Problem 1. Consider the function $f\colon [0, 2\pi] \to [-1, 1]$ given by $f(x) = \cos x$. Determine each of the following sets.

(i) $f([0, \pi])$
(ii) $f(\{\pi\})$
(iii) $f\left((0, \frac{\pi}{2})\right)$
(iv) $f((0, \pi))$
(v) $f^{-1}(\{-1, 1\}) = \{0, \pi, 2\pi\}$
(vi) $f^{-1}(\{0, 1\})$
(vii) $f^{-1}((-1, 0))$
(viii) $f^{-1}(\{0\})$

Solution.

(i) $[-1, 1]$

(ii) $\{-1\}$

(iii) $(0, 1)$

(iv) $(-1, 1)$

(v) $\{0, \pi, 2\pi\}$

(vi) $\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$

(vii) $(\frac{\pi}{2}, \frac{3\pi}{2})$

(viii) $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$\square$

Problem 2. Consider $f\colon A \to B$.

(i) Prove f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

(ii) Prove f is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

Solution.

(i)

Suppose the function f is injective with a range of B'.

Suppose $x, y \in X$ and $X \subseteq A$.

Then, $f(x) = f(y)$ if and only if $x = y$.

Observe that $f^{-1}(z)$ is not a function but behaves as an injective function if $z \in B'$.

Because $f(x), f(y) \in B'$, $f^{-1}(f(x)) = f^{-1}(f(y))$ if and only if $f(x) = f(y)$.

So, $f^{-1}(f(x)) = f^{-1}(f(y))$ if and only if $x = y$.

Thus, $X = f^{-1}(f(X))$.

Suppose that $X = f^{-1}(f(X))$ for all X where $X \subseteq A$.

So, for all singletons $\{x\} \subseteq X$, $\{x\} = f^{-1}(f(\{x\}))$.

NOTE: come up with some way to show that f(x) = f(y), so x = y.

□