Relaxed LMI-Based Stability Conditions for Takagi–Sugeno Fuzzy Control Systems Using Regional-Membership-Function-Shape-Dependent Analysis Approach

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***Abstract*—This paper presents relaxed stability conditions for Takagi–Sugeno (T-S) fuzzy-model-based control systems. It is assumed that the stability conditions are represented by some inequalities in the form of a p-dimensional fuzzy summation. To investigate the system stability, the inequalities are expanded to n-dimensional fuzzy summation $(n \geq p)$. The boundary and regional informations of membership functions are then utilized for relaxation of stability analysis results. Two analysis approaches are proposed in this paper. The first approach is called the global-membership-function-shape-dependent approach, in which the lower and upper bounds of the membership functions, and its products from 2 to n in the full operating domain, are considered in the stability analysis. The second approach is named as the regional-membership-function-shape-dependent approach in which the operating region is partitioned to subregions, and the boundary information of membership functions on each operating subregion is employed to facilitate the stability analysis. In both approaches, by the help of the boundary and/or regional information of the membership functions, some inequality constraints in the form of multidimensional fuzzy summation containing some slack matrices are constructed. Stability conditions in the form of linear matrix inequalities (LMIs) are derived. Numerical examples are given to demonstrate the effectiveness of the proposed stability conditions.**

***Index Terms*—Fuzzy control, linear matrix inequality (LMI), membership-function-shape-dependent (MFSD) stability condition, relaxed quadratic stability.**

I. Introduction

THE FUZZY-MODEL-BASED (FMB) control approach offers a promising platform to investigate the system stability and facilitate controller synthesis for nonlinear systems. The Takagi–Sugeno (T-S) fuzzy model provides an effective mathematical tool to describe complex nonlinear systems in a general framework [1]–[3]. Based on the T-S fuzzy model, a fuzzy controller is employed to close the feedback loop to form an FMB control system [4]. The most common stability analysis approach for the FMB control systems is based on the Lyapunov stability theory [5]. In addition, parallel distributed compensation (PDC) design technique [4], [6] and convex optimization techniques, such as linear matrix inequality (LMI) [7], play an important role to make the FMB control approach applicable.

In the FMB control, one of the issues that draws a great deal of attention is the stability analysis. An attempt was made to relax stability conditions of T-S FMB control system in [8] with consideration of membership functions subject to the PDC design. Thereafter, some promising approaches have been proposed to relax the conservatism in stability analysis [9]–[14]. Polya's theorem was applied in [14] to provide a set of less-conservative sufficient conditions in the form of LMIs. In the mentioned approach, the stability analysis was generalized by expanding inequalities of p-dimensional fuzzy summation to an n-dimensional ones $(p \leq n)$. In addition, this approach can be used to relax various fuzzy controls represented by the inequalities of multidimensional fuzzy summation. For instance, the problem of designing a fuzzy static output feedback controller [13], a twin PDC (TPDC) for fuzzy descriptor systems [15], and a robust static output feedback controller for nonlinear discrete time interval systems with time delays [16] are some cases with stability conditions in the form of 3-D fuzzy summation. In this case, further relaxation of stability conditions can be achieved via Polya's theorem.

The importance of membership functions for stability analysis of FMB control systems was revealed in [4]. It was shown in [17]–[21] that the shape of the membership functions plays an important role in relaxing the stability conditions. In general, two categories of membership-function-shape-dependent (MFSD) analysis approach, namely, the non-PDC [17]–[19] and PDC [20], [21] approaches, have been reported. The upper bound constraints on multiplication of each of the two membership functions were employed in [20] to relax the LMI stability conditions for FMB control systems of 2-D fuzzy summation. This approach has been generalized for FMB control systems with multidimensional fuzzy summation relaxed by polynomial constraints [21].

In this paper, the system stability of FMB control systems in the form of p-dimensional fuzzy summation is investigated. The principle stability conditions are achieved by expanding the inequalities of p-dimensional fuzzy summation to the ones of n-dimensional fuzzy summation [14]. In order to relax the stability conditions, two approaches are proposed, namely, the global-MFSD (GMFSD) and the regional-MFSD (RMFSD) approaches. In the GMFSD approach, both lower and upper bound constraints of each membership function and multiplication of two, three, four, ..., and n of them on the full operating region are constructed and brought to the stability analysis for relaxation of stability conditions. In the RMFSD approach, the operating region of membership functions is first partitioned into subregions. Corresponding to each subregion, similar to the GMFSD approach, the boundary information of the membership functions are employed in the stability analysis. It can be seen that the RMFSD approach has advantages over the GMFSD one. Due to consideration of the constraints of membership functions on different subregions, further information of membership functions is brought to the stability analysis, thus resulting in further relaxation of stability conditions. To illustrate the merits of the RMFSD approach, consider the operating region of $\mu_2(z(t))$ (one of the membership functions) in Fig. 1. Under the GMFSD approach, for instance, the constraints for this particular membership function are $\mu_2(z(t)) - \gamma_{(2)} \geq 0$ and $\mu_2(z(t)) - \beta_{(2)} \leq 0$. The values of $\gamma_{(2)}$ and $\beta_{(2)}$ are the lower and upper bounds of the membership function of $\mu_2(z(t))$ in the global range. Under the RMFSD approach, the operating regions for the membership functions are divided into a number of regions. Referring to Fig. 1, it is divided into eight regions. Considering region R_2, the lower and upper bound constraints are $\mu_2(z(t)) - \gamma_{(2),R_2} \geq 0$ and $\mu_2(z(t)) - \beta_{(2),R_2} \leq 0$, for $z(t) \in R_2$, respectively. In this case, compared with the GMFSD approach, more information of membership functions can be utilized to facilitate the stability analysis and lead to less-conservative stability conditions.

The rest of this paper is organized as follows. In Section II, necessary notations and some published stability analysis results for FMB control systems are represented. In Section III, the proposed GMFSD and RMFSD approaches are presented. In Section IV, numerical examples are given to illustrate the achieved improvements. In Section V, a conclusion is drawn.

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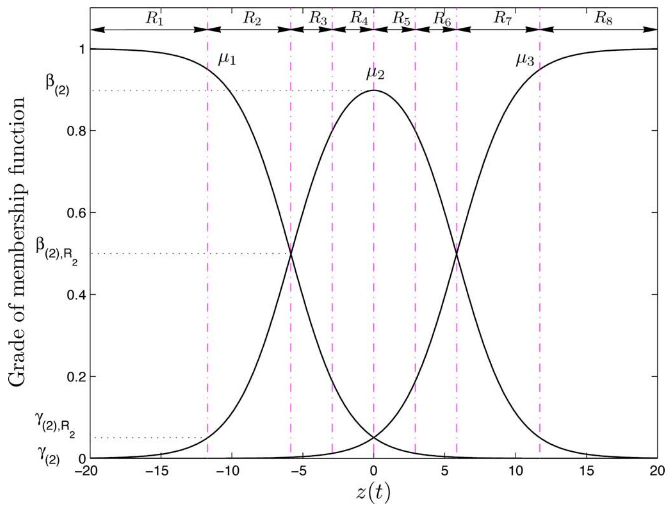

Fig. 1. Membership functions for Simulation Example 1.

II. Notations and Previous Stability Conditions

In this section, some existing notations [14] used in this paper are recalled, and some new ones are defined.

