§13

179(2) $y=|\sin x| = \operatorname{sgn}(\sin(x))\sin(x)$

$y' = \operatorname{sgn}'_{\sin x}\cdot \sin'(x)\sin x + \sin' x\,\operatorname{sgn}(\sin(x))$.

$\forall x: \sin x \neq 0 \quad \exists\, \operatorname{sgn}'_{\sin x} = 0 \Rightarrow \forall x: \sin x \neq 0 \mapsto \exists\, y'$ при этом $\nexists\, y'$ в $x: \sin x = 0$, т.к. $\nexists\, \operatorname{sgn}'_{\sin x}$ в $x: \sin x = 0$.

197(5) $y = 2x^2 - x^4,\ 0 < x < 1,\ y = 3/4$

$\frac{dy}{dx} = 4x - 4x^3,\ 0<x<1 \Rightarrow \frac{dx}{dy} = x'_y = (4x-4x^3)^{-1},\ 0<x<1$

$x'(y_0) = (4x_0 - 4x_0^3)^{-1},\quad 0 < x_0 < 1$

$x_0:\quad 3/4 - 1 = -1 + 2x^2 - x^4$

$-1/4 = -(x^2-1)^2$

$\pm\frac{1}{2} = x^2 - 1$

$x^2 = \frac{1}{2};\ \frac{3}{2}$

$x = \pm\frac{\sqrt2}{2};\ \pm\frac{\sqrt6}{2}$. $\pm\frac{\sqrt6}{2}, -\frac{\sqrt2}{2}$ не вход. в $(0;1) \Rightarrow x_0 = \frac{\sqrt2}{2}$

$x'(3/4) = (2\sqrt2 - \sqrt2)^{-1} = \frac{\sqrt2}{2}$

N201(3) $x = a\cos t,\ y = b\sin t,\ 0 < t < \pi \quad y'_x - ?$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = b\cos t\,(-a\sin t)^{-1} = -\frac{b}{a}\operatorname{ctg} t$

N214(2) $d\left(\operatorname{arctg}\frac{\ln x}{x}\right) = \operatorname{arctg}'_{\frac{\ln x}{x}}\, d\left(\frac{\ln x}{x}\right) =$

$= \operatorname{arctg}'_{\frac{\ln x}{x}} \left(\frac{\ln x}{x}\right)'_x dx = \frac{1}{1+\frac{\ln^2 x}{x^2}}\,\frac{1-\ln x}{x^2}\,dx =$

$= \frac{1-\ln x}{x^2+\ln^2 x}\,dx$. В т. $x_1 = \frac{1}{e}$: $\frac{2}{e^{-2}+1}dx = \frac{2e^2}{1+e^2}dx$. В т. $x_2 = e$: 0.

№ 173. α - ? $y = \begin{cases} |x|^\alpha \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ ← в т. $x=0$ 1) непр. 2) $\exists y'$ 3) $\exists$ непр. y'

1) $\lim\limits_{x\to 0} f(x) = f(0) = 0$

$\lim\limits_{x\to 0} |x|^\alpha \sin(1/x) = 0$

$\alpha > 0$: $|x|^\alpha$ - б.м., $\sin(\frac{1}{x})$ - огр. $\Rightarrow |x|^\alpha \sin(\frac{1}{x})$ - б.м.

$\alpha = 0$: $\nexists \lim\limits_{x\to 0} \sin(\frac{1}{x})$. $\square$ можно взять послед-ть Гейне $x'_n = \frac{1}{\pi n}$ и $x''_n = \frac{1}{\pi/2 + \pi n}$, $n \in \mathbb{N}$. $\lim\limits_{n\to\infty} f(x'_n) \neq \lim\limits_{n\to\infty} f(x''_n)$ $\blacksquare$

$\alpha < 0$: $|x|^\alpha$ - б.б., берём те же послед-ти получаем, что $\nexists$.

2) $f'(0) = \lim\limits_{\Delta x\to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x} = \Big| \lim\limits_{\substack{x\to 0 \\ \Delta x := x}} \frac{f(x)}{x} = |x|^{\alpha-1} \cdot \operatorname{sgn}(x) \cdot$
$\cdot \sin(\frac{1}{x})$. Аналогично 1) при $\alpha := \alpha - 1 \Rightarrow \alpha - 1 > 0$

3) $\lim\limits_{x\to 0} f'(x) = f'(0) = 0$

$\forall x \neq 0 \mapsto f'(x) = \begin{cases} (x^\alpha \sin\frac{1}{x})' = \alpha x^{\alpha-1} \sin\frac{1}{x} - x^{\alpha-2}\cos\frac{1}{x}, & x > 0 \\ ((-x)^\alpha \sin\frac{1}{x})' = \alpha(-x)^{\alpha-1}\sin\frac{1}{x} - (-x)^{\alpha-2}\cos\frac{1}{x}, & x < 0 \end{cases}$

$\lim\limits_{x\to 0-0} f'(x) = \lim\limits_{x\to 0-0} \alpha(-x)^{\alpha-1}\sin\frac{1}{x} - \lim\limits_{x\to 0+0} (-x)^{\alpha-2}\cos\frac{1}{x}$. т.к. для $\exists f'$ необходимо $\alpha > 1$ (2) $\Rightarrow \alpha(-x)^{\alpha-1}\sin\frac{1}{x}$ - б.м. на огр. - б.м. $\Rightarrow$

$\Rightarrow \lim\limits_{x\to 0+0} f'(x) = -\lim\limits_{x\to 0+0} (-x)^{\alpha-2}\cos\frac{1}{x} = 0 \Rightarrow \alpha > 2$ (аналогично (2))

Аналогично $\lim\limits_{x\to 0+0} f'(x) = 0 \Leftrightarrow \alpha > 2$.

1) $\alpha > 0$; 2) $\alpha > 1$; 3) $\alpha > 2$.

№10(3) §14

$y^4 - 4x^4 - 6xy = 0$, $M(1; 2)$

$$4y^3y' - 16x^3 - 6y - 6xy' = 0$$

в т. $M(1;2)$: $32y'(x_M) - 16 - 12 - 6y'(x_M) = 0$

$$26y'(x_M) = 28 \Rightarrow y'(x_M) = \frac{14}{13}$$

касат.: $y = y(x_M) + y'(x_M)(x - x_M) = 2 + \frac{14}{13}(x-1) = \frac{14}{13}x + \frac{12}{13}$

нормаль: $y = y(x_M) - \frac{1}{y'(x_M)}(x - x_M) = 2 - \frac{13}{14}(x-1) = -\frac{13}{14}x + \frac{41}{14}$

т.е. кас.: $14x - 13y + 12 = 0$, норм.: $13x + 14y - 41 = 0$

§15 №1(6) $y = \ln\left(x + \sqrt{x^2+1}\right)$, y'' – ?

$$y' = \frac{f'}{f},\ f = x + \sqrt{x^2+1},\ f' = 1 + \frac{x}{\sqrt{x^2+1}},\ f'' = \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} = (x^2+1)^{-\frac{3}{2}}$$

$$y'' = \frac{f''f - (f')^2}{f^2} = \frac{f''}{f} - \left(\frac{f'}{f}\right)^2 = \frac{(x^2+1)^{-3/2}}{x+(x^2+1)^{1/2}} - \left(\frac{1 + x(x^2+1)^{-1/2}}{x + (x^2+1)^{1/2}}\right)^2 =$$

$$= \left((x^2+1)^{1/2} - x\right)(x^2+1)^{-3/2} - \frac{1 + 2x(x^2+1)^{-1/2} + x^2(x^2+1)^{-1}}{x^2 + 2x(x^2+1)^{1/2} + x^2 + 1} =$$

$$= \left((x^2+1)^{1/2} - x\right)(x^2+1)^{-3/2} - (x^2+1)^{-1}\,\frac{x^2+1+2x(x^2+1)^{1/2}+x^2}{x^2+2x(x^2+1)^{1/2}+x^2+1} =$$

$$= \left((x^2+1)^{1/2} - x\right)(x^2+1)^{-3/2} - (x^2+1)^{-1} = -x(x^2+1)^{-3/2}$$

