Theorem 4.10. Let $\delta: M_{n\times n}(F) \to F$ be an alternating n-linear function.

(a) If $A \in M_{n\times n}(F)$ and B is a matrix from A by interchanging any two rows of A, then $\delta(B) = -\delta(A)$

(b) If $A \in M_{n\times n}(F)$ has two identical rows, then $\delta(A) = 0$.

Proof.

(a) ① Definition of n-linear function

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\cdot\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

② Def: An n-linear function $\delta: M_{n\times n}(F) \to F$ is called alternating if, for each $A \in M_{n\times n}(F)$, we have $\delta(A) = 0$ whenever two adjacent rows of A are identical.

③ We have $\delta: M_{n\times n}(F) \to F$, an alternating n-linear function.
$A \in M_{n\times n}(F)$,
B : matrix obtained from A by interchanging rows r and s

④ Let's say $s > r$, and since δ is an alternating (Whenever two adjacent rows of A are identical, $\delta(A) = 0$) n-linear function, we have

$$0 = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r + a_s \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix}$$

$$= \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a \\ \vdots \\ a_r \\ a_s \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a \\ \vdots \\ a_s \\ a_s \\ \vdots \\ a_n \end{pmatrix}$$

First term: δ: alternating, hence goes to 0.
Second term: $\delta(A)$
Third term: $\delta(B)$
Fourth term: δ: alternating, hence goes to 0.

$\Rightarrow \quad 0 = 0 + \delta(A) + \delta(B) + 0$

$\Rightarrow \quad -\delta(A) = \delta(B)$

Whatever the number is for s, since δ is alternating : $\delta\begin{pmatrix} a_1 \\ \vdots \\ a_r + a_s \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} \Rightarrow -\delta(A) = \delta(B)$

⑤ Let's say $r > s$

$$0 = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_s + a_r \\ a_s + a_r \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ a_s + a_r \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_s + a_r \\ \vdots \\ a_n \end{pmatrix}$$

$$= \underbrace{\delta\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ a_s \\ \vdots \\ a_n \end{pmatrix}}_{\delta\text{: alternating, hence goes to } 0.} + \underbrace{\delta\begin{pmatrix} a \\ \vdots \\ a_s \\ a_r \\ \vdots \\ a_n \end{pmatrix}}_{\delta(A)} + \underbrace{\delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_s \\ \vdots \\ a_n \end{pmatrix}}_{\delta(B)} + \underbrace{\delta\begin{pmatrix} a \\ \vdots \\ a_r \\ a_r \\ \vdots \\ a_n \end{pmatrix}}_{\delta\text{: alternating, hence goes to } 0.}$$

$\Rightarrow \quad 0 = 0 + \delta(A) + \delta(B) + 0$

$\Rightarrow \quad -\delta(A) = \delta(B)$

Whatever the number is for r, Since δ is alternating : $\delta\begin{pmatrix} a_1 \\ \vdots \\ a_s + a_r \\ a_s + a_r \\ \vdots \\ a_n \end{pmatrix} \Rightarrow -\delta(A) = \delta(B)$

Even if $r > s$ or $s > r$, $-\delta(A) = \delta(B)$.

(b) If $A \in M_{n \times n}(F)$ has two identical rows, then $\delta(A) = 0$.

: Let r, and s be the identical rows in A, and $r < s$.

① When $s = r+1$.

: Since δ is alternating, and two adjacent rows (r, j) are identical, $\delta(A) = 0$.

② When $s > r+1$.

: Let B be a matrix obtained from A by interchanging $r+1$ and s, then r and s will be located adjacently.

Then, Since δ is an alternating n-linear function, $\delta(B) = 0$.

And, Corollary 3 states when we obtain a matrix (in here, B) by interchanging rows, $\delta(B) = -\delta(A)$

Since $\delta(B) = 0$, $\delta(B) = 0 = -\delta(A)$

Which means, $\delta(A) = 0$

Corollary 1.

: Let $\delta: M_{n\times n}(F) \to F$ be an alternating n-linear function. If B is a matrix obtained from $A \in M_{n\times n}(F)$ by adding a multiple of some row of A to another row, then $\delta(B) = \delta(A)$.

