§17 №32.

$$\lim_{x\to1}\frac{x^{50}-50x+49}{x^{100}-100x+99}=\lim_{x\to1}\frac{50x^{49}-50}{100x^{99}-100}=\lim_{x\to1}\frac{50\cdot49x^{48}}{100\cdot99\cdot x^{98}}=$$

$$=\frac{50\cdot49}{100\cdot99}=\frac{49}{198}$$

№49. Докажем по индукции, что искомый предел равен 0.

База: $\lim\limits_{x\to+\infty}\frac{x^m}{e^{-x^3}}$, $\quad n\underset{3}{\equiv}m,\ 0<m<3$

Предположение: $\lim\limits_{x\to+\infty}\frac{x^{n-3}}{e^{-x^3}}=0$

Шаг: $$\lim_{x\to+\infty}x^ne^{-x^3}=\lim_{x\to+\infty}\frac{x^n}{e^{x^3}}=\lim_{x\to+\infty}\frac{n\,x^{n-1}}{3x^2e^{x^3}}=$$

$$=\lim_{x\to+\infty}\frac{n\,x^{n-3}}{3e^{x^3}}=\frac{n}{3}\lim_{x\to+\infty}\frac{x^{n-3}}{e^{x^3}}=0\ \blacksquare$$

№63.

$$\lim_{x\to+\infty}\left(\frac{2}{\pi}\operatorname{arctg}x\right)^x=\lim_{x\to+\infty}e^{\ln\left(\frac{2}{\pi}\operatorname{arctg}x\right)x}\ (=)$$

$$\lim_{x\to+\infty}\frac{\ln\left(\frac{2}{\pi}\operatorname{arctg}x\right)}{1/x}=\lim_{x\to+\infty}\frac{\frac{\pi}{2\operatorname{arctg}x}\cdot\frac{\pi/2}{1+x^2}}{-1/x^2}=\lim_{x\to+\infty}\frac{-\frac{x^2}{1+x^2}}{\operatorname{arctg}x}=$$

$$=\frac{\lim\limits_{x\to+\infty}\left(-\frac{x^2}{1+x^2}\right)}{\lim\limits_{x\to+\infty}(\operatorname{arctg}x)}=\frac{\lim\limits_{x\to+\infty}\left(-\frac{x^2}{1+x^2}\right)}{\pi/2}=-\frac{2}{\pi}\ \Rightarrow$$

$$\Rightarrow\ (=)\ e^{-\frac{2}{\pi}}$$

№76.

1) $\lim\limits_{x\to\infty} \frac{x+\cos x}{x-\cos x} \overset{?}{=} \lim\limits_{x\to\infty} \frac{1-\sin x}{1+\sin x}$ — $\nexists$, т.к. можно взять две послед. Гейне: $x_n' = \{\frac{\pi}{2}+2\pi n\}$ и $x_n'' = \{\pi n\}$. Тогда $\lim\limits_{n\to\infty} f(x_n') = \frac{1-1}{1+1} = 0 \neq$ $\neq \lim\limits_{n\to\infty} f(x_n'') = \frac{1-0}{1+0} = 1 \Rightarrow$ правило Лопиталя применять нельзя.

$$\lim_{x\to+\infty} \frac{x+\cos x}{x-\cos x} = \lim_{x\to+\infty} \frac{1+\frac{\cos x}{x}}{1-\frac{\cos x}{x}} = 1$$

2) $\lim\limits_{x\to 0} \frac{x^3 \sin(1/x)}{\sin^2 x} = \lim\limits_{x\to 0} \frac{f(x)}{g(x)} \overset{?}{=} \lim \frac{f'(x)}{g'(x)}$

$g'(x) = (\sin^2 x)' = \sin 2x \xrightarrow[x\to 0]{} 0 \Rightarrow$ правило Лопиталя применять нельзя.

$$\lim_{x\to 0} \frac{x^3\sin(1/x)}{\sin^2 x} = \lim_{x\to 0} \frac{x^2}{\sin^2 x}\, x\sin(1/x) =$$

$$= \lim_{x\to 0} \frac{x^2}{\sin^2 x} \lim_{x\to 0} x\sin(1/x) = 1\cdot 0 = 0$$

§19

№7(3) $\lim\limits_{x\to 0} \frac{\ln(1+x)\cos x - e^{\operatorname{tg} x} + \sqrt{1+2x^2}}{x-\sin x} =$