Suppose $\mathbb{I}_m$ is defined as

$$\mathbb{I}_m = \{i(m) = (i_1, i_2, i_3, \ldots, i_m) \in \mathbb{N}^m \mid 1 \le i_k \le r\} \quad (1)$$

for $k = 1, \ldots, m$, in which r is the number of rules in a fuzzy system. The notations of $i(m)$ and $\mathbb{I}_m^+$ are defined as follows:

$$i(m) = (i_1, i_2, i_3, \ldots, i_m) = (i(m-2), i_{m-1}, i_m) \quad (2)$$

$$\mathbb{I}_m^+ = \{i(m) \in \mathbb{I}_m \mid i_1 \le i_2 \le \cdots \le i_m\}. \quad (3)$$

In this paper, matrices are denoted in the form of $\mathbf{\Lambda}_{i(m)}$, where $i(m)$ denotes the index of the matrix. Considering $i(m) \in \mathbb{I}_m^+$, $\mathcal{P}(i(m))$ is defined as the set of all of the permutations of $i(m)$, and considering the integers $c \le m$ and $k \le m$, the following notions are defined:

$$j(c) \in \mathcal{P}(i(m)) \triangleq \{(j_1, j_2, \ldots, j_c) \mid j_1 = i_1, \ldots, j_c = i_c \\ (i_1, i_2, \ldots, i_c, i_{c+1}, \ldots, i_m) \in \mathcal{P}(i(m))\} \quad (4)$$

$$\mathcal{P}_k^+(i(m)) \triangleq \{j(k) \in \mathcal{P}(i(m)) \mid j(k) \in \mathbb{I}_k^+\}. \quad (5)$$

The product of m membership functions is denoted as

$$\mu_{i(m)} \triangleq \prod_{k=1}^{m} \mu_{i_k}. \quad (6)$$

The m-dimensional summation operators are defined as

$$\sum_{i(m)\in\mathbb{I}_m} \triangleq \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_m=1}^{r} \quad (7)$$

$$\sum_{i(m)\in\mathbb{I}_m^+} \triangleq \sum_{i_1=1}^{r}\sum_{i_1\le i_2\le r}\cdots\sum_{i_{m-1}\le i_m\le r}. \quad (8)$$

Suppose $\{\mu_1(\mathbf{z}(t)), \ldots, \mu_r(\mathbf{z}(t))\}$ is a set of fuzzy partition fulfilling the following conditions:

$$\sum_{i_1=1}^{r} \mu_{i_1}(\mathbf{z}(t)) = 1, \qquad \mu_{i_1}(\mathbf{z}(t)) \in [0, 1]. \quad (9)$$

With respect to (6), (7), and (9), the following equality is satisfied:

$$\sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}(\mathbf{z}(t)) = 1. \quad (10)$$

Now consider the following dynamic nonlinear system represented by a continuous-time FMB control system [1]:

$$\dot{\mathbf{x}}(t) = \sum_{i_1=1}^{r} \mu_{i_1}(\mathbf{z}(t))(\mathbf{A}_{i_1}\mathbf{x}(t) + \mathbf{B}_{i_1}\mathbf{u}(t)) \quad (11)$$

where $\mathbf{x}(t) \in \mathbb{R}^v$, $\mathbf{u}(t) \in \mathbb{R}^q$, r is a positive integer denoting the number of subsystems, $\mathbf{z}(t)$ is the premise vector, and $\mathbf{A}_{i_1} \in \mathbb{R}^{v\times v}$ and $\mathbf{B}_{i_1} \in \mathbb{R}^{v\times q}$ are the constant system and input matrices. To close the feedback loop of the nonlinear plant represented by the T-S fuzzy model in the form of (11), the PDC technique [4], [6] is employed to design the fuzzy controller. The PDC fuzzy controller is defined as follows:

$$\mathbf{u}(t) = -\sum_{i_2=1}^{r} \mu_{i_2}(\mathbf{z}(t))\mathbf{K}_{i_2}\mathbf{x}(t) \quad (12)$$

where $\mathbf{K}_{i_2} \in \mathbb{R}^{v\times q}$ is the feedback gain of the $i_2\mathrm{th}$ rule. From (11) and (12), the FMB control system is defined as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} \mu_{i_1}(\mathbf{z}(t))\mu_{i_2}(\mathbf{z}(t))(\mathbf{A}_{i_1} - \mathbf{B}_{i_1}\mathbf{K}_{i_2})\mathbf{x}(t). \quad (13)$$

The most common sufficient stability or performance conditions used in a wide range of literature [9], [13], [22] for (13) are in the form of the inequality of 2-D fuzzy summation given as

$$\Xi_2(t) = \sum_{i(2)\in\mathbb{I}_2} \mu_{i(2)}\mathbf{x}(t)^T\mathbf{Q}_{i(2)}\mathbf{x}(t) > 0. \quad (14)$$

Consider the quadratic Lyapunov function candidate $V(t) = \mathbf{x}(t)^T\mathbf{P}^{-1}\mathbf{x}(t)$ where $\mathbf{P} \in \mathbb{R}^{v\times v}$ is a positive symmetric definite matrix. Obtaining the first time derivative of $V(t)$ and with the FMB control system of (13), the inequality of (14) can be achieved, where $\mathbf{Q}_{i(2)} = \mathbf{Q}_{(i_1,i_2)} = -(\mathbf{A}_{i_1}\mathbf{P} + \mathbf{P}\mathbf{A}_{i_1}^T - \mathbf{B}_{i_1}\mathbf{H}_{i_2} - \mathbf{H}_{i_2}^T\mathbf{B}_{i_1}^T)$. Feedback gains are derived as $\mathbf{K}_{i_2} = \mathbf{H}_{i_2}\mathbf{P}^{-1}$ for all i_2, and $\mathbf{H}_{i_2} \in \mathbb{R}^{q\times v}$ are the arbitrary matrices [22].

In the following, some published relaxed LMI-based stability conditions are recalled. The system stability of the FMB control systems of (13) was investigated in [20], which was facilitated by the upper bound information of the membership functions. The stability conditions in terms of LMIs are summarized in the following lemma. For brevity, μ_i, $\mathbf{x}$, and $\mathbf{u}$ are used to denote $\mu_i(\mathbf{z}(t))$, $\mathbf{x}(t)$, and $\mathbf{u}(t)$, respectively.

Lemma 1 [20]: Given the upper bound constraints fulfilling $\mu_{(i_1,i_2)} \le \beta_{(i_1,i_2)}$, where $\beta_{(i_1,i_2)}$ are prior determined scalars for $i_1 \le i_2 \le r$ (overlapping for every two membership functions). The inequality of (14) holds if there exist matrices $\mathbf{X}_{(i_1,i_2)} = \mathbf{X}_{(i_2,i_1)}^T \in \mathbb{R}^{v\times v}$ and $\mathbf{N}_{(i_1,i_2)} = \mathbf{N}_{(i_1,i_2)}^T \in \mathbb{R}^{v\times v}$, where $i_1 \le i_2$, such that the following LMIs are satisfied:

$$\mathbf{Q}_{(i_1,i_1)} + \mathbf{N}_{(i_1,i_1)} - \sum_{i_1=1}^{r} \sum_{i_1 \le i_2 \le r} \beta_{(i_1,i_2)} \mathbf{N}_{(i_1,i_2)} \ge \mathbf{X}_{(i_1,i_1)}$$

$$\mathbf{Q}_{(i_1,i_2)} + \mathbf{Q}_{(i_2,i_1)} + \mathbf{N}_{(i_1,i_2)} - 2\sum_{i_1=1}^{r} \sum_{i_1 \le i_2 \le r} \beta_{(i_1,i_2)} \mathbf{N}_{(i_1,i_2)}$$
$$\ge \mathbf{X}_{(i_1,i_2)} + \mathbf{X}_{(i_2,i_1)}, \qquad i_1 < i_2$$
$$\mathbf{N}_{(i_1,i_2)} \ge 0, \qquad i_1 \le i_2$$
$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{(1,1)} & \cdots & \mathbf{X}_{(1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{(r,1)} & \cdots & \mathbf{X}_{(r,r)} \end{pmatrix} > 0 \tag{15}$$

