

1. Determinants of Order 2.

Definition

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix with entries from a field F , then we define the determinant of A , denoted $\det(A)$ or $|A|$, to be the scalar $ad - bc$.

Theorem 4.1: The function $\det: M_{2 \times 2}(F) \rightarrow F$ is a linear function of each row of a 2×2 matrix when the other row is held fixed.

That is, if v, w , and w are in F^2 and k is a scalar, then

$$\det \begin{pmatrix} v+kw \\ w \end{pmatrix} = \det \begin{pmatrix} v \\ w \end{pmatrix} + k \det \begin{pmatrix} v \\ w \end{pmatrix}$$

and, $\det \begin{pmatrix} w \\ v+kw \end{pmatrix} = \det \begin{pmatrix} w \\ v \end{pmatrix} + k \det \begin{pmatrix} w \\ v \end{pmatrix}$

Theorem 4.2. Let $A \in M_{2 \times 2}(F)$. Then the determinant of A is nonzero iff A is invertible. Moreover, if A is invertible, then

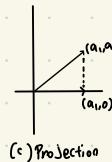
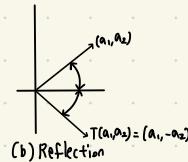
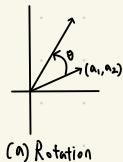
$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Recap:

Linear Transformation: Let V and W be vector spaces over the same field F . We call a function $T: V \rightarrow W$ a linear transformation from V to W , if $\forall x, y \in V$, and $c \in F$

$$(a) T(x+y) = T(x) + T(y)$$

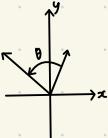
$$(b) T(cx) = cT(x)$$



One thing that turns out to be very useful to understanding one of these transformations is to measure exactly how much T stretches or squishes things. 'Determinant' is a scaling factor, the factor by which a linear transformation changes any area.

The Area of Parallelogram

: By the angle between two vectors in \mathbb{R}^2 , we mean the angle with measure θ ($0 \leq \theta < \pi$) that is formed by the vectors emanating from the origin.

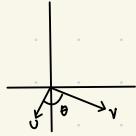


If $\beta = \{u, v\}$ is an ordered basis for \mathbb{R}^2 , we define the orientation of β to be the real number.

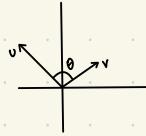
$$O(u) = \frac{\det(u)}{|\det(u)|}, \text{ and clearly } O(v) = \pm 1$$

Recall that a coordinate system $\{u, v\}$ is called "right handed" if v can be rotated in a counterclockwise direction through an angle θ to coincide with u . Otherwise $\{u, v\}$ is called "left handed" system.

Ex)

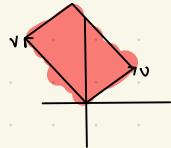


A right handed coordinate system



A right handed coordinate system

Any ordered set $\{u, v\}$ in \mathbb{R}^2 determines parallelogram in the following manner. Regarding u and v as arrows emanating from the origin of \mathbb{R}^2 , we call the parallelogram having u and v as adjacent sides the parallelogram determined by u and v .



$$A(u, v) = O(u) \cdot \det(v)$$

2. Determinants of order n

Definitions: Let $A \in M_{n \times n}(F)$. If $n=1$, so that $A=A_{11}$, we define $\det(A)=A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

The scalar $(-1)^{i+j} \det(\tilde{A}_{ij})$ is called cofactor of the entry of A in row i , column j .

Ex) Let $A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in M_{3 \times 3}(\mathbb{R})$

\Rightarrow A matrix without 1 row, 1 column

$$\begin{aligned} \det(A) &= (-1)^{1+1} \cdot A_{11} \cdot \det(\tilde{A}_{11}) + (-1)^{1+2} A_{12} \cdot \det(\tilde{A}_{12}) + (-1)^{1+3} A_{13} \cdot \det(\tilde{A}_{13}) \\ &= 1 \cdot 1 \cdot \det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + -1 \cdot 3 \cdot \det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + 1 \cdot -3 \cdot \det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} = 40. \end{aligned}$$

Theorem 4.3

: The determinant of an $n \times n$ matrix is a linear function of each row when the remaining rows are held fixed. That is, for $1 \leq r \leq n$, we have

$$\det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v+kv \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Theorem 4.4 : The determinant of a square matrix can be evaluated by cofactor expansion along any rows.

