Problem 1. Consider the function $f\colon [0, 2\pi] \to [-1, 1]$ given by $f(x) = \cos x$. Determine each of the following sets.

(i) $f([0, \pi])$
(ii) $f(\{\pi\})$
(iii) $f\left((0, \frac{\pi}{2})\right)$
(iv) $f((0, \pi))$
(v) $f^{-1}(\{-1, 1\}) = \{0, \pi, 2\pi\}$
(vi) $f^{-1}(\{0, 1\})$
(vii) $f^{-1}((-1, 0))$
(viii) $f^{-1}(\{0\})$

Solution.

(i) $[-1, 1]$

(ii) $\{-1\}$

(iii) $(0, 1)$

(iv) $(-1, 1)$

(v) $\{0, \pi, 2\pi\}$

(vi) $\{0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi\}$

(vii) $(\frac{\pi}{2}, \pi) \cup (\pi, \frac{3\pi}{2})$

(viii) $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$\square$

Problem 2. Consider $f\colon A \to B$.

(i) Prove f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

(ii) Prove f is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

Solution.

(i)

Suppose $X \subseteq A$.

Suppose the function f is injective.

Suppose $m \in X$.

Because f is a function, there is some $f(m)$ where $f(m) \in \{f(x) : x \in X\}$.

So, $f(m) \in f(X)$.

So, $m \in \{x : f(x) \in f(X)\}$.

Therefore, $m \in f^{-1}(f(X))$.

Suppose $m \in f^{-1}(f(X))$.

So, $m \in \{x : f(x) \in f(X)\}$.

So, it must be the case that $f(m) \in f(X)$.

Expanded, $f(m) \in \{f(x) : x \in X\}$.

So, there is some $x \in X$ where $f(m) = f(x)$.

Because f is injective, $m = x$.

So, $m \in X$.

So, $X = f^{-1}(f(X))$.

Suppose that $X = f^{-1}(f(X))$ for all $X \subseteq A$.

So, for all singletons $\{x\} \subseteq A$, $\{x\} = f^{-1}(f(\{x\}))$.

Assume there exists some $y, z \in A$ such that $f(y) = f(z)$ and $y \neq z$.

So, the set $f^{-1}(f(\{y\}))$ contains both y and z.

This is a contradiction. So, if $f(y) = f(z)$, then $y = z$, for all $y, z \in A$.

Thus, f is injective.

(ii)

Suppose $Y \subseteq B$.

Suppose the function f is surjective.

Suppose $n \in Y$.

Because f is surjective, there is some x_n for which $f(x_n) = n$.

So, $x_n \in \{x : f(x) \in Y\}$ and $\{x : f(x) \in Y\} = f^{-1}(Y)$.

So, $x_n \in f^{-1}(Y)$.

Then $n \in \{f(x) : x \in f^{-1}(Y)\}$ and $\{f(x) : x \in f^{-1}(Y)\} = f(f^{-1}(Y))$.

So, $n \in f(f^{-1}(Y))$.

Suppose $n \in f(f^{-1}(Y))$.

Then $n \in \{f(x) : x \in \{x : f(x) \in Y\}\}$.

So, there is some x_n where $f(x_n) = n$ and $x_n \in \{x : f(x) \in Y\}$.

So, $f(x_n) \in Y$, thus $n \in Y$.

Therefore, $Y = f(f^{-1}(Y))$ because an arbitrary element y in either set is always in both sets.

Suppose $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

So, for all singletons $\{y\} \subseteq B$, $\{y\} = f(f^{-1}(\{y\}))$.

Therefore, for every $y \in B$, there is some $x \in A$ such that $f(x) = y$ (particularly the only $x \in f^{-1}(\{y\})$).

Thus, f is surjective.

$\square$