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Analysis and design of fuzzy reduced-dimensional observer and fuzzy functional observer

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Abstract

On the basis of the T-S fuzzy model, this paper discusses the following three problems: the fuzzy reduced-dimensional observer; the general structure of the fuzzy observer; the fuzzy functional observer, for both the continuous and discrete case, respectively. The separation property, that is, the fuzzy controller and the fuzzy observer can be independently designed, is also addressed. The numerical simulation and the experiment on an inverted pendulum system are given to illustrate the soundness of these results. © 2001 Published by Elsevier Science B.V.

Keywords: T-S fuzzy model; Fuzzy reduced-dimensional observer; Fuzzy functional observer; Separation property; Nonlinear system

1. Introduction

During the last decade, fuzzy logic control has attracted great attention from both the academic and industrial communities. Many people have devoted a great deal of time and effort to both theoretical research and implementation techniques for fuzzy logic controllers. Up to now, fuzzy logic control has been suggested as an alternative approach to conventional control techniques for complex control systems.

Fuzzy logic control is one of the most useful approaches for utilizing the qualitative knowledge of a system to design a controller. Fuzzy logic control is generally applicable to plants that are mathematically poorly modelled and where the qualitative knowledge of experienced operators is available for providing qualitative control. The control techniques represent a means of both collecting human knowledge and expertise and dealing with uncertainties in the process of control. However, the control techniques suffer from drawbacks

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such as (i) the design of the fuzzy logic control is difficult because no theoretical basis is available; (ii) the performance of the fuzzy logic control can be inconsistent because the fuzzy logic control depends mainly on the individual operators' experience. Therefore, despite the fact that much progress has been made in successfully applying fuzzy logic control to industrial control systems, it has become evident that many basic problems remain to be further addressed. Stability analysis and systematic design are certainly among the most important issues for fuzzy control systems. Recently, there have been significant research efforts on these issues.

With the developments of neural networks and fuzzy systems, it is known that the qualitative knowledge of a system can also be represented in a nonlinear functional form. On the basis of the idea, some fuzzy control system design methods, based on fuzzy models, have appeared in the fuzzy control field [1,4,6,8,9]. These methods are conceptually simple and straightforward. Linear feedback control techniques can be utilized as in the case of feedback stabilization. The procedure is as follows: First, the nonlinear plant is represented by a Takagi–Sugeno type fuzzy model. In this type of fuzzy model, local dynamics in different state-space regions are represented by linear models. The overall model of the system is obtained by fuzzy "blending" of these linear models through nonlinear fuzzy membership functions. Second, the control design is completed on the basis of the fuzzy model via the so-called parallel distributed compensation scheme. The idea is that for each local linear model, a linear feedback control is designed. Finally, the resulting overall controller, which is nonlinear in general, is again a fuzzy "blending" of each individual linear controller. At the same time, the robust stabilization problem has been also considered [5,7]. The importance of the observer in control systems impels the observer design problem to be addressed [2]. The separation property has been developed, that is, the fuzzy controller and the fuzzy observer can be independently designed. On the basis of the research work, the following problems: (i) the design of the fuzzy reduced-dimensional observer; (ii) the general structure of the fuzzy observer; and (iii) the design of the fuzzy functional observer, will be further discussed in this paper.

The paper is organized as follows: Section 2 discusses the design of the fuzzy reduced-dimensional observer. Section 3 discusses the general structure of the fuzzy observer. Section 4 discusses the design of the fuzzy functional observer. In Section 5, the separation property, for both the fuzzy reduced-dimensional and fuzzy functional observer, is also addressed. Section 6 gives the detailed numerical simulation, namely the balancing of an inverted pendulum on a car. The experiment is further carried out in Section 7. Section 8 collects some concluding remarks.

2. Fuzzy reduced-dimensional observer

Many physical systems are very complex in practice so that accurate mathematical models can be very difficult to obtain if not impossible. However, many of these systems can be expressed in some form of mathematical model locally, or as an aggregation of a set of mathematical models. Takagi and Sugeno have proposed a fuzzy model to describe the complex systems [3]. Here we consider using the following fuzzy dynamic model to represent a complex multi-input multi-output system which includes both local analytic linear models and fuzzy membership functions.

In the continuous case, the Takagi–Sugeno fuzzy dynamic model is described by fuzzy IF-THEN rules, which locally represent linear input–output relations of nonlinear systems. The ith rule of the fuzzy model is of the following form:

Plant Rule i: IF $s_1(t)$ is $\boldsymbol{F}_{i1}$ and $\cdots$ and $s_g(t)$ is $\boldsymbol{F}_{ig}$,

$$\begin{aligned} \text{THEN } & \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t), \\ & \boldsymbol{y}_i(t) = \boldsymbol{C}_i\boldsymbol{x}(t), \quad i = 1,2,\ldots,r, \end{aligned} \tag{1}$$

where $\boldsymbol{F}_{ij}$ $(j=1,2,\ldots,g)$ are fuzzy sets. $\boldsymbol{x}(t)\in R^n$ is the state vector, $\boldsymbol{u}(t)\in R^m$ is the input vector, $\boldsymbol{y}_i(t)\in R^q$ is the output vector. $(\boldsymbol{A}_i\in R^{n\times n},\boldsymbol{B}_i\in R^{n\times m},\boldsymbol{C}_i\in R^{q\times n})$ is the matrix triplet. r is the number of IF-THEN rules. $s_1(t)$–$s_g(t)$ are some measurable system variables, i.e., the premise variables.

Given a pair $[\boldsymbol{x}(t),\boldsymbol{u}(t)]$, by using a standard fuzzy inference method, that is, using a singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the final state and output of the fuzzy system are inferred as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r}\mu_i\boldsymbol{A}_i\boldsymbol{x}(t)+\sum_{i=1}^{r}\mu_i\boldsymbol{B}_i\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \sum_{i=1}^{r}\mu_i\boldsymbol{C}_i\boldsymbol{x}(t),\end{aligned}\tag{2}$$

where μ_i satisfies

$$\mu_i=\mu_i[\boldsymbol{s}(t)]\geqslant 0,\ \ i=1,2,\ldots,r;\qquad \sum_{i=1}^{r}\mu_i[\boldsymbol{s}(t)]=1;\quad \boldsymbol{s}(t)=[s_1(t)\quad s_2(t)\quad\cdots\quad s_g(t)]$$

for all t.

In the discrete case, the Takagi–Sugeno fuzzy dynamic model is also described by fuzzy IF-THEN rules; the ith rule of the fuzzy model is of the following form:

$$\begin{aligned}&\text{Plant Rule } i\text{: IF } s_1(k) \text{ is } \boldsymbol{F}_{i1} \text{ and } \cdots \text{ and } s_g(k) \text{ is } \boldsymbol{F}_{ig},\\ &\qquad\text{THEN } \boldsymbol{x}(k+1)=\boldsymbol{A}_i^{\mathrm{d}}\boldsymbol{x}(k)+\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{u}(k),\\ &\qquad\qquad \boldsymbol{y}_i(k)=\boldsymbol{C}_i^{\mathrm{d}}\boldsymbol{x}(k),\quad i=1,2,\ldots,r,\end{aligned}\tag{3}$$

where every variable is a discrete case of that in the continuous-time fuzzy dynamic model (1).

Given a pair $[\boldsymbol{x}(k),\boldsymbol{u}(k)]$, in a similar way to the continuous-time case, the final state and output of the fuzzy system are inferred as follows:

$$\begin{aligned}\boldsymbol{x}(k+1) &= \sum_{i=1}^{r}\mu_i\boldsymbol{A}_i^{\mathrm{d}}\boldsymbol{x}(k)+\sum_{i=1}^{r}\mu_i\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{u}(k),\\ \boldsymbol{y}(t) &= \sum_{i=1}^{r}\mu_i\boldsymbol{C}_i^{\mathrm{d}}\boldsymbol{x}(k).\end{aligned}\tag{4}$$

First, we consider the design of the fuzzy reduced-dimensional observer in the continuous case, which will be used in the numerical simulation.

Assumption 2.1 The pairs $(\boldsymbol{A}_i,\boldsymbol{C}_i)$, $i=1,2,\ldots,r$, are observable, and $\operatorname{rank} C_i=q$ for $i=1,2,\ldots,r$.

The fuzzy reduced-dimensional observer in the continuous case, which is of $n-q$ dimensions, can be designed according to the following steps:

Step 1: Define $n\times n$ matrix

$$\boldsymbol{T}_i=\begin{bmatrix}\boldsymbol{C}_i\\ \boldsymbol{R}_i\end{bmatrix},$$

where $\boldsymbol{R}_i$ is $(n-q)\times n$ constant matrix such that $\boldsymbol{T}_i$ is nonsingular, so $\boldsymbol{R}_i$ is not unique and is arbitrary. The inverse matrix of $\boldsymbol{T}_i$ is represented by the following block matrix:

$$\boldsymbol{Q}_i=\boldsymbol{T}_i^{-1}=[\boldsymbol{Q}_{i1}\quad \boldsymbol{Q}_{i2}],$$

where $\boldsymbol{Q}_{i1}$ and $\boldsymbol{Q}_{i2}$ are $n \times q$ and $n \times (n-q)$ matrices, respectively. It is obvious that

$$\boldsymbol{I} = \boldsymbol{T}_i \boldsymbol{Q}_i = \begin{bmatrix} \boldsymbol{C}_i \\ \boldsymbol{R}_i \end{bmatrix} [\boldsymbol{Q}_{i1} \quad \boldsymbol{Q}_{i2}] = \begin{bmatrix} \boldsymbol{C}_i\boldsymbol{Q}_{i1} & \boldsymbol{C}_i\boldsymbol{Q}_{i2} \\ \boldsymbol{R}_i\boldsymbol{Q}_{i1} & \boldsymbol{R}_i\boldsymbol{Q}_{i2} \end{bmatrix},$$

that is,

$$\boldsymbol{C}_i\boldsymbol{Q}_{i1} = \boldsymbol{I}_q, \qquad \boldsymbol{C}_i\boldsymbol{Q}_{i2} = \boldsymbol{0}.$$

Step 2: For the estimated system (1), considering a linear nonsingular transformation $\bar{\boldsymbol{x}}(t) = \boldsymbol{T}_i\boldsymbol{x}(t)$, we can obtain

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}}(t) &= \boldsymbol{T}_i\boldsymbol{A}_i\boldsymbol{T}_i^{-1}\bar{\boldsymbol{x}}(t) + \boldsymbol{T}_i\boldsymbol{B}_i\boldsymbol{u}(t) = \bar{\boldsymbol{A}}_i\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{B}}_i\boldsymbol{u}(t), \\ \boldsymbol{y}_i(t) &= \boldsymbol{C}_i\boldsymbol{T}_i^{-1}\bar{\boldsymbol{x}}(t) = \boldsymbol{C}_i[\boldsymbol{Q}_{i1} \quad \boldsymbol{Q}_{i2}]\bar{\boldsymbol{x}}(t) = [\boldsymbol{C}_i\boldsymbol{Q}_{i1} \quad \boldsymbol{C}_i\boldsymbol{Q}_{i2}]\bar{\boldsymbol{x}}(t) = [\boldsymbol{I}_q \quad \boldsymbol{0}]\bar{\boldsymbol{x}}(t). \end{aligned}$$

If $\bar{\boldsymbol{x}}_1(t)$ and $\bar{\boldsymbol{x}}_2(t)$ are q and $n-q$ dimensions substates, respectively, the above formulas can be expressed by

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}}(t) &= \begin{bmatrix} \dot{\bar{\boldsymbol{x}}}_1(t) \\ \dot{\bar{\boldsymbol{x}}}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{A}}_{i11} & \bar{\boldsymbol{A}}_{i12} \\ \bar{\boldsymbol{A}}_{i21} & \bar{\boldsymbol{A}}_{i22} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{x}}_1(t) \\ \bar{\boldsymbol{x}}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{\boldsymbol{B}}_{i1} \\ \bar{\boldsymbol{B}}_{i2} \end{bmatrix} \boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= [\boldsymbol{I}_q \quad \boldsymbol{0}] \begin{bmatrix} \bar{\boldsymbol{x}}_1(t) \\ \bar{\boldsymbol{x}}_2(t) \end{bmatrix} = \bar{\boldsymbol{x}}_1(t), \end{aligned} \tag{5}$$

where $\bar{\boldsymbol{A}}_{i11}, \bar{\boldsymbol{A}}_{i12}, \bar{\boldsymbol{A}}_{i21}$ and $\bar{\boldsymbol{A}}_{i22}$ are $q \times q$, $q \times (n-q)$, $(n-q) \times q$ and $(n-q) \times (n-q)$ matrices, respectively. $\bar{\boldsymbol{B}}_{i1}$ and $\bar{\boldsymbol{B}}_{i2}$ are $q \times m$ and $(n-q) \times m$ matrices, respectively. According to formula (5), we can know, for the transformated state $\bar{\boldsymbol{x}}(t)$, its substate $\bar{\boldsymbol{x}}_1(t)$ is the output $\boldsymbol{y}(t)$ of the estimated system, which can be directly utilized and do not have to be reconstructed. There is only $n-q$ dimensions substate $\bar{\boldsymbol{x}}_2(t)$ to be reconstructed, so we only need $n-q$ dimensions state observer in order to achieve the goal of reconstructing all states.

Step 3: Using formula (5), we can obtain the state equation and the output equation corresponding to $\bar{\boldsymbol{x}}_2(t)$:

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}}_2(t) &= \bar{\boldsymbol{A}}_{i22}\bar{\boldsymbol{x}}_2(t) + [\bar{\boldsymbol{A}}_{i21}\boldsymbol{y}(t) + \bar{\boldsymbol{B}}_{i2}\boldsymbol{u}(t)], \\ \boldsymbol{v}_i(t) &= \dot{\boldsymbol{y}}(t) - \bar{\boldsymbol{A}}_{i11}\boldsymbol{y}(t) - \bar{\boldsymbol{B}}_{i1}\boldsymbol{u}(t) = \bar{\boldsymbol{A}}_{i12}\bar{\boldsymbol{x}}_2(t). \end{aligned} \tag{6}$$

It is obvious that $(\bar{\boldsymbol{A}}_{i22}, \bar{\boldsymbol{A}}_{i12})$ is observable if and only if $(\boldsymbol{A}_i, \boldsymbol{C}_i)$ is observable. The final state and output of the reduced-dimensional fuzzy system are inferred as follows:

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}}_2(t) &= \sum_{i=1}^{r} \mu_i\bar{\boldsymbol{A}}_{i22}\bar{\boldsymbol{x}}_2(t) + \left[\sum_{i=1}^{r} \mu_i\bar{\boldsymbol{A}}_{i21}\boldsymbol{y}(t) + \sum_{i=1}^{r} \mu_i\bar{\boldsymbol{B}}_{i2}\boldsymbol{u}(t)\right], \\ \boldsymbol{v}(t) &= \dot{\boldsymbol{y}}(t) - \sum_{i=1}^{r} \mu_i\bar{\boldsymbol{A}}_{i11}\boldsymbol{y}(t) - \sum_{i=1}^{r} \mu_i\bar{\boldsymbol{B}}_{i1}\boldsymbol{u}(t) = \sum_{i=1}^{r} \mu_i\bar{\boldsymbol{A}}_{i12}\bar{\boldsymbol{x}}_2(t). \end{aligned}$$

Step 4: Finally, we can construct the full-dimensional state observer for the $n-q$ dimensions subsystem (6). $(\bar{\boldsymbol{A}}_{i22}, \bar{\boldsymbol{A}}_{i12})$ is observable such that the $n-q$ dimensions state observer exists and is of the following form:

