0 Review and Miscellanea

This chapter adopts Hoffman's definition for clearer presentations.

0.1 Vector Spaces

A finite dimensional vector space is the fundamental setting for matrix analysis.

0.1.1 Scalar Field

Definition. *Field $\mathbb{F}$ is the set together with addition and multiplication. A field must satisfy the following properties:*

1. *Addition is commutative,*
$$x + y = y + x$$
for all x and y in $\mathbb{F}$.

2. *Addition is associative,*
$$x + (y + z) = (x + y) + z$$
for all x, y and z in $\mathbb{F}$.

3. *There is a unique element 0 (zero) in $\mathbb{F}$ such that $x + 0 = x$, for every x in $\mathbb{F}$.*

4. *To each x in $\mathbb{F}$ there corresponds a unique element $(-x)$ in $\mathbb{F}$ such that $x + (-x) = 0$.*

5. *Multiplication is commutative,*
$$xy = yx$$
for all x and y in $\mathbb{F}$.

6. *Multiplication is associative,*
$$x(yz) = (xy)z$$
for all x, y, and z in $\mathbb{F}$.

7. *There is a unique element 1 (one) in $\mathbb{F}$ such that $x1 = x$, for every x in $\mathbb{F}$.*

8. *To each non-zero x in $\mathbb{F}$ there corresponds a unique element x^{-1} (or $1/x$) in $\mathbb{F}$ such that $xx^{-1} = 1$.*

9. *Multiplication distributes over addition; that is, $x(y + z) = xy + xz$ for all x, y, and z in $\mathbb{F}$.*

Remarks. *If $\mathbb{F}$ is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0:*

$$1 + 1 + \cdots + 1 = 0$$

That does not happen in the complex field (or in any subfield thereof). If it does happen in a field $\mathbb{F}$, then the least n such that the sum of n 1's is 0 is called the characteristic of the field $\mathbb{F}$. If this does not happen in $\mathbb{F}$, then it it is called a field of characteristic zero.

Definition. *A vector space (or linear space) consists of the following:*

1. *a field $\mathbb{F}$ of scalars;*
2. *a set $\mathbb{V}$ of objects, called vectors;*
3. *a rule (or operation), called vector addition, which associates each pair of vectors $\vec{\alpha}$, $\vec{\beta}$ in $\mathbb{V}$ a vector $\vec{\alpha} + \vec{\beta}$ in $\mathbb{V}$, called the sum of $\vec{\alpha}$ and $\vec{\beta}$, in such a way that*
   (a) *addition is commutative, $\vec{\alpha} + \vec{\beta} = \vec{\beta} + \vec{\alpha}$;*
   (b) *addition is associative, $\vec{\alpha} + (\vec{\beta} + \vec{\gamma}) = (\vec{\alpha} + \vec{\beta}) + \vec{\gamma}$;*
   (c) *there is a unique vector 0 in $\mathbb{V}$, called the zero vector, such that $\vec{\alpha} + 0 = \vec{\alpha}$ for all $\vec{\alpha}$ in $\mathbb{V}$;*
   (d) *for each vector $\vec{\alpha}$ in $\mathbb{V}$ there exists a unique vector $-\vec{\alpha}$ in $\mathbb{V}$ such that $\vec{\alpha} + (-\vec{\alpha}) = 0$.*
4. *a rule (or operation), called scalar multiplication, which associates with each scalar c in $\mathbb{F}$ and vector $c\vec{\alpha}$ in $\mathbb{F}$, called the product of c and α, in such a way that*
   (a) *$1\vec{\alpha} = \vec{\alpha}$ for every $\vec{\alpha}$ in $\mathbb{V}$;*
   (b) *$(c_1c_2)\vec{\alpha} = c_1(c_2\vec{\alpha})$;*
   (c) *$c(\vec{\alpha} + \vec{\beta}) = c\vec{\alpha} + c\vec{\beta}$;*
   (d) *$(c_1 + c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha}$.*

