For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Symbolic logic can be used to find new expressions that are equivalent to old ones.

(i) Find an expression that is logically equivalent to the biconditional $P \Leftrightarrow Q$ that doesn't use $\Leftrightarrow$, $\Rightarrow$, or $\Leftarrow$.

(ii) Find an expression that is logically equivalent to the conditional $P \Rightarrow Q$ using only $\wedge$, $\vee$, and $\sim$.

(iii) Can you express $P \wedge Q$ using only $\vee$ and $\sim$? Justify your answer.

(iv) Can you express $P \vee Q$ using only $\wedge$ and $\sim$? Justify your answer.

This exercise shows that some of the symbols we use are redundant, but some are not. In any event, they are all useful.

Solution.

(i) $(Q \wedge P) \vee (\sim P \wedge \sim Q)$

Either both P and Q, or not P and not Q (but not both because that would lead to contradiction). This is really what $P \Leftrightarrow Q$ is saying.

(ii) $\sim P \vee Q$

This is material implication and is actually the definition of a logical conditional statement. It's derived from the only case when a conditional is false:

$\sim (P \wedge \sim Q) \iff (\sim P \vee Q)$

(iii) $\sim (\sim P \vee \sim Q)$

Applying De Morgan's law and double negation elimination:

$\sim (\sim P \vee \sim Q) \iff (\sim\sim P \wedge \sim\sim Q) \iff (P \wedge Q)$

(iv) $\sim (\sim P \wedge \sim Q)$

De Morgan's law works inversely as well, so the same argument applies:

$\sim (\sim P \wedge \sim Q) \iff (\sim\sim P \vee \sim\sim Q) \iff (P \vee Q)$

□

Problem 2. Consider the following statement.

$$\forall N \in \mathbb{N}, \exists X \in \mathscr{P}(\mathbb{N}), |X| \geq N$$

(i) Write the statement as an English sentence.
(ii) Give the negation of the statement in symbolic logic. (You answer should have no $\sim$ symbols.)
(iii) Write the negation of the statement as an English sentence.
(iv) Is the original statement true or false? Justify your answer.

Solution.

(i) "For every natural number x, there is at least one subset of the natural numbers that has a cardinality greater than or equal to x."

(ii) $\sim (\forall N \in \mathbb{N}, \exists X \in \mathscr{P}(\mathbb{N}), |X| \geq N)$

We can start by nesting the quantifiers to make simplification clearer:

$\sim \forall N((N \in \mathbb{N}) \Rightarrow (\exists X((X \in \mathscr{P}(\mathbb{N})) \wedge (|X| \geq N))))$

Then we can move the negation inward:

$\exists N \sim ((N \in \mathbb{N}) \Rightarrow (\exists X((X \in \mathscr{P}(\mathbb{N})) \wedge (|X| \geq N))))$

$\exists N((N \in \mathbb{N}) \wedge \sim (\exists X((X \in \mathscr{P}(\mathbb{N})) \wedge (|X| \geq N))))$

$\exists N((N \in \mathbb{N}) \wedge (\forall X \sim ((X \in \mathscr{P}(\mathbb{N})) \wedge (|X| \geq N))))$

$\exists N((N \in \mathbb{N}) \wedge (\forall X((X \in \mathscr{P}(\mathbb{N})) \Rightarrow \sim (|X| \geq N))))$

$\exists N((N \in \mathbb{N}) \wedge (\forall X((X \in \mathscr{P}(\mathbb{N})) \Rightarrow (|X| < N))))$

De-nest quantifiers:

$\exists N \in \mathbb{N}, \forall X \in \mathscr{P}(\mathbb{N}), N > |X|$

(iii) "There exists a natural number that is greater than cardinality of every subset of the natural numbers."

(iv) The original statement is true (and its negation is false). This is because for each natural number x, you can make a subset that contains all less than or equal to x. The cardinality of this set will be equal to x.

$\square$

Problem 3. Consider the following English sentence.

If r is a rational number and $r \neq 0$, then $\frac{M}{r}$ is an integer for some natural number M.

(i) Write the statement using symbolic logic.
(ii) Give the negation of the statement in symbolic logic. (You answer should have no $\sim$ symbols.)
(iii) Write the negation of the statement as an English sentence.
(iv) Is the original statement true or false? Justify your answer.

Solution.

(i) $\forall r \in \mathbb{Q}, \exists M \in \mathbb{N}, (r \neq 0) \Rightarrow (\frac{M}{r} \in \mathbb{Z})$

(ii) $\exists r \in \mathbb{Q}, \forall M \in \mathbb{N}, (r \neq 0) \land (\frac{M}{r} \notin \mathbb{Z})$

(iii) "There exists some non-zero rational number r such that there is no natural number M where $\frac{M}{r}$ is an integer."

(iv) The original statement is true (and its negation false).

Suppose $a, b \in \mathbb{Z}$, where $b \neq 0$, such that $\frac{a}{b} = r$.

$a \neq 0$ because $r \neq 0$.

Let M be some natural number equal to $|a|$.

$\frac{M}{\frac{a}{b}} = \frac{M}{a}b = \frac{|a|}{a}b = \pm b$

Because $\pm b \in \mathbb{Z}$ for all a and b, where $a \neq 0$ and $b \neq 0$, $\frac{M}{r}$ is also an integer.

$\square$