# 0 Review and Miscellanea

This chapter adopts Hoffman's definition for clearer presentations.

## 0.1 Vector Spaces

A finite dimensional vector space is the fundamental setting for matrix analysis.

### 0.1.1 Scalar Field

**Definition.** *Field $\mathbb{F}$ is the set together with addition and multiplication. A field must satisfy the following properties:*

1. *Addition is commutative,*
$$x + y = y + x$$
*for all $x$ and $y$ in $\mathbb{F}$.*

2. *Addition is associative,*
$$x + (y + z) = (x + y) + z$$
*for all $x$, $y$ and $z$ in $\mathbb{F}$.*

3. *There is a unique element 0 (zero) in $\mathbb{F}$ such that $x + 0 = x$, for every $x$ in $\mathbb{F}$.*

4. *To each $x$ in $\mathbb{F}$ there corresponds a unique element $(-x)$ in $\mathbb{F}$ such that $x + (-x) = 0$.*

5. *Multiplication is commutative,*
$$xy = yx$$
*for all $x$ and $y$ in $\mathbb{F}$.*

6. *Multiplication is associative,*
$$x(yz) = (xy)z$$
*for all $x$, $y$, and $z$ in $\mathbb{F}$.*

7. *There is a unique element 1 (one) in $\mathbb{F}$ such that $x1 = x$, for every $x$ in $\mathbb{F}$.*

8. *To each non-zero $x$ in $\mathbb{F}$ there corresponds a unique element $x^{-1}$ (or $1/x$) in $\mathbb{F}$ such that $xx^{-1} = 1$.*

9. *Multiplication distributes over addition; that is, $x(y + z) = xy + xz$ for all $x$, $y$, and $z$ in $\mathbb{F}$.*

**Remarks.** *If $\mathbb{F}$ is a field, it may be possible to add the unit 1 to itself a finite number of times and obtain 0:*
$$1 + 1 + \cdots + 1 = 0$$
*That does not happen in the complex field (or in any subfield thereof). If it does happen in a field $\mathbb{F}$, then the least $n$ such that the sum of $n$ 1's is 0 is called the characteristic of the field $\mathbb{F}$. If this does not happen in $\mathbb{F}$, then it it is called a field of characteristic zero.*

### 0.1.2 Vector spaces

**Definition.** *A vector space (or linear space) consists of the following:*

1. *a field $\mathbb{F}$ of scalars;*
2. *a set $\mathbb{V}$ of objects, called vectors;*
3. *a rule (or operation), called vector addition, which associates each pair of vectors $\vec{\alpha}$, $\vec{\beta}$ in $\mathbb{V}$ a vector $\vec{\alpha}+\vec{\beta}$ in $\mathbb{V}$, called the sum of $\vec{\alpha}$ and $\vec{\beta}$, in such a way that*
   (a) *addition is commutative, $\vec{\alpha}+\vec{\beta}=\vec{\beta}+\vec{\alpha}$;*
   (b) *addition is associative, $\vec{\alpha}+(\vec{\beta}+\vec{\gamma})=(\vec{\alpha}+\vec{\beta})+\vec{\gamma}$;*
   (c) *there is a unique vector 0 in $\mathbb{V}$, called the zero vector, such that $\vec{\alpha}+0=\vec{\alpha}$ for all $\vec{\alpha}$ in $\mathbb{V}$;*
   (d) *for each vector $\vec{\alpha}$ in $\mathbb{V}$ there exists a unique vector $-\vec{\alpha}$ in $\mathbb{V}$ such that $\vec{\alpha}+(-\vec{\alpha})=0$.*
4. *a rule (or operation), called scalar multiplication, which associates with each scalar $c$ in $\mathbb{F}$ and vector $c\vec{\alpha}$ in $\mathbb{F}$, called the product of $c$ and $\alpha$, in such a way that*
   (a) *$1\vec{\alpha}=\vec{\alpha}$ for every $\vec{\alpha}$ in $\mathbb{V}$;*
   (b) *$(c_1c_2)\vec{\alpha}=c_1(c_2\vec{\alpha})$;*
   (c) *$c(\vec{\alpha}+\vec{\beta})=c\vec{\alpha}+c\vec{\beta}$;*
   (d) *$(c_1+c_2)\vec{\alpha}=c_1\vec{\alpha}+c_2\vec{\alpha}$.*

