1. let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f(x)) = 2f(x) + 2(x+1)f'(x) + f''(x)$: Similar Problems : HW #5, 2.1 #3, #5, #9

State why T is not linear.

a) Prove that T is linear

2.1 : Linear Transformations, Null Spaces, and Ranges

★ We often simply call T linear. Reader should verify 4 properties of a function $T: V \to W$

① If T is linear, then $T(0) = 0$

② T is linear iff $T(cx+y) = cT(x) + T(y)$ $\forall x, y \in V$ and $c \in F$.

③ If T is linear, then $T(x-y) = T(x) - T(y)$ $\forall x, y \in V$

④ T is linear iff $x_1, x_2, \dots, x_n \in V$, and $a_1, a_2, \dots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

b) Compute the Null space $N(T)$

c) Compute rank and nullity of T

Theorem 2.3 (Dimension Theorem). Let V, W be V.S, and let $T: V \to W$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

d) Compute the matrix representation $[T]_\beta$, where $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$

Definition: Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in V$, let $a_1, a_2, \dots, a_n$ be the unique scalars s.t

$$x = \sum_{i=1}^{n} a_i v_i$$

we define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Example.

Let $V = P_2(\mathbb{R})$, $\beta = \{\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}\}$: Standard ordered basis, $f(x) = 4 + 6x - 7x^2$

$$[f]_\beta = \begin{array}{c} \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix} \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \end{array} \quad f(x) = \sum a_i v_i = 4(1) + 6(x) - 7(x^2)$$

#2. Compute the inverse of 2X2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad-bc \neq 0$, and $a,b,c,d \in \mathbb{R}$

Definition: Let V and W be vector spaces, and let $T: V \to W$ be linear. A function $U: W \to V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$.

The following facts hold for invertible functions T and U.

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$

We often use the fact that a function is invertible iff it is both one-to-one and onto.

3. Let $T: V \to W$ be a linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then, T is invertible iff $\operatorname{rank}(T) = \dim(V)$

Theorem 2.17. Let V and W be vector spaces, and let $T: V \to W$ be linear and invertible. Then $T^{-1}: W \to V$ is linear.

Theorem 2.18. Let $T: V \to W$ be linear. Then T is invertible iff $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

#3. Compute the following expressions : Similar problems in 2.3: Composition of linear transformations and Matrix Multiplication

a) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}$$

(c) $$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & -1 \end{pmatrix}^{-1}$$

#4. Consider the bases $\beta = \{2, x+1\}$, and $\gamma = \{-2x, 3x+1\}$ of the vector space $P_1(\mathbb{R})$: 2.5 : The Change of Coordinate Matrix

Theorem 2.22. Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$. Then

(a) Q is invertible

(b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$

Q is called a "Change of Coordinate matrix"

Q changes β'- coordinates into β- coordinates.

If $\beta = \{x_1, x_2, \dots, x_n\}$, and $\beta' = \{x_1', x_2', \dots, x_n'\}$

$x_j' = \sum_{i=1}^{n} Q_{ij} x_i$, for $j = 1, 2, \dots, n$; that is, the jth Column of Q is $[x_j']_\beta$

Example 1

In $\mathbb{R}^2$, let $\beta = \{(1,1), (1,-1)\}$, and $\beta' = \{(2,4), (3,1)\}$. Since

$(2,4) = 3(1,1) - 1(1,-1)$

$(3,1) = 2(1,1) + 1(1,-1)$

$$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

$[(2,4)_\beta] = Q[(2,4)]_{\beta'}$

(a) Find the change of coordinate matrix that changes β-Coordinates into γ- coordinates

$2 = 3(-2x) + 2(3x+1) = 2$

$x+1 = 1(-2x) + 1(3x+1) = x+1$

hence $Q = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$

(b) Find the change of coordinate matrix that changes γ-Coordinates into β- coordinates

(c) Verify that the matrices found in (a) and (b) are inverse to each other.

: Check 1) $[I_{P_1(\mathbb{R})}]_\beta^\gamma \cdot [I_{P_1(\mathbb{R})}]_\gamma^\beta = I$

2) $[I_{P_1(\mathbb{R})}]_\gamma^\beta \cdot [I_{P_1(\mathbb{R})}]_\beta^\gamma = I$

(d) Find the coordinate vector $[2x-4]_\gamma$ of $p(x) = 2x-4$ w.r.t the basis γ

: Since $(2x-4) = -7(-2x) - 4(3x+1)$, $[2x-4]_\gamma = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$

(e) Now use a change of coordinate matrix to calculate the coordinate vector $[2x-4]_\beta$ of $p(x) = 2x-4$ w.r.t the basis β.