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

Corollary: If $A \in M_{n \times n}(F)$ has two identical rows, then $\det(A)=0$

Ex) Let $A = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}$, $\det(A) = 3 \cdot 5 - 3 \cdot 5 = 0$.

$$\begin{aligned} \text{Let } B &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \det(B) = (-1)^4 \cdot 4 \cdot \det(\tilde{B}_{31}) + (-1)^5 \cdot 5 \cdot \det(\tilde{B}_{32}) + (-1)^6 \cdot 6 \cdot \det(\tilde{B}_{33}) \\ &= 1 \cdot 4 \cdot \det\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} + -1 \cdot 5 \cdot \det\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} + 1 \cdot 6 \cdot \det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0. \end{aligned}$$

Theorem 4.5: If $A \in M_{n \times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A , then $\det(B) = -\det(A)$

think $A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$, $B = \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ a_r \\ \vdots \\ a_n \end{pmatrix}$

Theorem 4.6 : Let $A \in M_{n \times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A . Then, $\det(B) = \det(A)$

Corollary : If $A \in M_{n \times n}(F)$ has rank less than n , then $\det(A) = 0$

Example for Theorem 4.6

$$\begin{aligned} \text{Let } M &= \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & 18 \end{pmatrix}, \text{ then } \det(M) = 1 \cdot 1 \cdot \det(\tilde{M}_{11}) + (-1) \cdot 3 \cdot \det(\tilde{M}_{21}) + 1 \cdot (-3) \cdot \det(\tilde{M}_{31}) \\ &= 1 \cdot \det\left(\begin{pmatrix} 4 & -7 \\ 16 & 18 \end{pmatrix}\right) + 0 + 0 = 40 \end{aligned}$$

If we add -4 times row 2 of M to row 3,

$$P = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 0 & 10 \end{pmatrix}, \det(P) = (-1)^2 + 1 \cdot \det\left(\begin{pmatrix} 4 & -7 \\ 0 & 10 \end{pmatrix}\right) = 40.$$

Hence, $\det(M) = \det(P)$

3. Properties of Determinants

Important facts about the determinants of elementary matrices.

- (a) If E is an elementary matrix obtained by interchanging any two rows of I , then $\det(E) = -1$
- (b) If E is an elementary matrix obtained by multiplying some row of I by the nonzero scalar k , then $\det(E) = k$
- (c) If E is an elementary matrix obtained by adding a multiple of some row of I to another row, then $\det(E) = 1$.

Theorem 4.7: For any $A, B \in M_{n \times n}(F)$, $\det(AB) = \det(A) \cdot \det(B)$

Corollary: If $A \in M_{n \times n}(F)$ is not invertible, then the rank of A is less than n . So, $\det(A) = 0$ by Corollary to Theorem 4.6

Theorem 4.8: For any $A \in M_{n \times n}(F)$, $\det(A^T) = \det(A)$

Theorem 4.9 (Cramer's Rule): Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where

$$x = (x_1, x_2, \dots, x_n)^T$$

If $\det(A) \neq 0$, then this system has a unique solution, and for each k ($k=1, 2, \dots, n$),

$$x_k = \frac{\det(M_k)}{\det(A)}$$

Example:

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_1 + x_3 = 3$$

$$x_1 + x_2 - x_3 = 1$$

The matrix form of this system of linear equations is $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Since $\det(A) = 6 \neq 0$, Cramer's rule applies.

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{\det\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}}{6} = \frac{15}{6} = \frac{5}{2}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}}{6} = \frac{-6}{6} = -1$$

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\text{Thus, } (x_1, x_2, x_3) = \left(\frac{5}{2}, -1, \frac{1}{2}\right)$$

* Recap: Upper triangular matrix has following properties:

$$\det(A) = \prod_{i=1}^n T_{ii}, \quad \text{tr}(A) = \sum_{i=1}^n T_{ii}$$

And, similar matrices have same trace and determinant.

