Problem 1. Let $\mathcal{C}$ be the unit circle, and let A be a collection of 20 evenly spaced points on $\mathcal{C}$. A straight segment connecting two points of A is called a *chord*. How many ways are there to draw 10 chords hitting all 20 points of A?

Solution.

Two different approaches:

- Pick between the $\binom{20}{2}$ options, then pick between the $\binom{18}{2}$ options, ...etc. Then because it didn't matter in what order you selected the 10 cords, divide by 10!.
- Arrange the 20 points in a list and count the number of permutations. Then conceptualize that their are 10 pairs in each string. Because a cord is directionless, we need to remove order from each pair, thus divide by 2! 10 times. Additionally, it doesn't matter in what order the cords appear in. So also divide by 10!.

$$\prod_{i=1}^{10} \binom{2 \cdot i}{2} \cdot \frac{1}{10!} = \frac{20!}{2!^{10} \cdot 10!} = 654,729,075$$

□

Problem 2. How many binary strings of length 10 contain somewhere within them the string `10001`?

Solution.

There are 6 different placements of `10001` with four positions that risk double count. We can count this by placing `10001` in each of the six positions and then fill the rest. Thus, $6 \cdot 2^5$ is the result before accounting for doubly count strings.

Now, there are four positions where we double count:

A = `10001?????`

B = `?????10001`

C = `?10001????`

D = `????10001?`

I will list only the non-zero intersections for brevity:

$|A \cap B| = 1 \implies$ `1000110001`

$|A \cap D| = 2 \implies$ `100010001?`

$|B \cap C| = 2 \implies$ `?100010001`

Thus, $6 \cdot 2^5 - 2 - 2 - 1 = 187$.

$\square$

Problem 3. Let $X = \{1, 2, 3, \ldots, n\}$.

(i) Determine the cardinality of the set

$$Z_k = \{A \in \mathscr{P}(X) \colon |A| = k\}$$

for $0 \leq k \leq n$.

(ii) Find a combinatorial proof showing that $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Hint: The parts of this problem are related.

Solution.

(i) The powers set contains all k-combinations of some set where $0 \leq k \leq n$. The indexed set builder is simply picking only subsets of X of a particular k. The number of k-combinations can be expressed as "n pick k".
Thus, $|Z_k| = \binom{n}{k}$.

(ii) Suppose we have sets A and B, where $A \subseteq B \subseteq X$ and $|A| = 1$. (X is already defined above.) How many different ways combinations of A and B are there for some X?

One way we can count this is by summing the count for each cardinality of B (which can be between 0 and n) where we pick one element from B and put into A. This consists of picking $|B|$ elements from X and 1 element from B. In symbols, this is $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\binom{k}{1} = \sum_{k=0}^{n} k\binom{n}{k}$.

Another way we can count this is by first picking 1 element from X to fill A, then fill B with the remaining $n-1$ elements. Now for each of these $n-1$ elements, we have a 2 choices: either put into B or leave out of B. To account for every possibility, we can use the multiplication principle. In symbols this is $n2^{n-1}$.

We have counted the same set two different ways which shows that their related expressions are equivalent. Thus, $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

$\square$