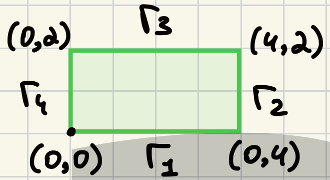
10.2(3)

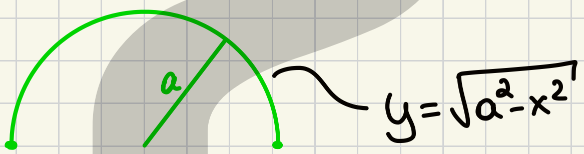
$$\int_{\Gamma} xy\,ds = \overset{0}{\int_{\Gamma_1} xy\,ds} + \int_{\Gamma_2} xy\,ds + \int_{\Gamma_3} xy\,ds + \overset{0}{\int_{\Gamma_4} xy\,ds} =$$



$$= \int_{\Gamma_2} xy\,ds + \int_{\Gamma_3} xy\,ds = 4\int_0^2 y\,dy + 2\int_0^4 x\,dx = 2y^2\Big|_0^2 + x^2\Big|_0^4 = 24$$

10.4

$$\int_{\Gamma} x^2 ds = \int_{-a}^{a} x^2\sqrt{1+y_x'^2}\,dx = \int_{-a}^{a} x^2 \frac{a}{\sqrt{a^2-x^2}}\,dx = a^2\int_{-a}^{a}\frac{\frac{x^2}{a^2}}{\sqrt{1-\frac{x^2}{a^2}}}\,dx = a^3\int_{-1}^{1}\frac{t^2}{\sqrt{1-t^2}}\,dt =$$



$$= a^3\int_{\pi}^{0}\frac{\cos^2\varphi}{\sin\varphi}\,d(\cos\varphi) = a^3\int_0^{\pi}\cos^2\varphi\,d\varphi =$$

$$= a^3\int_0^{\pi}\frac{1+\cos 2\varphi}{2}\,d\varphi = \frac{\pi a^3}{2}$$

10.10

$$\int_{\Gamma} f(x,y)\,ds, \quad \Gamma = \{(x,y) \mid x = a(t-\sin t),\ y = a(1-\cos t),\ 0 \le t \le 2\pi\}$$

1) $f(x,y) = y$

$$\int_{\Gamma} y\,ds = \int_0^{2\pi} y\sqrt{x_t'^2 + y_t'^2}\,dt = \int_0^{2\pi} ya\sqrt{(1-\cos t)^2+\sin^2 t}\,dt = \int_0^{2\pi} ya\sqrt{2-2\cos t}\,dt =$$

$$= a^2\sqrt{2}\int_0^{2\pi}(1-\cos t)^{3/2}\,dt = a^2\sqrt{2}\int_0^{2\pi}\left(\frac{1-\cos 2\cdot\frac{t}{2}}{2}\cdot 2\right)^{3/2} dt = 4a^2\int_0^{2\pi}\sin^3\frac{t}{2}\,dt =$$

$$= -8a^2\int_0^{\pi}\sin^2 p\,d(\cos p) = 8a^2\int_{-1}^{1}(1-q^2)\,dq = 16a^2 - \frac{16}{3}a^2 = \frac{32}{3}a^2$$

2) $f(x,y) = y^2$

$$\int_\Gamma y^2 ds = \int_0^{2\pi} y^2 a\sqrt{2-2\cos t}\,dt = a^3\sqrt{2}\int_0^{2\pi}(1-\cos t)^{5/2}\,dt = 8a^3\int_0^{2\pi}\sin^5\frac{t}{2}\,dt =$$

$$= -16a^3\int_0^{2\pi}\left(1-\cos^2\frac{t}{2}\right)^2 d\left(\cos\frac{t}{2}\right) = -16a^3\int_0^{\pi}\left(1-2\cos^2 p+\cos^4 p\right)d(\cos p) =$$

$$= 16a^3\int_{-1}^{1}\left(1-2q^2+q^4\right)dq = 16a^3\left[2-2\cdot\frac{2}{3}+\frac{2}{5}\right] = \frac{256a^3}{15}$$

10.19(2)

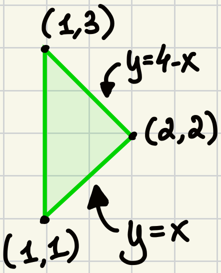
$$\int_\Gamma \left(x-\frac{1}{y}\right)dy,\ \Gamma=\{(x,y)\mid y=x^2,\ 1\le x\le 2\}$$

$$\int_\Gamma \left(x-\frac{1}{y}\right)dy = \int_1^2\left(x-\frac{1}{x^2}\right)2x\,dx = \int_1^2 2x^2dx - \int_1^2\frac{2}{x}dx = \frac{2}{3}(8-1)-2\ln 2 =$$

$$= \frac{14}{3}-2\ln 2$$

10.30(1)

$$\int_\Gamma 2(x^2+y^2)dx+(x+y)^2dy = \iint_G\left[2(x+y)-4y\right]dxdy =$$



$$= 2\iint_G(x-y)\,dxdy = 2\int_1^2dx\int_x^{4-x}(x-y)\,dy =$$

$$= -2\int_1^2dx\int_x^{4-x}(x-y)\,d(x-y) = -2\int_1^2dx\,\frac{(x-y)^2}{2}\Big|_x^{4-x} =$$

$$= -\int_1^2(2x-4)^2dx = -4\int_1^2(x-2)^2d(x-2) = -\frac{4(x-2)}{3}\Big|_1^2 = \frac{4}{3}$$

10.87(4)

$$\Gamma: x=a\cos^3t;\ y=a\sin^3t,\ 0\le t\le\frac{\pi}{2},\ \rho=\sqrt[3]{y}$$

$$m=\int_\Gamma\rho\,ds = \int_0^{\pi/2}\sqrt[3]{y}\cdot a\sqrt{(3\cos^2t\sin t)^2+(3\sin^2t\cos t)^2}\,dt =$$

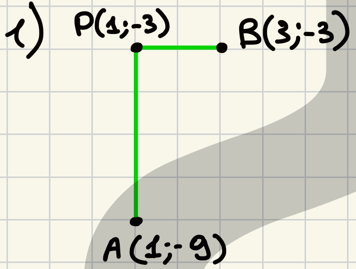
$$= 3a\int_0^{\pi/2} \sqrt[3]{y}\sqrt{\cos^4 t\sin^2 t+\sin^4 t\cos^2 t}\,dt = 3a\sqrt[3]{a}\int_0^{\pi/2}\sin t\cdot\cos t\sin t\,dt =$$

$$= 3a\sqrt[3]{a}\int_0^{\pi/2}\sin^2 t\,d(\sin t) = 3a\sqrt[3]{a}\,\frac{\sin^3 t}{3}\Big|_0^{\pi/2} = a\sqrt[3]{a}$$

10.110(1,2)

$$\vec{F} = \begin{pmatrix} 4x-5y \\ 2x+y \end{pmatrix}$$

1)

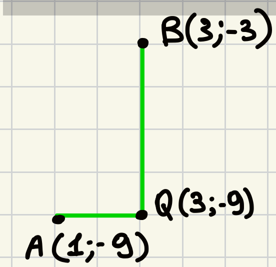


$$A = \int_\Gamma (\vec{F}, d\vec{r}) = \int_\Gamma (4x-5y)dx + (2x+y)dy =$$

$$= \int_{-9}^{-3}(2+y)dy + \int_1^3 (4x+15)dx = 12 + \tfrac{1}{2}y^2\Big|_{-9}^{-3} + 2x^2\Big|_1^3 + 30 =$$

$$= 12-36+16+30 = 22$$

2)



$$A = \int_\Gamma (\vec{F}, d\vec{r}) = \int_\Gamma (4x-5y)dx + (2x+y)dy =$$

$$= \int_1^3 (4x+45)dx + \int_{-9}^{-3}(6+y)dy = 2x^2\Big|_1^3 + 90 + 36 + \frac{y^2}{2}\Big|_{-9}^{-3} =$$

$$= 16+90+36-36 = 106$$

10.46

$$\int_\Gamma (2xy-y)dx + x^2dy = \iint_G \left[2x-(2x-1)\right]dxdy = \pi ab$$

$$\Gamma = \left\{(x,y) \,\Big|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$$

10.101(5)

G-огранич. $x = a\cos t,\ y = b\sin t$ для замыкания контура (следует из полярных коорд.)

$$\mu(G) = \frac{1}{2}\int_{\partial G^+} x\,dy - y\,dx = \frac{ab}{2}\int_0^{2\pi}(\cos^2 t + \sin^2 t)\,dt = \pi ab$$

10.58 $\int\limits_\Gamma (x+y)dx+(x-y)dy$, $A(2;-1)$, $B(1;0)$

$$(x+y)=\frac{\partial u}{\partial x} \to u=\frac{1}{2}x^2+yx+C(y)$$

$$\frac{\partial u}{\partial y}=x+C'(y)=x-y \to C'(y)=-y \Rightarrow C(y)=-\frac{y^2}{2}+C:=-\frac{1}{2}y^2$$

Т.е. $(x+y)dx+(x-y)dy=d\left(\frac{1}{2}x^2+yx-\frac{1}{2}y^2\right)\Rightarrow$

$\Rightarrow$ искомое: $\left(\frac{1}{2}x^2+yx-\frac{1}{2}y^2\right)\Big|_{A(2;-1)}^{B(1;0)}=\frac{1}{2}+\frac{1}{2}=1$

T1.