[Ex)

$$\det \begin{pmatrix} -\lambda & 1 & 2 & 0 \\ 0 & -\lambda & 2 & 6 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = (-\lambda)^4$$

5. A characterization of the Determinant.

Definition: A function $\delta: M_{n \times n}(F) \rightarrow F$ is called an n -linear function if it is a linear function of each row of a $n \times n$ matrix when the remaining $n-1$ rows are held fixed, that is δ is linear if, for every $r=1,2,\dots,n$, we have

$$\delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \cdot \delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Whenever k is a scalar and v, v , and each a_i are vectors in F^n .

Example 1: The function $\delta: M_{n \times n}(F) \rightarrow F$ defined by $\delta(A)=0$ for each $A \in M_{n \times n}(F)$ is an n -linear function.

Example 2: For $1 \leq j \leq n$, define $\delta_j: M_{n \times n}(F) \rightarrow F$ by $\delta_j(A) = A_{1j}A_{2j} \cdots A_{nj}$ for each $A \in M_{n \times n}$; that is, $\delta_j(A)$ equals the product of the entries of column j of A . Let $A \in M_{n \times n}(F)$, $a_i = (A_{i1}, A_{i2}, \dots, A_{in})$, and $v = (b_1, b_2, \dots, b_n) \in F^n$. Then each δ_j is an n -linear function because, for any scalar k , we have

$$\begin{aligned} \delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} &= A_{1j} \cdots A_{(r-1)j} \cdot (A_{rj} + kb_j) \cdot A_{(r+1)j} \cdots A_{nj} \\ &\quad \cancel{\star a \cdot (b+c)} \cdot e \\ &= a \cdot b \cdot e + a \cdot c \cdot e \\ &= A_{1j} \cdots A_{(r-1)j} \cdot A_{rj} A_{(r+1)j} \cdots A_{nj} + A_{1j} \cdots A_{(r-1)j} \cdot (kb_j) \cdot A_{rj} A_{(r+1)j} \cdots A_{nj} \\ &= A_{1j} \cdots A_{(r-1)j} \cdot A_{rj} A_{(r+1)j} \cdots A_{nj} + k(A_{1j} \cdots A_{(r-1)j} \cdot b_j \cdot A_{rj} A_{(r+1)j} \cdots A_{nj}) \\ &= \delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \cdot \delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \end{aligned}$$

Therefore, δ is a n -linear function.

Def: An n -linear function $\delta: M_{n \times n}(F) \rightarrow F$ is called alternating if, for each $A \in M_{n \times n}(F)$, we have $\delta(A)=0$ whenever two adjacent rows of A are identical.

Theorem 4.10. Let $\delta: M_{n \times n}(F) \rightarrow F$ be an alternating n -linear function.

- (a) If $A \in M_{n \times n}(F)$ and B is a matrix from A by interchanging any two rows of A , then $\delta(B) = -\delta(A)$
- (b) If $A \in M_{n \times n}(F)$ has two identical rows, then $\delta(A) = 0$.

Proof.

(a) ① Definition of n -linear function

$$\delta \begin{pmatrix} a_1 \\ \vdots \\ ar_{r-1} \\ ar_r \\ ar_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta \begin{pmatrix} a_1 \\ \vdots \\ ar_{r-1} \\ v \\ ar_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \cdot \delta \begin{pmatrix} a_1 \\ \vdots \\ ar_{r-1} \\ v \\ ar_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

② Def: An n -linear function $\delta: M_{n \times n}(F) \rightarrow F$ is called **alternating** if, for each $A \in M_{n \times n}(F)$, we have $\delta(A) = 0$ whenever two adjacent rows of A are identical.

③ We have $\delta: M_{n \times n}(F) \rightarrow F$, an **alternating n -linear function**,

$$A \in M_{n \times n}(F),$$

B : matrix obtained from A by interchanging rows r and s

④ (let's say $s > r$, and since δ is an alternating (Whenever two adjacent rows of A are identical, $\delta(A) = 0$) n -linear function, we have

