Gabor's 1946 Theory of Communication

Sevag Hanssian

MUMT 622, Winter 2021

January 28, 2021

Significance

Outcomes of Gabor's 1946 paper, "Theory of Communication"[1]:

1. First introduction of the time-frequency uncertainty principle, leading to an explosion of wavelet research in the 80s[2]
2. Proposed that any signal of finite energy can be decomposed into a linear combination of time-frequency shifts of the Gaussian function[3]
3. First use of the STFT, or the windowed Fourier transform with Gaussian windows[4]
4. Showed that Gaussian-windowed STFT minimizes time-frequency uncertainty

---

[1] Dennis Gabor. "Theory of Communication". In: *Journal of Institution of Electrical Engineers* 93.3 (1946), pp. 429–457. URL: http://www.granularsynthesis.com/pdf/gabor.pdf.

[2] Peter Hill. "Dennis Gabor - Contributions to Communication Theory Signal Processing". In: Oct. 2007, pp. 2632–2637. DOI: 10.1109/EURCON.2007.4400546.

[3] Vignon Sourou Oussa. *Why was the Nobel prize winner D. Gabor wrong?* URL: http://webhost.bridgew.edu/voussa/images/Presentations/Gabor.pdf.

[4] *Fourier and Wavelet Signal Processing*. Cambridge University Press, Jan. 2013. URL: http://www.fourierandwavelets.org/FWSP_a3.2_2013.pdf.

Heisenberg's Uncertainty Principle

Heisenberg's uncertainty principle in quantum physics[5]:

$$\sigma_x \sigma_p \geq \frac{h}{4\pi}$$

Heisenberg, 1927:

> *the more precisely the position [of an electron] is determined, the less precisely the momentum is known, and conversely*

[5] M. Hall. "Resolution and uncertainty in spectral decomposition". In: *First Break* 24.12 (2006). ISSN: 0263-5046. DOI: https://doi.org/10.3997/1365-2397.2006027. URL: https://www.earthdoc.org/content/journals/10.3997/1365-2397.2006027.

Gabor's Uncertainty Principle

Gabor's time-frequency uncertainty:

$$\sigma_t \sigma_f \geq \frac{1}{4\pi}, \qquad \Delta t \Delta f \geq 1$$

Gabor, 1946[6]:

> *although we can carry out the analysis [of the acoustic signal] with any degree of accuracy in the time direction or frequency direction, we cannot carry it out simultaneously in both beyond a certain limit*

[6] Dennis Gabor. "Theory of Communication". In: *Journal of Institution of Electrical Engineers* 93.3 (1946), pp. 429–457. URL: http://www.granularsynthesis.com/pdf/gabor.pdf.

Time-frequency tradeoff, intuition

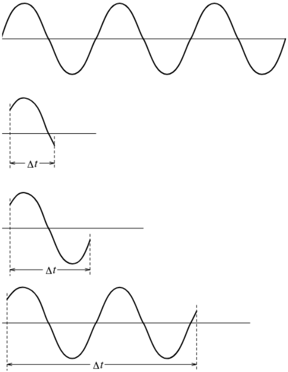


Figure: Improved frequency measurement over longer time intervals. The uncertainty in the frequency Δf decreases as the measurement interval Δt increases and vice versa[7]

[7] Bruce Maclennan. "Gabor Representations of Spatiotemporal Visual Images". In: (Nov. 1994). URL: http://www.its.caltech.edu/~matilde/GaborLocalization.pdf.

Gabor's Uncertainty Principle, visual

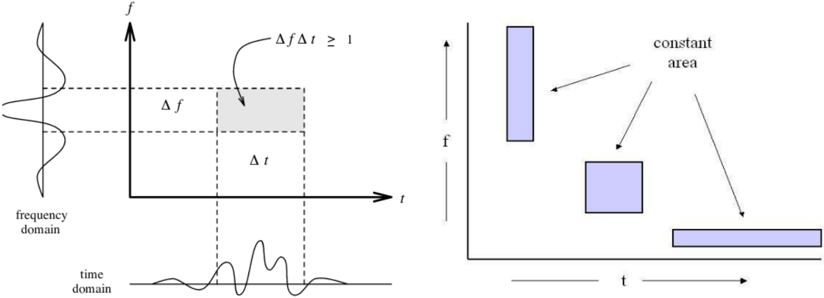


Figure: The most a signal can be localized in the Fourier domain is into rectangles of size $\Delta t \Delta f = 1$

