Problem 1. Consider the function $f\colon [0, 2\pi] \to [-1, 1]$ given by $f(x) = \cos x$. Determine each of the following sets.

(i) $f([0, \pi])$
(ii) $f(\{\pi\})$
(iii) $f\left((0, \frac{\pi}{2})\right)$
(iv) $f((0, \pi))$
(v) $f^{-1}(\{-1, 1\}) = \{0, \pi, 2\pi\}$
(vi) $f^{-1}(\{0, 1\})$
(vii) $f^{-1}((-1, 0))$
(viii) $f^{-1}(\{0\})$

Solution.

(i) $[-1, 1]$

(ii) -1

(iii) $(0, 1)$

(iv) $(-1, 1)$

(v) $\{0, \pi, 2\pi\}$

(vi) $\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$

(vii) $(\frac{\pi}{2}, \frac{3\pi}{2})$

(viii) $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$\square$

Problem 2. Consider $f\colon A \to B$.

(i) Prove f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

(ii) Prove f is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

Solution.

(i)

Suppose some function f is injective with a range of B'.

Then, for all $n, m \in A$ where $n \neq m$, $f(n) \neq f(m)$.

Additionally, the inverse relation f^{-1} can be restricted to the domain B' to yield the function f^{-1}_{res}.

This inverse function is trivial injective because that's what it means to be a function.

Consider each element $x \in X$. Because its unique, by definition of a set, $f(x)$ is also unique.

$\square$