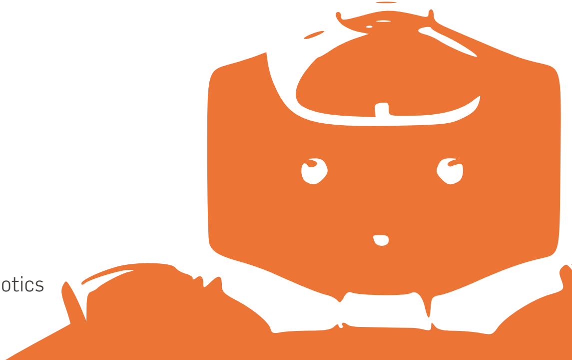
ROBOTICS
WITH
PLYMOUTH
UNIVERSITY

ROCO318
Mobile and Humanoid Robots
Part 3 - Kalman filters

Séverin Lemaignan

Centre for Neural Systems and Robotics
Plymouth University



For further reading, see:

- Welch and Bishop (2001) An introduction to the Kalman filter, SIGGRAPH 2001, ACM.
- Autonomous Mobile Robots, chapter 5.6.8

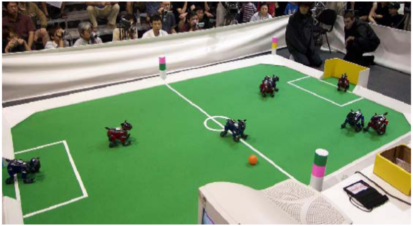
WHAT IS A KALMAN FILTER?

For example: speed, height, position, acceleration, ...

- Developed by Rudolph E. Kalman in 1960.
- Mathematical tool that estimates the real **state** of a system based on uncertain sensor readings.
- It assumes the system is **linear** and noise is **normal** (aka Gaussian).
- Gives past, present and **future estimations**.
- Still very effective and useful for all other classes of systems.
- Hugely popular in digital control systems.

APPLICATIONS OF KALMAN FILTERS

- Estimating critical flight parameters for guidance of missiles.
- Sensor fusion in aircraft.
- Fusion of localisation estimates in GPS.
- Estimating game controller sensor information.
- Prediction of ball position in robot football.
- Prediction of head and hands position and orientation in 3D body posture capture system.
- Prediction of the stock market.
- ...



SOME KALMAN FILTER FACTS

It is a filter? Not really, it does more than filters do

- Taking into account sensor measurements and process variables.
- Prediction forwards (and backwards if needed) in time.
- No explicit frequency response

Kalman Filter is **recursive**

- It start with initial estimates and continuously updates these estimates according to the process model and sensor measurements coming in.

Highly efficient: **Polynomial in measurement dimensionality** k and **state dimensionality** n: $O(k^{2.376} + n^2)$

Optimal, i.e. there is no way of doing better.

STOCHASTICITY

The true state of a system is unknown

- We don’t know the true speed of an air craft, or the true location of the robot.

This is due to **stochastic** (= random) noise in the measurements and the process.

- Measurement noise example: the air pressure meter reading fluctuates, even at the same altitude.
- Process noise example: even if we keep the accelerator in the same position the car never goes at exactly 60 mph.

LINEAR SYSTEM

A linear equation is a sum of input variables

- For example $f(x) = 5x + 3$ is linear, $f(x) = cos(x)$ is not.
- A linear system can be written in matrix form as $Y = A \cdot X$ or

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

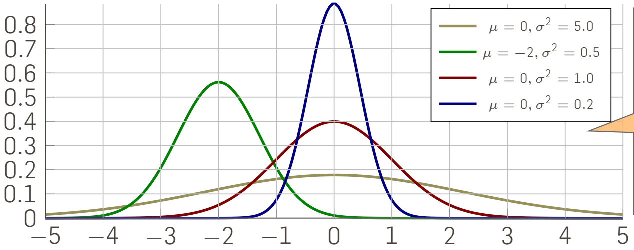
NORMAL

Normal or Gaussian

- Symmetrical distribution, captured with two values: **mean** μ **and variance** σ^2.
- Described by:

$$\varphi_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This factor keeps the integral (surface under the curve) equal to 1



If these would be measurement distributions of sensors, which sensor is the best?

