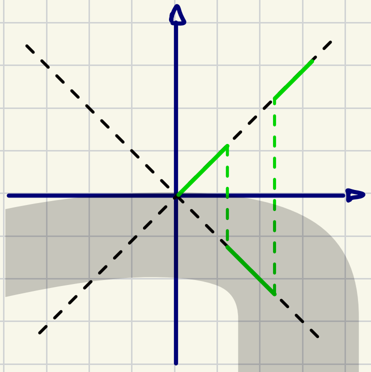
T1

$x^2=y^2$

а) $\exists$ беск. много ф-ций $y:\mathbb{R}\to\mathbb{R}$ удовл. $x^2=y^2$:



т.к. можно придумать бесконечно много конфигураций разрывов у ф-ции, похожей на ту, что нарисована.

б) $\exists$ 4 непрерывных ф-ций $y:\mathbb{R}\to\mathbb{R}$ удовл. $x^2=y^2$ — $y=x$, $y=|x|$, $y=-x$, $y=-|x|$, т.к. в каждой полуплоскости y лежит на одном из двух лучей в силу непрерывности $\Rightarrow$ всего $2\cdot 2=4$ варианта, которые указаны выше.

в) $\exists$ 2 непрерывных ф-ций $y:\mathbb{R}\to\mathbb{R}$ удовл. $\begin{cases} x^2=y^2 \\ y(1)=1 \end{cases}$

В силу б) и того, что $y=-x$ и $y=-|x|$ не подходят под условие $y(1)=1$, эти ф-ции — $y=x$, $y=|x|$.

г) $\exists$ 1 непрерывная ф-ция $y:[1,2]\to\mathbb{R}$ удовл. $\begin{cases} x^2=y^2 \\ y(1)=1 \end{cases}$

В силу в) и того, что на $[1,2]$ $x\equiv|x|$, эта ф-ция — $y=x$.

3.63(2)

$x-u=u\ln\frac{u}{y}$ в $(x,y)=(1,1)$

в $(1,1)$: $1-u=u\ln u$

$1=u(\ln u+1)\to u=1$

$$F(x,y,u) = x - u - u\ln\frac{u}{y} = 0$$

$$\left.\begin{aligned} &F'_x\big|_{(1,1,1)} = 1 \text{ - конеч.} \\ &F'_y\big|_{(1,1,1)} = \left(-u\cdot\frac{y}{u}\cdot\left(-\frac{u}{y^2}\right)\right)\Big|_{(1,1,1)} = \left(\frac{y}{u}\right)\Big|_{(1,1,1)} = 1 \text{ - конеч.} \\ &F'_u\big|_{(1,1,1)} = \left(-1 - \ln\frac{u}{y} - u\cdot\frac{y}{u}\cdot\frac{1}{y}\right)\Big|_{(1,1,1)} = -2 \neq 0 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ по Т. о неявной ф-ции в окр-ти $(1,1)$ $\exists\, u = u(x,y)$

$$\frac{\partial u}{\partial x}\Big|_{(1,1,1)} = -\frac{F'_x(1,1,1)}{F'_u(1,1,1)} = \frac{1}{2}$$

$$\frac{\partial u}{\partial y}\Big|_{(1,1,1)} = -\frac{F'_y(1,1,1)}{F'_u(1,1,1)} = \frac{1}{2}$$

$$du(x,y)\big|_{(1,1)} = \frac{\partial u}{\partial x}\Big|_{(1,1,1)}\cdot dx + \frac{\partial u}{\partial y}\Big|_{(1,1,1)}\cdot dy = \frac{1}{2}dx + \frac{1}{2}dy$$

Ответ: $\frac{1}{2}dx + \frac{1}{2}dy$

3.64(1)

$$F'(x,y,u(x,y)) = 3u^2u'_x - u - xu'_x = u'_{\cancel{x}}(\underbrace{3u^2 - x}_{0}) - u = -u$$

$$F(x,y,u) = u^3 - xu + y = 0$$

а) в $(x_0,y_0,u_0) = (3,-2,2)$:

$$\left.\begin{aligned} &F'_x = -u_0 = -2 \\ &F'_y = 1 \\ &F'_u = 3u_0^2 - x_0 = 9 \neq 0 \end{aligned}\right\}$$ по Т. о неявной ф-ции в окрестности $(3,-2,2)$ $\exists\, u = u(x,y)$

В этой точке:

$$\frac{\partial u}{\partial x} = -\frac{F'_x}{F'_u} = \frac{2}{9}$$

$$\frac{\partial u}{\partial y} = -\frac{F'_y}{F'_u} = -\frac{1}{9}$$

$\Rightarrow du = \frac{2}{9}dx - \frac{1}{9}dy$

б) в $(x_0,y_0,u_0) = (3,-2,-1)$ $F'_u = 3u_0^2 - x_0 = 0 \Rightarrow$ не выполняется Т. о неявной ф-ции, но она является достаточным условием $\Rightarrow$ нужно доказать, что $\nexists\, du$ в $(3,-2,-1)$)

Докажем, что $\nexists du(3,-2,-1)$

Пусть в этой точке $\exists du \Rightarrow \exists u=u(x,y)$. Тогда

для $\Phi(x,y)=F(x,y,u(x,y)) \mapsto \Phi'_x = 3u^2u'_x - u - xu'_x =$

$= u'_x(3u^2-x)-u \equiv 0$, т.к. $\Phi(x,y)\equiv 0$

Тогда $u'_x\cdot(3\cdot(-1)^2-3)+1=1\equiv 0$?! $\Rightarrow \nexists du$

Ответ: а) $\frac{2}{9}dx-\frac{1}{9}dy$; б) не существует.

3.71

$yf\left(\frac{z}{y}\right)=x^2+y^2+z^2$, f-дифф-ма определяет

дифф-емую $z(x,y) \Rightarrow (x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}=2xz$

□ $F(x,y,z)=yf\left(\frac{z}{y}\right)-x^2-y^2-z^2=0 \quad \exists z=z(x,y) \Rightarrow$

$\Rightarrow \Phi(x,y)=F(x,y,z(x,y))\equiv 0$

$$\begin{cases}\frac{\partial \Phi}{\partial x}=0\\ \frac{\partial \Phi}{\partial y}=0\end{cases} \quad \begin{cases} yf'_{\frac{z}{y}}\left(\frac{z}{y}\right)\frac{z'_x}{y}-2x-2zz'_x=0 \Rightarrow f'_{\frac{z}{y}}\left(\frac{z}{y}\right)z'_x=2x+2zz'_x\\ f\left(\frac{z}{y}\right)+yf'_{\frac{z}{y}}\left(\frac{z}{y}\right)\left(\frac{yz'_y-z}{y^2}\right)-2y-2zz'_y=0\end{cases} \Rightarrow$$

$$\Rightarrow f\left(\frac{z}{y}\right)z'_x+\left(z'_y-\frac{z}{y}\right)(2x+2zz'_x)-2yz'_x-2zz'_yz'_x=0$$

$$\left(\frac{x^2+y^2+z^2}{y}-\frac{2z^2}{y}-2y\right)z'_x+2xz'_y=\frac{2xz}{y}$$

$$(x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}=2xz \quad \blacksquare$$

§3:76

$$\begin{cases}F_1(x,y,u,v)=\sqrt{2}e^{u/x}\cos\frac{v}{y}-x=0\\ F_2(x,y,u,v)=\sqrt{2}e^{u/x}\sin\frac{v}{y}-y=0\end{cases}, \quad u(1,1)=0,\ v(1,1)=\frac{\pi}{4}$$

$$\begin{vmatrix}\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v}\\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{x}\sqrt{2}e^{u/x}\cos\frac{v}{y} & -\frac{1}{y}\sqrt{2}e^{u/x}\sin\frac{v}{y}\\ \frac{1}{x}\sqrt{2}e^{u/x}\sin\frac{v}{y} & \frac{1}{y}\sqrt{2}e^{u/x}\cos\frac{v}{y}\end{vmatrix}=\frac{2}{xy}e^{2u/x}\neq 0 \text{ в } (1,1)$$

и $\frac{\partial F_i}{\partial u}, \frac{\partial F_i}{\partial v}$ – непр. в $(1,1)$ $\Rightarrow$ выполняется Т. о системе неявных ф-ций и верно, что

$$\begin{Vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{Vmatrix} = \begin{Vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{Vmatrix}^{-1} \begin{Vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{Vmatrix}$$

В точке $(1, 1, 0, \frac{\pi}{4})$:

$$\begin{Vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{Vmatrix} = \frac{1}{2} \begin{Vmatrix} 1 & 1 \\ -1 & 1+\frac{\pi}{2} \end{Vmatrix}$$

<u>Ответ:</u> $du = \frac{1}{2}dx + \frac{1}{2}dy$, $dv = -\frac{1}{2}dx + \left(\frac{1}{2}+\frac{\pi}{4}\right)dy$

4.43(6)

$y - u = e^{xu}$, $u(1,1) = 0$

$F(x,y,u) = y - u - e^{xu}$

В точке $(1,1,0)$:

$F'_x = -ue^{xu} = 0$ – непр.