Observer Rule i: IF $s_1(t)$ is $\boldsymbol{F}_{i1}$ and $\cdots$ and $s_g(t)$ is $\boldsymbol{F}_{ig}$,

$$\begin{aligned} \text{THEN } \dot{\hat{\bar{\boldsymbol{x}}}}_2(t) &= \bar{\boldsymbol{A}}_{i22}\hat{\bar{\boldsymbol{x}}}_2(t) + [\bar{\boldsymbol{A}}_{i21}\boldsymbol{y}(t) + \bar{\boldsymbol{B}}_{i2}\boldsymbol{u}(t)] + \bar{\boldsymbol{L}}_i[\boldsymbol{v}(t) - \hat{\boldsymbol{v}}(t)], \\ \hat{\boldsymbol{v}}_i(t) &= \bar{\boldsymbol{A}}_{i12}\hat{\bar{\boldsymbol{x}}}_2(t), \quad i = 1, 2, \ldots, r. \end{aligned}$$

The final estimated state and output of the fuzzy observer are

$$\dot{\hat{\bar{x}}}_2(t) = \sum_{i=1}^{r} \mu_i \bar{A}_{i22} \hat{\bar{x}}_2(t) + \left[\sum_{i=1}^{r} \mu_i \bar{A}_{i21} y(t) + \sum_{i=1}^{r} \mu_i \bar{B}_{i2} u(t) \right] + \sum_{i=1}^{r} \mu_i \bar{L}_i [v(t) - \hat{v}(t)],$$

$$\hat{v}(t) = \sum_{i=1}^{r} \mu_i \bar{A}_{i12} \hat{\bar{x}}_2(t).$$

Utilizing some simple operations, we can obtain

$$\dot{\hat{\bar{x}}}_2(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j (\bar{A}_{i22} - \bar{L}_i \bar{A}_{j12}) \hat{\bar{x}}_2(t) + \sum_{i=1}^{r} \mu_i \bar{L}_i \dot{y}(t) + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j (\bar{A}_{i21} - \bar{L}_i \bar{A}_{j11}) y(t)$$
$$+ \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j (\bar{B}_{i2} - \bar{L}_i \bar{B}_{j1}) u(t).$$

According to the linear nonsingular transformation $\bar{x}(t) = T_i x(t)$ of Plant Rule i, the final state transformation can be described by

$$\bar{x}(t) = \sum_{i=1}^{r} \mu_i T_i x(t);$$

then

$$\hat{\bar{x}}(t) = \sum_{i=1}^{r} \mu_i T_i \hat{x}(t).$$

The estimated state $\hat{x}(t)$ can be given by

$$\hat{x}(t) = \left(\sum_{i=1}^{r} \mu_i T_i \right)^{-1} \hat{\bar{x}}(t) = T^{-1} \hat{\bar{x}}(t),$$

where T must be nonsingular and

$$\hat{\bar{x}}(t) = \begin{bmatrix} \hat{\bar{x}}_1(t) \\ \hat{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \hat{\bar{x}}_2(t) \end{bmatrix}.$$

Next, we consider the design of the fuzzy reduced-dimensional observer in the discrete case, which will be used in the following experiment.

Assumption 2.2 The pairs $(A_i^{\mathrm{d}}, C_i^{\mathrm{d}})$, $i = 1, 2, \ldots, r$, are observable, and rank $C_i^{\mathrm{d}} = q$ for $i = 1, 2, \ldots, r$.

The fuzzy reduced-dimensional observer in the discrete case, which is of $n - q$ dimensions, can be designed according to the same steps as those in the continuous case.

For the estimated system (3), considering a linear nonsingular transformation $\bar{x}(k) = T_i^{\mathrm{d}} x(k)$, we can obtain

$$\bar{x}(k+1) = T_i^{\mathrm{d}} A_i^{\mathrm{d}} (T_i^{\mathrm{d}})^{-1} \bar{x}(k) + T_i^{\mathrm{d}} B_i^{\mathrm{d}} u(k) = \bar{A}_i^{\mathrm{d}} \bar{x}(k) + \bar{B}_i^{\mathrm{d}} u(k),$$

$$y_i(k) = C_i^{\mathrm{d}} (T_i^{\mathrm{d}})^{-1} \bar{x}(k) = [\, I_q \quad 0 \,] \bar{x}(k).$$

If $\bar{x}_1(k)$ and $\bar{x}_2(k)$ are q and $n-q$ dimensions substates, respectively, the above formulas can be expressed by

$$\bar{x}(k+1)=\begin{bmatrix}\bar{x}_1(k+1)\\ \bar{x}_2(k+1)\end{bmatrix}=\begin{bmatrix}\bar{A}^{\mathrm{d}}_{i11} & \bar{A}^{\mathrm{d}}_{i12}\\ \bar{A}^{\mathrm{d}}_{i21} & \bar{A}^{\mathrm{d}}_{i22}\end{bmatrix}\begin{bmatrix}\bar{x}_1(k)\\ \bar{x}_2(k)\end{bmatrix}+\begin{bmatrix}\bar{B}^{\mathrm{d}}_{i1}\\ \bar{B}^{\mathrm{d}}_{i2}\end{bmatrix}u(k),$$

$$y(k)=[I_q\quad 0]\begin{bmatrix}\bar{x}_1(k)\\ \bar{x}_2(k)\end{bmatrix}=\bar{x}_1(k), \tag{7}$$

where $\bar{A}^{\mathrm{d}}_{i11},\bar{A}^{\mathrm{d}}_{i12},\bar{A}^{\mathrm{d}}_{i21},\bar{A}^{\mathrm{d}}_{i22},\bar{B}^{\mathrm{d}}_{i1}$ and $\bar{B}^{\mathrm{d}}_{i2}$ are appropriate dimensions matrices.

Using formula (7), we can obtain the state equation and the output equation corresponding to $\bar{x}_2(k)$:

$$\bar{x}_2(k+1)=\bar{A}^{\mathrm{d}}_{i22}\bar{x}_2(k)+[\bar{A}^{\mathrm{d}}_{i21}y(k)+\bar{B}^{\mathrm{d}}_{i2}u(k)],$$

$$y(k+1)-\bar{A}^{\mathrm{d}}_{i11}y(k)-\bar{B}^{\mathrm{d}}_{i1}u(k)=\bar{A}^{\mathrm{d}}_{i12}\bar{x}_2(k). \tag{8}$$

It is obvious that $(\bar{A}^{\mathrm{d}}_{i22},\bar{A}^{\mathrm{d}}_{i12})$ is observable if and only if $(A^{\mathrm{d}}_i,C^{\mathrm{d}}_i)$ is observable. The final state and output of the reduced-dimensional fuzzy system are inferred as follows:

$$\bar{x}_2(k+1)=\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i22}\bar{x}_2(k)+\left[\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i21}y(k)+\sum_{i=1}^{r}\mu_i\bar{B}^{\mathrm{d}}_{i2}u(k)\right],$$

$$y(k+1)-\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i11}y(k)-\sum_{i=1}^{r}\mu_i\bar{B}^{\mathrm{d}}_{i1}u(k)=\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i12}\bar{x}_2(k).$$

Finally, we can construct the full-dimensional state observer for the $n-q$ dimensions subsystem (8). $(\bar{A}^{\mathrm{d}}_{i22},\bar{A}^{\mathrm{d}}_{i12})$ is observable such that the $n-q$ dimensions state observer exists and is of the following form:

Observer Rule i: IF $s_1(k)$ is F_{i1} and $\cdots$ and $s_g(k)$ is F_{ig},

$$\text{THEN }\hat{\bar{x}}_2(k+1)=\bar{A}^{\mathrm{d}}_{i22}\hat{\bar{x}}_2(k)+[\bar{A}^{\mathrm{d}}_{i21}y(k)+\bar{B}^{\mathrm{d}}_{i2}u(k)]+\bar{L}^{\mathrm{d}}_i\left[y(k+1)-\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i11}y(k)-\sum_{i=1}^{r}\mu_i\bar{B}^{\mathrm{d}}_{i1}u(k)-\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i12}\hat{\bar{x}}_2(k)\right].$$

The final estimated state of the fuzzy observer is

$$\hat{\bar{x}}_2(k+1)=\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i22}\hat{\bar{x}}_2(k)+\left[\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i21}y(k)+\sum_{i=1}^{r}\mu_i\bar{B}^{\mathrm{d}}_{i2}u(k)\right]$$
$$+\sum_{i=1}^{r}\mu_i\bar{L}^{\mathrm{d}}_i\left[y(k+1)-\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i11}y(k)-\sum_{i=1}^{r}\mu_i\bar{B}^{\mathrm{d}}_{i1}u(k)-\sum_{i=1}^{r}\mu_i\bar{A}^{\mathrm{d}}_{i12}\hat{\bar{x}}_2(k)\right].$$

By some simple operations, we can obtain

$$\hat{\bar{x}}_2(k+1)=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\bar{A}^{\mathrm{d}}_{i22}-\bar{L}^{\mathrm{d}}_i\bar{A}^{\mathrm{d}}_{j12})\hat{\bar{x}}_2(k)+\sum_{i=1}^{r}\mu_i\bar{L}^{\mathrm{d}}_i y(k+1)$$
$$+\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\bar{A}^{\mathrm{d}}_{i21}-\bar{L}^{\mathrm{d}}_i\bar{A}^{\mathrm{d}}_{j11})y(k)+\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\bar{B}^{\mathrm{d}}_{i2}-\bar{L}^{\mathrm{d}}_i\bar{B}^{\mathrm{d}}_{j1})u(k).$$

The estimated state $\hat{\boldsymbol{x}}(k)$ can be given by

$$\hat{\boldsymbol{x}}(k)=\left(\sum_{i=1}^{r}\mu_i\boldsymbol{T}_i^{\mathrm{d}}\right)^{-1}\hat{\bar{\boldsymbol{x}}}(k)=(\boldsymbol{T}^{\mathrm{d}})^{-1}\hat{\bar{\boldsymbol{x}}}(k),$$

where $\boldsymbol{T}^{\mathrm{d}}$ must be nonsingular and

$$\hat{\bar{\boldsymbol{x}}}(k)=\begin{bmatrix}\hat{\bar{\boldsymbol{x}}}_1(k)\\ \hat{\bar{\boldsymbol{x}}}_2(k)\end{bmatrix}=\begin{bmatrix}\boldsymbol{y}(k)\\ \hat{\bar{\boldsymbol{x}}}_2(k)\end{bmatrix}.$$

3. General structure of fuzzy observer

The fuzzy observer and the fuzzy reduced-dimensional observer are respectively discussed in Ref. [2] and the section above, in which the structures of the fuzzy observer and the fuzzy reduced-dimensional observer are only special forms. In the section, we will address the general structure of the fuzzy observer.

First, consider the general structure of the fuzzy observer in the continuous case. The ith rule of the general fuzzy observer is of the following form:

Observer Rule i: IF $s_1(t)$ is $\boldsymbol{F}_{i1}$ and $\cdots$ and $s_g(t)$ is $\boldsymbol{F}_{ig}$,

$$\text{THEN } \dot{\boldsymbol{z}}(t)=\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}\boldsymbol{z}(t)+\boldsymbol{N}_i\boldsymbol{u}(t)+\boldsymbol{G}_i\sum_{j=1}^{r}\mu_j\boldsymbol{C}_j\boldsymbol{x}(t), \tag{9}$$

where $\boldsymbol{H}_{ij},\boldsymbol{N}_i$ and $\boldsymbol{G}_i$ are $l\times l, l\times m$ and $l\times q$ matrices, respectively. l is the dimension of the general fuzzy observer. The final state of the general fuzzy observer is inferred as follows:

$$\dot{\boldsymbol{z}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{H}_{ij}\boldsymbol{z}(t)+\sum_{i=1}^{r}\mu_i\boldsymbol{N}_i\boldsymbol{u}(t)+\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{G}_i\boldsymbol{C}_j\boldsymbol{x}(t). \tag{10}$$

Find, if possible, $\boldsymbol{R}_i$ $(i=1,2,\ldots,r)$ such that

$$\lim_{t\to\infty}\left[\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i\boldsymbol{x}(t)-\boldsymbol{z}(t)\right]=\boldsymbol{0}$$

for $\forall\boldsymbol{x}_0,\boldsymbol{z}_0,\boldsymbol{u}$.

Let $\boldsymbol{e}_i(t)=\boldsymbol{R}_i\boldsymbol{x}(t)-\boldsymbol{z}(t)$; then

$$\begin{aligned}\dot{\boldsymbol{e}}_i(t)&=\boldsymbol{R}_i\dot{\boldsymbol{x}}(t)-\dot{\boldsymbol{z}}(t)=\boldsymbol{R}_i[\boldsymbol{A}_i\boldsymbol{x}(t)+\boldsymbol{B}_i\boldsymbol{u}(t)]-\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}\boldsymbol{z}(t)-\boldsymbol{N}_i\boldsymbol{u}(t)-\boldsymbol{G}_i\sum_{j=1}^{r}\mu_j\boldsymbol{C}_j\boldsymbol{x}(t)\\&=\boldsymbol{R}_i\boldsymbol{A}_i\boldsymbol{x}(t)+\boldsymbol{R}_i\boldsymbol{B}_i\boldsymbol{u}(t)-\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}\boldsymbol{z}(t)-\boldsymbol{N}_i\boldsymbol{u}(t)-\sum_{j=1}^{r}\mu_j\boldsymbol{G}_i\boldsymbol{C}_j\boldsymbol{x}(t)\\&=\boldsymbol{R}_i\boldsymbol{A}_i\boldsymbol{x}(t)-\sum_{j=1}^{r}\mu_j\boldsymbol{G}_i\boldsymbol{C}_j\boldsymbol{x}(t)-\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}\boldsymbol{R}_i\boldsymbol{x}(t)+\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}\boldsymbol{R}_i\boldsymbol{x}(t)-\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}\boldsymbol{z}(t)\\&\quad+(\boldsymbol{R}_i\boldsymbol{B}_i-\boldsymbol{N}_i)\boldsymbol{u}(t)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{r} \mu_j \boldsymbol{R}_i \boldsymbol{A}_i \boldsymbol{x}(t) - \sum_{j=1}^{r} \mu_j \boldsymbol{G}_i \boldsymbol{C}_j \boldsymbol{x}(t) - \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij} \boldsymbol{R}_i \boldsymbol{x}(t) + \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij} [\boldsymbol{R}_i \boldsymbol{x}(t) - \boldsymbol{z}(t)] \\
&\quad + (\boldsymbol{R}_i \boldsymbol{B}_i - \boldsymbol{N}_i) \boldsymbol{u}(t) \\
&= \sum_{j=1}^{r} \mu_j (\boldsymbol{R}_i \boldsymbol{A}_i - \boldsymbol{G}_i \boldsymbol{C}_j - \boldsymbol{H}_{ij} \boldsymbol{R}_i) \boldsymbol{x}(t) + \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij} [\boldsymbol{R}_i \boldsymbol{x}(t) - \boldsymbol{z}(t)] + (\boldsymbol{R}_i \boldsymbol{B}_i - \boldsymbol{N}_i) \boldsymbol{u}(t).
\end{aligned}
$$