$$\begin{aligned} 0 &= \delta \begin{pmatrix} a_1 \\ \vdots \\ ar+as \\ ar+as \\ \vdots \\ a_n \end{pmatrix} = \delta \begin{pmatrix} a_1 \\ \vdots \\ ar \\ ar+as \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ as \\ ar+as \\ \vdots \\ a_n \end{pmatrix} \\ &\quad \downarrow \qquad \qquad \qquad \text{---} \\ &= \delta \begin{pmatrix} a_1 \\ \vdots \\ ar \\ ar \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ ar \\ as \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ as \\ ar \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ as \\ as \\ \vdots \\ a_n \end{pmatrix} \\ &\quad \text{by } \delta: \text{alternating, hence goes to 0.} \qquad \text{by } \delta: \text{alternating, hence goes to 0.} \end{aligned}$$

$$\Rightarrow 0 = 0 + \delta(A) + \delta(B) + 0$$

$$\Rightarrow -\delta(A) = \delta(B)$$

Whatever the number is for s , since δ is alternating : $\delta \begin{pmatrix} a_1 \\ \vdots \\ ar+as \\ ar+as \\ \vdots \\ a_n \end{pmatrix} \Rightarrow -\delta(A) = \delta(B)$

(5) Let's say $r > s$

$$0 = \delta \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix} = \delta \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix}$$

↓ ↓ ↓ ↓

$$= \delta \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix} + \delta \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix}$$

δ : alternating, hence goes to 0.

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$\delta(A)$

$\delta(B)$

$$\Rightarrow 0 = 0 + \delta(A) + \delta(B) + 0$$

$$\Rightarrow -\delta(A) = \delta(B)$$

Whatever the number is for r , since δ is alternating : $\delta \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix} \Rightarrow -\delta(A) = \delta(B)$

Even if $r > s$ or $s > r$, $-\delta(A) = \delta(B)$.

$$\delta \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \text{astar} \\ \text{astar} \\ \vdots \\ a_n \end{pmatrix}$$

(b) If $A \in M_{n \times n}(F)$ has two identical rows, then $\delta(A) = 0$.

: Let r and s be the identical rows in A , and $r \neq s$.

① When $s = r+1$.

: Since δ is alternating, and two adjacent rows (r, j) are identical, $\delta(A) = 0$.

② When $s > r+1$.

: Let B be a matrix obtained from A by interchanging r th and s th row, then r and s will be located adjacently.

Then, since δ is an alternating n -linear function, $\delta(B) = 0$.

And, Corollary 3 states when we obtain a matrix (in here, B) by interchanging rows, $\delta(B) = -\delta(A)$

Since $\delta(B) = 0$, $\delta(B) = 0 = -\delta(A)$

Which means, $\delta(A) = 0$

Corollary 1.

: Let $\delta: M_{n \times n}(F) \rightarrow F$ be an alternating n -linear function. If B is a matrix obtained from $A \in M_{n \times n}(F)$ by adding a multiple of some row of A to another row, then $\delta(B) = \delta(A)$.

B : Matrix obtained from $A \in M_{n \times n}(F)$ by adding $k \cdot r_i$ to r_j , where k is a scalar and r_i is a row i of A .

Ex)

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_r^r, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}: r_2 + 3r_3$$

r_j is a row j of A , and $r \neq j$

Let C be a matrix obtained from A by replacing r_i to r_j

$$\text{Ex)} C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We can see row i of B (in example r_2 in B) is equal to sum of row j of A and k times row j of C .

Then, $\delta(B) = \delta(A) + k \cdot \delta(C)$

Since C has two adjacent identical rows, $\delta(C) = 0$

Hence, $\delta(B) = \delta(A) + 0$

Therefore, $\delta(A) = \delta(B)$ if B is a matrix obtained from A by adding a multiple of some row of A to another row.

Corollary 2.

: Let $\delta: M_{n \times n}(F) \rightarrow F$ be an alternating n -linear function.

If $M \in M_{n \times n}(F)$ has rank less than n , $\delta(M) = 0$.

Since M has rank less than n , \exists row vectors of M are dependent.

Let's say v_i is one of dependent vectors.

Then, v_i can be obtained by linear combination of other vectors in M .

It means, \exists two identical rows in a matrix.

As we proved in Theorem 4.10 (b), Since M has two identical rows after linear combination of other vectors in M , $\delta(M) = 0$.

Thus, if $M \in M_{n \times n}(F)$ has rank less than n , then $\delta(M) = 0$.

Corollary 3.

