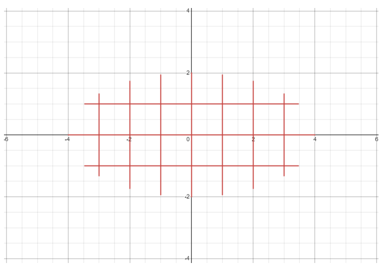
For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Let $A = \mathbb{Z} \times \mathbb{R}$, let $B = \mathbb{R} \times \mathbb{Z}$, and let $C = \{(x, y) : x^2 + 4y^2 \leq 16\}$. For each of the following sets, describe the set in set-builder notation and give a detailed sketch the set in $\mathbb{R}^2$.

(i) $C \cap (A \cup B)$
(ii) $C \cap A \cap B$
(iii) $C - (A \cup B)$
(iv) $C - (A \cap B)$

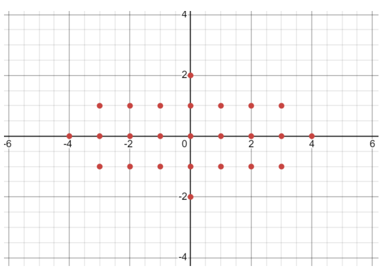
Solution.

(i)



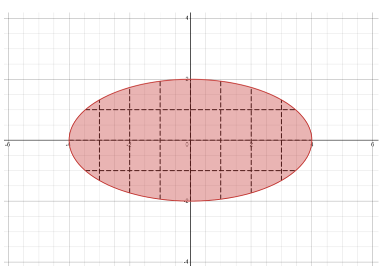
$\{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 16 \wedge (x \in \mathbb{Z} \vee y \in \mathbb{Z})\}$

(ii)



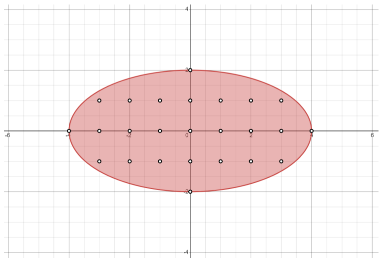
$\{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 16 \wedge (x \in \mathbb{Z} \wedge y \in \mathbb{Z})\}$

(iii)



$\{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 16 \;\wedge (x \notin \mathbb{Z} \wedge y \notin \mathbb{Z})\}$

(iv)



$$\{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 16 \ \wedge (x \notin \mathbb{Z} \vee y \notin \mathbb{Z})\}$$

□

Problem 2. Let $\mathcal{F} = \{A \subseteq \mathbb{N} : |A| \text{ is finite}\}$. Determine whether each of the following is true or false; justify your answer.

(i) $\mathbb{N} \in \mathcal{F}$.

(ii) If $A \in \mathcal{F}$, then $\overline{A} \in \mathcal{F}$.

(iii) $|\mathcal{F}|$ is finite.

(iv) If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

(v) If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

Solution.

(i) False. $\mathbb{N}$'s cardinality is $\aleph_0$ which is the smallest infinite cardinality.

(ii) False. In this context $\overline{A} = \mathbb{N} - A$. No matter how big $|A|$ is, $|A|$ is still finite and $|A| < |\mathbb{N}|$, so $|\overline{A}| = |\mathbb{N}|$ and $\overline{A} \notin \mathcal{F}$.

(iii) False. There are an infinite number of finite (and infinite) subsets of $\mathbb{N}$.

Consider $S = \{\{1\}, \{2\}, \{3\} \ldots\} = \{\{x\} : x \in \mathbb{N}\}$.

$S \subseteq \mathcal{F}$ and S's cardinality is clearly infinite, thus so is $\mathcal{F}$'s.

(iv) False. $\bigcup_{n=1}^{\infty} A_n$ *is* $\mathbb{N}$. Consider S from the previous subproblem, the union of the singletons as we reach all in $\mathbb{N}$ is equal to $\mathbb{N}$. No natural numbers are excluded.

(v) True. $\{1\}$ and $\{2\}$ are finite subsets of $\mathbb{N}$ and thus members of $\mathcal{F}$. Their intersection is $\varnothing$, thus $\bigcap_{n=1}^{\infty} A_n = \varnothing$. $\varnothing$ is also a subset of $\mathbb{N}$ and finite, and thus a member of $\mathcal{F}$.

$\square$

Problem 3. For each $\alpha \in \mathbb{R}$, define $A_\alpha = \{(x, \alpha \cos x) \in \mathbb{R}^2 : -2\pi \leq x \leq 2\pi\}$. Describe the following sets in set-builder notation and draw them in the plane $\mathbb{R}^2$.

(i) $A_{\frac{1}{2}}$

(ii) $\displaystyle\bigcup_{\alpha \in \mathbb{Z}} A_\alpha$

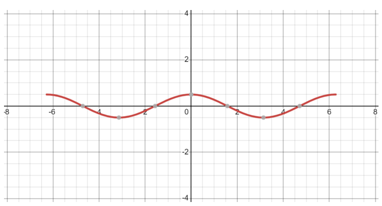
(iii) $\displaystyle\bigcap_{\alpha \in \mathbb{Z}} A_\alpha$

(iv) $\displaystyle\bigcup_{\alpha \in \mathbb{R}} A_\alpha$

(v) $\displaystyle\bigcap_{\alpha \in \mathbb{R}} A_\alpha$

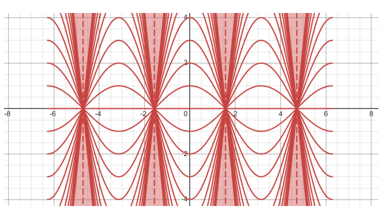
Solution.

(i)



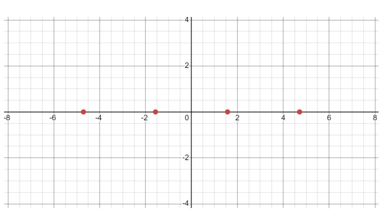
$\{(x, \frac{1}{2}\cos(x)) \in \mathbb{R}^2 : -2\pi \leq x \leq 2\pi\})$

(ii)



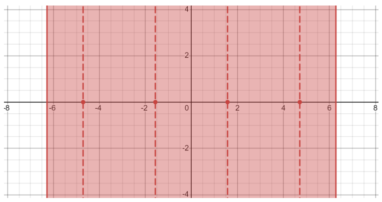
$\{(x, \alpha \cos(x) \in \mathbb{R}^2 : -2\pi \leq x \leq 2\pi, a \in \mathbb{Z}\}$

(iii)



$\{(-\frac{3\pi}{2}, 0), (-\frac{1\pi}{2}, 0), (\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0)\}$

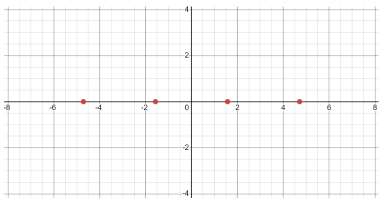
(iv)



$\{(x,y) \in \mathbb{R}^2 : x \in [-2\pi, -\frac{3\pi}{2}) \cup (-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]\}$

$\cup\{(-\frac{3\pi}{2}, 0), (-\frac{1\pi}{2}, 0), (\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0)\}$

(v)



$\{(-\frac{3\pi}{2}, 0), (-\frac{1\pi}{2}, 0), (\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0)\}$

$\square$