Матан Д/З 1

I.

№33. $y = \dfrac{\arcsin x}{e^x}$

$$y' = \frac{(\arcsin x)' e^x - (e^x)' \arcsin x}{e^{2x}} = \frac{e^{-x}}{\sqrt{1-x^2}} -$$

$$- e^{-x} \arcsin x = e^{-x}\left((1-x^2)^{-1/2} - \arcsin x\right)$$

№78. $y = \ln|\sin x|$

$$y' = \frac{1}{|\sin x|} \operatorname{sign}(\sin x) \cos x = \frac{\cos x}{\sin x} = \operatorname{ctg} x$$

№106. $y = 3^{\cos^2 x}$

$$y' = \ln 3 \; 3^{\cos^2 x} \; (\cos^2 x)' = \ln 3 \cdot 3^{\cos^2 x} \cdot$$

$$\cdot 2\cos x\,(-\sin x) = -\ln 3 \cdot 3^{\cos^2 x} \cdot \sin 2x$$

№146. $y = x^{x^2}$

$$y' = \left(e^{\ln x \cdot x^2}\right)' = e^{\ln x \cdot x^2}\left(\ln x \cdot x^2\right)' =$$

$$= x^{x^2}\left(\frac{1}{x} \cdot x^2 + \ln x \cdot 2x\right) = x^{x^2}\left(x + 2\ln x \cdot x\right) =$$

$$= x^{x^2+1}(1 + 2\ln x)$$

Т.1 $y = \left(\dfrac{\operatorname{tg}\sqrt{1-\log_3 2x}}{\operatorname{cth}(x^3+3e^{x^4})}\right)^{\arccos 2x^2} = a^b$

$$y' = \left(e^{\ln a \cdot b}\right)' = a^b \cdot (\ln a \cdot b)' =$$

$$= a^b\left((\ln a)' b + (b)' \ln a\right)$$

1) $b' = (\arccos 2x^2)' = \dfrac{-1}{\sqrt{1-(2x^2)^2}}\,(2x^2)' =$

$$= \frac{-4x}{\sqrt{1-4x^4}}$$

2) $(\ln a)' = \dfrac{a'}{a}$

3) $a' = \left(\dfrac{c}{d}\right)' = \dfrac{c'd - d'c}{d^2}$

4) $c' = \left(\operatorname{tg}\sqrt{1-\log_3 2x}\right)' = \cos^{-2}\left(\sqrt{1-\log_3 2x}\right) \cdot$

$\cdot \dfrac{1}{2\sqrt{1-\log_3 2x}} \dfrac{1}{2x\ln 3}$

5) $d' = \left(\operatorname{cth}\left(x^3+3e^{x^4}\right)\right)' = -\operatorname{sh}^{-2}\left(x^3+3e^{x^4}\right) \cdot$

$\cdot \left(3x^2+3e^{x^4} \cdot 4x^3\right)$

3) $a' = \dfrac{c'd - d'c}{d^2} = \dfrac{\left(4\cos^2\left(\sqrt{1-\log_3 2x}\right)\sqrt{1-\log_3 2x} \cdot x\ln 3 \cdot \operatorname{cth}\left(x^3+3e^{x^4}\right)\right)^{-1} + \operatorname{sh}^{-2}\left(x^3+3e^{x^4}\right) \cdot \left(3x^2+3e^{x^4} \cdot 4x^3\right)}{\operatorname{cth}^2\left(x^3+3e^{x^4}\right)}$

$y' = a^b\left(b\dfrac{a'}{a} + b'\ln a\right) = \left(\dfrac{\operatorname{tg}\sqrt{1-\log_3 2x}}{\operatorname{cth}\left(x^3+3e^{x^4}\right)}\right)^{\arccos(2x^2)} \cdot$

$\cdot \left(\arccos(2x^2)\dfrac{\left(4\cos^2\left(\sqrt{1-\log_3 2x}\right)\sqrt{1-\log_3 2x} \cdot x\ln 3 \cdot \operatorname{cth}\left(x^3+3e^{x^4}\right)\right)^{-1} + \operatorname{sh}^{-2}\left(x^3+3e^{x^4}\right)\left(3x^2+3e^{x^4} \cdot 4x^3\right)}{} \cdot\right.$

$\left.\cdot \dfrac{\operatorname{cth}\left(x^3+3e^{x^4}\right)}{\operatorname{tg}\sqrt{1-\log_3 2x}} - \dfrac{4x}{\sqrt{1-4x^4}} \ln\left(\dfrac{\operatorname{tg}\sqrt{1-\log_3 2x}}{\operatorname{cth}\left(x^3+3e^{x^4}\right)}\right)\right)$

II.

