Linear algebra (MIT)

Lecture 1

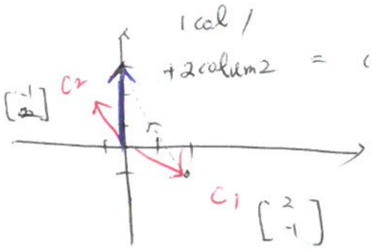
∘ coefficient matrix , Row picture

∘ $\begin{bmatrix} 2x - y = 0 \\ -x + 2y = 3 \end{bmatrix}$ col pic

$$x^{1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y^{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

linear combination of columns. 시각화

1 col 1 + 2 colum2 = (0, 3)



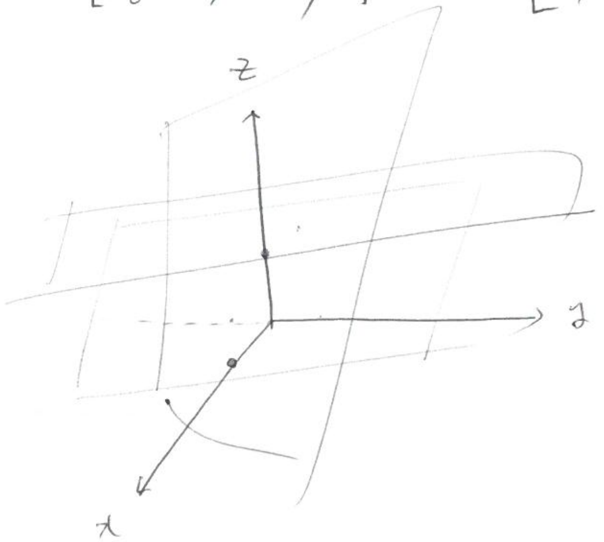
row pic

∘ $2x - y = 0$

$-x + 2y - z = -1$

$-3y + 4z = 4$

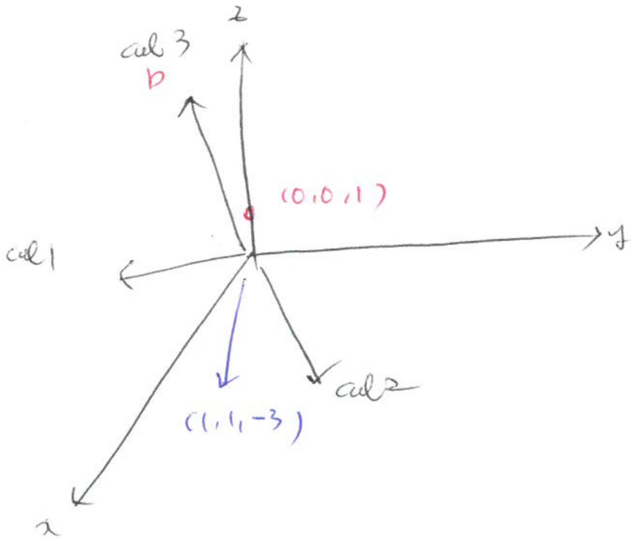
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}$$



∘ col pic

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

row pic 포기하고



if $\begin{matrix} 0 \\ -1 \\ 4 \end{matrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ $(x, y, z) = (1, 1, 0)$

∘ algebra 에서 $Ax = b$ 모든 b에 대해 solution (col matrix) 이 존재 할까? → invertable 이면 존재 non-singular

☆ 30m

∘ 9-dimension 일 때는 어떨까? 거의 성립할까? (solution 존재할까?)

matrix vector

∘ $Ax = b$ solution (ex) col 으로 생각해보기

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$Ax = b$ linear combination of col

A는 combination of colums of A.

Linear Algebra lecture 2.

1. Elimination [success / failure
2. Back substitution
3. Elimination matrices
4. Matrix multiplication

①

$x + 2y + z = 2$

$3x + 8y + z = 12$

$4y + z = 2$

$Ax = b$

1st pivot

(1)

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

A b

(2)

$$\rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

(3)

$$\rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{array}\right]$$

failure "

② Back substitution

$x = 2$
$y = 1$
$z = -2$ "

③ Elimination matrices

* Linear Algebra : Multiplication of matrices 할 때 각 row와 col 들이 어떻게 작동하는지 이해해야 함!,,

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3 \times col\,1 + 4 \times col\,2 + 5 \times col\,3$$

3x3 3x1 3x1

Matrix x col = cols

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = 1 \times row\,1 + 2 \times row\,2 + 7 \times row\,3$$

1x3 3x3 1x3

row x matrix = rows

④ Matrices : Substract 3 x row 1 from row 2

(1)

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

(2) subtract 2 x row 2 from row 3

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$E_{32}(E_{21}A) = U$

$(E_{32}E_{21})A = U$

Elimination 한방에 해주는 matrix는... $E_{32} \times E_{21}$

※ Permutation matrix

Exchange rows 1 and 2

$$\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

★ left : row operation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

★ right : col operation

※ Inverses

$$\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

subtract 3 x row 1 from row 2

add 3 x row 1 from row 2

반대: inverses

Lecture 3

○ Matrix multiplication

i)

$$\underset{A_{m\times n}}{\begin{bmatrix} \\ a_{31} \text{———} \\ \\ \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} & b_{14} & \\ & | & \\ & | & \end{bmatrix}} = \underset{C = AB}{\begin{bmatrix} & c(i,j) & \\ & \searrow \cdot & \\ & & \end{bmatrix}}$$

row3, col4, $c(i,j) \mapsto c(3,4)$, c_{34}

$$c_{34} = (\text{row3 of } A) \underset{\text{dot}}{\cdot} (\text{column 4 of } B)$$

$$= a_{31}b_{14} + a_{32}b_{24} + \cdots = \sum_{k=1}^{n} a_{3k}b_{k4}$$

columns

ii)

$$\underset{A_{m\times n}}{\begin{bmatrix} & \\ & \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} | \; | & | \\ | \; | & | \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} | \; | & | \\ | \; | & | \end{bmatrix}}$$

col1, A{col1}

col 으로 보기

having several columns

columns of C are combination of columns of A

rows

iii)

$$\underset{A_{m\times n}}{\begin{bmatrix} \text{——} \\ \text{——} \\ \text{——} \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} & \\ & \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} \text{——} \\ \text{——} \\ \text{——} \end{bmatrix}}$$

row

rows of C are combination of rows of B

Column of A x a row of B

iv)

$m\times 1$ $\quad 1\times p$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

AB = Sum of (Cols of A) x (rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

○ Block multiplication

$$\underset{A}{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}} \underset{B}{\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}} = \begin{bmatrix} A_1B_1 + A_2B_3 & \\ & \end{bmatrix}$$

○ Inverses (Square matrices)

- $A^{-1}A = I = AA^{-1}$
  ↑ if it exists

  invertable, nonsingular

- Singular case, non-invertable (no inverse)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Reason 1. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 와 $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$을 조합해서 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$을 만들수 없음!

같은 방향을 가리킴. 같은 선상에 있으므로 (1,0)이 나올수 없다!

Reason 2. (중요)

You can find a vector x with $Ax = 0$.

ex) $Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A^{-1}Ax = 0$

$x = 0$

$x \neq 0$ 모순됨!

$$- \underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A^{-1}는 row or col 로 구할수 있음.

col → A x columns j of A^{-1} = columns j of I

※ Gauss-Jordan (Solve 2 equations at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

└ elimination 으로 구하기!

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

A $\quad$ I

↓

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & +1 \end{array}\right]$$

I $\quad$ A^{-1}

$EA = I$ tells us $E = A^{-1}$

$E[A \;\; I] = [I \;\; A^{-1}]$

Lecture 4 ★

└ Factorization into $A = LU$

+ elimination 할 때 inverse 이용, ($A = LU$ 구할 때 $EA = U$ $A = E^{-1}U$) 과정.

∘ $AA^{-1} = I = A^{-1}A$

$(AB)B^{-1}A^{-1} = I$

$B^{-1}A^{-1}AB = I$

Basic fact

Inverse, Transpose

∘ $AA^{-1} = I$

$(A^{-1})^T A^T = I$

Inverse of A transpose

∘ A

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad 2\times2$$

$$\Rightarrow \overset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$A = LU$ (inverse E_{21})

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

[pivot 쪼개기
pivot만 분리해서 matrix로 만들기]

★ ∘ 3x3

$E_{32}E_{31}E_{21}A = U$ (no row exchanges)

$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$= LU$

☆☆☆ 19:00m,,

suppose that ...

★

p.96 설명

★ Jordan

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E$$

(left side of A)

$EA = U$

inverses
(reverse order)

$$\underset{E_{21}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_{32}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L$$

left of U

$A = LU$

$\boxed{A = LU}$

If no row exchanges,
multipliers go directly into L

(for elimination ...)

∘ How many operations on $n\times n$ matrix A ?

$\boxed{n = 100}$

first pivot

100^2 회.

actually $n^2 - n$ 이 맞음 $(100^2 - 100)$

$n, n^2, n^3, n!$ ... ?

about 99^2

count

$\therefore n^2 + (n-1)^2 + (n-2)^2 + \cdots + 3^2 + 2^2 + 1^2$

$\approx \frac{1}{3}n^3$

적분개념으로 생각해도 됨 $\int x^2 dx = \frac{x^3}{3}$

on A

∘ Permutations 3x3 ... row exchanges 6 P's

$$\underset{\text{identity}}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

자기자신 inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

서로 inverse

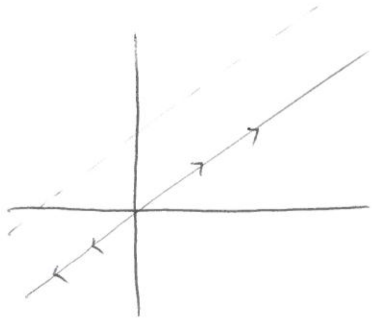
$P^{-1} = P^T$

* 4x4 → 4!

Lecture 5.

◦ Permutation

◦ Vector space



◦ Subspace of R^2 rule

① all of R^2 (plane)

② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ L

③ zero vector only Z

R^3도 동일

how to create subspace from a matrix

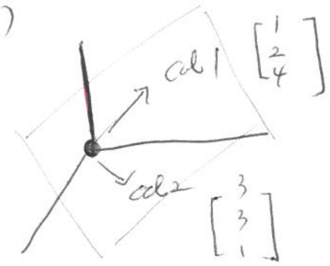
★

◦ $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$: columns in R^3

add their combinations form a subspace

called column space $C(A)$

C(A)



Lecture 6

(◦ Vector spaces & sub spaces
◦ column space of A : Solving $Ax=b$)

◦ Null space of A

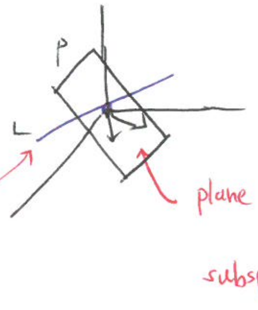
review ...

◦ vector space 의 정의 . requirements of vector space

all combs $\boxed{cv + dw}$ are in the space ☆☆

($v+w$ and cv are in the space)
sum multiple

- R^3



plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is subspace of R^3

line is a subspace

◦ 2 subspaces : P and L

- $P \cup L$ = all vectors in P or L

This (is / isn't) subspace

↳ will be outside of union P, L,

- $P \cap L$ = all vectors in both P and L

↳ subspace 임. (zero 한 점)

◦ Subspaces S and T

- intersection $S \cap T$ is a subspace

◦ Column space of $\underset{4\times3}{A}$ is subspace of R^4 $C(A)$ $\mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} = \text{all linear combination of columns}$$

Does $Ax=b$ have a solution for every b ? No.