$F'_y = 1$

$F'_u = -1 - xe^{xu} = -2$

Выполняются условия Т. о неявной ф-ции $\Rightarrow$ в т. $(1,1)$:

$$\begin{cases} u'_x = -\frac{F'_x}{F'_u} = 0 \\ u'_y = -\frac{F'_y}{F'_u} = \frac{1}{2} \end{cases} \Rightarrow du = \frac{1}{2}dy$$

$dy - du = e^{xu}(x\,du + u\,dx)$

$-d^2u = e^{xu}\left[(x\,du + u\,dx)^2 + x\,d^2u + 2dx\,du\right]$

$-d^2u = e^{xu}\left[x^2du^2 + 2(xu+1)dx\,du + u^2dx^2 + x\,d^2u\right]$

$$d^2u(-1-xe^{xu})=e^{xu}\left[x^2\left(\tfrac12 dy\right)^2+2(xu+1)\,dx\cdot\tfrac12 dy+u^2dx^2\right]$$

$$d^2u=-\frac{1}{e^{-xu}+x}\left[\frac14x^2dy^2+(xu+1)dxdy+u^2dx^2\right]$$

В точке $(x,y,u)=(1,1,0)$

$$d^2u=-\frac18dy^2-\frac12dxdy$$

Замечание: u - дважды дифф-ма, т.к. $F(x,y,u)$ дважды дифф-ма по x,y,u, т.е. $F'_x, F'_y, F'_u\in C(U_\delta(1,1))\Rightarrow u'_x, u'_y\in C(U_\delta(x,y))$ как композиции ф-ций $\in C(U_\delta(1,1))$ (причём $F'_u\neq 0$ в $U_\delta(1,1)$)

Ответ: $-\frac18dy^2-\frac12dxdy$

4.46(2)

$f\left(\frac{x}{u},\frac{y}{u}\right)\equiv 0$, f-дважды дифф-ма, определяет дважды дифф-ая $u(x,y)$

$$0\equiv df\left(\frac{x}{u},\frac{y}{u}\right)=\left|\begin{matrix} z=x/u\\ \omega=y/u\end{matrix}\right|=\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial\omega}d\omega=f'_z\frac{udx-xdu}{u^2}+f'_\omega\frac{udy-ydu}{u^2}=$$

$$=f'_z\frac{dx}{u}+f'_\omega\frac{dy}{u}-(xf'_z+yf'_\omega)\frac{du}{u^2}\quad(*)$$

$$(*)\cdot u:\ \frac{du}{u}=\frac{f'_zdx+f'_\omega dy}{xf'_z+yf'_\omega}\Rightarrow u=\frac{xf'_z+yf'_\omega}{f'_zdx+f'_\omega dy}du=\frac{xf'_z+yf'_\omega}{f'_zdx+f'_\omega dy}$$

$$(u'_xdx+u'_ydy)=-\frac{xf'_z+yf'_\omega}{f'_zdx+f'_\omega dy}\left(\frac{f'_x}{f'_u}dx+\frac{f'_y}{f'_u}dy\right)\quad(**)$$

$$\begin{cases}u'_x=-\dfrac{f'_x}{f'_u}\\ u'_y=-\dfrac{f'_y}{f'_u}\end{cases}$$ - из Т. об обратной ф-ции

$$(*)\cdot u^2:\ u(f'_zdx+f'_\omega dy)-(xf'_z+yf'_\omega)du=0\ (***)$$

$$d(***):\ du(f'_zdx+f'_\omega dy)+u(f''_{zz}dxdz+f''_{\omega\omega}dyd\omega)-$$

$$-d^2u(xf'_z+yf'_\omega)-du(f'_zdx+f'_\omega dy+f''_{zz}dz+f''_{\omega\omega}d\omega)=0$$

$$d^2u=(xf'_z+yf'_w)^{-1}\left(-du\left(f''_{zz}dz+f''_{ww}dw\right)+u\left(f''_{zz}dx\,dz+f''_{ww}\cdot dy\,dw\right)\right)$$

$$d^2u=(xf'_z+yf'_w)^{-1}\left((u\,dx-du)f''_{zz}dz+(u\,dy-du)f''_{ww}dw\right)$$

$$d^2u=u\cdot(xf'_z+yf'_w)^{-1}\left(\left(dx-\frac{du}{u}\right)f''_{zz}dz+\left(dy-\frac{du}{u}\right)f''_{ww}dw\right),\ (**)$$

$$\Rightarrow d^2u=(f'_zdx+f'_wdy)^{-1}\left(f''_{zz}dz\,dx+f''_{ww}dw\,dy-\left(f''_{zz}dz+f''_{ww}dw\right)\frac{f'_zdx+f'_wdy}{xf'_z+yf'_w}\right)\ (***)$$

$$dz=\frac{u\,dx-x\,du}{u^2}=\left(-\frac{xf'_z+yf'_w}{f'_zdx+f'_wdy}\left(\frac{f'_x}{f'_u}dx+\frac{f'_y}{f'_u}dy\right)\right)^{-1}\left[dx-x\frac{du}{u}\right]=$$

$$=\left(-\frac{xf'_z+yf'_w}{f'_zdx+f'_wdy}\left(\frac{f'_x}{f'_u}dx+\frac{f'_y}{f'_u}dy\right)\right)^{-1}\left[dx+x\frac{f'_zdx+f_wdy}{xf'_z+yf'_w}\right]\ (****)$$

$$dw=\frac{u\,dy-y\,du}{u^2}=\left(-\frac{xf'_z+yf'_w}{f'_zdx+f'_wdy}\left(\frac{f'_x}{f'_u}dx+\frac{f'_y}{f'_u}dy\right)\right)^{-1}\left[dx-x\frac{du}{u}\right]=$$

$$=\left(-\frac{xf'_z+yf'_w}{f'_zdx+f'_wdy}\left(\frac{f'_x}{f'_u}dx+\frac{f'_y}{f'_u}dy\right)\right)^{-1}\left[dy+y\frac{f'_zdx+f_wdy}{xf'_z+yf'_w}\right]\ (*****)$$

$(***), (****), (*****) \Rightarrow$

$$\Rightarrow (***)=(f'_zdx+f'_wdy)^{-1}\left(-\frac{xf'_z+yf'_w}{f'_zdx+f'_wdy}\left(\frac{f'_x}{f'_u}dx+\frac{f'_y}{f'_u}dy\right)\right)^{-1}\left\{f''_{zz}\cdot\left[dx+x\frac{f'_zdx+f'_wdy}{xf'_z+yf'_w}\right]\left(dx-\frac{f'_zdx+f'_wdy}{xf'_z+yf'_w}\right)+f''_{ww}\left[dy+y\frac{f'_zdx+f'_wdy}{xf'_z+yf'_w}\right]\cdot\left(dy-\frac{f'_zdx+f'_wdy}{xf'_z+yf'_w}\right)\right\}=\ldots=-\frac{(f'_w)^2f''_{zz}-2f'_zf'_wf''_{zw}+(f'_z)^2f''_{ww}}{(xf'_z+yf'_w)^2}(y\,dx-x\,dy)^2$$

<u>Ответ:</u> $-\dfrac{(f'_w)^2f''_{zz}-2f'_zf'_wf''_{zw}+(f'_z)^2f''_{ww}}{(xf'_z+yf'_w)^2}(y\,dx-x\,dy)^2$

3.104

$$\begin{cases} x = r\cos^p\varphi \\ y = r\sin^p\varphi \end{cases}, \; p \in \mathbb{N}$$

$$\frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos^p\varphi & -pr\cos^{p-1}\varphi\sin\varphi \\ \sin^p\varphi & pr\sin^{p-1}\varphi\cos\varphi \end{vmatrix} = pr(\sin\varphi\cos\varphi)^{p-1}$$

T3

$$\begin{cases} u = e^x\cos y \\ v = e^x\sin y \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u & -v \\ v & u \end{vmatrix} = u^2 + v^2 = e^{2x} > 0 \;\; \forall (x,y) \in \mathbb{R}^2$$

Но $(x, y+\varphi)$ и $(x, y+\varphi+2\pi k)$ $\forall \varphi \in \mathbb{R}, \forall k \in \mathbb{Z}$ соотвествует одинаковая пара (u,v) $\Rightarrow$ не взаимооднозначно.