: By using principles of change of coordinate matrices, we have

$[2x-4]_\beta = [I_{P_1(\mathbb{R})}]_\gamma^\beta [2x-4]_\gamma = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -7 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

#5. Consider the matrices : Similar problems 3.1: Elementary Matrix Operations and Elementary Matrices, #2,5

$$A=\begin{pmatrix} 4 & 2 & -1 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix},\quad B=\begin{pmatrix} 2 & 8 & 3 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix},\quad C=\begin{pmatrix} 2 & 2 & 4 \\ -2 & 5 & 3 \\ -4 & 3 & -1 \end{pmatrix}$$

(a) Find a 3X3 elementary matrix E s.t $B=EA$

(b) Find 3X3 elementary matrices E_1, E_2 s.t $C=AE_1E_2$

(c) Find the inverse of elementary matrix $E=\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$?

#6. T or F

#7. Express the matrix $B=\begin{pmatrix} 0 & 3 & 9 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ as a product of elementary matrices. : Similar Problems 3.2 #7

#8. Calculate the rank of the matrix $A=\begin{pmatrix} 1 & 3 & 5 & 1 & 5 \\ 1 & 4 & 7 & 3 & -2 \\ 1 & 5 & 9 & 5 & -9 \\ 0 & 3 & 6 & 2 & -1 \end{pmatrix}$: Similar Problems 3.2 #2.

#9. Constructing (or not constructing) examples

(a) Are $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$ isomorphic vector spaces? If "yes", then exhibit an isomorphism $P_3(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$. If "no", then explain why no such isomorphism can exist.

Definitions : Let V and W be vector spaces. We say that V is isomorphic to W if $\exists$ a linear transformation $T: V \to W$ that is invertible. Such a linear transformation is called isomorphism from V to W.

#3. Which of the following pairs of vector spaces are isomorphic?

(a) F^3 and $P_3(F)$

Not isomorphic since $\dim(F^3) \neq \dim(P_3(F))$

(d) $V=\{A\in M_{2\times2}(\mathbb{R}) : tr(A)=0\}$ and $\mathbb{R}^4$

(b) Does there exist a 3×2 matrix A and 2×3 matrix B so that $AB = I_3$?

#10. Find the solution set to the system of linear equations. : Similar problems 3.3 #23

$$
\begin{aligned}
2x_1 + x_2 - x_3 &= 5 \\
x_1 - x_2 + x_3 &= 1 \\
x_1 + 2x_2 - 2x_3 &= 4
\end{aligned}
$$

#11. Show that if A is a square matrix that has two identical rows, then $\det(A) = 0$.

#1. let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f(x)) = 2f(x) + 2(x+1)f'(x) + f''(x)$: Similar Problems : HW #5, 2.1 #3, #5, #9

State why T is not linear.

a) Prove that T is linear

2.1: Linear Transformations, Null Spaces, and Ranges

☆ We often simply call T linear. Reader should verify 4 properties of a function $T: V \to W$

① If T is linear, then $T(0) = 0$

② T is linear iff $T(cx+y) = cT(x) + T(y)\ \forall x, y \in V$ and $c \in F$. ☆

③ If T is linear, then $T(x-y) = T(x) - T(y)\ \forall x, y \in V$

④ T is linear iff $x_1, x_2, \dots, x_n \in V$, and $a_1, a_2, \dots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

2.1

#3. $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) \to (a_1 + a_2, 0, 2a_1 - a_2)$

Let's say $\exists\ c \in \mathbb{R}$, $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$

① $T(0,0) = (0+0, 0, 2\cdot 0 - 0) = (0,0,0) = 0.$

② $T(cx+y) = T(c(x_1, x_2) + (y_1, y_2)) = T(cx_1 + y_1, cx_2 + y_2)$

$$= (cx_1 + y_1 + cx_2 + y_2, 0, 2(cx_1 + y_1) - (cx_2 - y_2))$$
$$= (c(x_1 + x_2) + y_1 + y_2, 0, c(2x_1 - x_2) - 2y_1 - y_2)$$
$$= cT(x) + T(y)$$

Hence, T is linear

#5. $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$

Let $f(x) = a_1x^2 + a_2x + a_3$

$g(x) = b_1x^2 + b_2x + b_3$

and $f(x), g(x) \in P_2(\mathbb{R})$.

