Problem 1. Consider the function $f\colon [0,2\pi] \to [-1,1]$ given by $f(x) = \cos x$. Determine each of the following sets.

(i) $f([0,\pi])$
(ii) $f(\{\pi\})$
(iii) $f\left((0,\frac{\pi}{2})\right)$
(iv) $f((0,\pi))$
(v) $f^{-1}(\{-1,1\}) = \{0,\pi,2\pi\}$
(vi) $f^{-1}(\{0,1\})$
(vii) $f^{-1}((-1,0))$
(viii) $f^{-1}(\{0\})$

Solution.

(i) $[-1,1]$

(ii) $\{-1\}$

(iii) $(0,1)$

(iv) $(-1,1)$

(v) $\{0,\pi,2\pi\}$

(vi) $\{\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi\}$

(vii) $(\frac{\pi}{2},\frac{3\pi}{2})$

(viii) $\{\frac{\pi}{2},\frac{3\pi}{2}\}$

$\square$

Problem 2. Consider $f \colon A \to B$.

(i) Prove f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

(ii) Prove f is surjective if and only if $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

Solution.

(i)

Suppose $X \subseteq A$.

Suppose the function f is injective with a range of B'.

Consider the image of $f_{\text{img}} : \mathscr{P}(A) \to \mathscr{P}(B')$ where

$f_{\text{img}} = \{(X, \{f(x) \in B' : x \in X\}) \in \mathscr{P}(A) \times \mathscr{P}(B')\}$.

This relationship is a function because for every subset X of A there is exactly one related subset of B', particularly $\{f(x) \in B' : x \in X\}$.

f_{img} is also injective, as seen below:

Assume f_{img} is not injective.

Then, for some M and N, $f_{\text{img}}(M) = f_{\text{img}}(N)$ and $M \neq N$.

Then $\{f(x) \in B' : x \in M\} = \{f(x) \in B' : x \in N\}$.

Let $m \in M$ and $x \notin N$ without loss of generality.

Because f is injective, $f(m)$ is only in $f_{\text{img}}(M)$ and not $f_{\text{img}}(N)$.

So, $f_{\text{img}}(M) \neq f_{\text{img}}(N)$, which is a contradiction.

f_{img} is also surjective:

Let $M \in \mathscr{P}(B')$.

Then $M = f_{\text{img}}(\{x \in A : f(x) \in M\})$.

So f_{img} is bijective.

Recall that bijective functions are invertible, so $f_{\text{img}}^{-1} : \mathscr{P}(B') \to \mathscr{P}(A)$

So, the images composed: $f_{\text{img}}^{-1} \circ f_{\text{img}} = i_{\mathscr{P}(A)}$.

Thus, $f^{-1}(f(X)) = X$.

Suppose that $X = f^{-1}(f(X))$ for all $X \subseteq A$.

So, for all singletons $\{x\} \subseteq A$, $\{x\} = f^{-1}(f(\{x\}))$.

Assume there exists some $y, z \in A$ such that $f(y) = f(z)$ and $y \neq z$.

So, the set $f^{-1}(f(\{y\}))$ contains both y and z.

This is a contradiction. So, if $f(y) = f(z)$, then $y = z$, for all $y, z \in A$.

Thus, f is injective.

(ii)

Suppose $Y \subseteq B$.

Suppose the function f is surjective.

Suppose $y \in Y$.

Because f is surjective, there is some x_y for which $f(x_y) = y$.

So, $x_y \in \{x : f(x) \in Y\}$ and $\{x : f(x) \in Y\} = f^{-1}(Y)$.

So, $x_y \in f^{-1}(Y)$.

Then $y \in \{f(x) : x \in f^{-1}(Y)\}$ and $\{f(x) : x \in f^{-1}(Y)\} = f(f^{-1}(Y))$.

So, $y \in f(f^{-1}(Y))$.

Suppose $y \in f(f^{-1}(Y))$.

Then $y \in \{f(x) : x \in \{x : f(x) \in Y\}\}$.

So, there is some x_y where $f(x_y) = y$ and $x_y \in \{x : f(x) \in Y\}$.

So, $f(x_y) \in Y$, thus $y \in Y$.

Therefore, $Y = f(f^{-1}(Y))$ because an arbitrary element y in either set is always in both sets.

Suppose $Y = f(f^{-1}(Y))$ for all $Y \subseteq B$.

So, for all singletons $\{y\} \subseteq B$, $\{y\} = f(f^{-1}(\{y\}))$.

Therefore, for every $y \in B$, there is some $x \in A$ such that $f(x) = y$.

$\square$