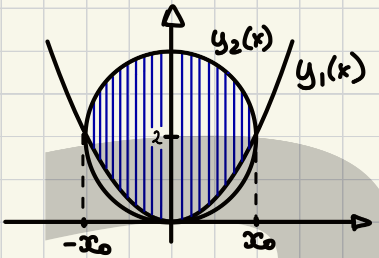
§7

4(5). Найти площадь фигуры, ограниченной кривыми.

$2y=x^2,\ x^2+y^2=4y,\ 2y\geq x^2$



Найдём x_0: $\begin{cases} x_0^2+y^2=4y \\ 2y=x_0^2 \end{cases} \Rightarrow x_0^2+\frac{x_0^4}{4}=2x_0^2$

$x_0>0 \Rightarrow x_0=2$

$S=2\int_0^2 (y_2(x)-y_1(x))dx \ominus$

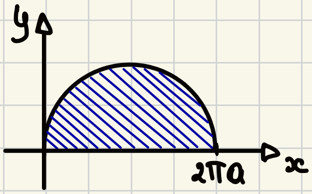
$y_2(x)$: $x^2+(y-2)^2=4 \Rightarrow y_2(x)=2+\sqrt{4-x^2}$; $y_1(x)=\frac{1}{2}x^2$

$\ominus\ 2\int_0^2\left(2+\sqrt{4-x^2}-\frac{1}{2}x^2\right)dx = 8+8\int_0^2\sqrt{1-\left(\frac{x}{2}\right)^2}\,d\left(\frac{x}{2}\right)-\int_0^2 x^2dx \ominus$

$\int\sqrt{1-t^2}\,dt=\int\cos^2u\,du=\int\frac{1+\cos2u}{2}du=\frac{u}{2}+\frac{\sin2u}{4}=\frac{\arcsin t}{2}+\frac{t\sqrt{1-t^2}}{2}$

$\ominus\ 8+8\left[\frac{\arcsin t}{2}+\frac{t\sqrt{1-t^2}}{2}\right]\Big|_0^1-\frac{x^3}{3}\Big|_0^2=\frac{16}{3}+2\pi$

26. Найти площадь фигуры, ограниченной аркой циклоиды $x=a(t-\sin t)$, $y=a(1-\cos t)$, $0\leq t\leq 2\pi$ и отрезком $[0;2\pi a]$ оси абсцисс

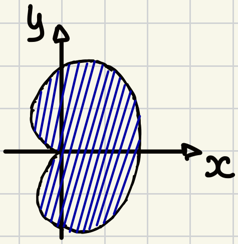


$S=\int_0^{2\pi a}y(x)dx=\int_0^{2\pi}y(t)x'(t)dt=\int_0^{2\pi}a^2(1-\cos t)^2dt=a^2\left[\int_0^{2\pi}dt-2\int_0^{2\pi}\cos t\,dt+\int_0^{2\pi}\cos^2t\,dt\right]=a^2\left[2\pi-2\sin t\Big|_0^{2\pi}+\int_0^{2\pi}\frac{1+\cos2t}{2}dt\right]=$

$=a^2\left[2\pi+\frac{t}{2}\Big|_0^{2\pi}+\frac{\sin2t}{4}\Big|_0^{2\pi}\right]=3\pi a^2$

33(3) Найти площадь фигуры, ограниченной кривыми, заданными в полярных координатах:

$r=a(1+\cos\varphi)$



$$S=\frac{1}{2}\int_0^{2\pi} r^2(\varphi)\,d\varphi = \frac{1}{2}\int_0^{2\pi} a^2(1+\cos\varphi)^2\,d\varphi = \pi a^2 +$$

$$+\,a^2\int_0^{2\pi}\cos\varphi\,d\varphi + \frac{a^2}{2}\int_0^{2\pi}\frac{1+\cos 2\varphi}{2}\,d\varphi = \pi a^2 + \frac{\pi a^2}{2} = \frac{3}{2}\pi a^2$$

