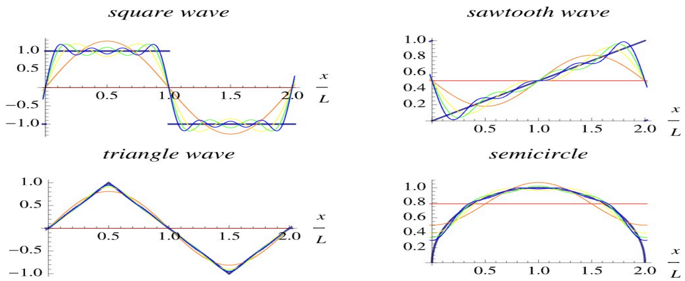
Definition: A Fourier Series is an expansion of a periodic function, $f(x)$, in terms of an infinite sum of sines and cosines. The computation and study of Fourier Series is known as harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or approximation.



Applications: Signal filtering, Noise removal, identifying the resonant frequency of a structure, compression of audio signals, and speech recognition.

Formula : $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$, where Period = $2L$

$$a_n = \frac{1}{L}\cdot\int_{-L}^{L} f(t)\cdot\cos\left(\frac{n\pi}{L}\cdot t\right)dt$$

$$b_n = \frac{1}{L}\cdot\int_{-L}^{L} f(t)\cdot\sin\left(\frac{n\pi}{L}\cdot t\right)dt$$

with $n > 0$

Why? ↓

★ 3 important integrals

① $\int_0^{2\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

Sin(mt) modulates amplitude of sin(nt)

② $\int_0^{2\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

③ $\int_0^{2\pi} \cos(mt)\sin(nt)\,dt = 0$

Then,

$$\int_0^{2\pi} f(t)\sin(mt)\,dt = \int_0^{2\pi}\left(\overset{0}{\frac{a_0}{2}} + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt)\right)\sin(mt)\,dt$$

★ Integration of constant times sin over its period is always 0

Since ③ $\int_0^{2\pi} \cos(mt)\sin(nt)\,dt = 0$, $\int_0^{2\pi} \sum_{n=1}^{a_n} a_n\cos(nt)\sin(mt)\,dt = 0$

Then, $\int_0^{2\pi} f(t)\sin(mt)\,dt = \int_0^{2\pi}\left(\sum_{n=1}^{\infty} b_n\cdot\sin(nt)\right)\sin(mt)\,dt$

And we know that $\int_0^{2\pi} \sin(nt)\sin(mt)\,dt = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

Then,

$$\int_0^{2\pi} \left(\sum_{n=1}^{\infty} b_n \cdot \sin(nt)\right) \sin(mt)\,dt$$

$$= \int_0^{2\pi} \Big(\underbrace{b_1 \sin t + b_2 \sin 2t}_{m \neq n \to 0} + \cdots b_m \sin mt \underbrace{+ \cdots + b_N \sin Nt) \sin(mt)}_{m = n \to \pi}\,dt$$

$$= \int_0^{2\pi} b_m \sin(mt)\sin(mt)\,dt$$

$$= b_m \cdot \pi$$

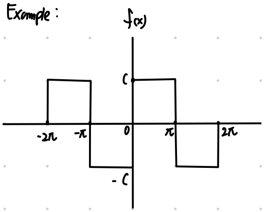
Therefore,

$$b_m \pi = \int_0^{2\pi} f(t)\sin(mt)\,dt$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(mt)\,dt$$

We can easily derive a_0, a_n also with same approach.

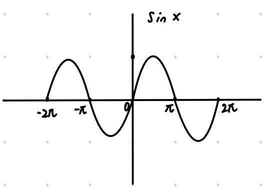
Example :



$$f(x) \approx a_0 + \sum_{n=1}^{N} a_n \cdot \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{N} b_n \cdot \sin\left(\frac{n\pi}{L}x\right)$$

☆ For modeling odd functions, use the sine terms

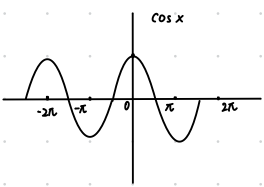
even functions, use the cosine terms



We have period $(P = 2L) = 2\pi$, $L = \pi$

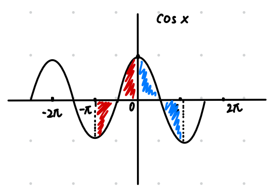
· $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right] = 0$



· $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{\pi}{\pi}x\right)dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos x\,dx + \int_0^{\pi} f(x)\cos x\,dx\right] =$



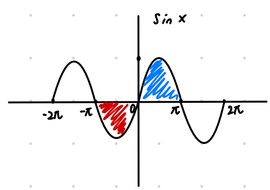
$= 0$

$a_2 = a_3 = a_4 \ldots = 0$

· $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{\pi}{\pi}x\right)dx$

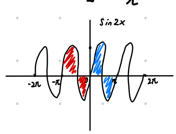
$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin x\,dx + \int_0^{\pi} f(x)\sin x\,dx\right]$

$= \frac{1}{\pi}\left[(-c)\cdot(-2) + c\cdot 2\right] = \frac{4c}{\pi}$



$b_2 = 0$

Why?



$b_3 = \frac{4c}{3\pi}$, $b_5 = \frac{4c}{5\pi}$, $b_7 = \frac{4c}{7\pi}$ ...

Hence, in Fourier Series,

$$f(x) \approx b_1 \sin(x) + b_3 \sin(3x) + b_5 \sin(5x) + \cdots$$

$$\approx \frac{4c}{\pi}\sin(x) + \frac{4c}{3\pi}\sin(3x) + \frac{4c}{5\pi}\sin(5x) + \cdots$$