cap V^c \subset \bigcup_{i=1}^{n} V(x_i).$$

As such,

$$\Gamma(\theta) \subset V \cup \left(\bigcup_{i=1}^{n} V(x_i) \right).$$

The right hand side is itself an open set, so by the upper hemicontinuity of Γ, there exists a neighborhood $U(\theta)$ of θ such that

$$\bigcup_{\theta' \in U(\theta)} \Gamma(\theta') \subset V \cup \left(\bigcup_{i=1}^{n} V(x_i) \right).$$

Define the neighborhood

$$U = U(\theta) \cap U(x_1) \cap \cdots \cap U(x_n)$$

of θ. For any $\theta' \in U$ and $x \in G(\theta')$, since $\theta' \in U(\theta)$,

$$x \in G(\theta') \subset \Gamma(\theta') \subset V \cup \left(\bigcup_{i=1}^{n} V(x_i) \right).$$

Suppose that $x \notin V$. Then, there exists some $1 \leq i \leq n$ such that $x \in V(x_i)$; therefore,

$$(x, \theta') \in V(x_i) \times U(x_i) \subset Gr(B)^c,$$

and we have $f(x, \theta') \neq v(\theta')$. However, since $x \in G(\theta') \subset B(\theta')$, it must be the case that $f(x, \theta') = v(\theta')$, a contradiction. Therefore, $x \in V$; this holds for any $x \in G(\theta')$, so

$$G(\theta') \subset V.$$

This in turn holds for any $\theta' \in U$, so $G(U) \subset V$.

In any case, we can find a neighborhood around θ such that the image of this neighborhood under G is a subset of V. By definition, G is upper hemicontinuous at θ, and since this holds for any $\theta \in \Theta$, G is upper hemicontinuous on Θ.

Q.E.D.

4.3 A Measurable Selection Theorem

Sometimes it is desirable to make a selection from a correspondence, that is, given a correspondence $\Gamma : E \to F$, to choose a function whose value at every $x \in E$ is contained in the set $\Gamma(x)$. In particular, it is of interest whether we can choose this function so that it is measurable relative to selected σ-algebras. This is the content of the next theorem:

Theorem 4.4 (Kuratowski–Ryll-Nardzewski Measurable Selection Theorem) Let $(E,\mathcal{E})$ be a measurable space and (F,d) a complete and separable metric space with metric topology τ and Borel σ-algebra $\mathcal{B}(F,\tau)$. Let $\Gamma : E \to F$ be a non-empty closed valued correspondence such that, for any closed set A in F,

$$\{x \in E \mid \Gamma(x) \cap A \neq \emptyset\} \in \mathcal{E}.$$

Then, there exists a measurable selection of Γ, that is, a function $g : E \to F$ measurable relative to $\mathcal{E}$ and $\mathcal{B}(F,\tau)$, and $g(x) \in \Gamma(x)$ for any $x \in E$.

Proof) By the separability of (F,d), there exists a countable set $F_0 \subset F$ that is dense in F. Arrange the elements of F_0 into the sequence $\{y_n\}_{n \in N_+}$, and define the functions $N_0 : E \to N_+$ and $\zeta_0 : E \to F$ as

$$\begin{aligned} N_0(x) &= \min\{n \in N_+ \mid \Gamma(x) \cap \overline{B}_d(y_n,1) \neq \emptyset\} \\ \zeta_0(x) &= y_{N_0(x)} \end{aligned}$$

for any $x \in E$, where $\overline{B}_d(y_n,1)$ is the closed ball of radius 1 around y_n. Note that, for any $x \in E$, the set $\{n \in N_+ \mid \Gamma(x) \cap \overline{B}_d(y_n,1) \neq \emptyset\}$ is non-empty since $\Gamma(x)$ is non-empty and the denseness of F_0 in F implies that 1-balls around points in F_0 cover F. The function ζ_0 constructed as above takes countably many values (its range is a subset of F_0), and for any $n \in N_+$,

$$\zeta_0^{-1}(\{y_n\}) = \left(\bigcap_{i=1}^{n-1} \{x \in E \mid \Gamma(x) \cap \overline{B}_d(y_i,1) \neq \emptyset\}^c \right) \cap \{x \in E \mid \Gamma(x) \cap \overline{B}_d(y_n,1) \neq \emptyset\} \in \mathcal{E},$$

where the last inclusion follows by the assumption that $\{x \in E \mid \Gamma(x) \cap A \neq \emptyset\}$ is $\mathcal{E}$-measurable for any closed set A in F. Thus, for any open set $V \in \tau$,

$$\zeta_0^{-1}(V) = \bigcup_{n \in N_+, y_n \in V} \zeta_0^{-1}(\{y_n\}) \in \mathcal{E},$$

and because τ generates $\mathcal{B}(F,\tau)$, ζ_0 is measurable relative to $\mathcal{E}$ and $\mathcal{B}(F,\tau)$.

