This assignment is due by 4pm on Friday, September 9th. Minimal credit will be given for incomplete solutions or solutions that do not provide details on how the solution is found. You are encouraged to discuss the problems with your classmates, but all work (analysis and code) must be your own.

1. In this problem you will explore the stability of various methods for computing the reduced QR Factorization of a matrix.

   (a) Write a function in your favorite prototyping language that takes an $m \times n$ matrix A and uses Gram-Schmidt to compute its reduced QR Factorization. Turn in a print-out of your code with this assignment.

   (b) Write a function in your favorite prototyping language that takes an $m \times n$ matrix A and uses Modified Gram-Schmidt to compute its reduced QR Factorization. Turn in a print-out of your code with this assignment.

   (c) Compute the QR Factorization of random $2n \times n$ matrices for $n = 2, 4, 8 \ldots, 128$ using (i) Gram-Schmidt, (ii) Modified Gram-Schmidt, and (iii) a canned QR function from your favorite scripting language (which almost certainly uses the Householder method that we'll learn in class next week). You will likely have to add a flag to get the canned QR code to return the reduced instead of the full factorization. Check your language's documentation. For each matrix and factorization, measure the orthogonality of the columns of $\hat{Q}$ using $\delta = \|\hat{Q}^T\hat{Q} - I\|_\infty$ (`norm( , inf)` in Matlab and `norm( , Inf)` in Julia). On a single set of axes, produce a log-log plot of n vs. δ (with n on the horizontal axis and δ on the vertical axis) for each factorization method. What can you conclude from your results?

   (d) Consider the matrix A from class (shown below). Repeat the general idea of part (c), but this time keep the size of the matrix fixed as shown, and let ϵ decrease like $\epsilon = 10^{-\ell}$ for $\ell = 2, 3, \ldots, 10$. Produce a log-log plot of ϵ vs. δ for each factorization. What can you conclude about your results this time?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix}$$

2. Let A be a matrix with the property that columns $1, 3, 5, 7, \ldots$ are orthogonal to columns $2, 4, 6, 8, \ldots$ In a reduced QR factorization $A = \hat{Q}\hat{R}$ (as computed by Gram-Schmidt, say), what special structure does $\hat{R}$ possess? You may assume that the columns of A are linearly independent.