Proof: See [20]. ∎

The aforementioned lemma is the relaxed stability conditions with 2-D fuzzy summation with consideration of upper bound information of membership functions. In the next section, the system stability of FMB control systems with p-dimensional fuzzy summation relaxed by consideration of boundary information of membership functions is considered. Hence, before proceeding further, the technique expanding a p-dimensional fuzzy summation to a higher dimensional one is presented. Consider $\Xi_p(t)$ as follows:

$$\Xi_p(t) = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_p=1}^{r} \mu_{i_1}\mu_{i_2}\cdots\mu_{i_p}\mathbf{x}^T\mathbf{Q}_{(i_1,i_2,\cdots,i_p)}\mathbf{x}$$
$$= \sum_{i(p)\in\mathbb{I}_p} \mu_{i(p)}\mathbf{x}^T\mathbf{Q}_{i(p)}\mathbf{x} \tag{16}$$

where $\mathbf{Q}_{i(p)}$ is a symmetric square matrix. From (10) and (16), $\Xi_p(t)$ can be cast in the following form:

$$\Xi_n(t) = \left(\sum_{i(n-p)\in\mathbb{I}_{n-p}} \mu_{i(n-p)}\right)\Xi_p(t) = \sum_{i(n)\in\mathbb{I}_n} \mu_{i(n)}\mathbf{x}^T\mathbf{Q}_{i(p)}\mathbf{x}$$
$$= \sum_{i(n)\in\mathbb{I}_n^+} \mu_{i(n)}\mathbf{x}^T\tilde{\mathbf{Q}}_{i(n)}\mathbf{x} \qquad \forall\, n \ge p \ge 2 \tag{17}$$

where $\tilde{\mathbf{Q}}_{i(n)}$ is defined as

$$\tilde{\mathbf{Q}}_{i(n)} = \sum_{j(p)\in\mathcal{P}(i(n))} \mathbf{Q}_{j(p)}, \qquad i(n) \in \mathbb{I}_n^+. \tag{18}$$

Lemma 2 [14]: Sufficient conditions for positivity of (17) are

$$\sum_{i(n-2)\in\mathbb{I}_{(n-2)}} \mu_{i(n-2)}\xi^T \begin{pmatrix} \mathbf{X}_{(i(n-2),1,1)} & \cdots & \mathbf{X}_{(i(n-2),1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{(i(n-2),r,1)} & \cdots & \mathbf{X}_{(i(n-2),r,r)} \end{pmatrix} \xi > 0 \tag{19}$$

$$\tilde{\mathbf{Q}}_{i(n)} > \sum_{j(n)\in\mathcal{P}(i(n))} (\mathbf{X}_{j(n)} + \mathbf{X}_{j(n)}^T)/2 \qquad \forall\, i(n) \in \mathbb{I}_n^+ \tag{20}$$

where $\mathbf{X}_{(i(n-2),i_{n-1},i_n)} = \mathbf{X}_{(i(n-2),i_n,i_{n-1})}^T$, $\xi^T = [\mu_{i_1}\mathbf{x}^T \cdots \mu_{i_r}\mathbf{x}^T]$, and $\tilde{\mathbf{Q}}_{i(n)}$ is defined in (18).

Proof: See [14]. ∎

Remark 1: It has been reported in [14] that the Kim and Lee [9] and Liu and Zang approaches [10] are special cases of Lemma 2 for $n = 2$ and $n = 3$, respectively.

III. Main Results

In this section, considering the shape of the membership functions, the system stability of the FMB control systems in the form of p-dimensional fuzzy summation is investigated. To facilitate the stability analysis, the inequality constraints containing the information of the lower and upper bounds of each, and multiplication (overlapping) of two, three, ..., and n membership functions in different operating subregions are constructed. Some slack matrices introduced by the inequality constraints carry the information of membership functions to the stability analysis for relaxation of stability conditions.

A. Introduction of Relaxation Expressions

Based on the lower and upper bound constraints, and the introduction of some slack matrices, some new relaxation expressions are defined. As the stability analysis is carried out in an inequality in the form of higher dimensional fuzzy summation, the inequality constraints in the form of different dimensional fuzzy summation have to be expanded to the same order for stability analysis. The following proposition is proposed for this purpose.

Proposition 1: Considering $\mu_{i(m)}$ defined in (6), the following m-dimensional fuzzy summations are equivalent:

$$\sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)} = \sum_{i(m)\in\mathbb{I}_m^+} \Delta_{i(m)}\mu_{i(m)} \tag{21}$$

where $\Delta_{i(m)} = m!/(m_1!\cdots m_r!)$ and where $z = 1, 2, \ldots, r$, and m_z is the number of the value z in $i(m)$.

Proof: Expanding the left-hand side of (21), the proof could be easily concluded. ∎

For instance, suppose $m = 4$, $r = 3$, and $i(4) = (1, 1, 2, 2)$. From (21), we have $\Delta_{i(4)} = m!/(m_1!\, m_2!\, m_3!)$. As the number of 1, 2, and 3 in $i(4)$ is 2, 2, and 0, respectively, we have $m_1 = 2, m_2 = 2$, and $m_3 = 0$, and then, $\Delta_{i(4)} = 6$.

In the following, $L_{j(m)}$ and $H_{j(m)}$ containing some arbitrary symmetric positive definite matrices $\mathbf{M}_{j(m)}$, $\mathbf{N}_{j(m)}$, and the lower and upper bound constraint information of the products of m membership functions are introduced. Suppose that the lower bound constraints imposed on the membership functions and their multiplications are given as $\mu_{i_1}\mu_{i_2}\ldots\mu_{i_m} \ge \gamma_{(i_1,i_2,\ldots,i_m)}$, which is denoted by

$$\mu_{j(m)} \ge \gamma_{j(m)}, \qquad m = 1, \ldots, n \tag{22}$$

where $\gamma_{j(m)}$ is a scalar to be determined, which is the lower bound of the multiplication of $\mu_{i_1}, \ldots, \mu_{i_m}$. Considering (10) and introducing $\mathbf{M}_{j(m)} = \mathbf{M}_{j(m)}^T \in \mathbb{R}^{v\times v}$ as a positive definite matrix, the inequality (22), can be written as

$$\mu_{j(m)} \ge \gamma_{j(m)} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}$$
$$\Rightarrow \mu_{j(m)}\mathbf{x}^T\mathbf{M}_{j(m)}\mathbf{x} \ge \gamma_{j(m)} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}\mathbf{x}^T\mathbf{M}_{j(m)}\mathbf{x}.$$

We denote

$$L_{j(m)} = -\mu_{j(m)}\mathbf{x}^T\mathbf{M}_{j(m)}\mathbf{x} + \gamma_{j(m)} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}\mathbf{x}^T\mathbf{M}_{j(m)}\mathbf{x} \le 0. \tag{23}$$

