This assignment is due by 11:55pm on Friday, October 21st. Submit your solutions as a **single pdf** to Moodle. Minimal credit will be given for incomplete solutions or solutions that do not provide details on how the solution is found. You are encouraged to discuss the problems with your classmates, but all work (analysis and code) must be your own.

1. The residents of Bullburg shop for groceries exactly once a week at one of three markets named 1,2, and 3. They are not faithful customers; they switch stores frequently. For $i, j = 1, 2, 3$, let p_{ij} be a number between 0 and 1 that represents the fraction of people who shopped at store j in a given week who will switch to store i in the following week. This can be interpreted as a probability that a given customer will switch from store j to store i from one week to the next. These numbers can be arranged into a matrix P. Suppose the values of the p_{ij} for Bullburg are

$$P = \begin{bmatrix} 0.63 & 0.18 & 0.14 \\ 0.26 & 0.65 & 0.31 \\ 0.11 & 0.17 & 0.55 \end{bmatrix}$$

Since $p_{23} = .31$, 31% of the people who shop at store 3 in a given week will switch to store 2 the following week. Fifty-five percent of the people who shopped at store 3 will return to store 3 the following week, etc. Notice that the sum of the entries in each column of p is 1.

   (a) A matrix $P \in \mathbb{R}^{n \times n}$ whose entries are nonnegative and whose column sums are all 1 is called *column-stochastic*. If Bullburg had n stores, the probabilities p_{ij} would form an $n \times n$ matrix. Show that $w^T P = w^T$, where $w^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$ (we call w^T a *left* eigenvector of P with associated eigenvalue 1).

   (b) Let $\mathbf{q}_k \in \mathbb{R}^3$ be a vector whose three components represent the fraction of Bullburg residents who shopped at stores 1, 2, and 3, respectively, in week k. For instance, if $\mathbf{q}_{20} = \begin{bmatrix} .24 & .34 & .42 \end{bmatrix}^T$, then 24% of the population shops at store 1, 34% at store 2, and 42% at store 3 during week 20. Give a simple probability argument that demonstrates the following relationship

$$\mathbf{q}_{k+1} = P\mathbf{q}_k$$

   for all k. The sequence $\{q_k\}$ is called a *Markov chain.*

   (c) The previous equation shows that we can simulate the Markov chain by applying the power method with the transition matrix P. Explain why there is no need (in exact arithmetic) to normalize the new vector $\mathbf{q}_k$ after each iteration.

   (d) Assuming that at week 1 one third of the population shopped at each store, calculate the fraction that visited the stores in subsequent weeks by simulating the Markov chain for many weeks. Use Julia/Python/Matlab to do the calculations. What is the long-term trend?

2. In this problem you will explore the Shifted-Inverse Power Method (SIPM).

   (a) Before your start coding, do some research and decide on a good stopping criterion for SIPM. Explain the motivation behind the method that you decide to implement.

   (b) Write a function SIPM that takes a matrix A and a shift μ and performs SIPM. Your function should return the associated eigenpair, wich the eigenvector scaled to have unit Euclidean length and a positive number in the first position. Turn in a print-out of your function.

   (c) Run your code with the matrix P from problem 1, a shift of $\mu = 0.55$, and a stopping tolerance of 10^{-9}. What eigenpair of P does your method converge to? How many iterations did it take?

(d) Modify your code to determine the rate of linear convergence of the eigenpair. One way to do this is to compute on each valid iteration the estimate

$$\gamma = \frac{|\lambda - \lambda^{(k)}|}{|\lambda - \lambda^{(k-1)}|}$$

If you know the eigenvalue that you're trying to get then you can hardcode this in and measure convergence. In a more realistic setting you don't know the eigenvalue you're after and you can instead use the following

$$\gamma = \frac{|\lambda^{(k)} - \lambda^{(k-1)}|}{|\lambda^{(k-1)} - \lambda^{(k-2)}|}$$

What convergence rate do you get for the given matrix and shift?

(e) Does this convergence estimate seem correct based on the theory stated in class? Justify your response.

3. Let A be an $n \times n$ upper Hessenberg Matrix. Show that an LU Factorization of A (with partial pivoting) can be accomplished in about n^2 flops. What special form does L have in this case?