$\int\limits_\gamma \frac{xdy-ydx}{x^2+y^2}$, γ - простая замкнутая кривая

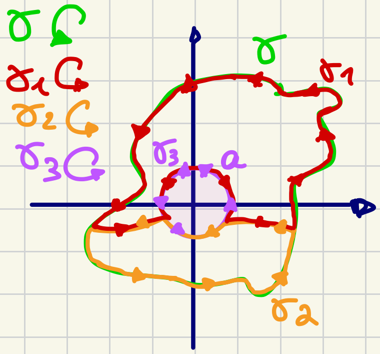
Пусть $(0,0)\notin G$, $\partial G=\gamma$, тогда G - односвязное

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)=\frac{x^2+y^2-2x^2}{x^2+y^2}=\frac{y^2-x^2}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)=\frac{-x^2-y^2+2y^2}{x^2+y^2}=\frac{y^2-x^2}{x^2+y^2}$$

$\Rightarrow$ поле потенциально $\Rightarrow$ ответ 0

Пусть $(0,0)\in G$. Выделим в G круг произвольного радиуса a.



Тогда искомый $\int\limits_\gamma \frac{xdy-ydx}{x^2+y^2}=$

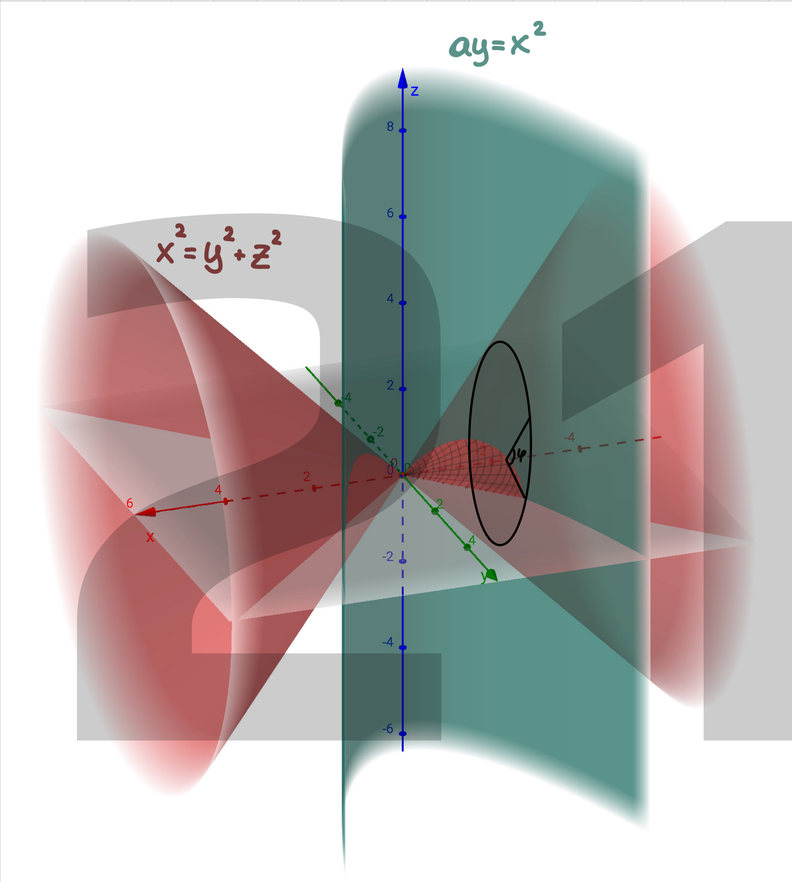
$$=\underbrace{\int\limits_{\gamma_1}\frac{xdy-ydx}{x^2+y^2}}_{0}+\underbrace{\int\limits_{\gamma_2}\frac{xdy-ydx}{x^2+y^2}}_{0}+\int\limits_{\gamma_3}\frac{xdy-ydx}{x^2+y^2}$$

в силу результата, полученного выше

$$\int\limits_{\gamma_3}\frac{xdy-ydx}{x^2+y^2}=\left|\begin{matrix}x=a\cos t\\ y=a\sin t\end{matrix}\right|\int\limits_0^{2\pi}dt=2\pi \Rightarrow \int\limits_\gamma \frac{xdy-ydx}{x^2+y^2}=2\pi$$

Итак, если $G \ni (0,0)$ $\oint\limits_{\gamma} \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi$, иначе $\oint\limits_{\gamma} \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

9.38



площадь $\Pi = \frac{1}{4}\tilde{\Pi}$, где $\tilde{\Pi}$ - площадь заштрихованной на рисунке поверхности, а Π - искомая площадь.

Замена $\begin{cases} x = \rho \\ y = \rho\cos\varphi \\ z = \rho\sin\varphi \end{cases}$

$$\iint\limits_{\tilde{\Pi}} ds = \iint\limits_{G} \left|[r'_\varphi, r'_\rho]\right| d\varphi\, d\rho$$

Пересечение поверх-тей:

$ay = y^2 + z^2$

При $z=0$: $y(y-a)=0 \Rightarrow$

$\Rightarrow$ точка A – $\rho = a$, т.е. $\rho \in [0,a]$. Отсечённая пов-ть задаётся

$$\begin{cases} x^2 = y^2 + z^2 \\ ay \geq x^2 \end{cases} \Rightarrow ay \geq y^2 + z^2, \text{ т.е. } a\rho\cos\varphi \geq \rho^2 \Rightarrow \cos\varphi \geq \frac{\rho}{a}$$

Т.е. $G = \begin{cases} 0 \leq \rho \leq a \\ \cos\varphi \geq \frac{\rho}{a} \end{cases}$

$\vec{r}(\rho,\varphi) = \|\rho \;\; \rho\cos\varphi \;\; \rho\sin\varphi\|^T$

$\vec{r}'_\varphi = \|0 \;\; -\rho\sin\varphi \;\; \rho\cos\varphi\|^T \;\rightarrow\; \left|\det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -\rho\sin\varphi & \rho\cos\varphi \\ 1 & \cos\varphi & \sin\varphi \end{pmatrix}\right| = \rho\sqrt{2}$

$\vec{r}'_\rho = \|1 \;\; \cos\varphi \;\; \sin\varphi\|^T$

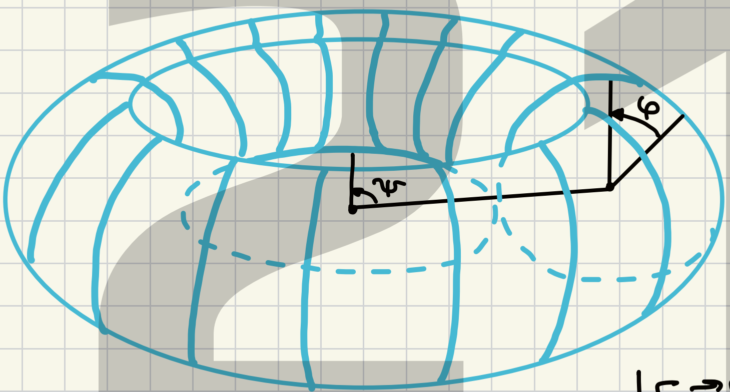
$$\Pi = 4\iint_{G_1} \rho\sqrt{2}\, d\rho\, d\varphi = 4\sqrt{2}\int_0^a \rho\, d\rho \int_0^{\arccos \rho/a} d\varphi = 4\sqrt{2}\int_0^a \rho \arccos\frac{\rho}{a}\, d\rho =$$

$$= 4\sqrt{2}a^2 \int_0^1 x\arccos x\, dx = a^2 \cdot 2\sqrt{2} \cdot \left(x^2 \arccos x \Big|_0^1 + \int_0^1 \frac{x^2}{\sqrt{1-x^2}} dx \right) =$$

$$= a^2 \cdot 2\sqrt{2} \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cos t\, dt = a^2 \cdot 2\sqrt{2} \cdot \frac{\pi}{4} = a^2 \frac{\pi\sqrt{2}}{2}$$

9.51

$x = (b + a\cos\psi)\cos\varphi$, $y = (b + a\cos\psi)\sin\varphi$, $z = a\sin\psi$, $0 < a \le b$



$$\vec{r}'_\varphi = \begin{pmatrix} -(b+a\cos\psi)\sin\varphi \\ (b+a\cos\psi)\cos\varphi \\ 0 \end{pmatrix}$$

$$\vec{r}'_\psi = \begin{pmatrix} -a\sin\psi\cos\varphi \\ -a\sin\psi\sin\varphi \\ a\cos\psi \end{pmatrix}$$

$$\beta = (\widehat{\vec{r}'_\varphi, \vec{r}'_\psi})$$

$$\left|[\vec{r}'_\varphi \times \vec{r}'_\psi]\right| = |\vec{r}'_\varphi||\vec{r}'_\psi||\sin\beta| =$$