### 0.1.3 Subspaces, span, and linear combinations

**Definition.** *A vector $\vec{\beta}$ in $\mathbb{V}$ is said to be a linear combination of the vectors $\vec{\alpha_1},\cdots,\vec{\alpha_n}$ in $\mathbb{V}$ provided there exists scalars $c_1,\cdots,c_n$ in $\mathbb{F}$ such that*

$$\begin{aligned}\vec{\beta} &= c_1\vec{\alpha_1}+\cdots+c_n\vec{\alpha_n}\\ &= \sum_{i=1}^{n} c_i\vec{\alpha}_i\end{aligned}$$

**Definition.** *Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. A subspace of $\mathbb{V}$ is a subset $\mathbb{W}$ of $\mathbb{V}$ which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication on $\mathbb{V}$.*

**Theorem 1.** *A non-empty subset $\mathbb{W}$ of $\mathbb{V}$ is a subspace of $\mathbb{V}$ if and only if for each pair of vectors $\vec{\alpha}$, $\vec{\beta}$ in $\mathbb{W}$ and each scalar $c$ in $\mathbb{F}$ the vector $c\vec{\alpha}+\vec{\beta}$ is again in $\mathbb{W}$.*

*Proof.* Suppose $\mathbb{W}$ is a non-empty subset of $\mathbb{V}$ such that $c\vec{\alpha}+\vec{\beta}$ belongs to $\mathbb{W}$ for all vectors $\vec{\alpha}$, $\vec{\beta}$ in $\mathbb{W}$ and all scalars $c$ in $\mathbb{F}$. Since $\mathbb{W}$ is non-empty, there is a vector $\vec{\rho}$ in $\mathbb{W}$, and hence $(-1)\vec{\rho}+\vec{\rho}=0$ is in $\mathbb{W}$. Then if $\vec{\alpha}$ is any vector in $\mathbb{W}$ and c any scalar, the vector $c\vec{\alpha}=c\vec{\alpha}+0$ is in $\mathbb{W}$. In particular, $(-1)\vec{\alpha}=-\vec{\alpha}$ is in $\mathbb{W}$. Finally, if $\vec{\alpha}$ and $\vec{\beta}$ are in $\mathbb{W}$, then $\vec{\alpha}+\vec{\beta}=1\vec{\alpha}+\vec{\beta}$ is in $\mathbb{W}$. Thus $\mathbb{W}$ is a subspace of $\mathbb{V}$. Conversely, if $\mathbb{W}$ is a subspace of $\mathbb{V}$, $\vec{\alpha}$ and $\vec{\beta}$ are in $\mathbb{W}$, and $c$ is a scalar, certainly $c\vec{\alpha}+\vec{\beta}$ is in $\mathbb{W}$. □

**Lemma.** *If $\mathbf{A}$ is an $m\times n$ matrix over $\mathbb{F}$ and $\mathbf{B}$, $\mathbf{C}$ are $n\times p$ matrices over $\mathbb{F}$, then*

$$\mathbf{A}(d\mathbf{B}+\mathbf{C})=d(\mathbf{AB})+\mathbf{AC} \tag{1}$$

*for each scalar $d$ in $\mathbb{F}$.*

*Proof.*

$$\begin{aligned}
[\mathbf{A}(d\mathbf{B}+\mathbf{C})]_{ij} &= \sum_{k=1}^{n} a_{ik}(d\mathbf{B}+\mathbf{C})_{kj} \\
&= \sum_{k=1}^{n} a_{ik}(db_{kj}+c_{kj}) \\
&= \sum_{k=1}^{n} (da_{ik}b_{kj}+a_{ik}c_{kj}) \\
&= d\sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \\
&= d(\mathbf{AB})_{ij} + (\mathbf{AC})_{ij} \\
&= (d(\mathbf{AB}) + \mathbf{AC})_{ij}
\end{aligned}$$