Let $c \in \mathbb{R}$, an arbitrary $\mathbb{R}$.

$T(f+g(x)) = x(f(x) + g(x)) + (f+g)'(x)$

$$= x(a_1x^2 + a_2x + a_3 + b_1x^2 + b_2x + b_3) + (2a_1x + a_2 + 2b_1 + b_2)$$
$$= x(a_1x^2 + a_2 + a_3) + (2a_1x + a_2) + x(b_1x^2 + b_2x + b_3) + (2b_1 + b_2)$$
$$= T(f(x)) + T(g(x))$$

$T(cf(x)) = T(ca_1x^2 + ca_2x + ca_3)$

$$= x(ca_1x^2 + ca_2x + ca_3) + (2ca_1x + ca_2)$$
$$= cx(a_1x^2 + a_2x + a_3) + c(2a_1x + a_2)$$
$$= cT(f(x))$$

#1. let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f(x)) = 2f(x) + 2(x+1)f'(x) + f''(x)$

a) Prove that T is linear

Let $f(x) = a_1x^2 + a_2x + a_3$

$g(x) = b_1x^2 + b_2x + b_3$

$c \in \mathbb{R}$, an arbitrary real number.

and, $f(x), g(x) \in P_2(\mathbb{R})$,

If T is linear, $T(f+g(x)) = T(f(x)) + T(g(x))$

and $T(cf(x)) = cT(f(x))$

$= 2f(x) + 2(x+1)f'(x) + f''(x) + 2g(x) + 2(x+1)g'(x) + g''(x)$

① $T(f+g(x)) = 2(f(x)+g(x)) + 2(x+1)(f+g)' + (f+g)''$

$= 2(a_1x^2 + a_2x + a_3 + b_1x^2 + b_2x + b_3) + 2(x+1)(2a_1x + a_2 + 2b_1x + b_2) + (2a_1 + 2b_1)$

$= 2a_1x^2 + 2a_2x + 2a_3 + 2b_1x^2 + 2b_2x + 2b_3 + 2(x+1)(2a_1x + a_2) + 2(x+1)(2b_1x + b_2) + 2a_1 + 2b_1$

$= \underbrace{2a_1x^2 + 2a_2 + 2a_3 + 2(x+1)(2a_1x + a_2) + 2a_1}_{T(f(x))} + \underbrace{2b_1x^2 + 2b_2x + 2b_3 + 2(x+1)(2b_1x + b_2) + 2b_1}_{T(g(x))}$

$= T(f(x)) + T(g(x))$

② $T(cf(x)) = T(ca_1x^2 + ca_2x + ca_3) = 2(ca_1x^2 + ca_2x + ca_3) + 2(x+1)(2ca_1x + ca_2) + (2ca_1)$

$= c2(a_1x^2 + a_2x + a_3) + c\,2(x+1)(2a_1x + a_2) + c(2a_1)$

$= c \cdot T(f(x))$

Hence, T is linear.

b) Compute the Null space $N(T)$

$N(T) = \{ f(x) \in P_2(\mathbb{R}) \mid T(f(x)) = 0 \}$

$0 = T(f(x)) = 2f(x) + 2(x+1)f'(x) + f''(x)$

$= 2(a_1x^2 + a_2x + a_3) + 2(x+1)(2a_1x + a_2) + (2a_1)$

$= 2a_1x^2 + a_2x + a_3 + (2x+2)(2a_1x + a_2) + 2a_1$

$= \cancel{2a_1x^2} + \cancel{a_2x} + a_3 + \cancel{4a_1x^2} + \cancel{2a_2x} + \cancel{4a_1x} + 2a_2 + 2a_1$

$= 6a_1x^2 + 3a_2x + 4a_1x + 2a_1 + 2a_2 + a_3 = 0$

$= a_1(6x^2 + 4x + 2) + a_2(3x + 2) + a_3 = 0$

which is possible when $a_1 = a_2 = a_3 = 0$.