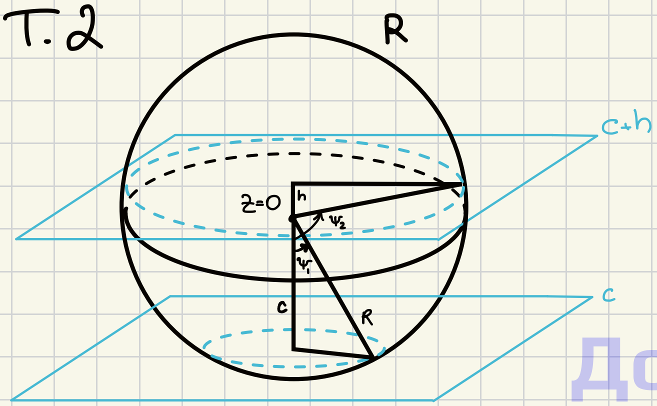
$$= |\vec{r}'_\varphi||\vec{r}'_\psi|\sqrt{1 - \frac{(\vec{r}'_\varphi, \vec{r}'_\psi)^2}{|\vec{r}'_\varphi||\vec{r}'_\psi|}} = \sqrt{|\vec{r}'_\varphi|^2 |\vec{r}'_\psi|^2 - (\vec{r}'_\varphi, \vec{r}'_\psi)^2} =$$

$$= \sqrt{(b + a\cos\psi)^2 a^2 - 0} = (b + a\cos\psi)a$$

$$\Pi = \iint_G (b + a\cos\psi) a\, d\varphi\, d\psi, \quad G = [2\pi, 2\pi]^2$$

$$\Pi = a\int_0^{2\pi} d\varphi \int_0^{2\pi} (b + \cos\psi)\, d\psi = 2\pi a\left[2\pi b + 0\right] = 4\pi^2 ab$$

T.2



$x = R\sin\psi\cos\varphi$ $\quad \varphi \in [0, 2\pi]$

$y = R\sin\psi\sin\varphi$ $\quad \psi \in [0, \pi]$

$z = R\cos\psi$

$$\left|[\vec{r}'_\varphi, \vec{r}'_\psi]\right| = \sqrt{|\vec{r}'_\varphi|^2 |\vec{r}'_\psi|^2 - (\vec{r}'_\varphi, \vec{r}'_\psi)^2} =$$

$$= \sqrt{R^4 \sin^2\psi - 0} = R^2|\sin\psi| = R^2\sin\psi$$

$$\Pi = \iint_G R^2 \sin\psi\, d\varphi\, d\psi$$

G: $\varphi \in [0, 2\pi]$, $R\cos\psi$ между c и $c+h$ $\to$ ψ между $\arccos\left(\frac{c+h}{R}\right)$ и $\arccos\left(\frac{c}{R}\right)$

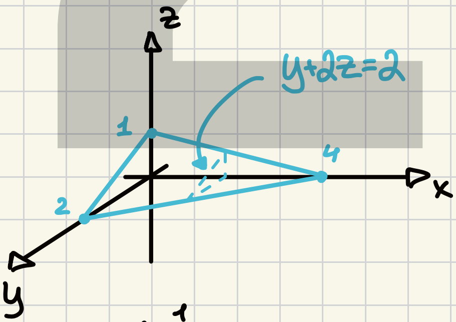
$$\Pi = R^2 \int_0^{2\pi} d\varphi \cdot \operatorname{sgn}(h) \int_{\arccos\left(\frac{c}{R}\right)}^{\arccos\left(\frac{c+h}{R}\right)} \sin\psi \, d\psi = 2\pi R^2 \operatorname{sgn}(h) \int_{\arccos\left(\frac{c+h}{R}\right)}^{\arccos\left(\frac{c}{R}\right)} (-d\cos\psi) =$$

$= 2\pi R^2 \operatorname{sgn}(h) \frac{h}{R} = 2\pi R|h| = f(R,h)$ — зависит только от R, h (в силу того, что дано, что обе пл-ти пересекают сферу)

11.1(1)

$$\iint_S (x+y+z)\,dS, \quad S = \{(x,y,z) \mid x+2y+4z=4,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$$

$x+2y+4z=4 \Rightarrow x+y+z = 4-y-3z \qquad \vec{r} = \|4-2y-4z \;\; y \;\; z\|^T$



$\vec{r}'_y = \|-2 \;\; 1 \;\; 0\|^T$

$\vec{r}'_z = \|-4 \;\; 0 \;\; 1\|^T$

$|[\vec{r}'_y, \vec{r}'_z]| = \sqrt{5 \cdot 17 - 64} = \sqrt{21}$

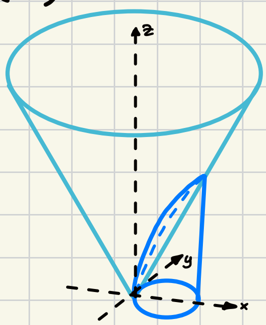
$\iint_G (4-y-3z)\sqrt{21}\,dy\,dz$, $G = \{y \in [0,2], z \in [0, 1-\frac{y}{2}]\}$

$$\sqrt{21}\int_0^2 dy \int_0^{1-\frac{1}{2}y} (4-y-3z)\,dz = \sqrt{21}\int_0^2 dy\left((4-y)\left(1-\tfrac{1}{2}y\right) - \tfrac{3}{2}\left(1-\tfrac{1}{2}y\right)^2\right) =$$

$$= \sqrt{21}\left[\int_0^2 \left(4-\tfrac{3}{2}\right)dy + \int_0^2 \left(-2y-y+\tfrac{3}{2}y\right)dy + \int_0^2 \left(\tfrac{1}{2}y^2 - \tfrac{3}{8}y^2\right)dy\right] =$$

$$= \sqrt{21}\left(5-3+\tfrac{1}{3}\right) = \frac{7\sqrt{21}}{3}$$

11.18(3)



$x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = \rho$

$x^2 + y^2 \leq x \Rightarrow \rho \leq \cos\varphi \Rightarrow \rho \in [0, \cos\varphi]$

$\varphi \in [-\pi/2, \pi/2]$

$|[\vec{r}'_\varphi, \vec{r}'_\rho]| = \sqrt{\rho^2(1+\sin^2\varphi+\cos^2\varphi)^2 - 0} = \rho\sqrt{2}$

$$m=\iint\limits_G d\rho\, d\varphi = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{\cos\varphi} \rho\sqrt{2}\, d\rho = \frac{1}{\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^2\varphi\, d\varphi = \frac{1}{2\sqrt2}\left[\int_{-\pi/2}^{\pi/2} d\varphi + \frac12\int_{-\pi/2}^{\pi/2}\cos 2\varphi\, d(2\varphi)\right] = \frac{\pi\sqrt2}{4}$$

$$M_x=\iint\limits_G x\, d\rho\, d\varphi = \int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi\int_0^{\cos\varphi}\rho^2\sqrt2\, d\rho = \frac{\sqrt2}{3}\int_{-\pi/2}^{\pi/2}\cos^4\varphi\, d\varphi = \frac{\sqrt2}{3}\int_{-\pi/2}^{\pi/2}\frac14(1+\cos 2\varphi)^2 d\varphi = \frac{\sqrt2}{12}\int_{-\pi/2}^{\pi/2}(1+2\cos2\varphi+\cos^2 2\varphi)\, d\varphi = \frac{\pi\sqrt2}{12}+\frac{\sqrt2}{12}\cdot\frac{\pi}{2} = \frac{\pi\sqrt2}{8}$$

$M_y=0$ в силу симметрии

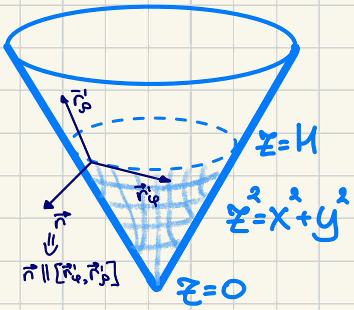
$$M_z=\iint\limits_G z\, d\rho\, d\varphi = \int_{-\pi/2}^{\pi/2} d\varphi\int_0^{\cos\varphi}\rho^2\sqrt2\, d\rho = \frac{\sqrt2}{3}\int_{-\pi/2}^{\pi/2}\cos^3\varphi\, d\varphi = \frac{\sqrt2}{3}\int_{-\pi/2}^{\pi/2}(1-\sin^2\varphi)\cdot d(\sin\varphi) = \frac{\sqrt2}{3}\int_{-1}^{1}(1-t^2)\, dt = \frac{\sqrt2}{3}\left(2-\frac23\right) = \frac{4\sqrt2}{9}$$

$$x_c=\frac{M_x}{m}=\frac12,\quad y_c=0,\quad z_c=\frac{M_z}{m}=\frac{16}{9\pi}$$

