Prove the following propositions. Format your proof so each step of the proof is on its own line; each line should still be a complete sentence.

Proposition 1. If $n \in \mathbb{N}$, then

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof.

Suppose $n \in \mathbb{N}$.

Basis Step: $n = 1$

$$\frac{1}{2!} = 1 - \frac{1}{(1)+1}.$$

$$= \frac{1}{2!} \quad \checkmark\text{True}$$

Inductive Step: $n = k + 1$

Assume $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$.

Show that:

$$\begin{aligned}
1 - \frac{1}{((k+1)+1)!} &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{((k+1)+1)!} \\
&= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!} \\
&= (1 - \frac{1}{(k+1)!}) + \frac{(k+1)}{((k+1)+1)!} \\
&= (1 - \frac{1}{(k+1)!}) + \frac{(k+1)}{(k+2)!} \\
&= 1 + (-\frac{1}{(k+1)!} \cdot \frac{k+2}{k+2}) + \frac{(k+1)}{(k+2)!} \\
&= 1 + (-\frac{k+2}{(k+2)!}) + \frac{(k+1)}{(k+2)!} \\
&= 1 + \frac{(k+1)-(k+2)}{(k+2)!} \\
&= 1 - \frac{1}{(k+2)!} \\
&= 1 - \frac{1}{((k+1)+1)!} \quad \checkmark\text{True}
\end{aligned}$$

$\square$

Problem 2. A chocolate bar consists of unit squares arranged in an $n \times m$ rectangular grid. You may split the bar into individual unit squares by breaking along the lines. What is the number of breaks required? Prove your answer is correct.

Solution.

Proposition: The number of breaks required is $nm - 1$.

Suppose $n, m \in \mathbb{N}$.

Basis Step: $n = 1, m = 1$

This is 1×1 unit square, it requires no splits.

$0 = (1)(1) - 1 \quad \checkmark$

Inductive Step for n (without loss of generality for m):

Assume that $n_k m - 1$ is the number of breaks required for a $n_k \times m$ chocolate bar.

Additionally, assume that $m-1$ is the number of breaks required for a $1 \times m$ chocolate bar. This is when $n = 1$.

A chocolate bar of $(n_k + 1) \times m$ size will have an additional row of length m.

If we first split this additional row off, we are left with a $n_k \times m$ piece and a $1 \times m$ piece.

These pieces have been assumed to require $n_k m - 1$ and $m - 1$ splits respectively.

This sums to $(1) + (n_k m - 1) + (m - 1) \implies (n_k + 1)m - 1$.

□

Proposition 3. Let $n \in \mathbb{N}$ with $n \geq 2$, and let $A_1, A_2, \ldots, A_n$ be sets. Let B be a set. Then,

$$B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n).$$

Proof.

Basis Step: $n = 2$

$$\begin{aligned}
B \cap (A_1 \cup A_2) &= \{x : x \in B \land x \in (A_1 \cup A_2)\} \\
&= \{x : x \in B \land x \in (A_1 \cup A_2)\} \\
&= \{x : x \in B \land (x \in A_1 \lor x \in A_2)\} \\
&= \{x : (x \in B \land x \in A_1) \lor (x \in B \land x \in A_2)\} \\
&= \{x : (x \in B \cap A_1) \lor (x \in B \cap A_2)\} \\
&= (B \cap A_1) \cup (B \cap A_2)
\end{aligned}$$

Inductive Step: $n = k + 1$

Assume that $B \cap (A_1 \cup A_2 \cup \cdots \cup A_k) = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k)$.

$$\begin{aligned}
&B \cap (A_1 \cup A_2 \cup \cdots \cup A_{k+1}) = \\
&= \{x : x \in B \land x \in (A_1 \cup A_2 \cup \cdots \cup A_{k+1})\} \\
&= \{x : x \in B \land x \in (A_1 \cup \cdots \cup A_k \cup A_{k+1})\} \\
&= \{x : x \in B \land (x \in (A_1 \cup \cdots \cup A_k) \lor x \in A_{k+1})\} \\
&= \{x : (x \in B \land x \in (A_1 \cup \cdots \cup A_k)) \lor (x \in B \land x \in A_{k+1})\} \\
&= \{x : (x \in (B \cap (A_1 \cup \cdots \cup A_k))) \lor (x \in B \land x \in A_{k+1})\} \\
&= \{x : (x \in (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k)) \lor (x \in B \land x \in A_{k+1})\} \\
&= \{x : (x \in (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k)) \lor (x \in (B \cap A_{k+1}))\} \\
&= (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k) \cup (B \cap A_{k+1}) \\
&= (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_{k+1})
\end{aligned}$$

$\square$