Problem Set 2

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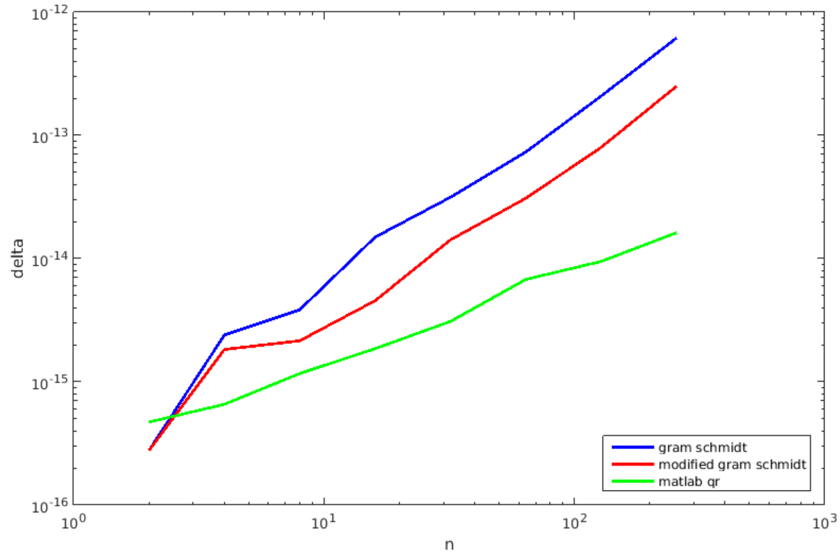
1. (a) Gram-Schmidt

```
function [ Q, R ] = gram_schmidt( A )
    [m, n] = size(A);
    Q = zeros(m,n);
    R = zeros(n,n);
    for j=1:n
        v_j = A(:,j);
        for i=1:j-1
            R(i,j) = dot(Q(:,i), A(:,j));
            v_j = v_j - R(i,j) * Q(:,i);
        end
        R(j,j) = norm(v_j);
        Q(:,j) = v_j/R(j,j);
    end
end
```

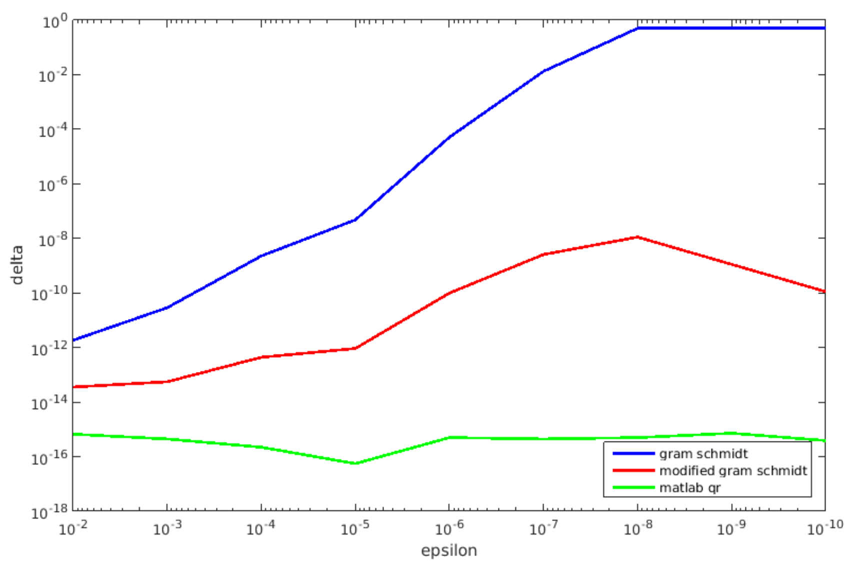
(b) Modified Gram-Schmidt

```
function [ Q, R ] = modified_gram_schmidt( A )
    [m, n] = size(A);
    Q = zeros(m,n);
    R = zeros(n,n);
    V = A;
    for i = 1:n
        R(i,i) = norm(V(:,i));
        Q(:,i) = V(:,i)/R(i,i);
        for j = i+1:n
            R(i,j) = dot(Q(:,i), V(:,j));
            V(:,j) = V(:,j) - R(i,j)*Q(:,i);
        end
    end
end
```

(c) The figure (next page) shows that when the matrix size increasing, modified Gram-Schmidt is numerically more accurate than traditional Gram-Schmidt, while Householder triangularization (canned QR function of Matlab) is the most accurate method.



(d) The figure shows that when the matrix is ill-conditioned, the modified Gram-Schmidt is numerically more stable than the traditional Gram-Schmidt, while Householder triangularization (canned QR function of Matlab) is most stable.



2. Use Gram-Schmidt method.

$$r_{ij} = q_i^T a_j$$

$$r_{jj} = \left\| a_j - \sum_{i=1}^{j-1} r_{ij} q_i \right\|_2$$

First, let's compute $r_{1,2i}$ $(i = 1, 2, \cdots)$

$$r_{1,2i} = q_1^T a_{2i} = \frac{a_1^T a_{2i}}{r_{11}} = 0 \quad (i = 1, 2, \cdots)$$

Then

$$q_2 = \frac{a_2 - r_{12} q_1}{r_{22}} = \frac{a_2}{r_{22}}$$

So

$$r_{2,2i+1} = q_2^T a_{2i+1} = \frac{a_2^T a_{2i+1}}{r_{22}} = 0 \quad (i = 1, 2, \cdots)$$

$$q_3 = \frac{a_3 - r_{13} q_1 - r_{23} q_2}{r_{33}} = \frac{a_3 - r_{13} q_1}{r_{22}} = \frac{a_3 - r_{13} \frac{a_1}{r_{11}}}{r_{22}}$$

Following these steps, we'll see that q_{2i-1} is a linear combination of $a_1, a_3, \cdots, a_{2i-1}$, where $i = 1, 2, \cdots$, and q_{2i} is a linear combination of $a_2, a_4, \cdots, a_{2i}$, where $i = 1, 2, \cdots$.

And in the upper triangular matrix R, if $i + j$ is odd, then $r_{ij} = 0$. The matrix R looks like this:

$$\begin{bmatrix} r_{11} & 0 & r_{13} & 0 & r_{15} & \cdots \\ 0 & r_{22} & 0 & r_{24} & 0 & \cdots \\ 0 & 0 & r_{33} & 0 & r_{35} & \cdots \\ 0 & 0 & 0 & r_{44} & 0 & \cdots \\ 0 & 0 & 0 & 0 & r_{55} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$