11.39

$$\iint\limits_S (y-z)\,dy\,dz+(z-x)\,dz\,dx+(x-y)\,dx\,dy =$$

$$x=\rho\cos\varphi,\quad y=\rho\sin\varphi,\quad z=\rho$$

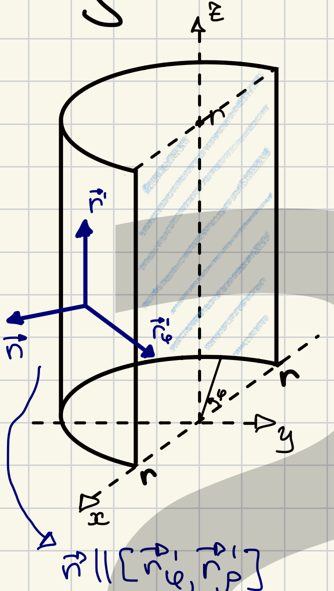


$$= \iint\limits_G \begin{vmatrix} y-z & z-x & x-y \\ x'_\varphi & y'_\varphi & z'_\varphi \\ x'_\rho & y'_\rho & z'_\rho \end{vmatrix} d\varphi\, d\rho =$$

$$= \iint\limits_G \begin{vmatrix} y-z & z-x & x-y \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \\ \cos\varphi & \sin\varphi & 1 \end{vmatrix} d\varphi\, d\rho = \iint\limits_G \Big[\rho(\sin\varphi-1)\rho\cos\varphi + \rho(1-\cos\varphi)\rho\sin\varphi - \rho(\cos\varphi-\sin\varphi)\rho\Big]\, d\varphi\, d\rho = \int_0^H \rho^2 d\rho\int_0^{2\pi}\Big[(\sin\varphi-1)\cos\varphi + (1-\cos\varphi)\sin\varphi - (\cos\varphi-\sin\varphi)\Big] d\varphi = \frac{2H^3}{3}\int_0^{2\pi}(\sin\varphi-\cos\varphi)\, d\varphi = 0$$

11.40

$$\iint\limits_S yz^2\,dx\,dz,\ S: x=-r\cos\varphi, y=r\sin\varphi, z=h, \varphi\in[0,\pi], h\in[0,r]$$



$$\left|[\vec{r}\,'_\varphi,\vec{r}\,'_h]\right| = \left|\det\begin{pmatrix} 0 & 1 & 0 \\ r\sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1\end{pmatrix}\right| = r|\sin\varphi| =$$

$= r\sin\varphi\ (\varphi\in[0,\pi]$

пов-ть внутр. => "–" перед $\left|[\vec{r}\,'_\varphi,\vec{r}\,'_h]\right|$

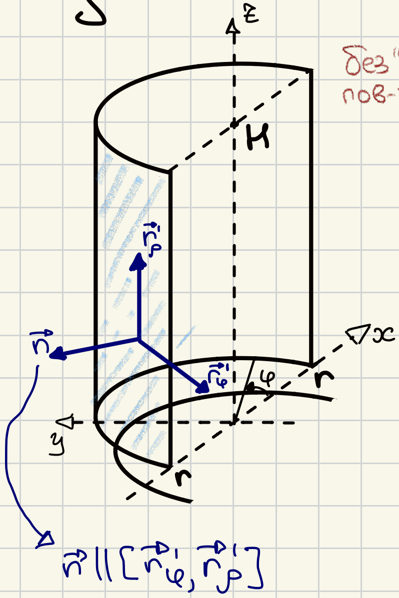
$$\iint\limits_S r\sin\varphi\, h^2(-r\sin\varphi)\,d\varphi\,dh =$$

$$= -r^2\int_0^\pi \sin^2\varphi\,d\varphi\int_0^r h^2\,dh = -\frac{r^5}{3}\int_0^\pi \frac{1-\cos 2\varphi}{2}\,d\varphi = -\frac{\pi r^5}{6}$$

11.41.

$$\iint\limits_S yz\,dx\,dy + zx\,dy\,dz + xy\,dz\,dx =$$

$x = r\cos\varphi,\ y = r\sin\varphi,\ z = h,\ \varphi\in[\frac{\pi}{2},\pi], h\in[0,H]$



без "–", т.к. пов-ть внешн.

$$= \iint\limits_S \begin{vmatrix} yz & zx & xy \\ x'_\varphi & y'_\varphi & z'_\varphi \\ x'_h & y'_h & z'_h \end{vmatrix} d\varphi\,d\rho =$$

$$= \iint\limits_S \begin{vmatrix} yz & zx & xy \\ x'_\varphi & y'_\varphi & z'_\varphi \\ x'_h & y'_h & z'_h \end{vmatrix} d\varphi\,dh = \iint\limits_S \begin{vmatrix} xz & xy & yz \\ -r\sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}\cdot$$

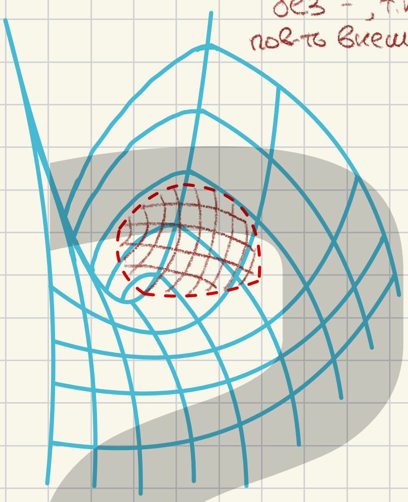
$$d\varphi\,dh = \iint\limits_S (hr^2\cos^2\varphi + r^3\sin^2\varphi\cos\varphi)\,d\varphi\,dh =$$

$$= r^2\int_{\pi/2}^{\pi}\cos^2\varphi\,d\varphi\int_0^H (h + r\sin\varphi)\,dh = r^2\int_{\pi/2}^{\pi}\left(\frac{H^2}{2} + Hr\sin\varphi\right)\cos^2\varphi\,d\varphi =$$

$$= r^2\left[\frac{H^2}{4}\int_{\pi/2}^{\pi}(1+\cos 2\varphi)\,d\varphi + Hr\int_{\pi/2}^{\pi}\cos^2\varphi\,d(\cos\varphi)\right] = r^2\left[\frac{\pi H^2}{8} - \frac{Hr}{3}\right]$$

11.43

$$\iint\limits_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy, \quad S = \{ z = x^2 - y^2,\ |y| \le x \le a \}$$



$$\vec{r} = \|x\ y\ z\|^T = \|x\ y\ x^2-y^2\|^T$$

без "−", т.к. пов-ть внешняя

$$\iint\limits_S \begin{vmatrix} x & y & x^2-y^2 \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} dx\,dy = \iint\limits_S (2y^2 - 2x^2 + x^2 - y^2)\,dx\,dy = \iint\limits_S (y^2 - x^2)\,dx\,dy \ =$$

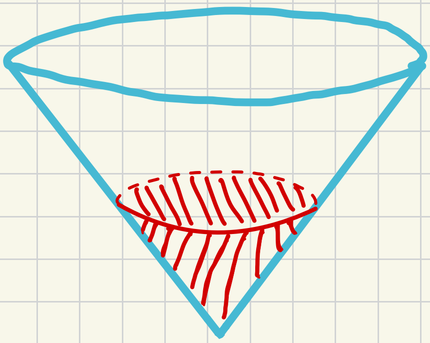
$y \in [-x, x] \quad x \in [0, a]$

$$= \int_0^a dx \int_{-x}^{x} (y^2 - x^2)\,dy = \int_0^a dx \left(\frac{2x^3}{3} - 2x^3 \right) = -\frac{4}{3} \int_0^a x^3\,dx = -\frac{a^4}{3}$$

11.45 (1,3)

$$\iint\limits_S z\,dx\,dy + (5x+y)\,dy\,dz$$

1) S - внеш. сторона конуса $x^2+y^2 \le z^2,\ 0 \le z \le 4$



$$\iint\limits_S z\,dx\,dy + (5x+y)\,dy\,dz = \iiint\limits_G \operatorname{div} \begin{pmatrix} 5x+y \\ 0 \\ z \end{pmatrix} dx\,dy\,dz$$

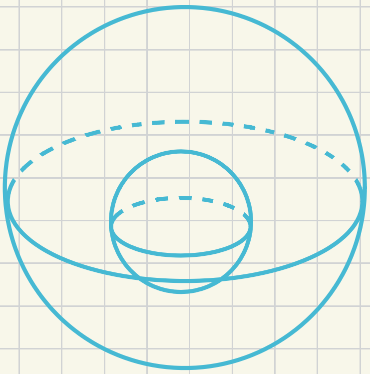
$$= 6 \iiint\limits_G dx\,dy\,dz = 6\mu(G) =$$

G - кусок конуса $x^2+y^2 \le z^2,\ 0 \le z \le 4$

$$\mu(G) = \frac{1}{3} hS = \frac{1}{3} \cdot 4 \cdot \pi \cdot 4^2 = \frac{64}{3}\pi$$

$= 128\pi$

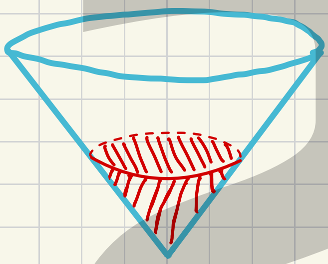
3) S - внеш. сторона границы области $1 < x^2+y^2+z^2 < 4$



$$\iint\limits_S z\,dx\,dy+(5x+y)\,dy\,dz=6\iiint\limits_G dx\,dy\,dz=$$

$$=6\mu(G)=6\left(\frac{4}{3}\pi\cdot 2^3-\frac{4}{3}\pi\cdot 1^3\right)=56\pi$$

11.46(2)



