Prove the following propositions. Format your proof so each step of the proof is on its own line; each line should still be a complete sentence.

Proposition 1. If $n \in \mathbb{N}$, then

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof.

Suppose $n \in \mathbb{N}$.

Basis Step: $n = 1$

$$\frac{1}{2!} = 1 - \frac{1}{(1)+1}.$$

$$= \frac{1}{2!} \quad \checkmark\text{True}$$

Inductive Step: $n = k + 1$

Assume $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$.

Show that:

$$1 - \frac{1}{((k+1)+1)!} = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{((k+1)+1)!}$$

$$= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{(k+1)}{((k+1)+1)!}$$

$$= (1 - \frac{1}{(k+1)!}) + \frac{(k+1)}{((k+1)+1)!}$$

$$= (1 - \frac{1}{(k+1)!}) + \frac{(k+1)}{(k+2)!}$$

$$= 1 + (-\frac{1}{(k+1)!} \cdot \frac{k+2}{k+2}) + \frac{(k+1)}{(k+2)!}$$

$$= 1 + (-\frac{k+2}{(k+2)!}) + \frac{(k+1)}{(k+2)!}$$

$$= 1 + \frac{(k+1)-(k+2)}{(k+2)!}$$

$$= 1 - \frac{1}{(k+2)!}$$

$$= 1 - \frac{1}{((k+1)+1)!} \quad \checkmark\text{True}$$

$\square$

Problem 2. A chocolate bar consists of unit squares arranged in an $n \times m$ rectangular grid. You may split the bar into individual unit squares by breaking along the lines. What is the number of breaks required? Prove your answer is correct.

Solution.

Proposition: The number of breaks required is $nm - 1$.

Suppose $n, m \in \mathbb{N}$.

Basis Step: $n = 1$

Basis Step: $m = 1$

The number of breaks required for an 1×1 square is trivially 0 breaks.

This is equal to $nm - 1 = (1)(1) - 1 = 0$.

Inductive Step: $m = k + 1$

Assume, for $m = k$, $nm - 1 = k - 1$ is the required number of breaks.

Show $nm - 1$ is the required number of breaks for $m = k + 1$.

Suppose we have already performed the breaking for the first $k - 1$ breaks.

At this point, when $m = k$, all the unit squares are separated. However $m = k + 1$, so we have one more unit square than previously. This means there is a unit square pair that has yet to be separated. So exactly 1 more split should occur.

This is $(k - 1) + 1$ splits.

$(k - 1) + 1 \implies n(k + 1) - 1$.

So, when $n = 1$, $nm - 1$ is the number of splits required.

Inductive Step: $n = k + 1$

Assume, for $n = k$, $nm - 1 = km - 1$ is the required number of breaks.

Show that $nm - 1$ is the required number of breaks for $n = k + 1$.

Suppose we have already performed the break for the first $km - 1$ breaks.

At this point, when $n = k$, all the unit squares are separated. However $n = k + 1$, so we have m more unit squares (because it's a $n \times m$ rectangle). This will be bound in a line

$\square$

Proposition 3. Let $n \in \mathbb{N}$ with $n \geq 2$, and let $A_1, A_2, \ldots, A_n$ be sets. Let B be a set. Then,

$$B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n).$$

Proof. $\square$