For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Let A and B be sets.

(i) Under what conditions do we have $A \times B = B \times A$?
(ii) When is it true that $|\mathscr{P}(A) \times \mathscr{P}(A)| = |\mathscr{P}(A \times A)|$?
(iii) What can you conclude if $A - B = \varnothing$?
(iv) Describe in words the set $X = (A \times A) - D$, where the subset $D \subseteq A \times A$ is given by $D = \{(a, a) : a \in A\}$.

Solution.

(i) $A \times B = B \times A$ if and only if $(A = B) \vee (A = \varnothing) \vee (B = \varnothing)$

Proof by contraposition:

Suppose $A \neq B \neq \varnothing$, where $x \in A$ but $x \notin B$.

Then, $\{(x, b) : b \in B\} \subseteq (A \times B)$ and $\{(x, b) : b \in B\} \nsubseteq (B \times A)$.

Because at least one member is present in $(A \times B)$ and not $(B \times A)$,

$(A \times B) \neq (B \times A)$.

The true cases are trivial,

Case 1:

If $A = B$, then $A \times B = A \times A = B \times A$.

Case 2 & 3:

A or B equals $\varnothing$. The cartesian product of any set and the empty set is the empty set, because their are no elements to iterate over.

(ii) $|\mathscr{P}(A) \times \mathscr{P}(A)| = |\mathscr{P}(A \times A)|$

$|\mathscr{P}(A)| \cdot |\mathscr{P}(A)| = 2^{(|A| \cdot |A|)}$

$2^{|A|} \cdot 2^{|A|} = 2^{(|A|^2)}$

$2^{2|A|} = 2^{(|A|^2)}$

$2|A| = |A|^2$

$0 = |A|^2 - 2|A| = |A|(|A| - 2)$

Thus, $|A| \in \{0, 2\}$

(iii) $A - B = \{x : x \in A, x \notin B\} = \varnothing$

No x in A that's not in B. This is the definition of a subset. Thus, $A \subseteq B$.

(iv) X is the empty set.

$D = \{(a, a) : a \in A\}$ is the simplified definition of the cartesian product if the left and right hand arguments are the same.

$(A \times A) - D \iff (A \times A) - (A \times A)$.

The difference of a set and itself is the empty set.

$\square$

Problem 2. Determine whether each of the following is true or false; justify your answer.

(i) $\mathbb{R}^2 \subseteq \mathbb{R}^3$
(ii) $A \times \varnothing = \varnothing$ for every set A.
(iii) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.
(iv) If $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$.

Solution.

(i) $\mathbb{R}^2 \subseteq \mathbb{R}^3$ is false because there is at least one element in $\mathbb{R}^2$ (take the ordered pair $(0, 0)$ for example), that is not in $\mathbb{R}^3$ (which has a similar but different math object $(0, 0, 0)$).

(ii) $A \times \varnothing = \varnothing$ for every set A is true and was demonstrated above. "The cartesian product of any set and the empty set is the empty set, because their are no elements to iterate over."

(iii) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ is true.

Each subset of A is also a subset of B and must be a member of a set of all subsets of B.

Thus, $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

(iv) If $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$ is true.

Contraposition:

If $A \nsubseteq B$, then $\mathscr{P}(A) \nsubseteq \mathscr{P}(B)$.

Let $x \in A$ and $x \notin B$.

Then, $\{x\} \in \mathscr{P}(A)$ and $\{x\} \notin \mathscr{P}(B)$.

Thus, $\mathscr{P}(A) \nsubseteq \mathscr{P}(B)$.

Direct:

$\{X : X \subseteq A\} \subseteq \mathscr{P}(B)$, thus, each $X \in \mathscr{P}(B)$ including where $X = A$.

Because $\mathscr{P}(B)$ contains A as an element, A must be a particular combination of all the elements in B.

All elements in A are in B, thus $A \subseteq B$.

$\square$