The smallest possible $\Delta t \Delta f$ rectangle is called the *logon*, a "unit of information"

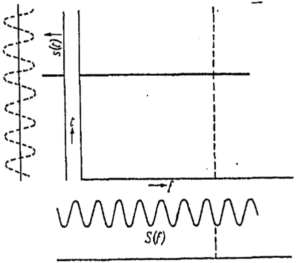
Consequence of the Fourier transform

In time-frequency analysis, it has been proven that linear operators cannot exceed the uncertainty bound [...] Nonlinearity does not by itself confer any acuity advantage, and in fact most nonlinearities are merely distortions and thus deleterious. However, by the above theorem, any carefully crafted analysis that can beat this limit must necessarily be nonlinear.[8]

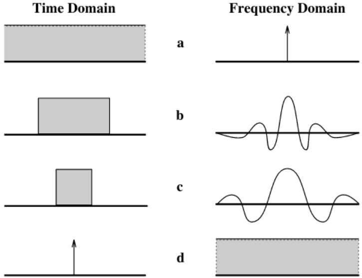
[8] Jacob Oppenheim and Marcelo Magnasco. "Human Time-Frequency Acuity Beats the Fourier Uncertainty Principle". In: *Physical Review Letters* 110 (Aug. 2012). DOI: 10.1103/PhysRevLett.110.044301.

Time-frequency tradeoff, visual

Arises from the linearity of the Fourier transform



(a) Unit impulse vs. infinite sine wave[9]



(b) Spread of a signal and its Fourier transform are inversely proportional[10]

Figure: Time vs. frequency

[9] Dennis Gabor. "Theory of Communication". In: *Journal of Institution of Electrical Engineers* 93.3 (1946), pp. 429–457. URL: http://www.granularsynthesis.com/pdf/gabor.pdf.

[10] Bruce Maclennan. "Gabor Representations of Spatiotemporal Visual Images". In:

Physical intuitions

The foregoing solutions [of the Fourier transform], though unquestionably mathematically correct, are somewhat difficult to reconcile with our physical intuitions and our physical concepts of such variable frequency mechanisms as, for instance, the siren

– Carson (quoted by Gabor)

Gabor came to the conclusion that the difficulty lay in our mutually exclusive formulations of time analysis and frequency analysis ... he suggested a new method of analyzing signals in which time and frequency play symmetrical parts.[11]

[11] A. Korpel. "Gabor: frequency, time, and memory". In: *Appl. Opt.* 21.20 (Oct. 1982), pp. 3624–3632. DOI: 10.1364/AO.21.003624. URL: http://ao.osa.org/abstract.cfm?URI=ao-21-20-3624.

Psychoacoustics

Psychoacoustic[12] studies have shown that humans can exhibit a better time-frequency resolution than Gabor's limit:

> *We have conducted the first direct psychoacoustical test of the Fourier uncertainty principle in human hearing, by measuring simultaneous temporal and frequency discrimination. Our data indicate that human subjects often beat the bound prescribed by the uncertainty theorem, by factors in excess of 10.*

Similarly to how Gabor was dissatisfied with time-frequency's inability to reconcile with physical intuitions:

> *most sound analysis and processing tools today continue to use models based on spectral theories. We believe it is time to revisit this issue.*

[12] Jacob Oppenheim and Marcelo Magnasco. "Human Time-Frequency Acuity Beats the Fourier Uncertainty Principle". In: *Physical Review Letters* 110 (Aug. 2012). DOI: 10.1103/PhysRevLett.110.044301.

Psychoacoustics

Brian C. J. Moore in 1973:[13]

> *It is concluded that models based on a place (spectral) analysis should be subject to a limitation of the type $\Delta f \cdot d \geq$ constant, where Δf is the frequency difference limen (DL) for a tone pulse of duration d. [...] It was found that at short durations the product of Δf and d was about one order of magnitude smaller than the minimum predicted from the place model*

[13] B. C. J. Moore. "Frequency difference limens for short-duration tones". In: *The Journal of the Acoustical Society of America* 54.3 (1973), pp. 610–619. DOI: 10.1121/1.1913640. eprint: https://doi.org/10.1121/1.1913640. URL: https://doi.org/10.1121/1.1913640.