№10(4) $y = \operatorname{arctg}\frac{2+x^2}{2-x^2}$, $x = 0$ d^2y – ?

$$y' = \frac{1}{1+\left(\frac{2+x^2}{2-x^2}\right)^2}\cdot\frac{2x(2-x^2)+2x(2+x^2)}{(2-x^2)^2} = \frac{8x}{(2-x^2)^2+(2+x^2)^2} =$$

$$= \frac{8x}{8 + 2x^2} = \frac{4x}{4+x^2}$$

$$y'' = \frac{4(4+x^2) - 2x\cdot 4x}{(4+x^2)^2} = \frac{16-4x^2}{(4+x^2)^2} \quad y''(0) = 1 \Rightarrow d^2y = dx^2$$

№13(2) $y = u(2+v)$ $\quad d^2y - ?$

$$y'' = 2u'' + (uv)'' = 2u'' + (u'v+v'u)' = 2u'' + u''v + 2u'v' + uv'' = \frac{d^2u}{dx^2}(2+v) + 2\frac{du\,dv}{d^2x} + u\frac{d^2v}{dx^2}$$

$$d^2y = y''dx^2 = d^2u(2+v) + 2du\,dv + u\,d^2v$$

№14(7) $x = \frac{e^t}{1+t}$, $y = (t-1)e^t$. $\frac{d^2y}{dx^2} - ?$

$$y'_x = y'_t t'_x = (e^t + (t-1)e^t)\frac{1}{x'_t} = te^t\frac{(1+t)^2}{e^t(1+t) - e^t} = (1+t)^2$$

$$y''_x = (y'_x)'_t t'_x = 2(1+t)\frac{(1+t)^2}{te^t} = \frac{2(1+t)^3}{te^t}$$

№22(4) d^2y в т. $(1;0)$ $y = y(x)$

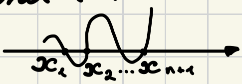
$$F(x,y) = 3(y-x+1) + \operatorname{arctg}(y/x) = 0$$

$$dF(x,y) = 3(dy-dx) + \frac{x\,dy - y\,dx}{x^2+y^2} = \left(3+\frac{x}{x^2+y^2}\right)dy - \left(3+\frac{y}{x^2+y^2}\right)dx = 0 \Rightarrow dy = \frac{3x^2+3y^2+y}{3x^2+3y^2+x}dx.$$ В т. $(1,0)$ $3x^2+3y^2+y=3$

$3x^2+3y^2+x=4 \Rightarrow dy = \frac{3}{4}dx.$ $\quad$ в т. $(1,0)$

$$d^2y = d\left(\frac{3x^2+3y^2+y}{3x^2+3y^2+x}\right)dx = \frac{4(6x\,dx+6y\,dy+dy) - 3(6x\,dx+6y\,dy+dx)}{16}dx =$$

$$= \frac{6x\,dx+6y\,dy+4dy-3dx}{16}dx = \frac{6\cdot 1dx+0+3dx-3dx}{16}dx = \frac{3}{8}dx^2$$

§16 №5. Мат. индукция: база - верна по Т. Ролля.
$\exists$ верно для $n-1$
шаг: 

применим Т. Ролля ко всем

(x_i, x_{i+1}), $1 \le i \le n$. Получим, что $\forall (x_i, x_{i+1})\ \exists\ c_i \in$

$\in (x_i, x_{i+1}) : f'(c_i) = 0 \Rightarrow$ т.к. отрезков n штук f' имеет $\geq n$ нулей. Т.к. f дифф n раз, то f' дифф. $n-1$ раз $\Rightarrow$ по предположению индукции $\exists\, \xi : (f')^{(n-1)}(\xi) = f^{(n)}(\xi) = 0$ ■

№15(4) Т. Лагранжа для $f = e^t$ на $(0; x-1)$ $(x > 1)$

$\exists\, \xi \in (0; x-1) : \dfrac{e^{x-1} - e^0}{x-1-0} = f'(\xi) = e^{\xi} > 1$, т.к. $\xi > 0 \Rightarrow$

$\Rightarrow e^{x-1} - 1 > x - 1 \Leftrightarrow e^{x-1} > x \Leftrightarrow e^x > ex$ ■

№19. $\exists\, f'(x)$ огр., т.е. $\exists\, M : |f'(x)| < M \ \forall x \in (a,b)$

Рассм f на $(a, \frac{a+b}{2})$, $(\frac{a+b}{2}, b)$. В $\frac{a+b}{2}$ f огр., т.к. f - дифф.

$\forall x \in (a, \frac{a+b}{2})$ по Т. Лагранжа для f на $(x, \frac{a+b}{2})$

$\exists\, \xi \in (a, \frac{a+b}{2}) : f'(\xi) = \dfrac{f(\frac{a+b}{2}) - f(x)}{\frac{a+b}{2} - x} \Rightarrow f(x) =$

$= f(\frac{a+b}{2}) - f'(\xi)(\frac{a+b}{2} - x)$.

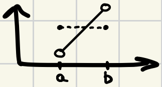
Тогда $|f(x)| = |f(\frac{a+b}{2}) - f'(\xi)(\frac{a+b}{2} - x)| < |f(\frac{a+b}{2})| +$

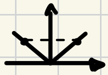
$+ |f'(\xi)(\frac{a+b}{2} - x)| < |f(\frac{a+b}{2})| + |f'(\xi)|\left(|\frac{a+b}{2}| + |x|\right) <$

$< |f(\frac{a+b}{2})| + M\left(|\frac{a+b}{2}| + \max(|a|, |\frac{a+b}{2}|)\right) =$

$= C$, т.е. $\exists\, C : \forall x \ |f(x)| < C \Rightarrow f(x)$ огр. на $(a, \frac{a+b}{2})$. Аналогично док-во, что $f(x)$ огр. на $(\frac{a+b}{2}, b)$

$f(x)$ огр. в $\frac{a+b}{2} \Rightarrow f(x)$ огр. на (a,b) $\overset{\circ}{X}! \Rightarrow f'$ неогр. на (a,b).

№33. а) если f прерывна на $[a,b]$, то возможна такая ситуация:  такой $\xi \nexists$.