Hence $N(T) = \{0\}$.

nullity$(T) = 0$, and T is one-to-one.

c) Compute rank and nullity of T

Theorem 2.3 (Dimension Theorem). Let V, W be V.S, and let $T: V \to W$ be linear. If V is finite-dimensional, then

$\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

$\dim(P_2(\mathbb{R})) = 3$.

$\text{nullity}(T) = 0$.

Hence, $\text{rank}(T) = 3$.

d) Compute the matrix representation $[T]_\beta$, where $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$

Definition: Let $\beta = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in V$,

let $a_1, a_2, \dots, a_n$ be the unique scalars s.t

$$x = \sum_{i=1}^{n} a_i v_i$$

we define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Example.

Let $V = P_2(\mathbb{R})$, $\underset{v_1 \, v_2 \, v_3}{\beta = \{1, x, x^2\}}$: standard ordered basis, $f(x) = 4 + 6x - 7x^2$

$$[f]_\beta = \begin{array}{c} \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix} \end{array}\begin{array}{l} a_1 \\ a_2 \\ a_3 \end{array} \quad f(x) = \sum a_i v_i = 4(1) + 6(x) - 7(x^2)$$

$V = P_2(\mathbb{R})$

$\beta = [1, x, x^2]$

$T(f(x)) = 2f(x) + 2(x+1)f'(x) + f''(x)$

$T(1) = 2(1) + 2(x+1) \cdot 0 + 0 = 2$

$T(x) = 2x + 2(x+1)(1) + 0 = 4x + 2$

$T(x^2) = 2x^2 + 2(x+1)(2x) = 2x^2 + 4x^2 + 4x = 6x^2 + 4x$

$$[T]_\beta = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{pmatrix}$$

* Matrix Representation: $[T]_\beta$

$T(1)$

$T(x)$ with given definition of $f(x)$.

$T(x^2)$

#2. Compute the inverse of 2X2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad-bc \neq 0$, and $a,b,c,d \in \mathbb{R}$

Definition: Let V and W be vector spaces, and let $T: V \to W$ be linear. A function $U: W \to V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$.

The following facts hold for invertible functions T and U.

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$

We often use the fact that a function is invertible iff it is both one-to-one and onto.

3. Let $T: V \to W$ be a linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then, T is invertible iff $\text{rank}(T) = \dim(V)$

Theorem 2.17. Let V and W be vector spaces, and let $T: V \to W$ be linear and invertible. Then $T^{-1}: W \to V$ is linear.

Theorem 2.18. Let $T: V \to W$ be linear. Then T is invertible iff $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

#4. Consider the bases $\beta = \{2, x+1\}$, and $\gamma = \{-2x, 3x+1\}$ of the vector space $P_1(\mathbb{R})$: 2.5 : The Change of Coordinate Matrix

Theorem 2.22. Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$. Then

(a) Q is invertible

(b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$

Q is called a "Change of Coordinate matrix"

Q changes β'- coordinates into β- coordinates.

If $\beta = \{x_1, x_2, \dots, x_n\}$, and $\beta' = \{x_1', x_2', \dots, x_n'\}$

$x_j' = \sum_{i=1}^{n} Q_{ij} x_i$, for $j = 1, 2, \dots, n$; that is, the jth Column of Q is $[x_j']_\beta$

Example 1

In $\mathbb{R}^2$, let $\beta = \{(1,1), (1,-1)\}$, and $\beta' = \{(2,4), (3,1)\}$. Since

$(2,4) = 3(1,1) - 1(1,-1)$

$(3,1) = 2(1,1) + 1(1,-1)$

$$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

$[(2,4)_\beta] = Q[(2,4)]_{\beta'}$

(a) Find the change of coordinate matrix that changes β-coordinates into γ- coordinates

$\beta = \{2, x+1\}$, $\gamma = \{-2x, 3x+1\}$. of the vector space $P_1(\mathbb{R})$

changes β-coordinates into γ-coordinates

$$2 = a(-2x) + b(3x+1)$$

$$= -2ax + 3bx + b.$$

$$a = 3, \quad b = 2$$

$$x+1 = a(-2x) + b(3x+1)$$

$$\Rightarrow a = 1, b = 1 \qquad \text{Hence } [Q]_\beta^\gamma = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

(b) Find the change of coordinate matrix that changes γ-coordinates into β-coordinates

$$-2x = a(2) + b(x+1)$$

$$\Rightarrow a = 1, \ b = -2$$

$$3x+1 = a(2) + b(x+1)$$

$$\Rightarrow a = -1, b = 3 \qquad \text{Hence, } [Q]_\gamma^\beta = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

(c) Verify that the matrices found in (a) and (b) are inverse to each other.

