Prove the following propositions. Format your proof so each step of the proof is on its own line; each line should still be a complete sentence.

**Proposition 1.** Suppose $a, b, p \in \mathbb{Z}$ and $p$ is prime. Prove that if $p|ab$, then $p|a$ or $p|b$. (Suggestion: Use the Proposition on page 152.)

*Proof.*

Suppose that $p|ab$.

Then $px = ab$ for some integer $x$.

Then $\text{sign}(a)\text{sign}(p)|p|x = |a|b \implies |p|y = |a|b$ where $y$ is the integer $\text{sign}(a)\text{sign}(p)x$.

**Proposition 7.1:**

If $a, b \in \mathbb{N}$, then there exists integers $k$ and $l$ for which $\gcd(a, b) = ak + bl$.

If $a = 0$, then $p|a$. Suppose $a \neq 0$.

So, $\gcd(|a|, |p|) = |a|k + |p|l$ where $k, l \in \mathbb{Z}$.

Additionally, because $p$ is prime, $|a|$ is either a multiple of $|p|$ or only shares the trivial divisor 1.

In the first case, $|p|||a| \implies p|a$, so assume $\gcd(|a|, |p|) = 1$.

Then $|a|k + |p|l = 1$.

Multiplying by $b$: $|a|bk + b|p|l = b$.

Subbing in for $|a|b$: $|p|yk + b|p|l = b \implies |p|(xk + pl) = b$.

So $|p||b$.

Therefore $p|b$. $\square$

**Proposition 2.** If $A$, $B$, and $C$ are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

*Proof.*

Suppose $A$, $B$, and $C$ are sets.

Consider the following sequence of equalities:

$$
\begin{aligned}
A \cup (B \cap C) &= \{x : (x \in A) \cup (B \cap C)\} && \text{(definition of set)} \\
&= \{x : (x \in A)\} \cup \{x : x \in B \wedge x \in C\} && \text{(definition of intersection)} \\
&= \{x : (x \in A) \vee (x \in B \wedge x \in C)\} && \text{(definition of union)} \\
&= \{x : (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} && \text{(logical `or' distributive property)} \\
&= \{x : (x \in A \vee x \in B)\} \cap \{x : (x \in A \vee x \in C)\} && \text{(definition of intersection)} \\
&= (A \cup B) \cap \{x : (x \in A \vee x \in C)\} && \text{(definition of union)} \\
&= (A \cup B) \cap (A \cup C) && \text{(definition of union)}
\end{aligned}
$$

□

**Problem 3.** Determine whether the following statement is true. If it is true, prove it; if it is false, give a disproof.

If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

*Solution.*

Suppose $A = \{1\}$, $B = \{2\}$, and $X = \{1, 2\}$.

Then $A \cup B = \{1, 2\}$.

So, $X = A \cup B$.

Thus, $X \subseteq A \cup B$.

But also, $X \nsubseteq A$ and $X \nsubseteq B$. □