№2 (16)

$$\int ctg^2 x\,dx = \int \frac{\cos^2 x + \sin^2 x - \sin^2 x}{\sin^2 x}\,dx = \int\left(\frac{1}{\sin^2 x} - 1\right)dx =$$

$$= \int \frac{dx}{\sin^2 x} - x \ominus$$

$$(ctg\,x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$\ominus -\int -\frac{dx}{\sin^2 x} - x = -ctg x - x + C$$

N12(2)

$$2)\ \int x\sqrt{1+x}\,dx \overset{\substack{x=t^2-1\\ \sqrt{1+x}=t}}{=\!=\!=} \int (t^2-1)\,t\,d(t^2-1) =$$

$$= \int (t^2-1)\cdot t\cdot 2t\,dt = 2\int (t^2-1)\,t^2\,dt =$$

$$= 2\left[\int t^4 dt - \int t^2 dt\right] = 2\left[\frac{t^5}{5} - \frac{t^3}{3}\right] + C =$$

$$= \frac{2}{5}t^5 - \frac{2}{3}t^3 + C = \frac{2}{5}\sqrt{1+x}^5 - \frac{2}{3}\sqrt{1+x}^3 + C =$$

$$= \frac{2}{5}(1+x)^2\sqrt{1+x} - \frac{2}{3}(1+x)\sqrt{1+x} + C$$

N13(7)

$$7)\ \int \frac{dx}{\sqrt{e^x-1}} \overset{x=\ln t}{=\!=} \int \frac{dt}{t\sqrt{t-1}} \overset{\substack{t=u^2+1\\ \sqrt{t-1}=u}}{=\!=\!=} \int \frac{2u\,du}{(u^2+1)\,u} =$$

$$= 2\int \frac{du}{1+u^2} = 2\,arctg\,u + C = 2\,arctg\sqrt{t-1} + C =$$

$$= 2\,arctg\sqrt{e^x-1} + C$$

N15(5)

$$5)\ \int \frac{dx}{\cos x} = \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{d(\sin x)}{1-\sin^2 x} =$$

$= -\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$ $\qquad -\frac{1}{\sqrt{2}}\ln\left|t + \sqrt{t^2-a^2}\right| + C$

№15.11.

11) $\int \frac{\sin x}{\sqrt{\cos 2x}}dx = \int \frac{-d(\cos x)}{\sqrt{2\cos^2 x - 1}} =$

$= -\frac{1}{\sqrt{2}}\int \frac{d(\sqrt{2}\cos x)}{\sqrt{(\sqrt{2}\cos x)^2 - 1}} = -\frac{1}{\sqrt{2}}\ln\left|\sqrt{2}\cos x + \sqrt{\cos 2x}\right| + C$

№17(4)

$d(x\ln x) = (\ln x + 1)dx$

4) $\int x\ln x dx = x^2\ln x - \int x\, d(x\ln x) =$

$= x^2\ln x - \int x(\ln x + 1)dx = x^2\ln x -$

$- \int x\ln x dx - \int x dx$

$2\int x\ln x dx = x^2\ln x - \frac{x^2}{2} + C$

$\int x\ln x\, dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C$

№23.(5)

5) $\int \arcsin^2 x\, dx \overset{x=\sin t}{=\!=} \int t^2 d(\sin t) =$

$= t^2\sin t - \int \sin t\, d(t^2) = t^2\sin t - 2\int t\sin t\, dt =$

$= t^2\sin t + 2\int t\, d(\cos t) = t^2\sin t + 2[t\cos t -$

$- \int \cos t\, dt] = t^2\sin t + 2t\cos t - 2\sin t + C =$

$= x\arcsin^2 x + 2\arcsin x\sqrt{1-x^2} - 2x + C$

№24(3)