1) $(x - \frac{x^2}{2} + \frac{x^3}{3} + \bar{o}(x^3))(1 - \frac{x^2}{2} + \bar{o}(x^3)) = x - \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^3}{3} + \bar{o}(x^3),\ x\to 0$

2) $1 + \operatorname{tg} x + \frac{\operatorname{tg}^2 x}{2} + \frac{\operatorname{tg}^3 x}{6} + \bar{o}(\operatorname{tg}^3 x) = 1 + x + \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^3}{6} + \bar{o}(x^3),\ x\to 0$

3) $1 + x^2 + \bar{o}(x^3),\ x\to 0$

Числ.: $x - \frac{x^2}{2} - \frac{x^3}{6} - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{2}\right) + 1 + x^2 + \bar{o}(x^3) =$

$= -\frac{2}{3}x^3 + \bar{o}(x^3),\ x \to 0$

<u>Знам:</u> $x - \left(x - \frac{x^3}{6}\right) + \bar{o}(x^3) = \frac{x^3}{6} + \bar{o}(x^3),\ x \to 0$

$$= \lim_{x\to 0} \frac{-\frac{2}{3}x^3 + \bar{o}(x^3)}{\frac{x^3}{6} + \bar{o}(x^3)} = -4$$

№9(6) $\lim\limits_{x\to 0} \dfrac{\operatorname{tg}(\sin x) - \ln\left(x + \sqrt[3]{1+x^2}\right) - x^2/6}{\operatorname{th}(x - x^3) - x} =$

3 $\operatorname{th}(x - x^3) - x = x - x^3 - \frac{1}{3}(x - x^3)^3 - x + \bar{o}(x^3) =$

$= -x^3 - \frac{1}{3}x^3 + \bar{o}(x^3) = -\frac{4}{3}x^3 + \bar{o}(x^3),\ x \to 0$

1 $\operatorname{tg}(\sin x) = x - \frac{x^3}{6} + \frac{1}{3}\left(x - \frac{x^3}{6}\right)^3 + \bar{o}(x^3) = x + \frac{x^3}{6} + \bar{o}(x^3)$

2 $\ln\left(x + \sqrt[3]{1+x^2}\right) + x^2/6 \ \boxed{=}$

$(1+x^2)^{1/3} = 1 + \frac{1}{3}x^2 + \bar{o}(x^3),\ x \to 0$

$\ln\left(1 + x + \frac{1}{3}x^2 + \bar{o}(x^3)\right) = x + \frac{1}{3}x^2 + \bar{o}(x^3) - \frac{\left(x + \frac{1}{3}x^2 + \bar{o}(x^3)\right)^2}{2} +$

$+ \frac{1}{3}\left(x + \frac{1}{3}x^2 + \bar{o}(x^3)\right)^3 + \bar{o}(x^3) = x + \frac{1}{3}x^2 - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^3}{3} + \bar{o}(x^3) =$

$= x - \frac{x^2}{6} + \bar{o}(x^3),\ x \to 0$

$\boxed{=}\ x + \bar{o}(x^3),\ x \to 0$

$$= \lim_{x\to 0} \frac{x + x^3/6 - x + \bar{o}(x^3)}{-\frac{4}{3}x^3 + \bar{o}(x^3)} = -\frac{1}{8}$$

№14(5)

$$\lim_{x\to 0} \frac{e^{x/(1-x)} - \operatorname{sh} x - \cos x}{\sqrt[6]{1+x} + \sqrt[6]{1-x} - 2} =$$

Знам.: $1+\frac{x}{6} - \frac{5}{72}x^2 + 1 - \frac{x}{6} - \frac{5}{72}x^2 - 2 + \bar{o}(x^2) =$

$= -\frac{5}{36}x^2 + \bar{o}(x^2), \ x\to 0$

Числ.: $x\frac{1}{1-x} = x + x^2 + \bar o(x^2), \ x\to 0$

$e^{\frac{x}{1-x}} = 1 + (x+x^2) + \frac12(x+x^2)^2 + \bar o(x^2) = 1 + x + \frac32 x^2 + \bar o(x^2), x\to 0$

$e^{x/(1-x)} - \operatorname{sh}x - \cos x = 1 + x + \frac32x^2 - x - \left(1-\frac{x^2}{2}\right) + \bar o(x^2) =$

