Prove the following propositions. Format your proof so each step of the proof is on its own line; each line should still be a complete sentence.

Proposition 1. Suppose $a, b, p \in \mathbb{Z}$ and p is prime. Prove that if $p|ab$, then $p|a$ or $p|b$. (Suggestion: Use the Proposition on page 152.)

Proof.

Suppose that $p \mid ab$.

Then $px = ab$ for some integer x.

To make the proposition work with our integers we need to first ensure they are strictly positive.

So, $|p|y = |a|b$ where y is the integer $\text{sign}(a)\text{sign}(p)x$.

Proposition 7.1:

If $a, b \in \mathbb{N}$, then there exists integers k and l for which $\gcd(a, b) = ak + bl$.

Case 1: $p \mid a$

Then we are done.

Case 2: $p \nmid a$

Then $a \neq 0$, so $|a|$ is a natural number.

So, $\gcd(|a|, |p|) = |a|k + |p|l$ where $k, l \in \mathbb{Z}$. (by proposition 7.1)

Additionally, because p is prime, $|a|$ is either a multiple of $|p|$ or only shares the trivial divisor 1. But, $p \nmid a$.

So assume $\gcd(|a|, |p|) = 1$.

Then $1 = |a|k + |p|l$.

Multiplying by b: $b = |a|bk + b|p|l$.

Substituting $|a|b$: $b = |p|yk + b|p|l \implies b = |p|(yk + bl)$.

So, $|p| \mid b$.

Therefore, $p \mid b$.

$\square$

Proposition 2. If A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

Suppose A, B, and C are sets.

Consider the following sequence of equalities:

$$A \cup (B \cap C) = \{x : (x \in A)\} \cup (B \cap C) \qquad \text{(definition of set)}$$
$$= \{x : (x \in A)\} \cup \{x : x \in B \wedge x \in C\} \qquad \text{(definition of intersection)}$$
$$= \{x : (x \in A) \vee (x \in B \wedge x \in C)\} \qquad \text{(definition of union)}$$
$$= \{x : (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \qquad \text{(logical 'or' distributive property)}$$
$$= \{x : (x \in A \vee x \in B)\} \cap \{x : (x \in A \vee x \in C)\} \qquad \text{(definition of intersection)}$$
$$= (A \cup B) \cap \{x : (x \in A \vee x \in C)\} \qquad \text{(definition of union)}$$
$$= (A \cup B) \cap (A \cup C) \qquad \text{(definition of union)}$$

□

Problem 3. Determine whether the following statement is true. If it is true, prove it; if it is false, give a disproof.

$$\text{If } X \subseteq A \cup B, \text{ then } X \subseteq A \text{ or } X \subseteq B.$$

Solution.

Suppose $A = \{1\}$, $B = \{2\}$, and $X = \{1, 2\}$.

Then $A \cup B = \{1, 2\}$.

So, $X = A \cup B$.

Thus, $X \subseteq A \cup B$.

But also, $X \not\subseteq A$ and $X \not\subseteq B$. □