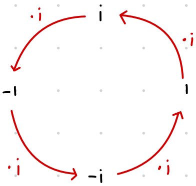
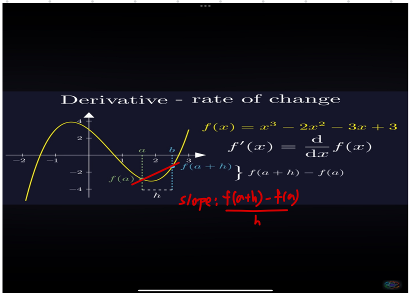
Imaginary number i



: Multiplying i by itself repeatedly, it creates cyclic pattern.

· To understand Euler's identity and Euler's constant number, e, you must know the concepts of derivative.

Derivative : Rate of change.



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Derivative rules:

1. Constant rule : $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$

2. Sum rule . $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

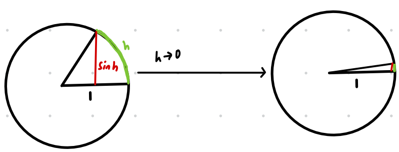
3. Power rule: $\frac{d}{dx}[x^a] = a \cdot x^{a-1}$

4. $\frac{d}{dx}[a^x] = \ln(a) \cdot a^x$. ex) $\frac{d}{dx}[2^x] = \ln(2) \cdot 2^x$

Trigonometric derivatives.

① $f(x) = \sin(x)$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} = \lim_{h \to 0} \sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h} = \cos(x)$$



: As $h \to 0$, $\sin h = h$

Hence, $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$

And,

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0$$

② $f(x) = \cos(x)$

$f'(x) = -\sin(x)$

Trigonometric function in Taylor series.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^n}{n!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Now, what's e?

$2^x(2^h - 1)$

① $\frac{d}{dx} 2^x = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h \to 0} \frac{2^x \cdot 2^h - 2^x}{h} = \lim_{h \to 0} 2^x \cdot \frac{2^h - 1}{h}$

As $\lim_{h \to 0}$, $\frac{2^h - 1}{h} \approx 0.693171 = \ln(2)$

② $\frac{d}{dx} 3^x = \lim_{h \to 0} 3^x \cdot \frac{3^h - 1}{h}$, As $\lim_{h \to 0}$, $\frac{3^h - 1}{h} \approx 1.1$

③ $\lim_{h \to 0} \frac{2.5^h - 1}{h} \approx 0.91$

e is a constant number that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, therefore derivative (rate of change) is equivalent to the function : $f(x) = e^x$, $f'(x) = e^x$

100% interest (per year) : $P(1+1) = 2P$

50% interest (per $\frac{1}{2}$ year) : $P + \frac{1}{2}P + \frac{1}{2}(P + \frac{1}{2}P) = P(1+\frac{1}{2})^2 = 2.25P$

33% interest (per $\frac{1}{3}$ year) : $P(1+\frac{1}{3})^3 = 2.37P$

$\frac{100}{n}$% interest (per $\frac{1}{n}$ year) : $P(1+\frac{1}{n})^n$

$\lim_{n\to\infty} (1+\frac{1}{n})^n = 2.71828\ldots = e$

'e^x' in Taylor Series :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots = \cos(x) + i \cdot \sin(x)$$

$$i \cdot \sin(x) = ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!}$$

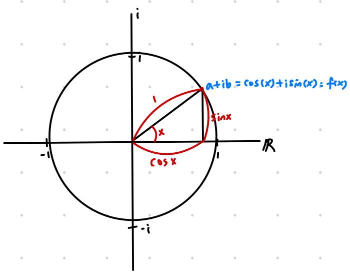
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Then, $e^{i\pi} = \cos\pi + i \cdot \sin\pi$

$= -1 + i \cdot 0$

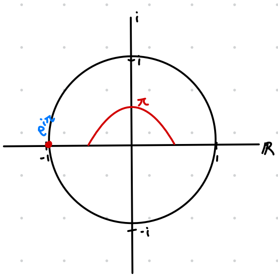
$e^{i\pi} + 1 = 0$: Euler's identity

Complex Plane



f(x) properties	e^x properties
1. $f(a) \times f(b) = f(a+b)$	1. $e^{ia} \times e^{ib} = e^{i(a+b)}$
2. $\{f(x)\}^2 = f(2x)$	2. $\{e^{ix}\}^2 = e^{i2x}$
3. $\frac{1}{f(x)} = f(-x)$	3. $\frac{1}{e^{ix}} = e^{-ix}$
4. $f(0) = 1$	4. $e^{i \cdot 0} = 1$
5. $f'(x) = if(x)$	5. $(e^{ix})' = ie^{ix}$

$\therefore e^{ix} = \cos(x) + i\sin(x)$



$\therefore e^{i\pi} = -1$

$e^{i\pi} + 1 = 0$: Euler's identity