$= 2x^2 + \bar o(x^2), \ x\to 0$

$= \frac{2}{-\frac{5}{36}} = -\frac{72}{5}$

№22(2) $\lim_{x\to 0}\left(\frac{\underset{(1)}{\sqrt{1-2x} - \sqrt[3]{1-3x}}}{\ln \operatorname{ch} x}\right)^{1/x} =$

$= \lim_{x\to 0} e^{\ln^{(3)}\left(\frac{\sqrt{1-2x}-\sqrt[3]{1-3x}}{\underset{(2)}{\ln \operatorname{ch} x}}\right)\frac{1}{x}} =$

(1) $\sqrt{1-2x} - \sqrt[3]{1-3x} = \left(1 - x - \frac12x^2 - \frac12x^3\right) - \left(1 - x - x^2 - \frac53x^3\right) + \bar o(x^3) = \frac12x^2 + \frac76x^3 + \bar o(x^3), \ x\to 0$

(2) $\ln(\operatorname{ch}x) = \ln\left(1+\frac{x^2}{2} + \bar o(x^2)\right) = \frac{x^2}{2} + \bar o(x^3), \ x\to 0$

(3) $\ln\left(\frac{1/2\, x^2 + 7/6\, x^3 + \bar o(x^3)}{1/2\, x^2 + \bar o(x^3)}\right) = \ln\left(1+\frac73x + \bar o(x)\right) = \frac73x + \bar o(x), x\to 0$

$= \lim_{x\to 0} e^{(7/3\, x + \bar o(x))\, 1/x} = e^{7/3}$

№29(4) $\lim_{x\to 0}\left(\frac{\operatorname{tg}(2x+x^3) - \operatorname{th}(x+2x^3)}{x}\right)^{1/(\sqrt[3]{1+x^3} - \sqrt{1+x^2})} =$

$$= \lim_{x\to 0} e^{\ln\left(\frac{\operatorname{tg}(2x+x^3)-\operatorname{th}(x+2x^3)}{x}\right)\cdot 1/\left(\sqrt[3]{1+x^3}-\sqrt{1+x^2}\right)} \,(=)$$

Числ.: $\operatorname{tg}(2x+x^3)-\operatorname{th}(x+2x^3) = 2x+x^3+\frac{8x^3}{3}-\left(x+2x^3-\frac{x^3}{3}\right)+\bar{o}(x^3) = x+x^3\left(1+\frac{8}{3}-2+\frac{1}{3}\right)+\bar{o}(x^3) = x+2x^3+\bar{o}(x^3),\ x\to 0$

$$\ln\left(\frac{x+2x^3+\bar{o}(x^3)}{x}\right) = \ln\left(1+2x^2+\bar{o}(x^2)\right) = 2x^2+\bar{o}(x^2),\ x\to 0$$

Знам.: $\sqrt[3]{1+x^3}-\sqrt{1+x^2} = 1+\frac{x^3}{3}-\left(1+\frac{x^2}{2}\right)+\bar{o}(x^2) = \frac{x^3}{3}-\frac{x^2}{2}+\bar{o}(x^2),\ x\to 0$

$$(=) \lim_{x\to 0} \exp\left(\frac{2x^2+\bar{o}(x^2)}{x^3/3-x^2/2+\bar{o}(x^2)}\right) =$$

$$= \lim_{x\to 0} \exp\left(\frac{2+\bar{o}(1)}{\frac{x}{3}-\frac{1}{2}+\bar{o}(1)}\right) = e^{-4}$$

№47(5) $\lim_{x\to +0}\left(\frac{\operatorname{sh} x}{\operatorname{arctg} x}\right)^{1/x^2+\ln x} = \lim_{x\to +0} \exp\left(\ln\left(\frac{\operatorname{sh} x}{\operatorname{arctg} x}\right)\left(\frac{1}{x^2}+\ln x\right)\right) \,(=)$

$$\frac{\operatorname{sh} x}{\operatorname{arctg} x} = \frac{x+\frac{x^3}{6}+\bar{o}(x^4)}{x-\frac{x^3}{3}+\bar{o}(x^4)} = \frac{1+x^2/6+\bar{o}(x^3)}{1-x^2/3+\bar{o}(x^3)} = \left(1+\frac{x^2}{6}+\bar{o}(x^3)\right)\cdot$$

$$\cdot\left(1+\frac{x^2}{3}+\bar{o}(x^3)\right) = 1+\frac{x^2}{2}+\bar{o}(x^3),\ x\to 0$$

$$\ln\left(1+\frac{x^2}{2}+\bar{o}(x^3)\right) = \frac{x^2}{2}+\bar{o}(x^3),\ x\to 0$$