Pivot columns or (2,3) (1,3) ...   3 equations, 4 unknown

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- Which b's allow this equation to be solved ?

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \cdots,$$

can solve $Ax=b$ exactly when b is in $C(A)$
∟ column space

◦ Null space of A = all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$
∈ in R^3

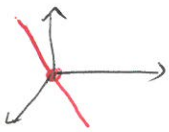
- Nullspace

• $$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

N(A) contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$

general sol.

N(A) : a line



a line in R^3

◦ Check that Solutions to $Ax=0$ always give a subspace (Nullspace 는 항상 subspace 이다.)

- proof) x 를 해라고 할 때 x + x′ 도 해임. cx′ 도 해!

$Ax=0$ $A(x+x')=0$
$Ax'=0$ $A(cx')=0$

- $Ax=0$ 에서 0이 아닐때 , solution of x 는 subspace 가 될수 없음. (0,0,0) 을 포함하지 않음! x로 이루어진 line or plane 이 (0,0,0)을 통과할 수 없기 때문에) subspace 가 될 수 없다.

∴ vector space 의 정의 , requirements
column space] two ways of representing vector space
Null space

Lecture 7 : Solving $Ax=0$: Pivot Variables, Special Solutions - $Rx=0$

└ $Ax=0$ 푸는 알고리즘.

∘ $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$ → not independent

↓

not independent

- Solving algorithm ...

→ $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$

Pivots

→ echelon $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = u$

∘ rank of A = # of pivots = 2

2 pivots columns (1, 3 col)

2, 4 col $\boxed{\text{free columns}}$

∘ $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$2x_3 + 4x_4 = 0$

$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ assign any number to free columns.

$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

→ r=2 일 때, n-r = 4-2 가 free variables 의 수.
(pivots)

- R = reduced row echelon form

zeros above & below pivots

$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R = \text{rref}(A)$

└ matlab 명령어.

✓ notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot rows/cols

$x_1 + 2x_2 + (-2)x_4 = 0$
$x_3 + 2x_4 = 0$) 로 바뀜.

∴ $Ax=0 \to Ux=0 \to Rx=0$ 모두 같음.

- 정리하면 (?) (magic 이 아님 ?)

$\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}$ $\boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}}$

Pivot col free cols

0 0 0 0

∘ rref form

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ typical rref form.
r pivot rows

↑ pivot cols ↑ n-r free cols

★ (29m ~ 33m)

$Rx=0$ $RN=0$

$N = \begin{bmatrix} -F \\ I \end{bmatrix}$

nullspace matrix
(columns = special solution!)
(not magic)

아래 (예제)에서 확인하자 :)

또한 $Rx=0$
즉 $[I \ F]\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$
$x_{pivot} = -Fx_{free}$

R의 column 들이 special solution matrix가 됨!

example

- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

pivot columns free column.

u

I F

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$

r=2 again!

n-r = 3-2 = 1 free column.

∴ $x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ free variable assign to 1.

$= c\begin{bmatrix} -F \\ I \end{bmatrix}$

↑ N

Lecture 8: Solving $Ax = b$: Row Reduced form R

○ $\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \xrightarrow{eli} \left[\begin{array}{ccccc} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - b_1 \\ 0 & 0 & \cancel{2} & \cancel{4} & b_3 - 3b_1 \end{array}\right]$

$b_3 + b_2 - b_1$

pivot columns

Augmentation matrix $= [A \; b]$

$0 = b_3 - b_2 - b_1$

- $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ 일 경우 $\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 성립,,

- Solvability Condition on b

• $Ax = b$ solvable when b is in Column space of A
$C(A)$

$\boxed{\text{(같은 말)}}$ ★

• If a combination of rows of A gives zero row, then the same combinations of entries of b must give 0.

○ To find complete sol'n to $Ax = b$

① $X_{particular}$: set all free variables to zero. $\left(\begin{matrix} x_2 = 0 \\ x_4 = 0 \end{matrix}\right)$

Solve $Ax = b$ for pivot variables

$x_1 + 2x_3 = 1 \qquad x_1 = -2$

$2x_3 = 3 \qquad x_3 = \frac{3}{2}$

$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

①

② $X_{nullspace}$

$X = X_p + X_n$ … 풀이) $\begin{array}{l} A x_p = b \\ + \; A x_n = 0 \end{array}$ ☆☆

$A(X_p + X_n) = b$

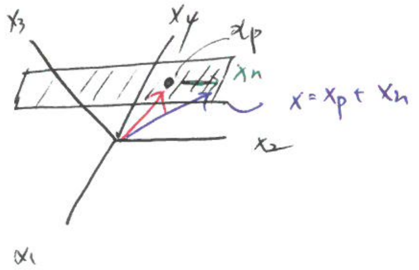
subspace, nullspace

all combinations of solutions

∴ $X_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$n - r = 2$
↓
c_1, c_2
2개

- Plot all solutions x in R^4



∴ Go through elimination and find $X_{particular}$ and $X_{nullspace}$.

○ m by n matrix A of rank r (knowing that $r \le m$, $r \le n$)

- Full column rank means $r = n$. No free variables

$N(A) = \{\text{zero vector}\}$

Solution to $Ax = b$: $x = x_p$, unique solution if it exists.

☆ full column rank (0 or 1 solutions)

• ex) $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(two independent column, two pivots, rank 2 col)

full column rank

계산 안하고 알 수 있다.

☆ • full row rank $r = m$,

can solve $Ax = b$ for every b. Exists.

Left with $n - r$ free variables.

$A^T = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$

I F

2 rank, two pivots

☆ • $r = m = n$, full rank

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, invertable $R = I$,

0 or 1 solution
∩
Solvable every b
↓
1 solution

★ ∴ numb of rank — tell you solutions

★

$r < m$, $r < n$	정리 $r = m = n$	$r = n < m$	$r = m < n$
$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$	$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = [I \; F]$
0 or ∞ solutions	1 solution	0 or 1 solution	1 or ∞ solutions

Lecture 9 : Independence, basis, and dimension

∘ Suppose A is m by n with $m<n$,

Then there are nonzero solutions to $Ax=0$

(more unknown than equations)

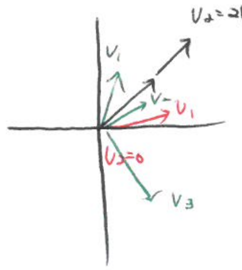
Reason : There will be free variables !!

background

∘ Independence : Vectors $x_1, x_2, \cdots, x_n$ are independent if no combinations gives zero vector (except the zero comb. all $c_i=0$)

$$c_1x_1+c_2x_2+\cdots+c_nx_n \neq 0$$

- ex)



$2v_1 - v_2 = 0$ dependent

$0\cdot v_1 + bv_2 = 0$ dependent

(둘 중 하나 zero vector → dependent)

v_1 v_2 independent

$+v_3$ → dependent

(from background)

ex)

$$A = \begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$A = \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Nullspace 에 뭔가 있기 때문에. dependent

∘ Repeat when $v_1, \cdots, v_n$ are column of A.

→ They are independent if nullspace of A is {zero vector}

↳ rank $= n$, $N(A) = \{0\}$, no free variables

→ They are dependent if $Ac=0$ for some nonzero c.

↳ rank $< n$, yes free variables

∘ Spanning a space

- Vector $v_1, \cdots, v_l$ span a space means : the space consists of all combs. of those vectors

∘ Basis for a space is a sequence of vectors $v_1, v_2, \cdots, v_d$ with two properties

1. They are independent.
2. They span the space.

- example :

① Space is $\mathbb{R}^3$ standard basis

One basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ☆ ✓

Another basis $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ → not invertable, dependent!, same row (1, 2)

└ matrix 에 넣고 elimination 해서 free variable 없음을 확인하면 됨.

② $\mathbb{R}^n$ n vectors give basis if the $n\times n$ matrix with these cols is invertable.

∘ Given a space, every basis for the space has the same number of vectors.

→ $\boxed{\text{dimension}}$

Def. Dimension of the space.

정리 : Independence, spanning, basis, dimension

∘ Space is $C(A)$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$
↑ ↑

ㄴ $N(A)$ 를 보면

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

→ dependent. $N(A)$ 에 0 말고 다른것이 있음.

→ Basis : 1, 2 col. or 1,3 or 2,3 ... or ... $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix}$

→ rank(A) = # pivot columns = 2 = dimension of $C(A)$ ★

rank of matrix.

③ Nullspace 다시 보기. $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ dim $N(A)$ = # free variables $= n - r$

special solutions

Lecture 10 : 4 fundamental subspaces

∘ columnspace $C(A)$ in R^m ※ A is $m \times n$

nullspace $N(A)$ in R^n

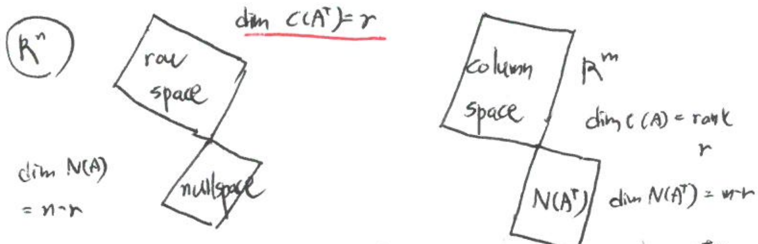
rowspace = all combinations of rows

= all combinations of columns of A^T

= $C(A^T)$ in R^n

nullspace of A^T = $N(A^T)$ = left nullspace of A in R^m

∘ 4 subspaces ★ from example $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{pmatrix}$



	$C(A)$	$C(A^T)$	$N(A)$	$N(A^T)$
- basis	pivot cols		special solutions	
- dimension	r	r	$n-r$	$m-r$

(num of free variables)

∘ example for row space $C(A^T)$

take combination

$$A=\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

→ Basis for row spaces is first r rows of R (of A)

example

★ ∘ 4th space : $N(A^T)$

$A^Ty = 0$ → $y^TA = 0^T$

$[\quad][\quad] = [0]$ $\quad [y^T]\begin{bmatrix} A \end{bmatrix} = [0]$

└ left nullspace

rref $\begin{bmatrix} \underset{m\times n}{A} & \underset{m\times m}{I} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m\times n} & \underset{m\times m}{E} \end{bmatrix}$

$EA = R$ in chap. 2 , R was I

Then E was A^{-1}.

다시 예로 돌아가면

$$A=\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

위로부터

$$E \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ 을 구할수 있음.

따라서 $EA = R$ 을 만족하는 E를 구함.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ \boxed{-1 \quad 0 \quad 1} \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

0을 만드는 comb

★ … how to produce a basis for the left nullspace

★★★? ∘ new vector space M ! All 3×3 matrices !!