□

**Theorem 2.** *Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. The intersection of any collection of subspcaces of $\mathbb{V}$ is a subspace of $\mathbb{V}$.*

*Proof.* Let $\mathbb{W}_a$ be a collection of subspaces of $\mathbb{V}$, and let $\mathbb{W} = \cap_a \mathbb{W}_a$ be their intersection. Recall that $\mathbb{W}$ is defined as the set of all elements belonging to every $\mathbb{W}_a$. Since each $\mathbb{W}_a$ is a subspace, each contains the zero vector. Thus zero is inthe intersection of $\mathbb{W}$, and $\mathbb{W}$ is non-empty. Let $\vec{\alpha}$ and $\vec{\beta}$ be vectors in $\mathbb{W}$ and let $c$ be a scalar. By definition of $\mathbb{W}$, both $\vec{\alpha}$ and $\vec{\beta}$ belong to each $\mathbb{W}_a$, and because each $\mathbb{W}_a$ is a subspace, the vector $(c\vec{\alpha}+\vec{\beta})$ is in every $\mathbb{W}_a$. Thus $c\vec{\alpha}+\vec{\beta}$ is in $\mathbb{W}$. By Theorem 1, $\mathbb{W}$ is a subspace of $\mathbb{V}$. □

**Definition.** *Let $\mathbb{S}$ be a set of vectors in a vector space $\mathbb{V}$. The subspace spanned by $\mathbb{S}$ is defined to be the intersection $\mathbb{W}$ of all subspaces of $\mathbb{V}$ which contain $\mathbb{S}$. When $\mathbb{S}$ is a finite set of vectors, $\mathbb{S} = \vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$, we shall simply call $\mathbb{W}$ the subspace spanned by the vectors $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$.*

**Theorem 3.** *The subspace spanned by a non-empty subset $\mathbb{S}$ of a vector space $\mathbb{V}$ is the set of all linear combinations of vectors in $\mathbb{S}$.*

*Proof.* Let $\mathbb{W}$ be the subspace spanned by $\mathbb{S}$. Then each linear combination

$$\vec{\alpha} = x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \cdots + x_m\vec{\alpha}_m$$

of vector $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_m$ in $\mathbb{S}$ is clearly in $\mathbb{W}$. Thus $\mathbb{W}$ contains the set $\mathbb{L}$ of all linear combinations of vectors in $\mathbb{S}$. The set $\mathbb{L}$, on the other hand, contains $\mathbb{S}$ and is non-empty. If $\vec{\alpha}$, $\vec{\beta}$ belong to $\mathbb{L}$ then $\vec{\alpha}$ is a linear combination,

$$\vec{\alpha} = x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \cdots + x_m\vec{\alpha}_m$$

of vectors $\vec{\alpha}_i$ in $\mathbb{S}$, and $\vec{\beta}$ is a linear combination,

$$\vec{\beta} = y_1\vec{\beta}_1 + y_2\vec{\beta}_2 + \cdots + y_n\vec{\beta}_n$$

of vectors $\vec{\beta}_i$ in $\mathbb{S}$. For each scalar $c$,

$$c\vec{\alpha}+\vec{\beta} = \sum_{i=1}^{m} cx_i\vec{\alpha}_i + \sum_{j=1}^{n} y_i\vec{\beta}_i$$