B: Matrix obtained from $A \in M_{n\times n}(F)$ by adding $k \cdot r_i$ to r_j, where k is a scalar and r_i is a row i of A,
r_j is a row j of A, and $r \neq j$

Ex)

Let $A = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \end{array}$ $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$: $r_2 + 3r_3$

Let C be a matrix obtained from A by replacing r_i to r_j

Ex) $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

We can see row j of B (in example r_2 in B) is equal to sum of row j of A and k times row j of C.

Then, $\delta(B) = \delta(A) + k \cdot \delta(C)$

Since C has two adjacent identical rows, $\delta(C) = 0$

Hence, $\delta(B) = \delta(A) + 0$

Therefore, $\delta(A) = \delta(B)$ if B is a matrix obtained from A by adding a multiple of some row of A to another row.

Corollary 2.

: Let $\delta: M_{n\times n}(F) \to F$ be an alternating n-linear function.
If $M \in M_{n\times n}(F)$ has rank less than n, $\delta(M) = 0$.

Since M has rank less than n, $\exists$ row vectors of M are dependent.

Let's say v_i is one of dependent vectors.

Then, v_i can be obtained by linear combination of other vectors in M.

It means, $\exists$ two identical rows in a matrix.

As we proved in Theorem 4.10 (b), Since M has two identical rows after linear combination of other vectors in M,
$\delta(M) = 0$.

Thus, if $M \in M_{n\times n}(F)$ has rank less than n, then $\delta(M) = 0$.

Corollary 3.

: Let $\delta : M_{n\times n}(F) \to F$ to be an alternating n-linear function, and let E_1, E_2, and E_3 in $M_{n\times n}(F)$ be elementary matrices of types 1, 2, and 3, respectively. Suppose that E_2 is obtained by multiplying some row of I by the nonzero scalar k. Then $\delta(E_1) = -\delta(I)$, $\delta(E_2) = k\cdot\delta(I)$, and $\delta(E_3) = \delta(I)$.

① We can obtain an E_1 from I by interchanging two rows. Then, by Theorem 4.10 Corollary 1, $\delta(E_1) = -\delta(I)$.

② We can obtain an E_2 from I by multiplying some row of I by the nonzero scalar k.
Then, since δ is a n-linear function, by using its definition,
$\delta(E_2) = k\cdot\delta(I)$

③ We can obtain E_3 by adding a multiple of some row of I to another row
Then, by Theorem 4.10 Corollary 1, $\delta(E_3) = \delta(I)$.

Theorem 4.11.

: Let $\delta : M_{n\times n}(F) \to$ be an alternating n-linear function such that $\delta(I) = 1$. For any $A, B \in M_{n\times n}(F)$, we have $\delta(AB) = \det(A)\cdot\delta(B)$

① If $A \in M_{n\times n}(F)$ has rank less than n, then it's obvious AB also will be a matrix with a rank less than n.
Then, from Theorem 4.10 Corollary 2, ↳ (Since $\text{rank}(A) < n$, $\exists$ at least one row filled with 0).
$\delta(AB) = 0 = \det(A)\cdot\delta(B)$

☆ Theorem 4.6 : Corollary : If $A \in M_{n\times n}(F)$ has rank less than n, then $\det(A) = 0$
Hence, if A has rank less than n, $\delta(AB) = \det(A)\cdot\delta(B)$

② If $\text{rank}(A) = n$,
When $\text{rank}(A) = n$, it means A can be invertible. (An $n\times n$ matrix is invertible iff its rank is n).
Hence, A can be written as $A = E_pE_{p-1}\cdots E_1$ (Steps for using Theorem 4.10 Corollaries).

Let $M \in M_{n\times n}(F)$, and E be all elementary matrices.
If $\delta(EM) = \det(E)\cdot\delta(M)$, it's proven.