Gabor elementary functions

> *What is the shape of the signal for which the product $\Delta t \Delta f$ actually assumes the smallest possible value? [... it is] the modulation product of a harmonic oscillation of any frequency with a pulse of the form of the probability function*

i.e. apply a Gaussian envelope to the signal

$$\psi(t) = \mathrm{e}^{-\alpha^2(t-t_0)}\mathrm{cis}(2\pi f_0 t + \phi)$$

$$\phi(f) = \mathrm{e}^{-\frac{\pi}{\alpha}^2(f-f_0)^2}\mathrm{cis}[-2\pi(f-f_0)+\phi)]$$

The constant α is connected with Δt and Δf as follows:

$$\Delta t = \sqrt{\frac{\pi}{2}}\frac{1}{\alpha}, \Delta f = \frac{1}{\sqrt{2\pi}}\alpha$$

Gabor elementary functions, alternative form

Alternate form in Bruce Maclennan. "Gabor Representations of Spatiotemporal Visual Images". In: (Nov. 1994). URL: http://www.its.caltech.edu/~matilde/GaborLocalization.pdf:

$$
\begin{aligned}
C_{jk}(t) &= \exp[-\pi(t - j\Delta t)^2/\alpha^2]\cos[2\pi k\Delta f\,(t - j\,\Delta t)], \\
S_{jk}(t) &= \exp[-\pi(t - j\Delta t)^2/\alpha^2]\sin[2\pi k\Delta f\,(t - j\,\Delta t)],
\end{aligned}
$$

$$\phi_{jk} = C_{jk} + iS_{jk}$$

As $\alpha \to \infty$, this reduces to the Fourier representation:

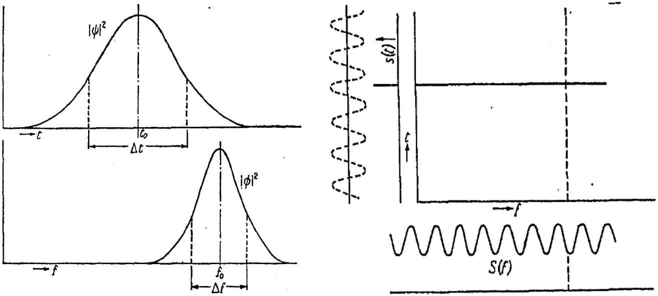
$$
\begin{aligned}
\phi_{jk}(t) &= \exp[2\pi ik\Delta f(t - j\,\Delta t)], \\
C_{jk}(t) &= \cos[2\pi k\Delta f(t - j\,\Delta t)], \\
S_{jk}(t) &= \sin[2\pi k\Delta f(t - j\,\Delta t)].
\end{aligned}
$$

As $\alpha \to 0$, this reduces to Dirac delta functions spaced at Δt:

$$
\begin{aligned}
\phi_{jk}(t) &= \delta(t - j\Delta t) + i\delta(t - j\Delta t), \\
C_{jk}(t) &= S_{jk}(t) = \delta(t - j\Delta t).
\end{aligned}
$$

Gabor elementary functions

The parameter α determines the locality (spread) of the Gaussian envelope



(a) Envelope of the elementary signal

(b) Unit impulse vs. infinite sine wave

Figure: Limits of α result in an "impulse" in time and frequency[14]

---

[14] Dennis Gabor. "Theory of Communication". In: *Journal of Institution of Electrical Engineers* 93.3 (1946), pp. 429–457. URL: http://www.granularsynthesis.com/pdf/gabor.pdf.