б) нет, пример $-|x|$ на $[-1;1]$  такой $\xi \nexists$

в) нет, пример x на $[0;1]$ такой $\xi \nexists$

№30. Используем Т. Коши для $f(x)$ и $g(x)=\frac{1}{x}$ на отрезке $[1;2]$: $\exists \xi \in (1;2)$: $\dfrac{f(2)-f(1)}{\frac{1}{2}-\frac{1}{1}} = \dfrac{f'(\xi)}{g'(\xi)}$

$$f(2)-f(1) = -\frac{1}{2} f'(\xi) \Big/ \left(-\frac{1}{\xi^2}\right) = \frac{\xi^2}{2} f'(\xi) \quad \blacksquare$$

Т.1. Заметим, что $th'(f(x)) = \left(\dfrac{sh\, f(x)}{ch\, f(x)}\right)' =$

$$= \frac{ch^2 f(x) - sh^2 f(x)}{ch^2 f(x)} f'(x) = \frac{f'(x)}{ch^2 f(x)}$$

По Т. Лагранжа $\exists \xi \in (2024, 2028)$:

$$\frac{th(f(2028)) - th(f(2024))}{2028-2024} = th'(f(\xi))$$

$$th(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} = 1 - \frac{2e^{-t}}{e^t + e^{-t}} = 1 - \frac{2}{e^{2t}+1}$$

$\forall\, t_1, t_2$ $\quad th(t_1) - th(t_2) = 2\left(\dfrac{1}{e^{2t_2}+1} - \dfrac{1}{e^{2t_1}+1}\right) < 2 \Rightarrow$

$\Rightarrow \exists \xi \in (2024, 2028)$: $th'(f(\xi)) = \dfrac{th(f(2028)) - th(f(2024))}{4} <$

$< \dfrac{2}{4} = \dfrac{1}{2}$. Т.е. $\exists \xi \in (2024, 2028)$: $th' f(\xi) =$

$$= \frac{f'(\xi)}{ch^2 f(\xi)} < \frac{1}{2} \Rightarrow f'(\xi) < \frac{ch^2 f(\xi)}{2} < ch^2 f(\xi) \quad \blacksquare$$

§15 №24 (9, 15)

9) $y=\sin^2 x \sin 2x = \frac{1-\cos 2x}{2}\sin 2x = \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x$

$$y^{(n)} = \frac{1}{2}\sin^{(n)} 2x - \frac{1}{4}\sin^{(n)} 4x = 2^{n-1}\sin\left(2x+\frac{\pi n}{2}\right) - 4^{n-1}\sin\left(4x+\frac{\pi n}{2}\right)$$

15) $y=\frac{3-2x^2}{2x^2+3x-2} = \frac{-2x^2-3x+2+3x+1}{2x^2+3x-2} = -1 + \frac{3x+1}{(x+2)(2x-1)} = -1+\frac{x+2+2x-1}{(x+2)(2x-1)} = -1+\frac{1}{2x-1}+\frac{1}{x+2}$

$$y^{(n)} = (-1)^n n!\left(2^n(2x-1)^{-1-n} + (x+2)^{-1-n}\right)$$

№25 (3, 7, 10)

3) $y=(3-2x)^2 e^{2-3x}$

$$y^{(n)}(x) = \sum_{k=0}^{n} C_n^k \left((3-2x)^2\right)^{(k)} \left(e^{2-3x}\right)^{(n-k)}$$

$\left((3-2x)^2\right)^{(k)} = 0,\ k>2$ $\Rightarrow$

$$\Rightarrow y^{(n)}(x) = C_n^0 (3-2x)^2 \left(e^{2-3x}\right)^{(n)} + C_n^1 \cdot (-2)2(3-2x) \cdot \left(e^{2-3x}\right)^{(n-1)} + C_n^2 \cdot 8\left(e^{2-3x}\right)^{(n-2)} = (3-2x)^2(-3)^n \cdot e^{2-3x} - 4n(3-2x)(-3)^{n-1}e^{2-3x} + 4n(n-1)(-3)^{n-2} \cdot e^{2-3x} = (-3)^{n-2}e^{2-3x}\left(9(9-12x+4x^2)+12n(3-2x)+4n(n-1)\right) = (-3)^{n-2}e^{2-3x}\left(81-108x+36x^2+36n-24nx+4n^2-4n\right) = (-3)^{n-2}e^{2-3x}\left(36x^2-12(9+2n)x+4n^2+32n+81\right)$$

7) $y = x\ln(x^2-3x+2)$

$$\left.\begin{aligned} & y^{(n)} = \sum_{k=0}^{n} C_n^k x^{(k)} \left(\ln(x^2-3x+2)\right)^{(n-k)} \\ & x^{(k)} = 0,\ k>1 \end{aligned}\right| \Rightarrow$$

$$\Rightarrow y^{(n)} = x\left(\ln(x^2-3x+2)\right)^{(n)} + n\left(\ln(x^2-3x+2)\right)^{(n-1)}$$

$$\left(\ln(x^2-3x+2)\right)' = \frac{2x-3}{(x^2-3x+2)} = \frac{x-2+x-1}{(x-2)(x-1)} =$$

$$= \frac{1}{x-1} + \frac{1}{x-2}$$

$$\left(\ln(x^2-3x+2)\right)^{(n-1+1)} = \left(\left(\ln(x^2-3x+2)\right)'\right)^{(n-1)} =$$

$$= \left(\frac{1}{x-1}\right)^{(n-1)} + \left(\frac{1}{x-2}\right)^{(n-1)} = (-1)^{n-1}(n-1)!\left((x-1)^{-n} + (x-2)^{-n}\right)$$

Итак, $y^{(n)} = x(-1)^{n-1}(n-1)!\left((x-1)^{-n}+(x-2)^{-n}\right) +$

$+ (-1)^{n-2}n(n-2)!\left((x-1)^{1-n} + (x-2)^{1-n}\right) = (-1)^{n-2}(n-2)!\cdot$

$\cdot\left(-x(n-1)(x-1)^{-n} - x(n-1)(x-2)^{-n} + n(x-1)^{1-n} + n(x-2)^{1-n}\right) =$

$= (-1)^{n-2}(n-2)!\left((-nx+x+nx-n)(x-1)^{-n} + (-xn+x+nx-2n)(x-2)^{-n}\right) =$

$= (-1)^{n-2}(n-2)!\left((x-n)(x-1)^{-n} + (x-2n)(x-2)^{-n}\right), n>1$

$= \ln(x^2-3x+2) + \dfrac{x(2x-3)}{x^2-3x+2}$, $n=1$

10) $y = (x^2+x)\cos^2 x = (x^2+x)\dfrac{1+\cos 2x}{2}$

$$\left.\begin{aligned} & y^{(n)} = \sum_{k=0}^{n} C_n^k (x^2+x)^{(k)} \left(\frac{1+\cos 2x}{2}\right)^{(n-k)} \\ & (x^2+x)^{(k)} = 0,\ k>2 \end{aligned}\right| \Rightarrow$$

$$\Rightarrow y^{(n)} = (x^2+x)\left(\frac{1+\cos 2x}{2}\right)^{(n)} + n(2x+1)\left(\frac{1+\cos 2x}{2}\right)^{(n-1)} +$$

$$+ n(n-1)\left(\frac{1+\cos 2x}{2}\right)^{(n-2)} = (x^2+x)\, 2^{n-1} \cos\left(2x+\frac{\pi n}{2}\right) +$$

$$+ n(2x+1)\, 2^{n-2} \cos\left(2x+\frac{\pi(n-1)}{2}\right) + 2^{n-3} n(n-1) \cos\left(2x+\frac{\pi(n-2)}{2}\right)$$

$$= 2^{n-3}\left(\left(4x^2+4x-n^2+n\right)\cos\left(2x+\frac{\pi n}{2}\right) + 2n(2x+1)\cdot\right.$$

$$\left.\cdot \sin\left(2x+\frac{\pi n}{2}\right)\right), \quad n>2$$

$$= 1+\cos 2x - (2x+1)\sin(2x) - 2(x^2+x)\cos 2x, \quad n=2$$

$$= (2x+1)\left(\frac{1+\cos 2x}{2}\right) - (x^2+x)\sin 2x, \quad n=1$$

№26(2) $y = \dfrac{x^2}{\sqrt{1-2x}}$

$$y^{(n)} = \sum C_n^k (x^2)^{(k)} \left((1-2x)^{-1/2}\right)^{(n-k)} \quad \Bigg| \quad \Rightarrow y^{(n)} =$$