0.1.2 Subspaces, span, and linear combinations

Definition. *A vector $\vec{\beta}$ in $\mathbb{V}$ is said to be a linear combination of the vectors $\vec{\alpha_1}, \cdots, \vec{\alpha_n}$ in $\mathbb{V}$ provided there exists scalars $c_1, \cdots, c_n$ in $\mathbb{F}$ such that*

$$\begin{aligned}\vec{\beta} &= c_1\vec{\alpha_1} + \cdots + c_n\vec{\alpha_n} \\ &= \sum_{i=1}^{n} c_i\vec{\alpha}_i\end{aligned}$$

Definition. *Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. A subspace of $\mathbb{V}$ is a subset $\mathbb{W}$ of $\mathbb{V}$ which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication on $\mathbb{V}$.*

Theorem 1. *A non-empty subset $\mathbb{W}$ of $\mathbb{V}$ is a subspace of $\mathbb{V}$ if and only if for each pair of vectors $\vec{\alpha}$, $\vec{\beta}$ in $\mathbb{W}$ and each scalar c in $\mathbb{F}$ the vector $c\vec{\alpha}+\vec{\beta}$ is again in $\mathbb{W}$.*

Proof. Suppose $\mathbb{W}$ is a non-empty subset of $\mathbb{V}$ such that $c\vec{\alpha}+\vec{\beta}$ belongs to $\mathbb{W}$ for all vectors $\vec{\alpha}$, $\vec{\beta}$ in $\mathbb{W}$ and all scalars c in $\mathbb{F}$. Since $\mathbb{W}$ is non-empty, there is a vector $\vec{\rho}$ in $\mathbb{W}$, and hence $(-1)\vec{\rho}+\vec{\rho}=0$ is in $\mathbb{W}$. Then if $\vec{\alpha}$ is any vector in $\mathbb{W}$ and c any scalar, the vector $c\vec{\alpha}=c\vec{\alpha}+0$ is in $\mathbb{W}$. In particular, $(-1)\vec{\alpha}=-\vec{\alpha}$ is in $\mathbb{W}$. Finally, if $\vec{\alpha}$ and $\vec{\beta}$ are in $\mathbb{W}$, then $\vec{\alpha}+\vec{\beta}=1\vec{\alpha}+\vec{\beta}$ is in $\mathbb{W}$. Thus $\mathbb{W}$ is a subspace of $\mathbb{V}$. Conversely, if $\mathbb{W}$ is a subspace of $\mathbb{V}$, $\vec{\alpha}$ and $\vec{\beta}$ are in $\mathbb{W}$, and c is a scalar, certainly $c\vec{\alpha}+\vec{\beta}$ is in $\mathbb{W}$. □

Lemma. *If $\mathbf{A}$ is an $m\times n$ matrix over $\mathbb{F}$ and $\mathbf{B}$, $\mathbf{C}$ are $n\times p$ matrices over $\mathbb{F}$, then*

$$\mathbf{A}(d\mathbf{B}+\mathbf{C})=d(\mathbf{AB})+\mathbf{AC} \tag{1}$$

for each scalar d in $\mathbb{F}$.

Proof.

$$\begin{aligned}
[\mathbf{A}(d\mathbf{B}+\mathbf{C})]_{ij} &= \sum_{k=1}^{n} a_{ik}(d\mathbf{B}+\mathbf{C})_{kj} \\
&= \sum_{k=1}^{n} a_{ik}(db_{kj}+c_{kj}) \\
&= \sum_{k=1}^{n} (da_{ik}b_{kj}+a_{ik}c_{kj}) \\
&= d\sum_{k=1}^{n} a_{ik}b_{kj}+\sum_{k=1}^{n} a_{ik}c_{kj} \\
&= d(\mathbf{AB})_{ij}+(\mathbf{AC})_{ij} \\
&= (d(\mathbf{AB})+\mathbf{AC})_{ij}
\end{aligned}$$

□

Theorem 2. *Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. The intersection of any collection of subspcaces of $\mathbb{V}$ is a subspace of $\mathbb{V}$.*