Let $\boldsymbol{R}_i\boldsymbol{A}_i - \boldsymbol{G}_i\boldsymbol{C}_j - \boldsymbol{H}_{ij}\boldsymbol{R}_i = \boldsymbol{0}$, i.e., $\boldsymbol{R}_i\boldsymbol{A}_i - \boldsymbol{H}_{ij}\boldsymbol{R}_i = \boldsymbol{G}_i\boldsymbol{C}_j$ $(i,j=1,2,\ldots,r)$ and $\boldsymbol{R}_i\boldsymbol{B}_i - \boldsymbol{N}_i = \boldsymbol{0}$, i.e., $\boldsymbol{R}_i\boldsymbol{B}_i = \boldsymbol{N}_i$ $(i = 1,2,\ldots,r)$, then

$$
\dot{\boldsymbol{e}}_i(t) = \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij} [\boldsymbol{R}_i \boldsymbol{x}(t) - \boldsymbol{z}(t)] = \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij} \boldsymbol{e}_i(t),
$$

that is,

$$
\begin{bmatrix} \dot{\boldsymbol{e}}_1(t) \\ \dot{\boldsymbol{e}}_2(t) \\ \vdots \\ \dot{\boldsymbol{e}}_r(t) \end{bmatrix}
= \begin{bmatrix} \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{1j} \boldsymbol{e}_1(t) \\ \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{2j} \boldsymbol{e}_2(t) \\ \vdots \\ \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{rj} \boldsymbol{e}_r(t) \end{bmatrix}
= \sum_{j=1}^{r} \mu_j \begin{bmatrix} \boldsymbol{H}_{1j} & & & \\ & \boldsymbol{H}_{2j} & & \\ & & \ddots & \\ & & & \boldsymbol{H}_{rj} \end{bmatrix}
\begin{bmatrix} \boldsymbol{e}_1(t) \\ \boldsymbol{e}_2(t) \\ \vdots \\ \boldsymbol{e}_r(t) \end{bmatrix}
= \sum_{j=1}^{r} \mu_j \boldsymbol{H}_j \begin{bmatrix} \boldsymbol{e}_1(t) \\ \boldsymbol{e}_2(t) \\ \vdots \\ \boldsymbol{e}_r(t) \end{bmatrix},
$$

where $\boldsymbol{H}_j = \text{block} - \text{diag}[\boldsymbol{H}_{1j} \;\; \boldsymbol{H}_{2j} \;\; \cdots \;\; \boldsymbol{H}_{rj}]$. Find a common positive-definite matrix $\boldsymbol{P}_2$ such that

$$
\boldsymbol{H}_j^{\mathrm{T}} \boldsymbol{P}_2 + \boldsymbol{P}_2 \boldsymbol{H}_j < \boldsymbol{0}, \quad j = 1,2,\ldots,r,
$$

then

$$
\lim_{t\to\infty} \boldsymbol{e}_i(t) = \boldsymbol{0}, \quad i = 1,2,\ldots,r.
$$

It is obvious that

$$
\boldsymbol{e}(t) = \sum_{i=1}^{r} \mu_i \boldsymbol{e}_i(t) = \sum_{i=1}^{r} \mu_i [\boldsymbol{R}_i \boldsymbol{x}(t) - \boldsymbol{z}(t)] = \sum_{i=1}^{r} \mu_i \boldsymbol{R}_i \boldsymbol{x}(t) - \boldsymbol{z}(t),
$$

therefore

$$
\lim_{t\to\infty} \left[\sum_{i=1}^{r} \mu_i \boldsymbol{R}_i \boldsymbol{x}(t) - \boldsymbol{z}(t) \right] = \lim_{t\to\infty} \boldsymbol{e}(t) = \lim_{t\to\infty} \sum_{i=1}^{r} \mu_i \boldsymbol{e}_i(t) = \boldsymbol{0}.
$$

To sum up, we can obtain the following theorem.

Theorem 3.1. *For the general fuzzy observer* (9) *and* (10), *find*, *if possible*, $\boldsymbol{R}_i$ $(i = 1,2,\ldots,r)$ *such that*

$$
\boldsymbol{R}_i \boldsymbol{A}_i - \boldsymbol{H}_{ij} \boldsymbol{R}_i = \boldsymbol{G}_i \boldsymbol{C}_j, \;\; i,j = 1,2,\ldots,r; \quad \boldsymbol{R}_i \boldsymbol{B}_i = \boldsymbol{N}_i, \;\; i = 1,2,\ldots,r,
$$

and there exists a common positive-definite matrix $\boldsymbol{P}_2$ *satisfying*

$$
\boldsymbol{H}_j^{\mathrm{T}} \boldsymbol{P}_2 + \boldsymbol{P}_2 \boldsymbol{H}_j < \boldsymbol{0}, \quad j = 1,2,\ldots,r;
$$

then

$$
\lim_{t\to\infty} \left[\sum_{i=1}^{r} \mu_i \boldsymbol{R}_i \boldsymbol{x}(t) - \boldsymbol{z}(t) \right] = \boldsymbol{0}
$$

for $\forall \boldsymbol{x}_0, \boldsymbol{z}_0, \boldsymbol{u}$.

Remark 3.1. If $l=n$, $\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i$ is nonsingular, the system (10) and $\hat{\boldsymbol{x}}(t)=(\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i)^{-1}\boldsymbol{z}(t)$ form a n dimensions fuzzy state observer.

Remark 3.2. If $l=n$, $\boldsymbol{R}_i$ $(i=1,2,\dots,r)$ are unit matrices, the system (10) and $\hat{\boldsymbol{x}}(t)=\boldsymbol{z}(t)$ form a n dimensions basic fuzzy state observer [2]. It is obvious that

$$\boldsymbol{A}_i-\boldsymbol{G}_i\boldsymbol{C}_j=\boldsymbol{H}_{ij},\ \ i,j=1,2,\dots,r;\qquad \boldsymbol{B}_i=\boldsymbol{N}_i,\ \ i=1,2,\dots,r.$$

Remark 3.3. If $l=n-q$, and there exist $\boldsymbol{R}_i$ $(i=1,2,\dots,r)$ such that

$$\boldsymbol{T}=\begin{bmatrix}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i\\ \sum_{i=1}^{r}\mu_i\boldsymbol{R}_i\end{bmatrix}$$

is nonsingular, the system (10) and $\boldsymbol{z}(t)=\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i\hat{\boldsymbol{x}}(t)$ form a $n-q$ dimensions fuzzy state observer. It is obvious that

$$\lim_{t\to\infty}\boldsymbol{y}(t)=\lim_{t\to\infty}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i\boldsymbol{x}(t)=\lim_{t\to\infty}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i\hat{\boldsymbol{x}}(t),$$

i.e., as $t\to\infty$, $\boldsymbol{y}(t)=\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i\hat{\boldsymbol{x}}(t)$, then

$$\begin{bmatrix}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i\\ \sum_{i=1}^{r}\mu_i\boldsymbol{R}_i\end{bmatrix}\hat{\boldsymbol{x}}(t)=\boldsymbol{T}\hat{\boldsymbol{x}}(t)=\begin{bmatrix}\boldsymbol{y}(t)\\ \boldsymbol{z}(t)\end{bmatrix};\quad \hat{\boldsymbol{x}}(t)=\boldsymbol{T}^{-1}\begin{bmatrix}\boldsymbol{y}(t)\\ \boldsymbol{z}(t)\end{bmatrix}.$$

Next, consider the general structure of the fuzzy observer in the discrete case. The ith rule of the general fuzzy observer is of the following form:

Observer Rule i: IF $s_1(k)$ is $\boldsymbol{F}_{i1}$ and $\cdots$ and $s_g(k)$ is $\boldsymbol{F}_{ig}$,

$$\text{THEN}\ \boldsymbol{z}(k+1)=\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{z}(k)+\boldsymbol{N}_i^{\mathrm{d}}\boldsymbol{u}(k)+\boldsymbol{G}_i^{\mathrm{d}}\sum_{j=1}^{r}\mu_j\boldsymbol{C}_j^{\mathrm{d}}\boldsymbol{x}(k),\tag{11}$$

where $\boldsymbol{H}_{ij}^{\mathrm{d}},\boldsymbol{N}_i^{\mathrm{d}}$ and $\boldsymbol{G}_i^{\mathrm{d}}$ are $l\times l, l\times m$ and $l\times q$ matrices, respectively. l is the dimension of the general fuzzy observer. The final state of the general fuzzy observer is inferred as follows:

$$\boldsymbol{z}(k+1)=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{z}(k)+\sum_{i=1}^{r}\mu_i\boldsymbol{N}_i^{\mathrm{d}}\boldsymbol{u}(k)+\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{G}_i^{\mathrm{d}}\boldsymbol{C}_j^{\mathrm{d}}\boldsymbol{x}(k).\tag{12}$$

Find, if possible, $\boldsymbol{R}_i^{\mathrm{d}}$ $(i=1,2,\dots,r)$ such that

$$\lim_{k\to\infty}\left[\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k)-\boldsymbol{z}(k)\right]=\boldsymbol{0}$$

for $\forall\boldsymbol{x}_0,\boldsymbol{z}_0,\boldsymbol{u}$.

Let $\boldsymbol{e}_i(k)=\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k)-\boldsymbol{z}(k)$; then

$$\begin{aligned}\boldsymbol{e}_i(k+1)&=\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k+1)-\boldsymbol{z}(k+1)\\ &=\boldsymbol{R}_i^{\mathrm{d}}[\boldsymbol{A}_i^{\mathrm{d}}\boldsymbol{x}(k)+\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{u}(k)]-\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{z}(k)-\boldsymbol{N}_i^{\mathrm{d}}\boldsymbol{u}(k)-\boldsymbol{G}_i^{\mathrm{d}}\sum_{j=1}^{r}\mu_j\boldsymbol{C}_j^{\mathrm{d}}\boldsymbol{x}(k)\\ &=\sum_{j=1}^{r}\mu_j(\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{G}_i^{\mathrm{d}}\boldsymbol{C}_j^{\mathrm{d}}-\boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{R}_i^{\mathrm{d}})\boldsymbol{x}(k)+\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}^{\mathrm{d}}[\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k)-\boldsymbol{z}(k)]+(\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{B}_i^{\mathrm{d}}-\boldsymbol{N}_i^{\mathrm{d}})\boldsymbol{u}(k).\end{aligned}$$

Let $\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{A}_i^{\mathrm{d}} - \boldsymbol{G}_i^{\mathrm{d}}\boldsymbol{C}_j^{\mathrm{d}} - \boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{R}_i^{\mathrm{d}} = \boldsymbol{0}$, i.e., $\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{A}_i^{\mathrm{d}} - \boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{R}_i^{\mathrm{d}} = \boldsymbol{G}_i^{\mathrm{d}}\boldsymbol{C}_j^{\mathrm{d}}$ $(i,j=1,2,\dots,r)$ and $\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{B}_i^{\mathrm{d}} - \boldsymbol{N}_i^{\mathrm{d}} = \boldsymbol{0}$, i.e., $\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{B}_i^{\mathrm{d}} = \boldsymbol{N}_i^{\mathrm{d}}$ $(i=1,2,\dots,r)$; then

$$\boldsymbol{e}_i(k+1) = \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij}^{\mathrm{d}}[\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k) - \boldsymbol{z}(k)] = \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{e}_i(k),$$

that is,

$$\begin{bmatrix} \boldsymbol{e}_1(k+1) \\ \boldsymbol{e}_2(k+1) \\ \vdots \\ \boldsymbol{e}_r(k+1) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{1j}^{\mathrm{d}}\boldsymbol{e}_1(k) \\ \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{2j}^{\mathrm{d}}\boldsymbol{e}_2(k) \\ \vdots \\ \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{rj}^{\mathrm{d}}\boldsymbol{e}_r(k) \end{bmatrix} = \sum_{j=1}^{r} \mu_j \begin{bmatrix} \boldsymbol{H}_{1j}^{\mathrm{d}} & & & \\ & \boldsymbol{H}_{2j}^{\mathrm{d}} & & \\ & & \ddots & \\ & & & \boldsymbol{H}_{rj}^{\mathrm{d}} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1(k) \\ \boldsymbol{e}_2(k) \\ \vdots \\ \boldsymbol{e}_r(k) \end{bmatrix}$$

$$= \sum_{j=1}^{r} \mu_j \boldsymbol{H}_j^{\mathrm{d}} \begin{bmatrix} \boldsymbol{e}_1(k) \\ \boldsymbol{e}_2(k) \\ \vdots \\ \boldsymbol{e}_r(k) \end{bmatrix},$$

where $\boldsymbol{H}_j^{\mathrm{d}} = \text{block} - \text{diag}[\boldsymbol{H}_{1j}^{\mathrm{d}} \ \boldsymbol{H}_{2j}^{\mathrm{d}} \ \cdots \ \boldsymbol{H}_{rj}^{\mathrm{d}}]$. Find a common positive-definite matrix $\boldsymbol{P}_2^{\mathrm{d}}$ such that

$$(\boldsymbol{H}_j^{\mathrm{d}})^{\mathrm{T}}\boldsymbol{P}_2^{\mathrm{d}}\boldsymbol{H}_j^{\mathrm{d}} - \boldsymbol{P}_2^{\mathrm{d}} < \boldsymbol{0}, \quad j=1,2,\dots,r,$$

then

$$\lim_{k\to\infty} \boldsymbol{e}_i(k) = \boldsymbol{0}, \quad i=1,2,\dots,r.$$

It is obvious that

$$\boldsymbol{e}(k) = \sum_{i=1}^{r} \mu_i \boldsymbol{e}_i(k) = \sum_{i=1}^{r} \mu_i [\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k) - \boldsymbol{z}(k)] = \sum_{i=1}^{r} \mu_i \boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k) - \boldsymbol{z}(k),$$

therefore

$$\lim_{k\to\infty} \left[\sum_{i=1}^{r} \mu_i \boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k) - \boldsymbol{z}(k) \right] = \lim_{k\to\infty} \boldsymbol{e}(k) = \lim_{k\to\infty} \sum_{i=1}^{r} \mu_i \boldsymbol{e}_i(k) = \boldsymbol{0}.$$

To sum up, we can obtain the following theorem.

Theorem 3.2. *For the general fuzzy observer* (11) *and* (12), *find, if possible,* $\boldsymbol{R}_i^{\mathrm{d}}$ $(i=1,2,\dots,r)$ *such that*

$$\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{A}_i^{\mathrm{d}} - \boldsymbol{H}_{ij}^{\mathrm{d}}\boldsymbol{R}_i^{\mathrm{d}} = \boldsymbol{G}_i^{\mathrm{d}}\boldsymbol{C}_j^{\mathrm{d}}, \quad i,j=1,2,\dots,r; \qquad \boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{B}_i^{\mathrm{d}} = \boldsymbol{N}_i^{\mathrm{d}}, \quad i=1,2,\dots,r,$$

and there exists a common positive-definite matrix $\boldsymbol{P}_2^{\mathrm{d}}$ *satisfying*

$$(\boldsymbol{H}_j^{\mathrm{d}})^{\mathrm{T}}\boldsymbol{P}_2^{\mathrm{d}}\boldsymbol{H}_j^{\mathrm{d}} - \boldsymbol{P}_2^{\mathrm{d}} < \boldsymbol{0}, \quad j=1,2,\dots,r,$$

then

$$\lim_{k\to\infty}\left[\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i^{\mathrm{d}}\boldsymbol{x}(k)-\boldsymbol{z}(k)\right]=\boldsymbol{0}$$

for $\forall\boldsymbol{x}_0,\boldsymbol{z}_0,\boldsymbol{u}$.

Remark 3.4. If $l=n$, $\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i^{\mathrm{d}}$ is nonsingular, the system (12) and $\hat{\boldsymbol{x}}(k)=(\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i^{\mathrm{d}})^{-1}\cdot\boldsymbol{z}(k)$ form a n dimensions fuzzy state observer.

