Kummer rigidity for IHS manifolds

— Seung uk Jang

1. Irreducible Holomorphic Symplectic (IHS) manifolds

Def A compact Kähler manifold X is an IHS manifold
if $\pi_1 X = 0$ & $H^0(X, \Omega_X^2) = \mathbb{C}\cdot\sigma$

$\boxed{\exists \text{ everywhere nondegenerate holomorphic 2-form } \sigma.}$

"X has a 'unique' holomorphic symplectic form σ" $\Rightarrow \dim_{\mathbb{C}} X = 2n$.

(eg) $\dim_{\mathbb{C}} X = 2 \Rightarrow X$ is a K3 surface.

Remark $H^0(K_X) \ni \sigma^n$, nowhere vanishing.

$\Rightarrow K_X = 0$, $c_1(X) = 0 \Rightarrow X$ has a Ricci-flat metric ω.

Prop A Kähler metric ω on IHS X is Ricci-flat $\iff \omega^{2n} = (\sigma\bar{\sigma})^n$.

◎ Examples of IHS manifolds

(eg) (Hilbert scheme of points in a K3 surface)

Y : K3 surface. $X = \mathrm{Hilb}^n(Y)$

$= \left\{ Z \subset Y : \begin{array}{l} \dim Z = 0 \\ \dim_{\mathbb{C}} \mathcal{O}_Z(Z) = n \end{array} \right\}$

For σ_Y : holomorphic symplectic form on Y, $\quad$ = ~~normalization~~ desingularization of $Y^n/\mathfrak{S}_n$

σ = holomorphic symplectic form on X : induced from $\sum_{i=1}^{n} \pi_i^* \sigma_Y$ on $Y^n \xrightarrow[\text{quotient}]{} Y^n/\mathfrak{S}_n \xleftarrow[\text{normalizer}]{} X$

$\sigma\{p_1, \cdots, p_n\} = \sigma_Y(p_1) + \cdots + \sigma_Y(p_n)$

(eg) (generalized Kummer construction)

$\mathbb{T}^2$: 2D complex torus.

$Z \longmapsto \{p_0, p_1, \cdots, p_n\} \longmapsto \sum p_i$

$$\begin{array}{ccccc} \mathrm{Hilb}^{n+1}(\mathbb{T}^2) & \longrightarrow & (\mathbb{T}^2)^{n+1}/\mathfrak{S}_{n+1} & \longrightarrow & \mathbb{T}^2 \\ \cup & & & & \\ K_n(\mathbb{T}^2) & & & & \end{array}$$

$(\mathbb{T}^2)^n/\mathfrak{S}_{n+1} \xleftarrow{\text{desing.}} K_n(\mathbb{T}^2) = \left\{ Z \in \mathrm{Hilb}^{n+1} : \sum_{p \in Z} p = 0 \right\}$

σ on $K_n(\mathbb{T}^2)$ induced from $\sum_{i=0}^{n} \pi_i^* \sigma_{\mathbb{T}^2}$ $\quad$ 'holom. vol. form' on $\mathbb{T}^2$

$K_n(\mathbb{T}^2)$ = (generalized Kummer variety of dim $2n$)

σ on $K_n(\mathbb{T}^2)$ induced from $\sum_{i=0} \pi_i^*(\sigma_{\mathbb{T}^2})$

$K_n(\mathbb{T}^2)$ = (generalized Kummer variety of dim $2n$.)

$K_1(\mathbb{T}^2)$ = Kummer surface.

◎ Cohomology of IHS manifolds

(Facts) ① $H^2(X) = \underbrace{\mathbb{C}\cdot\sigma}_{H^{2,0}} \oplus H^{1,1}(X) \oplus \underbrace{\mathbb{C}\cdot\bar\sigma}_{H^{0,2}}$ (Hodge decomp.)

② On $H^2(X)$, one has a quadratic form q_X (Beauville–Bogomolov–Fujiki form) analogous to the intersection forms on K3:

$H^2 \times H^{2n-2} \to \mathbb{C}$

$H^2 \times H^2 \to \mathbb{C}$

- $q_X(\beta) = c_n \cdot \int_X \beta^2 (\sigma\bar\sigma)^{n-1}$ for $\beta \in H^{1,1}$
- q_X is (up to scaling) primitive integral; (perhaps not unimodular) has signature $(1, h^{1,1}-1)$ on $H^{1,1}$.
- $q_X(\alpha)^n = c_n' \int_X \alpha^{2n}$. "Volume"