$A+B$, cA

(not AB for now)

subspaces of M ‖ upper triangulars ‖ symmetric matrices

‖ diagnal matrices ‖ dim of this subspace is 3

D

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Lecture 11 : Matrix spaces ; Rank 1 ; Small World graphs

· Basis for M = all 3×3's

: 총 9개 필요 $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \cdots, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$

: $\dim M = 9$

- 3×3 symmetric 인 경우 $\dim S = 6$
- 3×3 Uppertriangular 인 경우 $\dim U = 6$
- 3×3 $S \cap U$ diagonal 3×3, $\dim(S\cap U) = 3$
- 3×3 $S + U$ = any element of S + any element of U
  = all 3×3
  = $\dim(S+U) = 9$

∘ $\dfrac{d^2y}{dx^2} + y = 0$

- solutions: $y = \cos x, \sin x$ → basis
- $y = c_1 \cos x + c_2 \sin x$
- $\dim(\text{solution space}) = 2$

∘ Rank one matrices

- $\underset{2\times3}{A} = \begin{bmatrix}1&4&5\\2&8&10\end{bmatrix}$

$r = 1$

$= \underset{2\times1}{\begin{bmatrix}1\\2\end{bmatrix}} \underset{1\times3}{\begin{bmatrix}1&4&5\end{bmatrix}}$

$\dim C(A) = r = \dim C(A^T)$

→ Rank 1 matrix $A = uv^T$ 형태를 가짐
building block 과 같음.

- M = all 5×17 matrices
  subset of rank 1 matrices
  not a subspace

- In R^4 $v = \begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}$,

$Av = 0$

S = all v in R^4 with $v_1 + v_2 + v_3 + v_4 = 0$

S = nullspace of $A = \begin{bmatrix}1&1&1&1\end{bmatrix}$ $n = 4$

rank $= 1 = r$
$\dim N(A) = n - r$
$= 3$

(Bases of new vector spaces
Rank one matrices
Small world graphs)

row space $\dim = 1$] r n
nullspace $N(A)$ $\dim = 3$
column space $\dim = 1$ $C(A) = R^1$
$N(A^T) = \{0\}$, $\dim = 0$] 1 m

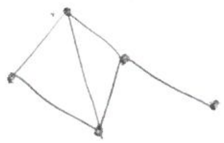
∗ Basis for S

$\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ basis for subspace (nullspace)

∘ Small world graphs

- graph = {nodes, edges}

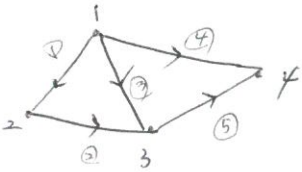
- 'distance' node to node.



Lecture 12 : Graphs, Networks, Incidence Matrices

- Graphs & Networks
- Incidence Matrices
- Kirchhoff's Laws

o Graph = Nodes, Edges



$n=4$ nodes
$m=5$ edges

o Incidence Matrix

$$A = \begin{array}{c|cccc|c} \text{node} & 1 & 2 & 3 & 4 & \text{edge} \\ \hline & -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & 0 & 2 \\ & -1 & 0 & 1 & 0 & 3 \\ & -1 & 0 & 0 & 1 & 4 \\ & 0 & 0 & -1 & 1 & 5 \end{array}$$

edges 1, 2, 3: loop

복습필요.
- Nullspace 찾아보면 ... $Ax=0$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2-x_1 \\ x_3-x_2 \\ x_3-x_1 \\ x_4-x_1 \\ x_4-x_3 \end{bmatrix}$$

- $x = x_1, x_2, x_3, x_4$ Potentials at nodes

ohm constant, current source

$A^T C A x = f$

potential differences

$A^T y = 0$ KCL $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

PD $e = Ax$ → current $y = Ce$ (OHM's LAW) → KCL

$x_2 - x_1$, etc potential differences

currents y_1, y_2, y_3, y_4, y_5 on edges

- $x = c\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ 이면 $Ax=0$, $\dim N(A) = 1$

↳ basis

- ($x_4 = 0$ 로 두면 (ground))

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

rank=3

- $A^T y = 0$ $\dim N(A^T) = m - r = 2$

$n \times m$
4×5

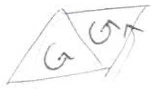
$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -y_1 - y_3 - y_4 &= 0 \\ y_1 - y_2 &= 0 \\ y_2 + y_3 - y_5 &= 0 \\ y_4 + y_5 &= 0 \end{aligned}\right\}$$ KCL을 의미함!

$\Sigma = 0$.

- Basis for $N(A^T)$

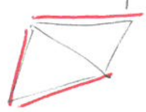
$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$



$$\cancel{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}$$

3가지 loop 이므로... 3개가 아니고 2개임. (전체 loop는 두개 합)

- column space of A^T



* 3 independent columns (1, 2, 4 th col of A^T)

* No loops → Tree (∴ graph with no loops)

※ $\dim N(A^T) = m - r$

\# loops = # edges − (# nodes − 1)

(∵ rank = $n-1$)

\# nodes − # edges + # loops = 1

Euler's formula

ex) $5 - 7 + 3 = 1$

ecture 13, Quiz 1 review

① u, v, w in $\mathbb{R}^7$

② 6x3 $\mathcal{U}$, pivot $r=3$

$\to$ nullspace = [0]

$N(\mathcal{U}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

③ $B = \begin{bmatrix} \mathcal{U} \\ 2\mathcal{U} \end{bmatrix}$ echelon form $\to \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix}$ rank of B = 3

$C = \begin{bmatrix} \mathcal{U} & \mathcal{U} \\ \mathcal{U} & 0 \end{bmatrix} \to \begin{bmatrix} \mathcal{U} & \mathcal{U} \\ 0 & -\mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{U} & 0 \\ 0 & +\mathcal{U} \end{bmatrix}$

rank of C = 6

~~dim(C) = 6~~

C 10x6

$\to \dim N(C^T) = 10-6 = 4$

$\cancel{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ rref

④ $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3x4 $\qquad$ X $\qquad$ Y

basis for $N(B) \subseteq \mathbb{R}^4$

X가 invertible 이면 $N(B) = N(Y)$

$\dim(N(B)) = 2$

basis for $N(B) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Complete solution $Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$x_p + x_N = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

⑤ T F

If A, B same 4 subspaces, then $A = cB$

example : A, B any invertible 6x6

⑥ $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ $\quad x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3x3

rank (A) = 1

dim N(A) = 2

$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

⑦ $Ax = b$ can be solved if b is in C(A).

b has the form $b = c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Don't forget other cases $r=m, r=n$

⑧ $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can't be in nullspace and ~~row space.~~ be a row of A

$\to$ False.

$\begin{bmatrix} 1 & 2 & 3 \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

⑨ False.

$B^2 = 0 \to B = 0$

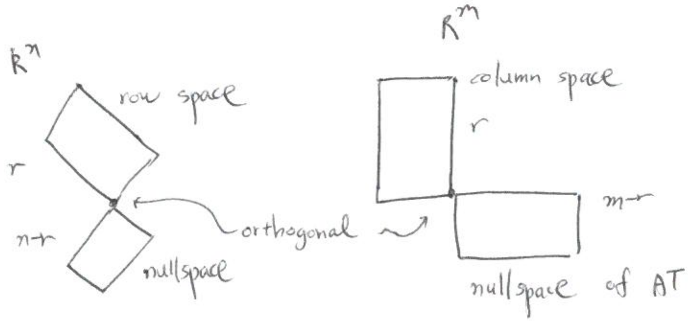
$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Lecture 14: Orthogonal vectors and subspaces

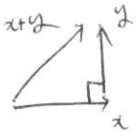
∘ Orthogonal vectors and subspaces

nullspace ⊥ row space

$N(A^TA) = N(A)$



∘ Orthogonal vectors



$x^Ty = 0$) equivalent to orthogonality

피타고라스

$\|x\|^2 + \|y\|^2 = \|x+y\|^2$

$x^Tx + y^Ty = (x+y)^T(x+y)$

$= x^Tx + y^Ty + x^Ty + y^Tx$

$0 = \cancel{2}x^Ty$

$\therefore x^Ty = 0$

∘ Subspace S is orthogonal to subspace T means

: every vector in S is orthogonal to every vector in T.

[단 intersection 은 원점 (point) 하나여야 할 듯]

∘ Row space is orthogonal to nullspace

why? $Ax = 0$
$$\begin{bmatrix} \text{row 1 of } A \\ \text{row 2 of } A \\ \vdots \\ \text{row } m \text{ of } A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

from equation ...

$(c_1 \text{row}_1 + c_2 \text{row}_2 + \dots)^T x = 0$

∘ χ. row space 와 nullspace 의 dimension 합은 항상 n 임 $(r + n - r = n)$

∘ nullspace and row space are orthogonal complements in R^n. Nullspace contains all vectors ⊥ row space.

∘ Coming : $Ax = b$ when there is no solution

($m > n$) ① many (a lot of) equations and few unknown ... too many ($m > n$)

해결

② Thousand measurement, 계속, 수천번 관측하여 equation 수가 많은 경우

- 해결법 ... ?

: A^TA 의 성질 : symmetric, 항상 invertible 하지는 않음!

: 양쪽에 A^T 곱하면 ...

$A^TA\hat{x} = A^Tb$ (hat)

best solution

☆ ∘ χ. ($N(A^TA) = N(A)$

rank of A^TA = rank of A)

A^TA is invertible exactly if A has independent columns

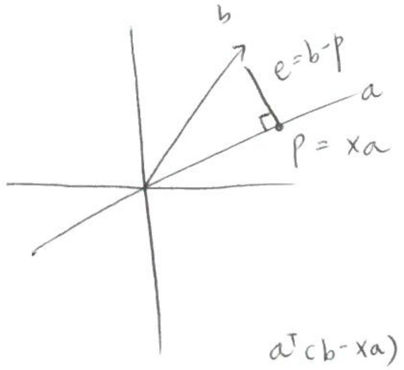
from

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

Lecture 15 : Projections onto subspaces

- Projections
  Least squares
  Projection matrix

- 

$a^T(b-xa)=0$ ... ①

formula 유도했음.

↳ ①로 부터 식 다 유도가능.

$xa^Ta = a^Tb$

$x = \frac{a^Tb}{a^Ta}$, $p = ax$

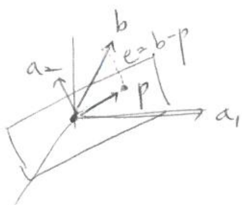
$p = a\frac{a^Tb}{a^Ta}$

$\text{Proj}\ p = Pb$, Matrix $P = \frac{aa^T}{a^Ta}$

(위의)

- P의 성질 $C(P)$ = line through a
  $\text{rank}(P) = 1$ (one dimension)
  $P^T = P$, $P^2 = P$

- Why project?
  Because $Ax=b$ may have no solution.
  [b가 C(A)에 없기 때문에 proj(b) 해서 근사 해 을 구함 (best solution)]
  Solve $A\hat{x} = p$ ~ instead proj of b onto column space

- 

- plane of a_1, a_2 (basis)
  = column space of A
  $= \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

- $e=b-p$ is perpendicular to plane.

- $p = \hat{x}_1 a_1 + \hat{x}_2 a_2$
  $p = A\hat{x}$

- $p = A\hat{x}$, Find $\hat{x}$

key : $\underset{e}{(b-A\hat{x})}$ is perp. to the plane.

- $a_1^T(b-A\hat{x}) = 0$ $\quad a_2^T(b-A\hat{x}) = 0$

- $\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b-A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

즉, $A^T\underbrace{(b-A\hat{x})}_{e} = 0$

- e in $N(A^T)$
  $e \perp C(A)$ → Yes!