Remark 3.5. If $l=n$, $\boldsymbol{R}_i^{\mathrm{d}}$ $(i=1,2,\ldots,r)$ are unit matrices, the system (12) and $\hat{\boldsymbol{x}}(k)=\boldsymbol{z}(k)$ form a n dimensions basic fuzzy state observer [2]. It is obvious that

$$\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{G}_i^{\mathrm{d}}\boldsymbol{C}_j^{\mathrm{d}}=\boldsymbol{H}_{ij}^{\mathrm{d}},\ \ i,j=1,2,\ldots,r;\qquad \boldsymbol{B}_i^{\mathrm{d}}=\boldsymbol{N}_i^{\mathrm{d}},\ \ i=1,2,\ldots,r.$$

Remark 3.6. If $l=n-q$, and there exist $\boldsymbol{R}_i^{\mathrm{d}}$ $(i=1,2,\ldots,r)$ such that

$$\boldsymbol{T}^{\mathrm{d}}=\begin{bmatrix}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i^{\mathrm{d}}\\ \sum_{i=1}^{r}\mu_i\boldsymbol{R}_i^{\mathrm{d}}\end{bmatrix}$$

is nonsingular, the system (12) and $\boldsymbol{z}(k)=\sum_{i=1}^{r}\mu_i\boldsymbol{R}_i^{\mathrm{d}}\hat{\boldsymbol{x}}(k)$ form a $n-q$ dimensions fuzzy state observer. It is obvious that

$$\lim_{k\to\infty}\boldsymbol{y}(k)=\lim_{k\to\infty}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i^{\mathrm{d}}\boldsymbol{x}(k)=\lim_{k\to\infty}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i^{\mathrm{d}}\hat{\boldsymbol{x}}(k),$$

i.e., as $k\to\infty$, $\boldsymbol{y}(k)=\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i^{\mathrm{d}}\hat{\boldsymbol{x}}(k)$, then

$$\begin{bmatrix}\sum_{i=1}^{r}\mu_i\boldsymbol{C}_i^{\mathrm{d}}\\ \sum_{i=1}^{r}\mu_i\boldsymbol{R}_i^{\mathrm{d}}\end{bmatrix}\hat{\boldsymbol{x}}(k)=\boldsymbol{T}^{\mathrm{d}}\hat{\boldsymbol{x}}(k)=\begin{bmatrix}\boldsymbol{y}(k)\\ \boldsymbol{z}(k)\end{bmatrix};\quad \hat{\boldsymbol{x}}(k)=(\boldsymbol{T}^{\mathrm{d}})^{-1}\begin{bmatrix}\boldsymbol{y}(k)\\ \boldsymbol{z}(k)\end{bmatrix}.$$

4. Fuzzy functional observer

It is well known that the ultimate aim of obtaining the state estimate $\hat{\boldsymbol{x}}(t)$ is the construction of the feedback control law $\sum_{i=1}^{r}\mu_i\boldsymbol{K}_i\hat{\boldsymbol{x}}(t)$ in the fuzzy control system design. Therefore, we can directly discuss more the estimate of the feedback control law $\sum_{i=1}^{r}\mu_i\boldsymbol{K}_i\hat{\boldsymbol{x}}(t)$ than the estimate of the state. In what follows, we introduce the concept of the fuzzy functional observer.

First, we discuss the fuzzy functional observer in the continuous case. The ith rule of the fuzzy functional observer is of the following form:

Observer Rule i: IF $s_1(t)$ is $\boldsymbol{F}_{i1}$ and $\cdots$ and $s_g(t)$ is $\boldsymbol{F}_{ig}$,

$$\text{THEN } \dot{\boldsymbol{z}}(t)=\sum_{j=1}^{r}\mu_j\boldsymbol{H}_{ij}\boldsymbol{z}(t)+\boldsymbol{N}_i\boldsymbol{u}(t)+\boldsymbol{G}_i\sum_{j=1}^{r}\mu_j\boldsymbol{C}_j\boldsymbol{x}(t), \tag{13}$$
$$\dot{\boldsymbol{w}}_i(t)=\boldsymbol{E}_i\boldsymbol{z}(t)+\boldsymbol{M}_i\boldsymbol{y}_i(t),\quad i=1,2,\ldots,r,$$

where $\boldsymbol{H}_{ij},\boldsymbol{N}_i,\boldsymbol{G}_i,\boldsymbol{E}_i$ and $\boldsymbol{M}_i$ are $l\times l$, $l\times m$, $l\times q$, $m\times l$ and $m\times q$ matrices, respectively. l is the dimension of the fuzzy functional observer. The final state and output of the fuzzy functional observer are inferred as follows:

$$\begin{aligned}\dot{\boldsymbol{z}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{H}_{ij}\boldsymbol{z}(t)+\sum_{i=1}^{r}\mu_i\boldsymbol{N}_i\boldsymbol{u}(t)+\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{G}_i\boldsymbol{C}_j\boldsymbol{x}(t),\\ \boldsymbol{w}(t) &= \sum_{i=1}^{r}\mu_i\boldsymbol{w}_i(t)=\sum_{i=1}^{r}\mu_i\boldsymbol{E}_i\boldsymbol{z}(t)+\sum_{i=1}^{r}\mu_i\boldsymbol{M}_i\boldsymbol{y}_i(t).\end{aligned} \tag{14}$$

Theorem 4.1. *For the fuzzy functional observer* (13) *and* (14), *find, if possible,* $\boldsymbol{R}_i$ $(i=1,2,\ldots,r)$ *such that*

$$\boldsymbol{R}_i\boldsymbol{A}_i-\boldsymbol{H}_{ij}\boldsymbol{R}_i=\boldsymbol{G}_i\boldsymbol{C}_j,\quad i,j=1,2,\ldots,r,$$

$$\boldsymbol{R}_i\boldsymbol{B}_i=\boldsymbol{N}_i,\quad \boldsymbol{K}_i=\boldsymbol{E}_i\boldsymbol{R}_i+\boldsymbol{M}_i\boldsymbol{C}_i,\quad i=1,2,\ldots,r,$$

and there exists a common positive-definite matrix $\boldsymbol{P}_2$ *satisfying*

$$\boldsymbol{H}_j^{\mathrm{T}}\boldsymbol{P}_2+\boldsymbol{P}_2\boldsymbol{H}_j<\boldsymbol{0},\quad j=1,2,\ldots,r,$$

then

$$\lim_{t\to\infty}\left[\sum_{i=1}^{r}\mu_i\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{w}(t)\right]=\boldsymbol{0}$$

for $\forall\boldsymbol{x}_0,\boldsymbol{z}_0,\boldsymbol{u}$.

Proof. According to the proof of Theorem 3.1, it is obvious that

$$\lim_{t\to\infty}\boldsymbol{z}(t)=\lim_{t\to\infty}\boldsymbol{R}_i\boldsymbol{x}(t).$$

Let $\boldsymbol{e}_i'(t)=\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{w}_i(t)$; then

$$\boldsymbol{e}_i'(t)=\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{E}_i\boldsymbol{z}(t)-\boldsymbol{M}_i\boldsymbol{y}_i(t)=\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{E}_i\boldsymbol{z}(t)-\boldsymbol{M}_i\boldsymbol{C}_i\boldsymbol{x}(t),$$

$$\begin{aligned}\lim_{t\to\infty}\boldsymbol{e}_i'(t)&=\lim_{t\to\infty}[\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{E}_i\boldsymbol{z}(t)-\boldsymbol{M}_i\boldsymbol{C}_i\boldsymbol{x}(t)]\\ &=\lim_{t\to\infty}[\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{E}_i\boldsymbol{R}_i\boldsymbol{x}(t)-\boldsymbol{M}_i\boldsymbol{C}_i\boldsymbol{x}(t)]\\ &=\lim_{t\to\infty}(\boldsymbol{K}_i-\boldsymbol{E}_i\boldsymbol{R}_i-\boldsymbol{M}_i\boldsymbol{C}_i)\boldsymbol{x}(t).\end{aligned}$$

If $\boldsymbol{K}_i=\boldsymbol{E}_i\boldsymbol{R}_i+\boldsymbol{M}_i\boldsymbol{C}_i$ $(i=1,2,\ldots,r)$, $\lim_{t\to\infty}\boldsymbol{e}_i'(t)=\boldsymbol{0}$. Let $\boldsymbol{e}'(t)=\sum_{i=1}^{r}\mu_i\boldsymbol{e}_i'(t)$ and $\boldsymbol{w}(t)=\sum_{i=1}^{r}\mu_i\boldsymbol{w}_i(t)$; then

$$\boldsymbol{e}'(t)=\sum_{i=1}^{r}\mu_i\boldsymbol{e}_i'(t)=\sum_{i=1}^{r}\mu_i[\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{w}_i(t)]=\sum_{i=1}^{r}\mu_i\boldsymbol{K}_i\boldsymbol{x}(t)-\sum_{i=1}^{r}\mu_i\boldsymbol{w}_i(t)=\sum_{i=1}^{r}\mu_i\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{w}(t),$$

therefore

$$\lim_{t\to\infty}\left[\sum_{i=1}^{r}\mu_i\boldsymbol{K}_i\boldsymbol{x}(t)-\boldsymbol{w}(t)\right]=\lim_{t\to\infty}\boldsymbol{e}'(t)=\lim_{t\to\infty}\sum_{i=1}^{r}\mu_i\boldsymbol{e}_i'(t)=\boldsymbol{0}.\qquad\square$$

Next, we consider the fuzzy functional observer in the discrete case. The ith rule of the fuzzy functional observer is of the following form:

Observer Rule i: IF $s_1(k)$ is $\boldsymbol{F}_{i1}$ and $\cdots$ and $s_g(k)$ is $\boldsymbol{F}_{ig}$,

$$\text{THEN } \boldsymbol{z}(k+1) = \sum_{j=1}^{r} \mu_j \boldsymbol{H}_{ij}^{\mathrm{d}} \boldsymbol{z}(t) + \boldsymbol{N}_i^{\mathrm{d}} \boldsymbol{u}(k) + \boldsymbol{G}_i^{\mathrm{d}} \sum_{j=1}^{r} \mu_j \boldsymbol{C}_j^{\mathrm{d}} \boldsymbol{x}(k), \tag{15}$$

$$\boldsymbol{w}_i(k) = \boldsymbol{E}_i^{\mathrm{d}} \boldsymbol{z}(k) + \boldsymbol{M}_i^{\mathrm{d}} \boldsymbol{y}_i(k), \quad i = 1,2,\ldots,r,$$

where $\boldsymbol{H}_{ij}^{\mathrm{d}}, \boldsymbol{N}_i^{\mathrm{d}}, \boldsymbol{G}_i^{\mathrm{d}}, \boldsymbol{E}_i^{\mathrm{d}}$ and $\boldsymbol{M}_i^{\mathrm{d}}$ are $l \times l$, $l \times m$, $l \times q$, $m \times l$ and $m \times q$ matrices, respectively. l is the dimension of the fuzzy functional observer. The final state and output of the fuzzy functional observer are inferred as follows:

$$\begin{aligned} \boldsymbol{z}(k+1) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{H}_{ij}^{\mathrm{d}} \boldsymbol{z}(k) + \sum_{i=1}^{r} \mu_i \boldsymbol{N}_i^{\mathrm{d}} \boldsymbol{u}(k) + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{G}_i^{\mathrm{d}} \boldsymbol{C}_j^{\mathrm{d}} \boldsymbol{x}(k), \\ \boldsymbol{w}(k) &= \sum_{i=1}^{r} \mu_i \boldsymbol{w}_i(k) = \sum_{i=1}^{r} \mu_i \boldsymbol{E}_i^{\mathrm{d}} \boldsymbol{z}(k) + \sum_{i=1}^{r} \mu_i \boldsymbol{M}_i^{\mathrm{d}} \boldsymbol{y}_i(k). \end{aligned} \tag{16}$$

Theorem 4.2. *For the fuzzy functional observer* (15) *and* (16), *find, if possible,* $\boldsymbol{R}_i^{\mathrm{d}}$ $(i = 1,2,\ldots,r)$ *such that*

$$\boldsymbol{R}_i^{\mathrm{d}} \boldsymbol{A}_i^{\mathrm{d}} - \boldsymbol{H}_{ij}^{\mathrm{d}} \boldsymbol{R}_i^{\mathrm{d}} = \boldsymbol{G}_i^{\mathrm{d}} \boldsymbol{C}_j^{\mathrm{d}}, \quad i,j = 1,2,\ldots,r,$$

$$\boldsymbol{R}_i^{\mathrm{d}} \boldsymbol{B}_i^{\mathrm{d}} = \boldsymbol{N}_i^{\mathrm{d}}, \qquad \boldsymbol{K}_i^{\mathrm{d}} = \boldsymbol{E}_i^{\mathrm{d}} \boldsymbol{R}_i^{\mathrm{d}} + \boldsymbol{M}_i^{\mathrm{d}} \boldsymbol{C}_i^{\mathrm{d}}, \quad i = 1,2,\ldots,r;$$

and there exists a common positive-definite matrix $\boldsymbol{P}_2^{\mathrm{d}}$ *satisfying*

$$(\boldsymbol{H}_j^{\mathrm{d}})^{\mathrm{T}} \boldsymbol{P}_2^{\mathrm{d}} \boldsymbol{H}_j^{\mathrm{d}} - \boldsymbol{P}_2^{\mathrm{d}} < \boldsymbol{0}, \quad j = 1,2,\ldots,r;$$

then

$$\lim_{k\to\infty} \left[\sum_{i=1}^{r} \mu_i \boldsymbol{K}_i^{\mathrm{d}} \boldsymbol{x}(k) - \boldsymbol{w}(k) \right] = \boldsymbol{0}$$

for $\forall \boldsymbol{x}_0, \boldsymbol{z}_0, \boldsymbol{u}$.

Proof. The proof is similar to the proof of Theorem 4.1, and hence omitted. □

5. Separation property

The fuzzy observer can reconstruct the unmeasurable states of a controlled system, thus it is possible to carry out the fuzzy state feedback in practice. The fuzzy state feedback is synthesized for the real states of the controlled system, therefore, it is natural that the designer and the analyzer of the control system will be concerned about the problem, which may occur when the fuzzy state feedback is realized by the estimated states instead of the real states. In Ref. [2], the basic characteristic of the fuzzy control system including the fuzzy full-dimensional observer, i.e., the separation property that the fuzzy controller and the fuzzy observer can be independently designed, is expounded. In the section, the separation property in both the continuous and discrete case is addressed for the fuzzy control system including either the fuzzy reduced-dimensional or fuzzy functional observer.

First, we address the separation property in the continuous case for the fuzzy control system including the fuzzy reduced-dimensional observer.

Let $\tilde{\bar{x}}_2(t) = \bar{x}_2(t) - \hat{\bar{x}}_2(t)$, then

$$\dot{\tilde{\bar{x}}}_2(t) = \dot{\bar{x}}_2(t) - \dot{\hat{\bar{x}}}_2(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j(\bar{A}_{i22} - \bar{L}_i\bar{A}_{j12})\tilde{\bar{x}}_2(t).$$

Substitute $u(t) = -\sum_{i=1}^{r}\mu_i K_i \hat{x}(t)$ into (2); then

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - B_iK_j)x(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j B_iK_j[x(t) - \hat{x}(t)] \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - B_iK_j)x(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j B_iK_j T^{-1}[\bar{x}(t) - \hat{\bar{x}}(t)] \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - B_iK_j)x(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j B_iK_j T^{-1}\left\{\begin{bmatrix} y(t) \\ \bar{x}_2(t)\end{bmatrix} - \begin{bmatrix} y(t) \\ \hat{\bar{x}}_2(t)\end{bmatrix}\right\} \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - B_iK_j)x(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j B_iK_j T^{-1}\begin{bmatrix} \mathbf{0} \\ \tilde{\bar{x}}_2(t)\end{bmatrix}.
\end{aligned}$$

Considering the system

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - B_iK_j)x(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j B_iK_j T^{-1}\begin{bmatrix} \mathbf{0} \\ \tilde{\bar{x}}_2(t)\end{bmatrix}, \\
\dot{\tilde{\bar{x}}}_2(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\bar{A}_{i22} - \bar{L}_i\bar{A}_{j12})\tilde{\bar{x}}_2(t),
\end{aligned} \tag{17}$$

we can obtain the following theorem.