③ (Verbitsky 1995) The subring $SH^2(X) \subset H^\bullet(X)$ generated by $H^2(X)$ is isomorphic to

$$SH^2(X) \cong \underbrace{\mathrm{Sym}^\bullet H^2(X)}_{\text{symmetric algebra}} / \langle \alpha^{n+1} \mid q_X(\alpha)=0 \rangle$$

deg $\geq 2(n+1) = 2n+2$

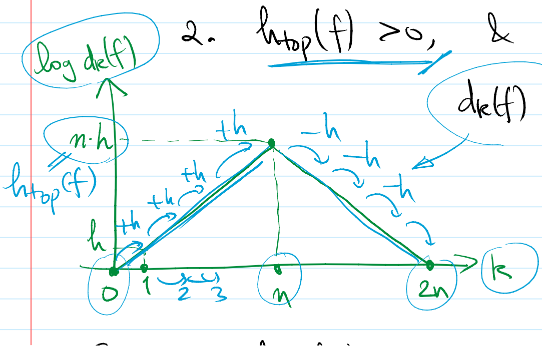
In particular,
$$SH^2(X)_{\deg \leq 2n} = \bigoplus_{k=0}^{n} \mathrm{Sym}^k H^2(X).$$

◎ Dynamics on IHS manifolds

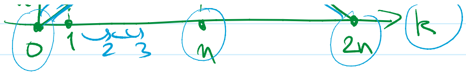
Thm (Oguiso 2009) Let X^{2n} be IHS & $f: X \to X$ be a holomorphic automorphism. Then either

1. $h_{top}(f) = 0$, & dynamical degrees $d_k(f) = \text{Spectral Radius}(f^*: H^{2k} \to H^{2k})$ are all 1.
2. $h_{top}(f) > 0$, & for $h = \frac{1}{n} h_{top}(f)$, $$d_k(f) = \begin{cases} e^{kh} & \text{if } k \leq n \\ e^{(2n-k)h} & \text{if } k > n \end{cases}.$$

De Thélin '08:
$\chi^-_{max} \leq -\frac{h}{2}$, $\chi^+_{min} \geq +\frac{1}{2}h$



$h = \log d_1(f)$



$h = \log d_1(f)$

Cor If $h_{top}(f) > 0$, then f admits eigenclasses $[\eta_+], [\eta_-] \in H^{1,1}(X)$ s.t. $f^*[\eta_\pm] = e^{\pm h}[\eta_\pm]$.

$e^h = d_1(f)$ - eigenclass $[\eta_+]$,
$e^h = d_1(f^{-1})$ - eigenclass $[\eta_-]$

Cor (Oguiso '09 + Dinh–Sibony '10) If $h_{top}(f) > 0$, we have Green $(1,1)$-currents $S^+ \in [\eta_+]$, $S^- \in [\eta_-]$ with ^{A} Hölder potentials s.t.

$$^B\ f^* S^+ = e^h S^+, \quad ^B\ f^* S^- = e^{-h} S^-.$$

(By (De Thélin – Dinh '12), implies: $(S^+)^n (S^-)^n$ is the m.m.e.)

as $S^\pm$ has continuous potentials, one can define the product of them, using $dd^c u \cdot T = dd^c(u \cdot T)$.

◎ Computable example: Cat map

Let $\mathbb{T}^2 = E \times E$, E an elliptic curve$/\mathbb{C}$.

Set $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$: $(x, y) \mapsto (x + x + y, x + y)$

$\Rightarrow$ A induces a map $A : K_1(\mathbb{T}^2) \to K_1(\mathbb{T}^2)$.

($A : \{p, -p\} \mapsto \{Ap, -Ap\}$ where $p \neq -p$ on $\mathbb{T}^2$.)

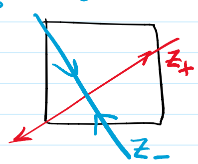
$(x, xw; [1:w])$ $\hookrightarrow$ "$w\, dw \wedge dx$" $=$ "$d(w^2) \wedge dx$"

ω : Ricci-flat metric on $K_1(\mathbb{T}^2)$, s.t. ω on $\{p, -p\}$ $(p \neq -p)$ is same as the Euclidean metric $\omega_{\mathbb{T}^2, p}$ at p.

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Then, $\frac{1}{\varphi^{2n}} (A^n)^* \omega \longrightarrow c \cdot dz_+ \wedge d\bar{z}_+ = S^+$ (z_+ : expanding direction)

φ^2



$\varphi = \frac{1+\sqrt{5}}{2}$ Golden ratio

$\frac{1}{\varphi^{2n}} (A^{-n})^* \omega \longrightarrow c' \cdot dz_- \wedge d\bar{z}_- = S^-$ (z_- : contracting direction)

φ^{-2}

(on $\{p, -p\}$, $p \neq -p$.)