3) $\int e^{ax}\sin bx\,dx = \int \sin bx \cdot \frac{1}{a}\,d(e^{ax}) =$

$= \frac{1}{a}\left[e^{ax}\sin bx - \int e^{ax}\,d(\sin bx)\right] =$

$= \frac{1}{a}\left[e^{ax}\sin bx - \int e^{ax}\cos bx \cdot b\,dx\right] =$

$= \frac{1}{a}\left[e^{ax}\sin bx - \frac{b}{a}\int \cos bx\,d(e^{ax})\right] =$

$= \frac{1}{a}\left[e^{ax}\sin bx - \frac{b}{a}\left[e^{ax}\cos bx - \int e^{ax}\,d(\cos bx)\right]\right] =$

$= \frac{1}{a}\left[e^{ax}\sin bx - \frac{b}{a}\left[e^{ax}\cos bx + b\int e^{ax}\sin bx\,dx\right]\right]$

$\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$

$\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx$

$\int e^{ax}\sin bx\,dx = \frac{a^2}{a^2+b^2}\left[\frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx\right] =$

$= e^{ax}\left(\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\right] + C =$

$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \arccos\left(\frac{a}{a^2+b^2}\right)\right) + C$

III 4) Д-ть: $\forall\, a,b \in \mathbb{Q} : a<b,\ \exists\, c \in I : a<c<b$

$\exists\, c = a + \frac{\sqrt{2}}{n},\ n\in\mathbb{N} \Rightarrow c > a$. Найдём $N \in \mathbb{N} : a + \frac{\sqrt{2}}{N} < b$

$N > \frac{\sqrt{2}}{b-a} \Rightarrow \forall\, n \geq N = \left[\frac{\sqrt{2}}{b-a}\right] + 1 : c = a + \frac{\sqrt{2}}{n} \mapsto$

$a < c < b$. $c \in I$ ∎.

№8. $X = \{x : x \in \mathbb{Q}, x^2 < 2\}$. Д-ть: X не имеет ни наименьш., ни наибольш. $\sup X, \inf X$ - ?

$\exists M = \max X$, $M \in \mathbb{Q}$. Возьмём число

$B = \frac{3M+4}{2M+3}$. $B - M = \frac{4 - 2M^2}{2M+3} > 0$, т.к. $M^2 < 2$

$$2 - B^2 = \frac{2(4M^2+12M+9) - 9M^2 - 24M - 16}{(2M+3)^2} =$$

$$= \frac{2 - M^2}{(2M+3)^2} > 0, \text{ т.к. } 2 > M^2 \Rightarrow B > M,$$

$B^2 < 2$ и $B \in \mathbb{Q} \Rightarrow M \neq \max X$! ∎

№10. X-огр. $\Rightarrow \exists M_x : \forall x \in X \mapsto x < M_x$

Y-огр. $\Rightarrow \exists M_y : \forall y \in Y \mapsto y < M_y$ $\Big\} \Rightarrow$

$\Rightarrow \exists M_{x+y} = M_x + M_y : \forall x \in X, y \in Y \mapsto x + y < M_{x+y} \Rightarrow$

$\Rightarrow X + Y$ - огр.

$\sup X = M_x$: 1) $\forall x \in X \mapsto x \le M_x$

2) $\forall B_x$: $\inf X < B_x < M_x \mapsto B_x \in X$

$\sup Y = M_y$: 1) $\forall y \in Y \mapsto y \le M_y$

2) $\forall B_y$: $\inf Y < B_y < M_y \mapsto B_y \in Y$

$\sup X + \sup Y = M_x + M_y = M_{x+y}$

M_{x+y}: 1) $\forall x+y: x \in X, y \in Y \mapsto x+y \le M_{x+y}$

2) $\forall B_x + B_y$: $\inf X + \inf Y < B_x + B_y < M_x + M_y \mapsto B_x + B_y \in X+Y \Rightarrow$

$\Rightarrow M_{x+y} = \sup X + \sup Y = \sup(X+Y)$

Аналогично $\inf X + \inf Y = \inf(X+Y)$

Т.2. $1 - x + x^2 - \ldots + (-1)^n x^n = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots$

$+ \ldots (-x)^n = \dfrac{(-x)^{n+1} - 1}{-x-1} = \dfrac{1-(-x)^{n+1}}{1+x} = \dfrac{1+(-1)^n x^{n+1}}{1+x}$