Theorem 5.1. *If there exist two scalar functions* $V(x): R^n \to R$ *and* $\tilde{V}(\tilde{\bar{x}}_2): R^{n-q} \to R$ *and positive real numbers* $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ *and* $\tilde{\gamma}_4$ *such that*

(1) $\gamma_1\|x\|^2 \leqslant V(x) \leqslant \gamma_2\|x\|^2$, $\tilde{\gamma}_1\|\tilde{\bar{x}}_2\|^2 \leqslant \tilde{V}(\tilde{\bar{x}}_2) \leqslant \tilde{\gamma}_2\|\tilde{\bar{x}}_2\|^2$,

(2) $(\partial V(x)/\partial x)\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(A_i - B_iK_j)x \leqslant -\gamma_3\|x\|^2$, $(\partial\tilde{V}(\tilde{\bar{x}}_2)/\partial\tilde{\bar{x}}_2)\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\bar{A}_{i22} - \bar{L}_i\bar{A}_{j12})\tilde{\bar{x}}_2 \leqslant -\tilde{\gamma}_3\|\tilde{\bar{x}}_2\|^2$,

(3) $\|\partial V(x)/\partial x\| \leqslant \gamma_4\|x\|$, $\|\partial\tilde{V}(\tilde{\bar{x}}_2)/\partial\tilde{\bar{x}}_2\| \leqslant \tilde{\gamma}_4\|\tilde{\bar{x}}_2\|$,

the system (17) *is globally uniformly asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 4.1 in Ref. [2], and omitted. □

Remark 5.1. If there exists a common positive-definite matrix P_1 such that

$$(A_i - B_iK_i)^{\mathrm{T}}P_1 + P_1(A_i - B_iK_i) < \mathbf{0}$$

for $i = 1, 2, \ldots, r$ and

$$\left(\frac{A_i - B_iK_j + A_j - B_jK_i}{2}\right)^{\mathrm{T}} P_1 + P_1\left(\frac{A_i - B_iK_j + A_j - B_jK_i}{2}\right) < \mathbf{0}$$

for $i<j\leqslant r$, the subsystem

$$\dot{\boldsymbol{x}}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}(\boldsymbol{A}_i-\boldsymbol{B}_i\boldsymbol{K}_j)\boldsymbol{x}(t)$$

is asymptotically stable in the large. If there exists a common positive-definite matrix $\boldsymbol{P}_2$ such that

$$(\bar{\boldsymbol{A}}_{i22}-\bar{\boldsymbol{L}}_i\bar{\boldsymbol{A}}_{i12})^{\mathrm{T}}\boldsymbol{P}_2+\boldsymbol{P}_2(\bar{\boldsymbol{A}}_{i22}-\bar{\boldsymbol{L}}_i\bar{\boldsymbol{A}}_{i12})<\boldsymbol{0}$$

for $i=1,2,\ldots,r$ and

$$\left(\frac{\bar{\boldsymbol{A}}_{i22}-\bar{\boldsymbol{L}}_i\bar{\boldsymbol{A}}_{j12}+\bar{\boldsymbol{A}}_{j22}-\bar{\boldsymbol{L}}_j\bar{\boldsymbol{A}}_{i12}}{2}\right)^{\mathrm{T}}\boldsymbol{P}_2+\boldsymbol{P}_2\left(\frac{\bar{\boldsymbol{A}}_{i22}-\bar{\boldsymbol{L}}_i\bar{\boldsymbol{A}}_{j12}+\bar{\boldsymbol{A}}_{j22}-\bar{\boldsymbol{L}}_j\bar{\boldsymbol{A}}_{i12}}{2}\right)<\boldsymbol{0}$$

for $i<j\leqslant r$, the subsystem

$$\dot{\tilde{\bar{\boldsymbol{x}}}}_2(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\bar{\boldsymbol{A}}_{i22}-\bar{\boldsymbol{L}}_i\bar{\boldsymbol{A}}_{j12})\tilde{\bar{\boldsymbol{x}}}_2(t)$$

is asymptotically stable in the large. Let $V(\boldsymbol{x})=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}$ and $\tilde{V}(\tilde{\bar{\boldsymbol{x}}}_2)=\tilde{\bar{\boldsymbol{x}}}_2^{\mathrm{T}}\boldsymbol{P}_2\tilde{\bar{\boldsymbol{x}}}_2$; it is obvious that the conditions (1)–(3) can be satisfied.

In the discrete case, let $\tilde{\bar{\boldsymbol{x}}}_2(k)=\bar{\boldsymbol{x}}_2(k)-\hat{\bar{\boldsymbol{x}}}_2(k)$; considering the system

$$\begin{aligned}\boldsymbol{x}(k+1)&=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_j^{\mathrm{d}})\boldsymbol{x}(k)+\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_j^{\mathrm{d}}(\boldsymbol{T}^{\mathrm{d}})^{-1}\begin{bmatrix}\boldsymbol{0}\\ \tilde{\bar{\boldsymbol{x}}}_2(k)\end{bmatrix},\\ \tilde{\bar{\boldsymbol{x}}}_2(k+1)&=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\bar{\boldsymbol{A}}_{i22}^{\mathrm{d}}-\bar{\boldsymbol{L}}_i^{\mathrm{d}}\bar{\boldsymbol{A}}_{j12}^{\mathrm{d}})\tilde{\bar{\boldsymbol{x}}}_2(k),\end{aligned}\tag{18}$$

we can obtain the following theorem.

Theorem 5.2. *If there exist common positive-definite matrices $\boldsymbol{P}_1^{\mathrm{d}}$ and $\boldsymbol{P}_2^{\mathrm{d}}$ such that*

$$\begin{aligned}&(\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}})^{\mathrm{T}}\boldsymbol{P}_1^{\mathrm{d}}(\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}})-\boldsymbol{P}_1^{\mathrm{d}}<\boldsymbol{0},\\ &(\bar{\boldsymbol{A}}_{i22}^{\mathrm{d}}-\bar{\boldsymbol{L}}_i^{\mathrm{d}}\bar{\boldsymbol{A}}_{i12}^{\mathrm{d}})^{\mathrm{T}}\boldsymbol{P}_2^{\mathrm{d}}(\bar{\boldsymbol{A}}_{i22}^{\mathrm{d}}-\bar{\boldsymbol{L}}_i^{\mathrm{d}}\bar{\boldsymbol{A}}_{i12}^{\mathrm{d}})-\boldsymbol{P}_2^{\mathrm{d}}<\boldsymbol{0}\end{aligned}$$

for $i=1,2,\ldots,r$ and

$$\left(\frac{\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_j^{\mathrm{d}}+\boldsymbol{A}_j^{\mathrm{d}}-\boldsymbol{B}_j^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}}}{2}\right)^{\mathrm{T}}\boldsymbol{P}_1^{\mathrm{d}}\left(\frac{\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_j^{\mathrm{d}}+\boldsymbol{A}_j^{\mathrm{d}}-\boldsymbol{B}_j^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}}}{2}\right)-\boldsymbol{P}_1^{\mathrm{d}}<\boldsymbol{0},$$

$$\left(\frac{\bar{\boldsymbol{A}}_{i22}^{\mathrm{d}}-\bar{\boldsymbol{L}}_i^{\mathrm{d}}\bar{\boldsymbol{A}}_{j12}^{\mathrm{d}}+\bar{\boldsymbol{A}}_{j22}^{\mathrm{d}}-\bar{\boldsymbol{L}}_j^{\mathrm{d}}\bar{\boldsymbol{A}}_{i12}^{\mathrm{d}}}{2}\right)^{\mathrm{T}}\boldsymbol{P}_2^{\mathrm{d}}\left(\frac{\bar{\boldsymbol{A}}_{i22}^{\mathrm{d}}-\bar{\boldsymbol{L}}_i^{\mathrm{d}}\bar{\boldsymbol{A}}_{j12}^{\mathrm{d}}+\bar{\boldsymbol{A}}_{j22}^{\mathrm{d}}-\bar{\boldsymbol{L}}_j^{\mathrm{d}}\bar{\boldsymbol{A}}_{i12}^{\mathrm{d}}}{2}\right)-\boldsymbol{P}_2^{\mathrm{d}}<\boldsymbol{0}$$

for $i<j\leqslant r$, the system (18) *is uniformly asymptotically stable.*

Proof. See Section 5 of Ref. [2]. □

Next, we discuss the separation property in the continuous case for the fuzzy control system including the fuzzy functional observer.

Substitute $\boldsymbol{u}(t) = -[\sum_{i=1}^{r} \mu_i \boldsymbol{E}_i \boldsymbol{z}(t) + \sum_{i=1}^{r} \mu_i \boldsymbol{M}_i \boldsymbol{y}_i(t)]$ into (2); then

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \mu_i \boldsymbol{A}_i \boldsymbol{x}(t) - \sum_{i=1}^{r} \mu_i \boldsymbol{B}_i \left[\sum_{j=1}^{r} \mu_j \boldsymbol{E}_j \boldsymbol{z}(t) + \sum_{j=1}^{r} \mu_j \boldsymbol{M}_j \boldsymbol{y}_j(t) \right] \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{A}_i \boldsymbol{x}(t) - \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{M}_j \boldsymbol{C}_j \boldsymbol{x}(t) - \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j \boldsymbol{z}(t) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (\boldsymbol{A}_i - \boldsymbol{B}_i \boldsymbol{M}_j \boldsymbol{C}_j) \boldsymbol{x}(t) - \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j \boldsymbol{z}(t) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j [\boldsymbol{A}_i - \boldsymbol{B}_i (\boldsymbol{K}_j - \boldsymbol{E}_j \boldsymbol{R}_j)] \boldsymbol{x}(t) - \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j \boldsymbol{z}(t) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (\boldsymbol{A}_i - \boldsymbol{B}_i \boldsymbol{K}_j) \boldsymbol{x}(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j \boldsymbol{R}_j \boldsymbol{x}(t) - \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j \boldsymbol{z}(t) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (\boldsymbol{A}_i - \boldsymbol{B}_i \boldsymbol{K}_j) \boldsymbol{x}(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j [\boldsymbol{R}_j \boldsymbol{x}(t) - \boldsymbol{z}(t)] \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (\boldsymbol{A}_i - \boldsymbol{B}_i \boldsymbol{K}_j) \boldsymbol{x}(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j \boldsymbol{e}_j(t).
\end{aligned}
$$

According to Section 3, we can obtain

$$
\dot{\boldsymbol{e}}_j(t) = \sum_{i=1}^{r} \mu_i \boldsymbol{H}_{ji} \boldsymbol{e}_j(t),
$$

that is,

$$
\begin{bmatrix} \dot{\boldsymbol{e}}_1(t) \\ \dot{\boldsymbol{e}}_2(t) \\ \vdots \\ \dot{\boldsymbol{e}}_r(t) \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{r} \mu_i \boldsymbol{H}_{1i} \boldsymbol{e}_1(t) \\ \sum_{i=1}^{r} \mu_i \boldsymbol{H}_{2i} \boldsymbol{e}_2(t) \\ \vdots \\ \sum_{i=1}^{r} \mu_i \boldsymbol{H}_{ri} \boldsymbol{e}_r(t) \end{bmatrix} = \sum_{i=1}^{r} \mu_i \begin{bmatrix} \boldsymbol{H}_{1i} & & & \\ & \boldsymbol{H}_{2i} & & \\ & & \ddots & \\ & & & \boldsymbol{H}_{ri} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1(t) \\ \boldsymbol{e}_2(t) \\ \vdots \\ \boldsymbol{e}_r(t) \end{bmatrix} = \sum_{i=1}^{r} \mu_i \boldsymbol{H}_i \begin{bmatrix} \boldsymbol{e}_1(t) \\ \boldsymbol{e}_2(t) \\ \vdots \\ \boldsymbol{e}_r(t) \end{bmatrix},
$$

where $\boldsymbol{H}_i = \text{block} - \text{diag}[\boldsymbol{H}_{1i} \;\; \boldsymbol{H}_{2i} \;\; \cdots \;\; \boldsymbol{H}_{ri}]$.

Considering the system

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (\boldsymbol{A}_i - \boldsymbol{B}_i \boldsymbol{K}_j) \boldsymbol{x}(t) + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \boldsymbol{B}_i \boldsymbol{E}_j \boldsymbol{e}_j(t), \\
\dot{\tilde{e}}(t) &= \sum_{i=1}^{r} \mu_i \boldsymbol{H}_i \tilde{e}(t), \\
\tilde{e}(t) &= [\boldsymbol{e}_1(t) \quad \boldsymbol{e}_2(t) \quad \cdots \quad \boldsymbol{e}_r(t)]^{\mathrm{T}},
\end{aligned} \tag{19}
$$

we can obtain the following theorem.

Theorem 5.3. *If there exist two scalar functions* $V(\boldsymbol{x}): R^n \to R$ *and* $\tilde{V}(\tilde{\boldsymbol{e}}): R^{l\times r} \to R$ *and positive real numbers* $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ *and* $\tilde{\gamma}_4$ *such that*

(1) $\gamma_1\|\boldsymbol{x}\|^2 \leqslant V(\boldsymbol{x}) \leqslant \gamma_2\|\boldsymbol{x}\|^2$, $\tilde{\gamma}_1\|\tilde{\boldsymbol{e}}\|^2 \leqslant \tilde{V}(\tilde{\boldsymbol{e}}) \leqslant \tilde{\gamma}_2\|\tilde{\boldsymbol{e}}\|^2$,
(2) $(\partial V(\boldsymbol{x})/\partial \boldsymbol{x}) \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j(\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_j)\boldsymbol{x} \leqslant -\gamma_3\|\boldsymbol{x}\|^2$, $(\partial \tilde{V}(\tilde{\boldsymbol{e}})/\partial \tilde{\boldsymbol{e}}) \sum_{i=1}^{r} \mu_i \boldsymbol{H}_i \tilde{\boldsymbol{e}} \leqslant -\tilde{\gamma}_3\|\tilde{\boldsymbol{e}}\|^2$,
(3) $\|\partial V(\boldsymbol{x})/\partial \boldsymbol{x}\| \leqslant \gamma_4\|\boldsymbol{x}\|$, $\|\partial \tilde{V}(\tilde{\boldsymbol{e}})/\partial \tilde{\boldsymbol{e}}\| \leqslant \tilde{\gamma}_4\|\tilde{\boldsymbol{e}}\|$,

the system (19) *is globally uniformly asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 4.1 in Ref. [2], and omitted. □

Remark 5.2. If there exists a common positive-definite matrix $\boldsymbol{P}_1$ such that

$$(\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_i)^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1(\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_i) < \boldsymbol{0}$$

for $i = 1, 2, \ldots, r$ and

$$\left(\frac{\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_j + \boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K}_i}{2}\right)^{\mathrm{T}} \boldsymbol{P}_1 + \boldsymbol{P}_1\left(\frac{\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_j + \boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K}_i}{2}\right) < \boldsymbol{0}$$

for $i < j \leqslant r$, the subsystem

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}(\boldsymbol{A}_i - \boldsymbol{B}_i\boldsymbol{K}_j)\boldsymbol{x}(t)$$

is asymptotically stable in the large. If there exists a common positive-definite matrix $\boldsymbol{P}_2$ such that

$$\boldsymbol{H}_i^{\mathrm{T}}\boldsymbol{P}_2 + \boldsymbol{P}_2\boldsymbol{H}_i < \boldsymbol{0}$$

for $i = 1, 2, \ldots, r$, the subsystem

$$\dot{\tilde{\boldsymbol{e}}}(t) = \sum_{i=1}^{r} \mu_i \boldsymbol{H}_i \tilde{\boldsymbol{e}}(t)$$

is asymptotically stable in the large. Let $V(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}$ and $\tilde{V}(\tilde{\boldsymbol{e}}) = \tilde{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{P}_2\tilde{\boldsymbol{e}}$; it is obvious that the conditions (1)–(3) can be satisfied.

