

For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Symbolic logic can be used to find new expressions that are equivalent to old ones.

- (i) Find an expression that is logically equivalent to the biconditional $P \Leftrightarrow Q$ that doesn't use \Leftrightarrow , \Rightarrow , or \Leftarrow .
- (ii) Find an expression that is logically equivalent to the conditional $P \Rightarrow Q$ using only \wedge , \vee , and \sim .
- (iii) Can you express $P \wedge Q$ using only \vee and \sim ? Justify your answer.
- (iv) Can you express $P \vee Q$ using only \wedge and \sim ? Justify your answer.

This exercise shows that some of the symbols we use are redundant, but some are not. In any event, they are all useful.

Solution.

(i) $(Q \wedge P) \vee (\sim P \wedge \sim Q)$

Either both P and Q , or not P and not Q (but not both because that would lead to contradiction). This is really what $P \Leftrightarrow Q$ is saying.

(ii) $\sim P \vee Q$

This is material implication and is actually the definition of a logical conditional statement. It's derived from the only case when a conditional is false:

$$\sim (P \wedge \sim Q) \iff (\sim P \vee Q)$$

(iii) $\sim (\sim P \vee \sim Q)$

Applying De Morgan's law and double negation elimination:

$$\sim (\sim P \vee \sim Q) \iff (\sim\sim P \wedge \sim\sim Q) \iff (P \wedge Q)$$

(iv) $\sim (\sim P \wedge \sim Q)$

De Morgan's law works inversely as well, so the same argument applies:

$$\sim (\sim P \wedge \sim Q) \iff (\sim\sim P \vee \sim\sim Q) \iff (P \vee Q)$$

□

Problem 2. Consider the following statement.

$$\forall N \in \mathbb{N}, \exists X \in \mathcal{P}(\mathbb{N}), |X| \geq N$$

- (i) Write the statement as an English sentence.
- (ii) Give the negation of the statement in symbolic logic. (Your answer should have no \sim symbols.)
- (iii) Write the negation of the statement as an English sentence.
- (iv) Is the original statement true or false? Justify your answer.

Solution.

- (i) “For every natural number x , there is at least one subset of the natural numbers that has a cardinality greater than or equal to x .”

- (ii) $\sim (\forall N \in \mathbb{N}, \exists X \in \mathcal{P}(\mathbb{N}), |X| \geq N)$

We can start by nesting the quantifiers to make simplification clearer:

$$\sim \forall N((N \in \mathbb{N}) \Rightarrow (\exists X((X \in \mathcal{P}(\mathbb{N})) \wedge (|X| \geq N))))$$

Then we can move the negation inward:

$$\exists N \sim ((N \in \mathbb{N}) \Rightarrow (\exists X((X \in \mathcal{P}(\mathbb{N})) \wedge (|X| \geq N))))$$

$$\exists N((N \in \mathbb{N}) \wedge \sim (\exists X((X \in \mathcal{P}(\mathbb{N})) \wedge (|X| \geq N))))$$

$$\exists N((N \in \mathbb{N}) \wedge (\forall X \sim ((X \in \mathcal{P}(\mathbb{N})) \wedge (|X| \geq N))))$$

$$\exists N((N \in \mathbb{N}) \wedge (\forall X((X \in \mathcal{P}(\mathbb{N})) \Rightarrow \sim (|X| \geq N))))$$

$$\exists N((N \in \mathbb{N}) \wedge (\forall X((X \in \mathcal{P}(\mathbb{N})) \Rightarrow (|X| < N))))$$

De-nest quantifiers:

$$\exists N \in \mathbb{N}, \forall X \in \mathcal{P}(\mathbb{N}), N > |X|$$

- (iii) “There exists a natural number that is greater than cardinality of every subset of the natural numbers.”
- (iv) The original statement is true (and its negation is false). This is because for each natural number x , you can make a subset that contains all less than or equal to x . The cardinality of this set will be equal to x .

□

Problem 3. Consider the following English sentence.

If r is a rational number and $r \neq 0$, then $\frac{M}{r}$ is an integer for some natural number M .

- (i) Write the statement using symbolic logic.
- (ii) Give the negation of the statement in symbolic logic. (Your answer should have no \sim symbols.)
- (iii) Write the negation of the statement as an English sentence.
- (iv) Is the original statement true or false? Justify your answer.

Solution.

(i) $\forall r \in \mathbb{Q}, \exists M \in \mathbb{N}, (r \neq 0) \Rightarrow (\frac{M}{r} \in \mathbb{Z})$

(ii) $\exists r \in \mathbb{Q}, \forall M \in \mathbb{N}, (r \neq 0) \wedge (\frac{M}{r} \notin \mathbb{Z})$

(iii) “There exists some non-zero rational number r such that there is no natural number M where $\frac{M}{r}$ is an integer.”

(iv) The original statement is true (and its negation false).

Suppose $a, b \in \mathbb{Z}$, where $b \neq 0$, such that $\frac{a}{b} = r$.

$a \neq 0$ because $r \neq 0$.

Let M be some natural number equal to $|a|$.

$$\frac{M}{\frac{a}{b}} = \frac{M}{a}b = \frac{|a|}{a}b = \pm b$$

Thus, $\pm b \in \mathbb{Z}$ for all a and b , where $a \neq 0$ and $b \neq 0$.

□