$A^TA\hat{x} = A^Tb$ 이므로

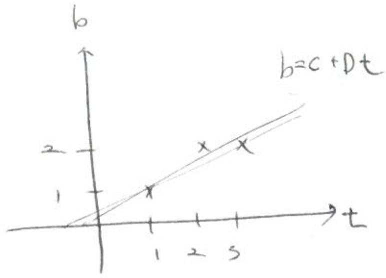
유도과정 ✓ ... ①과 동일한 의미인

- $\hat{x} = (A^TA)^{-1}A^Tb$
  $p = A\hat{x} = A(A^TA)^{-1}A^Tb$ ← [$(A^TA)^{-1}$은 존재한다고 가정하는듯]
  matrix $P = A(A^TA)^{-1}A^T$

*P의 성질 $P^T = P$, $P^2 = P$ 보일 수 있음.

- Least squares
  Fitting by a line

$(1,1)$ $(2,2)$ $(3,2)$



find the best line ...

$C + D = 1$
$C + 2D = 2$
$C + 3D = 3$

can't solved.

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

no solution. best solution은 b를 C(A)에 project 시켜 $\hat{x}$ 구한것임.

(양변에 A^T 곱하기 !)

$A^TA\hat{x} = A^Tb$

Lecture 16: Projection matrix and least squares

- Projections
- Least squares and best straight line

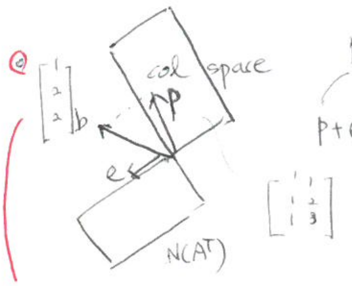
- Proj. Matrix

$P = A(A^TA)^{-1}A^T$

If b in column space $Pb = b$

If $b \perp$ column space $Pb = 0$

exactly same!



Pb

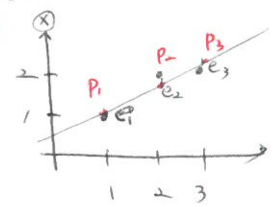
$p + e = b$ $(I-P)b$

↳ proj onto ⊥ space

Fig 1. Vectors

Fig 2. line eplane)

outlier



$C + Dt$

$C + D = 1$ b_1

$C + 2D = 2$ b_2

$C + 3D = 2$ b_3

- $Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- Minimize error $\|Ax-b\|^2 = \|e\|^2$ (least square)

└ 자주 사용 (not always)

$= e_1^2 + e_2^2 + e_3^2$

$= (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$ … ①

- ※ fitting straight line : regression problem (linear)

- Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$A^TA\hat{x} = A^Tb$, $P = A\hat{x}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

$3C + 6D = 5$

$6C + 14D = 11$ … ② normal equation

- ① : calculus 편미분으로 C, D 구하거나

② : linear algebra

derivative 하면 ① → ②

②로 부터 $2D = 1$

$D = 0.5$, $C = \frac{2}{3}$

→ $P_1 = \frac{7}{6}$, $P_2 = \frac{5}{3}$, $P_3 = \frac{13}{6}$

$-e_1 = \frac{1}{6}$, $e_2 = +\left(\frac{1}{3}\right)$, $-e_3 = \frac{1}{6}$

옆 그림 참조 Fig 1

$p + e = b$ … ①

$$\begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

- perpendicular

- e is perpendicular to column space of A.

- If A has independent columns, then A^TA is invertible

- Proof: Suppose $A^TAx = 0$, $x = 0$ 을 보이면 A^TA는 invertible 임을 증명할 수 있다. (Nullspace에 zero 만 있은경우, A^TA는 invertible 임.)

* inverse는 elimination 해서 n개의 pivot이 남았을 때 존재함. (nullspace에 0만 있는것과 동치) chap 2.5 참고

$x^TA^TAx = 0 \cdot x^T$

$= 0$

$\underbrace{(Ax)^TAx}_{} = 0$

Ax square 이므로 …

$Ax = 0$

A는 independent 한 column을 가지므로 … (free variable 없음)

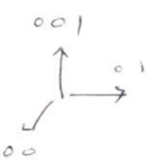
$x = 0$

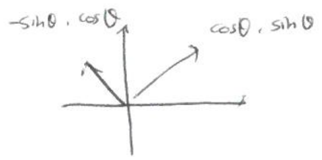
∴ $x = 0$, A^TA는 invertible. …②

- Next time's subject

: Columns definitely independent if they are perpendicular unit vectors. = orthonormal vectors

ex)





Lecture 17. Orthogonal matrices and gram-Schmidt.

ㄴ 책 보기 ?

- Orthogonal basis $q_1, \cdots, q_n$
  Orthogonal matrix Q (: square) ✓ not to be square matrix!
  Gram-Schmidt $A \to Q$

- orthonormal vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$- \; Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad Q^TQ = \begin{bmatrix} - q_1^T - \\ \vdots \\ - q_n^T - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = I$$

- If Q is square then $Q^TQ = I$ tells us $Q^T = Q^{-1}$

* Example

Permutation $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\underset{Q}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{Q^T}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}}$$

* $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

* $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

* rectangular one ... $Q = \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

- Q has orthonormal columns.

Project onto its column space.

$$P = Q(Q^TQ)^{-1}Q^T = QQ^T \; \{= I \text{ if } Q \text{ is square}\}$$

$A^TA\hat{x} = A^Tb$, Now A is Q

$$\underbrace{Q^TQ}_{I}\hat{x} = Q^Tb$$

* Inverse 필요없게됨! $(A^TA)^{-1}$

$$\hat{x} = Q^Tb$$

$$\boxed{\therefore \hat{x}_i = q_i^Tb}$$

dot product

- Gram-Schmidt 만들기 ?
- independent vectors $a, b, c \to$ orthogonal $\to$ orthonormal
  A, B $\qquad q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|}$
  $q_3 = \frac{C}{\|C\|}$

이해안되면 Lecture 15. Q 참조!

* $\underline{e} = $

$$\boxed{B = b - \frac{A^Tb}{A^TA}A} \;, \qquad A^TB = A^T\left(b - \frac{A^Tb}{A^TA}A\right) = 0$$

$$C = c - \underbrace{\frac{A^Tc}{A^TA}A}_{\text{A direction}} - \underbrace{\frac{B^Tc}{B^TB}B}_{\text{B direction}}$$

헷갈리면 ok.

$C \perp A$, $C \perp B$

- example

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\to B = b - \frac{a^Tb}{a^Ta}a$$

$$= \underset{b}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} - \underset{\frac{A^Tb}{A^TA}}{\frac{3}{3}} \underset{A}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\therefore B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad q_1 = \frac{A}{\|A\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\to Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

* elimination 을 Gram-Schmidt 에 적용.

$A = LU$, $A = QR$ (A, R은 서로의 linear combination)

$A = LU$ → elimination, $A = QR$ → expression of Gram-schmidt

책보기 : 236 page

$A = QR$

정사영이라 생각 내적 ?

모양 찾아보기

$$\underset{a \quad b}{\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} a_1^Tq_1 & a_2^Tq_1 \\ \underset{=0}{a_1^Tq_2} & a_2^Tq_2 \end{bmatrix}$$

ㄴ Triangular

p.233

Lecture 18 : Properties of Determinants

- Determinants $\det A = |A|$
- Properties 1, 2, 3, 4 - 10
- ± signs

- Properties ① $\det I = 1$

  ② Exchange rows ; reverse sign of det

  ↳↳ $\det P = 1$ or -1 임을 알수 있음.

- ① $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ (※. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$)

  ② $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

③. for any matrix ...

$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

linear comb of 1st row

$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

→ determinant is linear (function) for each row

④ 2 equal rows → $\det = 0$

└ Can be proved by ②. $\det A = \det A'$

⑤ Subtract $l \times$ row i from row k (elimination)

Det Doesn't change.

[$\det A = \det U$]

- $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ \frac{-la}{-l} & \frac{-lb}{-l} \end{vmatrix}$ (from ③)

↳ $-l \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$

⑥ Row of zeros → $\det A = 0$

(from ③ take $t = 0$, $\begin{vmatrix} 0 \cdot a & 0 \cdot b \\ c & d \end{vmatrix} = 0 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$)

⑦ $U = \begin{bmatrix} d_1 & \times & \times & \times \\ 0 & d_2 & \times & \times \\ 0 & 0 & \ddots & \times \\ 0 & 0 & 0 & d_n \end{bmatrix}$, $\det U = d_1 \times d_2 \times \cdots \times d_n$

product of pivots

... from ⑤, ③ $d_3 d_2 d_1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, ①

$\times\times\times \to 0$

I

⑧ $\det A = 0$ when A is singular

- (row = 0 일때 singular ... ?)

  └ singular 이면 0인 row 존재.

- $\det A \neq 0$ when A is invertible

  $A \to U \to D \to d_1 d_2 \cdots d_n$

  (invertible 이면 elimination 해서 n pivot 존재)

- 수식유도가능!

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - b \cdot \frac{c}{a} \end{vmatrix}$

$= a \cdot (d - b \cdot \frac{c}{a})$

$= ad - bc$

⑨ $\det AB = (\det A)(\det B)$

- $\det A^{-1} = (\det A)^{-1}$ 유도가능!

ex) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, → $A^{-1} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{vmatrix}$

⑨ $\det 2A = 2^n \det A$ ($n \times n$ matrix)

증명 P. 249 참고

⑩ $\det A^T = \det A$

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

- column 이 0일 때 $\det A = 0$ 임을 유도가능!

- Proof) $|A^T| = |A|$

$|U^T L^T| = |LU|$ ... chapter 2

$|U^T||L^T| = |L||U|$ ... ⑨

lower triangular matrix with diagonal 1

L, U

L U

elimination 으로 동일해질?

Lecture 19. Determinant formulas and Cofactors.

- Formula for det A ($n!$ terms)
- Cofactor formula
- Tridiagonal matrices

∘ ① det $I = 1$
② sign reverse with row exchange
③ determinant is linear for each row

- 2x2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$

$= \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{\text{col}=0} + \underbrace{\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}}_{ad} + \underbrace{\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}}_{-bc} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{\text{col}=0}$

col = 0 이므로 ad -bc col = 0

row exchange

※ p. 257 참고

- 3x3 에서도 가능

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

[n^n 중에 $n!$만 살아남음.
↓ → 인 것들 모두 0] — (1)

Permutation matrix

$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{31}a_{23}a_{12} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

∘ Big formula

- $\det A = \underbrace{\sum}_{n!\text{ terms}} \pm a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$ (half positive, half negative)

$(\alpha, \beta, \gamma, \cdots, \omega)$ = Permutation of $(1, 2, \cdots, n)$

- Example

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1 - 1 = 0$$

→ singular

$(4, 3, 2, 1) \to +1$ 2 exchange

$(3, 2, 1, 4) \to -1$ 1 exchange

∘ Cofactors 3x3 in parens

- $\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$
$\quad + a_{12}(-a_{21}a_{33} + a_{23}a_{31})$
$\quad + a_{13}(\qquad)$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

- Cofactor of $a_{ij} = C_{ij}$

$\pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \text{, col } j \text{ erased} \end{pmatrix}$

$\begin{bmatrix} + \text{ if } i+j \text{ even} \\ - \text{ if } i+j \text{ odd} \end{bmatrix} \rightarrow \begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$

∘ Cofactor formula (along row 1 / i)

- $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
$\qquad\quad i1 \qquad\qquad\qquad\qquad in$

- 적용 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d + b \cdot (-c)$

유도과정 책 참고

내용 중에 쓰면서 복습 필요. 3가지 방법

- Example $\xrightarrow{A_3}$ Tridiagonal matrix

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1, \ |A_2| = 0, \ |A_3| = -1$

$|A_4| = 1 \cdot |A_3| - 1 \cdot |A_2|$

$|A_n| = |A_{n-1}| - |A_{n-2}| \ \cdots$ ①

from ①

→ $|A_5| = 0, \ |A_6| = 1, \ |A_7| = 1 \ \cdots$

Lecture 20. Cramer's Rule, Inverse matrix, and volume

- Formula for A^{-1}
- Cramers Rule for $x = A^{-1}b$
- $|\text{Det } A| =$ Volume of box

app 1 //

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

→ $\boxed{A^{-1} = \frac{1}{\det A} C^T}$ ← products n-1 entries

products of n entries

- Check $AC^T = (\det A)\, I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & & \\ C_{12} & & \\ \vdots & & \\ C_{1n} & & \end{bmatrix} =$$

→ 곱하면 det A가 됨을 알 수 있다!