Similarly, consider the upper bound constraints imposed on the membership functions and multiplication of them as $\mu_{i_1}\mu_{i_2}\ldots\mu_{i_m} \leq \beta_{(i_1,i_2,\ldots,i_m)}$, which is denoted by

$$\mu_{j(m)} \leq \beta_{j(m)}, \qquad m = 1,\ldots,n \tag{24}$$

where $\beta_{j(m)}$ is a scalar to be determined, which is the upper bound of the multiplication of $\mu_{i_1},\mu_{i_1},\ldots,\mu_{i_m}$. Considering (10) and (24), and introducing $\mathbf{N}_{j(m)} = \mathbf{N}_{j(m)}^T \in \mathbb{R}^{v\times v}$ as a positive definite matrix, the aforementioned inequality can be written as

$$\mu_{j(m)} \leq \beta_{j(m)} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}$$
$$\Rightarrow \mu_{j(m)}\mathbf{x}^T\mathbf{N}_{j(m)}\mathbf{x} \leq \beta_{j(m)} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}\mathbf{x}^T\mathbf{N}_{j(m)}\mathbf{x}.$$
$$H_{j(m)} = \mu_{j(m)}\mathbf{x}^T\mathbf{N}_{j(m)}\mathbf{x} - \beta_{j(m)} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}\mathbf{x}^T\mathbf{N}_{j(m)}\mathbf{x} \leq 0. \tag{25}$$

From (23) and (25), we define

$$\begin{aligned}\Theta_m(t) &= \sum_{j(m)\in\mathbb{I}_m^+} (L_{j(m)} + H_{j(m)})\\ &= \mathbf{x}^T \sum_{j(m)\in\mathbb{I}_m^+} \Bigg(\mu_{j(m)}(\mathbf{N}_{j(m)} - \mathbf{M}_{j(m)}) + (\gamma_{j(m)}\\ &\quad \times \mathbf{M}_{j(m)} - \beta_{j(m)}\mathbf{N}_{j(m)}) \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}\Bigg)\mathbf{x} \leq 0. \end{aligned} \tag{26}$$

Based on Proposition 1, (26) can be written as

$$\begin{aligned}\Theta_m(t) &= \mathbf{x}^T \sum_{j(m)\in\mathbb{I}_m^+} \mu_{j(m)}(\mathbf{N}_{j(m)} - \mathbf{M}_{j(m)})\mathbf{x} + \mathbf{x}^T\\ &\quad \times \sum_{i(m)\in\mathbb{I}_m^+} \Delta_{i(m)}\mu_{i(m)} \sum_{j(m)\in\mathbb{I}_m^+} (\gamma_{j(m)}\mathbf{M}_{j(m)} - \beta_{j(m)}\mathbf{N}_{j(m)})\mathbf{x}\\ &= \mathbf{x}^T \sum_{i(m)\in\mathbb{I}_m^+} \mu_{i(m)} \Bigg((\mathbf{N}_{i(m)} - \mathbf{M}_{i(m)}) + \Delta_{i(m)}\\ &\quad \times \sum_{j(m)\in\mathbb{I}_m^+} (\gamma_{j(m)}\mathbf{M}_{j(m)} - \beta_{j(m)}\mathbf{N}_{j(m)})\Bigg)\mathbf{x} \leq 0. \end{aligned} \tag{27}$$

For brevity, (27) is represented in the following form:

$$\Theta_m(t) = \mathbf{x}^T \sum_{i(m)\in\mathbb{I}_m^+} \mu_{i(m)}\mathbf{\Omega}_{i(m)}\mathbf{x} \tag{28}$$

where

$$\begin{aligned}\mathbf{\Omega}_{i(m)} &= \mathbf{N}_{i(m)} - \mathbf{M}_{i(m)} + \Delta_{i(m)}\\ &\quad \times \sum_{j(m)\in\mathbb{I}_m^+} (\gamma_{j(m)}\mathbf{M}_{j(m)} - \beta_{j(m)}\mathbf{N}_{j(m)}). \end{aligned} \tag{29}$$

Remark 2: $\Theta_m(t)$ contains the slack matrices $M_{j(m)}$ and $N_{j(m)}$ corresponding to the product of $\mu_{i_1},\mu_{i_2},\ldots,\mu_{i_m}$. The expressions $\Theta_1(t)$ to $\Theta_{n-1}(t)$ should be cast in the form of n-dimensional summation so that the final new relaxed stability conditions can be expressed in the form of LMIs. The following proposition helps achieve this expansion.

Proposition 2: Consider $\Theta_{m.l}(t)$, which is the expanded form of $\Theta_m(t)$ and is defined as

$$\Theta_{m.l}(t) = \sum_{i(l-m)\in\mathbb{I}_{l-m}} \mu_{i(l-m)}\Theta_m(t), \qquad 1 \leq m < l \leq n \tag{30}$$

and then, $\Theta_{m.l}(t) = \Theta_m(t)$.

Proof: Using the property of (10) and following the same logic leading to (17), it can be concluded easily. ■

According to Proposition 2, it can be said that

$$\Theta_m(t) = \Theta_{m.m+1}(t) = \cdots = \Theta_{m.n}(t) \qquad \forall n > m. \tag{31}$$

Indeed, when all $\Theta_{m.n}(t)$ with different values of m have the same dimension of summation, they can be added together to facilitate the stability analysis. For instance, assuming $m = 2, n = 3$, and considering Proposition 2, in the following, $\Theta_2(t)$ is cast in the form of 3-D fuzzy summation. In other words, constraints on every two fuzzy membership functions that are in the form of 2-D summation ($\Theta_2(t)$) is cast in the form of 3-D fuzzy summation as

$$\begin{aligned}\Theta_{2.3}(t) &= \sum_{i_3=1}^{r} \mu_{i_3}\Theta_2(t) = \sum_{i_3=1}^{r} \mu_{i_3}\mathbf{x}^T \sum_{i_1=1}^{r}\sum_{i_1\leq i_2\leq r} \mu_{i_1}\mu_{i_2}\mathbf{\Omega}_{(i_1,i_2)}\mathbf{x}\\ &= \mathbf{x}^T \sum_{i_1=1}^{r}\sum_{i_1\leq i_2\leq r}\sum_{i_3=1}^{r} \mu_{(i_1,i_2,i_3)}\mathbf{\Omega}_{(i_1,i_2)}\mathbf{x}\\ &= \mathbf{x}^T \sum_{i(3)\in\mathbb{I}_3^+} \mu_{i(3)}\tilde{\mathbf{\Omega}}_{i(3)}^2\mathbf{x} \end{aligned} \tag{32}$$

where

$$\tilde{\mathbf{\Omega}}_{i(3)}^2 = \sum_{j(2)\in\mathcal{P}_2^+(i(3))} \mathbf{\Omega}_{j(2)}.$$

According to Proposition 2, $\Theta_2(t)$ can be cast in the form of $\Theta_{2.3}(t),\ldots,\Theta_{2.n}(t)$. In addition, by following the same approach, $\Theta_m(t)$ can be cast in the following form containing n-dimensional fuzzy summation as

$$\Theta_{m.n}(t) = \mathbf{x}^T \sum_{i(n)\in\mathbb{I}_n^+} \mu_{i(n)}\tilde{\mathbf{\Omega}}_{i(n)}^m\mathbf{x} \tag{33}$$

where

$$\tilde{\mathbf{\Omega}}_{i(n)}^m = \sum_{j(m)\in\mathcal{P}_m^+(i(n))} \mathbf{\Omega}_{j(m)} \qquad \forall\, n > m. \tag{34}$$