Furthermore, the scaling factors $\log \varphi^2$, $\log \varphi^{-2}$ are Lyapanov spectra of the system $(K_1(\mathbb{T}^2), A)$.

2. Kummer Rigidity

Thm (Cantat – Dupont 2020; Filip – Tosatti 2021)

Let X be a projective or K3 (perhaps nonprojective) surface, and $f : X \to X$ a holomorphic automorphism of $h_{top}(f) > 0$.

If m.m.e. $\mu \ll vol_X$, then $(X, f) = (K_1(\mathbb{T}^2), A)$.

m.m.e. = measure of maximal entropy

(X, f): called Kummer example

$K_1(\mathbb{T}^2)$: induced from A; A: affine-linear $A : \mathbb{T}^2 \to \mathbb{T}^2$.

- Background
  - "locally homogeneous rigidity" of m.m.e. $\ll$ vol:
    (cf. Besson – Courtois – Gallot '95
    $\Leftarrow$ A conjecture of Katok)
  - m.m.e. $\ll$ vol + complex manifold:
    i) Zdunik 1990: rational $\mathbb{P}^1 \to \mathbb{P}^1$ + m.m.e. $\ll$ vol
    $\Rightarrow$ Lattès map
    ii) Berteloot – Dupont 2005: sim., rational $\mathbb{P}^k \to \mathbb{P}^k$.

    "Essentially comes from torus"

Lattès: $E/_{\pm 1} \to E/_{\pm 1}$

$E \to E$

$z \mapsto az + b$

The proof (for the surface case) uses Green (1,1)-currents, which are natural in surface AND IHS case.

Q Do we have an analogous result for IHS?

That is:

Conjecture Let X^{2n} be IHS and $f : X \to X$ be a holomorphic automorphism, with $h_{top}(f) > 0$.

If m.m.e. $\mu \ll vol_X$, then

"Kummer example"

- X normalizes a finite torus quotient $\mathbb{T}^{2n}/\Gamma$,
- f lifts an affine-linear $A : \mathbb{T}^{2n}/\Gamma \to \mathbb{T}^{2n}/\Gamma$.

Thm (J. '21) OK for X projective and $\mu = vol_X$.

(J., WIP) Can $\mu \ll vol_X$?

(eg) Let $\mathbb{T}^2$ be a complex torus.

$\mathbb{T}^2 \xrightarrow{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{T}^2$

$(\mathbb{T}^2)^n \longrightarrow (\mathbb{T}^2)^n$

(eg) Let $\mathbb{T}^2$ be a complex torus.

$$\boxed{K_n(\mathbb{T}^2)} \xrightarrow{\text{desing.}} (\mathbb{T}^2)^n / \underbrace{S_{n+1}}_{\mathrm{Weyl}(A_n)}$$

$(01).(t_1, \dots, t_n) = (-t_1 - t_2 - \dots - t_n, t_2, \dots, t_n)$

$(ij).(t_1, \dots, t_n) = (t_1, \dots, t_j, \dots, t_i, \dots, t_n)$, $0 < i < j$

$\circlearrowright A$ on $K_n(\mathbb{T}^2)$ $\leftarrow$ $\circlearrowright A$ on $(\mathbb{T}^2)^n/S_{n+1}$

Kummer example

S_{n+1}- ~~invariant~~ "equivariant" $\boxed{\text{affine map } (\mathbb{T}^2)^n \to (\mathbb{T}^2)^n}$

$$\begin{array}{ccc} (\mathbb{T}^2)^n & \longrightarrow & (\mathbb{T}^2)^n \\ & \downarrow & \\ K_n(\mathbb{T}^2) & \xrightarrow{K_n([\cdot])} & K_n(\mathbb{T}^2) \end{array}$$

(eg) Let $S = K_1(\mathbb{T}^2)$ be a Kummer surface.

$$\mathrm{Hilb}^n(S) \xrightarrow{\text{normalize}} (\mathbb{T}^2)^n / \underbrace{\langle \iota, S_n \rangle}_{= \mathrm{Weyl}(B_n)}$$

$\iota.(t_1, \dots, t_n) = (-t_1, t_2, \dots, t_n)$

$\sigma.(t_1, \dots, t_n) = (t_{\sigma(1)}, \dots, t_{\sigma(n)})$, $\sigma \in S_n$

$\circlearrowright A$ $\leftarrow$ $\circlearrowright A : \langle \iota, S_n \rangle$ ~~invariant~~ equivariant affine map $(\mathbb{T}^2)^n \to (\mathbb{T}^2)^n$

What about $\mathrm{Weyl}(D_n)$? $\mathrm{Weyl}(E_k)$? ...