: Let $\delta: M_{nn}(F) \rightarrow F$ to be an alternating n -linear function, and let E_1, E_2 , and E_3 in $M_{nn}(F)$ be elementary matrices of types 1, 2, and 3, respectively. Suppose that E_2 is obtained by multiplying some row of I by the nonzero scalar k . Then $\delta(E_1) = -\delta(I)$, $\delta(E_2) = k \cdot \delta(I)$, and $\delta(E_3) = \delta(I)$.

① We can obtain an E_1 from I by interchanging two rows. Then, by Theorem 4.10 Corollary 1, $\delta(E_1) = -\delta(I)$.

② We can obtain an E_2 from I by multiplying some row of I by the nonzero scalar k .

Then, since δ is a n -linear function, by using its definition,

$$\delta(E_2) = k \cdot \delta(I)$$

③ We can obtain E_3 by adding a multiple of some row of I to another row

Then, by Theorem 4.10 Corollary 1, $\delta(E_3) = \delta(I)$.

Theorem 4.11.

: Let $\delta: M_{nn}(F) \rightarrow F$ be an alternating n -linear function such that $\delta(I) = 1$. For any $A, B \in M_{nn}(F)$, we have $\delta(AB) = \det(A) \cdot \delta(B)$

① If $A \in M_{nn}(F)$ has rank less than n , then it's obvious AB also will be a matrix with a rank less than n .

Then, from Theorem 4.10 Corollary 2,

$\delta(AB) = 0$ (Since $\text{rank}(A) < n, \exists$ at least one row filled with 0).

$$\delta(AB) = 0 = \det(A) \cdot \delta(B)$$

★ Theorem 4.6 : Corollary: If $A \in M_{nn}(F)$ has rank less than n , then $\det(A) = 0$

Hence, if A has rank less than n , $\delta(AB) = \det(A) \cdot \delta(B)$

② If $\text{rank}(A) = n$,

When $\text{rank}(A) = n$, it means A can be invertible. (An $n \times n$ matrix is invertible iff its rank is n).

Hence, A can be written as $A = E_p E_{p+1} \cdots E_n$ (Stays for using Theorem 4.10 Corollaries).

Let $M \in M_{nn}(F)$, and E be all elementary matrices.

If $\delta(EM) = \det(E) \cdot \delta(M)$, it's proven.

① Let E be an elementary matrix of type 1: Obtained from I by interchanging row of i and j of I .

Then, by applying Theorem 4.10 Corollary 3, $\delta(E) = -\delta(I)$, hence $\delta(EM) = -\delta(I)\delta(M) = \delta(E)\delta(M)$

And, since $\delta(I) = 1 = \det(I)$, $\delta(E)\delta(M) = \det(E) \cdot \delta(M)$

② Let E be an elementary matrix of type 2: multiplying some row of I by the nonzero scalar k .

Then, from Theorem 4.10 Corollary 3, $\delta(E) = k \cdot \delta(I) = \det(E) = k \cdot \det(I)$

$$\begin{aligned}\text{Therefore, } \delta(EM) &= k \cdot \delta(I) \cdot \delta(M) = \delta(E) \delta(M) \\ &= \det(E) \cdot \det(M)\end{aligned}$$

③ Let E be an elementary matrix of type 3: adding a multiple of some row of I to another row.

: Then, from Theorem 4.10 Corollary 3, $\delta(E) = \delta(I) = \det(E)$

$$\text{Then, } \delta(EM) = \delta(I) \cdot \delta(M) = \delta(M) = \delta(E) \delta(M) = \det(E) \delta(M).$$

Therefore, for any $A, B \in M_{n \times n}(F)$, we have $\delta(AB) = \delta(A)\delta(B)$.

Theorem 4.12: If $\delta: M_{n \times n}(F) \rightarrow F$ is an alternating n -linear function such that $\delta(I) = 1$, then $\delta(A) = \det(A) \forall A \in M_{n \times n}(F)$.

Proof:

We have $\delta: M_{n \times n}(F) \rightarrow F$, an alternating n -linear function, and $\delta(I) = 1$.
 $A \in M_{n \times n}(F)$.

① When $\text{rank}(A) < n$

: From Theorem 4.10 Corollary 2, $\delta(A) = 0$.

