

0 Review and Miscellanea

This chapter adopts Hoffman's definition for clearer presentations.

0.1 Vector Spaces

A finite dimensional vector space is the fundamental setting for matrix analysis.

0.1.1 Scalar Field

Definition. *Field \mathbb{F} is the set together with addition and multiplication. A field must satisfy the following properties:*

1. *Addition is commutative,*

$$x + y = y + x$$

for all x and y in \mathbb{F} .

2. *Addition is associative,*

$$x + (y + z) = (x + y) + z$$

for all x , y and z in \mathbb{F} .

3. *There is a unique element 0 (zero) in \mathbb{F} such that $x + 0 = x$, for every x in \mathbb{F} .*

4. *To each x in \mathbb{F} there corresponds a unique element $(-x)$ in \mathbb{F} such that $x + (-x) = 0$.*

5. *Multiplication is commutative,*

$$xy = yx$$

for all x and y in \mathbb{F} .

6. *Multiplication is associative,*

$$x(yz) = (xy)z$$

for all x , y , and z in \mathbb{F} .

7. *There is a unique element 1 (one) in \mathbb{F} such that $x1 = x$, for every x in \mathbb{F} .*

8. *To each non-zero x in \mathbb{F} there corresponds a unique element x^{-1} (or $1/x$) in \mathbb{F} such that $xx^{-1} = 1$.*

9. *Multiplication distributes over addition; that is, $x(y + z) = xy + xz$ for all x , y , and z in \mathbb{F} .*

Remarks. *If \mathbb{F} is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0 :*

$$1 + 1 + \cdots + 1 = 0$$

That does not happen in the complex field (or in any subfield thereof). If it does happen in a field \mathbb{F} , then the least n such that the sum of n 1 's is 0 is called the characteristic of the field \mathbb{F} . If this does not happen in \mathbb{F} , then it is called a field of characteristic zero.

0.1.2 Vector spaces

Definition. A vector space (or linear space) consists of the following:

1. a field \mathbb{F} of scalars;
2. a set \mathbb{V} of objects, called vectors;
3. a rule (or operation), called vector addition, which associates each pair of vectors $\vec{\alpha}, \vec{\beta}$ in \mathbb{V} a vector $\vec{\alpha} + \vec{\beta}$ in \mathbb{V} , called the sum of $\vec{\alpha}$ and $\vec{\beta}$, in such a way that
 - (a) addition is commutative, $\vec{\alpha} + \vec{\beta} = \vec{\beta} + \vec{\alpha}$;
 - (b) addition is associative, $\vec{\alpha} + (\vec{\beta} + \vec{\gamma}) = (\vec{\alpha} + \vec{\beta}) + \vec{\gamma}$;
 - (c) there is a unique vector 0 in \mathbb{V} , called the zero vector, such that $\vec{\alpha} + 0 = \vec{\alpha}$ for all $\vec{\alpha}$ in \mathbb{V} ;
 - (d) for each vector $\vec{\alpha}$ in \mathbb{V} there exists a unique vector $-\vec{\alpha}$ in \mathbb{V} such that $\vec{\alpha} + (-\vec{\alpha}) = 0$.
4. a rule (or operation), called scalar multiplication, which associates with each scalar c in \mathbb{F} and vector $c\vec{\alpha}$ in \mathbb{F} , called the product of c and α , in such a way that
 - (a) $1\vec{\alpha} = \vec{\alpha}$ for every $\vec{\alpha}$ in \mathbb{V} ;
 - (b) $(c_1c_2)\vec{\alpha} = c_1(c_2\vec{\alpha})$;
 - (c) $c(\vec{\alpha} + \vec{\beta}) = c\vec{\alpha} + c\vec{\beta}$;
 - (d) $(c_1 + c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha}$.

0.1.3 Subspaces, span, and linear combinations

Definition. A vector $\vec{\beta}$ in \mathbb{V} is said to be a linear combination of the vectors $\vec{\alpha}_1, \dots, \vec{\alpha}_n$ in \mathbb{V} provided there exists scalars c_1, \dots, c_n in \mathbb{F} such that

$$\begin{aligned}\vec{\beta} &= c_1\vec{\alpha}_1 + \dots + c_n\vec{\alpha}_n \\ &= \sum_{i=1}^n c_i\vec{\alpha}_i\end{aligned}$$

Definition. Let \mathbb{V} be a vector space over the field \mathbb{F} . A subspace of \mathbb{V} is a subset \mathbb{W} of \mathbb{V} which is itself a vector space over \mathbb{F} with the operations of vector addition and scalar multiplication on \mathbb{V} .

Theorem 1. A non-empty subset \mathbb{W} of \mathbb{V} is a subspace of \mathbb{V} if and only if for each pair of vectors $\vec{\alpha}, \vec{\beta}$ in \mathbb{W} and each scalar c in \mathbb{F} the vector $c\vec{\alpha} + \vec{\beta}$ is again in \mathbb{W} .