BASIC CONCEPTS: GAUSSIAN OR NORMAL

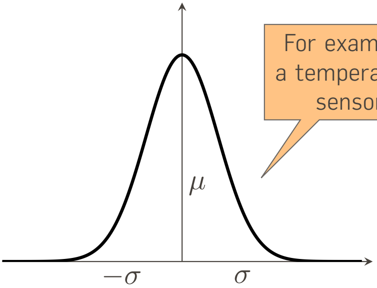
The Kalman Filter assumes that **measurement and process noise are normal** (also known as Gaussian) and **independent**.

Univariate

$$p(x) \sim \mathcal{N}(\mu, \sigma^2) :$$

Probability distribution

$$p(x) \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



For example, a temperature sensor

BASIC CONCEPTS: GAUSSIAN OR NORMAL

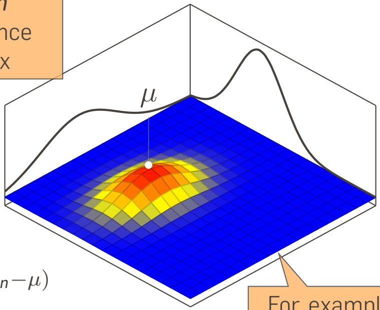
The Kalman Filter assumes that **measurement and process noise are normal** (also known as Gaussian) and **independent**.

Multivariate

$n \times n$ covariance matrix

$$p(x_n) = p\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \sim \mathcal{N}_n(\mu, \Sigma) :$$

$$p(x_n) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}(x_n-\mu)^\top \Sigma^{-1} (x_n-\mu)}$$



For example, the x and y position of a robot

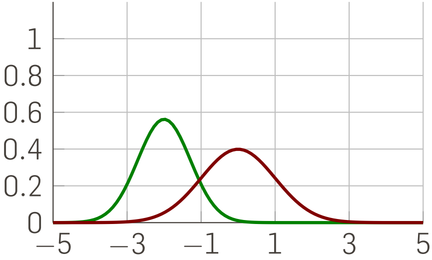
GAUSSIAN FUNCTION (2)

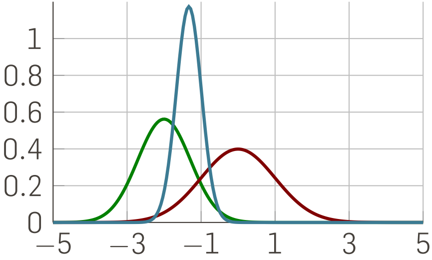
Combining Gaussians (μ, σ^2) and (ν, r^2):

$$\mu' = \frac{1}{\sigma^2 + r^2}(r^2\mu + \sigma^2\nu)$$

$$\sigma^{2\prime} = \frac{1}{\frac{1}{\sigma^2} + \frac{1}{r^2}}$$

Combining two Gaussians results in a Gaussian that has a ***smaller standard deviation***.





BASIC CONCEPTS: STATE

The **state** of a process is a vector of real numbers capturing the relevant information describing the process. $x = \mathbb{R}^n$

For example

- The position and speed of a wheeled robot: $x = [x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}]$
- The speed of a missile: $x = [v, t_{thurst}]$

THE BASICS

The Kalman filter needs a number of parameters to run.

These come from the **process equations**: equations that describe how the state of the system in the next time step depends on the current state and any changes that happen to the system.

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For example: a car drives down the road. Its position at time $t+1$ depends on its position at time t, the control input at t (is the car braking or accelerating) and system dynamics (it slows down due to friction).

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Instead of t, we use k to denote *discrete time steps*.

THE PROCESS EQUATIONS (1)

The process is governed by a linear difference equation:

The state at time step *k*

The state one time step ago

$$x_k = F \cdot x_{k-1} + B \cdot u_{k-1} + w_{k-1}$$

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$n \times n$ matrix changing the previous state into the current state

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Control input, a vector of size *l*

n × *n* matrix changing the previous state into the current state

n × *l* matrix that maps the control input onto the state

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n × ***n*** matrix changing the previous state into the current state

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The process noise, a vector of size ***n***.