It should be noted that there is no need for the expansion of $m = n$, as it is already in the form of an n-dimensional summation

$$\Theta_n(t) = \mathbf{x}^T \sum_{i(n)\in\mathbb{I}_n^+} \mu_{i(n)}\mathbf{\Omega}_{i(n)}\mathbf{x}. \tag{35}$$

B. GMFSD Conditions

Considering the full operating region of the membership functions, $\Theta_{1.n}(t),\ldots,\Theta_{n-1.n}(t),\ \Theta_n(t)$ carrying the lower and upper bound information of the membership functions are added to $\Xi_n(t)$ for relaxation of stability conditions. From (26), (31), and (35), we have

$$\Theta_{1.n}(t),\ldots,\Theta_{n-1.n}(t),\Theta_n(t) \leq 0$$

and it can be concluded that

$$\Xi_n(t) \geq \Xi_n(t) + \Theta_{1.n}(t) + \cdots + \Theta_{n-1.n}(t) + \Theta_n(t). \tag{36}$$

It can be seen that the positivity of the right-hand side implies that of the left-hand side and further implies the system stability. As the slack matrices carry the membership function information, it is possible to produce relaxed stability conditions, rather than only investigating the $\Xi_n(t)$. The following theorem is given to guarantee the positivity of the inequality (36).

Theorem 1: A multidimensional fuzzy summation $\Xi_p(t)$ expanded as $\Xi_n(t)$ in (17) is positive ($n \geq p$), if there exist positive definite matrices $\mathbf{X}_{(i(n-2),i_{n-1},i_n)} = \mathbf{X}^T_{(i(n-2),i_n,i_{n-1})}$ and symmetric positive semidefinite matrices $\mathbf{N}_{i(1)},\ldots,\mathbf{N}_{i(k)}$ and $\mathbf{M}_{i(1)},\ldots,\mathbf{M}_{i(k)}$, $i(k) \in \mathbb{I}^+_k$, where $k = 1,2,\ldots,n$, satisfying the following LMI conditions:

$$\sum_{i(n-2)\in\mathbb{I}_{(n-2)}} \mu_{i(n-2)} \times \xi^T \begin{pmatrix} \mathbf{X}_{(i(n-2),1,1)} & \cdots & \mathbf{X}_{(i(n-2),1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{(i(n-2),r,1)} & \cdots & \mathbf{X}_{(i(n-2),r,r)} \end{pmatrix} \xi > 0 \quad (37)$$

$$\tilde{\mathbf{Q}}_{i(n)} + \tilde{\mathbf{\Omega}}^1_{i(n)} + \cdots + \tilde{\mathbf{\Omega}}^{n-1}_{i(n)} + \mathbf{\Omega}_{i(n)} > \sum_{j(n)\in\mathcal{P}(i(n))} \frac{(\mathbf{X}_{j(n)} + \mathbf{X}^T_{j(n)})}{2} \qquad \forall\, i(n) \in \mathbb{I}^+_n \quad (38)$$

in which $\tilde{\mathbf{Q}}_{i(n)}$ is defined in (18), $\tilde{\mathbf{\Omega}}^1_{i(n)},\ldots,\tilde{\mathbf{\Omega}}^{n-1}_{i(n)}$ and $\mathbf{\Omega}_{i(n)}$ are defined based on the constraints on membership functions in (29) and (34), respectively, and $\xi^T = [\,\mu_{i_1}\mathbf{x}^T \quad \mu_{i_2}\mathbf{x}^T \quad \cdots \quad \mu_{i_r}\mathbf{x}^T\,]$.

Proof: Based on (36) and considering (33)–(35), we have

$$\begin{aligned} \Xi_n(t) &\geq \Xi_n(t) + \Theta_{1.n}(t) + \cdots + \Theta_{n-1.n}(t) + \Theta_n(t) \\ &= \mathbf{x}^T \sum_{i(n)\in\mathbb{I}^+_n} \mu_{i(n)}\tilde{\mathbf{Q}}_{i(n)}\mathbf{x} + \mathbf{x}^T \sum_{i(n)\in\mathbb{I}^+_n} \mu_{i(n)}\tilde{\mathbf{\Omega}}^1_{i(n)}\mathbf{x} + \cdots \\ &= \mathbf{x}^T \sum_{i(n)\in\mathbb{I}^+_n} \mu_{i(n)} \left(\tilde{\mathbf{Q}}_{i(n)} + \tilde{\mathbf{\Omega}}^1_{i(n)} + \cdots + \tilde{\mathbf{\Omega}}^{n-1}_{i(n)} + \mathbf{\Omega}_{i(n)}\right)\mathbf{x}. \end{aligned}$$

If (38) is satisfied, we have

$$\begin{aligned} &\mathbf{x}^T \sum_{i(n)\in\mathbb{I}^+_n} \mu_{i(n)} \left(\tilde{\mathbf{Q}}_{i(n)} + \tilde{\mathbf{\Omega}}^1_{i(n)} + \cdots + \tilde{\mathbf{\Omega}}^{n-1}_{i(n)} + \mathbf{\Omega}_{i(n)}\right)\mathbf{x} \\ &\quad > \mathbf{x}^T \sum_{i(n)\in\mathbb{I}^+_n} \mu_{i(n)} \sum_{j(n)\in\mathcal{P}(i(n))} \frac{(\mathbf{X}_{j(n)} + \mathbf{X}^T_{j(n)})}{2}\mathbf{x} \\ &\quad = \mathbf{x}^T \sum_{i(n)\in\mathbb{I}_n} \mu_{i(n)}\mathbf{X}_{i(n)}\mathbf{x} \\ &\quad = \sum_{i(n-2)\in\mathbb{I}_{n-2}} \mu_{i(n-2)}\mathbf{x}^T \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} \mu_{i_1}\mu_{i_2}\mathbf{X}_{(i(n-2),i_1,i_2)}\mathbf{x} \\ &\quad = \sum_{i(n-2)\in\mathbb{I}_{(n-2)}} \mu_{i(n-2)} \times \xi^T \begin{pmatrix} \mathbf{X}_{(i(n-2),1,1)} & \cdots & \mathbf{X}_{(i(n-2),1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{(i(n-2),r,1)} & \cdots & \mathbf{X}_{(i(n-2),r,r)} \end{pmatrix} \xi. \end{aligned}$$

It can be seen that if (37) holds, then $\Xi_n(t) > 0$. ∎

Remark 3: To apply this theorem, in practice, or in other words, to derive computable LMI conditions, the algorithm in Theorem 5 [14] can be employed.

Remark 4: Considering $n = p = 2$ and choosing $\mathbf{M}_{i(1)} = \mathbf{M}_{i(2)} = \mathbf{N}_{i(1)} = \mathbf{0}$ in Theorem 1, it will be reduced to Lemma 1. Hence, it can be concluded that the solution for Lemma 1 is also the solution of Theorem 1 but may not be the other way round. Similarly, choosing $\mathbf{M}_{i(1)} = \cdots = \mathbf{M}_{i(n)} = \mathbf{N}_{i(1)} = \cdots = \mathbf{N}_{i(n)} = \mathbf{0}$, Theorem 1 is reduced to Lemma 2. Hence, it can be concluded that the solution for Lemma 2 is also the solution of Theorem 1 but may not be the other way round.