In the discrete case, considering the system

$$\begin{aligned}
\boldsymbol{x}(k+1) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j(\boldsymbol{A}_i^{\mathrm{d}} - \boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_j^{\mathrm{d}})\boldsymbol{x}(k) + \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j \boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{E}_j^{\mathrm{d}}\boldsymbol{e}_j(k),\\
\tilde{\boldsymbol{e}}(k+1) &= \sum_{i=1}^{r} \mu_i \boldsymbol{H}_i^{\mathrm{d}}\tilde{\boldsymbol{e}}(k),\\
\tilde{\boldsymbol{e}}(k) &= [\boldsymbol{e}_1(k)\ \ \boldsymbol{e}_2(k)\ \ \cdots\ \ \boldsymbol{e}_r(k)]^{\mathrm{T}},\\
\boldsymbol{H}_i^{\mathrm{d}} &= \text{block} - \text{diag}[\boldsymbol{H}_{1i}^{\mathrm{d}}\quad \boldsymbol{H}_{2i}^{\mathrm{d}}\quad \cdots\quad \boldsymbol{H}_{ri}^{\mathrm{d}}],
\end{aligned} \tag{20}$$

we can obtain the following theorem.

Theorem 5.4. *If there exist common positive-definite matrices* $\boldsymbol{P}_1^{\mathrm{d}}$ *and* $\boldsymbol{P}_2^{\mathrm{d}}$ *such that*

$$(\boldsymbol{A}_i^{\mathrm{d}} - \boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}})^{\mathrm{T}}\boldsymbol{P}_1^{\mathrm{d}}(\boldsymbol{A}_i^{\mathrm{d}} - \boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}}) - \boldsymbol{P}_1^{\mathrm{d}} < \boldsymbol{0}, \quad (\boldsymbol{H}_i^{\mathrm{d}})^{\mathrm{T}}\boldsymbol{P}_2^{\mathrm{d}}\boldsymbol{H}_i^{\mathrm{d}} - \boldsymbol{P}_2^{\mathrm{d}} < \boldsymbol{0}$$

for $i=1,2,\ldots,r$ *and*

$$\left(\frac{\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_j^{\mathrm{d}}+\boldsymbol{A}_j^{\mathrm{d}}-\boldsymbol{B}_j^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}}}{2}\right)^{\mathrm{T}}\boldsymbol{P}_1^{\mathrm{d}}\left(\frac{\boldsymbol{A}_i^{\mathrm{d}}-\boldsymbol{B}_i^{\mathrm{d}}\boldsymbol{K}_j^{\mathrm{d}}+\boldsymbol{A}_j^{\mathrm{d}}-\boldsymbol{B}_j^{\mathrm{d}}\boldsymbol{K}_i^{\mathrm{d}}}{2}\right)-\boldsymbol{P}_1^{\mathrm{d}}<\boldsymbol{0}$$

for $i<j\leqslant r$, *the system* (20) *is uniformly asymptotically stable.*

Proof. See Section 5 of Ref. [2]. □

Remark 5.3. Theorems 5.1–5.4 show that the separation property, in both the continuous and discrete case, hold for the fuzzy control system including either the fuzzy reduced-dimensional or fuzzy functional observer. These results are the developments of the results in Ref. [2].

6. Numerical simulation

To illustrate the performance of the fuzzy reduced-dimensional observer and the fuzzy functional observer, consider a problem of balancing of an inverted pendulum on a cart. In the section, using MATLAB software package, we finish the numerical simulation at first. The equations of motion for the inverted pendulum device are

$$\begin{aligned}
\dot{x}_1&=x_2,\\
\dot{x}_2&=\frac{1.0}{[(M+m)(J+ml^2)-m^2l^2\cos^2 x_1]}[-f_1(M+m)x_2-m^2l^2x_2^2\sin x_1\cos x_1\\
&\quad+f_0\,mlx_4\cos x_1+(M+m)mgl\sin x_1-ml\cos x_1u],\\
\dot{x}_3&=x_4,\\
\dot{x}_4&=\frac{1.0}{[(M+m)(J+ml^2)-m^2l^2\cos^2 x_1]}[f_1mlx_2\cos x_1+(J+ml^2)mlx_2^2\sin x_1\\
&\quad-f_0(J+ml^2)x_4-m^2gl^2\sin x_1\cos x_1+(J+ml^2)u],
\end{aligned}\tag{21}$$

where x_1 denotes the angle (rad) of the pendulum from the vertical, x_2 is the angular velocity (rad/s), x_3 is the displacement (m) of the cart, and x_4 is the velocity (m/s) of the cart. $g=9.8$ m/s^2 is the gravity constant, m is the mass (kg) of the pendulum, M is the mass (kg) of the cart, f_0 is the friction factor (N/m/s) of the cart, f_1 is the friction factor (N/rad/s) of the pendulum, l is the length (m) from the center of mass of the pendulum to the shaft axis, J is the moment of inertia (kg m^2) of the pendulum round its center of mass, and u is the force (N) applied to the cart. We choose $M=1.3282$ kg, $m=0.22$ kg, $f_0=22.915$ N/m/s, $f_1=0.007056$ N/rad/s, $l=0.304$ m, $J=0.004963$ kg m^2 in the numerical simulation and the experiment.

To design the fuzzy reduced-dimensional observer and the fuzzy functional observer, we must have a fuzzy model which represents the dynamics of the nonlinear plant. Therefore, we first represent the system by a Takagi–Sugeno fuzzy model. To minimize the design effort and complexity, we try to use as few rules as possible. Notice that when $x_1=\pm\pi/2$, the system is uncontrollable. Here we approximate the system by the following two-rule fuzzy model:

Plant Rule 1: IF $x_1(t)$ is about 0,

THEN $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}_1\boldsymbol{x}(t)+\boldsymbol{B}_1u(t)$,

$\boldsymbol{y}_1(t)=\boldsymbol{C}_1\boldsymbol{x}(t)$, Plant Rule 2: IF $x_1(t)$ is about $\pm\pi/3$,

THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{B}_2u(t)$,

$\boldsymbol{y}_2(t) = \boldsymbol{C}_2\boldsymbol{x}(t)$,

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ a_{21} & a_{22} & 0.0 & a_{24} \\ 0.0 & 0.0 & 0.0 & 1.0 \\ a_{41} & a_{42} & 0.0 & a_{44} \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0.0 \\ b_2 \\ 0.0 \\ b_4 \end{bmatrix}, \quad \boldsymbol{C}_1 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix},$$

$$a_{21} = (M+m)mgl/a, \quad a_{22} = -f_1(M+m)/a, \quad a_{24} = f_0ml/a,$$

$$a_{41} = -m^2gl^2/a, \quad a_{42} = f_1ml/a, \quad a_{44} = -f_0(J+ml^2)/a,$$

$$b_2 = -ml/a, \quad b_4 = (J+ml^2)/a, \quad a = (M+m)(J+ml^2) - m^2l^2,$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ a'_{21} & a'_{22} & 0.0 & a'_{24} \\ 0.0 & 0.0 & 0.0 & 1.0 \\ a'_{41} & a'_{42} & 0.0 & a'_{44} \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0.0 \\ b'_2 \\ 0.0 \\ b'_4 \end{bmatrix}, \quad \boldsymbol{C}_2 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix},$$

$$a'_{21} = \frac{3\sqrt{3}}{2\pi}(M+m)mgl/a', \quad a'_{22} = -f_1(M+m)/a', \quad a'_{24} = f_0\,ml\,\cos 60^\circ/a',$$

$$a'_{41} = -\frac{3\sqrt{3}}{2\pi}m^2gl^2\cos 60^\circ/a', \quad a'_{42} = f_1ml\cos 60^\circ/a', \quad a'_{44} = -f_0(J+ml^2)/a',$$

$$b'_2 = -ml\cos 60^\circ/a', \quad b'_4 = (J+ml^2)/a', \quad a' = (M+m)(J+ml^2) - m^2l^2(\cos 60^\circ)^2.$$

Membership functions for Plant Rules 1 and 2 are

$$\mu_1[x_1(t)] = \left\{1.0 - \frac{1.0}{1.0 + \mathrm{e}^{-7.0[x_1(t)-\pi/6]}}\right\}\frac{1.0}{1.0 + \mathrm{e}^{-7.0[x_1(t)+\pi/6]}},$$

$$\mu_2[x_1(t)] = 1.0 - \mu_1[x_1(t)].$$

Choosing the closed-loop eigenvalues $[-7.0\ \ -3.0\ \ -6.0\ \ -1.0]$ for $\boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{K}_1$ and $\boldsymbol{A}_2 - \boldsymbol{B}_2\boldsymbol{K}_2$, we have

$$\boldsymbol{K}_1 = [\,-69.1679 \quad -12.8245 \quad -6.6685 \quad -33.9422],$$
$$\boldsymbol{K}_2 = [-145.2252 \quad -30.0898 \quad -8.8434 \quad -37.5585].$$

In the system, the states $x_1(t)$ and $x_3(t)$ are measurable, but the states $x_2(t)$ and $x_4(t)$ are unmeasurable, which makes us to design the state observer observing these states in order to realize the fuzzy state feedback control law $u(t) = -\sum_{i=1}^{2}\mu_i\boldsymbol{K}_i\hat{\boldsymbol{x}}(t)$.

6.1. Fuzzy reduced-dimensional observer

Consider linear nonsingular transformation matrices

$$\boldsymbol{T}_i = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}, \quad i = 1, 2;$$

then

$$\bar{\boldsymbol{A}}_{111} = \bar{\boldsymbol{A}}_{211} = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}, \quad \bar{\boldsymbol{A}}_{112} = \bar{\boldsymbol{A}}_{212} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \quad \bar{\boldsymbol{A}}_{122} = \begin{bmatrix} -0.3149 & 44.1811 \\ 0.0136 & -16.7096 \end{bmatrix},$$

$$\bar{\boldsymbol{A}}_{121} = \begin{bmatrix} 29.2529 & 0.0 \\ -1.2637 & 0.0 \end{bmatrix}, \quad \bar{\boldsymbol{A}}_{221} = \begin{bmatrix} 22.0587 & 0.0 \\ -0.4765 & 0.0 \end{bmatrix}, \quad \bar{\boldsymbol{A}}_{222} = \begin{bmatrix} -0.2872 & 20.1425 \\ 0.0062 & -15.2361 \end{bmatrix},$$

$$\bar{\boldsymbol{B}}_{11} = \bar{\boldsymbol{B}}_{21} = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}, \quad \bar{\boldsymbol{B}}_{12} = \begin{bmatrix} -1.9280 \\ 0.7292 \end{bmatrix}, \quad \bar{\boldsymbol{B}}_{22} = \begin{bmatrix} -0.8790 \\ 0.6649 \end{bmatrix}.$$

Choosing the closed-loop eigenvalues $[-32.0 - 30.0]$ for $\bar{\boldsymbol{A}}_{122} - \bar{\boldsymbol{L}}_1\bar{\boldsymbol{A}}_{112}$ and $\bar{\boldsymbol{A}}_{222} - \bar{\boldsymbol{L}}_2\bar{\boldsymbol{A}}_{212}$, we can obtain

$$\bar{\boldsymbol{L}}_1 = \begin{bmatrix} 31.6851 & 44.1811 \\ 0.0136 & 13.2904 \end{bmatrix}, \quad \bar{\boldsymbol{L}}_2 = \begin{bmatrix} 31.7128 & 20.1425 \\ 0.0062 & 14.7639 \end{bmatrix}.$$

In order to check the stability of a fuzzy system with a fuzzy controller and a fuzzy reduced-dimensional observer, we must find two common positive-definite matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. LMI optimization algorithm is a effective method. In this fuzzy system, matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are as follows:

$$\boldsymbol{P}_1 = \begin{bmatrix} 1.93554895090464 & 0.35986524787917 & 0.22362030149497 & 0.35248948967165 \\ 0.35986524787917 & 0.07627576776849 & 0.04387910868787 & 0.06840971854849 \\ 0.22362030149497 & 0.04387910868787 & 0.03834011388736 & 0.05137789625508 \\ 0.35248948967165 & 0.06840971854849 & 0.05137789625508 & 0.08002908164611 \end{bmatrix},$$

$$\boldsymbol{P}_2 = \begin{bmatrix} 0.05015632119223 & 0.18060090474080 \\ 0.18060090474080 & 0.92230091079124 \end{bmatrix}.$$

The resulting control law is

Controller Rule 1: IF $x_1(t)$ is about 0,
THEN $u(t) = -\boldsymbol{K}_1\hat{\boldsymbol{x}}(t)$,

Controller Rule 2: IF $x_1(t)$ is about $\pm\pi/3$,
THEN $u(t) = -\boldsymbol{K}_2\hat{\boldsymbol{x}}(t)$,

that is,

$$u(t) = -\mu_1\boldsymbol{K}_1\hat{\boldsymbol{x}}(t) - \mu_2\boldsymbol{K}_2\hat{\boldsymbol{x}}(t),$$

where μ_1 and μ_2 are the membership values of Rules 1 and 2, respectively. This nonlinear control law guarantees the stability of the fuzzy control system (fuzzy model + fuzzy controller + fuzzy reduced-dimensional

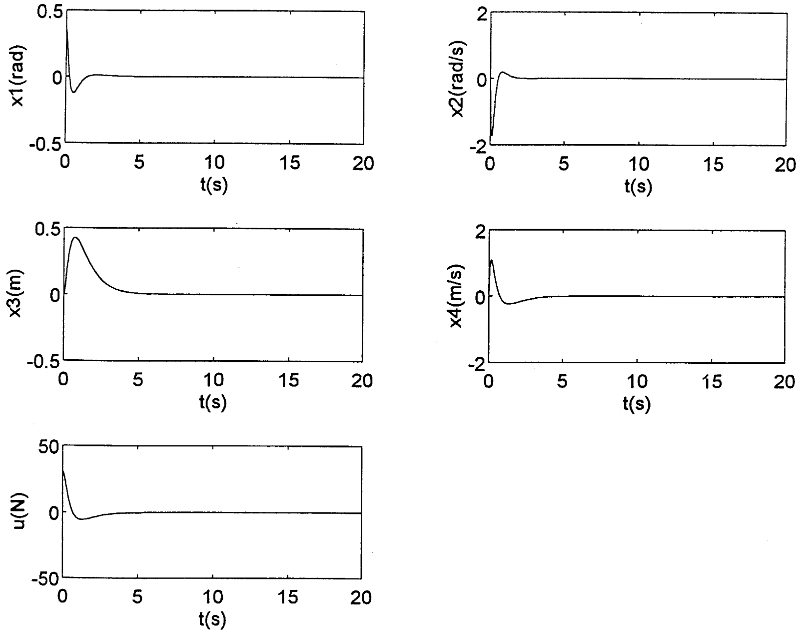

Fig. 1. Closed-loop response for initial conditions $x_1(0) = 20.0°$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$. x_1: the angle (rad) of the pendulum from the vertical; x_2: the angular velocity (rad/s); x_3: the displacement (m) of the cart; x_4: the velocity (m/s) of the cart; u: the force (N) applied to the cart; t: time (s).

observer). To assess the effectiveness of the nonlinear control law, we apply the controller to the original system (21).

Simulation results indicate the nonlinear control law can balance the pendulum for initial conditions $x_1(0) \in [-60.0° \ +60.0°](x_2(0) = x_3(0) = x_4(0) = 0.0)$. Figs. 1, 2 and 3 illustrate the closed-loop behavior of the system with the fuzzy controller and the fuzzy reduced-dimensional observer for initial conditions $x_1(0) = 20.0, 40.0, 60.0°$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$.