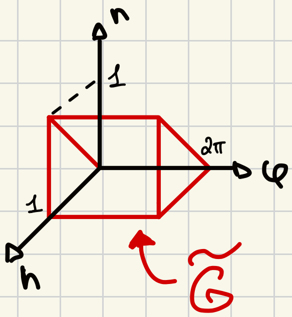
$\iint\limits_S x^2dy\,dz+y^2dz\,dx+z^2dx\,dy$, S – внешняя сторона $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le\frac{z^2}{c^2},\ 0\le z\le c$

$$\iiint\limits_G \operatorname{div}\begin{pmatrix}x^2\\ y^2\\ z^2\end{pmatrix}dx\,dy\,dz=2\iiint\limits_G (x+y+z)\,dx\,dy\,dz \ominus$$

Переход к $x=ar\cos\varphi,\ y=br\sin\varphi,\ z=ch$ $\ J=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial y}{\partial r}&\frac{\partial z}{\partial r}\\ \frac{\partial x}{\partial \varphi}&\frac{\partial y}{\partial \varphi}&\frac{\partial z}{\partial \varphi}\\ \frac{\partial x}{\partial h}&\frac{\partial y}{\partial h}&\frac{\partial z}{\partial h}\end{vmatrix}=$

$=abc[\cos\varphi\cdot r\cos\varphi-(-r\sin\varphi)\sin\varphi]=abcr$

$$\ominus\ 2abc\iiint\limits_{\tilde G}(abr(\cos\varphi+\sin\varphi)+ch)\,r\,dr\,d\varphi\,dh=2abc\int_0^{2\pi}d\varphi\,\cdot$$

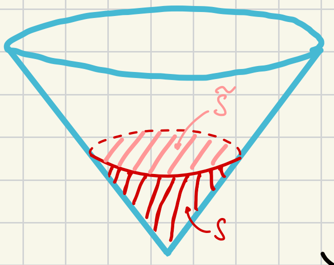


$$\cdot\int_0^1 dh\int_0^h(abr(\cos\varphi+\sin\varphi)+ch)\,r\,dr=2abc\int_0^{2\pi}d\varphi\int_0^1 dh\left[\frac{h^3}{3}(\cos\varphi+\right.$$

$$\left.+\sin\varphi)ab+c\frac{h^3}{2}\right]=2abc\int_0^{2\pi}d\varphi\left[\frac{1}{12}(\cos\varphi+\sin\varphi)ab+\frac{c}{8}\right]=$$

$$=2abc^2\cdot\frac{\pi}{4}=\frac{\pi abc^2}{2}$$

11.52(3)



$\iint\limits_S x^2dy\,dz+y^2dz\,dx+z^2dx\,dy$, S – нижняя сторона конической пов-ти $x^2+y^2=z^2$, $0\le z\le H$

$$\iint\limits_{S\cup\tilde S} x^2dy\,dz+y^2dz\,dx+z^2dx\,dy=$$

$$\iint\limits_{S} x^2dydz+y^2dzdx+z^2dxdy+\iint\limits_{\tilde{S}} x^2dydz+y^2dzdx+z^2dxdy$$

$$=\iiint\limits_{G} \operatorname{div}\begin{pmatrix}x^2\\y^2\\z^2\end{pmatrix}dxdydz=2\int_0^{2\pi}d\varphi\int_0^H dh\left(\tfrac{1}{3}h^3(\cos\varphi+\sin\varphi)+\tfrac{h^3}{2}\right)=$$

$$=2H^4\int_0^{2\pi}d\varphi\left[\tfrac{1}{12}(\cos\varphi+\sin\varphi)+\tfrac{1}{8}\right]=\frac{H^4}{2}$$ (см. выкладки 11.46(2))

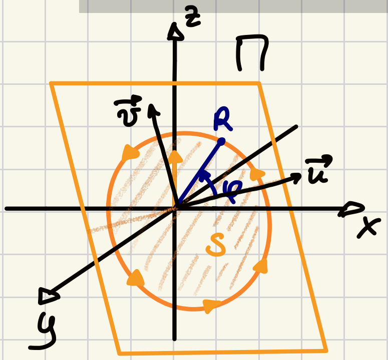
$$\iint\limits_{\tilde{S}} x^2dydz+y^2dzdx+z^2dxdy=\iint\limits_{\tilde{S}}H^2dxdy=H^2\mu(\tilde{S})=\pi H^4$$

$z=const=H$

$$\iint\limits_{S} x^2dydz+y^2dzdx+z^2dxdy=\frac{\pi H^4}{2}-\pi H^4=-\frac{\pi H^4}{2}$$

11.63(2)

$$\int\limits_L \frac{xdy-ydx}{x^2+y^2}+zdz,\quad L\text{ - окр-ть }\begin{cases}x^2+y^2+z^2=R^2\\x+y+z=0\end{cases}$$ - ориентированная полож. отн. $(0,1,0)$



S - контур, замкнутый L. $\{0,0,0\}\in S$, а поле не опр. в $\{0,0,0\}$ => Т. Стокса не применима.

$$\int\limits_L zdz=\int\limits_L \frac{1}{2}d(z^2)=0$$

Пусть $\vec{r}(\varphi)$ - радиус-вектор точки на L в зависимости от угла φ между $\vec{r}$ и $\vec{u}$. $(\vec{u},\vec{v})$ - базис в плоскости П.

$$\vec{r}(\varphi)=R(\vec{u}\cos\varphi+\vec{v}\sin\varphi)$$

Спроецируем $\vec{r}(\varphi)$ на плоскость xy

$$pr(\vec{r}(\varphi))=R(pr(\vec{u})\cos\varphi+pr(\vec{v})\sin\varphi)$$

$$\begin{pmatrix} x(\varphi) \\ y(\varphi) \end{pmatrix} = R \begin{pmatrix} \tilde{u}_x & \tilde{v}_x \\ \tilde{u}_y & \tilde{v}_y \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = R\mathcal{A} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\frac{d}{d\varphi} \begin{pmatrix} x(\varphi) \\ y(\varphi) \end{pmatrix} = R\mathcal{A} \frac{d}{d\varphi} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = R\mathcal{A} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = R\mathcal{A} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} d\varphi$$

$$x\,dy - y\,dx = \det \begin{pmatrix} x & dx \\ y & dy \end{pmatrix} = R^2 \det\mathcal{A} \det \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} d\varphi =$$

$$= R^2 \det\mathcal{A}\, d\varphi$$

$$x^2 + y^2 = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = R(\cos\varphi\ \sin\varphi)\, \mathcal{A}^T R \mathcal{A} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} =$$

$$= R^2 (\cos\varphi\ \sin\varphi)\, \mathcal{A}^T \mathcal{A} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \circeq$$

$\mathcal{A}^T\mathcal{A}$ – симметричка => можно ортогонально диагонализировать, т.е.

$$\mathcal{A}^T\mathcal{A} = Q^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Q, \text{ где } Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\circeq R^2 \begin{pmatrix} \cos\varphi\cos\theta - \sin\varphi\sin\theta \\ \cos\varphi\sin\theta + \sin\varphi\cos\theta \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\theta - \sin\varphi\sin\theta \\ \cos\varphi\sin\theta + \cos\theta\sin\varphi \end{pmatrix} =$$

$$= R^2 \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \end{pmatrix} = R^2 \begin{pmatrix} \lambda_1\cos(\varphi+\theta) \\ \lambda_2\sin(\varphi+\theta) \end{pmatrix}^T \cdot$$

$$\cdot \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \end{pmatrix} = R^2 \left(\lambda_1\cos^2(\varphi+\theta) + \lambda_2\sin^2(\varphi+\theta)\right)$$

$$\oint_L \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{R^2\det\mathcal{A}}{R^2(\lambda_1\cos^2(\varphi+\theta) + \lambda_2\sin^2(\varphi+\theta))} d\varphi = \left| t = \operatorname{tg}(\varphi+\theta) \right|$$

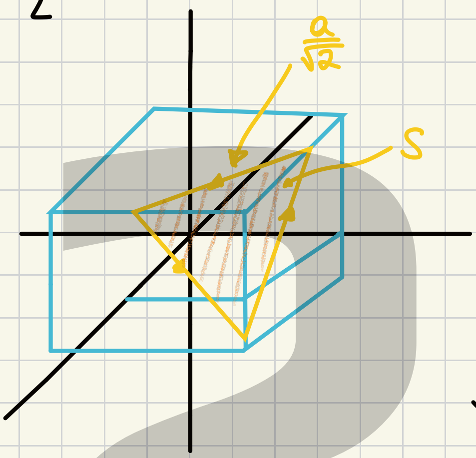
$$= 2\det\mathcal{A} \int_{-\infty}^{+\infty} \frac{dt}{\lambda_1 + \lambda_2 t^2} = \left| \omega = \sqrt{\frac{\lambda_2}{\lambda_1}}\, t \right| = 2\det\mathcal{A} \frac{1}{\sqrt{\lambda_1\lambda_2}} \int_{-\infty}^{+\infty} \frac{d\omega}{1+\omega^2} = \det\mathcal{A} \frac{2\pi}{\sqrt{\lambda_1\lambda_2}} \circeq$$

$$\sqrt{\lambda_1\lambda_2} = \sqrt{\det\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} = |\det\mathcal{A}| \Rightarrow \circeq \operatorname{sgn}(\det\mathcal{A}) \cdot 2\pi$$