Proof. Let $\mathbb{W}_a$ be a collection of subspaces of $\mathbb{V}$, and let $\mathbb{W}=\cap_a \mathbb{W}_a$ be their intersection. Recall that $\mathbb{W}$ is defined as the set of all elements belonging to every $\mathbb{W}_a$. Since each $\mathbb{W}_a$ is a subspace, each contains the zero vector. Thus zero is inthe intersection of $\mathbb{W}$, and $\mathbb{W}$ is non-empty. Let $\vec{\alpha}$ and $\vec{\beta}$ be vectors in $\mathbb{W}$ and let c be a scalar. By definition of $\mathbb{W}$, both $\vec{\alpha}$ and $\vec{\beta}$ belong to each $\mathbb{W}_a$, and because each $\mathbb{W}_a$ is a subspace, the vector $(c\vec{\alpha}+\vec{\beta})$ is in every $\mathbb{W}_a$. Thus $c\vec{\alpha}+\vec{\beta}$ is in $\mathbb{W}$. By Theorem 1, $\mathbb{W}$ is a subspace of $\mathbb{V}$. □

Definition. *Let $\mathbb{S}$ be a set of vectors in a vector space $\mathbb{V}$. The subspace spanned by $\mathbb{S}$ is defined to be the intersection $\mathbb{W}$ of all subspaces of $\mathbb{V}$ which contain $\mathbb{S}$. When $\mathbb{S}$ is a finite set of vectors, $\mathbb{S} = \vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$, we shall simply call $\mathbb{W}$ the subspace spanned by the vectors $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$.*

Theorem 3. *The subspace spanned by a non-empty subset $\mathbb{S}$ of a vector space $\mathbb{V}$ is the set of all linear combinations of vectors in $\mathbb{S}$.*

Proof. Let $\mathbb{W}$ be the subspace spanned by $\mathbb{S}$. Then each linear combination

$$\vec{\alpha} = x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \cdots + x_m\vec{\alpha}_m$$

of vector $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_m$ in $\mathbb{S}$ is clearly in $\mathbb{W}$. Thus $\mathbb{W}$ contains the set $\mathbb{L}$ of all linear combinations of vectors in $\mathbb{S}$. The set $\mathbb{L}$, on the other hand, contains $\mathbb{S}$ and is non-empty. If $\vec{\alpha}$, $\vec{\beta}$ belong to $\mathbb{L}$ then $\vec{\alpha}$ is a linear combination,

$$\vec{\alpha} = x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \cdots + x_m\vec{\alpha}_m$$

of vectors $\vec{\alpha}_i$ in $\mathbb{S}$, and $\vec{\beta}$ is a linear combination,

$$\vec{\beta} = y_1\vec{\beta}_1 + y_2\vec{\beta}_2 + \cdots + y_n\vec{\beta}_n$$

of vectors $\vec{\beta}_i$ in $\mathbb{S}$. For each scalar c,

$$c\vec{\alpha} + \vec{\beta} = \sum_{i=1}^{m} cx_i\vec{\alpha}_i + \sum_{j=1}^{n} y_i\vec{\beta}_i$$

Hence $c\vec{\alpha}+\vec{\beta}$ belongs to $\mathbb{L}$. Thus $\mathbb{L}$ is a subspace of $\mathbb{V}$. Now we have shown that $\mathbb{L}$ is a subspace of $\mathbb{V}$ which contains $\mathbb{S}$, and also that any subspace which contains $\mathbb{S}$ contains $\mathbb{L}$. It follows that $\mathbb{L}$ is the intersection of all subspaces containing $\mathbb{S}$, i.e., that $\mathbb{L}$ is the subspace spanned by the set $\mathbb{S}$. □

Definition. *If $\mathbb{S}_1, \mathbb{S}_2, \cdots, \mathbb{S}_n$ are subsets of a vector space $\mathbb{V}$, the set of all sums*

$$\vec{\alpha}_1 + \vec{\alpha}_2 + \cdots + \vec{\alpha}_k$$

of vectors $\vec{\alpha}_i$ in $\mathbb{S}_i$ is called the sum of the subsets $\mathbb{S}_1, \mathbb{S}_2, \cdots, \mathbb{S}_k$