<u>69(11)</u> Найти длину дуги кривой $y=\ln(1+\sin x),\ 0\le x\le \frac{\pi}{2}$

$$L=\int_0^{\pi/2}\sqrt{1+y'^2}\,dx = \int_0^{\pi/2}\sqrt{1+\left(\frac{\cos x}{1+\sin x}\right)^2}\,dx = \int_0^{\pi/2}\sqrt{\frac{(1+\sin x)^2+\cos^2 x}{(1+\sin x)^2}}\,dx =$$

$$=\int_0^{\pi/2}\sqrt{\frac{2}{1+\sin x}}\,dx = \int_0^{\pi/2}\sqrt{\frac{2}{1+\cos(\frac{\pi}{2}-x)}}\,dx = \int_0^{\pi/2}\frac{dx}{|\cos(\frac{\pi}{4}-\frac{x}{2})|} = -2\int_0^{\pi/2}\frac{d(\frac{\pi}{4}-\frac{x}{2})}{\cos(\frac{\pi}{4}-\frac{x}{2})} =$$

$$= -2\int_{\pi/4}^{0}\frac{dt}{\cos t} = 2\int_0^{\pi/4}\frac{d(\sin t)}{1-\sin^2 t} = 2\int_0^{\frac{\sqrt{2}}{2}}\frac{du}{1-u^2} = \ln\left|\frac{1+u}{1-u}\right|\Bigg|_0^{\frac{\sqrt{2}}{2}} = 2\ln(\sqrt{2}+1)$$

<u>72(3)</u> Найти длину дуги кривой $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\le t\le 2\pi$ (циклоида)

$$L=\int_0^{2\pi}\sqrt{x'^2+y'^2}\,dt = \int_0^{2\pi}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt = a\int_0^{2\pi}\sqrt{2-2\cos t}\,dt =$$

$$=2a\int_0^{2\pi}|\sin t|\,dt = 8a\int_0^{\pi/2}\sin t\,dt = 8a$$

<u>82(3)</u> Найти длину дуги кривой $r=a(1-\cos\varphi)$ (кардиоида)

$$L=\int_0^{2\pi}\sqrt{r'^2+r^2}\,d\varphi = \int_0^{2\pi}a\sqrt{\sin^2\varphi+(1-\cos\varphi)^2}\,d\varphi = 2a\int_0^{2\pi}\sqrt{\frac{1-\cos\varphi}{2}}\,d\varphi =$$

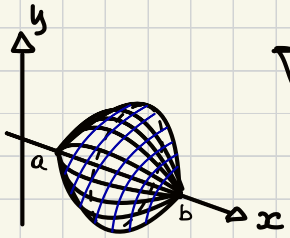
$$=2a\int_0^{2\pi}|\sin t|\,dt = 8a\int_0^{\pi/2}\sin t\,dt = 8a$$

§8

<u>12(1)</u> Найти объём тела, образованного при вращении фигуры, ограниченной данными кривыми, вокруг а) оси Ox; б) оси Oy:

$y=(x-a)(x-b),\ y=0,\ b>a\ge 0$

а)

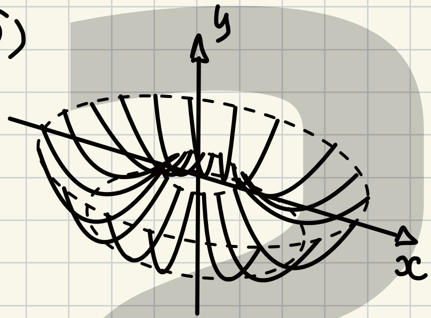


$$V=\pi\int_a^b y^2(x)\,dx=\pi\int_a^b (x-a)^2(x-b)^2\,dx=\left|\begin{matrix}c=b-a\\ t=x-a\end{matrix}\right.$$

