For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Let A and B be sets.

(i) Under what conditions do we have $A \times B = B \times A$?
(ii) When is it true that $|\mathscr{P}(A) \times \mathscr{P}(A)| = |\mathscr{P}(A \times A)|$?
(iii) What can you conclude if $A - B = \varnothing$?
(iv) Describe in words the set $X = (A \times A) - D$, where the subset $D \subsetneq A \times A$ is given by $D = \{(a, a) : a \in A\}$.

Solution.

(i) When, $A = B$, when $A = \varnothing$, or when $B = \varnothing$. Or rephrased: $A \times B = B \times A \iff (A = B) \vee (A = \varnothing) \vee (B = \varnothing)$ Proof by contraposition: Assume $A \neq B \neq \varnothing$, where $x \in A$ but $x \notin B$. Then, $\{(x, b) : b \in B\} \subseteq (A \times B)$ and $\{(x, b) : b \in B\} \nsubseteq (B \times A)$. Because at least one member is present in $(A \times B)$ and not $(B \times A)$, $(A \times B) \neq (B \times A)$. The other cases are trivial: if $A = B$, then $A \times B = A \times A = B \times A$. The cartesian product of any set and the emptyset is the emptyset, because their are no elements to iterate over.

(ii) $|\mathscr{P}(A) \times \mathscr{P}(A)| = |\mathscr{P}(A \times A)|$

$|\mathscr{P}(A)| \cdot |\mathscr{P}(A)| = 2^{(|A| \cdot |A|)}$

$2^{|A|} \cdot 2^{|A|} = 2^{(|A|^2)}$

$2^{2|A|} = 2^{(|A|^2)}$

$2|A| = |A|^2$

$0 = |A|^2 - 2|A| = |A|(|A| - 2)$

Thus, $|A| \in \{0, 2\}$

(iii) $A - B = \{x : x \in A, x \notin B\} = \varnothing$

No x in A that's not in B, thus, $\forall x(\neg(x \in B) \to \neg(x \in A))$. Applying logical inference of contraposition: $\forall x((x \in A) \to (x \in B))$. For every x in A, it is also in B. This is the definition of a subset. Thus, $A \subseteq B$.

(iv) X is the emptyset. $D = \{(a, a) : a \in A\}$ is the simplified definition of the cartesian product if the left and right hand arguments are the same. $D = (A \times A)$.

$(A \times A) - D \iff (A \times A) - (A \times A)$. If we take the conclusion of the above item, then, because $(A \times A) \subseteq (A \times A)$, $(A \times A) - (A \times A) = \varnothing = X$.

□

Problem 2. Determine whether each of the following is true or false; justify your answer.

(i) $\mathbb{R}^2 \subseteq \mathbb{R}^3$
(ii) $A \times \varnothing = \varnothing$ for every set A.
(iii) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.
(iv) If $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$.

Solution.

(i) $\mathbb{R}^2 \subseteq \mathbb{R}^3$ is false because there is at least one element in $\mathbb{R}^2$ (take the ordered pair $(0, 0)$ for example), that is not in $\mathbb{R}^3$ (which has a similar but different math object $(0, 0, 0)$).

(ii) $A \times \varnothing = \varnothing$ for every set A is true and was demonstrated above. "The cartesian product of any set and the emptyset is the emptyset, because their are no elements to iterate over." Or rephrased, there are no pairs of elements we can make because $\varnothing$ has no elements.

(iii) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ is true. $\mathscr{P}(A) = \{X : X \subseteq A\}$. $X \subseteq A \subseteq B \implies X \subseteq B$. $\mathscr{P}(A)$ thus contains only subsets of B. Because, $\mathscr{P}(B)$ contains *all* subsets of B, $\mathscr{P}(A) \subseteq \mathscr{P}(B)$

(iv) If $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$ is true. Contraposition: $A \nsubseteq B \implies \mathscr{P}(A) \nsubseteq \mathscr{P}(B)$. Let $x \in A$ and $x \notin B$. Then, $\{x\} \in \mathscr{P}(A)$ and $\{x\} \notin \mathscr{P}(B)$. Thus, $\mathscr{P}(A) \nsubseteq \mathscr{P}(B)$.

□