Hence $c\vec{\alpha} + \vec{\beta}$ belongs to $\mathbb{L}$. Thus $\mathbb{L}$ is a subspace of $\mathbb{V}$. Now we have shown that $\mathbb{L}$ is a subspace of $\mathbb{V}$ which contains $\mathbb{S}$, and also that any subspace which contains $\mathbb{S}$ contains $\mathbb{L}$. It follows that $\mathbb{L}$ is the intersection of all subspaces containing $\mathbb{S}$, i.e., that $\mathbb{L}$ is the subspace spanned by the set $\mathbb{S}$. $\square$

**Definition.** *If $\mathbb{S}_1, \mathbb{S}_2, \cdots, \mathbb{S}_n$ are subsets of a vector space $\mathbb{V}$, the set of all sums*

$$\vec{\alpha}_1 + \vec{\alpha}_2 + \cdots + \vec{\alpha}_k$$

*of vectors $\vec{\alpha}_i$ in $\mathbb{S}_i$ is called the sum of the subsets $\mathbb{S}_1, \mathbb{S}_2, \cdots, \mathbb{S}_k$ and is denoted by*

$$\mathbb{S}_1 + \mathbb{S}_2 + \cdots + \mathbb{S}_k$$

*or by*

$$\sum_{i=1}^{k} \mathbb{S}_i$$

### 0.1.4 Linear dependence and linear independence

**Definition.** *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. A subset $\mathbb{S}$ of $\mathbb{V}$ is said to be linearly dependent (or simply, dependent) if there exists distinct vectors $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$ in $\mathbb{S}$ and scalars $c_1, c_2, \cdots, c_n$ in $\mathbb{F}$, not all of which are $0$, such that*

$$c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + \cdots + c_n\vec{\alpha}_n = 0$$

*A set which is not linearly dependent is called linearly independent. If the set $\mathbb{S}$ contains only finitely many vectors $\vec{\alpha}_1, \vec{\alpha}_2, \cdots \vec{\alpha}_n$, we someimtes say that $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$ is dependent (or independent) instead of saying $\mathbb{S}$ is dependent (or independent).*

### 0.1.5 Basis

**Definition.** *Let $\mathbb{V}$ be a vector space. A basis for $\mathbb{V}$ is linearly independent set of vectors in $\mathbb{V}$ which spans the space $\mathbb{V}$. The space $\mathbb{V}$ is finite-dimensional if it has a finite basis.*

**Theorem 4.** *Let $\mathbb{V}$ be a vector space which is spanned by a finite set of vectors $\vec{\beta}_1, \vec{\beta}_2, \cdots, \vec{\beta}_m$. Then any independent set of vectors in $\mathbb{V}$ is finite and contains no more than $m$ elements.*

*Proof.* To prove the theorem it suffices to show that every subset $\mathbb{S}$ of $\mathbb{V}$ which contains more than $m$ vectors is linearly dependent. Let $\mathbb{S}$ be such a set. In $\mathbb{S}$ there are distinct vectors $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$ where $n > m$. Since $\vec{\beta}_1, \cdots, \vec{\beta}_m$ span $\mathbb{V}$, there exist scalars $A_{ij}$ in $\mathbb{F}$ such that

$$\vec{\alpha}_j = \sum_{i=1}^{m} A_{ij}\vec{\beta}_i$$

For any $n$ scalars $x_1, x_2, \cdots, x_n$, we have

$$\begin{aligned} x_1\vec{\alpha}_1 + \cdots + x_n\vec{\alpha}_n &= \sum_{j=1}^{n} x_j\vec{\alpha}_j \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\vec{\beta}_i \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m}(A_{ij}x_j)\vec{\beta}_i \\ &= \sum_{i=1}^{m}(\sum_{j=1}^{n} A_{ij}x_j)\vec{\beta}_i \end{aligned}$$

Since $n > m$, it implies there exist scalars $x_1, x_2, \cdots, x_n$ not all 0 such that

$$\sum_{j=i}^{n} A_{ij}x_j = 0, \qquad 1 \le i \le m.$$

Hence $x_1\vec{\alpha}_1 + x_2\vec{\alpha}_2 + \cdots + x_n\vec{\alpha}_n = 0$. This allows that $\mathbb{S}$ is a linearly dependent set. □