THE PROCESS EQUATIONS (2)

We take *m* measurements which will be related to the state *x* according to:

$$z_k = H \cdot x_k + v_k$$

Measurements, a vector of size *m*

$m \times n$ matrix mapping the state to the measurements

Measurement noise, a vector of size *m*

THE PROCESS EQUATIONS: RECAP OF MAIN MODELS

- **State transition model** F: matrix $n \times n$ that describes how the state changes from $k-1$ to k without controls or noise.
- **Control input model** B: matrix $n \times l$ that describes how the control u_{k-1} changes the state from $k-1$ to k.
- **Observation model** H: Matrix $m \times n$ that describes how to map the state x_k to the measurements z_k.

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- **Process noise model** w_k: a vector of size n
- **Measurement noise model** v_k: a vector of size m

COVARIANCE

Covariance: measure of how two variables change together.

Two series X and Y of values, each of size n.

$$cov(X,Y) = \overline{(X-\bar{X})(Y-\bar{Y})} = \sum_{i=1}^{n} \frac{(x_i-\bar{X})(y_i-\bar{Y})}{n}$$

If $cov(X,Y) > 0$, then X and Y tend move together.
If $cov(X,Y) < 0$ then X and Y have an opposite effect on eachother.
And $cov(X,Y) = 0$?

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Example:

$$\begin{aligned}
X &= [4,2,3,4,5,5,3,1,1,2] \\
Y &= [6,4,3,5,7,8,5,3,3,2] \\
\bar{X} &= 3 \\
\bar{Y} &= 4.6 \\
X-\bar{X} &= [1,-1,0,1,2,2,0,-2,-2,-1] \\
Y-\bar{Y} &= [1.4,-0.6,-1.6,0.4,2.4,3.4,0.4,-1.6,-1.6,-2.6] \\
(X-\bar{X})(Y-\bar{Y}) &= [-0.84,1.56,2.56,-0.24,0.96,1.36,-0.64,5.76,5.76,6.76] \\
cov(X,Y) &= 2.3
\end{aligned}$$

COVARIANCE

A covariance matrix is a matrix showing the covariance of two or more variables to each other.

- If one variable changes, does the other variable change as well and in what direction?
- Example: altitude, temperature and air pressure

altitude	T °C	pressure
0	20	1
1000	10	0.9
2000	0	0.8
3000	-10	0.7
4000	-20	0.5
5000	-30	0.3

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5000	-30	0.3

	alt.	**T**	**P**
alt.	2916667	-29166.7	-400
T	-29166.7	291.667	4
P	-400	4	0.057

THE PROCESS EQUATIONS (3)

The variables $\boldsymbol{w_k}$ and $\boldsymbol{v_k}$ contain the random noise on the state and measurements. They (are assumed to) have a **normal** distribution.

$$p(w) \sim \mathcal{N}(0, Q)$$
$$p(v) \sim \mathcal{N}(0, R)$$

With a **covariance matrix** of Q and R; this reflects the width of the normal distribution

p means probability distribution

$\mathcal{N}$ is the notation for a normal distribution

ROUND AND ROUND GOES THE KALMAN FILTER

The goal of a Kalman filter is to **estimate** the state *x* at each time step given

- noisy measurements,
- control input,
- the process equations.

The state of the filter is represented by two variables:

- $\hat{\mathbf{x}}_{k|k}$: the state estimate at time k given observations up to and including at time k
- $\mathbf{P}_{k|k}$: the *error covariance matrix* (a measure of the estimated accuracy of the state estimate)

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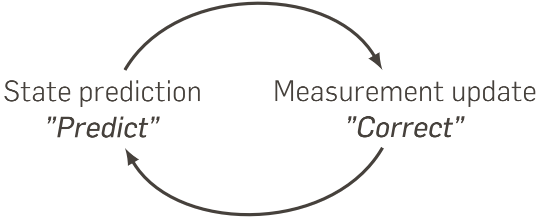
ROUND AND ROUND GOES THE KALMAN FILTER

The Kalman filter continuously loops through two steps

- The **state prediction** step.
- The **measurement update** step.