C. RMFSD Conditions

In the RMFSD approach, the operating region of the membership functions is divided into Y regions denoted as $R_1, R_2, \ldots, R_Y$, where $Y \geq 1$ is an integer to be determined. As it is depicted in Fig. 1, each region has individual constraints (e.g., in region R_2, $\gamma_{(2),R_2} \leq \mu_2 \leq \beta_{(2),R_2}$, where $\gamma_{(2),R_2}$ and $\beta_{(2),R_2}$ are the lower and upper bounds of the membership function μ_2) that are employed to introduce some slack matrices for bringing regional information of membership functions to stability analysis for the relaxation of stability conditions. In the following analysis, R_α denotes the αth region.

Following the GMFSD approach, and introducing slack matrices $M_{j(m),R_\alpha}$ and $N_{j(m),R_\alpha}$ corresponding to the product of $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_m}$ in the region R_α, the following expressions are proposed:

$$\begin{aligned} L_{j(m),R_\alpha} &= -\mu_{j(m)}\mathbf{x}^T\mathbf{M}_{j(m),R_\alpha}\mathbf{x} \\ &\quad + \gamma_{j(m),R_\alpha} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}\mathbf{x}^T\mathbf{M}_{j(m),R_\alpha}\mathbf{x} \leq 0 \quad (39) \end{aligned}$$

$$\begin{aligned} H_{j(m),R_\alpha} &= \mu_{j(m)}\mathbf{x}^T\mathbf{N}_{j(m),R_\alpha}\mathbf{x} \\ &\quad - \beta_{j(m),R_\alpha} \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)}\mathbf{x}^T\mathbf{N}_{j(m),R_\alpha}\mathbf{x} \leq 0. \quad (40) \end{aligned}$$

Then, $\Theta_{m,R_\alpha}(t)$ is defined as follows:

$$\begin{aligned} \Theta_{m,R_\alpha}(t) &= \sum_{j(m)\in\mathbb{I}^+_m} (L_{j(m),R_\alpha} + H_{j(m),R_\alpha}) \\ &= \mathbf{x}^T \Bigg(\sum_{j(m)\in\mathbb{I}^+_m} \mu_{j(m)}(\mathbf{N}_{j(m),R_\alpha} - \mathbf{M}_{j(m),R_\alpha}) \\ &\quad + (\gamma_{j(m),R_\alpha}\mathbf{M}_{j(m),R_\alpha} - \beta_{j(m),R_\alpha}\mathbf{N}_{j(m),R_\alpha}) \\ &\quad \times \sum_{i(m)\in\mathbb{I}_m} \mu_{i(m)} \Bigg)\mathbf{x} \leq 0. \quad (41) \end{aligned}$$

Similar to the GMFSD approach, it can be expanded as

$$\Theta_{m,R_\alpha}(t) = \mathbf{x}^T \sum_{i(m)\in\mathbb{I}^+_m} \mu_{i(m)}\mathbf{\Omega}_{i(m),R_\alpha}\mathbf{x}$$

where

$$\begin{aligned} \mathbf{\Omega}_{i(m),R_\alpha} &= \mathbf{N}_{i(m),R_\alpha} - \mathbf{M}_{i(m),R_\alpha} + \Delta_{i(m)} \\ &\quad \times \sum_{j(m)\in\mathbb{I}^+_m} (\gamma_{j(m),R_\alpha}\mathbf{M}_{j(m),R_\alpha} - \beta_{j(m),R_\alpha}\mathbf{N}_{j(m),R_\alpha}) \end{aligned} \quad (42)$$

in which $\Delta_{i(m)}$ was defined in Proposition 1. Based on Proposition 2, the same approach is used to expand $\Theta_{m,R_\alpha}(t)$ to $\Theta_{m.n,R_\alpha}(t)$ in the form of n-dimensional fuzzy summation. Hence, we have

$$\Theta_{m,R_\alpha}(t) = \cdots = \Theta_{m.n-1,R_\alpha}(t) = \Theta_{m.n,R_\alpha}(t) \quad \forall n > m \tag{43}$$

in which $\Theta_{m.n,R_\alpha}(t) = \mathbf{x}^T \sum_{i(n)\in\mathbb{I}_n^+} \mu_{i(n)} \tilde{\mathbf{\Omega}}_{i(n),R_\alpha}^m \mathbf{x}$, where

$$\tilde{\mathbf{\Omega}}_{i(n),R_\alpha}^m = \sum_{j(m)\in\mathcal{P}_m^+(i(n))} \mathbf{\Omega}_{j(m),R_\alpha} \qquad \forall m < n \tag{44}$$

and

$$\Theta_{n,R_\alpha}(t) = \mathbf{x}^T \sum_{i(n)\in\mathbb{I}_n^+} \mu_{i(n)} \mathbf{\Omega}_{i(n),R_\alpha} \mathbf{x}. \tag{45}$$

Theorem 2: A multidimensional fuzzy summation $\Xi_p(t)$ expanded as $\Xi_n(t)$ in (17) is positive, if there exist positive definite matrices $\mathbf{X}_{(i(n-2),i_{n-1},i_n)} = \mathbf{X}_{(i(n-2),i_n,i_{n-1})}^T$, and symmetric positive semidefinite matrices $\mathbf{N}_{i(1),R_\alpha}, \ldots, \mathbf{N}_{i(k),R_\alpha}$ and $\mathbf{M}_{i(1),R_\alpha}, \ldots, \mathbf{M}_{i(k),R_\alpha}$, $i(k) \in \mathbb{I}_k^+$, where $k = 1, 2, \ldots, n$, and $\alpha = 1, 2, \ldots, Y$, satisfying the following LMI conditions:

$$\sum_{i(n-2)\in\mathbb{I}_{(n-2)}} \mu_{i(n-2)} \xi^T \begin{pmatrix} \mathbf{X}_{(i(n-2),1,1)} & \cdots & \mathbf{X}_{(i(n-2),1,r)} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{(i(n-2),r,1)} & \cdots & \mathbf{X}_{(i(n-2),r,r)} \end{pmatrix} \xi > 0 \tag{46}$$

$$\tilde{\mathbf{Q}}_{i(n)} + \tilde{\mathbf{\Omega}}_{i(n),R_\alpha}^1 + \cdots + \tilde{\mathbf{\Omega}}_{i(n),R_\alpha}^{n-1} + \mathbf{\Omega}_{i(n),R_\alpha} > \sum_{j(n)\in\mathcal{P}(i(n))} \frac{(\mathbf{X}_{j(n)} + \mathbf{X}_{j(n)}^T)}{2} \qquad \forall\, i(n) \in \mathbb{I}_n^+ \tag{47}$$

where it is supposed that membership functions' operating domain is divided into Y regions denoted as $R_1, R_2, \ldots, R_Y$, in which $\tilde{\mathbf{Q}}_{i(n)}$ is defined in (18), $\tilde{\mathbf{\Omega}}_{i(n),R_\alpha}^1, \ldots, \tilde{\mathbf{\Omega}}_{i(n),R_\alpha}^{n-1}$ and $\mathbf{\Omega}_{i(n),R_\alpha}$ are defined based on the constraints on membership functions in region R_α, as in (44) and (45), respectively, and $\xi^T = [\,\mu_{i_1}\mathbf{x}^T \quad \mu_{i_2}\mathbf{x}^T \quad \cdots \quad \mu_{i_r}\mathbf{x}^T\,]$.