**Corollary 1.** *If $\mathbb{V}$ is a finite-dimensional vector space, then any two bases of $\mathbb{V}$ have the same (finite) number of elements.*

*Proof.* Since $\mathbb{V}$ is finite-dimensional, it has the finite basis

$$\vec{\beta}_1, \vec{\beta}_2, \cdots, \vec{\beta}_m$$

By Theorem 4 every basis of $\mathbb{V}$ is finite and contains no more than $m$ elements. Thus if $\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$ is a basis, $n \le m$. By the same argument, $m \le n$. Hence $m = n$. □

**Corollary 2.** *Let $\mathbb{V}$ be a finite-dimensional vector space and let $n = \dim V$. Then*

1. *any subset of $\mathbb{V}$ which contains more than $n$ vectors is linearly dependent;*
2. *no subset of $\mathbb{V}$ which contains fewer than $n$ vectors can span $\mathbb{V}$.*

**Lemma.** *Let $\mathbb{S}$ be a linearly independent subset of a vector space $\mathbb{V}$. Suppose $\vec{\beta}$ is a vector in $\mathbb{V}$ which is not in the subspace spanned by $\mathbb{S}$. Then the set obtained by adjoining $\vec{\beta}$ to $\mathbb{S}$ is linearly independent.*

*Proof.* Suppose $\vec{\alpha}_1, \cdots, \vec{\alpha}_m$ are distinct vectors in $\mathbb{S}$ and that

$$c_1\vec{\alpha}_1 + \cdots + c_m\vec{\alpha}_m + b\vec{\beta} = 0$$

Then $b = 0$; for otherwise,

$$\vec{\beta} = (-c_1/b)\vec{\alpha}_1 + \cdots + (-c_m/b)\vec{\alpha}_m$$

and $\vec{\beta}$ is in the subspace spanned by $\mathbb{S}$. Thus $c_1\vec{\alpha}_1 + \cdots + c_m\vec{\alpha}_m = 0$, and since $\mathbb{S}$ is linearly independent set each $c_i = 0$. □

**Theorem 5.** *If $\mathbb{W}$ is a subspace of a finite-dimensional vector space $\mathbb{V}$, every linearly independent subset of $\mathbb{W}$ is finite and is part of a (finite) basis for $\mathbb{W}$.*

*Proof.* Suppose $\mathbb{S}_0$ is a linearly independent subset of $\mathbb{W}$. If $\mathbb{S}$ is a linearly independent subset of $\mathbb{W}$ containing $\mathbb{S}_0$, then $\mathbb{S}$ is also a linearly independent subset of $\mathbb{V}$; since $\mathbb{V}$ is finite-dimensional, $\mathbb{S}_0$ contains no more than $\dim V$ elements.

We extend $\mathbb{S}_0$ to a basis for $\mathbb{W}$, as follows. If $\mathbb{S}_0$ spans $\mathbb{W}$, then $\mathbb{S}_0$ is a basis for $\mathbb{W}$ and we are done. If $\mathbb{S}_0$ does not span $\mathbb{W}$, we use the preceding lemma to find a vector $\vec{\beta}_1$ in $\mathbb{W}$ such that the set $\mathbb{S}_1 = \mathbb{S}_0 \cup \{\vec{\beta}_1\}$ is independent. If $\mathbb{S}_0$ spans $\mathbb{W}$, fine. If not, apply the lemma to obtain a vector $\vec{\beta}_2$ in $\mathbb{W}$ such that $\mathbb{S}_2 = \mathbb{S}_1 \cup \{\vec{\beta}_2\}$ is independent. If we continue in this way, then (in not more than $\dim \mathbb{V}$ steps) we reach a set

$$\mathbb{S}_m = \mathbb{S}_0 \cup \{\vec{\beta}_1, \cdots, \vec{\beta}_m\}$$

□