Now suppose that, for some $k \geq 1$, we have constructed functions $\zeta_0, \cdots, \zeta_{k-1}$ taking values in F_0 such that

$$d(\zeta_i(x), \zeta_{i-1}(x)) \leq 2^{-i+1} \quad \text{and} \quad d(\zeta_i(x), \Gamma(x)) \leq 2^{-i}$$

for any $1 \leq i \leq k-1$ and $x \in E$, where $d(\cdot, \Gamma(x))$ is the distance function for the closed set $\Gamma(x)$. Construct the functions $N_k : E \to N_+$ and $\zeta_k : E \to F$ as

$$\begin{aligned} N_k(x) &= \min\left\{n \in N_+ \mid \Gamma(x) \cap \overline{B}_d\left(\zeta_{k-1}(x), 2^{-k}\right) \cap \overline{B}_d\left(y_n, 2^{-k}\right) \neq \emptyset\right\} \\ \zeta_k(x) &= y_{N_k(x)} \end{aligned}$$

for any $x \in E$. Again, $N_k(x) \in N_+$ due to the denseness of F_0 in F, and ζ_k is measurable relative to $\mathcal{E}$ and $\mathcal{B}(F, \tau)$ by the same reason as ζ_0. Finally, for any $x \in E$, by construction there exists a $z \in \Gamma(x)$ such that

$$d(z, \zeta_{k-1}(x)) \leq 2^{-k} \quad \text{and} \quad d\left(z, y_{N_k(x)}\right) \leq 2^{-k}.$$

Therefore,

$$d(\zeta_k(x), \zeta_{k-1}(x)) = d(y_{N_k(x)}, \zeta_{k-1}(x)) \leq d(z, \zeta_{k-1}(x)) + d\left(z, y_{N_k(x)}\right) \leq 2^{-k+1}$$

and

$$d(\zeta_k(x), \Gamma(x)) \leq d(\zeta_k(x), z) = d\left(z, y_{N_k(x)}\right) \leq 2^{-k}.$$

For any $x \in E$ and $m, k \in N_+$, assuming without loss of generality that $m > k$, we have

$$\begin{aligned} d(\zeta_m(x), \zeta_k(x)) &\leq \sum_{i=0}^{m-k-1} d(\zeta_{m-i}(x), \zeta_{m-i-1}(x)) \\ &\leq \sum_{i=0}^{m-k-1} 2^{-m+i+1} = 2^{-m+1}\left(\sum_{i=0}^{m-k-1} 2^{-i}\right) = 2^{-m+1}\left(2 - 2^{-m-k+1}\right). \end{aligned}$$

Taking $m, k \to \infty$ on both sides reveals that

$$\lim_{m,k\to\infty} d(\zeta_m(x), \zeta_k(x)) = 0,$$

implying that $\{\zeta_k(x)\}_{k \in N_+}$ is a Cauchy sequence in F. By the completeness of (F, d), this sequence converges to some $\zeta_x \in F$. This holds for any $x \in E$, so we can define the function $g : E \to F$ as

$$g(x) = \zeta_x$$

for any $x \in E$. Then, $\{\zeta_k\}_{k \in N_+}$ is a sequence of measurable functions converging pointwise to g, which tells us that g is also measurable relative to $\mathcal{E}$ and $\mathcal{B}(F, d)$. Furthermore, for any $x \in E$, since

$$d(\zeta_k(x), \Gamma(x)) \leq 2^{-k}$$

and the distance function $d(\cdot, \Gamma(x))$ is continuous on E, taking $k \to \infty$ on both sides yields

$$d(g(x), \Gamma(x)) = 0.$$

By the closedness of $\Gamma(x)$, $g(x) \in \Gamma(x)$; this holds for any $x \in E$, so the proof is complete.

Q.E.D.

Corollary to Theorem 4.4 Let (E, τ) be a topological space with Borel σ-algebra $\mathcal{B}(E, \tau)$ and (F, ρ) a complete and separable metric space with metric topology s and Borel σ-algebra $\mathcal{B}(F, s)$. Let $\Gamma : E \to F$ be a non-empty closed valued and upper hemicontinuous correspondence. Then, there exists a measurable selection of Γ.

Proof) In light of the above theorem, we need only show that, for any closed set A in F,

$$\{x \in E \mid \Gamma(x) \cap A \neq \emptyset\}$$

is contained in $\mathcal{B}(E, \tau)$. We will show the stronger result that any such set is a closed subset of E.

Fix a closed set A and define

$$V = \{x \in E \mid \Gamma(x) \cap A = \emptyset\} = \{x \in E \mid \Gamma(x) \subset A^c\}.$$

For any $x \in V$, $\Gamma(x)$ is contained in the open set A^c. By the upper hemicontinuity of Γ, there exists a neighborhood U of x such that $\Gamma(x') \subset A^c$ for any $x' \in U$. Therefore, $U \subset V$, and since this holds for any $x \in V$, V is an open set. This means that

$$\{x \in E \mid \Gamma(x) \cap A \neq \emptyset\} = V^c$$

is a closed set.

Q.E.D.

4.4 Fixed Point Theorems