From Theorem 4.6 : Corollary: If $A \in M_{n \times n}(F)$ has rank less than n , then $\det(A) = 0$

Therefore, when $\text{rank}(A) < n$, $\det(A) = 0 = \delta(A)$.

② When $\text{rank}(A) = n$, : A is invertible.

$$A = E_p \cdots E_2 E_1.$$

Since A is a matrix obtained from $E_p \cdots E_2 E_1$, from Theorem 4.10 Corollary 1 ($\delta(B) = \delta(A)$)

$$\delta(A) = \delta(E_p \cdots E_2 E_1)$$

We know that $\delta(AB) = \det(A) \cdot \delta(B)$ from Theorem 4.11, hence

$$\begin{aligned}\delta(A) &= \det(E_p) \cdot \delta(E_{p+1} \cdots E_2 E_1) \\ &= \det(E_p) \cdot \det(E_{p+1}) \cdot \delta(E_{p+2} \cdots E_2 E_1) \\ &= \dots \\ &= \det(E_p) \cdot \det(E_{p+1}) \cdots \det(E_1)\end{aligned}$$

We know : Theorem 4.7: For any $A, B \in M_{n \times n}(F)$, $\det(AB) = \det(A) \cdot \det(B)$, therefore

$$\delta(A) = \det(E_p E_{p+1} \cdots E_1) = \det(A)$$

Therefore, if $\delta: M_{n \times n}(F) \rightarrow F$ is an alternating n -linear function such that $\delta(I) = 1$, then $\delta(A) = \det(A) \forall A \in M_{n \times n}(F)$.

Practice

#6, 7, 8: Determine which of the functions $\delta: M_{3 \times 3}(F) \rightarrow F$ are 3-linear functions

*: δ is n-linear function if it's linear of each row of an $n \times n$ matrix when the remaining $n-1$ rows are held.

$$\#6. \delta(A) = A_{11} + A_{23} + A_{32}$$

Definition:

$$\delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \cdot \delta \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

$$\text{Let } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

① First row

$$\begin{aligned} \delta \begin{pmatrix} A_{11} + kB_{11} & A_{12} + kB_{12} & A_{13} + kB_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} &= A_{11} + kB_{11} + A_{23} + A_{32} = A_{11} + A_{23} + A_{32} + kB_{11} \\ &= \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k \delta \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

It's not in a form of

$$\delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k \delta \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Hence, it's not linear

For example,

$$\text{If } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Then, } \delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \neq 2 = 2 \cdot \delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, given δ isn't a 3-linear function.

$$\#7. \delta(A) = A_{11}A_{21}A_{32}$$

Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

① first row

$$\begin{aligned} \delta \begin{pmatrix} A_{11} + kB_{11} & A_{12} + kB_{12} & A_{13} + kB_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} &= (A_{11} + kB_{11}) A_{21} \cdot A_{32} \\ &= A_{11}A_{21}A_{32} + K(B_{11}A_{21}A_{32}) \\ &= \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + K \delta \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \end{aligned}$$

② Second row

$$\begin{aligned} \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} + kB_{21} & A_{22} + kB_{22} & A_{23} + kB_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} &= A_{11} \cdot (A_{21} + kB_{21}) \cdot A_{32} \\ &= A_{11}A_{21}A_{32} + K(A_{11}B_{21}A_{32}) \\ &= \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + K \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ B_{21} & B_{22} & B_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \end{aligned}$$

③ third row

$$\begin{aligned} \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} + kB_{31} & A_{32} + kB_{32} & A_{33} + kB_{33} \end{pmatrix} &= A_{11} \cdot A_{21} \cdot (A_{32} + kB_{32}) \\ &= A_{11}A_{21}A_{32} + K(A_{11}A_{21}B_{32}) \\ &= \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + K \delta \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \end{aligned}$$

Therefore, given δ is a 3-linear function.

$$\#8. \quad f(A) = A_{11}A_{31}A_{32}$$

Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

① first row

$$\begin{aligned} f \begin{pmatrix} A_{11} + kB_{11} & A_{12} + kB_{12} & A_{13} + kB_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} &= (A_{11} + kB_{11}) A_{31} \cdot A_{32} \\ &= A_{11}A_{31}A_{32} + k(B_{11}A_{31}A_{32}) \\ &= f \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} + k f \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \end{aligned}$$

② Second row

$$\begin{aligned} f \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} + kB_{21} & A_{22} + kB_{22} & A_{23} + kB_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} &= A_{11} \cdot A_{32} \cdot A_{33} \\ &= f \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \end{aligned}$$

It's not linear.