$$(x^2)^{(k)} = 0, \quad k>2$$

$$= x^2\left((1-2x)^{-1/2}\right)^{(n)} + 2x\left((1-2x)^{-1/2}\right)^{(n-1)} + 2\cdot$$

$$\cdot\left((1-2x)^{-1/2}\right)^{(n-2)} = x^2(-2)^n\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots(1-2x)^{-1/2-n} +$$

$$+ 2x(-2)^{n-1}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots(1-2x)^{1/2-n} + (-2)^{n-2}\left(-\frac{1}{2}\right)\cdot$$

$$\cdot\left(-\frac{3}{2}\right)\ldots(1-2x)^{3/2-n} = x^2(2n-1)!!\,(1-2x)^{-1/2-n} +$$

$$+ 2x(2n-3)!!\,(1-2x)^{1/2-n} + (2n-5)!!\,(1-2x)^{3/2-n} =$$

$$= (2n-5)!!\,(1-2x)^{-1/2-n}\left(x^2(2n-1)(2n-3) + 2x(2n-3)\cdot\right.$$

$$\left.\cdot(1-2x) + (1-2x)^2\right) = (2n-5)!!\,(1-2x)^{-1/2-n}\left(x^2(4n^2-8n+\right.$$

$$\left.+3-8n+12+4) + (4n-6-4)x+1\right) = (2n-5)!!\,(1-2x)^{-1/2-n}\cdot$$

$\cdot\left((4n^2-16n+19)x^2+(4n-10)x+1\right),\ n>2$

$$= \frac{3x^2-4x+2}{(1-2x)^{5/2}},\quad n=2$$

$$= \frac{2x-3x^2}{(1-2x)^{3/2}},\quad n=1$$

§9 №50(1,2)

1) $x^2=\bar{o}(x) \Rightarrow \lim\limits_{x\to x_0}\frac{x^2}{x}=\lim\limits_{x\to x_0} x=0 \Rightarrow x_0=0 \Rightarrow$

$\Rightarrow\ x^2=\bar{o}(x),\ x\to 0$

2) $x=\bar{o}(x^2) \Rightarrow \lim\limits_{x\to x_0}\frac{x}{x^2}=\lim\limits_{x\to x_0}\frac{1}{x}=0 \Rightarrow x_0=\infty \Rightarrow$

$\Rightarrow\ x=\bar{o}(x^2),\ x\to\infty$

№51(1) $x\to 0,\ n\in\mathbb{N},\ k\in\mathbb{N},\ n\geq k$

1) $\bar{o}(x^n)+\bar{o}(x^k)=\bar{o}(x^k)$ - Показать

Рассм. $\lim\limits_{x\to 0}\frac{\bar{o}(x^n)+\bar{o}(x^k)}{x^k}=\lim\limits_{x\to 0}\bar{o}(x^{n-k})+\bar{o}(1)=$

$=\lim\limits_{x\to 0}\bar{o}(x^{n-k})$, т.к. $n-k\geq 0$, то $\lim\limits_{x\to 0}\bar{o}(x^{n-k})=0 \Rightarrow$

$\Rightarrow\ \bar{o}(x^n)+\bar{o}(x^k)=\bar{o}(x^k)$

Т.3 Д-ть: при $x\to 0$ верно $f(x)=\bar{o}(g(x))$ и

$g(x)\sim h(x) \Rightarrow f(x)=\bar{o}(h(x)),\ x\to 0.$

$g(x)\sim h(x) \Rightarrow h(x)=g(x)+\bar{o}(g(x)),\ x\to 0$

Рассмотрим $\lim\limits_{x\to 0}\frac{f(x)}{h(x)}=\lim\limits_{x\to 0}\frac{f(x)}{g(x)+\bar{o}(g(x))}=$

$=\lim\limits_{x\to 0}\frac{f(x)/g(x)}{1+\bar{o}(g(x))/g(x)}=\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=0$, т.к. $f(x)=\bar{o}(g(x)) \Rightarrow$

$\Rightarrow f(x)=\bar{o}(h(x)),\ x\to 0$

T.4. $x\to 0$

$(2x-3x^4+\bar o(x^4))(1-x+2x^2-x^3+\bar o(x^3)) =$

$=2x-2x^2+4x^3-2x^4-3x^4+\bar o(x^4)=2x-2x^2+4x^3-5x^4+$

$+\bar o(x^4),\ x\to 0$

§18 №2(6)

6) $\ln\frac{1+2x}{1-x} = \ln|1+2x| - \ln|1-x| = 2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}+\ldots+$

$+\frac{(-1)^{n-1}(2x)^n}{n} - \left(-x-\frac{(-x)^2}{2}-\frac{(-x)^3}{3}-\ldots+\frac{(-1)^{2n-1}x^n}{n}\right)+\bar o(x^n)=$

$=x(2+1)-x^2\left(\frac{2^2}{2}-\frac12\right)+x^3\left(\frac{2^3}{3}+\frac13\right)-x^4\left(\frac{2^4}{4}-\frac14\right)+\ldots+$

$+x^n\left(\frac{2^n}{n}+(-1)^{n-1}\frac1n\right)+\bar o(x^n)=\sum_{k=1}^n\left(\frac{(-1)^{k-1}2^k+1}{k}\right)x^k+\bar o(x^n),$

$x\to 0$

№3(5)

5) $\frac{x}{\sqrt[3]{9-6x+x^2}} = x(3-x)^{-\frac23} = 3^{-\frac23}x\left(1-\frac{x}{3}\right)^{-\frac23}=$

$=3^{-\frac23}x\left(\sum_{k=0}^{n-1}C^k_{-\frac23}\left(-\frac x3\right)^k+\bar o(x^{n-1})\right)=3^{-\frac23}\sum_{k=0}^{n-1}C^k_{-\frac23}(-1)^k3^{-k}x^{k+1}+$

$+\bar o(x^n)=\sum_{k=1}^nC^{k-1}_{-\frac23}(-1)^{k-1}3^{\frac13-k}x^k+\bar o(x^n),\ x\to 0$

№4(7)

7) $\frac{x^2+4x-1}{x^2+2x-3}=1+\frac{2x+2}{x^2+2x-3}=1+\frac{2x-2+4}{(x-1)(x+3)}=1+\frac{2}{x+3}+$

$+4\cdot\frac14\left(\frac1{x-1}-\frac1{x+3}\right)=1+\frac1{x-1}+\frac1{x+3}=1-\sum_{k=0}^n x^k+3\sum_{k=0}^n\frac{(-1)^k}{3^k}x^k+$

$+\bar o(x^n)=1+\sum_{k=0}^n x^k\left[\frac{(-1)^k}{3^{k-1}}-1\right]+\bar o(x^n),\ x\to 0$

№5(3)

$$3)\ x\sin^2 2x = x\,\frac{1-\cos 4x}{2} = \frac{x}{2}\left(1-\sum_{k=0}^{n-1}(-1)^k\frac{(4x)^{2k}}{(2k)!}+\bar{o}(x^{2n-1})\right)=$$

$$=\frac{x}{2}-\frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{4^{2k}}{(2k)!}x^{2k+1}+\bar{o}(x^{2n}) = \frac{x}{2}-\frac{x}{2}+\sum_{k=1}^{n-1}(-1)^k\frac{2^{4k-1}}{(2k)!}x^{2k+1}+$$

$$+\bar{o}(x^{2n}) = \sum_{k=1}^{n-1}\frac{(-1)^{k-1}2^{4k-1}}{(2k)!}x^{2k+1}+\bar{o}(x^{2n}),\ x\to 0$$