① Let E be an elementary matrix of type 1: Obtained from I by interchanging row of i and j of I.
Then, by applying Theorem 4.10 Corollary 3, $\delta(E) = -\delta(I)$, hence $\delta(EM) = -\delta(I)\delta(M) = \delta(E)\delta(M)$

And, since $\delta(I) = 1 = \det(I)$, $\delta(E)\,\delta(M) = \det(E)\cdot\delta(M)$

② Let E be an elementary matrix of type 2: Multiplying some row of I by the nonzero scalar k.

Then, from Theorem 4.10 Corollary 3, $\delta(E) = k\cdot\delta(I) = \det(E) = k\cdot\det(I)$

Therefore, $\delta(EM) = k\cdot\delta(I)\cdot\delta(M) = \delta(E)\delta(M)$

$= \det(E)\cdot\delta(M)$

③ Let E be an elementary matrix of type 3: adding a multiple of some row of I to another row

: Then, from Theorem 4.10 Corollary 3, $\delta(E) = \delta(I) = \det(E)$

Then, $\delta(EM) = \delta(I)\cdot\delta(M) = \delta(M) = \delta(E)\delta(M) = \det(E)\delta(M)$.

Therefore, for any $A, B \in M_{n\times n}(F)$, we have $\delta(AB) = \delta(A)\delta(B)$.

Theorem 4.12: If $\delta: M_{n\times n}(F) \to F$ is an alternating n-linear function such that $\delta(I) = 1$, then $\delta(A) = \det(A)$ $\forall A \in M_{n\times n}(F)$.

Proof:

We have $\delta: M_{n\times n}(F) \to F$, an alternating n-linear function, and $\delta(I) = 1$.

$A \in M_{n\times n}(F)$.

① When $\operatorname{rank}(A) < n$

: From Theorem 4.10 Corollary 2, $\delta(A) = 0$.

From Theorem 4.6: Corollary: If $A \in M_{n\times n}(F)$ has rank less than n, then $\det(A) = 0$

Therefore, when $\operatorname{rank}(A) < n$, $\det(A) = 0 = \delta(A)$.

② When $\operatorname{rank}(A) = n$, : A is invertible.

$A = E_p \cdots E_2E_1$.

Since A is a matrix obtained from $E_p \cdots E_2E_1$, from Theorem 4.10 Corollary 1 ($\delta(B) = \delta(A)$)

$\delta(A) = \delta(E_p \cdots E_2E_1)$

We know that $\delta(AB) = \det(A)\cdot\delta(B)$ from Theorem 4.11, hence

$\delta(A) = \det(E_p)\cdot\delta(E_{p-1}\cdots E_2E_1)$

$= \det(E_p)\cdot\det(E_{p-1})\cdot\delta(E_{p-2}\cdots E_2E_1)$

$= \cdots$

$= \det(E_p)\cdot\det(E_{p-1})\cdot \ldots \cdot\det(E_1)$

We know: Theorem 4.7: For any $A, B \in M_{n\times n}(F)$, $\det(AB) = \det(A)\cdot\det(B)$, therefore

$\delta(A) = \det(E_p\cdot E_{p-1}\cdots E_1) = \det(A)$

Therefore, if $\delta: M_{n\times n}(F) \to F$ is an alternating n-linear function such that $\delta(I) = 1$, then $\delta(A) = \det(A)$ $\forall A \in M_{n\times n}(F)$.

Practice

#6,7,8: Determine which of the functions $\delta : M_{3\times3}(F) \to F$ are 3-linear functions

☆ : δ is n-linear function if it's linear of each row of an $n\times n$ matrix when the remaining $n-1$ rows are held.

#6. $\delta(A) = A_{11} + A_{23} + A_{32}$

Definition:

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\cdot\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

① First row

$$\delta\begin{pmatrix} A_{11}+kB_{11} & A_{12}+kB_{12} & A_{13}+kB_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = A_{11}+kB_{11}+A_{23}+A_{32} = A_{11}+A_{23}+A_{32}+kB_{11}$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k\,\delta\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It's not in a form of

$$\delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k\,\delta\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{23} \end{pmatrix}$$

Hence, it's not linear

For example,

let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then, $\delta\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \neq 2 = 2\cdot\delta\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Therefore, given δ isn't a 3-linear function.

