Problem Set 4

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1. (a) $A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/\alpha & 0 \\ 0 & 1/\alpha \end{bmatrix}$

   $\|A\|_1 = \max\{\alpha + 0, 0 + \alpha\} = \alpha$

   $\|A\|_\infty = \max\{\alpha + 0, 0 + \alpha\} = \alpha$

   $\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho(\begin{bmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{bmatrix})} = \sqrt{\alpha^2} = \alpha$ (Since α is positive)

   $\|A^{-1}\|_1 = \max\{1/\alpha + 0, 0 + 1/\alpha\} = 1/\alpha$

   $\|A^{-1}\|_\infty = \max\{1/\alpha + 0, 0 + 1/\alpha\} = 1/\alpha$

   $\|A^{-1}\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho(\begin{bmatrix} 1/\alpha^2 & 0 \\ 0 & 1/\alpha^2 \end{bmatrix})} = \sqrt{1/\alpha^2} = 1/\alpha$ (Since α is positive)

   $\kappa(A) = \|A\| \, \|A^{-1}\| = 1$

   $\det(A) = \alpha^2 - 0 = \alpha^2$

   (b) Let's say $A \in \mathbb{R}^{n \times n}$, then

   $\kappa(\alpha A) = \|\alpha A\| \, \|(\alpha A)^{-1}\| = |\alpha| \, \|A\| \, |1/\alpha| \, \|A^{-1}\| = \|A\| \, \|A^{-1}\| = \kappa(A)$

   $\det(\alpha A) = \alpha^n \det(A)$

   If $\alpha = \epsilon$ ($\epsilon \ll 1$) and n is large, $\det(\alpha A) \ll \det(A)$, but the condition number $\kappa(\alpha A) = \kappa(A)$.

2. (a)

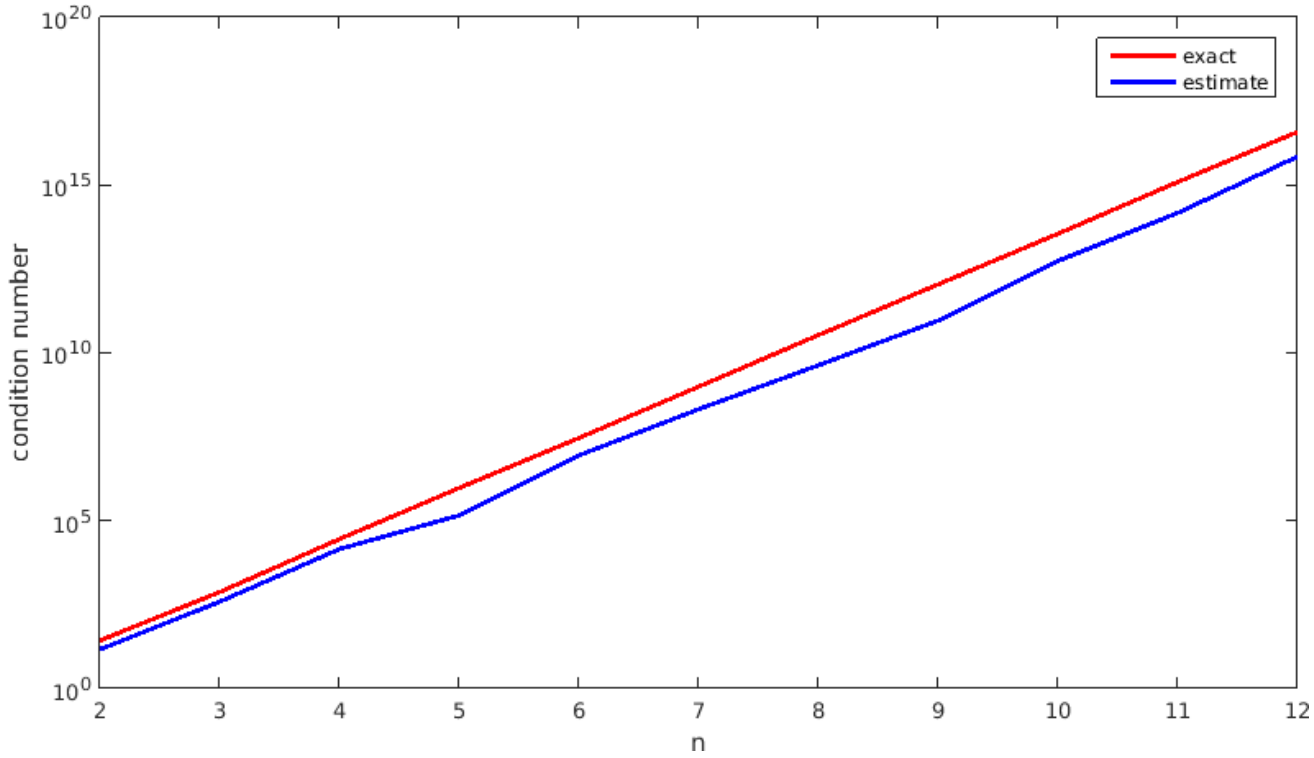
$$\begin{aligned} \left\|A^{-1}\right\|_1 &= \max_{\mathbf{w} \neq 0} \frac{\|A^{-1}\mathbf{w}\|_1}{\|\mathbf{w}\|_1} \\ &\geq \frac{\|A^{-1}\mathbf{w}\|_1}{\|\mathbf{w}\|_1} \ (\forall \ \mathbf{w} \neq \mathbf{0}) \end{aligned}$$

   (b)

$$\begin{aligned} \kappa_1(A) &= \|A\|_1 \left\|A^{-1}\right\|_1 \\ &\geq \|A\|_1 \frac{\|A^{-1}\mathbf{w}\|_1}{\|\mathbf{w}\|_1} \ (\forall \ \mathbf{w} \neq \mathbf{0}) \end{aligned}$$

   (c) Estimate condition number:

```
function [ kappa ] = estimate_cond_no( A )
    [n,~] = size(A);
    norm_A = norm(A,1);
    norm_inv_A = NaN;
    for i=1:5
        w = rand(n,1);
        norm_inv_A = max(norm_inv_A, norm(A\w,1) / norm(w,1));
    end
    kappa = norm_A * norm_inv_A;
end
```




```
c1 = zeros(1,11);
c2 = zeros(1,11);
for i=2:12
    A = hilb(i);
    c1(1,i-1) = norm(A,1)*norm(inv(A),1);
    c2(1,i-1) = estimate_cond_no(A);
end
x = (2:12);
figure;
```

```
10 semilogy(x, c1,'r', x, c2, 'b', 'LineWidth', 2);
11 legend('exact', 'estimate');
12 xlabel('n');
13 ylabel('condition number');
```

The accuracy of the estimation decreases when the size of matrix increases.