For these problems, you should justify your answers. You do not need to provide a rigorous mathematical proof, but rather an informal argument.

Problem 1. Give a combinatorial proof of the fact that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

Solution.

Suppose, $C \subseteq B \subseteq A$, where $|A| = n$, $|B| = k$, and $|C| = m$. Each combination is equal to another if both B and C have the exact same elements.

The left side counts each unique combination of B and C by first filling B with k elements from n, then picking m elements from B to fill C.

Let's try counting this another way. Suppose we first pick the elements that should fill C, so $\binom{n}{m}$. Then, because B has a cardinality k and is a subset of C, we should first, fill B with all elements from C, then pick the remaining $k - m$ elements from the rest of A. A has $n - m$ remaining elements. So, $\binom{n-m}{k-m}$. Thus, the total count will be $\binom{n}{m}\binom{n-m}{k-m}$.

It has been shown that you can count the same set of combinations of B and C two different ways. Thus, $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

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Problem 2. Give a combinatorial proof of the fact that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$. Then, give a proof using the binomial theorem.

Solution.

Combinatorial Proof:

The left side the equality is counting the number of different ways we can put n elements into 2 exclusive subsets A and B, where we do not have to use all n elements. For example with 1 element, we can put it in into A or B. With 2 elements x and y, we can put them both in A, both in B, x in A and y in B, or x in B and y in A.

The right side of the equality is counting the number of different ways we can put n objects into 3 bins. We have three choices for each element, with a unique thing to count every time we move an element between bins.

Now, this is actually really what we're doing on the left side. For each element it is either being picked to be placed in subset A or B, or it is being left out of both. This left out of both can simply be subset C. $A \cup B \cup C$ contain all elements, and $A \cap B = \varnothing$, $B \cap C = \varnothing$, $A \cap C = \varnothing$. This is exactly what we are doing on the right side.

Binomial Theorem Proof:

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$$

Looking at the given equality, $\displaystyle\sum_{k=0}^{n} 2^k \binom{n}{k}$ is very similar to the binomial theorem's left side.

Setting $x = 1$ and $y = 2$, the binomial theorem shows that:

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = (1+2)^n = 3^n.$$

$\square$