Relation to Shannon's 1948 Sampling Theorem

Most important outcome of Shannon's seminal communications paper[15] in 1948, the Sampling Theorem, states that "to reconstruct ψ we must take equally spaced samples at a minimum of the Nyquist frequency, which is twice the maximal frequency"[16]

Recall that in Gabor's representation, as $\alpha \to 0$, this reduces to Dirac delta functions spaced at Δt:

$$\begin{aligned} \phi_{jk}(t) &= \delta(t - j\Delta t) + i\delta(t - j\Delta t), \\ C_{jk}(t) &= S_{jk}(t) = \delta(t - j\Delta t). \end{aligned}$$

The $\alpha = 0$ limit represents two samples, a_{jk}, b_{jk} for each Δt interval, as required by Shannon's Sampling Theorem.

---

[15] C. E. Shannon. "A mathematical theory of communication". In: *The Bell System Technical Journal* 27.3 (1948), pp. 379–423. DOI: 10.1002/j.1538-7305.1948.tb01338.x.

[16] Bruce Maclennan. "Gabor Representations of Spatiotemporal Visual Images". In: (Nov. 1994). URL: http://www.its.caltech.edu/~matilde/GaborLocalization.pdf.

Gaussian window

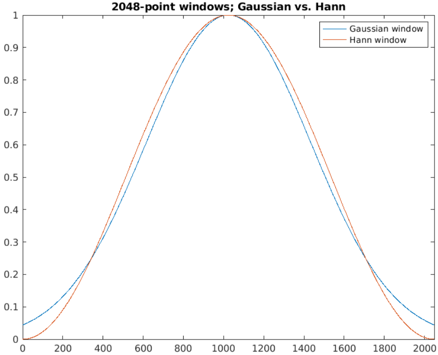


Figure: gausswin and hann windows in MATLAB

Note that Hann window is exactly 0 outside of the specified range
Gaussian asymptotically approaches 0 but never reaches it

Problems with the Gabor functions

Summarized from Maclennan, “Gabor Representations of Spatiotemporal Visual Images”

1. The Gabor functions are **not strictly local** – along with their infinite Gaussian envelope, they stretch out to infinity
   Biologically problematic, but the Gaussian envelope is well-localized (99.7% of its area is within 3 standard deviations of the mean), so can be a “good enough approximation” of biology
2. The Gabor representation is **nonorthogonal**. This means computing the coefficients is *possible* but not with the simple inner product. Inner products can lead to a good estimation of the Gabor coefficients with iterative refinement (Daugman's 1993 algorithm[17])

Daugman unified Gabor elementary functions and wavelets by defining Gabor wavelets

[17] J. G. Daugman. “High confidence visual recognition of persons by a test of statistical independence”. In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 15.11 (1993), pp. 1148–1161. DOI: 10.1109/34.244676.

Problems with the Gabor functions

3 Despite being nonorthogonal in $L_2(\mathbb{R})$, they can **still be a frame**[18]

However, the requirements of orthogonality and the basis property are very stringent, making it difficult as a rule to find a good orthonormal basis. As an alternative to orthonormal bases, we present a generalization known as frames.

Also:

Nonorthogonality is ubiquitous in biological systems – we should learn how nature lives with it and even exploits it [...] orthogonality is a rather delicate property – functions either are or aren't orthogonal; there are no degrees of orthogonality – and so it is probably too fragile for biology to be able to depend on it.

[18] Christopher Heil and David Walnut. "Continuous and discrete wavelet transforms. SIAM Review, 31, 628-666". In: *SIAM Review* 31 (Dec. 1989), pp. 628–666. DOI: 10.1137/1031129.

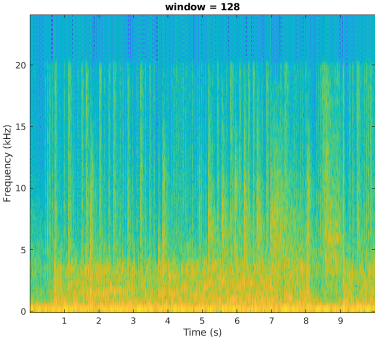
Problems with the Gabor functions

Vignon Sourou Oussa. *Why was the Nobel prize winner D. Gabor wrong?* URL: http://webhost.bridgew.edu/voussa/images/Presentations/Gabor.pdf for further reading

