**Problem 1.** Consider the function $f\colon [0, 2\pi] \to [-1, 1]$ given by $f(x) = \cos x$. Determine each of the following sets.

(i) $f\left([0, \pi]\right)$
(ii) $f\left(\{\pi\}\right)$
(iii) $f\left(\left(0, \frac{\pi}{2}\right)\right)$
(iv) $f\left((0, \pi)\right)$
(v) $f^{-1}\left(\{-1, 1\}\right) = \{0, \pi, 2\pi\}$
(vi) $f^{-1}\left(\{0, 1\}\right)$
(vii) $f^{-1}\left((-1, 0)\right)$
(viii) $f^{-1}\left(\{0\}\right)$

*Solution.*

(i) $[-1, 1]$

(ii) $-1$

(iii) $(0, 1)$

(iv) $(-1, 1)$

(v) $\{0, \pi, 2\pi\}$

(vi) $\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$

(vii) $(\frac{\pi}{2}, \frac{3\pi}{2})$

(viii) $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$\square$

**Problem 2.** Consider $f \colon A \to B$.

(i) Prove $f$ is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

(ii) Prove $f$ is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

*Solution.*

(i)

Suppose the function $f$ is injective with a range of $B'$.

Then, for all $n, m \in A$ where $n \neq m$, $f(n) \neq f(m)$ and $f(n), f(m) \in B'$.

Observe that the inverse relation $f'$ may not be a function because $B'$ is not always $B$. However, if we restrict it's domain to $B'$ so that $f'_{\text{res}} : B' \to A$, then it is both a function and bijective.

Similarly, if we go back and restrict $f$'s codomain such that $f_{\text{res}} : A \to B'$ then it is also bijective.

Thus, $f_{\text{res}}^{-1}(f_{\text{res}}(x)) = i_A$.

This ensures that for every $x \in X$, $x = f^{-1}(f(x))$.

So, $X = f^-1(f(X))$

□