Prove the following propositions. Format your proof so each step of the proof is on its own line; each line should still be a complete sentence.

Proposition 1. Suppose $a, b, p \in \mathbb{Z}$ and p is prime. Prove that if $p|ab$, then $p|a$ or $p|b$. (Suggestion: Use the Proposition on page 152.)

Proof.

Suppose $p \nmid a$ and $p \nmid b$.

Then p is not a prime factor of a or b.

ab is simply adding the multiplicities of each prime factor of a and of b together.

p's multiplicity in a is 0, same with b.

Because $0 + 0 = 0$, p's multiplicity in ab is 0.

So, $p \nmid ab$. □

Proposition 2. If A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.

Suppose A, B, and C are sets.

Consider the following sequence of equalities:

$$
\begin{aligned}
A \cup (B \cap C) &= \{x : (x \in A) \cup (B \cap C)\} && \text{(definition of set)} \\
&= \{x : (x \in A)\} \cup \{x : x \in B \wedge x \in C\} && \text{(definition of intersection)} \\
&= \{x : (x \in A) \vee (x \in B \wedge x \in C)\} && \text{(definition of union)} \\
&= \{x : (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} && \text{(logical `or' distributive property)} \\
&= \{x : (x \in A \vee x \in B)\} \cap \{x : (x \in A \vee x \in C)\} && \text{(definition of intersection)} \\
&= (A \cup B) \cap \{x : (x \in A \vee x \in C)\} && \text{(definition of union)} \\
&= (A \cup B) \cap (A \cup C) && \text{(definition of union)}
\end{aligned}
$$

□

Problem 3. Determine whether the following statement is true. If it is true, prove it; if it is false, give a disproof.

$$\text{If } X \subseteq A \cup B, \text{ then } X \subseteq A \text{ or } X \subseteq B.$$

Solution.

Suppose $A = \{1\}$, $B = \{2\}$, and $X = \{1, 2\}$.

Then $A \cup B = \{1, 2\}$.

So, $X = A \cup B$.

Thus, $X \subseteq A \cup B$.

But also, $X \nsubseteq A$ and $X \nsubseteq B$. $\square$