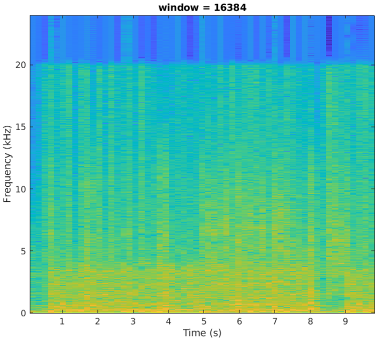
> *In 1932 John von Neumann conjectured without proof that the time-frequency shifts of the Gaussian span a dense subspace in the space of signals of finite energy. In 1946 Gabor conjectured that the time-frequency shifts of the Gaussian is a basis for the space of signals of finite energy. [...] In colloquial terms, the expansions are numerically unstable and cannot be used in practice.*

Time-frequency resolution in the STFT

Using default MATLAB spectrogram parameters[19] (Hamming window)



(a) Good Δt, bad Δf

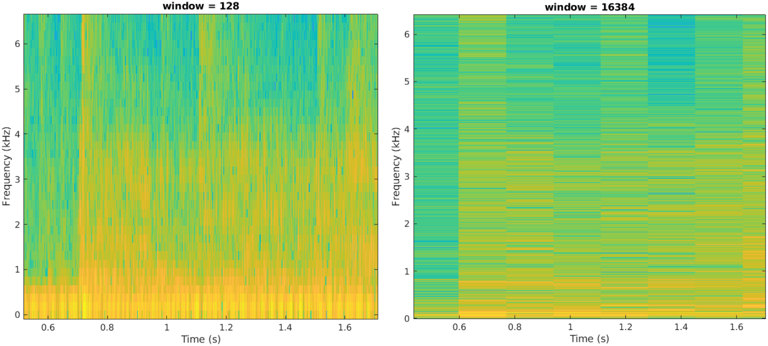


(b) Good Δf, bad Δt

Figure: STFT, small vs. big window

[19] *Spectrogram using short-time Fourier transform - MATLAB spectrogram*. URL: https://www.mathworks.com/help/signal/ref/spectrogram.html.

Time-frequency resolution in the STFT

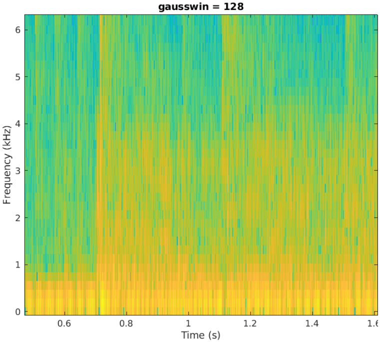


(a) Good Δt, bad Δf

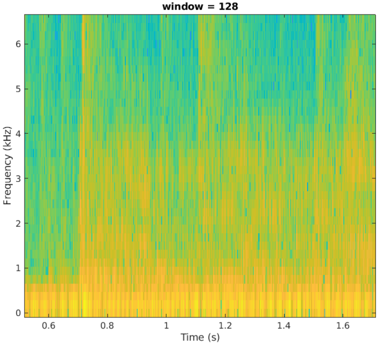
(b) Good Δf, bad Δt

Figure: STFT, small vs. big window

STFT with gausswin (i.e. Gabor transform)



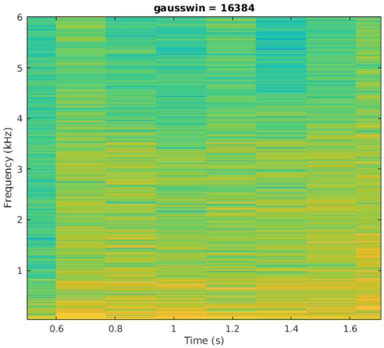
(a) Gausswin



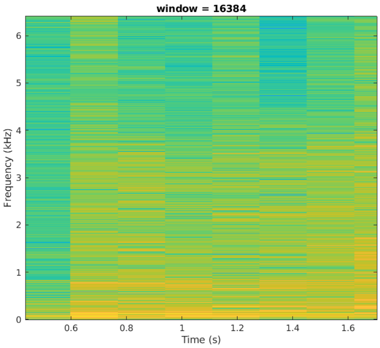
(b) Hamming

Figure: STFT, small window = 128, Gaussian vs. Hamming

STFT with gausswin (i.e. Gabor transform)



(a) Gausswin



(b) Hamming

Figure: STFT, large window = 16384, Gaussian vs. Hamming