№2(4)

$$4)\ (2x+1)\sqrt{1-x} = (2x+1)\left[\sum_{k=0}^{n} C_{\frac{1}{2}}^{k}(-1)^k x^k+\bar{o}(x^n)\right]=$$

$$=\sum_{k=0}^{n-1} C_{\frac{1}{2}}^{k}(-1)^k x^k\cdot 2x+\sum_{k=0}^{n} C_{\frac{1}{2}}^{k}(-1)^k x^k+\bar{o}(x^n)=$$

$$=\sum_{k=1}^{n} C_{\frac{1}{2}}^{k-1}(-1)^{k-1}2x^k+\sum_{k=0}^{n} C_{\frac{1}{2}}^{k}(-1)^k x^k+\bar{o}(x^n)=$$

$$=1+\sum_{k=1}^{n}(-1)^k\left[2C_{\frac{1}{2}}^{k}-C_{\frac{1}{2}}^{k-1}\right]x^k+\bar{o}(x^n)\ \circled{=}$$

$$2C_{1/2}^{k}-C_{1/2}^{k-1} = \frac{2\cdot 1/2\,(1/2-1)(1/2-2)\ldots(1/2-k)}{k!}-$$

$$-\frac{k/2\,(1/2-1)\ldots(1/2-k+1)}{k\,(k-1)!} = \frac{1}{k!}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-k+1\right)\cdot$$

$$\cdot\left[(1/2-k)-\frac{k}{2}\right] = 2C_{1/2}^{k}\left[\frac{1}{2}-\frac{3k}{2}\right]\frac{1}{\frac{1}{2}-k} = C_{1/2}^{k}\,\frac{1-3k}{\frac{1}{2}-k}=$$

$$=C_{1/2}^{k}\,\frac{2-6k}{1-2k}\ \Rightarrow\ \circled{=}\ 1+\sum_{k=1}^{n}(-1)^k C_{1/2}^{k}\frac{2-6k}{1-2k}x^k+\bar{o}(x^n),\ x\to 0$$

№14(3)

$$3)\ x_0=-2,\ x\ln(2-3x+x^2).\ t:=x-x_0=x+2\Rightarrow$$

$$\Rightarrow x\ln(2-3x+x^2) = (t-2)\ln(2-3t+6+t^2-4t+4) = (t-2)\cdot$$

в окрестности $t_0=0$

$$\ln(12-7t+t^2) = (t-2)\left[\ln(6-t)+\ln(2-t)\right] =$$

$$= (t-2)\left[\ln 6 - \sum_{k=1}^{n} \frac{t^k}{6^k k} + \ln 2 - \sum_{k=1}^{n} \frac{t^k}{2^k k} + \bar{o}(t^n)\right] =$$

$$= (t-2)\ln 12 - \sum_{k=1}^{n} \frac{t^k}{2^k k}\left[\frac{1}{3^k}+1\right] + \bar{o}(t^n) =$$

$$= x\ln 12 - \sum_{k=1}^{n} \frac{(x+2)^k}{2^k k}\,\frac{1+3^k}{3^k} + \bar{o}((x+2)^n) =$$

$$= x\ln 12 - \sum_{k=1}^{n} \frac{(1+3^k)}{6^k k}(x+2)^k + \bar{o}((x+2)^n) =$$

$$= x\ln 12 - \frac{4}{6}(x+2) - \sum_{k=2}^{n} \frac{(1+3^k)}{6^k k}(x+2)^k + \bar{o}((x+2)^n) =$$

$$= -\frac{4}{3} + \left(\ln 12 - \frac{2}{3}\right)x - \sum_{k=2}^{n} \frac{(1+3^k)}{6^k k}(x-2)^k + \bar{o}((x+2)^n),\quad x\to -2$$

№20(б) x_0 до $\bar{o}((x-x_0)^{2n+1})$

б) $x_0=1$, $x(x-2)2^{x^2-2x-1}$. $t := x-x_0 = x-1 \Rightarrow$

$$\Rightarrow (t^2-1)2^{t^2-2} = \frac{1}{4}(t^2-1)2^{t^2} = \frac{1}{4}t^2 2^{t^2} - \frac{1}{4}2^{t^2} =$$

$$= \frac{1}{4}t^2 e^{t^2\ln 2} - \frac{1}{4}e^{t^2\ln 2} = \frac{1}{4}t^2 \sum_{k=0}^{n-1} \frac{t^{2k}(\ln 2)^k}{k!} - \frac{1}{4}\sum_{k=0}^{n} \frac{t^{2k}(\ln 2)^k}{k!} +$$

$$+ \bar{o}(t^{2n+1}) = \frac{1}{4}t^2 \sum_{k=1}^{n} \frac{t^{2k-2}(\ln 2)^{k-1}}{(k-1)!} - \frac{1}{4}\sum_{k=1}^{n} \frac{t^{2k}(\ln 2)^k}{k!} -$$

$$- \frac{1}{4} + \bar{o}(t^{2n+1}) = -\frac{1}{4} + \frac{1}{4}\sum_{k=1}^{n}\left[\frac{(\ln 2)^{k-1}}{(k-1)!} - \frac{(\ln 2)^k}{k!}\right]t^{2k} + \bar{o}(t^{2n+1})$$

$$= -\frac{1}{4} + \frac{1}{4}\sum_{k=1}^{n} \frac{(\ln 2)^{k-1}}{k!}(k-\ln 2)(x-1)^{2k} + \bar{o}((x-1)^{2n+1}),\quad x\to 1$$

№30(1)

1) $x^3|x| + \cos^2 x =$

$f(x) = x^3|x|$

$f' = 4x^3 \operatorname{sgn} x,\ x \neq 0.\quad \lim\limits_{x\to 0} \dfrac{x^3|x|}{x} = 0\ ,\ x = 0$

$f'' = 12x^2 \operatorname{sgn} x,\ x \neq 0\quad \lim\limits_{x\to 0} \dfrac{4x^2|x|}{x} = 0\ ,\ x = 0$

$f''' = 24x \operatorname{sgn} x,\ x \neq 0\quad \lim\limits_{x\to 0} \dfrac{12x|x|}{x} = 0,\ x = 0$

$f'''' = 24 \operatorname{sgn} x,\ x \neq 0\quad \lim\limits_{x\to 0} \dfrac{24|x|}{x}$ – не сущ. $\Rightarrow n = 3.$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x = \frac{1}{2} + \frac{1}{2}\left(1 - 2x^2 + \overline{o}(x^3)\right) = 1 - x^2 + \overline{o}(x^3)$

$= 1 - x^2 + \overline{o}(x^3),\ x \to 0$

№38(6) $\overline{o}(x^4)$

6) $\sqrt{\cos x} = \sqrt{1 + (\cos x - 1)} = \sum\limits_{k=0}^{n} C_{\frac{1}{2}}^{k} (\cos x - 1)^k + \overline{o}((\cos x - 1)^n) = \sum\limits_{k=0}^{2} C_{\frac{1}{2}}^{k} \left(\frac{x^4}{4!} - \frac{x^2}{2!}\right)^k + \overline{o}\left(\left(\frac{x^4}{4!} - \frac{x^2}{2!}\right)^2\right) =$

$= 1 + \frac{1}{2}\left(\frac{x^4}{4!} - \frac{x^2}{2!}\right) - \frac{1}{8}\frac{x^4}{(2!)^2} + \overline{o}(x^4) = 1 - \frac{x^2}{4} - \frac{x^4}{96} + \overline{o}(x^4),\ x \to 0$