ROUND AND ROUND GOES THE KALMAN FILTER

The **Prediction** step uses the state estimate from the previous timestep to produce an estimate of the state **at the current timestep**. This is called the **a priori** estimate

- *A priori* means that the estimate is taken before any new sensor measurements have come in.
- **Notation**: $\hat{\mathbf{x}}_{k|k-1}$

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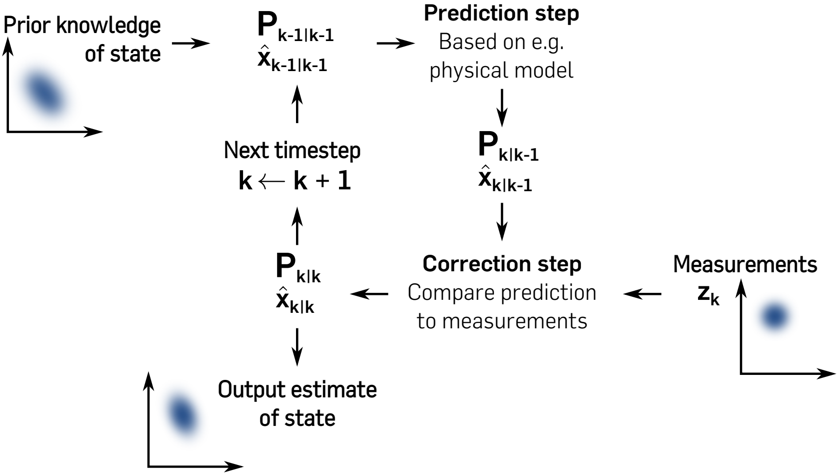
The **Correction** step is run after new measurements have come in an provide the **a posteriori** estimate.

- *A posteriori* because the estimate is made after new sensor measurements have come in.
- **Notation**: $\hat{\mathbf{x}}_{k|k}$



Prior knowledge of state
$P_{k-1|k-1}$
$\hat{x}_{k-1|k-1}$
Prediction step
Based on e.g. physical model
$P_{k|k-1}$
$\hat{x}_{k|k-1}$
Next timestep
$k \leftarrow k+1$
Correction step
Compare prediction to measurements
Measurements
z_k
$P_{k|k}$
$\hat{x}_{k|k}$
Output estimate of state

PREDICT STEP EQUATIONS

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F} \cdot \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B} \cdot u_{k-1}$$

A priori estimate of the state at time k

A posteriori estimate of the state at time $k-1$

$$\mathbf{P}_{k|k-1} = \mathbf{F} \cdot \mathbf{P}_{k-1|k-1} \cdot \mathbf{F}^{\top} + \mathbf{Q}$$

A priori estimate of the error co-variance at time k

A posteriori estimate of the error covari-ance at time $k-1$

Process noise

CORRECT STEP EQUATIONS (MEASUREMENT UPDATE)

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \cdot \mathbf{H}^T \cdot (\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^T + \mathbf{R})$$

The **Kalman gain**, this needs to calculated first

Sensor noise

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot (\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1})$$

The ***a posteriori*** estimated state

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k \cdot \mathbf{H}) \cdot \mathbf{P}_{k|k-1}$$

The ***a posteriori*** estimated covariance of our state

CORRECT STEP EQUATIONS (MEASUREMENT UPDATE)

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} \cdot \left(\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} + \mathbf{R}\right)$$

estimated state at last step

$$\hat{\mathbf{x}}_{k|k} = \boxed{\hat{\mathbf{x}}_{k|k-1}} + \mathbf{K}_k \cdot \left(\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1}\right)$$

$$\mathbf{P}_{k|k} = \left(I - \mathbf{K}_k \cdot \mathbf{H}\right) \cdot \mathbf{P}_{k|k-1}$$

CORRECT STEP EQUATIONS (MEASUREMENT UPDATE)

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} \cdot (\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} + \mathbf{R})$$

(actual measurements - expected measurement) × Kalman gain

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot (\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1})$$

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k \cdot \mathbf{H}) \cdot \mathbf{P}_{k|k-1}$$

THE KALMAN FILTER LOOP

Prediction

1. Project the state ahead
   $\hat{\mathbf{x}}_{k|k-1} = \mathbf{F} \cdot \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B} \cdot u_{k-1}$
2. Project the error covariance ahead
   $\mathbf{P}_{k|k-1} = \mathbf{F} \cdot \mathbf{P}_{k-1|k-1} \cdot \mathbf{F}^{\mathsf{T}} + \mathbf{Q}$

