For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Let A and B be sets.

(i) Under what conditions do we have $A \times B = B \times A$?
(ii) When is it true that $|\mathscr{P}(A) \times \mathscr{P}(A)| = |\mathscr{P}(A \times A)|$?
(iii) What can you conclude if $A - B = \varnothing$?
(iv) Describe in words the set $X = (A \times A) - D$, where the subset $D \subseteq A \times A$ is given by $D = \{(a, a) : a \in A\}$.

Solution. (i) When, $A = B$, when $A = \varnothing$, or when $B = \varnothing$. Or rephrased: $A \times B = B \times A \iff (A = B) \vee (A = \varnothing) \vee (B = \varnothing)$ Proof by contraposition: Suppose $A \neq B \neq \varnothing$, where $x \in A$ but $x \notin B$. Then, $\{(x, b) : b \in B\} \subseteq (A \times B)$ and $\{(x, b) : b \in B\} \nsubseteq (B \times A)$. Because at least one member is present in $(A \times B)$ and not $(B \times A)$, $(A \times B) \neq (B \times A)$. The other cases are trivial: if $A = B$, then $A \times B = A \times A = B \times A$. The cartesian product of any set and the emptyset is the emptyset, because their are no elements to iterate over.

(ii) $|\mathscr{P}(A) \times \mathscr{P}(A)| = |\mathscr{P}(A \times A)|$

$|\mathscr{P}(A)| \cdot |\mathscr{P}(A)| = 2^{(|A| \cdot |A|)}$

$2^{|A|} \cdot 2^{|A|} = 2^{(|A|^2)}$

$2^{2|A|} = 2^{(|A|^2)}$

$2|A| = |A|^2$

$0 = |A|^2 - 2|A| = |A|(|A| - 2)$

Thus, $|A| \in \{0, 2\}$

(iii) $A - B = \{x : x \in A, x \notin B\} = \varnothing$

No x in A that's not in B, thus, $\forall x(\neg(x \in B) \to \neg(x \in A))$. Applying logical inference of contraposition: $\forall x((x \in A) \to (x \in B))$. For every x in A, it is also in B. This is the definition of a subset. Thus, $A \subset B$.

$\square$

Problem 2. Determine whether each of the following is true or false; justify your answer.

(i) $\mathbb{R}^2 \subseteq \mathbb{R}^3$
(ii) $A \times \varnothing = \varnothing$ for every set A.

(iii) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.
(iv) If $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$.

Solution. $\square$