№39(7) $\overline{o}(x^5)$

7) $(1+x)^{\sin x} = e^{\ln(1+x)\sin x} =$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \overline{o}(x^4)$

$\sin x = x - \frac{x^3}{3!} + \overline{o}(x^4)$

$\ln(1+x)\sin x = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} - \frac{x^4}{6} + \frac{x^5}{12} + \overline{o}(x^5) =$

$= x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{6}$

$$= 1+x^2-\frac{x^3}{2}+\frac{x^4}{6}-\frac{x^5}{6}+\frac{(x^2-x^3/2+x^4/6-x^5/6)^2}{2}+$$
$$+\overline{o}(x^5)=1+x^2-\frac{x^3}{2}+\frac{2}{3}x^4-\frac{2}{3}x^5+\overline{o}(x^5),\ x\to 0$$

Т.5. До $\overline{o}(x^6),\ x\to 0$

а) $y=\operatorname{tg} x=\dfrac{\sin x}{\cos x} =$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\overline{o}(x^6),\ x\to 0$$
$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\overline{o}(x^5),\ x\to 0$$
$$\frac{1}{\cos x}=\frac{1}{1-(1-\cos x)}=1+(1-\cos x)+(1-\cos x)^2+$$
$$+\overline{o}((1-\cos x)^2)=1+\frac{x^2}{2!}-\frac{x^4}{4!}+\left(\frac{x^2}{2!}-\frac{x^4}{4!}\right)^2+\overline{o}(x^5)=$$
$$=1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^4}{4}+\overline{o}(x^5)=1+\frac{x^2}{2!}+\frac{5x^4}{24}+\overline{o}(x^5),\ x\to 0$$
$$=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\overline{o}(x^6)\right)\left(1+\frac{x^2}{2!}+\frac{5x^4}{24}+\overline{o}(x^5)\right)=x+\frac{x^3}{2!}+\frac{5x^5}{24}-$$
$$-\frac{x^3}{3!}+-\frac{x^5}{12}+\frac{x^5}{120}=x+\frac{x^3}{3}+\frac{2x^5}{15}+\overline{o}(x^6),\ x\to 0$$

б) $y=\operatorname{arctg} x$

$$y'=\operatorname{arctg}' x=\frac{1}{1+x^2}=\sum_{k=0}^{2}(-1)^k x^{2k}+\overline{o}(x^5),\ x\to 0$$
$$y=\int y'\,dx+C=C+\sum_{k=0}^{2}(-1)^k\int x^{2k}dx+\int\overline{o}(x^5)dx=$$
$$=C+\sum_{k=0}^{2}(-1)^k\frac{x^{2k+1}}{2k+1}+\overline{o}(x^6),\ x\to 0$$
$$y(0)=\operatorname{arctg}(0)=0 \Rightarrow C=0$$
$$\operatorname{arctg} x = x-\frac{x^3}{3}+\frac{x^5}{5}+\overline{o}(x^6),\ x\to 0$$

в) $y=\arcsin x$

$$y' = \frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{2} \frac{(2k-1)!!}{2^k k!} x^{2k} + \bar{o}(x^4), \ x \to 0$$

$$y = \int y' dx + \arcsin(0) = x + \sum_{k=1}^{2} \frac{(2k-1)!!}{2^k k!} \int x^{2k} dx + \bar{o}(x^6) =$$

$$= x + \frac{x^3}{6} + \frac{3x^5}{40} + \bar{o}(x^6), \ x \to 0$$

2) $$y = th\, x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$e^{2x} - 1 = 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \frac{(2x)^4}{4} + \frac{(2x)^5}{5} + \frac{(2x)^6}{6} + \bar{o}(x^6), \ x \to 0$$

$$\frac{1}{e^{2x}+1} = \frac{1}{2 + (e^{2x} - 1)} = \frac{1/2}{1 + (\frac{e^{2x}-1}{2})} = \frac{1}{2} \sum_{k=0}^{2} (-1)^k \cdot$$

$$\cdot \left(x + x^2 + \frac{4x^3}{3} + 2x^4 + \frac{16}{5}x^5 + \frac{16}{3}x^6\right)^k + \bar{o}(x^6) =$$

$$= \frac{1}{2} - \frac{1}{2}\left(x + x^2 + \frac{4x^3}{3} + 2x^4 + \frac{16}{5}x^5 + \frac{16}{3}x^6\right) + \frac{1}{2} \cdot$$

$$\cdot \left(x^2 + x^4 + \frac{16}{9}x^6 + 2\left(x^3 + \frac{4x^4}{3} + 2x^5 + \frac{16}{5}x^6 + \frac{4}{3}x^5 + 2x^6\right)\right) +$$

$$+ \bar{o}(x^6) = \frac{1}{2} - \frac{1}{2}x + \frac{1}{6}x^3 - \frac{1}{15}x^5 + \bar{o}(x^6)$$

$$th\, x = (e^{2x} - 1)\frac{1}{e^{2x}+1} = \left(2x + 2x^2 + \frac{8x^3}{3} + 4x^4 + \frac{32}{5}x^5 +\right.$$

$$\left. + \frac{32}{3}x^6\right)\left(\frac{1}{2} - \frac{1}{2}x + \frac{1}{6}x^3 - \frac{1}{15}x^5\right) + \bar{o}(x^6) =$$

$$= x - \frac{x^3}{3} + \frac{2x^5}{15} + \bar{o}(x^6)$$

§17 №32.

$$\lim_{x\to 1} \frac{x^{50}-50x+49}{x^{100}-100x+99} = \lim_{x\to 1} \frac{50x^{49}-50}{100x^{99}-100} = \lim_{x\to 1} \frac{50\cdot 49x^{48}}{100\cdot 99\cdot x^{98}} =$$

$$= \frac{50\cdot 49}{100\cdot 99} = \frac{49}{198}$$

№49. Докажем по индукции, что искомый предел равен 0.

База: $\lim\limits_{x\to+\infty} \frac{x^m}{e^{-x^3}}$, $\quad n \underset{3}{\equiv} m,\ 0<m<3$

Предположение: $\lim\limits_{x\to+\infty} \frac{x^{n-3}}{e^{-x^3}} = 0$

Шаг: $\lim\limits_{x\to+\infty} x^n e^{-x^3} = \lim\limits_{x\to+\infty} \frac{x^n}{e^{x^3}} = \lim\limits_{x\to+\infty} \frac{n x^{n-1}}{3x^2 e^{x^3}} =$

$$= \lim_{x\to+\infty} \frac{n x^{n-3}}{3e^{x^3}} = \frac{n}{3} \lim_{x\to+\infty} \frac{x^{n-3}}{e^{x^3}} = 0 \ \blacksquare$$

№63.

$$\lim_{x\to+\infty} \left(\frac{2}{\pi}\operatorname{arctg} x\right)^x = \lim_{x\to+\infty} e^{\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right) x} \ \text{=}$$

$$\lim_{x\to+\infty} \frac{\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right)}{1/x} = \lim_{x\to+\infty} \frac{\frac{\pi}{2\operatorname{arctg} x}\cdot\frac{\pi/2}{1+x^2}}{-1/x^2} = \lim_{x\to+\infty} \frac{-\frac{x^2}{1+x^2}}{\operatorname{arctg} x} =$$

$$= \frac{\lim\limits_{x\to+\infty}\left(-\frac{x^2}{1+x^2}\right)}{\lim\limits_{x\to+\infty}(\operatorname{arctg} x)} = \frac{\lim\limits_{x\to+\infty}\left(-\frac{x^2}{1+x^2}\right)}{\pi/2} = -\frac{2}{\pi} \Rightarrow$$

$$\Rightarrow \ = \ e^{-\frac{2}{\pi}}$$

№76.