В силу ориентации контура ответ $> 0 \Rightarrow \det\mathcal{A} > 0$, тогда искомый ответ: 2π

11.64

$$\int\limits_L (y^2-z^2)dx + (z^2-x^2)dy + (x^2-y^2)dz$$

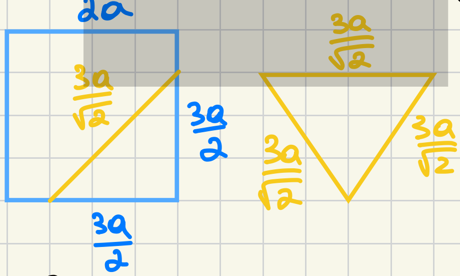


$$\operatorname{rot}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2-z^2 & z^2-x^2 & x^2-y^2 \end{vmatrix} = \begin{pmatrix} -2y-2z \\ -2x-2z \\ -2x-2y \end{pmatrix}$$

$$\vec{n}_S = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\iint\limits_S (\operatorname{rot}\vec{a}, \vec{n})\,ds = \frac{-4}{\sqrt{3}}\iint\limits_S (x+y+z)\,ds =$$

$$= \frac{-4}{\sqrt{3}}\iint\limits_S \frac{3a}{2}ds = -2a\sqrt{3}\iint\limits_S ds = -2a\sqrt{3}\,S_\Delta =$$

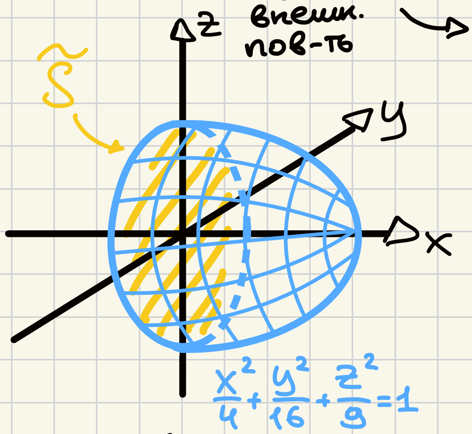


$$S_\Delta = \frac{1}{2}\cdot\frac{9a}{2}\cdot\frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{8}a$$

$$= -2a\sqrt{3}\,\frac{9\sqrt{3}}{8}a = -\frac{27}{4}a^2$$

Т3.

$$\iint\limits_S (4x+3y)dydz + (x+3y+z)dzdx + (6x+2y+3z)dxdy$$



без "-", т.к. внешн. пов-ть

$$\iiint\limits_G \operatorname{div}\begin{pmatrix}4x+3y\\x+3y+z\\6x+2y+3z\end{pmatrix}dxdydz = 10\iiint\limits_G dxdydz =$$

$$= 10\cdot\frac{1}{2}\mu(G) = 5\cdot\frac{4}{3}\pi\cdot 2\cdot 4\cdot 3 = 160\pi$$

искомый: $160\pi - \iint\limits_{\tilde{S}} (4x+3y)dydz +$

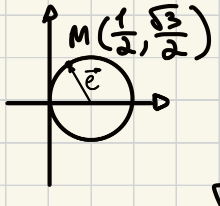
$+ (x+3y+z)dzdx + (6x+2y+3z)dxdy =$ x=const=0

$$= 160\pi - \iint\limits_{\tilde{S}} 3y\,dydz = 160\pi - \int\limits_0^{2\pi} d\varphi \int\limits_0^3 3r^2\cos\varphi\,dr =$$

$$= 160\pi - \int\limits_0^{2\pi} 27\cos\varphi\,d\varphi = 160\pi$$

3.44(2)

$$f(x,y)=\operatorname{arctg}\frac{y}{x},\quad \vec{e}=\begin{pmatrix}\frac{1}{2}-1\\ \frac{\sqrt{3}}{2}-0\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$$



$$\frac{\partial f}{\partial \vec{e}}=(\vec{e},\nabla)f=-\frac{1}{2}f'_x+\frac{\sqrt{3}}{2}f'_y=-\frac{1}{2}\frac{y}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)+$$

$$+\frac{\sqrt{3}}{2}\frac{1/x}{1+\left(\frac{y}{x}\right)^2}=\Big|_{M\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)}\Big|=-\frac{1}{2}\frac{\sqrt{3}/2}{4}\cdot(-4)+\frac{\sqrt{3}}{2}\frac{2}{4}=\frac{\sqrt{3}}{2}$$

3.48(3)

$$f(x,y,z)=\ln xyz,\quad M(1,-2,-3)$$

$$\frac{\partial f}{\partial \vec{e}}=(\vec{e}\nabla)f=\frac{l_x}{x}+\frac{l_y}{y}+\frac{l_z}{z}=\Big|_{M(1,-2,-3)}\Big|=l_x-\frac{l_y}{2}-\frac{l_z}{3}=g(l_x,l_y,l_z)$$

Ищем локальные экстремумы при условии $l_x^2+l_y^2+l_z^2-1=0$

$$L(l_x,l_y,l_z)=g(l_x,l_y,l_z)-\lambda(l_x^2+l_y^2+l_z^2-1)$$

$$\begin{cases}L'_{l_x}=1-2\lambda l_x=0\\ L'_{l_y}=-\frac{1}{2}-2\lambda l_y=0\\ L'_{l_z}=-\frac{1}{3}-2\lambda l_z=0\\ l_x^2+l_y^2+l_z^2=1\end{cases}\qquad \begin{cases}l_x=\frac{1}{2\lambda}\\ l_y=-\frac{1}{4\lambda}\\ l_z=-\frac{1}{6\lambda}\\ \frac{1}{\lambda^2}\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{36}\right)=1\ \to\ \lambda=\pm\frac{7}{12}\end{cases}$$

max при $\lambda=\frac{1}{12}\ \to\ l_x=\frac{6}{7},\ l_y=-\frac{3}{7},\ l_z=-\frac{2}{7}$

$$\frac{\partial f}{\partial \vec{e}}_{max}\Big|_M=\frac{6}{7}+\frac{3}{14}+\frac{2}{21}=\frac{49}{42}=\frac{7}{6}$$

12.14

Д-ть: $\operatorname{grad} f(u;v)=\frac{\partial f}{\partial t}(u;v)\operatorname{grad}u+\frac{\partial f}{\partial s}(u;v)\operatorname{grad}v$

в точке $(t,s)=(u(x,y,z),v(x,y,z))$

$$\operatorname{grad}f(u(x,y,z),v(x,y,z))=\begin{pmatrix}\partial f/\partial x\\ \partial f/\partial y\\ \partial f/\partial z\end{pmatrix}=\begin{pmatrix}\partial f/\partial t\,\partial u/\partial x+\partial f/\partial s\,\partial v/\partial x\\ \partial f/\partial t\,\partial u/\partial y+\partial f/\partial s\,\partial v/\partial y\\ \partial f/\partial t\,\partial u/\partial z+\partial f/\partial s\,\partial v/\partial z\end{pmatrix}=$$

$= \frac{\partial f}{\partial t}(u;v)\,\mathrm{grad}\,u + \frac{\partial f}{\partial s}(u;v)\,\mathrm{grad}\,v$

12.19

Д-ть: $\nabla f(r) = f'(r)\frac{\vec{r}}{r}$, $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} \partial f/\partial r \cdot \partial r/\partial x \\ \partial f/\partial r \cdot \partial r/\partial y \\ \partial f/\partial r \cdot \partial r/\partial z \end{pmatrix} = f'(r)\,\mathrm{grad}(r) = f'(r)\,\mathrm{grad}\left(\sqrt{x^2+y^2+z^2}\right) =$$

$$= f'(r)\begin{pmatrix} x/r \\ y/r \\ z/r \end{pmatrix} = f'(r)\frac{\vec{r}}{r}$$

12.15

$\vec{a} = \mathrm{const}$, $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2) $\mathrm{grad}(r^2) = (r^2)'_r \frac{\vec{r}}{r} = 2\vec{r}$

3) $\mathrm{grad}(1/r) = (1/r)'_r \frac{\vec{r}}{r} = -\frac{\vec{r}}{r^3}$

4) $\mathrm{grad}(\ln r) = (\ln r)'_r \frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$

12.37(2)

$$\mathrm{div}(u\vec{a}) = (\nabla, \overset{\downarrow}{u}\vec{a}) + (\nabla, u\overset{\downarrow}{\vec{a}}) = (\mathrm{grad}\,u, \vec{a}) + u\,\mathrm{div}\,\vec{a}$$

12.39

$$\mathrm{div}\,\mathrm{grad}\,u = (\nabla, \nabla u) = (\nabla, \nabla)u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