6.2. Fuzzy functional observer

In order to design the fuzzy functional observer in the continuous case, the following matrix equations

$$\boldsymbol{R}_i\boldsymbol{A}_i - \boldsymbol{H}_{ii}\boldsymbol{R}_i = \boldsymbol{G}_i\boldsymbol{C}_i \quad (\boldsymbol{C}_i = \boldsymbol{C}_j, \boldsymbol{H}_{ii} = \boldsymbol{H}_{ij},\ i = 1, 2),$$

$$\boldsymbol{R}_i\boldsymbol{B}_i = \boldsymbol{N}_i, \quad \boldsymbol{K}_i = \boldsymbol{E}_i\boldsymbol{R}_i + \boldsymbol{M}_i\boldsymbol{C}_i, \quad i = 1, 2$$

must be solved. For a single output system, these matrix equations can be easily solved, that is, the following matrix equations

$$\boldsymbol{R}_i\bar{\boldsymbol{A}}_i - H_{ii}\boldsymbol{R}_i = \boldsymbol{G}_i\bar{\boldsymbol{C}}, \quad \boldsymbol{R}_i\bar{\boldsymbol{B}}_i = N_i, \quad \bar{\boldsymbol{K}}_i = E_i\boldsymbol{R}_i + \boldsymbol{M}_i\bar{\boldsymbol{C}}, \quad i = 1, 2$$

will be solved, where

$$\bar{\boldsymbol{A}}_i = \boldsymbol{T}_i\boldsymbol{A}_i\boldsymbol{T}_i^{-1} = \begin{bmatrix} \bar{\boldsymbol{A}}_{i11} & \bar{\boldsymbol{A}}_{i12} \\ \bar{\boldsymbol{A}}_{i21} & \bar{\boldsymbol{A}}_{i22} \end{bmatrix}, \qquad \bar{\boldsymbol{B}}_i = \boldsymbol{T}_i\boldsymbol{B}_i = \begin{bmatrix} \bar{\boldsymbol{B}}_{i1} \\ \bar{\boldsymbol{B}}_{i2} \end{bmatrix},$$

$$\bar{\boldsymbol{C}} = \boldsymbol{C}_i\boldsymbol{T}_i^{-1} = [\boldsymbol{I}_q \quad \boldsymbol{0}], \qquad \bar{\boldsymbol{K}}_i = \boldsymbol{K}_i\boldsymbol{T}_i^{-1} = [\bar{\boldsymbol{K}}_{i1} \quad \bar{\boldsymbol{K}}_{i2}].$$

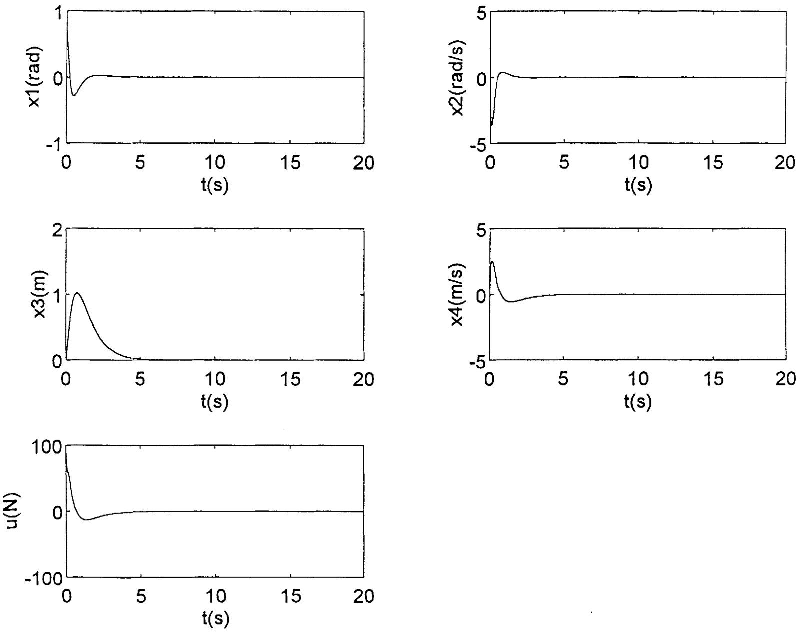

Fig. 2. Closed-loop response for initial conditions $x_1(0) = 40.0°$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$. $x_1, x_2, \ldots$ as in Fig. 1.

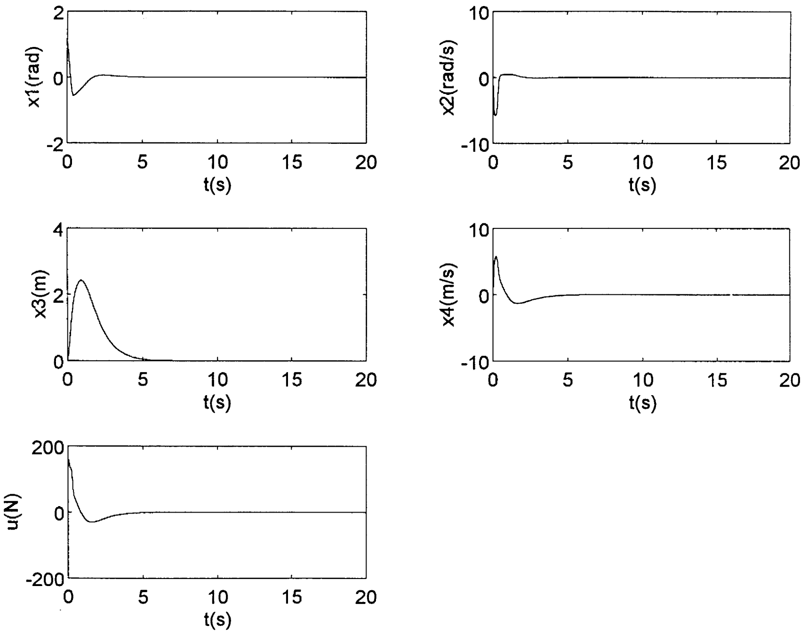

Fig. 3. Closed-loop response for initial conditions $x_1(0) = 60.0°$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$. $x_1, x_2, \ldots$ as in Fig. 1.

Let $\boldsymbol{R}_i = [\boldsymbol{R}_{i1} \;\; \boldsymbol{R}_{i2}]$, and substituting $\bar{\boldsymbol{A}}_i, \bar{\boldsymbol{B}}_i, \bar{\boldsymbol{C}}, \bar{\boldsymbol{K}}_i$ and $\boldsymbol{R}_i$ into the above matrix equations, we can obtain

$$\boldsymbol{R}_{i1}\bar{\boldsymbol{A}}_{i11} + \boldsymbol{R}_{i2}\bar{\boldsymbol{A}}_{i21} - H_{ii}\boldsymbol{R}_{i1} = \boldsymbol{G}_i, \qquad \boldsymbol{R}_{i1}\bar{\boldsymbol{A}}_{i12} + \boldsymbol{R}_{i2}\bar{\boldsymbol{A}}_{i22} - H_{ii}\boldsymbol{R}_{i2} = \boldsymbol{0},$$

$$\boldsymbol{R}_{i1}\bar{\boldsymbol{B}}_{i1} + \boldsymbol{R}_{i2}\bar{\boldsymbol{B}}_{i2} = N_i, \qquad \bar{\boldsymbol{K}}_{i1} = E_i\boldsymbol{R}_{i1} + \boldsymbol{M}_i, \qquad \bar{\boldsymbol{K}}_{i2} = E_i\boldsymbol{R}_{i2}, \quad i = 1, 2.$$

Choosing $E_1 = E_2 = 1.0$, $H_{11} = H_{12} = H_{21} = H_{22} = -30.0$, and substituting the variables into the matrix equations above, we can get

$$\boldsymbol{G}_1 = [11102.4426 \quad 30531.0892], \qquad \boldsymbol{G}_2 = [26182.7608 \quad 34817.8419],$$

$$\boldsymbol{M}_1 = [-450.3247 \quad -1024.3715], \qquad \boldsymbol{M}_2 = [-1039.5122 \quad -1169.4381],$$

$$N_1 = -0.0245, \qquad N_2 = 1.4767.$$

In order to check the stability of a fuzzy system with a fuzzy controller and a fuzzy functional observer, we must find two common positive-definite matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. $\boldsymbol{P}_1$ is the same as the one in Section 6.1. On the other hand, here

$$\boldsymbol{H}_1 = \begin{bmatrix} H_{11} & \\ & H_{21} \end{bmatrix} = \begin{bmatrix} -30.0 & 0.0 \\ 0.0 & -30.0 \end{bmatrix}, \qquad \boldsymbol{H}_2 = \begin{bmatrix} H_{12} & \\ & H_{22} \end{bmatrix} = \begin{bmatrix} -30.0 & 0.0 \\ 0.0 & -30.0 \end{bmatrix},$$

so $\boldsymbol{P}_2$ can be easily found.

The resulting control law is

$$u(t) = -\left[\sum_{i=1}^{2} \mu_i E_i z(t) + \sum_{i=1}^{2} \mu_i \boldsymbol{M}_i \boldsymbol{y}_i(t)\right],$$

where μ_1 and μ_2 are the membership values of Rules 1 and 2, respectively. This nonlinear control law guarantees the stability of the fuzzy control system (fuzzy model + fuzzy controller + fuzzy functional observer). To assess the effectiveness of the nonlinear control law, we apply the controller to the original system (21).

Simulation results indicate the nonlinear control law can balance the pendulum for initial conditions $x_1(0) \in [-60.0^\circ + 60.0^\circ](x_2(0) = x_3(0) = x_4(0) = 0.0)$. Figs. 4, 5 and 6 illustrate the closed-loop behavior of the system with the fuzzy controller and the fuzzy functional observer for initial conditions $x_1(0) = 20.0, 40.0, 60.0^\circ$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$.

7. Experiment

On the basis of the numerical simulation above, using C language, we do an experiment on an inverted pendulum system including an inverted pendulum on a cart moving along a slideway 1.2 m in length and a PC with an Intel 80486DX operating at 33 MHz.

First, we decentralize Plant Rules 1 and 2 with a sampling period $T = 0.005$ s and a zero-order holder, and obtain the discrete-time Takagi–Sugeno fuzzy dynamic model:

Plant Rule 1: IF $x_1(k)$ is about 0,

THEN $\boldsymbol{x}(k+1) = \boldsymbol{A}_1^{\mathrm{d}}\boldsymbol{x}(k) + \boldsymbol{B}_1^{\mathrm{d}}u(k)$,

$\boldsymbol{y}_1(k) = \boldsymbol{C}_1^{\mathrm{d}}\boldsymbol{x}(k)$,

Plant Rule 2: IF $x_1(k)$ is about $\pm\pi/3$,

THEN $\boldsymbol{x}(k+1) = \boldsymbol{A}_2^{\mathrm{d}}\boldsymbol{x}(k) + \boldsymbol{B}_2^{\mathrm{d}}u(k)$,

$\boldsymbol{y}_2(k) = \boldsymbol{C}_2^{\mathrm{d}}\boldsymbol{x}(k)$,

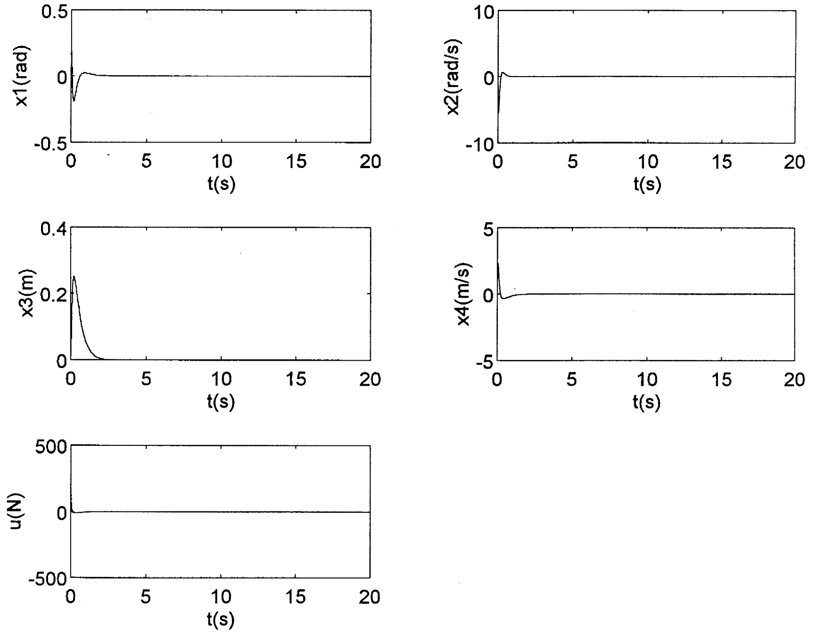

Fig. 4. Closed-loop response for initial conditions $x_1(0) = 20.0°$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$. $x_1, x_2, \ldots$ as in Fig. 1.

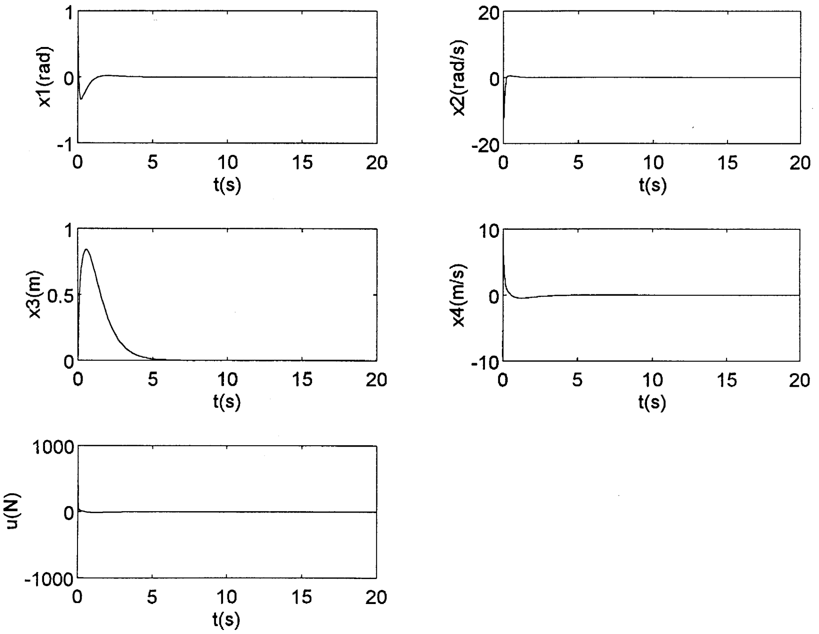

Fig. 5. Closed-loop response for initial conditions $x_1(0) = 40.0°$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$. $x_1, x_2, \ldots$ as in Fig. 1.