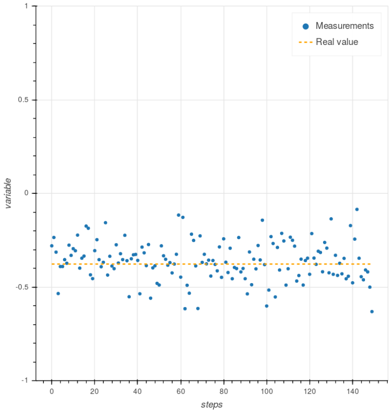
Correction

1. Compute Kalman gain
   $\mathbf{K}_k = \mathbf{P}_{k|k-1} \cdot \mathbf{H}^{\mathsf{T}} \cdot (\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^{\mathsf{T}} + \mathbf{R})^{-1}$
2. Update estimate with measurements $\mathbf{z}_k$
   $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot (\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1})$
3. Update the error covariance
   $\mathbf{P}_{k|k} = (I - \mathbf{K}_k \cdot \mathbf{H}) \cdot \mathbf{P}_{k|k-1}$

Initial estimates
for $\hat{\mathbf{x}}_{k-1}$ and $\mathbf{P}_{k-1}$

EXAMPLE 1: MEASURING A NOISY YET CONSTANT VALUE

A Kalman filter to estimate the state of a system with **one variable**. For this demonstration, the variable remains **constant** (for example, measuring a voltage or a temperature).



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Prediction:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F} \cdot \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B} \cdot u_{k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F} \cdot \mathbf{P}_{k-1|k-1} \cdot \mathbf{F}^{\top} + \mathbf{Q}$$

Prediction:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F} \cdot \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B} \cdot u_{k-1}$$

$\mathbf{x} = \begin{bmatrix} x \end{bmatrix}$

$\mathbf{F} = \begin{bmatrix} 1 \end{bmatrix}$

$u = \begin{bmatrix} 0 \end{bmatrix}$

$$\mathbf{P}_{k|k-1} = \mathbf{F} \cdot \mathbf{P}_{k-1|k-1} \cdot \mathbf{F}^{\mathsf{T}} + \mathbf{Q}$$

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Prediction:

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$$\mathbf{P}_{k|k-1} = \mathbf{F} \cdot \mathbf{P}_{k-1|k-1} \cdot \mathbf{F}^{\mathsf{T}} + \mathbf{Q} = \mathbf{P}_{k-1|k-1}$$

EXAMPLE 1: MEASURING A NOISY YET CONSTANT VALUE

Correction:

observation matrix $\mathbf{H} = \begin{bmatrix}1\end{bmatrix}$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\top \cdot \left(\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\top + \mathbf{R}\right)^{-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot \left(\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1}\right)$$

$$\mathbf{P}_{k|k} = \left(\mathcal{I} - \mathbf{K}_k \cdot \mathbf{H}\right) \cdot \mathbf{P}_{k|k-1}$$

Correction:

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\top \cdot (\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\top + \mathbf{R})^{-1}$$
$$= \mathbf{P}_{k|k-1} \cdot (\mathbf{P}_{k|k-1} + \mathbf{R})^{-1}$$

$\mathbf{R}$?
$\mathbf{R} = \begin{bmatrix} 0.01 \end{bmatrix}$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot (\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1})$$

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k \cdot \mathbf{H}) \cdot \mathbf{P}_{k|k-1}$$

EXAMPLE 1: MEASURING A NOISY YET CONSTANT VALUE

Correction:

$$\begin{aligned}\mathbf{K}_k &= \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} \cdot \left(\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} + \mathbf{R}\right)^{-1} \\ &= \mathbf{P}_{k|k-1} \cdot \left(\mathbf{P}_{k|k-1} + \mathbf{R}\right)^{-1} \\ &= \mathbf{P}_{k|k-1} \cdot \left(\mathbf{P}_{k|k-1} + [0.01]\right)^{-1}\end{aligned}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot \left(\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1}\right)$$

$$\mathbf{P}_{k|k} = \left(I - \mathbf{K}_k \cdot \mathbf{H}\right) \cdot \mathbf{P}_{k|k-1}$$

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$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot (\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1}) = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot (\mathbf{z}_k - \hat{\mathbf{x}}_{k|k-1})$$

$$\mathbf{P}_{k|k} = (I - \mathbf{K}_k \cdot \mathbf{H}) \cdot \mathbf{P}_{k|k-1}$$