$$(=) \lim_{x\to 0} \exp\left(\frac{\frac{x^2}{2}+\bar{o}(x^3)}{x^2}+\ln x\left(\frac{x^2}{2}+\bar{o}(x^3)\right)\right) =$$

$$= e^{1/2}\lim_{x\to 0} \exp\left(\ln x\left(\frac{x^2}{2}+\bar{o}(x^3)\right)\right) = e^{1/2}\lim_{x\to 0} \exp\left(\frac{x^2\ln x}{2}\right) =$$

$$= e^{1/2}\exp\left(\lim_{x\to 0}\frac{\ln x}{\frac{2}{x^2}}\right) = e^{1/2}\exp\left(\lim_{x\to 0}\frac{1/x}{-1/x^3}\right) =$$

$$= e^{1/2}\exp(0) = e^{1/2}$$

№58(3) $\lim\limits_{x\to+\infty}\dfrac{x^2(\sqrt[3]{x^3+x}-x)+\sin x\ln(1+x)}{\ln(1+x+e^{5x})} =$

$t := 1/x,\ t\to+0$

$$t^{-2}\left(\sqrt[3]{t^{-3}+t^{-1}}-t^{-1}\right) = t^{-2}\left(t^{-1}\sqrt[3]{1+t^2}-t^{-1}\right) =$$

$$= t^{-3}\left(1+\frac{1}{3}t^2-1+\bar{o}(t^3)\right) = \frac{1}{3}t^{-1}+\bar{o}(1),\ t\to+0 =$$

$$= \frac{1}{3}t^{-1},\ t\to+0$$

$$\sin\left(\frac{1}{t}\right)\ln\left(1+\frac{1}{t}\right) = \left(t^{-1}+\bar{o}(t^{-1})\right)\left(t^{-1}+\bar{o}(t^{-1})\right) =$$

$$= t^{-2}+\bar{o}(t^{-2}),\ t\to+0$$

$$\ln\left(1+t^{-1}+e^{5t^{-1}}\right) = \ln\left(1+t^{-1}+1+5t^{-1}+\frac{25t^{-2}}{2}\right) =$$

$$= \ln\left(2+6t^{-1}+\frac{25}{2}t^{-2}\right) = \ln 2+\ln\left(1+3t^{-1}+\frac{25}{4}t^{-2}\right) =$$

$$= \ln 2+3t^{-1}+\frac{25}{4}t^{-2}+\frac{1}{2}\left(3t^{-1}+\frac{25}{4}t^{-2}\right)^2+\bar{o}(t^{-2}) =$$

$$= \ln 2+3t^{-1}+\frac{43}{4}t^{-2}+\bar{o}(t^{-2}),\ t\to+0$$

$$= \lim_{t\to+0}\frac{\frac{1}{3}t^{-1}+t^{-2}+\bar{o}(t^{-2})}{\ln 2+3t^{-1}+\frac{43}{4}t^{-2}+\bar{o}(t^{-2})} = \lim_{x\to 0}\frac{\frac{1}{3}t+1+\bar{o}(1)}{\ln 2\cdot t^2+3t+\frac{43}{4}+\bar{o}(1)} =$$

$$= \frac{4}{43}$$

T.7. $e^x = \sum\limits_{k=0}^{n}\dfrac{x^k}{k!}+\dfrac{\exp^{(n+1)}(\xi)}{(n+1)!}x^{n+1},\ \xi\in[-1;2]$

погрешность - $R_n(x) = e^x - \sum\limits_{k=0}^{n} \frac{x^k}{k!} = e^x - P_n(x) =$

$$= \frac{\exp^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{e^{\xi}}{(n+1)!} x^{n+1}$$

$$|R_n(x)| = \left| \frac{e^{\xi}}{(n+1)!} x^{n+1} \right| < \frac{2^{n+1}}{(n+1)!} e^2 < 10^{-3}$$

$$\frac{2^{n+1}}{(n+1)!} < \frac{10^{-3}}{e^2}$$

$$\left.\begin{array}{l} n=9: \quad \frac{2^{10}}{10!} > \frac{10^{-3}}{e^2} \\ n=10: \quad \frac{2^{11}}{11!} < \frac{10^{-3}}{e^2} \end{array}\right| \Rightarrow e^x = \sum_{k=0}^{10} \frac{x^k}{k!}$$ с точностью до 10^{-3}.