$(= a_{i1}C_{i1} + a_{i2} + \cdots + a_{in}C_{in})$

$$= \begin{bmatrix} \det A & & & \\ & \det A & & \\ & & \ddots & \\ & & & \det A \end{bmatrix}$$

→ 나머지는 0이 됨.
다른 cofactor를 곱하는것은 중복된 column을 선택하는것이기 때문.

$= \frac{1}{(\det A)} I$

- $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

$\det|A_s| = ab - ab$
$= 0$

= ! (first and last rows are identical 이기 때문에) '0')

P.264 증명 → 앞에 쓰여져 있음 (?) → 책 보면서 정리

app2 ✓

- $Ax = b$

$x = A^{-1}b = \frac{1}{\det A} C^T b$

→ $C_{11}b_1 + C_{21}b_2 + \cdots$

- Cramer's rule

$x_1 = \frac{\det B_1}{\det A}$

$x_2 = \frac{\det B_2}{\det A}$

$x_j = \frac{\det B_j}{\det A}$

$B_1 = \begin{bmatrix} | & n-1 \\ b & \text{columns} \\ | & \text{of } A \end{bmatrix}$

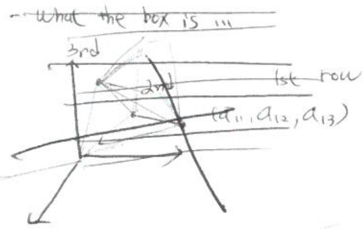
A with column) replaced by b

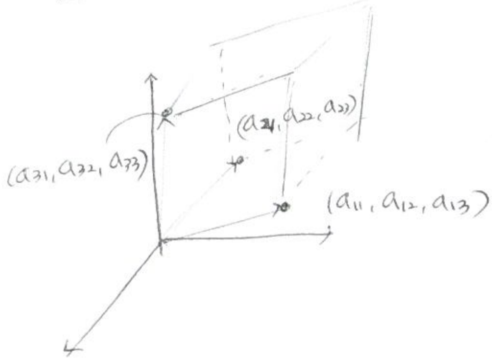
$B_j = A$ with column j replaced by b

다른 증명도 이용 (책)

app3

- absolute value
$|\text{Det } A| =$ Volume of box





- A가 Identity matrix 일 경우 box = unit cube (Property 1)
- Orthogonal matrix Q 일 경우, A = Q
  → another cube
  unit cube 이고 volume = 1.

  └ columns are orthonormal.

  Q^T 로 생각 각각 column을 row로 보고 생각해도됨!

→ $Q^TQ = I$ 이므로

$\det|Q^TQ| = \det|I|$

$\det|Q^T|\det|Q| = 1$) property 10

$|Q|^2 = 1$

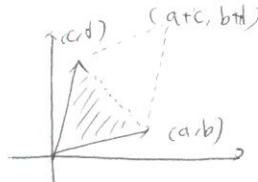
$\therefore |Q| = \pm 1$

$\det Q = \pm 1$

(Volume 도 Property ①, ② 만족)

⇒ cube 일 경우, det A = volume 임을 증명.

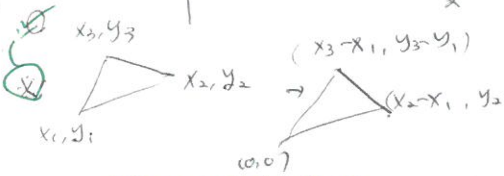
- Rectangular 일 경우 ...
  - 하나의 row가 2배가 된 경우, $\det|A'| = 2\det|A|$
  → volume은 property ③-ⓐ 도 만족.
  - $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ ... Prop ③-b



$\det|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

area $= \det|A| = ad - bc$

$\frac{1}{2}(ad-bc)$ for triangle



(원점으로 이동해서 계산)

Lecture 21: Eigenvalues and Eigenvectors

- Eigenvalues - Eigenvectors
  - $\det[A - \lambda I] = 0$
  - TRACE $= \lambda_1 + \lambda_2 + \cdots + \lambda_n$

Eigenvectors

- Ax parallel to x
  matrix vector
  function으로 보기

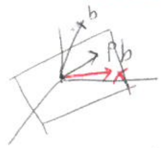
실수값 vector x가 존재하지 않는

정리 A가 singular 이면 $Ax=b$ 인 x가 존재하지 않고 $Ax=0$ 를 만족시키는 x가 $\neq [0]$ nullspace 에 존재한다 ?
즉 $Ax=0$

- $\boxed{Ax = \lambda x}$
  e value, e vector

동치인
singular : takes x into 0, $Ax=0$
3강 참고

- If A is singular, $\lambda = 0$ is eigenvalue



- what are x's and λ's for projection matrix
  - Any x in plane : $Px = x$, $\lambda = 1$
  - Any $x \perp$ plane : $Px = 0x$, $\lambda = 0$

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ... ①
  $\lambda = 1$, $\lambda = -1$

reverse two components
↳ row operation. 2강 참고.

- Fact : sum of λ's $= a_{11} + a_{22} + \cdots + a_{nn}$
  $n \times n$ matrix → n개의 λ 존재
  (추가공부 이부내용 찾아보기)

- How to solve $Ax = \lambda x$
- Rewrite : $(A - \lambda I)x = 0$
  ↳ 0이 아닌 vector를 구하기.
  must be singular   0 → useless
  → $\det(A - \lambda I) = 0$, find λ first
  λ 구한뒤 elimination 으로 x 구하면 됨
  freevariables 1, 0 으로 두는 과정 ...

- $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)^2 - 1 = 0$

$= \lambda^2 - 6\lambda + 8$

$\lambda_1 + \lambda_2 = 6$ - trace

$\lambda_1 \lambda_2 = 8$ - det A

$\lambda_1 = 2$, $\lambda_2 = 4$

$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \lambda_1 = 2$

$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \leftarrow \lambda_2 = 4$ ... ②

elimination 후 free variable 1로 두기, back substitution !

- ①, ② 를 비교해보면
  - λ 에 3I를 더하면 λ_1', λ_2' 이 됨 !
  - eigen vector 는 변화없음
  matrix 에 3I를 더했음.

- $(A + 3I)x = \lambda x + 3x = (\lambda + 3)x$

- Not so great, A+B, AB
  If $Ax = \lambda x$, B has eigenvalues α,
  $B x = \alpha x$
  y y 일 수 있음.
  $(A+B)x \neq (\lambda + \alpha)x$

extreame case ...
- Example $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ trace : $0 + 0 = \lambda_1 + \lambda_2$
  90° rotation   det $= 1 = \lambda_1 \lambda_2$

- $\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

complex number !

$\therefore \begin{bmatrix} \lambda = i \\ \lambda = -i \end{bmatrix}$ & conjugate each other.

⓪ $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}$

$(3-\lambda)^2 = 0$

$\begin{bmatrix} \lambda_1 = 3 \\ \lambda_2 = 3 \end{bmatrix}$

triangular 인 경우 eigenvalue가 diagonal에 있음 !

- $(A - \lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 =$ No 2nd independent eigenvector.

Lecture 22. Diagonalization and Powers of A. → 추가복습하기

- Diagonalizing a matrix $S^{-1}AS = \Lambda$

  Powers of A / equation $u_{k+1} = Au_k$

- $A-\lambda I$ singular, $Ax=\lambda x$

- Suppose n indep. eigenvectors of A — S^{-1} 존재!

  Put them in column of S

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & \lambda_n \end{bmatrix}$$

diagonal eigenvalue matrix Λ

$$= S\Lambda$$

- $AS = S\Lambda$

  $S^{-1}AS = \Lambda$

  ($A = S\Lambda S^{-1}$) new factorization (LU or QR ...)

If

- $Ax=\lambda x$,

i) $A^2x = \lambda Ax = \lambda^2 x$

ii) $A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1})$

$= S\Lambda^2 S^{-1}$

iii) $A^k = S\Lambda^k S^{-1}$

- Theorem

  $A^k \to 0$ as $k\to\infty$

  if all $|\lambda_i| < 1$

— 책에 증명있음. P. 300 (이해필요)

- Diagonalizable, A is sure to have n indep e-vectors (and be diagonalizable) ✓ 예로부터 이해. proof?

  if all the λ's are different (no repeated λ's) // 증명하기 쉬움

- Repeated eigen values // may or may not have n indep e-vectors

- $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\det|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$

  $\lambda = 2, 2$

즉 $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

∴ There are some matrices don't cover "$S\Lambda S^{-1}$"

◎ Equation $u_{k+1} = Au_k$ (p. 284, 285 읽기)

Start with given vector u_0

$u_1 = Au_0$, $u_2 = A^2u_0$, $u_3 = A^3u_0$, ..., $\boxed{u_k = A^k u_0}$

✓ — To really solve: write — expand to eigen vector

$u_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n = Sc$ P. 314

initial value — ~~column~~ linear combination 이기 때문에 c_1, c_2, 등 붙여줌.

$Au_0 = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n$

$A^{100}u_0 = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2 + \cdots + c_n\lambda_n^{100}x_n$

$u_{100} = \Lambda^{100}Sc$

- Example : Fibonacci sequence ; 0, 1, 1, 2, 3, 5, ...,

  $F_{k+2} = F_{k+1} + F_k$ ... ① $F_{k+1} = F_{k+1}$ ... ②

- Trick $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$

$= \lambda^2 - \lambda - 1 = 0$

$\lambda = \dfrac{1\pm\sqrt{5}}{2}$, ($\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \approx 1.618...$, $\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \approx -0.618...$)

- How fast Fibonacci increases?

  $F_{100} \approx c_1\left(\frac{1+\sqrt{5}}{2}\right)^{100}$ ($+ c_2\frac{1}{2}(1-\sqrt{5})^{100}$) extremely small → disappear!

- $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} \lambda \\ 1 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

  ↳ $x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

  ↳ $u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1x_1 + c_2x_2$

Lecture 23: Differential Equations and exp(At)

Differential Eqns $\frac{du}{dt} = Au$

Exponential e^{At} of a matrix

∘ Example

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\left(\begin{array}{l} \frac{du_1}{dt} = -u_1 + 2u_2 \\ \frac{du_2}{dt} = u_1 - 2u_2 \end{array} \right.$

$\rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ singular!

$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix}$

$= \lambda^2 + 3\lambda = 0$

$\lambda = 0, -3$

$\lambda_1 = 0$, $\lambda_2 = -3$

$t \to \infty$

steady state e^0 | disappear e^{-3}

- $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $Ax_1 = 0x_1$ $\left(A - \lambda I = \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} \right)$
- $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Ax_2 = -3x_2$ $\left(A - \lambda I = \begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} \right)$

★★★ p.313 참고 2가지 받아들이기.