T4

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \; r > 0 \quad (*)$$

$$\frac{\partial}{\partial x}(*): \begin{cases} 1 = r'_x\cos\varphi - r\varphi'_x\sin\varphi \\ 0 = r'_x\sin\varphi + r\varphi'_x\cos\varphi \end{cases}$$

$$\begin{cases} \sin\varphi = -r\varphi'_x(\cos^2\varphi + \sin^2\varphi) \Rightarrow \varphi'_x = -\frac{\sin\varphi}{r} \\ r'_x\sin\varphi\cos\varphi = -r\varphi'_x\cos^2\varphi \Rightarrow r'_x = -\frac{r}{\sin\varphi}\left(-\frac{\sin\varphi}{r}\right)\cos\varphi = \cos\varphi \end{cases}$$

$$\frac{\partial}{\partial y}(*): \begin{cases} 0 = r'_y\cos\varphi - r\varphi'_y\sin\varphi \\ 1 = r'_y\sin\varphi + r\varphi'_y\cos\varphi \end{cases}$$

$$\begin{cases} r'_y\sin\varphi\cos\varphi = r\varphi'_y\sin^2\varphi \Rightarrow r'_y = \sin\varphi \\ \cos\varphi = r\varphi'_y(\sin^2\varphi + \cos^2\varphi) \Rightarrow \varphi'_y = \frac{\cos\varphi}{r} \end{cases}$$

3.86

$$x\frac{\partial u}{\partial y}-y\frac{\partial u}{\partial x}=0 \qquad \begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \end{cases}$$

$$r\cos\varphi\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial y}\right)-r\sin\varphi\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \varphi}\frac{\partial \varphi}{\partial x}\right)=0$$

$$r\cos\varphi\left(u'_r\sin\varphi+u'_\varphi\frac{\cos\varphi}{r}\right)-r\sin\varphi\left(u'_r\cos\varphi-u'_\varphi\frac{\sin\varphi}{r}\right)=0$$

$$u'_\varphi\cdot\frac{1}{r}(\cos^2\varphi+\sin^2\varphi)=0$$

$$u'_\varphi\cdot\frac{1}{r}=0$$

$u'_\varphi=0 \Rightarrow$ решение $u=u(r)=u(x^2+y^2)$

Ответ: $u=u(x^2+y^2)$ – произвольная дифф-ая ф-ция.

3.88(2)

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z, \quad x=u,\ y=uv$$

$$u\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)+uv\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)=z$$

$$u\left(\frac{\partial z}{\partial u}\cdot 1+\frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right)\right)+uv\left(\frac{\partial z}{\partial u}\cdot 0+\frac{\partial z}{\partial v}\cdot\frac{1}{u}\right)=z$$

$$u\left(\frac{\partial z}{\partial u}-\frac{v}{u}\frac{\partial z}{\partial v}\right)+v\frac{\partial z}{\partial v}=z$$

$$u\frac{\partial z}{\partial u}=z$$

$$\ln|z|=\ln|u|+C(v)$$

$$z=C(v)\cdot u$$

Ответ: $z=x\cdot C\left(\frac{y}{x}\right)$, C-дифф-ма.

3.90

$$(z-x)\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0$$

$$dx(y,z)=\frac{\partial x}{\partial y}dy+\frac{\partial x}{\partial z}dz \Rightarrow dz=\left(\frac{\partial x}{\partial z}\right)^{-1}\left(dx-\frac{\partial x}{\partial y}dy\right) \Rightarrow$$

$$\Rightarrow \frac{\partial z}{\partial x} = \left(\frac{\partial x}{\partial z}\right)^{-1}; \quad \frac{\partial z}{\partial y} = \left(\frac{\partial x}{\partial z}\right)^{-1}\left(-\frac{\partial x}{\partial y}\right)$$

$$(z-x)\left(\frac{\partial x}{\partial z}\right)^{-1} + y\left(\frac{\partial x}{\partial z}\right)^{-1}\left(\frac{\partial x}{\partial y}\right) = 0$$

$$x = \frac{\partial x}{\partial y} y + z$$

<u>Ответ:</u> $\frac{\partial x}{\partial y} = \frac{x-z}{y}$

4.49

$$\begin{cases} u + v^2 = x, \\ u^2 - v^3 = y, \end{cases} \quad u(3,3) = 2,\ v(3,3) = 1 \quad (*)$$

в точке $(3,3,2,1)$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 2v \\ 2u & -3v^2 \end{vmatrix} = -3v^2 - 4uv = -3 - 8 \neq 0 \Rightarrow \exists \text{ дифф. } f_1, f_2: \begin{cases} u = f_1(x,y) \\ v = f_2(x,y) \end{cases}$$

$$d(*): \begin{cases} du + 2v\,dv = dx, \\ 2u\,du - 3v^2 dv = dy, \end{cases} \quad \begin{cases} \frac{v}{2} du + v^2 dv = \frac{v}{2} dx \\ \frac{2}{3} u\,du - v^2 dv = \frac{1}{3} dy \end{cases}$$

$$\begin{cases} \left(\frac{1}{2}v + \frac{2}{3}u\right) du = \frac{v}{2} dx + \frac{1}{3} dy \\ \frac{1}{2} du + v\,dv = \frac{1}{2} dx \end{cases} \quad \begin{cases} du = \left(\frac{1}{2}v + \frac{2}{3}u\right)^{-1}\left(\frac{v}{2}dx + \frac{1}{3}dy\right) \\ dv = \frac{1}{2v}(dx - du) \end{cases}$$

В точке $(3,3,2,1)$:

$$\begin{cases} du = \frac{6}{11}\left(\frac{1}{2}dx + \frac{1}{3}dy\right) = \frac{3dx + 2dy}{11} \\ dv = \frac{1}{2}\left(dx - \frac{3dx + 2dy}{11}\right) = \frac{4dx - dy}{11} \end{cases} \Rightarrow \text{ В } (3,3,2,1): \begin{matrix} u'_x = 3/11, \\ u'_y = 2/11, \\ v'_x = 4/11, \\ v'_y = -1/11, \end{matrix}$$

$$d^2u = \frac{d\left(\frac{v}{2}dx + \frac{1}{3}dy\right)\left(\frac{1}{2}v + \frac{2}{3}u\right) - d\left(\frac{1}{2}v + \frac{2}{3}u\right)\left(\frac{v}{2}dx + \frac{1}{3}dy\right)}{\left(\frac{1}{2}v + \frac{2}{3}u\right)^2}$$

$$d^2v = \frac{d(dx - du)\cdot 2v - 2dv(dx - du)}{4v^2}$$

В точке $(3,3,2,1)$:

$$\begin{cases} d^2u = -\dfrac{\left(\frac{1}{2}\frac{4dx-dy}{11}+\frac{2}{3}\,\frac{3dx+2dy}{11}\right)\left(\frac{1}{2}dx+\frac{1}{3}dy\right)}{(11/6)^2} \\ d^2v = \dfrac{-2d^2u-2\frac{4dx-dy}{11}\left(dx-\frac{3dx+2dy}{11}\right)}{4} \end{cases} \Rightarrow$$

$$\Rightarrow u''_{xy} = -\frac{48}{11^3},\ v''_{xy} = \frac{68}{11^3}$$

$$\omega = u(x,y)v(x,y)$$

$$\frac{\partial^2\omega(3,3)}{\partial x\partial y} = (u'_x v + u v'_x)'_y = u''_{xy}v + u'_x v'_y + u'_y v'_x + u v''_{xy} =$$

$$= -\frac{48}{11^3} - \frac{3}{11^2} + \frac{8}{11^2} + \frac{136}{11^3} = \frac{13}{121}$$

