Tiles

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Problem restatement

We model this problem as a graph. We consider each garden space to be a vertex, and we add an edge for each pair of garden spaces that are adjacent horizontally or vertically. With this formulation, the problem reduces to finding a maximum matching.

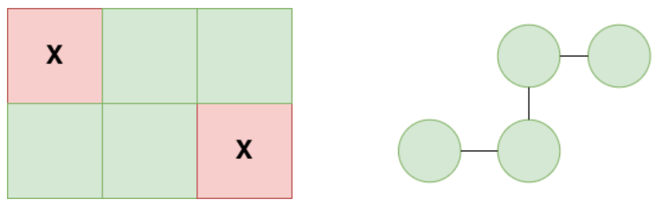

Figure 1: Graph reconstruction

For this exercise, we need to go one step further and model it as a flow, so we transform the matching instance into a flow network. The standard reduction is the following: Every garden tile v is split into two vertices: an *in-vertex* v_{in} and an *out-vertex* v_{out}. In addition, we add two other vertices *source* s and *sink* t. For each vertex v, we add edges (s, v_{in}) and (v_{out}, t) each with capacity 1. Intuitively, this ensures that each tile can be used by at most one domino.

Next, for every pair of adjacent tiles u and v (sharing an edge in the grid), we add two directed edges

$$u_{\text{in}} \to v_{\text{out}}, \qquad v_{\text{in}} \to u_{\text{out}},$$

each with capacity 1. These edges encode the possibility of placing a domino that covers u and v.

A valid s–t flow selects a set of disjoint tile pairs, each realised as a path

$$s \to v_{\text{in}} \to u_{\text{out}} \to t,$$

corresponding exactly to placing a domino on the pair $\{u, v\}$.

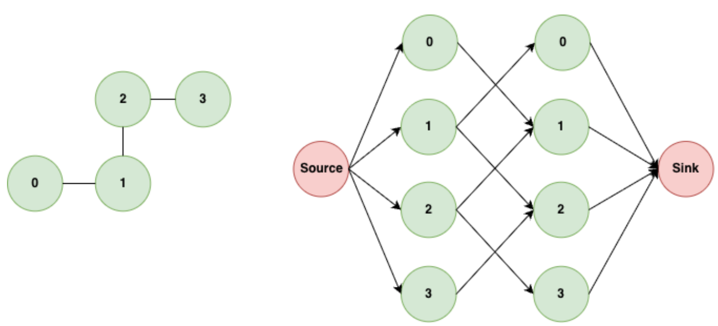

Figure 2: Flow construction. Each edge has capacity 1.

Proof of Correctness

Observe that the maximum possible flow value is $|V|$, since each tile contributes at most one unit of flow entering at v_{in} and one unit leaving at v_{out}. If a perfect matching exists, we can realise this upper bound directly: for every matched pair (u, v), we route one unit of flow along both paths

$$s \to u_{\text{in}} \to v_{\text{out}} \to t \qquad \text{and} \qquad s \to v_{\text{in}} \to u_{\text{out}} \to t,$$

thereby saturating all source and sink edges. This constructs an s–t flow of value $|V|$ whenever a perfect matching is present.

Conversely, assume that the garden does *not* admit a perfect matching. Then there exists at least one tile v that cannot be matched with any of its neighbours in any maximum matching. In the flow network, this means that every adjacency edge incident to either v_{in} or v_{out} must remain unused in any feasible s–t flow, because using such an edge would correspond to matching v with some neighbour. But the only way for flow to pass through v is along a path

$$s \to v_{\text{in}} \to u_{\text{out}} \to t$$

for one of its neighbours u. Since no such path can be used, both the edge (s, v_{in}) and the edge (v_{out}, t) remain unsaturated. Thus at least one unit of the total available capacity from s to t is unused, and the value of any flow must satisfy $|f| < |V|$.

Combining both directions, we conclude that the flow has value $|V|$ if and only if a perfect matching exists in the original adjacency graph of tiles. Therefore, computing the maximum flow in this network correctly determines whether the garden can be completely tiled by dominoes.