Proof: The proof is similar to Theorem 1. ■

Remark 5: For Theorem 2, the stability conditions are reduced to those in Lemma 1 if Y (the number of regions) is chosen as 1.

IV. Simulation Examples

The effectiveness of the proposed stability conditions is shown through two simulation examples.

Simulation Example 1: Consider a three-rule ($r = 3$) T-S fuzzy model in the form of (11), where $\mathbf{z}(t) = x_1$ with the following subsystems:

$$\mathbf{A}_1 = \begin{pmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{pmatrix}, \qquad \mathbf{B}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{pmatrix}, \qquad \mathbf{B}_2 = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} -a & -4.33 \\ 0 & 0.05 \end{pmatrix}, \qquad \mathbf{B}_3 = \begin{pmatrix} -b+6 \\ -5 \end{pmatrix}. \tag{48}$$

The membership functions for the T-S fuzzy model are as follows and are shown graphically in Fig. 1:

$$\mu_1 = \frac{1}{1 + e^{-((x_1 - 5.85)/2)}}, \quad \mu_3 = \frac{1}{1 + e^{((x_1 + 5.85)/2)}}$$

and $\mu_2 = 1 - \mu_1 - \mu_3$.

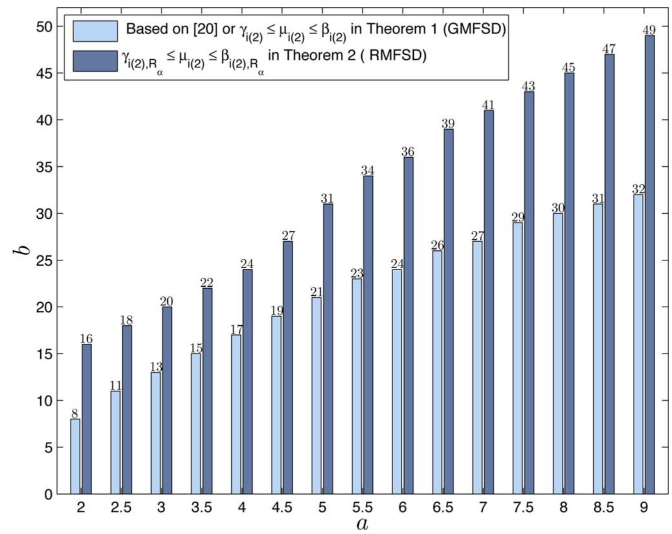

Fig. 2. Stability regions of Simulation Example 1 given by stability conditions in Lemma 1 [20] or Theorem 1 (GMFSD) with $\gamma_{i(2)} \leq \mu_{i(2)} \leq \beta_{i(2)}$ and Theorem 2 (RMFSD) with $\gamma_{i(2),R} \leq \mu_{i(2)} \leq \beta_{i(2),R}$ for $n = 2$.

A three-rule fuzzy controller in the form of (12) is employed to close the feedback loop leading to the fuzzy control system in the form of (13). The proposed relaxed stability conditions in Theorem 1 and Theorem 2 are employed to check the system stability for $2 \leq a \leq 9$ with interval of 0.5. The maximum value of b corresponding to the value of a is determined. To apply Theorem 2 in this example, the operating region of membership functions is divided into eight regions $R_1, \ldots, R_8$, as shown in Fig. 1. The solution of stability conditions is found numerically by the MATLAB LMI toolbox [23].

In the first step, the expansion degree n is selected as 2 (the same as degree of original system). For comparison purposes, the stability conditions in Lemma 1 [20] are employed to check the system stability under the same settings. To release the conservatism, the regional lower and upper bounds of $\mu_{i(2)}$ are used. The stability regions are shown in Fig. 2. It can be found that stability conditions in the proposed GMFSD stability conditions in Theorem 1 with $n = 2$ are offered the same stability region, as the lower bound information is trivial (i.e., $\gamma_{i(2)} = 0$). The RMFSD stability conditions with $n = 2$ in Theorem 2 are then employed to check the system stability for the same fuzzy control system. It can be seen that the stability region offered by Theorem 2 is larger.

To investigate the effectiveness of the stability conditions in Theorem 1 and Theorem 2 for higher expansion degree, we consider $n = 4$. For both theorems, the lower and upper bounds of μ_{i_1} (which are denoted as $\mu_{i(1)}$) are considered. The stability regions are shown in Fig. 3. For comparison purpose, the stability conditions in Lemma 2 [14] are considered. It can be found that stability conditions in Lemma 2 [14] offers the same stability region as provided by Theorem 1. However, the RMFSD stability conditions in Theorem 2 offer a larger stability region. Stability regions for stability conditions in Theorems 1 and 2 with the lower and upper bounds of $\mu_{i(4)}$ are shown in Fig. 4. From these figures, it can be seen that compared with Lemma 2, the stability conditions in Theorems 1 and 2 offer larger stability regions with the lower and upper bounds of $\mu_{i(4)}$. Furthermore, the RMFSD stability conditions in Theorem 2 outperform GMFSD ones in Theorem 1 in terms of larger stability regions. Comparing the simulation results, it can be seen that using lower and upper bounds with higher degree of product of membership functions results in a larger stability region.

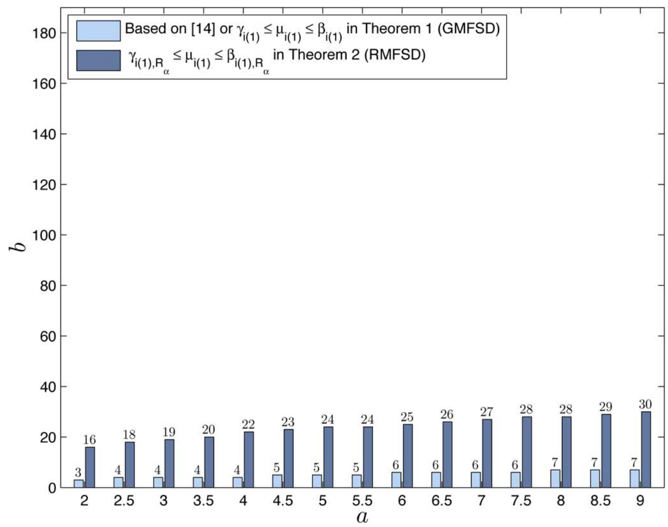

Fig. 3. Stability regions of Simulation Example 1 given by stability conditions in Lemma 1 [14] or Theorem 1 (GMFSD) with $\gamma_{i(1)} \le \mu_{i(1)} \le \beta_{i(1)}$ and Theorem 2 (RMFSD) with $\gamma_{i(1),R} \le \mu_{i(1)} \le \beta_{i(1),R}$ for $n = 4$.