4.52(1)

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2},\ u = x-at,\ v = x+at$$

$$z(x,t) \to z(u,v)$$

$$z'_x = z'_u u'_x + z'_v v'_x = z'_u + z'_v$$

$$z'_t = z'_u u'_t + z'_v v'_t = a(z'_v - z'_u)$$

$$z''_{xx} = (z'_u + z'_v)'_x = z''_{ux} + z''_{vx} = z''_{uu}u'_x + z''_{uv}v'_x + z''_{vu}u'_x +$$

$$+ z''_{vv}v'_x = z''_{uu} + 2z''_{uv} + z''_{vv}$$

$$z''_{tt} = (a(z'_v - z'_u))'_t = a(z'_{vt} - z'_{ut}) = a\left(z''_{vv}v'_t + z''_{vu}u'_t -\right.$$

$$\left.-(z''_{uv}v'_t + z''_{uu}u'_t)\right) = a\left(az''_{vv} - 2az''_{vu} + az''_{uu}\right) =$$

$$= a^2(z''_{vv} - 2z''_{vu} + z''_{uu})$$

$$a^2(z''_{vv} - 2z''_{vu} + z''_{uu}) = a^2(z''_{uu} + 2z''_{uv} + z''_{vv})$$

$$z''_{uv} = 0 \Rightarrow z(u,v) = \Phi(u) + \Psi(v)$$

Ответ: $z(u,v) = \Phi(u) + \Psi(v)$, Φ, Ψ - дифф-ые

5.21(4)

$$u(x,y)=2x^2+12xy+y^2,\quad x^2+4y^2=25$$

$$L(x,y)=2x^2+12xy+y^2+\lambda(x^2+4y^2-25)=0$$

$$\begin{cases} L'_x=4x+12y+2\lambda x=0 \\ L'_y=12x+2y+8\lambda y=0 \\ x^2+4y^2=25 \end{cases} \qquad \begin{cases} 16xy+48y^2+8\lambda xy=0 \\ 12x^2+2xy+8\lambda xy=0 \\ x^2+4y^2=25 \end{cases}$$

$$\begin{cases} 12x^2-14xy-48y^2=0 \\ 12x+2y+8\lambda y=0 \\ x^2+4y^2=25 \end{cases} \qquad \begin{cases} 6\left(\frac{x}{y}\right)^2-7\frac{x}{y}-24=0 \Rightarrow \left(2\frac{x}{y}+3\right)\left(3\frac{x}{y}-8\right)=0 \\ 12x+2y+8\lambda y=0 \\ x^2+4y^2=25 \end{cases}$$

$$\left[\begin{array}{l} \begin{cases} 2x=-3y \\ -18y+2y+8\lambda y=0 \Rightarrow \lambda=\frac{7}{4} \\ y=\pm 2 \end{cases} \\ \begin{cases} 3x=8y \\ 32y+2y+8\lambda y=0 \Rightarrow \lambda=-\frac{17}{4} \\ y=\pm\frac{3}{2} \end{cases} \end{array}\right. \qquad \left[\begin{array}{l} \begin{cases} x=\mp 3 \\ y=\pm 2 \\ \lambda=7/4 \end{cases} \\ \begin{cases} x=\pm 4 \\ y=\pm\frac{3}{2} \\ \lambda=-17/4 \end{cases} \end{array}\right.$$

$$L''_{xx}=4+2\lambda$$

$$L''_{yy}=2+8\lambda \quad \Rightarrow d^2L=(4+2\lambda)dx^2+12dxdy+(2+8\lambda)dy^2$$

$$L''_{xy}=12$$

В $(\pm 3,\mp 2)$: $d^2L=\frac{19}{2}dx^2+12dxdy+16dy^2>0$

В $(\pm 4,\pm\frac{3}{2})$: $d^2L=-\frac{9}{2}dx^2+12dxdy-32dy^2<0$

Ответ: $u(\pm 3,\mp 2)=-50$ – локальный минимум,

$u(\pm 4,\pm\frac{3}{2})=\frac{425}{4}$ – локальный максимум.

5.25(2)

$$u(x,y,z)=xy^2z^3,\ x+y+z=12,\ x>0,\ y>0,\ z>0$$

$$x+y+z=12 \Rightarrow z=12-x-y$$

$$u(x,y,z(x,y))=f(x,y)=xy^2(12-x-y)^3$$

$$f'_x=y^2(12-x-y)^3+xy^2\cdot 3(12-x-y)^2\cdot(-1)=y^2(12-x-y)^2(12-x-y-$$
$$-3x)=y^2(12-x-y)^2(12-4x-y)=0 \Rightarrow y=0,\ x+y=12,\ 4x+y=12$$

$$f'_y=x(12-x-y)^2(2y(12-x-y)-3y^2)=xy(12-x-y)^2(24-2x-$$
$$-5y)=0 \Rightarrow x=0,\ x+y=12,\ y=0,\ 2x+5y=24$$

Стационарные точки $f(x,y)$: $(0,0),(12,0),(0,12),(2,4)$

$$f''_{xx}=-2y^2(12-x-y)(12-4x-y)-4y^2(12-x-y)^2$$

$$f''_{yy}=x(12-x-y)^2(24-2x-5y)-2xy(12-x-y)(24-2x-5y)-5xy(12-x-y)^2$$

$$f''_{xy}=2y(12-x-y)^2(12-4x-y)-2y^2(12-x-y)(12-4x-y)-y^2(12-x-y)^2$$

$(0,0),(12,0),(0,12)$: $d^2f\equiv 0$ – полуопр.

$(2,4)$: $d^2f=-2304dx^2-1152dxdy-1440dy^2<0$

<u>Ответ</u>: $u(2,4,6)=6912$ – точка локального максимума.

26(1)

$$u(x,y,z)=xyz,\ x+y-z=3,\ x-y-z=8$$

$$x+y-3=x-y-8 \Rightarrow y=-\frac{5}{2},\ z=x-\frac{11}{2}$$

$$u(x,y,z)=f(x)=x\left(-\frac{5}{2}\right)\left(x-\frac{11}{2}\right)$$

$$f'_x=-5x+\frac{55}{4}=0 \Rightarrow x=\frac{11}{4}$$

Стационарная точка – $\left(\frac{11}{4},-\frac{5}{2},-\frac{11}{4}\right)$

$f''_{xx} = -5 < 0 \Rightarrow$ экстремум

Ответ: $u\left(\frac{11}{4}, -\frac{5}{2}, -\frac{11}{4}\right) = \frac{605}{32}$ – локальный максимум.

30(1)

$u = 3 + 2xy$ а) $x^2+y^2 \le 1$; б) $4 \le x^2+y^2 \le 9$

$$\begin{cases} u'_x = 2y = 0 \\ u'_y = 2x = 0 \end{cases} \Rightarrow (0,0) \text{ – стационарная точка } u$$

$$L(x,y) = 3 + 2xy + \lambda(x^2+y^2-a^2)$$

$$\begin{cases} L'_x = 2y + 2\lambda x \\ L'_y = 2x + 2\lambda y \\ x^2+y^2 = a^2 \end{cases} \begin{cases} \lambda x = -y \\ \lambda y = -x \\ x^2 = \frac{a^2}{\lambda^2+1} \end{cases} \Rightarrow \lambda^2 y^2 = \lambda^4 x^2 = x^2 \Rightarrow \lambda = \pm 1 \Rightarrow$$

$\Rightarrow (x,y) = \left(\pm\frac{a}{\sqrt{2}}, \pm\frac{a}{\sqrt{2}}\right)$ (4 точки)

а) Стационарные точки: внутри – $(0,0)$, на границе – $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$u(0,0) = 3$, $u\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \{2, 4\} \Rightarrow$ min: 2, max: 4

б) Стационарные точки: внутри – нет, на границе – $(\pm\sqrt{2}, \pm\sqrt{2})$, $\left(\pm\frac{3}{\sqrt{2}}, \pm\frac{3}{\sqrt{2}}\right)$

$u(\pm\sqrt{2}, \pm\sqrt{2}) = \{-1, 7\}$

$u\left(\pm\frac{3}{\sqrt{2}}, \pm\frac{3}{\sqrt{2}}\right) = \{-6, 12\} \Rightarrow$ min: −6, max: 12

Ответ: а) min: 2, max: 4; б) min: −6, max: 12.