IV. 275.(4)

4) $\left\{\dfrac{n^2+4n+8}{(n+1)^2}\right\} = \left\{\dfrac{2(n^2+2n+1)+6-n^2}{(n+1)^2}\right\} =$

$= \left\{\dfrac{2(n+1)^2+6-n^2}{(n+1)^2}\right\} = \left\{2 + \dfrac{6-n^2}{(n+1)^2}\right\}$

Проверим утв.: $M = 4 \mapsto \forall n \quad 2 + \dfrac{6-n^2}{(n+1)^2} < M$

$2 \overset{?}{>} \dfrac{6-n^2}{(n+1)^2} \Leftrightarrow 2n^2+4n+2 \overset{?}{>} 6 - n^2 \Leftrightarrow$

$\Leftrightarrow 3n^2 + 4n - 4 \overset{?}{>} 0 \quad D = 16 + 16 \cdot 3 = 8^2 \Leftrightarrow$

$\Leftrightarrow 3\left(n-\frac{2}{3}\right)(n+2) \overset{?}{>} 0$

$\forall n\in\mathbb{N}$ верно $\Rightarrow$ утв. верно $\Rightarrow$ ограничена ■

№276(5)

5) $\left\{n^{(-1)^n}\right\}$

$]\ \exists M: \forall n: \ n^{(-1)^n} < M \Leftrightarrow$

$\Leftrightarrow (-1)^n \ln n < \ln M$ (ln – монотонн. ↑)

Возьмём $n=2k$, тогда $\forall k \ \ln(2k) < \ln M$.

при $2k > M$ это неверно $\Rightarrow$ нач. утв. неверно $\Rightarrow$ $\left\{n^{(-1)^n}\right\}$ неограничена.

№279(1)

1) $x_n = \sum\limits_{k=1}^{n} \frac{1}{k(k+1)}, \ n\in\mathbb{N}$

$x_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \dots - \frac{1}{n+1} = 1 - \frac{1}{n+1}$

$\forall n: x_n = 1 - \frac{1}{n+1} < 1 \Rightarrow$ огр.

$\forall \varepsilon > 0 \ \exists N(\varepsilon)\in\mathbb{N}: \forall n > N, n\in\mathbb{N} \mapsto \cancel{1} - \frac{1}{n+1} > \cancel{1} - \varepsilon \Rightarrow$

$\Rightarrow n+1 > \varepsilon \Rightarrow n > \frac{1}{\varepsilon} - 1 \Rightarrow N = \left[\frac{1}{\varepsilon} - 1\right] + 1 \Rightarrow$

$\Rightarrow \sup\{x_n\} = 1$

$\frac{x_{n+1}}{x_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{(n+1)^2}{n(n+2)} = \frac{n^2+2n+1}{n^2+2n} > 1 \Rightarrow \{x_n\}$ –

возр. $\Rightarrow x_1 = \inf\{x_n\} = \frac{1}{2}$

№299(2)

2) $\left\{\frac{3n+4}{n+2}\right\} = \left\{\frac{n+2(n+2)}{n+2}\right\} = \left\{\frac{n}{n+2}+2\right\}$

$x_n = \frac{n}{n+2}+2$

$x_{n+1} = \frac{n+1}{n+3}+2$

$x_{n+1}-x_n = \frac{n+1}{n+3} - \frac{n}{n+2} = \frac{(n+1)(n+2)-n(n+3)}{(n+2)(n+3)} =$

$= \frac{n^2+3n+2-n^2-3n}{(n+2)(n+3)} = \frac{2}{(n+2)(n+3)} > 0 \quad \forall n \Rightarrow$

$\Rightarrow \{x_n\}$ – монотонная

№300(3)

$\left\{\sqrt[3]{n^3-1} - n\right\} = \left\{\frac{\left(\sqrt[3]{n^3-1}-n\right)\left(\sqrt[3]{n^3-1}^2+n\sqrt[3]{n^3-1}+n^2\right)}{\sqrt[3]{n^3-1}+n\sqrt[3]{n^3-1}+n^2}\right\} =$

$= \left\{\frac{-1}{\sqrt[3]{n^3-1}+n\sqrt[3]{n^3-1}+n^2}\right\}$

монотонный знам. $\Rightarrow$

$\Rightarrow$ монотонная дробь.