#7. $\delta(A) = A_{11}A_{21}A_{32}$

Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

① first row

$$\delta\begin{pmatrix} A_{11}+kB_{11} & A_{12}+kB_{12} & A_{13}+kB_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = (A_{11}+kB_{11})A_{21}\cdot A_{32}$$

$$= A_{11}A_{21}A_{32} + k(B_{11}A_{21}A_{32})$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k\,\delta\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

② Second row

$$\delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21}+kB_{21} & A_{22}+kB_{22} & A_{23}+kB_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = A_{11}\cdot(A_{21}+kB_{21})\cdot A_{32}$$

$$= A_{11}A_{21}A_{32} + k(A_{11}\cdot B_{21}\cdot A_{32})$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k\,\delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ B_{11} & B_{12} & B_{13} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

③ third row

$$\delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31}+kB_{31} & A_{32}+kB_{32} & A_{33}+kB_{33} \end{pmatrix} = A_{11}\cdot A_{21}\cdot(A_{32}+kB_{32})$$

$$= A_{11}A_{21}A_{32} + k(A_{11}\cdot A_{21}\cdot B_{32})$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k\,\delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

Therefore, given δ is a 3-linear function.

#8. $\delta(A) = A_{11} A_{31} A_{32}$

Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

① first row

$$\delta\begin{pmatrix} A_{11}+kB_{11} & A_{12}+kB_{12} & A_{13}+kB_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = (A_{11}+kB_{11}) A_{31} \cdot A_{32}$$

$$= A_{11}A_{31}A_{32} + k(B_{11}A_{31}A_{32})$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k\,\delta\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

② Second row

$$\delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21}+kB_{21} & A_{22}+kB_{22} & A_{23}+kB_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = A_{11} \cdot A_{32} \cdot A_{33}$$

$$= \delta\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

It's not linear.

Hence, given δ isn't a 3-linear function.

#13. Prove that $\det : M_{2\times2}(F) \to F$ is a 2-linear function of the columns of a matrix.

☆ Theorem 4.8. For any $A \in M_{n\times n}(F)$, $\det(A^T) = \det(A)$

Since we have $A \in M_{n\times n}(F)$, by using Theorem 4.8, we do know $\det(A^T) = \det(A)$

Hence, if we prove A is a 2-linear function, since row of A^T are columns in A, we prove A is a 2-linear function of the columns of a matrix.

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

① First row

$$\det\begin{pmatrix} A_{11}+kB_{11} & A_{12}+kB_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + k \cdot \det\begin{pmatrix} B_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

② Second row

$$\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21}+kB_{21} & A_{22}+kB_{22} \end{pmatrix} = \det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + k\cdot\det\begin{pmatrix} A_{11} & A_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Therefore, $\det: M_{2\times2}(F)\to F$ is a 2-linear function of the columns of a matrix.

#18. Prove that the set of all n-linear functions over a field F is a vector space over F under the operations of function addition and scalar multiplication as defined in Example 3 of Section 1.2.

We need to prove that all n-linear functions over F is a vector space over F under the operations of function addition and scalar multiplication

① $(\delta_1+\delta_2)(A) = \delta_1(A)+\delta_2(A)$

② $(c\cdot\delta_1)(A) = c\cdot\delta_1(A)$

where $c\in F$, and $A\in M_{n\times n}(F)$

And we need to check VS1 ~ 8 to verify set of all n-linear functions over F is a vector space over F

VS 1 : for all $\delta_1,\delta_2 \in M_{n\times n}(F)\to F$, and $A\in M_{n\times n}(F)$,

$\delta_1(A)+\delta_2(A) = \delta_2(A)+\delta_1(A)$: True since δ is linear

VS 2: $\forall \delta_1,\delta_2,\delta_3 \in M_{n\times n}(F)\to F$,

$\delta_1(A)+(\delta_2(A)+\delta_3(A)) = (\delta_1(A)+\delta_2(A))+\delta_3(A)$: True, δ is linear.