12.40(1)

$$\mathrm{div}(u\,\mathrm{grad}\,u) = (\nabla, u\nabla u) = (\nabla, \overset{\downarrow}{u}(\nabla u)) + (\nabla, u(\overset{\downarrow}{\nabla u})) =$$

$$= (\nabla u, \nabla u) + u\,\mathrm{div}(\nabla u) = |\nabla u|^2 + u\Delta u$$

12.41

6) $\mathrm{div}(f(r)\overset{\text{const}}{\vec{c}}) \overset{12.37(2)}{=} (\mathrm{grad}\,f(r), \vec{c}) + f(r)\,\overset{0}{\mathrm{div}\,\vec{c}} \overset{12.19}{=} \left(f'(r)\frac{\vec{r}}{r}, \vec{c}\right) = \frac{f'(r)}{r}(\vec{r}, \vec{c})$

7) $\mathrm{div}[\vec{c}, \vec{r}] = (\nabla, [\vec{c}, \vec{r}]) = (\nabla, \overset{\downarrow}{\vec{c}}, \vec{r}) + (\nabla, \vec{c}, \overset{\downarrow}{\vec{r}}) = (\vec{r}, \nabla, \overset{\downarrow}{\vec{c}}) - (\vec{c}, \nabla, \overset{\downarrow}{\vec{r}}) =$

$= (\vec{r}, [\nabla, \vec{c}]) - (\vec{c}, [\nabla, \vec{r}]) = (\vec{r}, \mathrm{rot}\,\vec{c}) - (\vec{c}, \mathrm{rot}\,\vec{r}) = 0 - 0 = 0$

8) $\operatorname{div}[\vec r,[\vec c,\vec r]]=(\nabla,[\vec r,[\vec c,\vec r]])=(\nabla,\overset{\downarrow}{\vec r},[\vec c,\vec r])+(\nabla,\vec r,[\overset{\downarrow}{\vec c},\vec r])=$

$=([\nabla,\vec r],[\vec c,\vec r])-(\vec r,[\vec\nabla,[\vec c,\vec r]])=(\overset{\vec 0}{\operatorname{rot}\vec r},[\vec c,\vec r])-(\vec r,[\vec\nabla,[\overset{\downarrow}{\vec c},\vec r]]+$

$+[\vec\nabla,[\vec c,\overset{\downarrow}{\vec r}]])\ \circleddash$

$[\vec\nabla,[\overset{\downarrow}{\vec c},\vec r]]=\vec c\,\cancel{(\nabla,\vec r)}\ (\vec r,\nabla)\vec c-\vec r(\nabla,\vec c)=0-0$

$[\vec\nabla,[\vec c,\overset{\downarrow}{\vec r}]]=\vec c(\nabla,\vec r)-\vec r\cancel{(\nabla,\vec c)}\ (\vec c,\nabla)\vec r=3\vec c-\vec c=2\vec c$

$\circleddash\ -(\vec r,2\vec c)=-2(\vec r,\vec c)$

12.49

3) $\operatorname{rot}(u\vec a)=[\nabla,u\vec a]=[\nabla,u\overset{\downarrow}{\vec a}]+[\nabla,\overset{\downarrow}{u}\vec a]=[\nabla,\vec a]u+$

$+[\nabla u,\vec a]=u\operatorname{rot}\vec a+[\operatorname{grad}u,\vec a]$

5) $\operatorname{rot}[\vec a,\vec b]=[\nabla,[\vec a,\vec b]]=[\nabla,[\overset{\downarrow}{\vec a},\vec b]]+[\nabla,[\vec a,\overset{\downarrow}{\vec b}]]=$

12.41

$\overset{}{=}(\vec b,\nabla)\vec a-\vec b(\nabla,\vec a)+\vec a(\nabla,\vec b)-(\vec a,\nabla)\vec b=\vec a\operatorname{div}\vec b-\vec b\operatorname{div}\vec a+$

$+(\vec b,\nabla)\vec a-(\vec a,\nabla)\vec b$

6) $\operatorname{div}[\vec a,\vec b]=(\nabla,[\vec a,\vec b])=(\nabla,\overset{\downarrow}{\vec a},\vec b)+(\nabla,\vec a,\overset{\downarrow}{\vec b})=(\vec b,[\nabla,\vec a])-$

$-(\vec a,[\nabla,\vec b])=(\vec b,\operatorname{rot}\vec a)-(\vec a,\operatorname{rot}\vec b)$

12.54(a)

$\operatorname{rot}[\vec r,[\vec c,\vec r]]=\operatorname{rot}(\vec c r^2-\vec r(\vec r,\vec c))=r^2\overset{\vec 0}{\operatorname{rot}\vec c}-\operatorname{rot}(\vec r(\vec r,\vec c))=$

$=-[\nabla,\vec r(\vec r,\vec c)]=-[\nabla,\overset{\downarrow}{\vec r}(\vec r,\vec c)]-[\nabla,\vec r(\overset{\downarrow}{\vec r},\vec c)]=-(\vec r,\vec c)\overset{\vec 0}{\operatorname{rot}\vec r}-$

$-[\nabla(\vec r,\vec c),\vec r]=[\vec r,\nabla(\overset{\downarrow}{\vec r},\vec c)+\underset{\vec 0}{\nabla(\vec r,\overset{\downarrow}{\vec c})}]\ \circleddash$

$[\vec c,[\nabla,\vec r]]=\nabla(\overset{\downarrow}{\vec r},\vec c)-(\vec c,\nabla)\vec r\Rightarrow\nabla(\overset{\downarrow}{\vec r},\vec c)=[\vec c,[\nabla,\vec r]]+(\vec c,\nabla)\vec r$

$\circleddash\ [\vec r,[\vec c,[\nabla,\vec r]]+(\vec c,\nabla)\vec r]=[\vec r,\vec 0+\vec c\operatorname{div}\vec r]=3[\vec r,\vec c]$

12.62.

$$\operatorname{rot}(u(r)\vec{a}(r)) = [\nabla, u\vec{a}] = [\nabla, \overset{\downarrow}{u}\vec{a}] + [\nabla, u\overset{\downarrow}{\vec{a}}] =$$

$$= [\nabla u, \vec{a}] + u[\nabla, \vec{a}] \overset{12.19}{=} \left[u'(r)\frac{\vec{r}}{r}, \vec{a}\right] + u \operatorname{rot}\vec{a} \;=$$

$$\operatorname{rot}\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \quad \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \quad \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)^T = \left(\frac{\partial a_z}{\partial r}\frac{\partial r}{\partial y} - \frac{\partial a_y}{\partial r}\frac{\partial r}{\partial z}\right.$$

$$\left.\frac{\partial a_x}{\partial r}\frac{\partial r}{\partial z} - \frac{\partial a_z}{\partial r}\frac{\partial r}{\partial x} \quad \frac{\partial a_y}{\partial r}\frac{\partial r}{\partial x} - \frac{\partial a_x}{\partial r}\frac{\partial r}{\partial y}\right)^T = \frac{1}{r}[\vec{r}, \vec{a}']$$

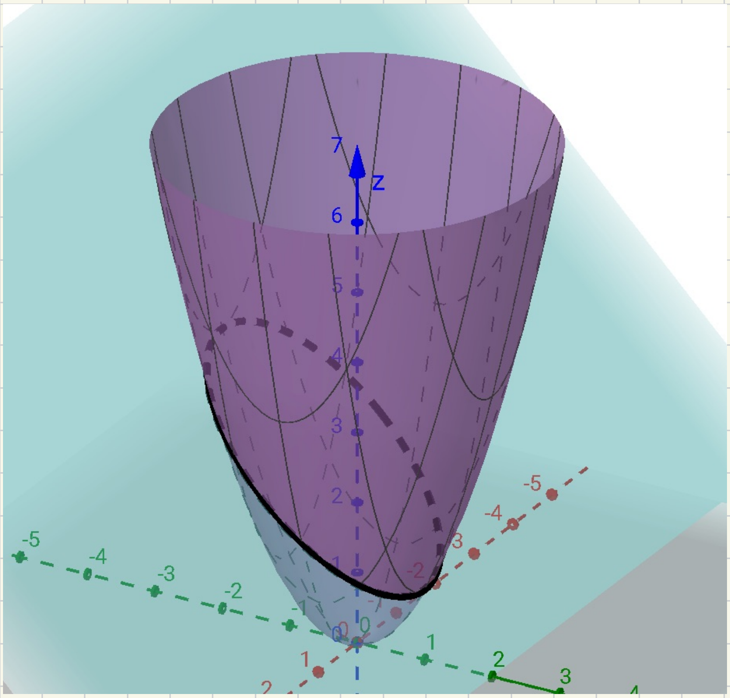
$$= \frac{1}{r}\left(u'(r)[\vec{r}, \vec{a}] + u(r)[\vec{r}, \vec{a}']\right)$$

12.68(5)

$$\oiint(\vec{a}, \vec{n})dS = \oiint(f(r)\vec{r}, \vec{n})dS = f(R)\oiint(\vec{r}, \vec{n})dS = f(R)\oiint\left(\vec{r}, \frac{\vec{r}}{r}\right)dS =$$

$$= f(R)\cdot\frac{1}{R}\oiint R^2 dS = f(R)R\cdot 4\pi R^2 = 4\pi R^3 f(R)$$

12.94(3)



$$\int(\vec{a}, d\vec{r}) = \iint_S (\operatorname{rot}\vec{a}, \vec{n})dS \;=$$

$$\operatorname{rot}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & y^3 & z^3 \end{vmatrix} = \vec{0}$$

$$= 0$$

12.104

$$\vec{H} = 2I\frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$$

Воспользуемся критерием потенциальности: $\forall \Gamma \oint\limits_{\Gamma} (\vec{H}, d\vec{r}) = 0$, Γ – замкнутый контур.