Thm (Ginzberg – Kaledin, Bellamy) There are not many finite groups Γ that admits a "symplectic resolution" for $\mathbb{T}^{2n}/\Gamma$:

$$\Gamma \cong \underbrace{\mathrm{Weyl}(A_n)}_{S_{n+1}}, \quad \underbrace{(\mathbb{Z}/m\mathbb{Z})^n \times S_n}_{= \mathrm{Weyl}(B_n) \text{ if } m=2}, \quad Q_8 \times \mathbb{Z}/3\mathbb{Z}.$$

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3. The Proof

- Surface Case

$$(\underset{\text{K3 surface}}{X}, \underset{h_{top}>0}{f}) \longrightarrow S^+, S^- \quad \text{Green currents (nef; } S^+ + S^- \text{ big)}$$

i) Contract away the singularity

$$\begin{array}{ccc} \underset{K3}{X} & \xrightarrow{\nu} & \underset{\text{orbifold}}{Y} \\ \circlearrowright & & \circlearrowright \\ f & \dashrightarrow & f_Y \end{array}$$

contraction, sending $\mathrm{Null}(S^+ + S^-)$ to 0-dim'l set

union of curves C where $\int_C S^+ + S^- = 0$

$\Rightarrow$ $\exists$ Ricci-flat orbifold metrics ω_t on Y,

s.t. $\nu^* \omega_t \sim e^t \cdot S^+ + e^{-t} \cdot S^-$ (cohomologous) $\Rightarrow \nu^* \omega_t^2 = \sigma\bar{\sigma} = e^{\cdot} \cdot S^+ \wedge S^-$

[So $(f_Y^N)^* \omega_t = \omega_{t + h \cdot N}$.]

ii) Green currents as metric + volume

$$\text{m.m.e. } \mu \ll \mathrm{vol}_X \longrightarrow \nu_* S^+, \ \nu_* S^- \text{ each}$$

is metric & volume along $W^u(x)$ $W^s(x)$ rspt.

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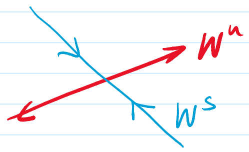
iii) Uniform hyperbolicity

$$f_Y^* \, \nu_* S^\pm = e^{\pm h} \, \nu_* S^\pm$$

$\longrightarrow$ $W^u(x)$ & $W^s(x)$ have invariant metrics

$\longrightarrow$



uniform stretch & shrink

$\longrightarrow$ Y flat, a torus quotient, &c.

Q How much of this sketch extend to IHS?

A Let's see ...

$$(X, f) \longrightarrow S^+, S^-$$

X: projective IHS; f: $h_{top} > 0$

S^+, S^-: (1,1)-Green currents (nef; $S^+ + S^-$ big)

i) Contract away the singularity

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[loop below] & Y \arrow[loop below] \end{tikzcd}$$

$f \dashrightarrow f_Y$

1, projective
2, birational.

contraction ~~sending~~
~~$\mathrm{Null}(S^+ + S^-)$~~
~~to 0-dim'l set~~
isomorphic outside of $\mathrm{Null}(S^+ + S^-)$

$\uparrow$ union of varieties V in which $\int_V (S^+ + S^-)^{\dim V} = 0$

$\Rightarrow$ (COLLINS–TOSATTI 2015)

$\exists$ Ricci-flat metric ω_0 on $\boxed{X \setminus E}$ — $\mathrm{Null}(S^+ + S^-) = E$

$\omega_0 \approx S^+ + S^-$ (C^∞_{loc}) as forms

$(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ $\omega_0 \sim$

$[\omega_0]$ "=" $[S^+] + [S^-]$.

ii) Green currents as metric + volume

m.m.e. $\mu \ll \mathrm{vol}_X$ $\longrightarrow$ $(S^+)^n$, $(S^-)^n$ each

is ~~metric &~~ volume along

$W^u(x)$, $W^s(x)$ rspt.

$\overset{?}{\Longrightarrow}$ S^+, S^- are metrics?

$S^{\pm} \text{ ``}\ll\text{'' } \omega$

$\nwarrow$ singular parts?