8.23

$f(x) = \frac{1}{x}$ непрерывна на измеримом множестве $(0,1]$, но не интегрируема на нём.

8.36(1)

$$f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ -1, & x\in\mathbb{I}\end{cases}$$

$|f|$ - непрерывная на замкнутом мн-ве => интегрируема на нём

$$\lim_{|\tau(x)|\to 0}\sum_{i=1}^{N}\omega(f;X_i)\mu(X_i)=2\sum_{i=1}^{N}\mu(X_i)=2\mu([-1,1])=4\neq 0 \Rightarrow$$ f не интегрируема на $[-1,1]$.

Т6

а) Рассмотрим $f(x,y)=\begin{cases}\dfrac{x-y}{(x+y)^3}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

$$I(x)=\int_0^1\frac{x-y}{(x+y)^3}dy=\left|\begin{matrix}x+y=t\\ dy=dt\\ [0,1]\to[x,x+1]\end{matrix}\right|=\int_x^{x+1}\frac{2x-t}{t^3}dt=2x\left(-\frac{1}{2}t^{-2}\right)\Big|_x^{x+1}-\left(-\frac{1}{t}\right)\Big|_x^{x+1}=\frac{1}{(1+x)^2}$$

интегрируема

$$I'(y)=-I(x)$$

$$\iint\limits_P f(x,y)dxdy=\int_0^1 I(x)dx=\int_0^1 I'(y)dy$$

не верно => $\nexists \iint\limits_P f(x,y)dxdy$

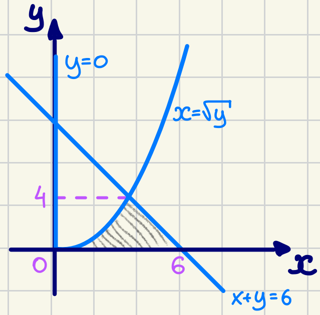
Ответ: нет.

б) Рассмотрим функцию $f(x,y)=\begin{cases}1, & x\neq 0\\ \dfrac{1}{y}, & x=0\end{cases}$ $[a,b]=[0,1]$, $[c,d]=(0,1]$

$\iint\limits_P f(x,y)dxdy=1$, но при $x=0$ $\int_0^1 f(x,y)dy=\int_0^1\frac{dy}{y}$ - не интегрируема

Ответ: Нет.

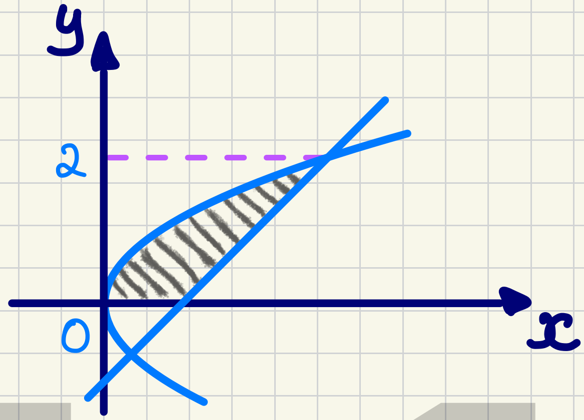
8.80(3)



$$\iint\limits_G f(x,y)dxdy=\int_0^6 dx\int_{x^2}^{6-x}f(x,y)dy=$$

$$=\int_0^4 dy\int_{\sqrt{y}}^{6-y}f(x,y)dx$$

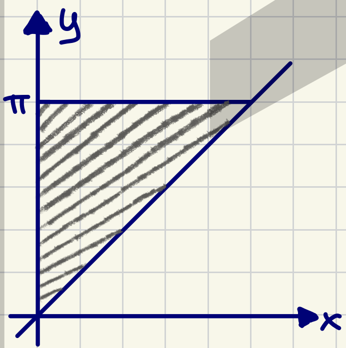
8.82(4)



$$\int_0^2 dy \int_{y^2=\psi(y)}^{y+2=\varphi(y)} f(x,y)\,dx = \iint_G f(x,y)\,dx\,dy$$

$$G=\{(x,y)\mid 0<y<2,\ y+2<x<y^2\}$$

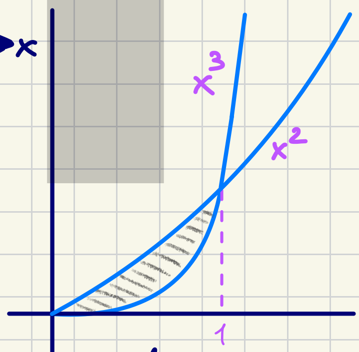
85(2)



$$\int_0^{\pi} dy \int_y^{\pi} \frac{\sin x}{x}\,dx = \int_0^{\pi} dx \int_0^{x} \frac{\sin x}{x}\,dy =$$

$$= \int_0^{\pi} \sin x\,dx = (-\cos x)\Big|_0^{\pi} = 2$$

8.90(2)



$$\iint_G \frac{y}{x^2}\,dx\,dy,\quad G=\{0<x,\ x^3\le y\le x^2\}$$

$$\iint \frac{y}{x^2}\,dx\,dy = \int_0^1 dx \int_{x^3}^{x^2} \frac{y}{x^2}\,dy = \int_0^1 \frac{dx}{x^2}\cdot\frac{1}{2}\left(x^4-x^6\right) = \frac{1}{2}\int_0^1 \left(x^2-x^4\right)dx = \frac{1}{15}$$

8.90(6)

$$\iint_G (x+2y)\,dx\,dy,\quad G=\{2<x<3,\ x<y<2x\}$$

$$\iint_G (x+2y)\,dx\,dy = \int_2^3 dx \int_x^{2x} (x+2y)\,dy = \int_2^3 dx\,\left(x(2x-x)+(2x)^2-x^2\right) =$$

$$= \int_2^3 4x^2\,dx = \frac{4}{3}(27-8) = \frac{76}{3}$$

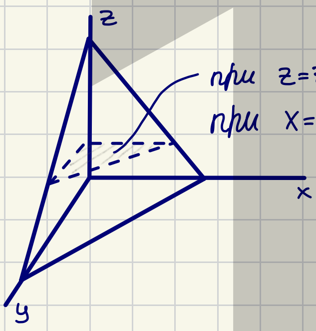
8.133(1)

$$1)\int_{-a}^{a}dx\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dy\int_0^h f(x,y,z)\,dz=\iiint_G f(x,y,z)\,dx\,dy\,dz =$$

G-цилиндр $\Rightarrow$ $$= \int_0^h dz\int_{-a}^{a}dy\int_{-\sqrt{x^2-a^2}}^{\sqrt{x^2-a^2}} f(x,y,z)\,dx$$

8.139(1)

$$\iiint_G y\,dx\,dy\,dz,\quad G=\{x>0,\ y>0,\ z>0,\ 2x+y+z<4\}$$



при $z=z_0$ $x\in[0,\frac{4-z_0}{2}]$

при $x=x_0$ $y\in[0,4-z_0-2x_0]$

$$\iiint_G y\,dx\,dy\,dz=\int_0^4 dz\iint_{S(z)} y\,dx\,dy=$$

$$=\int_0^4 dz\int_0^{\frac{4-z}{2}}dx\int_0^{4-z-2x}y\,dy=\int_0^4 dz\int_0^{\frac{4-z}{2}}dx\,\frac12(4-z-2x)^2=$$

$$=\int_0^4 dz\left(-\frac14\right)\frac{(4-z-2x)^3}{3}\Bigg|_{x=0}^{x=\frac{4-z}{2}}=\int_0^4 dz\left(-\frac14\right)\left(-\frac{(4-z)^3}{3}\right)=\frac1{12}\int_0^4(4-z)^3dz=\frac1{12}\,\frac{(4-z)^4}{4}\Bigg|_4^0=\frac{16}{3}$$

