

## 0 Review and Miscellanea

This chapter adopts Hoffman's definition for clearer presentations.

### 0.1 Vector Spaces

A finite dimensional vector space is the fundamental setting for matrix analysis.

#### 0.1.1 Scalar Field

**Definition.** *Field  $\mathbb{F}$  is the set together with addition and multiplication. A field must satisfy the following properties:*

1. *Addition is commutative,*

$$x + y = y + x$$

*for all  $x$  and  $y$  in  $\mathbb{F}$ .*

2. *Addition is associative,*

$$x + (y + z) = (x + y) + z$$

*for all  $x, y$  and  $z$  in  $\mathbb{F}$ .*

3. *There is a unique element 0 (zero) in  $\mathbb{F}$  such that  $x + 0 = x$ , for every  $x$  in  $\mathbb{F}$ .*

4. *To each  $x$  in  $\mathbb{F}$  there corresponds a unique element  $(-x)$  in  $\mathbb{F}$  such that  $x + (-x) = 0$ .*

5. *Multiplication is commutative,*

$$xy = yx$$

*for all  $x$  and  $y$  in  $\mathbb{F}$ .*

6. *Multiplication is associative,*

$$x(yz) = (xy)z$$

*for all  $x, y$ , and  $z$  in  $\mathbb{F}$ .*

7. *There is a unique element 1 (one) in  $\mathbb{F}$  such that  $x1 = x$ , for every  $x$  in  $\mathbb{F}$ .*

8. *To each non-zero  $x$  in  $\mathbb{F}$  there corresponds a unique element  $x^{-1}$  (or  $1/x$ ) in  $\mathbb{F}$  such that  $xx^{-1} = 1$ .*

9. *Multiplication distributes over addition; that is,  $x(y + z) = xy + xz$  for all  $x, y$ , and  $z$  in  $\mathbb{F}$ .*

**Remarks.** If  $\mathbb{F}$  is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0:

$$1 + 1 + \cdots + 1 = 0$$

That does not happen in the complex field (or in any subfield thereof). If it does happen in a field  $\mathbb{F}$ , then the least  $n$  such that the sum of  $n$  1's is 0 is called the characteristic of the field  $\mathbb{F}$ . If this does not happen in  $\mathbb{F}$ , then it is called a field of characteristic zero.

**Definition.** A vector space (or linear space) consists of the following:

1. a field  $\mathbb{F}$  of scalars;
2. a set  $\mathbb{V}$  of objects, called vectors;
3. a rule (or operation), called vector addition, which associates each pair of vectors  $\vec{\alpha}, \vec{\beta}$  in  $\mathbb{V}$  a vector  $\vec{\alpha} + \vec{\beta}$  in  $\mathbb{V}$ , called the sum of  $\vec{\alpha}$  and  $\vec{\beta}$ , in such a way that
  - (a) addition is commutative,  $\vec{\alpha} + \vec{\beta} = \vec{\beta} + \vec{\alpha}$ ;
  - (b) addition is associative,  $\vec{\alpha} + (\vec{\beta} + \vec{\gamma}) = (\vec{\alpha} + \vec{\beta}) + \vec{\gamma}$ ;
  - (c) there is a unique vector  $0$  in  $\mathbb{V}$ , called the zero vector, such that  $\vec{\alpha} + 0 = \vec{\alpha}$  for all  $\vec{\alpha}$  in  $\mathbb{V}$ ;
  - (d) for each vector  $\vec{\alpha}$  in  $\mathbb{V}$  there exists a unique vector  $-\vec{\alpha}$  in  $\mathbb{V}$  such that  $\vec{\alpha} + (-\vec{\alpha}) = 0$ .
4. a rule (or operation), called scalar multiplication, which associates with each scalar  $c$  in  $\mathbb{F}$  and vector  $c\vec{\alpha}$  in  $\mathbb{V}$ , called the product of  $c$  and  $\alpha$ , in such a way that
  - (a)  $1\vec{\alpha} = \vec{\alpha}$  for every  $\vec{\alpha}$  in  $\mathbb{V}$ ;
  - (b)  $(c_1c_2)\vec{\alpha} = c_1(c_2\vec{\alpha})$ ;
  - (c)  $c(\vec{\alpha} + \vec{\beta}) = c\vec{\alpha} + c\vec{\beta}$ ;
  - (d)  $(c_1 + c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha}$ .

