This assignment is due by 11:55pm on Friday, September 23rd. Minimal credit will be given for incomplete solutions or solutions that do not provide details on how the solution is found. You are encouraged to discuss the problems with your classmates, but all work (analysis and code) must be your own.

1. **Determinant vs. Condition Number**: We know that a matrix A is singular if and only if $\det(A) = 0$. We might expect that a very small determinant indicates a poorly conditioned matrix. This turns out not to be true, and you will prove it in this exercise.

   (a) Let α be a **positive** real number and consider the 2×2 diagonal matrix $A = \text{diag}(\alpha, \alpha)$. Show that for any induced matrix norm, $\|A\| = \alpha$, $\|A^{-1}\| = 1/\alpha$ and $\kappa(A) = 1$, while $\det(A) = \alpha^2$.

   (b) More generally, given any nonsingular matrix A, discuss the condition number and determinant of αA, where α is any positive real number.

2. **Estimating condition numbers**: The direct computation of the condition number of a matrix A is expensive because it requires the computation of A^{-1}. In this problem we will explore an inexpensive method of approximating the condition number based on the 1-norm, i.e. $\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1$.

   (a) Since computing $\|A\|_1$ is pretty easy, we really only need a cheap way of approximating $\|A^{-1}\|_1$. Let A be an $n \times n$ matrix and show that for any nonzero $\mathbf{w} \in \mathbb{R}^n$ we have

   $$\frac{\|A^{-1}\mathbf{w}\|_1}{\|\mathbf{w}\|_1} \leq \|A^{-1}\|_1$$

   (b) Use the result of part (a) to show that

   $$\kappa_1(A) \geq \frac{\|A\|_1 \|A^{-1}\mathbf{w}\|_1}{\|\mathbf{w}\|_1}. \qquad \left(\text{Based on this we will use: } \kappa_1(A) \approx \frac{\|A\|_1 \|A^{-1}\mathbf{w}\|_1}{\|\mathbf{w}\|_1} \right)$$

   In general we want to choose a $\mathbf{w}$ that makes the inequality in part (a) as tight as possible so that the lower bound on $\kappa_1(A)$ will be a good approximation. In practice, choosing a few random $\mathbf{w}$'s and picking the largest resulting estimate tends to work pretty well.

   (c) Your task in this problem is to estimate the condition number $\kappa_1(A)$ of the notoriously illconditioned Hilbert matrix in two ways. The $n \times n$ Hilbert matrix is defined by $A_{ij} = 1/(i+j-1)$. For $n = 2, 3, \ldots, 12$ compute the condition number of the Hilbert matrix directly by $\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1$ and also by the estimation procedure described above. You should use 5 random vectors $\mathbf{w}$ and pick the one that yields the largest estimate of $\kappa_1(A)$. For the application of A^{-1} to $\mathbf{w}$ in the estimate you should use the canned backslash command. For the exact value of the condition number go ahead and compute the inverse of A explicitly using, say, Julia's `inv()` function. Make a semi-log plot showing both sets of experiments with n on the horizontal axis and log-base-10 of the condition number on the vertical axis. Comment on the accuracy of the condition number estimate vs. the exact value.