Д/З 13

№1. а) $x+2x^2+3x^3+4x^4+\ldots nx^n = x(1+2x+3x^2+\ldots+nx^{n-1}) =$

$= x(x+x^2+x^3+\ldots+x^n)' = x\left(\frac{1-x^{n+1}}{1-x}\right)' = \frac{x}{(1-x)^2}\cdot$

$\cdot\left(-(n+1)x^n+1-x^{n+1}\right) \underset{\substack{|x|<1\\ n\to\infty}}{=\!=} \frac{x}{(1-x)^2}$

б) Ряд Тейлора => искомое – e^x

№2.

а) $\sum\limits_{k\geq 0}\binom{n+k}{k}x^k$

Заметим, что $(1+x)^{-(n+1)} = \sum\limits_{k=0}^{\infty} C^k_{-(n+1)}(-x)^k \ominus$

$C^k_{-(n+1)} = \frac{(-n)(-n-1)\ldots(-n-k)}{k!} = (-1)^k\frac{n(n+1)\ldots(n+k)}{k!} =$

$= (-1)^k C^k_{n+k}$

$\ominus \sum\limits_{k=0}^{\infty} C^k_{n+k}x^k = \sum\limits_{k=0}^{\infty}\binom{n+k}{k}x^k \Rightarrow \boxed{(1+x)^{-(n+1)}}$

б) $\sum\limits_{k\geq 0}\binom{k}{n}x^k = \sum\limits_{k\geq n}\binom{k}{n}x^k \overset{i=k-n}{=\!=} \sum\limits_{i\geq 0}C^n_{i+n}$

$x^{i+n} = x^n\sum\limits_{i\geq 0}C^n_{i+n}x^i = \boxed{x^n(1+x)^{-(n+1)}}$ (см. пункт а)

в) $\sum\limits_{k=1}^{n}(2k+1)x^k = 2\sum\limits_{k=1}^{n}kx^k + \sum\limits_{k=1}^{n}x^k = \frac{2x}{(1-x)^2}+\frac{1}{1-x}-$

$-1 = \boxed{\frac{3x-x^2}{(1-x)^2}}$

№3. а) Рассм. $f(x)=\sum_{k=1}^{n}\binom{n}{k+1}x^{k+1}$

$$\int f(x)\,dx=\sum_{k=1}^{n}\binom{n}{k+1}\frac{1}{k+2}x^{k+2}+C_1=g(x)$$

$$\int g(x)\,dx=\sum_{k=1}^{n}\binom{n}{k+1}\frac{1}{(k+2)(k+3)}x^{k+3}+C_1x+C_2=h(x)$$

$\exists\ h(1)=\sum_{k=1}^{n}\frac{\binom{n}{k+1}}{(k+2)(k+3)}$, т.е. $C_1=-C_2$

$$f(x)=(1+x)^n-1-nx$$

$$\int f(x)\,dx=\frac{1}{n+1}(1+x)^{n+1}-x-n\frac{x^2}{2}+C_1=g(x)$$

$$\int g(x)\,dx=\frac{(1+x)^{n+2}}{(n+1)(n+2)}-\frac{x^2}{2}-n\frac{x^3}{6}+C_1x+C_2=h(x)$$

$$h(1)=\boxed{\frac{2^{n+2}}{(n+1)(n+2)}-\frac{1}{2}-\frac{n}{6}}$$

б) $\sum_{k=0}^{n}k\binom{n}{k}2^k$

$$f(x)=(1+x)^n=\sum_{k=0}^{n}C_n^kx^k=(1+x)^n$$

$$f'(x)=\sum_{k=1}^{n}kC_n^kx^{k-1}=n(1+x)^{n-1}$$

$$f'(2)=\sum_{k=1}^{n}kC_n^k2^{k-1}$$

$$2f'(2)=2n\cdot3^{n-1}=\sum_{k=1}^{n}kC_n^k2^k=\sum_{k=0}^{n}kC_n^k2^k$$

Искомое - $\boxed{2n\cdot3^{n-1}}$

в) $\sum_{k=0}^{99}\binom{99}{k}\binom{100}{k}$

$$(1+x)^{2n+1} = (1+x)^n (1+x)^{n+1} = \sum_{k=0}^{n}\binom{n}{k}x^k \sum_{k=0}^{n+1}\binom{n+1}{k}.$$

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}x^k \sum_{k=0}^{n+1}\binom{n+1}{k}x^k$$

коэф. при x^n: $\binom{2n+1}{n} = \sum_{k=0}^{n}\binom{n}{n-k}\binom{n+1}{k} =$

$$= \sum_{k=0}^{n}\binom{n}{k}\binom{n+1}{k} \Rightarrow$$

$$\Rightarrow \sum_{k=0}^{99}\binom{99}{k}\binom{100}{k} = \boxed{\binom{199}{99}}$$

№4. $f(x) = (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$

$f'(x) = n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}$

$f''(x) = n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k}x^{k-2}$

$f''(1) = n(n-1)2^{n-2} = \sum_{k=0}^{n} k(k-1)\binom{n}{k} =$

$$= \sum_{k=2}^{n} k(k-1)\binom{n}{k} \quad \blacksquare$$

№5. $g(x) = a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + \ldots =$

$= a_0(1+x+\ldots+x^n) + a_1x(1+\ldots+x^n) + a_2x^2(1+\ldots+x^n) + \ldots =$

$= a_0\frac{1-x^{n+1}}{1-x} + a_1x\frac{1-x^{n+1}}{1-x} + \ldots a_nx^n\frac{1-x^{n+1}}{1-x} + \ldots =$

$$\underset{\substack{|x|<1\\ n\to\infty}}{=} \frac{1}{1-x}(a_0+a_1x+\ldots+a_nx^n) \Rightarrow$$

$\Rightarrow$ искомая $a_0+a_1x+\ldots+a_nx^n = \boxed{g(x)(1-x)}$