Proof. Suppose \mathbb{W} is a non-empty subset of \mathbb{V} such that $c\vec{\alpha} + \vec{\beta}$ belongs to \mathbb{W} for all vectors $\vec{\alpha}, \vec{\beta}$ in \mathbb{W} and all scalars c in \mathbb{F} . Since \mathbb{W} is non-empty, there is a vector $\vec{\rho}$ in \mathbb{W} , and hence $(-1)\vec{\rho} + \vec{\rho} = 0$ is in \mathbb{W} . Then if $\vec{\alpha}$ is any vector in \mathbb{W} and c any scalar, the vector $c\vec{\alpha} = c\vec{\alpha} + 0$ is in \mathbb{W} . In particular, $(-1)\vec{\alpha} = -\vec{\alpha}$ is in \mathbb{W} . Finally, if $\vec{\alpha}$ and $\vec{\beta}$ are in \mathbb{W} , then $\vec{\alpha} + \vec{\beta} = 1\vec{\alpha} + \vec{\beta}$ is in \mathbb{W} . Thus \mathbb{W} is a subspace of \mathbb{V} . Conversely, if \mathbb{W} is a subspace of \mathbb{V} , $\vec{\alpha}$ and $\vec{\beta}$ are in \mathbb{W} , and c is a scalar, certainly $c\vec{\alpha} + \vec{\beta}$ is in \mathbb{W} . \square

Lemma. If \mathbf{A} is an $m \times n$ matrix over \mathbb{F} and \mathbf{B}, \mathbf{C} are $n \times p$ matrices over \mathbb{F} , then

$$\mathbf{A}(d\mathbf{B} + \mathbf{C}) = d(\mathbf{A}\mathbf{B}) + \mathbf{A}\mathbf{C} \tag{1}$$

for each scalar d in \mathbb{F} .

Proof.

$$\begin{aligned}
[\mathbf{A}(d\mathbf{B} + \mathbf{C})]_{ij} &= \sum_{k=1}^n a_{ik}(d\mathbf{B} + \mathbf{C})_{kj} \\
&= \sum_{k=1}^n a_{ik}(db_{kj} + c_{kj}) \\
&= \sum_{k=1}^n (da_{ik}b_{kj} + a_{ik}c_{kj}) \\
&= d \sum_{k=1}^n a_{ik}b_{kj} + \sum_{k=1}^n a_{ik}c_{kj} \\
&= d(\mathbf{AB})_{ij} + (\mathbf{AC})_{ij} \\
&= (d(\mathbf{AB}) + \mathbf{AC})_{ij}
\end{aligned}$$

□

Theorem 2. Let \mathbb{V} be a vector space over the field \mathbb{F} . The intersection of any collection of subspaces of \mathbb{V} is a subspace of \mathbb{V} .

Proof. Let \mathbb{W}_a be a collection of subspaces of \mathbb{V} , and let $\mathbb{W} = \cap_a \mathbb{W}_a$ be their intersection. Recall that \mathbb{W} is defined as the set of all elements belonging to every \mathbb{W}_a . Since each \mathbb{W}_a is a subspace, each contains the zero vector. Thus zero is in the intersection of \mathbb{W} , and \mathbb{W} is non-empty. Let $\vec{\alpha}$ and $\vec{\beta}$ be vectors in \mathbb{W} and let c be a scalar. By definition of \mathbb{W} , both $\vec{\alpha}$ and $\vec{\beta}$ belong to each \mathbb{W}_a , and because each \mathbb{W}_a is a subspace, the vector $(c\vec{\alpha} + \vec{\beta})$ is in every \mathbb{W}_a . Thus $c\vec{\alpha} + \vec{\beta}$ is in \mathbb{W} . By Theorem 1, \mathbb{W} is a subspace of \mathbb{V} . □

Definition. Let \mathbb{S} be a set of vectors in a vector space \mathbb{V} . The subspace spanned by \mathbb{S} is defined to be the intersection \mathbb{W} of all subspaces of \mathbb{V} which contain \mathbb{S} . When \mathbb{S} is a finite set of vectors, $\mathbb{S} = \vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_n$, we shall simply call \mathbb{W} the subspace spanned by the vectors $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_n$.

Theorem 3. The subspace spanned by a non-empty subset \mathbb{S} of a vector space \mathbb{V} is the set of all linear combinations of vectors in \mathbb{S} .