→ **Solution**: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 \approx c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 : u_{k+1} = Au_k$

check: $\frac{du}{dt} = Au$ Plug in $e^{\lambda_1 t} x_1$, $\lambda_1 e^{\lambda_1 t} x_1 = A e^{\lambda_1 t} x_1$

$(\lambda_1 x_1 = Ax_1)$

$= c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

→ Use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t=0$

↳ $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{3}$

steady state $u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

∘ ① Stability $u(t) \to 0$ / need $e^{\lambda t} \to 0$ / $\text{Re}\,\lambda < 0$

$|e^{(-3+6i)t}| = e^{-3t}$ $(|e^{6it}| = 1)$

② Steady state

$\lambda_1 = 0$ and other $\text{Re}\,\lambda < 0$

③ Blow up if any $\text{Re}\,\lambda > 0$

∘ 2x2 stability $\text{Re}\,\lambda_1 < 0$, $\text{Re}\,\lambda_2 < 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: trace $a + d = \lambda_1 + \lambda_2 < 0$

det > 0

$(= \lambda_1 \lambda_2)$

- trace < 0, still blow up

$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ e^{-2t}, e^{t}

∘ $\frac{du}{dt} = Au$, Set $u = Sv$ ← evector matrix

$S \frac{dv}{dt} = ASv$

$\frac{dv}{dt} = S^{-1} A S v$

$= \Lambda v$

$\frac{dv_1}{dt} = \lambda_1 v_1$, ⋮

$v(t) = e^{\Lambda t} v(0)$

$u(t) = S e^{\Lambda t} S^{-1} u(0)$ $\quad e^{At} = S e^{\Lambda t} S^{-1}$

∘ Matrix exponential $e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$

similar thing in matrix

※ $e^x = \sum_0^\infty \frac{x^n}{n!}$, $\frac{1}{1-x} = \sum_0^\infty x^n$

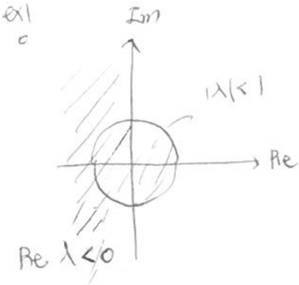
$e^{At} = I + \underbrace{S\Lambda S^{-1}}_{A} t + \frac{S\Lambda^2 S^{-1}}{2} t^2 + \cdots$

$e^{At} = \boxed{S e^{\Lambda t} S^{-1}}$

(assumption: A can be diagonalize)

$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$

ex1



ex2

∘ $y'' + by' + ky = 0$

$u = \begin{bmatrix} y' \\ y \end{bmatrix}$, $u' = \begin{bmatrix} y'' \\ y' \end{bmatrix}$

$\begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$

Lecture 24a : Markov matrices , Fourier Series

- Markov matrices
(steady state : $\underline{\lambda = 1}$
Fourier series Projections

- $A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & 0 & .4 \end{bmatrix}$

- ① All entries ≥ 0 (Probability)

  ② All columns add to 1
  ↳ gurantee eigenvalue is 1.

- key
  1. $\lambda = 1$ is an eigenvalue
  2. All other $|\lambda_c| < 1$

  ($*$ $u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \cdots$)

  if $\lambda = 1$, $|\lambda_c| < 1$, $u_k \to c_1 x_1$
  steady state
  (x_1 part of u_0)

→ $A - 1I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$

①을 증명하기

└ All colums add to zero → $A - I$ is singular
(row1 + row2 + row3 = 0 이므로 row는 dependent 하다. left nullspace 에 (1, 1, 1) 존재)
$n(A^T)$
then x_1 (eigenvector) is in $n(A)$
(singular 이므로)

중요한 것

- eigenvalues of A
  eigenvalues of A^T) are the same ?

Proof) $\det(A - \lambda I) = 0$
$\det|A| = \det|A^T|$ 이므로 (Property 10)
$\det(A^T - \lambda I) = 0$

- 또다른 eigenvector 찾기

$A - \lambda \cdot I = \begin{bmatrix} -.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix} \begin{bmatrix} .6 \\ 33 \\ .7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

계산으로 찾음!

- $u_{k+1} = A u_k$, A is Markov

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_k$$

- $\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

- after 1 time step,

$$\begin{bmatrix} u_{cal} \\ u_{mas} \end{bmatrix}_{k,1} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

- Use eigenvectors , eigenvalues to solve u_{100} .

$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 0.7$

$\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ($\lambda_1 = 1$ 일 때)

x_1 - eigenvector .

$\begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ($\lambda_2 = 0.7$ 일 때)

x_2 eigenvector

- $u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ 이고

$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$c_1 = \frac{1000}{3}$, $c_2 = \frac{2000}{3}$

$\therefore u_k = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} (.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

steady state

- Projections with orthonormal basis (Before Fourier ...)
expantion $q_1, \cdots, q_n$

Any vector v; $v = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
Q $\quad$ X

$q_1^T v = x_1 q_1^T q_1 = x_1$

dot product

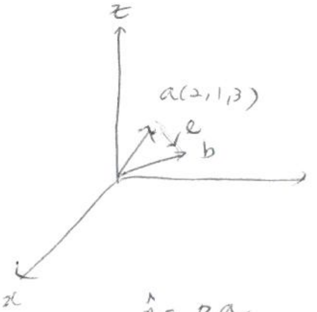
$X = Q^{-1} v = Q^T v$

$x_1 = q_1^T v$

Lecture 24b : Quiz 2 Review

1.

(a) Find matrix P that projects every vector b in R^3 onto the line in the direction of $a = (2,1,3)$



$\hat{x} = pa$

$e = b - pa$

$a^T(b - pa) = 0$

$a^T(b - pa) = 0$

$a^T(b - \hat{x}) = 0$

$a^T\hat{x} = a^Tb$

$pa^Ta = a^Tb$

$p = \frac{a^Tb}{a^Ta}$

$\hat{x} = \frac{a^Tb}{a^Ta}a$

$= a\frac{a^Tb}{a^Ta}$

$= \underbrace{\frac{aa^T}{a^Ta}}_{P} b$

실수

$\hat{x} = pa$

$e = b - \hat{x}$

$= b - pa$

$a^T(b - \hat{x}) = 0$

$a^Tb = a^T\hat{x} = a^Tpa$

$\hat{x} = \frac{a^Tb}{a^T}$

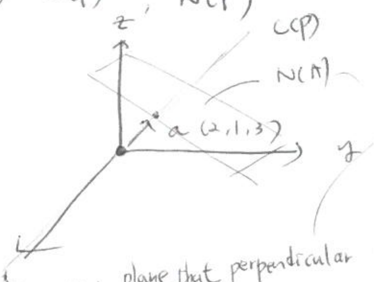
$\therefore P = \frac{\cancel{4+1+9}}{\cancel{4+1+9}} = \frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

$aa^T = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

$\therefore P = \frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

(b) $C(P)$, $N(P)$



N(P) : plane that perpendicular to a

$PX = 0$

$\begin{bmatrix} 4 & 2 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}X = 0$

$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}X$

$2X_1 + X_2 + 3X_3 = 0$.

• basis

$C(P)$: $(2,1,3)$

$N(P)$: $(-\frac{3}{2}, 0, 1)$, $(-\frac{1}{2}, 1, 0)$

(c) What are all eigenvectors of P and their corresponding eigenvalues? The diagonal entries of P add up to <u>1</u>.

$Px = \lambda x$ 의 vector x → eigenvector

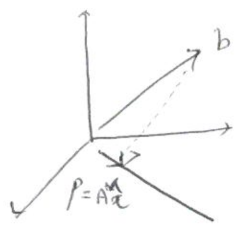
lambda λ → eigenvalue

• $\left(\begin{array}{ll} x_1 = (2,1,3) & x_2 = (-3,0,2) \\ \lambda_1 = 1 & \lambda_2 = 0 \end{array} \right)$

• $\lambda_1 + \lambda_2 = 1$

2.

(a) $p = A\hat{x}$



$e = b - A\hat{x}$, $p = A\hat{x}$

Q $\hat{x} = \cancel{AA^T(A^TA)^{-1}b}$

$\hat{x} = A(A^TA)^{-1}A^Tb$

① All the columns in matrix A are perpendicular (or in $C(A)$) to error vectors.

② If the columns of A are dependent,

(b) $A = QR$

orthonormal columns

upper triangle

$\hat{x} = A(A^TA)^{-1}A^Tb$

$p = A\hat{x}$

$A = QR$ 이므로 $\hat{x} = QR(R^TQ^TQR)^{-1}(QR)^Tb$

$A^T = (QR)^T$ $= QR(R^TR)^{-1}R^TQ^Tb$

$=$ $=$

$$R^TR = \begin{bmatrix} q_1^Ta & 0 & 0 \\ q_1^Tb & q_2^Tb & 0 \\ q_1^Tc & \cancel{q_2^Tc} & q_3^Tc \end{bmatrix}\begin{bmatrix} q_1^Ta & q_1^Tb & q_1^Tc \\ 0 & q_2^Tb & q_2^Tc \\ 0 & 0 & q_3^Tc \end{bmatrix}$$

$$= \begin{matrix} q_1^Ta & \cancel{ab} & ac \\ ab & \cancel{2b^2} & \cancel{2bc} \\ ac & \cancel{2bc} & 2c^2 \end{matrix}$$

(c) q_1, q_2 - orthonormal vectors in R^5

3. $A_n = \text{eye}(n) - \text{ones}(n)$

(a) Find the determinant of A_n

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} = A_3$$

$-2 \quad -2 \quad -2$

Lecture 25 : Symmetric matrices and positive definiteness

- Symmetric matrices
  - Eigenvectors / Eigenvalues
  - Start : Positive Definite Matrices

- $A = A^T$

① The eigenvalues are REAL

② The eigenvectors are PERPENDICULAR — 책 증명 참조
can be chosen → orthonormal eigenvectors, columns of Q

- Usual case $A = S\Lambda S^{-1}$

symmetric case $A = Q\Lambda Q^{-1} = \boxed{Q\Lambda Q^T}$ — p. 330 증명.

Proof

- why real eigenvalues? $(\overline{a+ib} = a - ib)$

$Ax = \lambda x \xrightarrow{\text{always}} A\bar{x} = \bar{\lambda}\bar{x} \rightarrow \bar{x}^T A^T = \bar{x}^T \bar{\lambda}$

$\bar{x}^T A x = \lambda \bar{x}^T x \cdots$ ①

$A^T = A$ (if A is symmetric)

$\bar{x}^T A x = \bar{x}^T \bar{\lambda} x \cdots$ ②

①, ②에 의해 $\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$ → (length)^2

$\therefore \lambda = \bar{\lambda}$, λ is real.