1) $\lim\limits_{x\to\infty} \dfrac{x+\cos x}{x-\cos x} \overset{?}{=} \lim\limits_{x\to\infty} \dfrac{1-\sin x}{1+\sin x}$ — $\nexists$, т.к. можно взять две послед. Гейне: $x_n' = \left\{\frac{\pi}{2} + 2\pi n\right\}$ и $x_n'' = \{\pi n\}$. Тогда $\lim\limits_{n\to\infty} f(x_n') = \dfrac{1-1}{1+1} = 0 \neq$ $\neq \lim\limits_{n\to\infty} f(x_n'') = \dfrac{1-0}{1+0} = 1 \Rightarrow$ правило Лопиталя применять нельзя.

$$\lim_{x\to+\infty} \frac{x+\cos x}{x-\cos x} = \lim_{x\to+\infty} \frac{1+\frac{\cos x}{x}}{1-\frac{\cos x}{x}} = 1$$

2) $\lim\limits_{x\to 0} \dfrac{x^3\sin(1/x)}{\sin^2 x} = \lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} \overset{?}{=} \lim \dfrac{f'(x)}{g'(x)}$

$g'(x) = (\sin^2 x)' = \sin 2x \xrightarrow[x\to 0]{} 0 \Rightarrow$ правило Лопиталя применять нельзя.

$$\lim_{x\to 0} \frac{x^3\sin(1/x)}{\sin^2 x} = \lim_{x\to 0} \frac{x^2}{\sin^2 x}\, x\sin(1/x) =$$

$$= \lim_{x\to 0} \frac{x^2}{\sin^2 x} \lim_{x\to 0} x\sin(1/x) = 1\cdot 0 = 0$$

§19

№7(3) $\lim\limits_{x\to 0} \dfrac{\overset{1}{\ln(1+x)\cos x} - \overset{2}{e^{\operatorname{tg} x}} + \overset{3}{\sqrt{1+2x^2}}}{\underset{4}{x-\sin x}}$ $=$

1 $\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \bar{o}(x^3)\right)\left(1 - \frac{x^2}{2} + \bar{o}(x^3)\right) = x - \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^3}{3} + \bar{o}(x^3),\ x\to 0$

2 $1 + \operatorname{tg} x + \frac{\operatorname{tg}^2 x}{2} + \frac{\operatorname{tg}^3 x}{6} + \bar{o}(\operatorname{tg}^3 x) = 1 + x + \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^3}{6} + \bar{o}(x^3),\ x\to 0$

3 $1 + x^2 + \bar{o}(x^3),\ x\to 0$

Числ.: $x-\frac{x^2}{2}-\frac{x^3}{6}-\left(1+x+\frac{x^2}{2}+\frac{x^3}{2}\right)+1+x^2+\bar{o}(x^3)=$

$=-\frac{2}{3}x^3+\bar{o}(x^3),\ x\to 0$

Знам.: $x-\left(x-\frac{x^3}{6}\right)+\bar{o}(x^3)=\frac{x^3}{6}+\bar{o}(x^3),\ x\to 0$

$$=\lim_{x\to 0}\frac{-\frac{2}{3}x^3+\bar{o}(x^3)}{\frac{x^3}{6}+\bar{o}(x^3)}=-4$$

№9(б) $$\lim_{x\to 0}\frac{tg(\sin x)-\ln(x+\sqrt[3]{1+x^2})-x^2/6}{th(x-x^3)-x}=$$

(3) $th(x-x^3)-x=x-x^3-\frac{1}{3}(x-x^3)^3-x+\bar{o}(x^3)=$

$=-x^3-\frac{1}{3}x^3+\bar{o}(x^3)=-\frac{4}{3}x^3+\bar{o}(x^3),\ x\to 0$

(1) $tg(\sin x)=x-\frac{x^3}{6}+\frac{1}{3}\left(x-\frac{x^3}{6}\right)^3+\bar{o}(x^3)=x+\frac{x^3}{6}+\bar{o}(x^3)$

(2) $\ln(x+\sqrt[3]{1+x^2})+x^2/6\ \boxed{=}$

$(1+x^2)^{1/3}=1+\frac{1}{3}x^2+\bar{o}(x^3),\ x\to 0$

$\ln\left(1+x+\frac{1}{3}x^2+\bar{o}(x^3)\right)=x+\frac{1}{3}x^2+\bar{o}(x^3)-\frac{\left(x+\frac{1}{3}x^2+\bar{o}(x^3)\right)^2}{2}+$

$+\frac{1}{3}\left(x+\frac{1}{3}x^2+\bar{o}(x^3)\right)^3+\bar{o}(x^3)=x+\frac{1}{3}x^2-\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^3}{3}+\bar{o}(x^3)=$

$=x-\frac{x^2}{6}+\bar{o}(x^3),\ x\to 0$

$\boxed{=}\ x+\bar{o}(x^3),\ x\to 0$

$$=\lim_{x\to 0}\frac{x+x^3/6-x+\bar{o}(x^3)}{-\frac{4}{3}x^3+\bar{o}(x^3)}=-\frac{1}{8}$$

№14(5)

$$\lim_{x\to 0} \frac{e^{x/(1-x)} - \operatorname{sh}x - \cos x}{\sqrt[6]{1+x}+\sqrt[6]{1-x}-2} =$$

Знам.: $1+\frac{x}{6}-\frac{5}{72}x^2+1-\frac{x}{6}-\frac{5}{72}x^2-2+\bar{o}(x^2) =$

$= -\frac{5}{36}x^2+\bar{o}(x^2),\ x\to 0$

Числ.: $x\frac{1}{1-x} = x+x^2+\bar{o}(x^2),\ x\to 0$

$e^{\frac{x}{1-x}} = 1+(x+x^2)+\frac{1}{2}(x+x^2)^2+\bar{o}(x^2) = 1+x+\frac{3}{2}x^2+\bar{o}(x^2),\ x\to 0$

$e^{x/(1-x)} - \operatorname{sh}x - \cos x = 1+x+\frac{3}{2}x^2 - x - \left(1-\frac{x^2}{2}\right)+\bar{o}(x^2) =$

$= 2x^2+\bar{o}(x^2),\ x\to 0$

$= \frac{2}{-\frac{5}{36}} = -\frac{72}{5}$

№22(2) $\lim_{x\to 0}\left(\frac{\sqrt{1-2x}-\sqrt[3]{1-3x}}{\ln \operatorname{ch} x}\right)^{1/x} =$

$= \lim_{x\to 0} e^{\ln\left(\frac{\sqrt{1-2x}-\sqrt[3]{1-3x}}{\ln \operatorname{ch} x}\right)\frac{1}{x}} =$

(1) $\sqrt{1-2x}-\sqrt[3]{1-3x} = \left(1-x-\frac{1}{2}x^2-\frac{1}{2}x^3\right)-\left(1-x-x^2-\frac{5}{3}x^3\right)+$

$+\bar{o}(x^3) = \frac{1}{2}x^2+\frac{7}{6}x^3+\bar{o}(x^3),\ x\to 0$

(2) $\ln(\operatorname{ch}x) = \ln\left(1+\frac{x^2}{2}+\bar{o}(x^2)\right) = \frac{x^2}{2}+\bar{o}(x^3),\ x\to 0$