175(3)

$$\int_{Q_n}\sum_{k=1}^n x_k^p\,dx,\quad Q_n=[0,a]^n$$

$$\int_{Q_n}\sum_{k=1}^n x_k^p\,dx=\int_0^a dx_n\int_0^a dx_{n-1}\dots\int_0^a\sum_{k=1}^n x_k^p\,dx_1=$$

$$=\int_0^a dx_n\int_0^a dx_{n-1}\dots\int_0^a\left(\frac{a^{p+1}}{p+1}+a\sum_{k=2}^n x_k^p\right)dx_2=\int_0^a dx_n\int_0^a dx_{n-1}\dots\int_0^a\left(\frac{2a^{p+2}}{p+1}+a\sum_{k=2}^n x_k^p\right)dx_3=$$

$$=\frac{na^{p+n}}{p+1}$$

176(1) $$\int_{\Pi_n}dx,\quad \Pi_n=\{0\le x_n\le x_{n-1}\le\dots\le x_2\le x_1\le a\}$$

$$\int\limits_{\Pi_n} dx = \int_0^a dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-1}} dx_n = \int_0^a dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-2}} x_{n-1}\, dx_{n-1} = \int_0^a dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-3}} \frac{x_{n-2}^2}{2} dx_{n-2} =$$

$$= \int_0^a \frac{x_n^{n-1}}{n-1} dx_n = \frac{a^n}{n!}$$

106(2)

$$\iint\limits_G \frac{dx dy}{x^2+y^2-1}, \quad G = \{9 \le x^2+y^2 \le 25\}$$

$$\iint\limits_G \frac{dx dy}{x^2+y^2-1} = \iint\limits_{\tilde{G}} \frac{|J|\, dr d\varphi}{r^2-1} = \iint\limits_{\tilde{G}} \frac{r\, dr d\varphi}{r^2-1} = \int_0^{2\pi} d\varphi \int_3^5 \frac{r dr}{r^2-1} = \pi \ln|r^2-1| \Big|_3^5 = \pi \ln\frac{24}{8} = \pi \ln 3$$

107(1)

$$\iint\limits_G \frac{y^2}{x^2+y^2} dx dy, \quad G = \{x^2+y^2 \le ax\}, \quad a>0$$

$$\iint\limits_G \frac{y^2}{x^2+y^2} dx dy = \iint\limits_{\tilde{G}} \sin^2\varphi\, r dr d\varphi, \quad \tilde{G} = \{r \le a\cos\varphi\}$$

$$\int_0^{2\pi} \sin^2\varphi d\varphi \int_0^{a\cos\varphi} r dr = \int_0^{2\pi} \sin^2\varphi d\varphi \frac{a^2\cos^2\varphi}{2} = \frac{a^2}{16}\int_0^{2\pi} \sin^2 2\varphi\, d(2\varphi) = \frac{a^2}{32}\int_0^{2\pi} (1-\cos 4\varphi)\, d(2\varphi) =$$

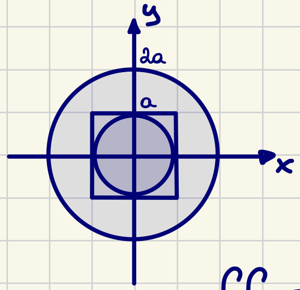
$$= \frac{\pi a^2}{16}$$

110

1) $$\iint\limits_G e^{-(x^2+y^2)} dx dy, \quad G = \{x^2+y^2 \le R^2,\ x \ge 0,\ y \ge 0\}$$

$$\int_0^R e^{-r^2} r dr \int_0^{\pi/2} d\varphi = \frac{\pi}{4}(1-e^{-R^2})$$

2) $$\int_0^a e^{-x^2} dx = I \ (*) \qquad I^2 = \iint\limits_{\Pi} e^{-(x^2+y^2)} dx dy, \quad \Pi = \{0 \le x \le a,\ 0 \le y \le a\}$$



$G_1=\{x^2+y^2\leq a^2,\ x\geq 0, y\geq 0\}$

$G_2=\{x^2+y^2\leq (2a)^2,\ x\geq 0, y\geq 0\}$

Т.к. $e^{-x^2+y^2}>0$, а $G_1\subset \Pi \subset G_2$, то

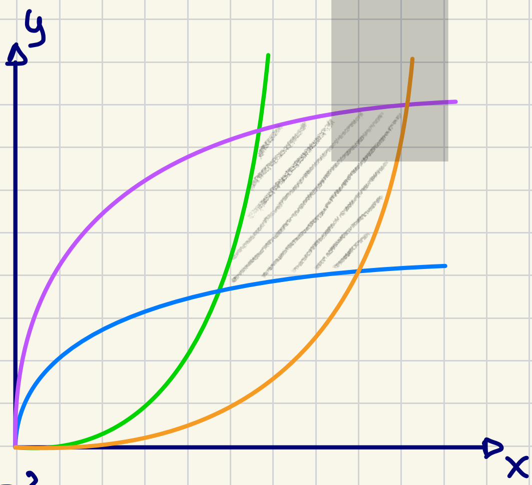
$$\iint\limits_{G_1} e^{-(x^2+y^2)}dxdy \leq I^2 \leq \iint\limits_{G_2} e^{-(x^2+y^2)}dxdy$$

Используя результат 1) и (*) получаем искомое

3) $\int\limits_0^{+\infty} e^{-x^2}dx = I \qquad I^2=\lim\limits_{R\to\infty}\frac{\sqrt{\pi}}{2}(1-e^{-R^2})=\frac{\sqrt{\pi}}{2}$

124(4)

$\iint\limits_G \frac{x}{y}dxdy$, G-ограничено $y=x^2,\ 8y=x^2,$ $x=y^2,\ 8x=y^2$



Замена $u=\frac{x^2}{y},\ v=\frac{y^2}{x}$

$x=\sqrt[3]{u^2v} \quad y=\sqrt[3]{v^2u}$

$$J=\begin{vmatrix}\frac{2}{3}\sqrt[3]{\frac{v}{u}} & \frac{1}{3}\sqrt[3]{\frac{u^2}{v^2}}\\ \frac{1}{3}\sqrt[3]{\frac{v^2}{u^2}} & \frac{2}{3}\sqrt[3]{\frac{u}{v}}\end{vmatrix}=\frac{4}{9}-\frac{1}{9}=\frac{1}{3}$$

$$\iint\limits_G \frac{x}{y}dxdy=\iint\limits_{\tilde{G}}\sqrt[3]{\frac{u}{v}}\cdot\frac{1}{3}\cdot du\,dv,\quad \tilde{G}=[1,8]^2$$

$$\iint\limits_{\tilde{G}}\sqrt[3]{\frac{u}{v}}\cdot\frac{1}{3}\cdot du\,dv=\frac{1}{3}\int\limits_1^8\sqrt[3]{u}\,du\int\limits_1^8\frac{1}{\sqrt[3]{v}}dv=\frac{1}{3}\left(\frac{u^{4/3}}{4/3}\right)\Big|_1^8\left(\frac{v^{2/3}}{2/3}\right)\Big|_1^8=\frac{135}{8}$$

125(4)

$\iint\limits_G x\,dxdy,\ G=\{x>0, y>0, \left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}<1\}$

Замена $u=\sqrt[3]{\frac{x}{a}},\ v=\sqrt[3]{\frac{y}{b}}$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}3au^2 & 0\\ 0 & 3bv^2\end{vmatrix}=9abu^2v^2$$

$$\iint\limits_{\tilde G} x\,dx\,dy=\iint\limits_{\tilde G} au^3\cdot 9abu^2v^2\,du\,dv,\quad \tilde G=\{u>0,\ v>0,\ u^2+v^2<1\}$$