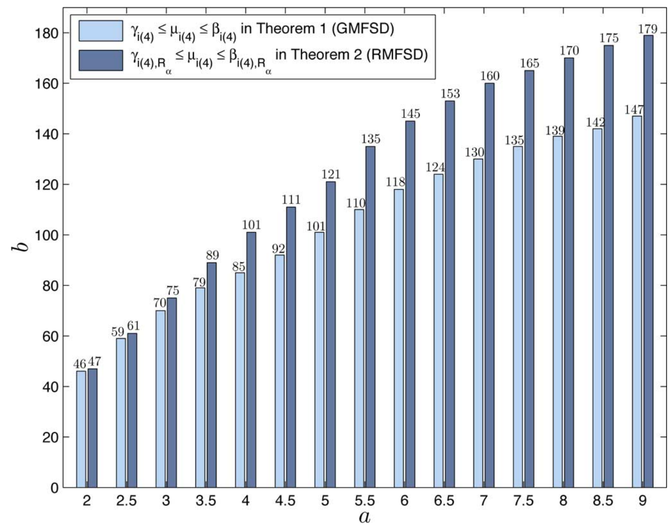

Fig. 4. Stability regions of Simulation Example 1 given by stability conditions in Theorem 1 (GMFSD) with $\gamma_{i(4)} \le \mu_{i(4)} \le \beta_{i(4)}$ and Theorem 2 (RMFSD) with $\gamma_{i(4),R} \le \mu_{i(4)} \le \beta_{i(4),R}$ for $n = 4$.

Simulation Example 2: In this simulation example, the ability of the proposed method to stabilize an inverted pendulum on a cart is illustrated. First, a two-rule T-S fuzzy model [24] is employed to represent the nonlinear system for which the subsystems are considered as follows:

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)mgl}{a_1} & -\frac{f_1(M+m)}{a_1} & 0 & \frac{f_0 ml}{a_1} \\ 0 & 0 & 0 & 1 \\ -\frac{m^2gl^2}{a_1} & \frac{f_1 ml}{a_1} & 0 & \frac{f_0(J+ml^2)}{a_1} \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{3\sqrt{3}}{2\pi}\frac{(M+m)mgl}{a_2} & -\frac{f_1(M+m)}{a_2} & 0 & \frac{f_0 ml\cos(\pi/3)}{a_2} \\ 0 & 0 & 0 & 1 \\ -\frac{3\sqrt{3}}{2\pi}\frac{m^2gl^2\cos(\pi/3)}{a_2} & \frac{f_1 ml\cos(\pi/3)}{a_2} & 0 & \frac{f_0(J+ml^2)}{a_2} \end{pmatrix}$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 \\ -\frac{ml}{a_1} \\ 0 \\ \frac{(J+ml^2)}{a_1} \end{pmatrix} \qquad \mathbf{B}_2 = \begin{pmatrix} 0 \\ -\frac{ml\cos(\pi/3)}{a_2} \\ 0 \\ \frac{(J+ml^2)}{a_2} \end{pmatrix} \tag{49}$$

where $a_1 = (M+m)(J+ml^2) - m^2l^2$, and $a_2 = (M+m)(J+ml^2) - m^2l^2\cos(\pi/3)^2$, in which x_1 denotes the angle (in radians) of the pendulum from the vertical, x_2 is the angular velocity (in radians per second), x_3 is the displacement (in meters) of the cart, and x_4 is the velocity (in meters per second) of the cart. The variable $g = 9.8$ m/s^2 is the gravity constant, m is the mass (in kilograms) of the pendulum, M is the mass (in kilograms) of the cart, f_0 is the friction factor (in newtons per meter per second) of the cart, f_1 is the friction factor (in newtons per radian per second) of the pendulum, l is the length (in meters) from the center of mass of the pendulum to the shaft axis, J is the moment of inertia (in kilograms square meter) of the pendulum round its center of mass, and u is the force (in newtons) applied to the cart. We choose $M = 1.3282$ kg, $m = 0.22$ kg, $f_0 = 22.915$ N/(m·s)$^{-1}$, $f_1 = 0.007056$ N/(rad·s)$^{-1}$, $l = 0.304$ m, $J = 0.004963$ kg·m^2. Membership functions are defined as $\mu_1 = \left(1 - (1/1 + e^{-7.0(x_1 - \pi/6)})\right)\left(1/1 + (e^{-7.0(x_1+\pi/6)})\right)$ and $\mu_2 = 1 - \mu_1$, and their operating domain is partitioned into eight regions.

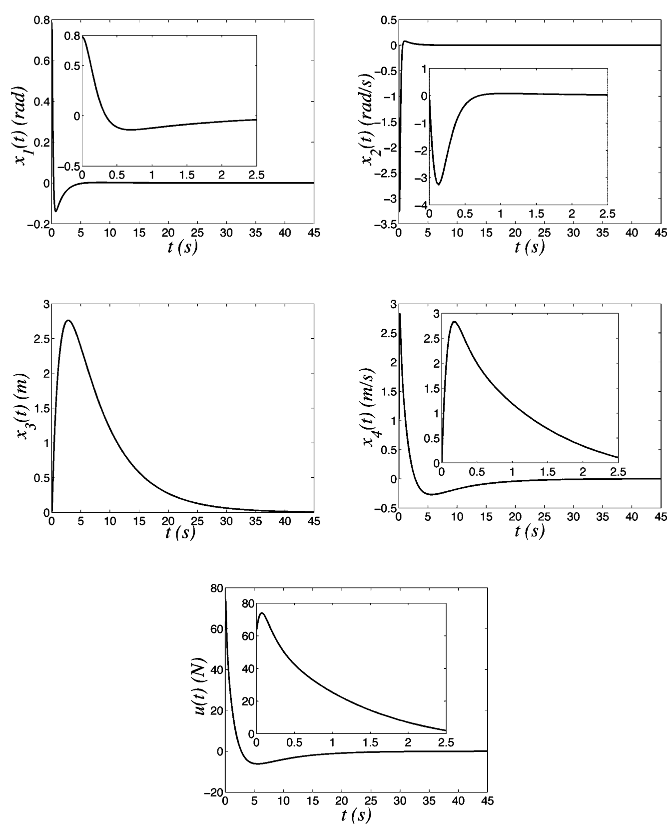

Fig. 5. Closed-loop response for initial conditions $x_1(0) = \pi/4$ and $x_2(0) = x_3(0) = x_4(0) = 0$, where x_1 is the angle (in radians) of the pendulum from the vertical, x_2 is the angular velocity (in radians per second), x_3 is the displacement (in meters) of the cart, x_4 is the velocity (in meters per second) of the cart, u is the force (in newtons) applied to the cart, and t is time (in seconds).

A two-rule fuzzy controller in the form of (12) is employed to close the feedback loop and stabilize the nonlinear system. According to the stability conditions in Theorem 2, the feedback gains are obtained as $\mathbf{G}_1 = [-77.6706 \; -13.0921 \; -0.5622 \; -27.6665]$

and $\mathbf{G}_2 = [-81.3463 - 12.6199 - 0.4683 - 25.9985]$, with the MATLAB LMI toolbox. The system responses for initial system state of $[(\pi/4)\ 0\ 0\ 0]$ are shown in Fig. 5. It can be seen that the fuzzy controller can successfully stabilize the nonlinear plant.

V. Conclusion

This paper has presented the relaxed LMI-based stability conditions for the fuzzy-model-based control systems in the form of p-dimensional fuzzy summation. Under the PDC design, the boundary and regional information of the membership functions have been employed to facilitate the stability analysis. Based on the lower and upper bounds of the membership functions in different suboperating regions, some inequality constraints in the form of multidimensional fuzzy summation have been constructed. With the introduction of the slack matrices by the inequality constraints, the information of membership functions have brought in the stability analysis for relaxation of stability conditions. It has been shown that some published stability conditions are considered as special cases of the proposed ones. Simulation examples have been given to verify the stability analysis results and illustrate the effectiveness of the proposed approach.

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