※ $\bar{x}^T x = [\bar{x}_1 \ \bar{x}_2 \ \cdots \ \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots$

$\bar{x}_1 x_1$: $(a-ib)(a+ib) = a^2 + b^2$

- Good matrices
  - real λ's
  - perpendicular x's

$A = \bar{A}^T$ (if complex)
$A = A^T$ (if real : symmetric)
— 예쁘한 real 일. symmetric이 가장 좋은 property 중 하나임

- $A = A^T \rightarrow A = Q\Lambda Q^T$ p. 332

$= \begin{bmatrix} q_1 & q_2 & \cdots \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \end{bmatrix} = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots$

→ Every symm matrix is a comb of Perpendicular Projection matrices. (p. 334)

┌ Symmetric 일 경우에만 성립. det $= \lambda_1 \cdots \lambda_n = d_1 \cdots d_n$

* Useful tips for computing eigenvalues
- Signs of pivots same as signs of λ's
- # pos pivots = # positive λ's

- Positive definite matrix (symmetric)

  - all eigenvalues are positive.
  - all pivots are positive
  - all subdeterminants are positive

ex) $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$ pivots $5, \frac{11}{5}$

eigenvalues
$\lambda^2 - 8\lambda + 11 = 0$
$\lambda = 4 \pm \sqrt{5}$

$\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$

Lecture 26 : Complex Matrices ; Fast Fourier Transform

- Complex vectors, matrices
  - inner products
  - Discrete Fourier
  - Fast Transform = FFT
  - Fourier matrix F_n

- $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ lenth in C^n $\quad \bar{z}^T z$ is ↑ no good

  $z^H z$ (Hermitian)

$[\bar{z}_1 \ \bar{z}_2 \ \cdots \ \bar{z}_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

(※ $\bar{z}_1 z_1 = |z_1|^2$)

ex) $[1 \ -i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 1+1 = 2$

→ inner product $\bar{y}^T x = y^H x$

- Symmetric $A^T = A$ no good if A complex

  $\bar{A}^T = A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$

  Hermitian $A^H = A$ ... Complex 에서는 symmetric 대신 Hermitian 필요.

- Perpendicular

  $q_1, q_2, \cdots, q_n$

  $\bar{q}_i^T q_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$

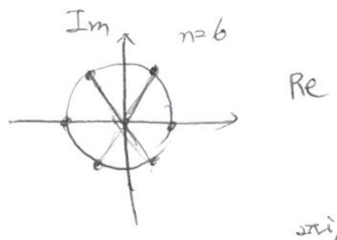
  $Q = [q_1 \ q_2 \ \cdots \ q_n]$

  $\bar{Q}^T Q = I = Q^H Q$ Orthogonal Unitary

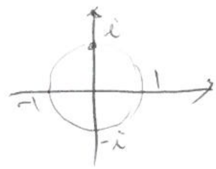
- $F_n = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & & w^{(n-1)^2} \end{bmatrix}$

- $(F_n)_{ij} = w^{ij}$ $\quad$ $(i, j = 0, \cdots, n-1)$

- $w^n = 1$, $w = e^{i2\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$



- $n=4$ 일 경우 $w^4 = 1$, $w = e^{2\pi i/4} = i$



- $F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & -i \end{bmatrix}$

  - columns are orthogonal, orthonormal

  → $F_4^H F_4 = I$

  $(F_4)^{-1} = F_4^H$

- FFT $\quad$ P. 509 ~ 514 $\quad$ (P 512) 공식임...

- $(W_{64})^2 = W_{32}$ $\quad$ (from $w = e^{i2\pi/n}$)

- $[F_{64}] = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$ Permutation

  └→ 64^2 $\quad$ $2(32)^2 + 32$ $\qquad$ $D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \ w^{31} \end{bmatrix}$

→ ∴ $\frac{1}{2} n \log_2 n$ steps

$n = 1024 = 2^{10}$

($n^2 > 1{,}000{,}000$ $\quad$ $1024 \cdot 1024$

Lecture 27 : Positive definite Matrices and minima

- Positive Definite Matrix (Tests)
- Tests for minimum ($x^TAx > 0$)
- Ellipsoids in R^n

Test ...

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

① $\lambda_1 > 0$, $\lambda_2 > 0$

② $a > 0$, $ac - b^2 > 0$

③ pivots $a > 0$, $\frac{ac-b^2}{a} > 0$

★ ④ $x^TAx > 0$

- Examples

$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$

- Pivots

2

- border line ... pos semidefinite , $\lambda = 0, 20$

$\underset{x^T}{[x_1\ x_2]} \underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = [x_1 \quad x_2] \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$

$= 2x_1^2 + 12x_1x_2 + 18x_2^2 > 0$

$(ax^2 + 2bxy + cy^2)$

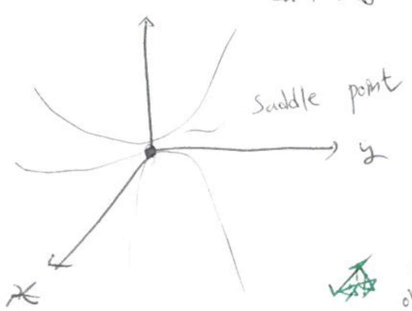
quadratic form

- Graph of $f(x, y) = \vec{x}^TA\vec{x}$

$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

$= ax^2 + 2bxy + cy^2$

$2x^2 + 12xy + 7y^2$ not positive definite matrix



이해필요

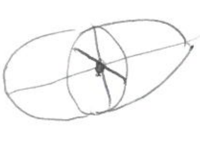
if $f(x_1, x_2, x_3) = 1$, ellipsoid (football)

$= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

$Q \Lambda Q^T = A$

eigenvector tells direction

eigenvalues " the length



- Graph of $f(x, y) = \vec{x}^TA\vec{x}$

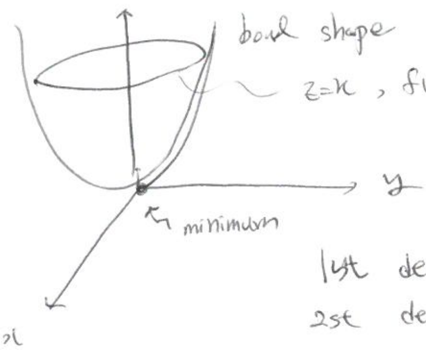
$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ → positive definite matrix

- $\lambda_1 > 0$ $\lambda_2 > 0$
- $a > 0$ $ac - b^2 > 0$
- pivots > 0
- ① $x^TAx > 0$ except minimum point at $x = 0$

→ $2x_1^2 + 12x_1x_2 + 20x_2^2 = f(x_1, x_2)$

$f(x, y) = 2x^2 + 12xy + 20y^2$

$= 2(x + 3y)^2 + 2y^2$



1st derivative $= 0$

2st derivative > 0

(from Calculus)

※ Calculus : Min - $\frac{d^2u}{dx^2} > 0$

18.06 Min ~ Matrix of 2nd $f(x_1, x_2, \cdots)$ derivs is pos def

→ $A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ → $\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

복습필요 $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$f(x, y) = 2x^2 + 12xy + 20y^2$

$= 2(x + 3y)^2 + 2y^2$

pivots

- Matrix of Second derivative

$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

cross derivative

Second derivative of x direction

Second derivative of y direction

- 3x3 example 복습필요

$A = \begin{bmatrix} +2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

dets $2, 3, 4$ linearly ↑

pivots $2, \frac{3}{2}, \frac{4}{3}$

eigenvalues $2-\sqrt{2}, 2, 2+\sqrt{2}$

$f = x^TAx = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$

Lecture 28 : Similar Matrices and Jordan form

- A^TA is positive definite !

Similar matrices A, B / Jordan form

$B = M^{-1}AM$

- Positive definite means $x^TAx > 0$ (except for $x=0$)

If A, B are pos def, A+B ~~are~~ is pos def.

$x^T(A+B)x > 0$, $x^TAx > 0$ $x^TBx > 0$

$\boxed{\text{rank}=n}$ independent column

- Now A m by n, A^TA square, symmetric

Pos def ... 일 수 밖에 없음 !

$x^TA^TAx = (Ax)^TAx = \|Ax\|^2 > 0$

(only if $x=0$, $Ax=0$)

n×n matrices

- A and B are similar means ; for some M

$B = M^{-1}AM$

- Example : A is similar to Λ

$S^{-1}AS = \Lambda$

eigenvector matrix

- $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}_{M^{-1}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{M} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix}$$

inverse의 직관적 이해. 이해하도록

$$= \underbrace{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}_{B}$$

A and B have same eigenvalues ! $\lambda = 3, 1$

→ Similar matrices have same λ's ???

family 도 있음. $\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$. ⊕ same number of independent eigenvectors

- Why... ? B has also same eigenvalues.

Proof) $Ax = \lambda x$ $(B = M^{-1}AM)$

$AMM^{-1}x = \lambda x$

$(M^{-1}AM)M^{-1}x = M^{-1}\lambda x = \lambda M^{-1}x$

$BM^{-1}x = \lambda M^{-1}x$

eigenvector of B is M^{-1} (eigenvector of A)

- Bad case $\lambda_1 = \lambda_2$, Can't be diagonalizable

- $\lambda_1 = \lambda_2 = 4$, One family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

→ $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = 4I$ 인 경우 family 는 자기 자신밖에 없음...

$M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}M = 4I$ One small family.

→ $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ ← Jordan form ☆☆☆ 개념 ⊕ 문제 풀어보면서 이해해야함...

→ More members of family

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}$,

$\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & 8-a \end{bmatrix}$

det = 16, trace = 8 } family of matrices similar

- Ex) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank = 2, dimension of nullspace = 2, dim N(A)

$\lambda = 0, 0, 0, 0$ 2 eigenvectors

Jordan block J_i =

Ax = 0 eigenvector is in the nullspace.

Ex) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank = 2, dim N(A) = 2, 2 eigenvectors

$\lambda = 0, 0, 0, 0$

Not similar to each other.

blocks = # eigenvectors

Jordan's theorem — diagonalize 의 일반화 버전.

: Every square matrix A is similar to a Jordan matrix J. $J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_d \end{bmatrix}$

Good Case : J is Λ

Lecture 29: Singular Value decomposition

○ Singular Value Decomposition = SVD

$A = U\Sigma V^T$ // Σ diagonal, U, V orthogonal

{ Symmetric, Positive, similar
$A = Q\Lambda Q^T$ (symm), ($\Lambda > 0$ if A pos)
~~$A = S\Lambda S^{-1}$~~

[illegible]

○ R^n row space column space R^m

∴ Find orthogonal basis
- start with Gram-Schmit

· $A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \end{bmatrix}$

Goal

· $AV = U\Sigma$

ex) $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

Find
orthonormal
v_1, v_2 in row space R^2
u_1, u_2 in col space R^2
$\sigma_1 > 0$, $\sigma_2 > 0$
orthonormal

$Av_1 = \sigma_1 u_1$
$Av_2 = \sigma_2 u_2$

- How to find v, u, σ

$A = U\Sigma V^{-1} = U\Sigma V^T$ p.363

$A^TA = V\Sigma^T U^T U\Sigma V^{-1}$

$A^TA = V\Sigma^2 V^{-1}$ can find V eigenvector

$\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix}$ $\Sigma^2 \rightarrow$ eigenvalues

· $A^TA = \begin{bmatrix} 4 & -3 \\ 4 & +3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & +3 \end{bmatrix}$

$= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

eigenvector $\frac{1}{2}$ (orthonormal)
$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ($\frac{32}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = Ax$)
$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ($\frac{18}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = Ax$)

$$\rightarrow \underset{A}{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}} \underset{V^T}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}$$

Σ

→ Find u's (u_1, u_2)

p.363 $AA^T = U\Sigma V^T V \Sigma^T U^T$
$= U \Sigma\Sigma^T U^{-1}$ eigenvalues
eigenvectors

· $AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$

eigenvector $AA^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$AA^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

※ (eigenvalues are same AB, BA)

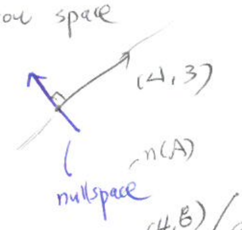
$$\rightarrow \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & \\ & \sqrt{18} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

