

For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Let A and B be sets.

- (i) Under what conditions do we have $A \times B = B \times A$?
- (ii) When is it true that $|\mathcal{P}(A) \times \mathcal{P}(A)| = |\mathcal{P}(A \times A)|$?
- (iii) What can you conclude if $A - B = \emptyset$?
- (iv) Describe in words the set $X = (A \times A) - D$, where the subset $D \subseteq A \times A$ is given by $D = \{(a, a) : a \in A\}$.

Solution.

- (i) $A \times B = B \times A$ if and only if $(A = B) \vee (A = \emptyset) \vee (B = \emptyset)$

Proof by contraposition:

Suppose $A \neq B \neq \emptyset$, where $x \in A$ but $x \notin B$.

Then, $\{(x, b) : b \in B\} \subseteq (A \times B)$ and $\{(x, b) : b \in B\} \not\subseteq (B \times A)$.

Because at least one member is present in $(A \times B)$ and not $(B \times A)$,

$(A \times B) \neq (B \times A)$.

The true cases are trivial,

Case 1:

If $A = B$, then $A \times B = A \times A = B \times A$.

Case 2 & 3:

A or B equals \emptyset . The cartesian product of any set and the empty set is the empty set, because there are no elements to iterate over.

- (ii) $|\mathcal{P}(A) \times \mathcal{P}(A)| = |\mathcal{P}(A \times A)|$

$$|\mathcal{P}(A)| \cdot |\mathcal{P}(A)| = 2^{(|A| \cdot |A|)}$$

$$2^{|A|} \cdot 2^{|A|} = 2^{(|A|^2)}$$

$$2^{2|A|} = 2^{(|A|^2)}$$

$$2|A| = |A|^2$$

$$0 = |A|^2 - 2|A| = |A|(|A| - 2)$$

Thus, $|A| \in \{0, 2\}$

$$(iii) A - B = \{x : x \in A, x \notin B\} = \emptyset$$

No x in A that's not in B . This is the definition of a subset. Thus, $A \subseteq B$.

(iv) X is equal to every possible pairing of elements in A where an element is not paired with itself.

$D = \{(a, a) : a \in A\}$ is a subset of $A \times A$ because the cartesian product is equal to $\{(x, y) : x \in A, y \in A\}$, and D is particularly for every time $x = y$.

$$(A \times A) - D = \{p : p \in (A \times A), p \notin D\}$$

$$(A \times A) - D = \{(x, y) : x \in A, y \in A, x \neq y\}$$

□

Problem 2. Determine whether each of the following is true or false; justify your answer.

- (i) $\mathbb{R}^2 \subseteq \mathbb{R}^3$
- (ii) $A \times \emptyset = \emptyset$ for every set A .
- (iii) If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- (iv) If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

Solution.

- (i) $\mathbb{R}^2 \subseteq \mathbb{R}^3$ is false because there is at least one element in \mathbb{R}^2 (take the ordered pair $(0, 0)$ for example), that is not in \mathbb{R}^3 (which has a similar but different math object $(0, 0, 0)$).
- (ii) $A \times \emptyset = \emptyset$ for every set A is true and was demonstrated above. "The cartesian product of any set and the empty set is the empty set, because there are no elements to iterate over."
- (iii) If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ is true.

Each subset of A is also a subset of B and must be a member of a set of all subsets of B .

Thus, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

(iv) If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$ is true.

Contraposition:

If $A \not\subseteq B$, then $\mathcal{P}(A) \not\subseteq \mathcal{P}(B)$.

Let $x \in A$ and $x \notin B$.

Then, $\{x\} \in \mathcal{P}(A)$ and $\{x\} \notin \mathcal{P}(B)$.

Thus, $\mathcal{P}(A) \not\subseteq \mathcal{P}(B)$.

Direct:

$\{X : X \subseteq A\} \subseteq \mathcal{P}(B)$, thus, each $X \in \mathcal{P}(B)$ including where $X = A$.

Because $\mathcal{P}(B)$ contains A as an element, A must be a particular combination of all the elements in B .

All elements in A are in B , thus $A \subseteq B$.

□