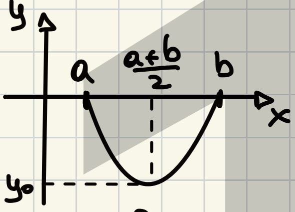
$$=\pi\int_0^c t^2(t-c)^2\,dt=\pi\int_0^c\left(t^4-2t^3c+t^2c^2\right)dt=$$

$$=\pi\left(\frac{c^5}{5}-\frac{c^5}{2}+\frac{c^5}{3}\right)=\frac{\pi}{30}(b-a)^5$$

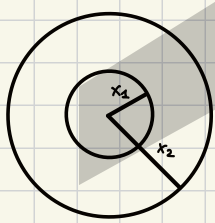
б)



Верт. сечение



горизонт. сечение



$$V=\int_{y_0}^0 S(y)\,dy=\int_{y_0}^0\pi\left(x_2^2-x_1^2\right)dy=$$

$$=\int_{y_0}^0\pi(x_2-x_1)(x_2+x_1)\,dy=\int_{y_0}^0\pi\sqrt{D(y)}\,(a+b)\,dy=$$

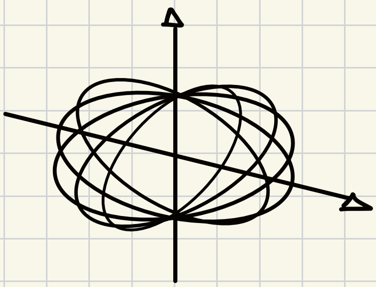
$$x^2-(a+b)x+ab=y\to D(y)=(a+b)^2-4(ab-y)$$

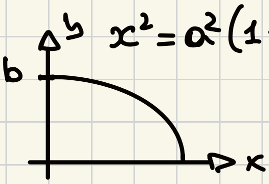
$$=\int_{y_0}^0\pi\sqrt{(a+b)^2-4ab+4y}\,(a+b)\,dy=\pi(a+b)\int_{y_0}^0\sqrt{(a-b)^2+4y}\,dy=$$

$$=2\pi(a+b)\int_{y_0}^0\sqrt{y-y_0}\,d(y-y_0)=2\pi(a+b)\cdot\frac{2}{3}\sqrt{y-y_0}^{\,3}\Big|_{y_0}^0=\frac{4}{3}\pi(a+b)\sqrt{-y_0}^{\,3}=$$

$$=\frac{4}{3}\pi(a+b)\sqrt{\frac{(b-a)^2}{4}}^{\,3}=\frac{1}{6}\pi(a+b)(b-a)^3$$

13(2). Найти объём эллипсоида, образованного при вращении эллипса $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ вокруг оси Oy.





$$x^2=a^2\left(1-\frac{y^2}{b^2}\right)\qquad V=2\pi\int_0^b x^2(y)\,dy=$$

$$=2\pi\int_0^b a^2\left(1-\frac{y^2}{b^2}\right)dy=2\pi a^2\left(b-\frac{1}{3}b\right)=$$

$$=\frac{4}{3}\pi a^2 b$$

82(4,5) Найти площадь поверхности, образованной при вращении вокруг оси Oy данной кривой:

4) $3x=4\cos y,\ -\pi/2 \le y \le 0$

$$S = 2\pi\int_{-\pi/2}^{0}|x(y)|\sqrt{1+x'^2(y)}\,dy = 2\pi\int_{-\pi/2}^{0}\frac{4}{3}|\cos y|\sqrt{1+\frac{16}{9}\sin^2 y}\,dy =$$

$$= \frac{8\pi}{3}\int_{-\pi/2}^{0}\sqrt{1+\frac{16}{9}\sin^2 y}\,d(\sin y) = 2\pi\int_{-4/3}^{0}\sqrt{1+t^2}\,dt =$$

$$\int\sqrt{1+t^2}\,dt = t\sqrt{1+t^2} - \int\frac{t^2\,dt}{\sqrt{1+t^2}} \Rightarrow \int\frac{t^2\,dt}{\sqrt{1+t^2}} = t\sqrt{1+t^2} - \int\sqrt{1+t^2}\,dt$$

