Problem 1. Consider the function $f\colon [0, 2\pi] \to [-1, 1]$ given by $f(x) = \cos x$. Determine each of the following sets.

   (i) $f\left([0, \pi]\right)$
  (ii) $f\left(\{\pi\}\right)$
 (iii) $f\left(\left(0, \frac{\pi}{2}\right)\right)$
 (iv) $f\left((0, \pi)\right)$
  (v) $f^{-1}\left(\{-1, 1\}\right) = \{0, \pi, 2\pi\}$
 (vi) $f^{-1}\left(\{0, 1\}\right)$
(vii) $f^{-1}\left((-1, 0)\right)$
(viii) $f^{-1}\left(\{0\}\right)$

Solution.

   (i) $[-1, 1]$

  (ii) $\{-1\}$

 (iii) $(0, 1)$

 (iv) $(-1, 1)$

  (v) $\{0, \pi, 2\pi\}$

 (vi) $\left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$

(vii) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

(viii) $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

$\square$

Problem 2. Consider $f \colon A \to B$.
   (i) Prove f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.
  (ii) Prove f is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

Solution.

    (i)

Suppose the function f is injective with a range of B'.

Suppose $x, y \in X$ and $X \subseteq A$.

Then, $f(x) = f(y)$ if and only if $x = y$.

Observe that f^{-1} is not a function but behaves as an injective function for all singletons that are subsets of B'.

Because $f(\{x\}), f(\{y\}) \subseteq B'$, $f^{-1}(f(\{x\})) = f^{-1}(f(\{y\}))$ if and only if $f(x) = f(y)$.

So, $f^{-1}(f(\{x\})) = f^{-1}(f(\{y\}))$ if and only if $x = y$.

So, $f^{-1}(f(\{x\})) = \{x\}$.

Thus, $f^{-1}(f(\bigcup_{x \in X}\{x\})) = X$

Thus, $X = f^{-1}(f(X))$.

Suppose that $X = f^{-1}(f(X))$ for all X where $X \subseteq A$.

So, for all singletons $\{x\} \subseteq X$, $\{x\} = f^{-1}(f(\{x\}))$.

   NOTE: come up with some way to show that f(x) = f(y), so x = y.

□