For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Give a combinatorial proof of the fact that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

Solution.

Suppose $C \subseteq B \subseteq A$, where $|A| = n$, $|B| = k$, and $|C| = m$. Each combination is equal to another if both B and C have the exact same elements.

The left side counts each unique combination of B and C by first filling B with k elements from n, then picking m elements from B to fill C.

Let's try counting this another way. Suppose we first pick the elements that should fill C, so $\binom{n}{m}$. Then, because B has a cardinality k and is a subset of C, we should first, fill B with all elements from C, then pick the remaining $k - m$ elements from the rest of A. A has $n - m$ remaining elements. So, $\binom{n-m}{k-m}$. Thus, the total count will be $\binom{n}{m}\binom{n-m}{k-m}$.

It has been shown that you can count the same set of combinations of B and C two different ways. Thus, $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

$\square$

Problem 2. Give a combinatorial proof of the fact that $\displaystyle\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$. Then, give a proof using the binomial theorem.

Solution.

Combinatorial Proof:

Suppose we have set S of cardinality n and want to know how many different ways we can make $A \subseteq S$ and $B \subseteq S$, where $A \cap B = \varnothing$.

One way we can count this is by counting the number of ways for some k elements of S to be placed in A or B. Then summing the total from each value of k where $0 \leq k \leq n$. 2^k can be used to count how many different ways we can put k elements into A or B (but not both). This is because for each of the k elements, there are 2 choices. The multiplication principle says that each diverging choice should be multiplied to count the total number of variations. In symbols, this is $\sum_{k=0}^{n} 2^k \binom{n}{k}$.

Another way we can count this is by saying that there are really 3 disjoint (and exhaustive) subsets. A, B, and $S - (A \cup B)$. Because they are all disjoint and all subsets, we can count the number of unique combinations of A and B by using the multiplication principle again. For each element in S, we have 3 choices, to place it in A, B, or neither. In symbols, this is 3^n.

It has been shown that you can count the same set of combinations of A and B two different ways. Thus, $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$

Binomial Theorem Proof:

$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$

Looking at the given equality, $\sum_{k=0}^{n} 2^k \binom{n}{k}$ is very similar to the binomial theorem's left side.

Setting $x = 1$ and $y = 2$, the binomial theorem shows that:

$\sum_{k=0}^{n} 2^k \binom{n}{k} = (1+2)^n = 3^n$.

$\square$