### 0.1.2 Subspaces, span, and linear combinations

**Definition.** A vector  $\vec{\beta}$  in  $\mathbb{V}$  is said to be a linear combination of the vectors  $\vec{\alpha}_1, \dots, \vec{\alpha}_n$  in  $\mathbb{V}$  provided there exists scalars  $c_1, \dots, c_n$  in  $\mathbb{F}$  such that

$$\begin{aligned}\vec{\beta} &= c_1\vec{\alpha}_1 + \cdots + c_n\vec{\alpha}_n \\ &= \sum_{i=1}^n c_i\vec{\alpha}_i\end{aligned}$$

**Definition.** Let  $\mathbb{V}$  be a vector space over the field  $\mathbb{F}$ . A subspace of  $\mathbb{V}$  is a subset  $\mathbb{W}$  of  $\mathbb{V}$  which is itself a vector space over  $\mathbb{F}$  with the operations of vector addition and scalar multiplication on  $\mathbb{V}$ .

**Theorem 1.** *A non-empty subset  $\mathbb{W}$  of  $\mathbb{V}$  is a subspace of  $\mathbb{V}$  if and only if for each pair of vectors  $\vec{\alpha}, \vec{\beta}$  in  $\mathbb{W}$  and each scalar  $c$  in  $\mathbb{F}$  the vector  $c\vec{\alpha} + \vec{\beta}$  is again in  $\mathbb{W}$ .*

*Proof.* Suppose  $\mathbb{W}$  is a non-empty subset of  $\mathbb{V}$  such that  $c\vec{\alpha} + \vec{\beta}$  belongs to  $\mathbb{W}$  for all vectors  $\vec{\alpha}, \vec{\beta}$  in  $\mathbb{W}$  and all scalars  $c$  in  $\mathbb{F}$ . Since  $\mathbb{W}$  is non-empty, there is a vector  $\vec{\rho}$  in  $\mathbb{W}$ , and hence  $(-1)\vec{\rho} + \vec{\rho} = 0$  is in  $\mathbb{W}$ . Then if  $\vec{\alpha}$  is any vector in  $\mathbb{W}$  and  $c$  any scalar, the vector  $c\vec{\alpha} = c\vec{\alpha} + 0$  is in  $\mathbb{W}$ . In particular,  $(-1)\vec{\alpha} = -\vec{\alpha}$  is in  $\mathbb{W}$ . Finally, if  $\vec{\alpha}$  and  $\vec{\beta}$  are in  $\mathbb{W}$ , then  $\vec{\alpha} + \vec{\beta} = 1\vec{\alpha} + \vec{\beta}$  is in  $\mathbb{W}$ . Thus  $\mathbb{W}$  is a subspace of  $\mathbb{V}$ . Conversely, if  $\mathbb{W}$  is a subspace of  $\mathbb{V}$ ,  $\vec{\alpha}$  and  $\vec{\beta}$  are in  $\mathbb{W}$ , and  $c$  is a scalar, certainly  $c\vec{\alpha} + \vec{\beta}$  is in  $\mathbb{W}$ .  $\square$

**Lemma.** *If  $\mathbf{A}$  is an  $m \times n$  matrix over  $\mathbb{F}$  and  $\mathbf{B}, \mathbf{C}$  are  $n \times p$  matrices over  $\mathbb{F}$ , then*

$$\mathbf{A}(d\mathbf{B} + \mathbf{C}) = d(\mathbf{A}\mathbf{B}) + \mathbf{A}\mathbf{C} \quad (1)$$

for each scalar  $d$  in  $\mathbb{F}$ .

*Proof.*

$$\begin{aligned} [\mathbf{A}(d\mathbf{B} + \mathbf{C})]_{ij} &= \sum_{k=1}^n a_{ik}(d\mathbf{B} + \mathbf{C})_{kj} \\ &= \sum_{k=1}^n a_{ik}(db_{kj} + c_{kj}) \\ &= \sum_{k=1}^n (da_{ik}b_{kj} + a_{ik}c_{kj}) \\ &= d \sum_{k=1}^n a_{ik}b_{kj} + \sum_{k=1}^n a_{ik}c_{kj} \\ &= d(\mathbf{A}\mathbf{B})_{ij} + (\mathbf{A}\mathbf{C})_{ij} \\ &= (d(\mathbf{A}\mathbf{B}) + \mathbf{A}\mathbf{C})_{ij} \end{aligned}$$

$\square$

**Theorem 2.** *Let  $\mathbb{V}$  be a vector space over the field  $\mathbb{F}$ . The intersection of any collection of subspaces of  $\mathbb{V}$  is a subspace of  $\mathbb{V}$ .*

*Proof.* Let  $\mathbb{W}_a$  be a collection of subspaces of  $\mathbb{V}$ , and let  $\mathbb{W} = \cap_a \mathbb{W}_a$   $\square$