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Correction:

$$
\begin{aligned}
\mathbf{K}_k &= \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} \cdot \left(\mathbf{H} \cdot \mathbf{P}_{k|k-1} \cdot \mathbf{H}^\mathsf{T} + \mathbf{R}\right)^{-1} \\
&= \mathbf{P}_{k|k-1} \cdot \left(\mathbf{P}_{k|k-1} + \mathbf{R}\right)^{-1} \\
&= \mathbf{P}_{k|k-1} \cdot \left(\mathbf{P}_{k|k-1} + [0.01]\right)^{-1}
\end{aligned}
$$

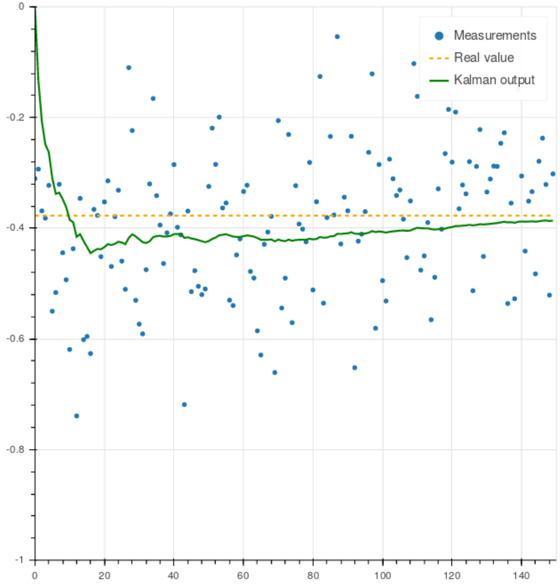
$$
\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot \left(\mathbf{z}_k - \mathbf{H} \cdot \hat{\mathbf{x}}_{k|k-1}\right) = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \cdot \left(\mathbf{z}_k - \hat{\mathbf{x}}_{k|k-1}\right)
$$

$$
\mathbf{P}_{k|k} = \left(I - \mathbf{K}_k \cdot \mathbf{H}\right) \cdot \mathbf{P}_{k|k-1} = \left([1] - \mathbf{K}_k\right) \cdot \mathbf{P}_{k|k-1}
$$

IN PYTHON...

```
from numpy.matlib import matrix

# initial state estimate
x = [matrix([0.])]
P = [matrix([0.001])]

k = 1 # step

R = matrix([0.01]) # estimate of measurement noise

def kalman(k):
    # Prediction phase
    x_prior = x[k-1]
    P_prior = P[k-1]

    # Correction phase
    K = P_prior * ( P_prior + R ).I
    x.append(x_prior + K * (z[k] - x_prior))
    P.append((matrix([1.]) - K) * P_prior)
```

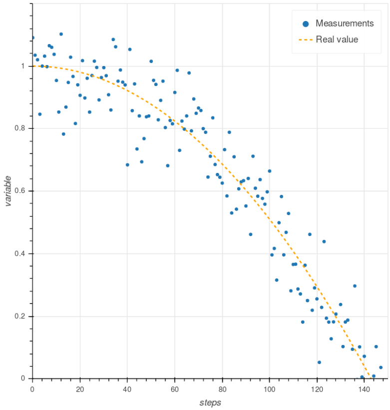
The complete examples are online.



Measurements
Real value
Kalman output
0
-0.2
-0.4
-0.6
-0.8
-1
0
20
40
60
80
100
120
140

EXAMPLE 2: FREE FALL

A Kalman filter to estimate the state of a system with **one variable** (height). The variable **changes** under the effect of an external force (gravity).



EXAMPLE 2: FREE FALL

Free fall equations

$$
\begin{aligned}
&\ddot{y}(t) = -g \\
\Rightarrow\ &\dot{y}(t) = \dot{y}(t_0) - g(t - t_0) \\
\Rightarrow\ &y(t) = y(t_0) + \dot{y}(t_0)(t - t_0) - \frac{g}{2}(t - t_0)^2
\end{aligned}
$$

EXAMPLE 2: FREE FALL

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\end{aligned}
$$

As a discrete time system, with time increment $t - t_0 = 1$:

$$y_k = y_{k-1} + \dot{y}_{k-1} - \tfrac{g}{2}$$

How to fit it into our process equations?