Замена $u=r\cos\varphi,\ v=r\sin\varphi$

$$\iint\limits_{G^*} 9a^2b\,r^7\cos^5\varphi\sin^2\varphi\,r\,dr\,d\varphi,\quad G^*=\{0<\varphi<\tfrac{\pi}{2},\ 0<r<1\}$$

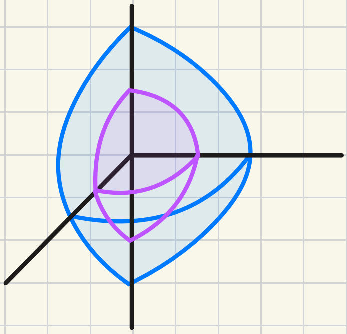
$$9a^2b\int_0^{\pi/2}\cos^5\varphi\sin^2\varphi\,d\varphi\int_0^1 r^8\,dr=a^2b\left(\frac{\sin^3\varphi}{3}-\frac{2\sin^5\varphi}{5}+\frac{\sin^7\varphi}{7}\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{8a^2b}{105}$$

144(3)

$$\iiint\limits_G (x^2+y^2-z^2)\,dx\,dy\,dz,\quad G=\{1\le x^2+y^2+z^2\le 4,\ x\ge 0,\ y\ge 0\}$$

$$\begin{cases}x=r\cos\theta\cos\varphi\\ y=r\cos\theta\sin\varphi\\ z=r\sin\theta\end{cases}\qquad J=r^2\cos\theta\qquad \begin{array}{l}r\in[1,2]\\ \varphi\in[0,\pi/2]\\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]\end{array}$$

$$\iiint\limits_G (x^2+y^2-z^2)\,dx\,dy\,dz=$$



$$=\int_1^2 dr\int_0^{\pi/2}d\varphi\int_{-\pi/2}^{\pi/2} r^4(\cos^2\theta-\sin^2\theta)\cos\theta\,d\theta=\frac{31\pi}{15}$$

145(2)

$$\iiint\limits_G f\left(\frac{z}{\sqrt{x^2+y^2}}\right)dx\,dy\,dz,\quad G=\{z^2\le x^2+y^2,\ x^2+y^2+z^2\le R^2\}$$

$$\begin{cases}x=r\cos\theta\cos\varphi\\ y=r\cos\theta\sin\varphi\\ z=r\sin\theta\end{cases}\qquad J=r^2\cos\theta\qquad \begin{array}{l}r\in[0,R]\\ \varphi\in[0,2\pi]\\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]\end{array}$$

$$\iiint\limits_G f\left(\frac{z}{\sqrt{x^2+y^2}}\right)dx\,dy\,dz = \int_0^{2\pi}d\varphi\int_0^R r^2dr\int_{-\pi/2}^{\pi/2}f(\operatorname{tg}\theta)\cos\theta\,d\theta = \frac{2\pi R^3}{3}\int_{-1}^{1}f(\operatorname{tg}\theta)\cos\theta\,d\theta$$

146(2)

$$\iiint\limits_G (x+y+z)dx\,dy\,dz,\quad G\text{-ограничена } x^2+y^2=1,\ z=0,\ x+y+z=2$$

$$G\ \begin{cases} x=r\cos\varphi\\ y=r\sin\varphi\\ z=h\end{cases}\quad J=1\quad \begin{array}{l} r\in[0,1]\\ \varphi\in[0,2\pi]\\ h\in[0,\,2-f(r,\varphi)]\quad f(r,\varphi)=r(\cos\varphi+\sin\varphi)\end{array}$$

$$\iiint\limits_G (x+y+z)dx\,dy\,dz = \int_0^{2\pi}d\varphi\int_0^1 dr\int_0^{2-f(r,\varphi)}(f(r,\varphi)+h)\,dh =$$

$$= \int_0^{2\pi}d\varphi\int_0^1 dr\left(f(r,\varphi)(2-f(r,\varphi))+\frac{1}{2}(2-f(r,\varphi))^2\right) =$$

$$= \int_0^{2\pi}d\varphi\int_0^1 dr\,\frac{1}{2}(4-f^2) = \int_0^{2\pi}d\varphi\int_0^1 dr\left(2-\frac{r^2}{2}(1+\sin 2\varphi)\right) = \int_0^{2\pi}d\varphi\int_0^1 dr\left(2-\frac{r^2}{2}(1+\sin 2\varphi)\right) =$$

$$= \int_0^{2\pi}d\varphi\left(2-\frac{1}{6}(1+\sin 2\varphi)\right) = 4\pi-\frac{\pi}{3} = \frac{11\pi}{3}$$

147

$$x=ar\cos\varphi\cos\psi,\ y=br\sin\varphi\cos\psi,\ z=cr\sin\psi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \psi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \psi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \psi}\end{vmatrix} = \begin{vmatrix} a\cos\varphi\cos\psi & -ar\sin\varphi\cos\psi & -ar\cos\varphi\sin\psi\\ b\sin\varphi\cos\psi & br\cos\varphi\cos\psi & -br\sin\varphi\sin\psi\\ c\sin\psi & 0 & cr\cos\psi\end{vmatrix} =$$

$$= c\sin\psi(abr^2\sin^2\varphi\sin\psi\cos\psi+abr^2\cos^2\varphi\sin\psi\cos\psi)+cr\cos\psi(abr\cos^2\varphi\cdot$$

$$\cdot\cos^2\psi+abr\sin^2\varphi\cos^2\psi) = abcr^2\sin^2\psi\cos\psi+abcr^2\cos^3\psi = abcr^2\cos\psi$$

148(1)

$$\iiint\limits_G \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz,\quad G=\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1\right\}$$

$$\begin{cases} x=ar\cos\varphi\cos\psi \\ y=br\sin\varphi\cos\psi \\ z=cr\sin\psi \end{cases}\qquad J=abcr^2\cos\psi\qquad \begin{array}{l} r\in[0,1] \\ \varphi\in[0,2\pi] \\ \psi\in[-\frac{\pi}{2},\frac{\pi}{2}] \end{array}$$

$$\iiint\limits_G \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz=\iiint\limits_{\tilde{G}} r^2abcr^2\cos\psi\, dr d\varphi d\psi=$$

$$=abc\int\limits_{-\pi/2}^{\pi/2}\cos\psi d\psi\int\limits_0^{2\pi}d\varphi\int\limits_0^1 r^4dr=\frac{4\pi}{5}abc$$

150(8)

$$\iiint\limits_G z^2dxdydz,\quad G=\{x^2+y^2+z^2\leq R^2,\ x^2+y^2+z^2\leq 2Rz\}$$

$$\begin{cases} x=r\cos\theta\cos\varphi \\ y=r\cos\theta\sin\varphi \\ z=R+r\sin\theta \end{cases}\qquad J=r^2\cos\theta\qquad \begin{array}{l} r\in[0,R] \\ \varphi\in[0,2\pi] \\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}] \end{array}$$

$$\iiint\limits_G z^2dxdydz=\iiint\limits_{\tilde{G}}(R+r\sin\theta)^2r^2\cos\theta\, drd\varphi d\theta=$$

$$=\int\limits_0^{2\pi}d\varphi\int\limits_{-\pi/2}^{\pi/2}d\theta\int\limits_0^R(R+r\sin\theta)^2r^2\cos\theta\, dr=\int\limits_0^{2\pi}d\varphi\int\limits_{-\pi/2}^{\pi/2}d\theta\left(\frac{R^5}{3}+\frac{R^5}{2}\sin\theta+\frac{R^5}{5}\sin^2\theta\right)\cos\theta=$$