$$\int\sqrt{1+t^2}\,dt = \int\frac{1+t^2}{\sqrt{1+t^2}}\,dt = \int\frac{dt}{\sqrt{1+t^2}} + \int\frac{t^2\,dt}{\sqrt{1+t^2}} = \int\frac{dt}{\sqrt{1+t^2}} + t\sqrt{1+t^2}\,dt - \int\sqrt{1+t^2}\,dt$$

$$\Rightarrow \int\sqrt{1+t^2}\,dt = \frac12 t\sqrt{1+t^2} + \int\frac{dt}{\sqrt{1+t^2}} = \frac12 t\sqrt{1+t^2} + \frac12\ln|t+\sqrt{1+t^2}|$$

$$= 2\pi\left(\frac12 t\sqrt{1+t^2}+\frac12\ln|t+\sqrt{1+t^2}|\right)\Big|_{-4/3}^{0} = -2\pi\left(\frac12\left(-\frac43\right)\frac53+\frac12\left|-\frac43+\frac53\right|\right) =$$

$$=\frac{20\pi}{9}+\pi\ln 3$$

5) $y^2=2(x-1),\ 0\le y\le 1$

$$S=2\pi\int_0^1|x(y)|\sqrt{1+x'^2(y)}\,dy = 2\pi\int_0^1\left|\frac12y^2+1\right|\sqrt{1+y^2}\,dy =$$

$$=\pi\int_0^1(y^2+2)\sqrt{1+y^2}\,dy = \pi\int_0^1(1+y^2)^{3/2}\,dy+\pi\int_0^1\sqrt{1+y^2}\,dy =$$

$\int\sqrt{1+y^2}\,dy = \frac12y\sqrt{1+y^2}+\frac12\ln|y+\sqrt{1+y^2}|$ (из прошлого пункта)

$$\int(1+y^2)^{3/2}\,dy = \left|_{y=\operatorname{sh}t}\right| = \int\operatorname{ch}^3t\,d(\operatorname{sh}t) = \int\operatorname{ch}^4t\,dt = \int(1+\operatorname{sh}^2t)^2\,dt =$$

$$=\int\left(1+\frac{\operatorname{ch}^2t+\operatorname{sh}^2t}{2}+\frac12\operatorname{sh}^2t-\frac12\operatorname{ch}^2t\right)^2dt = \int\left(\frac{1+\operatorname{ch}2t}{2}\right)^2dt = \int\frac14\,dt+\int\frac12\operatorname{ch}2t\,dt+$$

$$+\int\frac14\operatorname{ch}^2 2t\,dt = \frac14t+\frac14\operatorname{sh}2t+\int\frac18(1+\operatorname{ch}4t)\,dt = \frac14t+\frac14\operatorname{sh}2t+\frac18t+\frac1{32}\operatorname{sh}4t =$$

$$=\frac38t+\frac12\operatorname{sh}t\operatorname{ch}t+\frac1{16}\operatorname{sh}2t\operatorname{ch}2t = \frac38t+\frac12\operatorname{sh}t\operatorname{ch}t+\frac18\operatorname{sh}t\operatorname{ch}t(1+2\operatorname{sh}^2t) =$$

$$=\frac38\operatorname{arcsh}y+\frac12y\sqrt{1+y^2}+\frac18y\sqrt{1+y^2}(1+2y^2) = \frac38\ln|y+\sqrt{1+y^2}|+\frac58y\sqrt{1+y^2}+\frac14y^3\sqrt{1+y^2}$$

$$=\pi\left[\frac{\sqrt2}{2}+\frac12\ln|1+\sqrt2|+\frac38\ln|1+\sqrt2|+\frac{5\sqrt2}{8}+\frac{\sqrt2}{4}\right] = \frac\pi8\left[7\ln(1+\sqrt2)+11\sqrt2\right]$$