Differential equation : An equation that relates one or more unknown functions and their derivatives (rate of change)

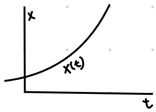
- Big idea: Often, when we are trying to mathematically model a situation, we know more about how something is changing than being able to exactly describe the state of the object at every point in time.

Example: The amount of money in a bank account is x, and the amount of money increases at a rate λ (interest rate) proportional to the $x(t)$

: $\dot{x} = \frac{dx}{dt} = \lambda x$: Differential equation (first order ODE)

x is an unknown function

What is the amount of money as a function of time?



Finding a solution

$\frac{dx}{dt} = \lambda x$

$\Rightarrow \frac{dx}{x} = \lambda \cdot dt \Rightarrow \int \frac{1}{x} dx = \int \lambda \cdot dt$

$\Rightarrow \ln(x(t)) = \lambda t + C$

$\Rightarrow x(t) = e^{\lambda t + C} = e^{\lambda t} e^{C}$

Constant

$x(0) = e^0 e^C = X_0$

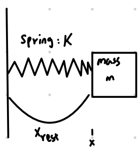
$\therefore x(t) = X_0 \cdot e^{\lambda t}$: Solution of first order simple ODE

Ordinary Differential Equation (ODE): An equation which consists of one or more functions of one independent variable along with their derivatives

Partial Differential Equation (PDE): An equation that depends on partial derivatives of several variables.

Second-Order Ordinary Differential Equations

e.g. Harmonic Oscillator



Newton's 2^{nd} Law : $F = ma$

v: velocity $= \frac{dx}{dt}$

a: acceleration $= \frac{dv}{dt} = \frac{d^2x}{dt^2}$

x: Position

$F = m \cdot \frac{d^2x}{dt^2}$

According to Hooke's Law : $F_{spring} = -Kx$

Since, $a = \frac{F}{m}$, $\frac{d^2x}{dt^2} = -\frac{K}{m}x$

$\ddot{x} = -\frac{K}{m}x$

Finding Solution of ODE : $\ddot{x} = -\frac{K}{m}x$ with $x(0) = X_0$, $\dot{x}(0) = V_0$

Suppose $m = 1$, $K = 1$. Then, ODE: $\ddot{x} = -x$

① Method 1: Guess!

Since $\ddot{x} = -x$

if $x(t) = \cos(t)$

$\dot{x}(t) = -\sin(t)$

$\ddot{x}(t) = -\cos(t)$, then $\ddot{x} = -x$

for general m, K:

$x(t) = \cos(\sqrt{K/m}\, t) X_0$

$w = \sqrt{K/m}$

② Method II: Taylor Series

$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \cdots$

$\dot{x}(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \cdots$

$\ddot{x}(t) = 2c_2 + 6c_3 t + 12c_4 t^2 + 20c_5 t^3 + \cdots$

Since $x(0) = c_0 + 0 + 0^2 + \cdots = c_0$

$\dot{x}(0) = c_1 = v_0$

And $\ddot{x} = -x$,

$2c_2 = -c_0 = -x_0$

$c_2 = -\frac{1}{2}x_0 = -\frac{1}{2!}x_0$

$6c_3 t = -c_1 t$

$c_3 = -\frac{1}{6}c_1 = -\frac{1}{3!}v_0$

$12c_4 = -c_2$

$12c_4 = -(-\frac{1}{2!}x_0)$

$c_4 = \frac{1}{4\times3\times2\times1}x_0 = \frac{1}{4!}x_0$

$20c_5 = -c_3 = -(-\frac{1}{3!}v_0)$

$c_5 = \frac{1}{5!}v_0$

$\vdots$

$\therefore x(t) = x_0 + v_0 t - \frac{x_0}{2!}t^2 - \frac{v_0}{3!}t^3 + \frac{x_0}{4!}t^4 + \frac{v_0}{5!}t^5 + \cdots$

$$= x_0\underbrace{\left[1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots\right]}_{\text{Taylor Series of } \cos(t)} + v_0\underbrace{\left[t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots\right]}_{\text{Taylor series of } \sin(t)} = \cos(t)x_0 + \sin(t)v_0$$

③ Method 3: Guess Again!

: What function, when taking multiple derivatives, is similar to itself, up to a constant?

$x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t}$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} = -e^{\lambda t} \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$

$x(t) = c_1 e^{it} + c_2 e^{-it}$

Since $e^{it} = \cos(t) + i\sin(t)$

$e^{-it} = \cos(t) - i\cdot\sin(t)$

And $x(0) = c_1 + c_2 = x_0$

$\dot{x}(0) = i(c_1 - c_2) = v_0$

Then, $x(t) = c_1(\cos(t) + i\sin(t)) + c_2(\cos(t) - i\cdot\sin(t))$

$= c_1\cos(t) + ic_1\cdot\sin(t) + c_2\cos(t) - ic_2\cdot\sin(t))$

$= (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t)$

$= x_0\cos(t) + v_0\sin(t)$

Higher Order ODE Systems

- $a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_2 \cdot \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$

or

$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$

$F = ma$: typical second order ODE