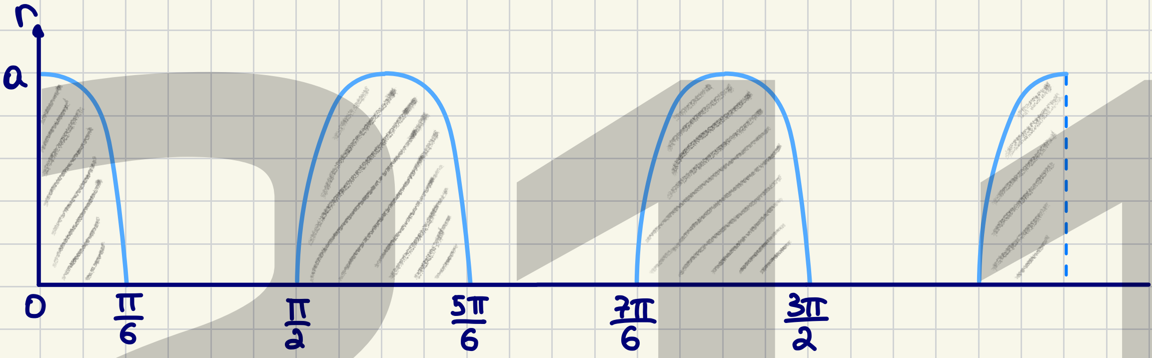
$$=\int\limits_0^{2\pi}d\varphi\int\limits_{-\pi/2}^{\pi/2}d\theta\left(\frac{R^5}{3}+\frac{R^5}{2}\sin\theta+\frac{R^5}{5}\sin^2\theta\right)\cos\theta=2\pi R^5\left(2+\frac{2}{3}\right)=\frac{16}{3}\pi R^5$$

9.6(6)

$$(x^2+y^2)^2=a(x^3-3xy^2),\ a>0$$

$$r^4=ar^3(\cos^3\varphi-3\cos\varphi\sin^2\varphi)$$

$$r=a(\cos^3\varphi-3\cos\varphi+3\cos^3\varphi)=a\cos3\varphi$$



$$6\iint\limits_{\tilde G} r\,dr\,d\varphi,\quad \tilde G=\left\{0<r<a\cos3\varphi,\ 0<\varphi<\frac{\pi}{6}\right\}$$

$$6\int_0^{\frac{\pi}{6}}d\varphi\int_0^{a\cos3\varphi}r\,dr=6\int_0^{\frac{\pi}{6}}d\varphi\,\frac{a^2\cos^2 3\varphi}{2}=3a^2\int_0^{\frac{\pi}{6}}d\varphi\,\frac{1+\cos6\varphi}{2}=\frac{\pi a^2}{4}$$

9.8(5)

$$(x+y)^4=a^2(x^2+y^2),\ x=0,\ y=0\ (x>0,\ y>0)$$

$$\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \end{cases}\qquad J=r$$

$$r^4(\cos\varphi+\sin\varphi)^4=a^2r^2$$

$$r=\frac{a}{2\sin(\varphi+\frac{\pi}{4})}$$

$$\int_0^{\pi/2}d\varphi\int_0^{\frac{a}{2\sin^2(\varphi+\pi/4)}}r\,dr=\int_0^{\pi/2}d\varphi\,\frac{a^2}{8\sin^4(\varphi+\frac{\pi}{4})}=\int_0^{\pi/2}d\varphi\,\frac{a^2}{8\sin^4(\varphi+\frac{\pi}{4})}=\frac{a^2}{8}\int_{\pi/4}^{3\pi/4}\frac{d\theta}{\sin^4\theta}=$$

$$= \frac{a^2}{8}\int\limits_{\pi/4}^{3\pi/4}(1+\operatorname{ctg}^2\theta)\,d\operatorname{ctg}\theta = \frac{a^2}{3}$$

9.10

$$(a_1x+b_1y+c_1)^2+(a_2x+b_2y+c_2)^2=1$$

Замена $\begin{Vmatrix} a_1 & b_1\\ a_2 & b_2\end{Vmatrix}\begin{Vmatrix}x\\y\end{Vmatrix}+\begin{Vmatrix}c_1\\c_2\end{Vmatrix} = S\begin{Vmatrix}x\\y\end{Vmatrix}+\begin{Vmatrix}c_1\\c_2\end{Vmatrix} \to J=\det S^{-1}=\frac{1}{|\Delta|}$

$$\iint\limits_G \frac{1}{|\Delta|}\,du\,dv = \frac{\pi}{|\Delta|}$$

9.13(б)

$$y^2+z^2=x,\quad x=y$$

$$\begin{cases} y=r\cos\varphi\\ z=r\sin\varphi\\ x=h\end{cases}\qquad J=r\qquad \begin{aligned} &r\in[0,\cos\varphi]\\ &\varphi\in[-\pi/2,\pi/2]\\ &h\in[r^2, r\cos\varphi]\end{aligned}$$

$$\int\limits_{-\pi/2}^{\pi/2}d\varphi\int\limits_0^{\cos\varphi}r\,dr\int\limits_{r^2}^{r\cos\varphi}dh = \int\limits_{-\pi/2}^{\pi/2}d\varphi\int\limits_0^{\cos\varphi}r\,dr\int\limits_{r^2}^{r\cos\varphi}dh = \int\limits_{-\pi/2}^{\pi/2}d\varphi\int\limits_0^{\cos\varphi}r^2(\cos\varphi-r)\,dr = \frac{1}{12}\int\limits_{-\pi/2}^{\pi/2}\cos^4\varphi\,d\varphi =$$

$$= \frac{1}{12}\int\limits_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi = \frac{1}{48}\int\limits_{-\pi/2}^{\pi/2}(1+\cos^2 2\varphi)\,d\varphi = \frac{1}{48}\int\limits_{-\pi/2}^{\pi/2}\frac{3}{2}\,d\varphi = \frac{\pi}{32}$$

9.16(1)

$$x^2+y^2+z^2=4,\quad z=\sqrt{x^2+y^2}\quad (z<\sqrt{x^2+y^2})$$

$r\sin\theta < r\cos\theta$

$$\begin{cases} x=r\cos\theta\cos\varphi\\ y=r\cos\theta\sin\varphi\\ z=r\sin\theta\end{cases}\qquad J=r^2\cos\theta\qquad \begin{aligned} &r\in[0,2]\\ &\varphi\in[0,2\pi]\\ &\theta\in[-\tfrac{\pi}{2},\tfrac{\pi}{4}]\end{aligned}$$

$$\int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/4} \cos\theta \, d\theta \int_0^2 r^2 dr = \frac{8\pi}{3}(2+\sqrt{2})$$

9.21

G - параллелепипед, ограниченный $a_i x + b_i y + c_i z = \pm d_i$, $i=1,2,3$

$$\begin{Vmatrix} u \\ v \\ w \end{Vmatrix} = \begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix} \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = S \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} \rightarrow J = \det S^{-1} = \frac{1}{|\Delta|}$$

$$\frac{1}{|\Delta|} \int_{-d_1}^{d_1} du \int_{-d_2}^{d_2} dv \int_{-d_3}^{d_3} dw = \frac{8 d_1 d_2 d_3}{|\Delta|}$$

9.63(1)

$$\rho(x,y,z) = \rho_0 (x+y+z)^2, \quad G = [0,a]^3$$

$$M = \rho_0 \int_0^a dx \int_0^a dy \int_0^a (x+y+z)^2 dz = \rho_0 \int_0^a dx \int_0^a dy \left[\frac{(x+y+a)^3}{3} - \frac{(x+y)^3}{3} \right] =$$

$$= \rho_0 \int_0^a dx \left[\frac{(x+2a)^4 - (x+a)^4}{12} - \frac{(x+a)^4 - x^4}{12} \right] = \rho_0 \left[\frac{(3a)^5 - (2a)^5}{60} - 2\frac{(2a)^5 - a^5}{60} + \frac{a^5}{60} \right] =$$

$$= \rho_0 \cdot \frac{5}{2} a^5$$

$$x_c = \frac{1}{M} \rho_0 \int_0^a x dx \int_0^a dy \int_0^a (x+y+z)^2 dz = \frac{1}{30a^5} \int_0^a x dx \left[(x+2a)^4 - 2(x+a)^4 + x^4 \right] =$$

$$= \frac{1}{30a^5} \int_0^a x dx \left[(x+2a)^4 - 2(x+a)^4 + x^4 \right] = \frac{1}{30a^5} \int_0^a x (12x^2a^2 + 24xa^3 + 14a^4) dx =$$

$$= \frac{1}{30a^5} 18a^6 = \frac{3}{5} a$$

$$x_c = y_c = z_c = \frac{3}{5} a$$