Problem 1. Consider the function $f\colon [0, 2\pi] \to [-1, 1]$ given by $f(x) = \cos x$. Determine each of the following sets.

(i) $f\left([0, \pi]\right)$
(ii) $f\left(\{\pi\}\right)$
(iii) $f\left((0, \frac{\pi}{2})\right)$
(iv) $f\left((0, \pi)\right)$
(v) $f^{-1}\left(\{-1, 1\}\right) = \{0, \pi, 2\pi\}$
(vi) $f^{-1}\left(\{0, 1\}\right)$
(vii) $f^{-1}\left((-1, 0)\right)$
(viii) $f^{-1}\left(\{0\}\right)$

Solution.

(i) $[-1, 1]$

(ii) $\{-1\}$

(iii) $(0, 1)$

(iv) $(-1, 1)$

(v) $\{0, \pi, 2\pi\}$

(vi) $\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$

(vii) $(\frac{\pi}{2}, \frac{3\pi}{2})$

(viii) $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$\square$

Problem 2. Consider $f\colon A \to B$.

(i) Prove f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

(ii) Prove f is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

Solution.

(i)

Suppose $X \subseteq A$.

Suppose the function f is injective with a range of B'.

Consider the image of $f_{\text{img}} : \mathscr{P}(A) \to \mathscr{P}(B')$ where

$f_{\text{img}} = \{(X, \{f(x) \in B' : x \in X\}) \in \mathscr{P}(A) \times \mathscr{P}(B')\}$.

This relationship is a function because for every subset X of A there is exactly one related subset of B', particularly $\{f(x) \in B' : x \in X\}$.

f_{img} is also injective, as seen below:

Assume f_{img} is not injective.

Then, for some M and N, $f_{\text{img}}(M) = f_{\text{img}}(N)$ and $M \neq N$.

Then $\{f(x) \in B' : x \in M\} = \{f(x) \in B' : x \in N\}$.

Let $m \in M$ and $x \notin N$ without loss of generality.

Because f is injective, $f(m)$ is only in $f_{\text{img}}(M)$ and not $f_{\text{img}}(N)$.

So, $f_{\text{img}}(M) \neq f_{\text{img}}(N)$, which is a contradiction.

f_{img} is also surjective:

Let $M \in \mathscr{P}(B')$.

Then $M = f_{\text{img}}(\{x \in A : f(x) \in M\})$.

So f_{img} is bijective.

Recall that bijective functions are invertible, so $f_{\text{img}}^{-1} : \mathscr{P}(B') \to \mathscr{P}(A)$

So, the images composed: $f_{\text{img}}^{-1} \circ f_{\text{img}} = i_{\mathscr{P}(A)}$.

Thus, $f^{-1}(f(X)) = X$.

Suppose that $X = f^{-1}(f(X))$ for all $X \subseteq A$.

So, for all singletons $\{x\} \subseteq A$, $\{x\} = f^{-1}(f(\{x\}))$.

Assume there exists some $y, z \in A$ such that $f(y) = f(z)$ and $y \neq z$.

So, the set $f^{-1}(f(\{y\}))$ contains both y and z.

This is a contradiction. So, if $f(y) = f(z)$, then $y = z$, for all $y, z \in A$.

Thus, f is injective.

(ii)

Suppose $Y \subseteq B$.

Suppose the function f is surjective.

Then, for every $b \in B$, $f^{-1}(\{b\}) \neq \varnothing$.

Because this is true for every $b \in B$, it is certainly true for all elements of Y.

In symbols:

$\{f(x) : x \in \{x \in A : f(x) \in Y\}\} = Y$

Every element of Y is reachable by some x, which is included in the set $\{x \in A : f(x) \in Y\}$. This set is then used to get f's image.

So, $f(f^{-1}(Y)) = Y$.

Suppose $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

So, for all singletons $\{y\} \subseteq B$, $\{y\} = f(f^{-1}(\{y\}))$.

Therefore, for every $y \in B$, there is some $x \in A$ such that $f(x) = y$.

$\square$