Proof. Let \mathbb{W} be the subspace spanned by \mathbb{S} . Then each linear combination

$$\vec{\alpha} = x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \dots + x_m\vec{\alpha}_m$$

of vector $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_m$ in \mathbb{S} is clearly in \mathbb{W} . Thus \mathbb{W} contains the set \mathbb{L} of all linear combinations of vectors in \mathbb{S} . The set \mathbb{L} , on the other hand, contains \mathbb{S} and is non-empty. If $\vec{\alpha}, \vec{\beta}$ belong to \mathbb{L} then $\vec{\alpha}$ is a linear combination,

$$\vec{\alpha} = x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \dots + x_m\vec{\alpha}_m$$

of vectors $\vec{\alpha}_i$ in \mathbb{S} , and $\vec{\beta}$ is a linear combination,

$$\vec{\beta} = y_1\vec{\beta}_1 + y_2\vec{\beta}_2 + \dots + y_n\vec{\beta}_n$$

of vectors $\vec{\beta}_i$ in \mathbb{S} . For each scalar c ,

$$c\vec{\alpha} + \vec{\beta} = \sum_{i=1}^m cx_i\vec{\alpha}_i + \sum_{j=1}^n y_j\vec{\beta}_j$$

Hence $c\vec{\alpha} + \vec{\beta}$ belongs to \mathbb{L} . Thus \mathbb{L} is a subspace of \mathbb{V} . Now we have shown that \mathbb{L} is a subspace of \mathbb{V} which contains \mathbb{S} , and also that any subspace which contains \mathbb{S} contains \mathbb{L} . It follows that \mathbb{L} is the intersection of all subspaces containing \mathbb{S} , i.e., that \mathbb{L} is the subspace spanned by the set \mathbb{S} . \square

Definition. If $\mathbb{S}_1, \mathbb{S}_2, \dots, \mathbb{S}_n$ are subsets of a vector space \mathbb{V} , the set of all sums

$$\vec{\alpha}_1 + \vec{\alpha}_2 + \dots + \vec{\alpha}_k$$

of vectors $\vec{\alpha}_i$ in \mathbb{S}_i is called the sum of the subsets $\mathbb{S}_1, \mathbb{S}_2, \dots, \mathbb{S}_k$ and is denoted by

$$\mathbb{S}_1 + \mathbb{S}_2 + \dots + \mathbb{S}_k$$

or by

$$\sum_{i=1}^k \mathbb{S}_i$$

0.1.4 Linear dependence and linear independence

Definition. Let \mathbb{V} be a vector space over \mathbb{F} . A subset \mathbb{S} of \mathbb{V} is said to be linearly dependent (or simply, dependent) if there exists distinct vectors $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_n$ in \mathbb{S} and scalars c_1, c_2, \dots, c_n in \mathbb{F} , not all of which are 0, such that

$$c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \dots + c_n\vec{\alpha}_n = 0$$

A set which is not linearly dependent is called linearly independent. If the set \mathbb{S} contains only finitely many vectors $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_n$, we sometimes say that $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_n$ is dependent (or independent) instead of saying \mathbb{S} is dependent (or independent).

0.1.5 Basis

Definition. Let \mathbb{V} be a vector space. A basis for \mathbb{V} is linearly independent set of vectors in \mathbb{V} which spans the space \mathbb{V} . The space \mathbb{V} is finite-dimensional if it has a finite basis.

Theorem 4. Let \mathbb{V} be a vector space which is spanned by a finite set of vectors $\vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_m$. Then any independent set of vectors in \mathbb{V} is finite and contains no more than m elements.

Proof. To prove the theorem it suffices to show that every subset \mathbb{S} of \mathbb{V} which contains more than m vectors is linearly dependent. Let \mathbb{S} be such a set. In \mathbb{S} there are distinct vectors $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_n$ where $n > m$. Since $\vec{\beta}_1, \dots, \vec{\beta}_m$ span \mathbb{V} , there exist scalars A_{ij} in \mathbb{F} such that

$$\vec{\alpha}_j = \sum_{i=1}^m A_{ij} \vec{\beta}_i$$

For any n scalars x_1, x_2, \dots, x_n , we have

$$\begin{aligned}
 x_1 \vec{\alpha}_1 + \dots + x_n \vec{\alpha}_n &= \sum_{j=1}^n x_j \vec{\alpha}_j \\
 &= \sum_{j=1}^n x_j \sum_{i=1}^m A_{ij} \vec{\beta}_i \\
 &= \sum_{j=1}^n \sum_{i=1}^m (A_{ij} x_j) \vec{\beta}_i \\
 &= \sum_{i=1}^m \left(\sum_{j=1}^n A_{ij} x_j \right) \vec{\beta}_i
 \end{aligned}$$

Since $n > m$, it implies there exist scalars x_1, x_2, \dots, x_n not all 0 such that

$$\sum_{j=1}^n A_{ij} x_j = 0, \quad 1 \leq i \leq m.$$

Hence $x_1 \vec{\alpha}_1 + x_2 \vec{\alpha}_2 + \dots + x_n \vec{\alpha}_n = 0$. This allows that \mathbb{S} is a linearly dependent set. □

Corollary. *If \mathbb{V} is*