Hence, given f isn't a 3-linear function.

#13. Prove that $\det: M_{n \times n}(\mathbb{F}) \rightarrow \mathbb{F}$ is a 2-linear function of the columns of a matrix.

* Theorem 4.8. For any $A \in M_{n \times n}(\mathbb{F})$, $\det(A^T) = \det(A)$

Since we have $A \in M_{n \times n}(\mathbb{F})$, by using Theorem 4.8, we do know $\det(A^T) = \det(A)$

Hence, if we prove A is a 2-linear function, since row of A^T are columns in A , we prove A is a 2-linear function of the columns of a matrix.

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

① First row

$$\det \begin{pmatrix} A_{11} + kB_{11} & A_{12} + kB_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + k \cdot \det \begin{pmatrix} B_{11} & B_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

② Second row

$$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} + kB_{21} & A_{22} + kB_{22} \end{pmatrix} = \det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + k \cdot \det \begin{pmatrix} A_{11} & A_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Therefore, $\det: M_{2x2}(F) \rightarrow F$ is a 2-linear function of the columns of a matrix.

#18. Prove that the set of all n -linear functions over a field F is a vector space over F under the operations of function addition and scalar multiplication as defined in Example 3 of Section 1.2.

We need to prove that all n -linear functions over F is a vector space over F under the operations of function addition and scalar multiplication

$$① (\delta_1 + \delta_2)(A) = \delta_1(A) + \delta_2(A)$$

$$② (c \cdot \delta)(A) = c \cdot \delta(A)$$

where $c \in F$, and $A \in M_{n \times n}(F)$

And we need to check VS1~VS6 to verify set of all n -linear functions over F is a vector space over F

VS 1: for all $\delta_1, \delta_2 \in M_{n \times n}(F) \rightarrow F$, and $A \in M_{n \times n}(F)$,

$$\delta_1(A) + \delta_2(A) = \delta_2(A) + \delta_1(A) : \text{True since } \delta \text{ is linear}$$

VS 2: $\forall \delta_1, \delta_2, \delta_3 \in M_{n \times n}(F) \rightarrow F$,

$$\delta_1(A) + (\delta_2(A) + \delta_3(A)) = (\delta_1(A) + \delta_2(A)) + \delta_3(A) : \text{True, } \delta \text{ is linear}$$

VS 3: $\exists \mathbf{0}$ s.t. $\delta_1 + \mathbf{0} = \delta_1$

$$\delta \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0, \text{ and } \delta_1 + 0 = \delta_1 : \text{True}$$

VS 4: For each element $\delta_1 \in M_{n \times n}(F) \rightarrow F$, $\exists \delta_2 \in M_{n \times n}(F) \rightarrow F$ s.t. $\delta_1 + \delta_2 = 0$

$$\text{Since } \delta_1 + \delta_2 = 0, \quad \delta_1 = -\delta_2 \Rightarrow \delta_1(A) + \delta_2(A) = -\delta_2(A) + \delta_2(A) = 0 : \text{True}$$

VS 5: For each element $\delta_1 \in M_{n \times n}(F) \rightarrow F$, $\exists \delta_1(A) = \delta_1(A) : \text{True}$

VS 6: For each element $a, b \in F$ and each pair of elements $\delta_1 \in M_{n \times n}(F) \rightarrow F$,

$$(ab)(\delta_1(A)) = a(b\delta_1(A)) : \text{True, since } \delta_1 \text{ is linear.}$$

V S η : For each element $a \in F$, and each pair of elements $f_1, f_2 \in M_{n \times n}(F) \rightarrow F$,
 $a(f_1 + f_2)(A) = af_1(A) + af_2(A)$: True, f is linear.

V S β : For each pair of elements $a, b \in F$, and each element $f \in M_{n \times n}(F) \rightarrow F$,
 $(af + bf)(A) = af(A) + bf(A)$: True. f is linear.

Hence, it's true that all n -linear functions over F is a vector space over F under the operations of function addition and scalar multiplication.