VS 3: $\exists\ 0$ s.t $\delta_1+0=\delta_1$

$\delta\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix} = 0$, and $\delta_1+0=\delta_1$: True

VS 4: For each element $\delta_1\in M_{n\times n}(F)\to F$, $\exists\ \delta_2\in M_{n\times n}(F)\to F$ s.t $\delta_1+\delta_2=0$

Since $\delta_1+\delta_2=0$, $\delta_1=-\delta_2 \Rightarrow \delta_1(A)+\delta_2(A) = -\delta_2(A)+\delta_2(A)=0$: True

VS5 : For each element $\delta_1\in M_{n\times n}(F)\to F$, $1\delta_1(A)=\delta_1(A)$: True

VS6: For each element $a,b\in F$ and each pair of elements $\delta_1 \in M_{n\times n}(F)\to F$,

$(ab)(\delta_1(A)) = a(b\delta_1(A))$: True, since δ_1 is linear.

VS 7 : For each element $a \in F$, and each pair of elements $\delta_1, \delta_2 \in M_{n\times n}(F) \to F$,
$a(\delta_1+\delta_2)(A) = a\delta_1(M) + a\delta_2(M)$: True, δ is linear

VS 8 : For each pair of elements $a, b \in F$, and each element $\delta_1 \in M_{n\times n}(F) \to F$,
$(a+b)\delta_1(A) = a\delta_1(A) + b\delta_1(A)$: True. δ is linear.

Hence, it's true that all n-linear functions over F is a vector space over F under the operations of function addition and scalar multiplication

1. Determinants of Order 2.

Definition

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2x2 matrix with entries from a field F, then we define the determinant of A, denoted $\det(A)$ or $|A|$, to be the scalar $ad-bc$.

Theorem 4.1 : The function $\det : M_{2\times2}(F) \to F$ is a linear function of each row of a 2x2 matrix when the other row is held fixed.

That is, if $u, v,$ and w are in F^2 and k is a scalar, then

$$\det\begin{pmatrix} u+kv \\ w \end{pmatrix} = \det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix}$$

and, $$\det\begin{pmatrix} w \\ u+kv \end{pmatrix} = \det\begin{pmatrix} w \\ u \end{pmatrix} + k\det\begin{pmatrix} w \\ v \end{pmatrix}$$

Theorem 4.2. Let $A \in M_{2\times2}(F)$. Then the determinant of A is nonzero iff A is invertible. Moreover, if A is invertible, then

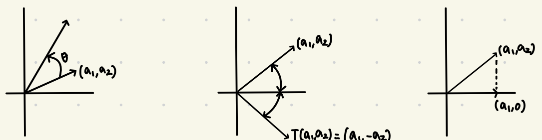
$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Recap:

Linear Transformation : Let V and W be vector space over the same field F. We call a function $T: V \to W$ a linear transformation from V to W, if $\forall x, y \in V$, and $c \in F$

(a) $T(x+y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

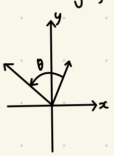


(a) Rotation (b) Reflection (c) Projection

☆ One thing that turns out to be very useful to understanding one of these transformations is to measure exactly how much T stretches or squishes things. 'Determinant' is a scaling factor, the factor by which a linear transformation changes any area.

The Area of Parallelogram

: By the angle between two vectors in $\mathbb{R}^2$, we mean the angle with measure θ $(0 \le \theta < \pi)$ that is formed by the vectors emanating from the origin.

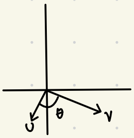


If $\beta = \{u, v\}$ is an ordered basis for $\mathbb{R}^2$, we define the orientation of β to be the real number.

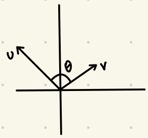
$$\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|},$$ and clearly $\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1$

Recall that a coordinate system $\{u, v\}$ is called "right handed" if u can be rotated in a counterclockwise direction through an angle θ to coincide with v. Otherwise $\{u, v\}$ is called "left handed" system.