: Check 1) $[I_{P_1(\mathbb{R})}]_\beta^\gamma \cdot [I_{P_1(\mathbb{R})}]_\gamma^\beta = I$

2) $[I_{P_1(\mathbb{R})}]_\gamma^\beta \cdot [I_{P_1(\mathbb{R})}]_\beta^\gamma = I$

$$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, they're invertible to each other.

(d) Find the coordinate vector $[2x-4]_\gamma$ of $p(x) = 2x-4$ w.r.t the basis γ

$$2x - 4 = a(-2x) + b(3x+1)$$

$$= -2ax + 3bx + b$$

$$\Rightarrow a = -7, \ b = -4$$

$$\text{Hence, } [2x-4]_\gamma = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$$

(e) Now use a change of coordinate matrix to calculate the coordinate vector $[2x-4]_\beta$ of $p(x) = 2x-4$ w.r.t the basis β.

: By using principles of change of coordinate matrices, we have

$$[2x-4]_\beta = [Q]_\gamma^\beta \cdot [2x-4]_\gamma$$
$$= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -7 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

#5. Consider the matrices : Similar problems 3.1: Elementary Matrix Operations and Elementary Matrices, #2,5

$$A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 8 & 3 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & 4 \\ -2 & 5 & 3 \\ -4 & 3 & -1 \end{pmatrix}$$

Similar Problems 3.1 #2.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$$

elementary operation that transforms A into B.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}$$

$C_2 \Rightarrow C_2 - 2C_1$

$$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}$$

elementary operation that transforms B into C

$$B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$$

$r_2 = r_2 - r_1$

$(1,-2,1) - (1,0,3) = (0,-2,-2)$

Now, #5(a): Find a 3X3 elementary matrix E s.t $B = EA$

① Find elementary operation that transforms A into B

$$A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 8 & 3 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix}$$

$r_1 = r_1 + 2r_3$

$(4,2,-1) + (-2,6,4) = (2,8,3)$

② Perform $r_1 = r_1 + 2r_3$ to the identity matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_1 = r_1 + 2r_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E.$$

Then, $B = EA$.

(b) Find 3x3 elementary matrices E_1, E_2, s.t $C = AE_1E_2$

$$A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 2 & 4 \\ -2 & 5 & 3 \\ -4 & 3 & -1 \end{pmatrix}$$

$C_1 \leftrightarrow C_3$, then

$$A = \begin{pmatrix} -1 & 2 & 4 \\ 1 & 5 & 3 \\ 2 & 3 & -1 \end{pmatrix}$$

$C_1 = -2C_1$, then

$$A = \begin{pmatrix} 2 & 2 & 4 \\ -2 & 5 & 3 \\ -4 & 3 & 1 \end{pmatrix}$$

Hence, $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1 = -2C_1} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$C = AE_1E_2 = \begin{pmatrix} 4 & 2 & -1 \\ 3 & 5 & 1 \\ -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Find the inverse of elementary matrix $E = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_1 = r_1 + 5r_3} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Perform reverse operation : $r_1 = r_1 - 5r_3$ to I_3 thus produce E^{-1}.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_1 = r_1 - 5r_3} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E^{-1}.$$

#7. Express the matrix $B=\begin{pmatrix}0&3&9\\1&2&0\\1&2&1\end{pmatrix}$ as a product of elementary matrices.

$$B=\begin{pmatrix}0&3&9\\1&2&0\\1&2&1\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&3&9\\1&2&0\\1&2&1\end{pmatrix}\xrightarrow{r_3=r_3-r_2}\begin{pmatrix}0&3&9\\1&2&0\\0&0&1\end{pmatrix}\xrightarrow{r_1=\frac{1}{3}r_1}\begin{pmatrix}0&1&3\\1&2&0\\0&0&1\end{pmatrix}\xrightarrow{r_1=r_1-3r_3}\begin{pmatrix}0&1&0\\1&2&0\\0&0&1\end{pmatrix}\xrightarrow{r_2=r_2-2r_1}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\xrightarrow{r_1\leftrightarrow r_2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$E_1=\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix},\quad E_2=\begin{pmatrix}\frac{1}{3}&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad E_3=\begin{pmatrix}1&0&-3\\0&1&0\\0&0&1\end{pmatrix},\quad E_4=\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix},\quad E_5=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