(3) $\ln\left(\frac{1/2\,x^2+7/6\,x^3+\bar{o}(x^3)}{1/2\,x^2+\bar{o}(x^3)}\right) = \ln\left(1+\frac{7}{3}x+\bar{o}(x)\right) = \frac{7}{3}x+\bar{o}(x),\ x\to 0$

$= \lim_{x\to 0} e^{(7/3\,x+\bar{o}(x))\,1/x} = e^{7/3}$

№29(4) $\lim_{x\to 0}\left(\frac{\operatorname{tg}(2x+x^3)-\operatorname{th}(x+2x^3)}{x}\right)^{1/(\sqrt[3]{1+x^3}-\sqrt{1+x^2})} =$

$$= \lim_{x\to 0} e^{\ln\left(\frac{tg(2x+x^3)-th(x+2x^3)}{x}\right) 1/\left(\sqrt[3]{1+x^3}-\sqrt{1+x^2}\right)} =$$

Числ.: $tg(2x+x^3)-th(x+2x^3) = 2x+x^3+\frac{8x^3}{3} - (x+2x^3-\frac{x^3}{3}) + \bar{o}(x^3) = x + x^3(1+\frac{8}{3}-2+\frac{1}{3}) + \bar{o}(x^3) = x+2x^3+\bar{o}(x^3),\ x\to 0$

$\ln\left(\frac{x+2x^3+\bar{o}(x^3)}{x}\right) = \ln(1+2x^2+\bar{o}(x^2)) = 2x^2+\bar{o}(x^2),\ x\to 0$

Знам.: $\sqrt[3]{1+x^3}-\sqrt{1+x^2} = 1+\frac{x^3}{3} - (1+\frac{x^2}{2}) + \bar{o}(x^2) = \frac{x^3}{3}-\frac{x^2}{2}+\bar{o}(x^2),\ x\to 0$

$$= \lim_{x\to 0} \exp\left(\frac{2x^2+\bar{o}(x^2)}{x^3/3 - x^2/2 + \bar{o}(x^2)}\right) =$$

$$= \lim_{x\to 0} \exp\left(\frac{2+\bar{o}(1)}{\frac{x}{3}-\frac{1}{2}+\bar{o}(1)}\right) = e^{-4}$$

№47(5) $\lim_{x\to +0} \left(\frac{sh\, x}{arctg\, x}\right)^{1/x^2+\ln x} = \lim_{x\to +0} \exp\left(\ln\left(\frac{sh\, x}{arctg\, x}\right)\left(\frac{1}{x^2}+\ln x\right)\right) =$

$$\frac{sh\, x}{arctg\, x} = \frac{x+\frac{x^3}{6}+\bar{o}(x^4)}{x-\frac{x^3}{3}+\bar{o}(x^4)} = \frac{1+x^2/6+\bar{o}(x^3)}{1-x^2/3+\bar{o}(x^3)} = \left(1+\frac{x^2}{6}+\bar{o}(x^3)\right)\cdot$$

$$\cdot\left(1+\frac{x^2}{3}+\bar{o}(x^3)\right) = 1+\frac{x^2}{2}+\bar{o}(x^3),\ x\to 0$$

$$\ln\left(1+\frac{x^2}{2}+\bar{o}(x^3)\right) = \frac{x^2}{2}+\bar{o}(x^3),\ x\to 0$$

$$= \lim_{x\to 0} \exp\left(\frac{\frac{x^2}{2}+\bar{o}(x^3)}{x^2} + \ln x\left(\frac{x^2}{2}+\bar{o}(x^3)\right)\right) =$$

$$= e^{1/2}\lim_{x\to 0} \exp\left(\ln x\left(\frac{x^2}{2}+\bar{o}(x^3)\right)\right) = e^{1/2}\lim_{x\to 0} \exp\left(\frac{x^2\ln x}{2}\right) =$$

$= e^{1/2}\exp\left(\lim\limits_{x\to 0}\frac{\ln x}{\frac{2}{x^2}}\right) = e^{1/2}\exp\left(\lim\limits_{x\to 0}\frac{1/x}{-1/x^3}\right) =$

$= e^{1/2}\exp(0) = e^{1/2}$

№58(3) $\lim\limits_{x\to+\infty}\frac{x^2(\sqrt[3]{x^3+x}-x)+\sin x\ln(1+x)}{\ln(1+x+e^{5x})} =$

$t := 1/x,\ t\to+0$

$t^{-2}(\sqrt[3]{t^{-3}+t^{-1}}-t^{-1}) = t^{-2}(t^{-1}\sqrt[3]{1+t^2}-t^{-1}) =$

$= t^{-3}\left(1+\frac{1}{3}t^2-1+\bar{o}(t^3)\right) = \frac{1}{3}t^{-1}+\bar{o}(1),\ t\to+0 =$

$= \frac{1}{3}t^{-1},\ t\to+0$

$\sin\left(\frac{1}{t}\right)\ln\left(1+\frac{1}{t}\right) = (t^{-1}+\bar{o}(t^{-1}))(t^{-1}+\bar{o}(t^{-1})) =$

$= t^{-2}+\bar{o}(t^{-2}),\ t\to+0$

$\ln(1+t^{-1}+e^{5t^{-1}}) = \ln\left(1+t^{-1}+1+5t^{-1}+\frac{25t^{-2}}{2}\right) =$

$= \ln\left(2+6t^{-1}+\frac{25}{2}t^{-2}\right) = \ln 2+\ln\left(1+3t^{-1}+\frac{25}{4}t^{-2}\right) =$

$= \ln 2+3t^{-1}+\frac{25}{4}t^{-2}+\frac{1}{2}\left(3t^{-1}+\frac{25}{4}t^{-2}\right)^2+\bar{o}(t^{-2}) =$

$= \ln 2+3t^{-1}+\frac{43}{4}t^{-2}+\bar{o}(t^{-2}),\ t\to+0$

$= \lim\limits_{t\to+0}\frac{\frac{1}{3}t^{-1}+t^{-2}+\bar{o}(t^{-2})}{\ln 2+3t^{-1}+\frac{43}{4}t^{-2}+\bar{o}(t^{-2})} = \lim\limits_{x\to 0}\frac{\frac{1}{3}t+1+\bar{o}(1)}{\ln 2\cdot t^2+3t+\frac{43}{4}+\bar{o}(1)} =$

$= \frac{4}{43}$

Т.7. $e^x = \sum\limits_{k=0}^{n}\frac{x^k}{k!}+\frac{\exp^{(n+1)}(\xi)}{(n+1)!}x^{n+1},\ \xi\in[-1;2]$

погрешность - $R_n(x) = e^x - \sum\limits_{k=0}^{n} \frac{x^k}{k!} = e^x - P_n(x) =$

$$= \frac{\exp^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{e^{\xi}}{(n+1)!} x^{n+1}$$

$$|R_n(x)| = \left| \frac{e^{\xi}}{(n+1)!} x^{n+1} \right| < \frac{2^{n+1}}{(n+1)!} e^2 < 10^{-3}$$

$$\frac{2^{n+1}}{(n+1)!} < \frac{10^{-3}}{e^2}$$

$$\left.\begin{array}{l} n=9: \quad \dfrac{2^{10}}{10!} > \dfrac{10^{-3}}{e^2} \\ n=10: \quad \dfrac{2^{11}}{11!} < \dfrac{10^{-3}}{e^2} \end{array}\right| \Rightarrow e^x = \sum_{k=0}^{10} \frac{x^k}{k!} \text{ с точностью до } 10^{-3}.$$