Ex)

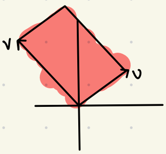


A right handed coordinate system



A right handed coordinate system

Any ordered set $\{u, v\}$ in $\mathbb{R}^2$ determines parallelogram in the following manner. Regarding u and v as arrows emanating from the origin of $\mathbb{R}^2$, we call the parallelogram having u and v as adjacent sides the parallelogram determined by u and v.



$$A\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix}$$

2. Determinants of order n

Definitions : Let $A \in M_{n\times n}(F)$. If $n=1$, so that $A = A_{11}$, we define $\det(A) = A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j})$$

The scalar $(-1)^{i+j} \det(\tilde{A}_{ij})$ is called cofactor of the entry of A in row i, column j.

Ex) Let $A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$

A matrix without 1 row, 1 column

$$\det(A) = (-1)^{1+1} \cdot A_{11} \cdot \det(\tilde{A}_{11}) + (-1)^{1+2} A_{12} \cdot \det(\tilde{A}_{12}) + (-1)^{1+3} A_{13} \cdot \det(\tilde{A}_{13})$$

$$= 1 \cdot 1 \cdot \det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + -1 \cdot 3 \cdot \det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + 1 \cdot -3 \cdot \det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} = 40.$$

Theorem 4.3

: The determinant of an $n\times n$ matrix is a linear function of each row when the remaining rows are held fixed. That is, for $1 \leq r \leq n$, we have

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \cdot \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Theorem 4.4 : The determinant of a square matrix can be evaluated by cofactor expansion along any rows.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

Corollary : If $A \in M_{n\times n}(F)$ has two identical rows, then $\det(A) = 0$

Ex) Let $A = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}$, $\det(A) = 3\cdot 5 - 3\cdot 5 = 0$.

Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $\det(B) = (-1)^4 \cdot 4 \cdot \det(\tilde{B}_{31}) + (-1)^5 \cdot 5 \cdot \det(\tilde{B}_{32}) + (-1)^6 \cdot 6 \cdot \det(\tilde{B}_{33})$

$$= 1 \cdot 4 \cdot \det\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} + -1 \cdot 5 \cdot \det\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} + 1 \cdot 6 \cdot \det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0.$$

Theorem 4.5 : If $A \in M_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then $\det(B) = -\det(A)$

think $A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix}$, $B = \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$

Theorem 4.6 : Let $A \in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A.
Then, $\det(B) = \det(A)$

Corollary : If $A \in M_{n\times n}(F)$ has rank less than n, then $\det(A) = 0$

Example for Theorem 4.6

: Let $M = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & 18 \end{pmatrix}$, then $\det(M) = 1 \cdot 1 \cdot \det(\tilde{M}_{11}) + -1 \cdot 3 \cdot \overbrace{\det(\tilde{M}_{12})}^{0} + 1 \cdot -3 \cdot \overbrace{\det(\tilde{M}_{13})}^{0}$

$= 1 \cdot \det\begin{pmatrix} 4 & -7 \\ 16 & 18 \end{pmatrix} + 0 + 0 = 40$

If we add -4 times row 2 of M to row 3,

$P = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 0 & 10 \end{pmatrix}$, $\det(P) = (-1)^2 + 1 \cdot \det\begin{pmatrix} 4 & -7 \\ 0 & 10 \end{pmatrix} = 40.$

Hence, $\det(M) = \det(P)$

3. Properties of Determinants

Important facts about the determinants of elementary matrices.

(a) If E is an elementary matrix obtained by interchanging any two rows of I, then $\det(E) = -1$

(b) If E is an elementary matrix obtained by multiplying some row of I by the nonzero scalar k, then $\det(E) = k$

(c) If E is an elementary matrix obtained by adding a multiple of some row of I to another row, then $\det(E) = 1$.

Theorem 4.7: For any $A, B \in M_{n\times n}(F)$, $\det(AB) = \det(A) \cdot \det(B)$

Corollary: If $A \in M_{n\times n}(F)$ is not invertible, then the rank of A is less than n. So, $\det(A) = 0$ by Corollary to Theorem 4.6

Theorem 4.8. For any $A \in M_{n\times n}(F)$, $\det(A^T) = \det(A)$

Theorem 4.9 (Cramer's Rule). Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \dots, x_n)^T$.