EXAMPLE 2: FREE FALL

The trick consists in embedding the velocity in our state: $\mathbf{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

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$$\begin{aligned} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k|k-1} &= \mathbf{F} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k-1|k-1} + \mathbf{B} \cdot u_{k-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k-1|k-1} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \cdot (-g) \end{aligned}$$

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Measurements:

$$\begin{aligned} z_k &= \mathbf{H} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k|k} + \mathbf{w}_k \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k|k} + \mathbf{w}_k \end{aligned}$$

EXAMPLE 2: FREE FALL

$$\Rightarrow \mathbf{F} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$u = \begin{bmatrix} -g \end{bmatrix}$$

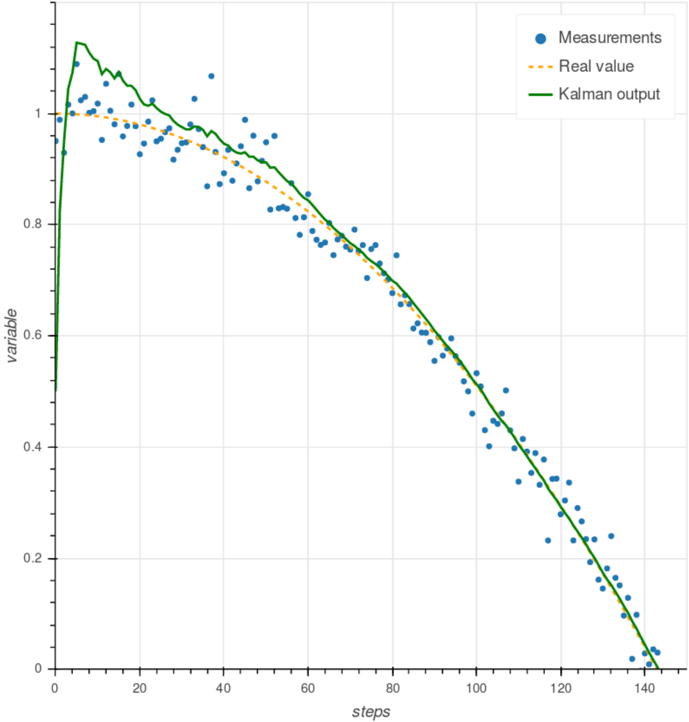
```
from numpy.matlib import matrix

x = [matrix([[0.5],[0.]])] # [y, dy]
P = [matrix([[0.001, 0.],[0., 0.001]])]
k = 1

F = matrix([[1.,1.],[0.,1.]]) # based on the free fall equations
B = matrix([[0.5],[1.]]) # the contribution of the gravity g
H = matrix([1.,0.]) # we only measure the height, not the velocity
Q = matrix([[0.],[0.]]) # no process noise
R = matrix([0.001]) # estimate of our measurement noise
u = matrix([-g]) # control input: gravity

def kalman(k):
    # Prediction phase
    x_prior = F * x[k-1] + B * u
    P_prior = F * P[k-1] * F.T + Q

    # Correction phase
    K = P_prior * H.T * ( H * P_prior * H.T + R ).I
    x.append(x_prior + K * (z[k] - H * x_prior))
    P.append((matrix([[1,0],[0,1]]) - K * H) * P_prior)
```



Measurements
Real value
Kalman output
variable
steps
0
20
40
60
80
100
120
140
0
0.2
0.4
0.6
0.8
1

FAQ

Is a Kalman Filter similar to ***complementary filters***?

- Complementary filters are often used to combine accelerometer and gyro readings on an IMU.
- CF is simple (few lines of code, no matrices) and combines a high and low pass filter.
- CF does not predict states into the future.

More about complementary filters

What if my problem is non-linear?

- There are alternative versions out there such as the Extended Kalman Filter (EKF) or Unscented Kalman Filter (UKF).

FURTHER READING

- A good presentation on Kalman filter
- Lesson 2 of Artificial Intelligence for Robotics at Udacity

Video demonstrations

Kalman filter on accelerometer and gyro to read stable angle

- **http://www.youtube.com/watch?v=MJ71V_wxtuU**
- **http://www.youtube.com/watch?v=Y3TzhXYF0Lg**

Kalman filter tracking an airplane

- **http://www.youtube.com/watch?v=0GSIKwfkFCA**

That's all, folks!

Questions:
Portland Square A216 or **severin.lemaignan@plymouth.ac.uk**

Slides:
github.com/severin-lemaignan/module-mobile-and-humanoid-robots