Возьмём контур и результат из 11.63(2). Из этого становится понятно, что в а) ответ: да, т.к. никакой контур Oz пронизывать не может, а значит Т. Стокса применима => потенц. (т.к. $\operatorname{rot} \vec{H} = \vec{0}$), а в б) ответ: нет, т.к. $\exists$ контур, который Oz будет пронизывать и по результату 11.63(2) получим, что $\oint\limits_{\Gamma} (\vec{H}, d\vec{r}) \neq 0$ => не потенциальное.

12.112(1)

$\vec{a} = \frac{\vec{r}}{r^3}$ потенциальность: $\operatorname{rot} \vec{a} = \operatorname{rot} \frac{\vec{r}}{r^3} = \frac{1}{r^3} \operatorname{rot} \vec{r} +$

$+ \left[\operatorname{grad} \frac{1}{r^3}, \vec{r}\right] = 0 + \left[-\frac{3}{r^4} \frac{\vec{r}}{r}, \vec{r}\right] = 0$ => в $\mathbb{R}^3 \setminus \{0,0,0\}$ потенциально

соленоидальность: при $r>0$ для $G = \{x^2+y^2+z^2=1\}$:

$\oiint\limits_{G} (\vec{a}, \vec{n})\, dS = \oiint\limits_{G} \left(\frac{\vec{r}}{r^3}, \frac{\vec{r}}{r}\right) dS = 4\pi \neq 0$ => не соленоидально

соленоидальность: при $z>0$ применима формула Остроградского-Гаусса: $\operatorname{div} \vec{a} = \operatorname{div} \frac{\vec{r}}{r^3} = \left(\operatorname{grad} \frac{1}{r^3}, \vec{r}\right) + \frac{1}{r^3} \operatorname{div} \vec{r} =$

$= -\frac{3}{r^3} + \frac{3}{r^3} = 0$ => соленоидально

12.115.

$\operatorname{div}(f(\vec{r})\vec{r}) = f(\vec{r}) \operatorname{div} \vec{r} + (\operatorname{grad} f(\vec{r}), \vec{r}) = 3f(\vec{r}) + f'(r) \frac{1}{r} (\vec{r}, \vec{r}) =$

$= 3f(\vec{r}) + f'(r) r = 0 \mid \cdot r^2$

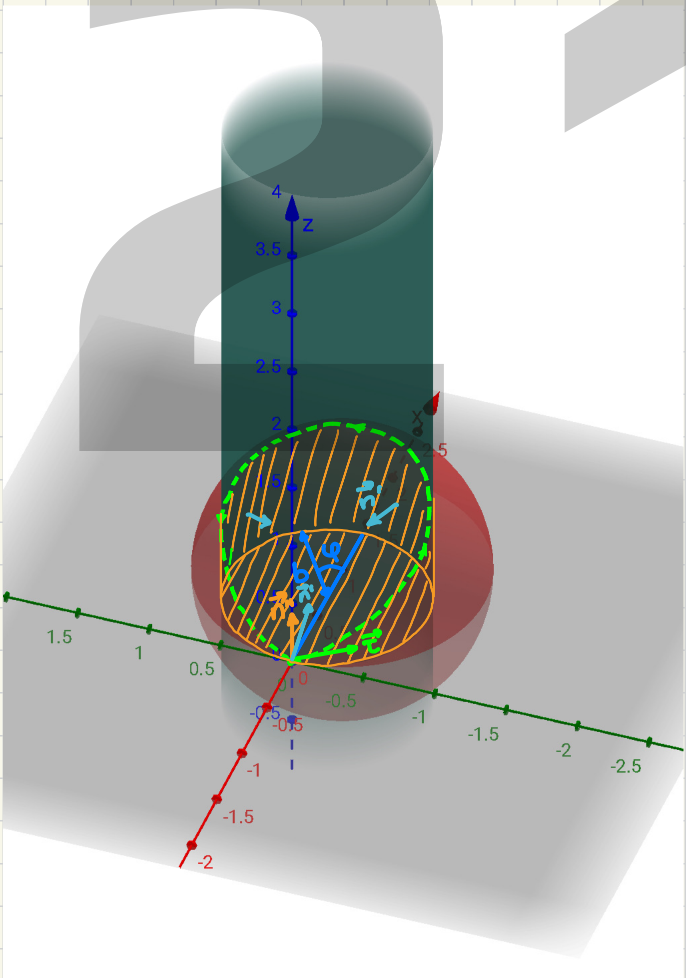
$3r^2 f(\vec{r}) + f'(r) r^3 = 0 \to (r^3 f(r))' = 0 \Rightarrow f(r) = \frac{C}{r^3} \Rightarrow \varphi = \frac{C}{r^4}$

Доп задача

11.67

$$\int_L (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz,$$

L - кривая $x^2+y^2+z^2=2ax$, $x^2+y^2=2bx$, $z>0$, $0<b<a$, ориентированная положительно отн. $(0\ 0\ 1)^T$



$$2\iint_S (y-z)dydz+(z-x)dxdz+(x-y)dxdy$$

$S=S'\cup S''$, S' - боковая сторона цилиндра под кривой, выше пл-ти xy, S'' - круг в пл-ти xy

Для диска нормаль $(0\ 0\ 1)^T$

$$2\iint_{S''}(y-z)dydz+(z-x)dxdz+(x-y)dxdy=$$

$$=2\iint_{S''}(x-y)dxdy \ominus$$

$$\begin{aligned} x&=r(1+\cos\varphi)\\ y&=r\sin\varphi \end{aligned} \rightarrow J=\begin{vmatrix} 1+\cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}=$$

$$=r(1+\cos\varphi)$$

$$\ominus\, 2\int_0^b r^2dr\int_0^{2\pi}(1+\cos\varphi)(1+\cos x-\sin x)d\varphi=\frac{2}{3}b^3\int_0^{2\pi}(1+\cos^2\varphi)d\varphi=$$

$$=\frac{2}{3}b^3\cdot 3\pi=2b^3$$

Для S' нормаль определяется из $\vec{n}\times\vec{\tau}$ - внутрь цилиндра

$$\begin{cases} x = b(1+\cos\varphi) \\ y = b\sin\varphi \\ z = t \end{cases} \qquad z^2 = 2ax - (x^2+y^2) = 2ax - 2bx \Rightarrow z^2(\varphi) = 2(a-b)b(1+\cos\varphi)$$

Нормаль к цилиндру – $(b\cos\varphi \;\; b\sin\varphi \;\; 0)^T$, внутренняя – $(-b\cos\varphi \;\; -b\sin\varphi \;\; 0)^T \Rightarrow$

$$\Rightarrow 2\iint\limits_{S''}\left((y-z)\cdot(-b\cos\varphi) + (z-x)\cdot(-b\sin\varphi) + (x-y)\cdot 0\right)d\varphi\, dt =$$

$$= -2b\iint\limits_{S''}\left[(b\sin\varphi - t)(-b\cos\varphi) + (t - b(1+\cos\varphi))(-b\sin\varphi)\right]d\varphi\, dt =$$

$$= -2b\iint\limits_{S''}\left(t(\sin\varphi - \cos\varphi) - b\sin\varphi\right)d\varphi\, dt = -2b\int\limits_{-\pi}^{\pi}\int\limits_{0}^{z(\varphi)} -2b\left(t(\sin\varphi - \cos\varphi) - b\sin\varphi\right)dt\, d\varphi = -2b\int\limits_{-\pi}^{\pi}\left[-2b\sin\varphi\frac{z^2(\varphi)}{2} + 2b\cos\varphi\frac{z^2(\varphi)}{2} - z(\varphi)b\sin\varphi\right]\cdot$$

$$\cdot d\varphi = -2b\int\limits_{-\pi}^{\pi} b\cos\varphi\, z^2(\varphi)\, d\varphi \circeq$$

т.к. $z(\varphi)$ – чёт, а $\sin\varphi$ нечёт, то $\int\limits_{-\pi}^{\pi}\left[-2b\sin\varphi\frac{z^2(\varphi)}{2} - z(\varphi)\cdot b\sin\varphi\right]d\varphi = 0$

$$\circeq 2b^2(a-b)\int\limits_{-\pi}^{\pi}(\cos\varphi + \cos^2\varphi)\,d\varphi = 2\pi b^2(a-b)$$

Сумма по S' и S'': $2\pi b^2(a-b) + 2\pi b^3 = \boxed{2\pi a b^2}$