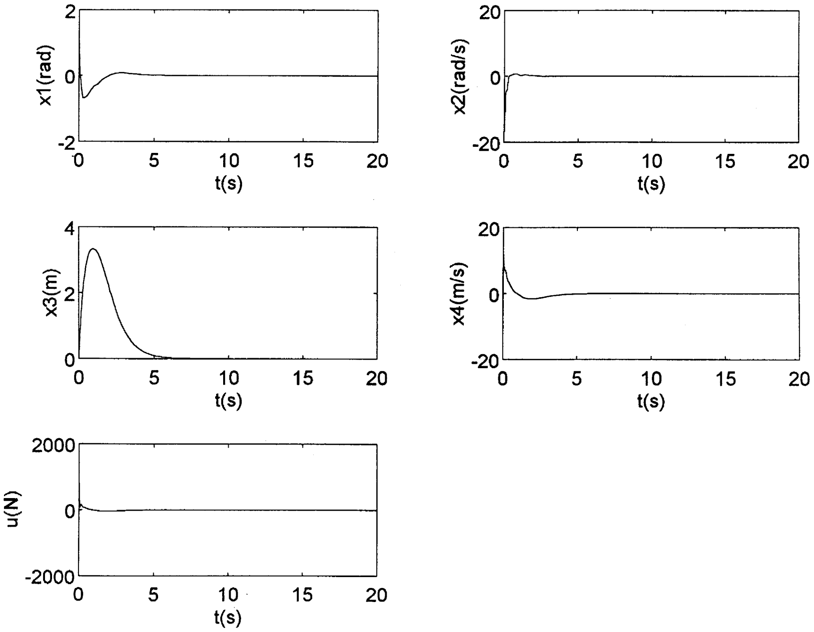

Fig. 6. Closed-loop response for initial conditions $x_1(0) = 60.0°$, and $x_2(0) = x_3(0) = x_4(0) = 0.0$. $x_1, x_2, \ldots$ as in Fig. 1.

where

$$A_1^{\mathrm{d}} = \begin{bmatrix} 1.00036435382220 & 0.00499668532032 & 0.0 & 0.00053694917482 \\ 0.14548923599960 & 0.99879808214278 & 0.0 & 0.21178674758507 \\ -0.00001535803517 & 0.00000013956098 & 1.0 & 0.00479683778240 \\ -0.00605760930610 & 0.00004985546214 & 0.0 & 0.91985283848196 \end{bmatrix},$$

$$B_1^{\mathrm{d}} = \begin{bmatrix} -0.00002343221361 \\ -0.00924227569649 \\ 0.00000886590520 \\ 0.00349758505425 \end{bmatrix}, \qquad C_1^{\mathrm{d}} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix},$$

$$A_2^{\mathrm{d}} = \begin{bmatrix} 1.00027541825024 & 0.00499687384590 & 0.0 & 0.00024540136153 \\ 0.11010754204057 & 0.99884207145830 & 0.0 & 0.09691078779658 \\ -0.00000580472257 & 0.00000006582749 & 1.0 & 0.00481429623406 \\ -0.00229232728470 & 0.00002403610304 & 0.0 & 0.92665011300373 \end{bmatrix},$$

$$B_2^{\mathrm{d}} = \begin{bmatrix} -0.00001070920190 \\ -0.00422914195054 \\ 0.00000810402644 \\ 0.00320095513839 \end{bmatrix}, \qquad C_2^{\mathrm{d}} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}.$$

Next, choosing the closed-loop eigenvalues $[\mathrm{e}^{-7.0\mathrm{T}}\ \ \mathrm{e}^{-3.0\mathrm{T}}\ \ \mathrm{e}^{-6.0\mathrm{T}}\ \ \mathrm{e}^{-1.0\mathrm{T}}]$ for $\boldsymbol{A}_1^{\mathrm{d}} - \boldsymbol{B}_1^{\mathrm{d}}\boldsymbol{K}_1^{\mathrm{d}}$ and $\boldsymbol{A}_2^{\mathrm{d}} - \boldsymbol{B}_2^{\mathrm{d}}\boldsymbol{K}_2^{\mathrm{d}}$, we have

$$\boldsymbol{K}_1^{\mathrm{d}} = [-69.9891 \quad -12.9866 \quad -6.6672 \quad -33.9567],$$

$$\boldsymbol{K}_2^{\mathrm{d}} = [-146.2793 \quad -30.3157 \quad -8.8091 \quad -37.5238].$$

7.1. Fuzzy reduced-dimensional observer

Consider linear nonsingular transformation matrices

$$\boldsymbol{T}_i^{\mathrm{d}} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}, \quad i = 1, 2;$$

then

$$\bar{\boldsymbol{A}}_{111}^{\mathrm{d}} = \begin{bmatrix} 1.00036435382220 & 0.0 \\ -0.00001535803517 & 1.0 \end{bmatrix}, \qquad \bar{\boldsymbol{A}}_{112}^{\mathrm{d}} = \begin{bmatrix} 0.00499668532032 & 0.00053694917482 \\ 0.00000013956098 & 0.00479683778240 \end{bmatrix},$$

$$\bar{\boldsymbol{A}}_{121}^{\mathrm{d}} = \begin{bmatrix} 0.14548923599960 & 0.0 \\ -0.00605760930610 & 0.0 \end{bmatrix}, \qquad \bar{\boldsymbol{A}}_{122}^{\mathrm{d}} = \begin{bmatrix} 0.99879808214278 & 0.21178674758507 \\ 0.00004985546214 & 0.91985283848196 \end{bmatrix},$$

$$\bar{\boldsymbol{A}}_{211}^{\mathrm{d}} = \begin{bmatrix} 1.00027541825024 & 0.0 \\ -0.00000580472257 & 1.0 \end{bmatrix}, \qquad \bar{\boldsymbol{A}}_{212}^{\mathrm{d}} = \begin{bmatrix} 0.00499687384590 & 0.00024540136153 \\ 0.00000006582749 & 0.00481429623406 \end{bmatrix},$$

$$\bar{\boldsymbol{A}}_{221}^{\mathrm{d}} = \begin{bmatrix} 0.11010754204057 & 0.0 \\ -0.00229232728470 & 0.0 \end{bmatrix}, \qquad \bar{\boldsymbol{A}}_{222}^{\mathrm{d}} = \begin{bmatrix} 0.99884207145830 & 0.09691078779658 \\ 0.00002403610304 & 0.92665011300373 \end{bmatrix},$$

$$\bar{\boldsymbol{B}}_{11}^{\mathrm{d}} = \begin{bmatrix} -0.000023432213607527 \\ 0.000008865905197381 \end{bmatrix}, \qquad \bar{\boldsymbol{B}}_{12}^{\mathrm{d}} = \begin{bmatrix} -0.00924227569649 \\ 0.00349758505425 \end{bmatrix},$$

$$\bar{\boldsymbol{B}}_{21}^{\mathrm{d}} = \begin{bmatrix} -0.000010709201899648 \\ 0.000008104026443029 \end{bmatrix}, \qquad \bar{\boldsymbol{B}}_{22}^{\mathrm{d}} = \begin{bmatrix} -0.00422914195054 \\ 0.00320095513839 \end{bmatrix}.$$

Choosing the closed-loop eigenvalues $[\mathrm{e}^{-32.0\mathrm{T}}\ \ \mathrm{e}^{-30.0\mathrm{T}}]$ for $\bar{\boldsymbol{A}}_{122}^{\mathrm{d}} - \bar{\boldsymbol{L}}_1^{\mathrm{d}}\bar{\boldsymbol{A}}_{112}^{\mathrm{d}}$ and $\bar{\boldsymbol{A}}_{222}^{\mathrm{d}} - \bar{\boldsymbol{L}}_2^{\mathrm{d}}\bar{\boldsymbol{A}}_{212}^{\mathrm{d}}$, we can obtain

$$\bar{\boldsymbol{L}}_1^{\mathrm{d}} = \begin{bmatrix} 29.3492 & 40.8660 \\ 0.0096 & 12.3289 \end{bmatrix}, \qquad \bar{\boldsymbol{L}}_2^{\mathrm{d}} = \begin{bmatrix} 29.3578 & 18.6333 \\ 0.0046 & 13.6969 \end{bmatrix}.$$

The resulting control law is

Controller Rule 1: IF $x_1(k)$ is about 0,
THEN $u(k) = -\boldsymbol{K}_1^{\mathrm{d}}\hat{\boldsymbol{x}}(k)$,

Controller Rule 2: IF $x_1(k)$ is about $\pm\pi/3$,
THEN $u(k) = -\boldsymbol{K}_2^{\mathrm{d}}\hat{\boldsymbol{x}}(k)$,

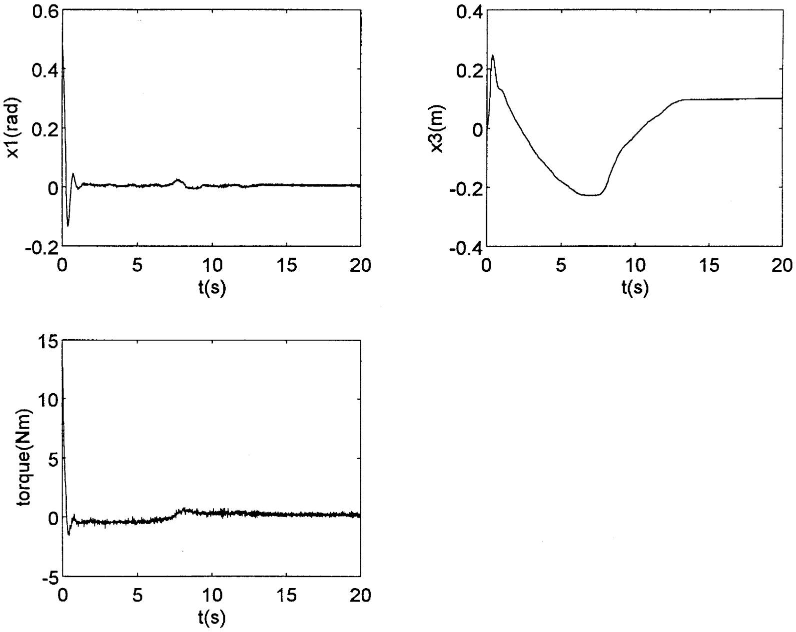

Fig. 7. Closed-loop response for initial conditions $x_1(0)=0.4685$ rad, and $x_2(0)=x_3(0)=x_4(0)=0.0$; torque: the torque (N m) applied to the cart; $x_1, x_2, \ldots$ as in Fig. 1.

that is,

$$u(k) = -\mu_1 \boldsymbol{K}_1^{\mathrm{d}} \hat{\boldsymbol{x}}(k) - \mu_2 \boldsymbol{K}_2^{\mathrm{d}} \hat{\boldsymbol{x}}(k),$$

where μ_1 and μ_2 are the same weights as those in the numerical simulation.

Experiment results indicate that the closed-loop behavior of the system has the excellent performance. The limitation of the length of the slideway and the power of the motor restricts the initial angle $x_1(0)$ which is not too large. Fig. 7 illustrates the closed-loop behavior of the system for initial conditions $x_1(0)=0.4685$ rad, and $x_2(0)=x_3(0)=x_4(0)=0.0$.

7.2. Fuzzy functional observer

In order to design the fuzzy functional observer in the discrete case, the following matrix equations

$$\boldsymbol{R}_{i1}^{\mathrm{d}} \bar{\boldsymbol{A}}_{i11}^{\mathrm{d}} + \boldsymbol{R}_{i2}^{\mathrm{d}} \bar{\boldsymbol{A}}_{i21}^{\mathrm{d}} - H_{ii}^{\mathrm{d}} \boldsymbol{R}_{i1}^{\mathrm{d}} = \boldsymbol{G}_i^{\mathrm{d}}, \qquad \boldsymbol{R}_{i1}^{\mathrm{d}} \bar{\boldsymbol{A}}_{i12}^{\mathrm{d}} + \boldsymbol{R}_{i2}^{\mathrm{d}} \bar{\boldsymbol{A}}_{i22}^{\mathrm{d}} - H_{ii}^{\mathrm{d}} \boldsymbol{R}_{i2}^{\mathrm{d}} = \boldsymbol{0},$$

$$\boldsymbol{R}_{i1}^{\mathrm{d}} \bar{\boldsymbol{B}}_{i1}^{\mathrm{d}} + \boldsymbol{R}_{i2}^{\mathrm{d}} \bar{\boldsymbol{B}}_{i2}^{\mathrm{d}} = N_i^{\mathrm{d}}, \qquad \bar{\boldsymbol{K}}_{i1}^{\mathrm{d}} = E_i^{\mathrm{d}} \boldsymbol{R}_{i1}^{\mathrm{d}} + \boldsymbol{M}_i^{\mathrm{d}}, \qquad \bar{\boldsymbol{K}}_{i2}^{\mathrm{d}} = E_i^{\mathrm{d}} \boldsymbol{R}_{i2}^{\mathrm{d}}, \quad i = 1, 2$$

will be solved. Choosing $E_1^{\mathrm{d}} = E_2^{\mathrm{d}} = 1.0$, $H_{11}^{\mathrm{d}} = H_{12}^{\mathrm{d}} = H_{21}^{\mathrm{d}} = H_{22}^{\mathrm{d}} = \mathrm{e}^{-30.0\mathrm{T}}$, and substituting the variables into the matrix equations above, we can obtain

$$\boldsymbol{G}_1^{\mathrm{d}} = [48.4682 \quad 132.5851], \qquad \boldsymbol{G}_2^{\mathrm{d}} = [113.7294 \quad 150.6431],$$

$$\boldsymbol{M}_1^{\mathrm{d}} = [-429.2028 \quad -958.5174], \qquad \boldsymbol{M}_2^{\mathrm{d}} = [-984.4949 \quad -1090.3002],$$

$$N_1^{\mathrm{d}} = 0.0013, \qquad N_2^{\mathrm{d}} = 0.0079.$$

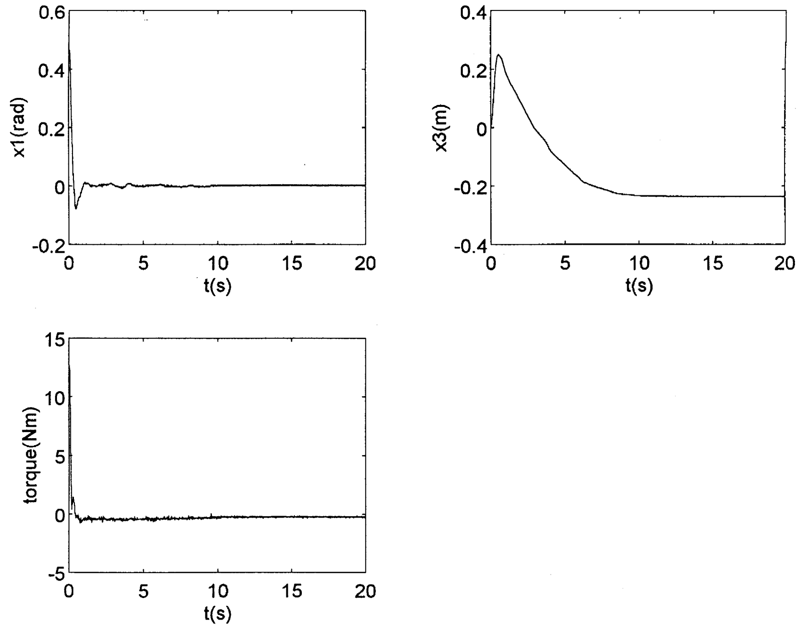

Fig. 8. Closed-loop response for initial conditions $x_1(0)=0.4649$ rad, and $x_2(0)=x_3(0)=x_4(0)=0.0$; torque: the torque (N m) applied to the cart; $x_1, x_2, \ldots$ as in Fig. 1.

The resulting control law is

$$u(t) = -\left[\sum_{i=1}^{2} \mu_i E_i^{\mathrm{d}} z(k) + \sum_{i=1}^{2} \mu_i \boldsymbol{M}_i^{\mathrm{d}} \boldsymbol{y}_i(k)\right],$$

where μ_1 and μ_2 are the same weights as those in the numerical simulation.

Fig. 8 illustrates the closed-loop behavior of the system for initial conditions $x_1(0)=0.4649$ rad, and $x_2(0)=x_3(0)=x_4(0)=0.0$. The limitation of the length of the slideway and the power of the motor restricts the initial angle $x_1(0)$ which is not too large.

8. Conclusions and discussion

On the basis of the fuzzy full-dimensional observer in Ref. [2], the fuzzy reduced-dimensional observer and the fuzzy functional observer are deeply researched in the paper. The dimensions of either the fuzzy reduced-dimensional or fuzzy functional observer are smaller than those of the fuzzy full-dimensional observer. Generally speaking, the fuzzy functional observer may possess much smaller dimensions. But the fuzzy reduced-dimensional observer and the fuzzy functional observer are sensitive to noises, specially, it is not too easy to solve the matrix equations for designing the fuzzy functional observer.

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