With these elementary matrices, we therefore have $E_5E_4E_3E_2E_1=I_3$

Hence, $B=E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$

: Similar Problems 3.2 #7

Express the invertible matrix $\begin{pmatrix}1&2&1\\1&0&1\\1&1&2\end{pmatrix}$ as a product of elementary matrices.

Let $A=\begin{pmatrix}1&2&1\\1&0&1\\1&1&2\end{pmatrix}$, and we need $A\to I_3$

$$\begin{pmatrix}1&2&1\\1&0&1\\1&1&2\end{pmatrix}\xrightarrow{r_2=r_2-r_1}\begin{pmatrix}1&2&1\\0&-2&0\\1&1&2\end{pmatrix}\xrightarrow{R_3=R_3-R_1}\begin{pmatrix}1&2&1\\0&-2&0\\0&-1&1\end{pmatrix}\xrightarrow{r_2=-\frac{1}{2}r_2}\begin{pmatrix}1&2&1\\0&1&0\\0&-1&1\end{pmatrix}\xrightarrow{r_3=r_3+r_2}\begin{pmatrix}1&2&1\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{r_1=r_1-2r_2}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{r_1=r_1-r_3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$E_1=\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix},\quad E_2=\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix},\quad E_3=\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&0\\0&0&1\end{pmatrix},\quad E_4=\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix},\quad E_5=\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix},\quad E_6=\begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}$$

$E_6E_5E_4E_3E_2E_1=I_3$

Hence $A=E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$$

#8. Calculate the rank of the matrix $A=\begin{pmatrix} 1 & 3 & 5 & 1 & 5 \\ 1 & 4 & 7 & 3 & -2 \\ 1 & 5 & 9 & 5 & -9 \\ 0 & 3 & 6 & 2 & -1 \end{pmatrix}$

: Similar Problems 3.2 #2.

#2

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ rank $=2$

(f) $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 8 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank=3

$A=\begin{pmatrix} 1 & 3 & 5 & 1 & 5 \\ 1 & 4 & 7 & 3 & -2 \\ 1 & 5 & 9 & 5 & -9 \\ 0 & 3 & 6 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & 1 & 5 \\ 0 & 1 & 2 & 2 & -7 \\ 0 & 2 & 4 & 4 & -14 \\ 0 & 3 & 6 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & 1 & 5 \\ 0 & 1 & 2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & 1 & 5 \\ 0 & 1 & 2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 & 1 & 5 \\ 0 & 1 & 2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rank=3.

#9. Constructing (or not constructing) examples

(a) Are $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$ isomorphic vector spaces? If "yes", then exhibit an isomorphism $P_3(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$. If "no", then explain why no such isomorphism can exist.

Definitions : Let V and W be vector spaces. We say that V is isomorphic to W if $\exists$ a linear transformation $T: V \to W$ that is invertible. Such a linear transformation is called isomorphism from V to W.

#3. Which of the following pairs of vector spaces are isomorphic?

(a) F^3 and $P_3(F)$

Not isomorphic since $\dim(F^3) \neq \dim(P_3(F))$

(d) $V=\{A \in M_{2\times2}(\mathbb{R}) : tr(A)=0\}$ and $\mathbb{R}^4$

$\dim(P_3(\mathbb{R})) = \dim(M_{2\times2}(\mathbb{R}))$. Hence, they're isomorphic vector spaces.

For an explicit isomorphism, $T(ax^3+bx^2+cx+d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

since T is both onto and one-to-one, it's invertible

(b) Does there exist a 3×2 matrix A and 2×3 matrix B so that $AB = I_3$?

False.

if A is 3×2, B is 2×3

$L_{AB} = L_A \cdot L_B : F^3 \to F^3$

where $L_A : F^3 \to F^2$ and $L_B = F^2 \to F^3$

If $AB = I_3$, then AB should be invertible, and hence $L_{AB} = L_A L_B$ is invertible.

But $L_A : F^2 \to F^3$ is not onto.