$\mu = \mathrm{vol}_X$ $\longrightarrow$ $\int_X f^*\omega_0 \wedge \omega_0^{2n-1}$ computes

(i) cohomologically,

$\cosh(Nh) [\omega_0]^{2n}$

$f^*[\omega_0]$

$(e^h[S^+] + e^{-h}[S^-]) \cdot ([S^+] + [S^-])^{2n-1}$

$[\omega_0]$

$\mathcal{V}_X$

$$\left(e^h[S^+]+e^{-h}[S^-]\right)\frac{f^*[\omega_0]\,([S^+]+[S^-])}{[\omega_0]}$$
$$= \left(\frac{e^h+e^{-h}}{2}\right)\binom{2n}{n}[S^+]^n[S^-]^n$$

(i) cohomologically, $\cosh(Nh)[\omega_0]^{2n}$

(ii) as differential forms,
$$o(1) + \int_X \frac{1}{n}\sum \cosh(\sigma_i(x,N))\,dx$$
(stretch factors of $f^*\omega_0$ rel ω_0)

ω_0

$f^*\omega_0 = \varphi^2\, dz_1\wedge d\bar z_1 + \varphi^{-2}\, dz_2\wedge d\bar z_2$

JENSEN + LEDRAPPIER – YOUNG $\Rightarrow$ $\sigma_i \equiv Nh$ a.e.

$\Rightarrow$ S^+, S^- replaced by $\omega_0|W^u$, $\omega_0|W^s$.

iii) Uniform hyperbolicity (* Not an issue if $\mu = \mathrm{vol}$)

$$f^* S^\pm = e^{\pm h} S^\pm$$

(Assume that they are metrics on W^u, W^s)

$\longrightarrow$ $W^u(x)$ & $W^s(x)$ have invariant metrics

$\overset{?}{\longrightarrow}$ uniform stretch & shrink

OK for $\mu = \mathrm{vol}$

- Lyapunov spectrum: $\pm\frac{h}{2}$, multiplicity n.

$\Rightarrow$ $f|W^s(x) \to W^s(fx)$, etc. are normalized to a $\mathbb{C}$-linear $A: \mathbb{C}^n \xrightarrow{\ \ } \mathbb{C}^n$. ($\mathbb{C}^n \approx W^s(x)$, $\mathbb{C}^n \approx W^s(fx)$)

- Local expression: $S = \Phi^1_{ij^*}(z)\, dz^i \wedge d\bar z^j$ on $W^s(fx)$

$$\Rightarrow f^*S^- = A_{ki}\,\Phi^1_{ij^*}(A.w)\,\bar A_{lj}\, dw^k\wedge d\bar w^l \text{ on } W^s(x)$$
$$= e^{-h}\Phi^0_{kl^*}(w)\, dw^k\wedge d\bar w^l$$

$W^s(x)$

$W^s(fx)$

ie, $A\,\Phi^1(A.w)\,A^* = e^{-h}\cdot\Phi^0(w)$ as matrices.

(K3 surface case) $\hookrightarrow$ $AA^*\Phi^1(A.w) = e^{-h}\Phi^0(w)$ + Lebesgue differentiation $\Rightarrow$ $\Phi^0(w)$ const, $A = e^{-h/2}\cdot(\text{unitary})$

$\overset{?}{\longrightarrow}$ $\Phi^0(w)$ const. $A = e^{-h/2}\cdot U(n)$ $\rightarrow$ ($A^{(N)}: W^s(x)\to W^s(f^Nx)$ is $e^{-Nh/2}\cdot(\text{unitary})$ w/ bdd. conjugation) $\Rightarrow$ uniformly hyperbolic.

$\longrightarrow$ Y flat, a torus quotient, &c $\longrightarrow$ Kummer example

Symplectic form + BENOIST – FOULON – LABOURIE

GREB – KEBEKUS – PETERNELL

$X \overset{\Phi}{\dashrightarrow} Y$, $X \setminus E$, $X\setminus E$ ω_0, $\cup$, Y_{reg}, flat

Local torelli : $\sigma \leftarrow q_X(\sigma) = 0$

$$\mathcal{M} \ni [X] \xrightarrow[\text{local Bom.}]{\text{Per}} \{ [\sigma] \in H^2 \mid q_X(\sigma) = 0 \}$$

vs. $\mathcal{M}_{K3}$?

$$\left(\mu \ll \cancel{\text{vol}} \Rightarrow \mu = \text{vol} \right) ?$$

$\mu(x) = \rho(x)\, d\text{vol}(x)$