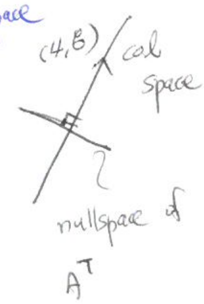
ex 2

Rank 1 matrix

Rectangular

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$





$$v_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

?? − p.363

$$\rightarrow \underset{A}{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}} = \underset{u}{\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V_2}{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}}$$

※ Σ 구하기

$$A^TA = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

$\lambda_1 = 0$, $\lambda_2 = 125$

(singular 이므로)

- $v_1, \cdots, v_r$ orthonormal basis for row space
- $u_1, \cdots, u_r$ orthonormal basis for column space
- $v_{r+1} \cdots v_n$ orthonormal basis for null space $n(A)$
- $u_{r+1} \cdots u_n$ orthonormal basis for $n(A^T)$

&

- $Av_i = \sigma_i u_i$

Lecture 30, Linear Transformations and Their Matrices

◦ Linear Transformations T

without coordinates : no matrix

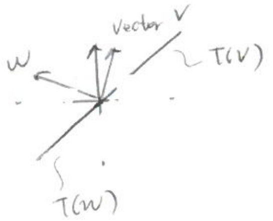
with coordinates → MATRIX

$T(v+w) = T(v) + T(w)$
$T(cv) = cT(v)$
} Rule of linear transformation

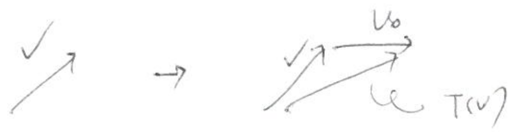
Transformation의 예

◦ Example 1. Projection

$T: R^2 \to R^2$ (function 이라 볼 수 있음)



◦ Example 2. Shift whole plane by v_0



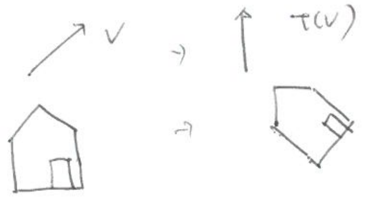
Rule 1,2 둘다 안맞음 !

◦ Other Non-example

$T(v) = \|v\|$, $T: R^3 \to R^1$

◦ Example 2. Rotation by 45° → (o)

$T: R^2 \to R^2$



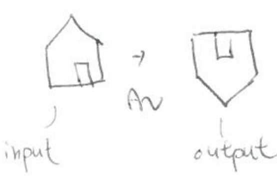
◦ Example 3!! Matrix A → (o)

$T(v) = Av$

$T(v+w) = Av + Aw$

$T(cv) = cAv$

if $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



* linear transformations

└ matrix muliplication

p. 375

◦ Start : $T: R^3 \to R^2$ p. 384

Example : $T(v) = Av$ ← 2×3 matrix

output in R^2 input in R^3

* Information needed to know $T(v)$ for all inputs

$T(v_1), T(v_2), \cdots, T(v_n)$ for any basis $v_1, \cdots, v_n$ └ input

[Because every $v = c_1v_1 + \cdots + c_nv_n$ (Comb of basis vectors)]

Know $T(v) = c_1T(v_1) + \cdots + c_nT(v_n)$ // p. 384

basis에 T가 어떻게 적용되는지 알면 모든 vector에 대해 알수 있음.

* Coordinates come from a basis

" of $v = c_1v_1 + \cdots + c_nv_n$

ex) $v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

coordinate

◦ Construct matrix A that represents lin tr T.

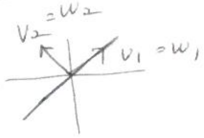
$T: R^n \to R^m$

Choose basis $v_1, \cdots, v_n$ for inputs R^n

" $w_1, \cdots, w_m$ for outputs R^m

WANT matrix A

ex) Projection



$v = c_1v_1 + c_2v_2$

$T(v) = c_1v_1$

inputs (c_1, c_2) ⇓ outputs $(c_1, 0)$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{\text{input coords}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\text{output coords}}{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}$

∘ Eigenvector basis leads to diagonal matrix Λ.

∘ Proj onto 45° line

use standard $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$

$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$

Proj $P = \dfrac{aa^T}{a^Ta} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

~ 이해안됨. 몇 장 맨마지막 ex 보면됨.

∘ Rule to find A. Given bases $v_1 - v_n$, $w_1 - w_m$

1st column of A ; Write $T(v_1) = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$

2nd column of A ; $T(v_2) = a_{12} w_1 + \cdots + a_{m2} w_m$

$\left(A \begin{pmatrix} \text{input} \\ \text{coords} \end{pmatrix} = \begin{pmatrix} \text{output} \\ \text{coords} \end{pmatrix} \right)$

∘ $T = \dfrac{d}{dx}$ (Takes derivative)

Input : $c_1 + c_2 x + c_3 x^2$ basis : $1, x, x^2$

output : $c_2 + 2c_3 x$ basis : $1, x$

deriv

linear !

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$$

A should be 2 by 3

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\left(\because \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix} \right)$$

◎ Linear transformation (derivative, inverse, rotation 등) 은 모두 matrix multiplication 으로 표현될수 있다.

Lecture 31. Change of Basis, Image Compression

- Change of Basis
- Compression of Images
- Transformation $\Leftrightarrow$ Matrix

- Pixel x_i (ith), image 512×512

$0 \le x_i \le 255$

8 bits

$x \in \mathbb{R}^n$, $n = (512)^2$

(3×512^2 if color RGB img)

- JPEG

$x = \begin{bmatrix} 121 \\ 124 \\ \vdots \end{bmatrix}$ 512^2, $x \in \mathbb{R}^n$, $n = 512^2$

「 basis 바꾼 뒤
불필요한 정보들
날리고
일부분으로 전체와 유사한
이미지 (또는 음성) 표현 가능.
signal processing 분야 」

- Standard basis

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

- Better basis

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}}_{\text{half 1, half -1}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$$

~ FFT 복습하기 (이전 lecture 일…)

- Fourier bais 8×8

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ w \\ w^2 \\ \vdots \\ w^{n-1} \end{bmatrix}$$

64 coeff

512 × 512

signal x (p)

$\downarrow$ change basis (lossless)

coeffs c

$\downarrow$ compression (lossy)

$\downarrow$

$\hat{c}$ (many zeros)

$\downarrow$

$\hat{x} = \sum \hat{c}_i v_i$ ($x = Fc$)

* video : sequence of images

higly correlated

smooth changes

P. 391~392 개념, 원리, 설명

- Wavelets 8×8 다시읽은.

$$c_1\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} c_2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} c_3\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdots$$

= Standard basis

$$\underbrace{\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_8 \end{bmatrix}}_{\text{Pixel values}}$$

lossless step

$\rightarrow p = c_1 w_1 + \cdots + c_8 w_8$

$$\underset{p}{p} = \underset{W}{\begin{bmatrix} 1 & 1 & & \\ 1 & 1 & & \\ 1 & 1 & \cdots & \\ 1 & -1 & & \\ 1 & -1 & & \\ 1 & -1 & & \end{bmatrix}} \underset{c}{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}}$$

c: coefficient

* $p = Wc$

$c = W^{-1}p$

GOOD BASIS

① Fast FFT, FWT (wavelet)

wavelets : { columns are orthogonal each other
binary values. easy to compute }

② Few is enough (to reproduce image)

- Change of basis

Columns of W = new basis vectors

$[x] \longrightarrow [c]$

old basis new basis

$x = Wc$ (matrix, basis)

- T with respect to $v_1, \cdots, v_8$

it has matrix A

with respect to $w_1, \cdots, w_8$

it has matrix B

similar

Similar $B = M^{-1}AM$

of basis ?

* what is A? Using basis $v_1, \cdots, v_8$.

Know T completely from $T(v_1), T(v_2), \cdots, T(v_8)$

Because every $x = c_1v_1 + \cdots + c_8v_8$

Then $T(x) = c_1T(v_1) + c_2T(v_2) + \cdots + c_8T(v_8)$

$$\left.\begin{aligned} \text{Write } T(v_1) &= a_{11}v_1 + a_{21}v_2 + \cdots + a_{81}v_8 \\ T(v_2) &= a_{12}v_1 + a_{22}v_2 + \cdots + a_{82}v_8 \end{aligned}\right\} 2$$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & \\ | & | & \cdots \\ a_{18} & a_{28} & \end{bmatrix}$$

* Suppose $T(v_i) = \lambda_i v_i$ (Eigenvector basis)

$$A = \begin{bmatrix} \lambda_1 & 0 & & & \\ 0 & \lambda_2 & & & \\ 0 & 0 & \lambda_3 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \lambda_8 \end{bmatrix}$$

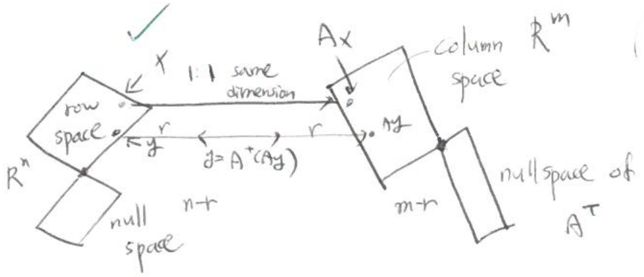
1st input is v_1
its output is $\lambda_1 v_1$

2nd input is v_2
" output " $\lambda_2 v_2$

Lecture 33. Left and Right Inverses, Pseudoinverse

역행렬을 모두 찾을 수 있다

- 4 subspaces
  - Left-inverses
  - Right-inverses
  - Pseudo-inverses



- 2-sided inverse
  - $AA^{-1} = I = A^{-1}A$ $(r = m = n)$ full rank (Chap 2)

- left inverse
  - full column rank $r = n < m$
    - nullspace $= \{0\}$
    - indep. cols / 0 or 1 solns to $Ax = b$

→ A^TA invertible, full rank

- $\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}} A = I$ (Crucial to least squares)

$\underset{n\times m}{A^{-1}_{left}}\ \underset{m\times n}{A} = I_{n\times n}$

- left 에 위치.

- right inverse
  - full row rank $r = m < n$
    - $N(A^T) = \{0\}$ indep. rows
    - ∞ sols to $Ax = b$, $n-m$ free variables

→ AA^T invertible

- $A\underbrace{A^T(AA^T)^{-1}}_{A^{-1}_{right}} = I$

* $A(A^TA)^{-1}A^T$: Projection onto column space
  $A^T(AA^T)^{-1}A$: " " row space

→ If x, y in row space then $Ax \neq Ay$. (from Chap 3)

Proof) $x \neq y$, if x, y is in row space,

$Ax - Ay = A(x-y) = 0$

$x - y$ is in null space

but $x - y$ is in row space. (paradoxx)

∴ x, y in row space

→ $Ax \neq Ay$.

How to

- Find the pseudoinverse A^+

$$\underset{m\times n}{\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & \sigma_r & \\ 0 & & & 0 \end{bmatrix}}$$ rank r, m rows

① Start from SVD : $A = U\Sigma V^T$

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & 0 \\ & \ddots & 1/\sigma_r & \\ 0 & & & 0 \end{bmatrix}_{n\times m}$$

- $\Sigma\Sigma^+ = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}$ $m\times m$

- $\Sigma^+\Sigma = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}$ $n\times n$

- $A^+ = V\Sigma^+U^T$

* pseudoinverse : 7.4

※ 32, 34, Quiz 2 들어야 할 것.