#10. Find the solution set to the system of linear equations. : Similar problems 3.3 #23

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 + x_3 &= 1 \\ x_1 + 2x_2 - 2x_3 &= 4 \end{aligned}$$

: Similar problems 3.3 #23

#2.

(a) $\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 6x_2 = 0 \end{cases}$

Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. Then, $\operatorname{rank}(A) = 1$, so that $\dim(\text{solution set}) = \overset{x_1, x_2}{\overset{\uparrow}{2}} - 1 = 1$.

And vector $s = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is clearly a solution set.

Hence, $\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$. is a basis of the solution set.

(b) $\{ x_1 + 2x_2 - 3x_3 + x_4 = 0$.

$A = \begin{pmatrix} 1 & 2 & -3 & 1 \end{pmatrix}$, $\operatorname{rank}(A) = 1$.

$\dim(\text{Solution set}) = 4 - 1 = 3$.

$S_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$, $S_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $S_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Hence $\{ s_1, s_2, s_3 \}$.

#3. Using the results of Exercise 2, find all solutions to the following systems

a) $\begin{cases} x_1+3x_2=5 \\ 2x_1+6x_2=10 \end{cases}$ clearly, $s=\begin{pmatrix}2\\1\end{pmatrix}$

Since $\left\{\begin{pmatrix}3\\-1\end{pmatrix}\right\}$ is a basis for the solution set,

full solution is $\left\{\begin{pmatrix}2\\1\end{pmatrix}+t\begin{pmatrix}3\\-1\end{pmatrix} : t\in\mathbb{R}\right\}$.

b) $\{x_1+2x_2-3x_3+x_4=1$

$s=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+t_1\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}+t_2\begin{pmatrix}3\\0\\1\\0\end{pmatrix}+t_3\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} : t_1,t_2,t_3\in\mathbb{R}\right\}$

$2x_1+x_2-x_3=5$

$x_1-x_2+x_3=1$

$x_1+2x_2-2x_3=4$

$\begin{cases} 2x_1+x_2-x_3=0 \\ x_1-x_2+x_3=0 \\ x_1+2x_2-2x_3=0 \end{cases}$ for basis

Let $A=\begin{pmatrix}2&1&-1\\1&-1&1\\1&2&-2\end{pmatrix}$ $\operatorname{rank}\begin{pmatrix}2&1&-1\\1&-1&1\\1&2&-2\end{pmatrix}=\operatorname{rank}\begin{pmatrix}3&0&0\\1&-1&1\\1&2&-2\end{pmatrix}=\operatorname{rank}\begin{pmatrix}1&0&0\\1&-1&1\\1&2&-2\end{pmatrix}=\operatorname{rank}\begin{pmatrix}1&0&0\\0&-1&1\\0&2&-2\end{pmatrix}=\operatorname{rank}\begin{pmatrix}1&0&0\\0&-1&1\\0&0&0\end{pmatrix}=2$

$\dim(\text{solution set})=3-\operatorname{rank}(A)=1$.

$s=\begin{pmatrix}0\\1\\1\end{pmatrix}$, basis of our solution set is $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.

And, solution for given system is

$\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\1\\1\end{pmatrix} : t\in\mathbb{R}\right\}$.

#11. Show that if A is a square matrix that has two identical rows, then $\det(A)=0$.

$$\text{let } A=\begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$$\det(A)=\left((-1)^{1+1}\cdot a_1\cdot \det\left(\tilde{A}_{11}\right)\right)+\left((-1)^{1+2}\cdot a_2\cdot \det\left(\tilde{A}_{12}\right)\right)+\left((-1)^{1+3} a_3\cdot \det\left(\tilde{A}_{13}\right)\right)$$

$$=\left(1\cdot a_1\cdot \det\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}\right)+\left(-a_2\cdot \det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}\right)+\left(a_3\cdot \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right)$$

$$=a_1(a_2b_3-a_3b_2)-a_2(a_1b_3-a_3b_1)+a_3\cdot(a_1b_2-a_2b_1)$$

$$=\cancel{a_1a_2b_3}-\cancel{a_1a_3b_2}-\cancel{a_1a_2b_3}+\cancel{a_2a_3b_1}+\cancel{a_1a_3b_2}-\cancel{a_2a_3b_1}$$

$$=0.$$

$\det(A)=0.$