If $\det(A) \neq 0$, then this system has a unique solution, and for each k $(k = 1, 2, \dots, n)$,

$$x_k = \frac{\det(M_k)}{\det(A)}$$

Example:

$$x_1 + 2x_2 + 3x_3 = 2$$
$$x_1 \quad + x_3 = 3$$
$$x_1 + x_2 - x_3 = 1$$

The matrix form of this system of linear equations is $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Since $\det(A) = 6 \neq 1$, Cramer's rule applies.

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{\det \begin{array}{c} \begin{array}{ccc} b & & \end{array} \\ \begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \end{array}}{6} = \frac{15}{6} = \frac{5}{2}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{\det \begin{array}{c} \begin{array}{ccc} & b & \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \end{array}}{6} = \frac{-6}{6} = -1$$

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{\det \begin{array}{c} \begin{array}{ccc} & & b \end{array} \\ \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix} \end{array}}{6} = \frac{3}{6} = \frac{1}{2}$$

Thus, $(x_1, x_2, x_3) = \left(\frac{5}{2}, -1, \frac{1}{2}\right)$

★ Recap: Upper triangular matrix has following properties:

$$\det(A) = \prod_{i=1}^{n} T_{ii}, \quad \mathrm{tr}(A) = \sum_{i=1}^{n} T_{ii}$$

And, similar matrices have same trace and determinant.

Ex)

$$\det\begin{pmatrix} -\lambda & 1 & 2 & 0 \\ 0 & -\lambda & 2 & 6 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = (-\lambda)^4$$

5. A characterization of the Determinant.

Definition: A function $\delta : M_{n\times n}(F) \to F$ is called an n-linear function if it is a linear function of each row of a $n\times n$ matrix when the remaining $n-1$ rows are held fixed, that is δ is linear if, for every $r = 1, 2, \dots, n$, we have

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\cdot\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

whenever k is a scalar and u, v, and each a_i are vectors in F^n.

Example 1: The function $\delta : M_{n\times n}(F) \to F$ defined by $\delta(A) = 0$ for each $A \in M_{n\times n}(F)$ is an n-linear function.

Example 2: For $1 \le j \le n$, define $\delta_j : M_{n\times n}(F) \to F$ by $\delta_j(A) = A_{1j}A_{2j}\cdots A_{nj}$ for each $A \in M_{n\times n}$; that is, $\delta_j(A)$ equals the product of the entries of column j of A. Let $A \in M_{n\times n}(F)$, $a_i = (A_{i1}, A_{i2}, \dots, A_{in}$, and $v = (b_1, b_2, \dots, b_n) \in F^n$. Then each δ_j is an n-linear function because, for any scalar k, we have

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = A_{1j}\cdots A_{(r-1)j}\cdot(A_{rj}+kb_j)\cdot A_{(r+1)j}\cdot\cdots A_{nj}$$

$$\star\; a\cdot(b+cx)\cdot e = a\cdot b\cdot e + a\cdot cx\cdot e$$

$$= A_{1j}\cdot\cdots\cdot A_{(r-1)j}\cdot A_{rj}A_{(r+1)j}\cdot\cdots\cdot A_{nj} + A_{1j}\cdot\cdots A_{(r-1)j}\cdot(kb_j)\cdot A_{rj}\cdot A_{(r+1)j}\cdot\cdots\cdot A_{nj}$$

$$= A_{1j}\cdot\cdots\cdot A_{(r-1)j}\cdot A_{rj}A_{(r+1)j}\cdot\cdots\cdot A_{nj} + k(A_{1j}\cdot\cdots A_{(r-1)j}\cdot b_j\cdot A_{rj}\cdot A_{(r+1)j}\cdot\dots\cdot A_{nj})$$

$$= \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\cdot\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Therefore, δ is a n-linear function

Def: An n-linear function $\delta : M_{n\times n}(F) \to F$ is called alternating if, for each $A \in M_{n\times n}(F)$, we have $\delta(A) = 0$ whenever two adjacent rows of A are identical.