DRP Presentation

The Conway Knot is Not Slice

Sewon Myung

November 2025

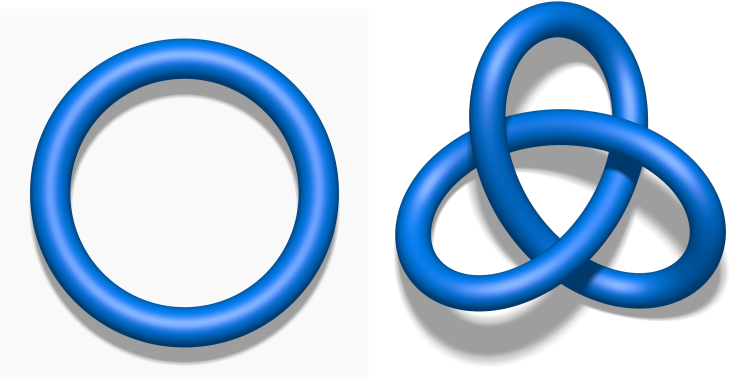
Knots

Definition

A knot is an embedding of the circle S^1 into $\mathbb{R}^3$.

- Knot theory studies when two knots can be continuously transformed into one another without cutting or letting strands pass through each other.
- Reidemeister moves generate all such transformations in S^3.
- Examples: the unknot and the trefoil.

Examples of Knots



Unknot (left) and trefoil (right).

Knot Invariants

- Knot invariants help us distinguish knots.
- Examples:
    - Crossing number
    - Alexander polynomial
    - Conway polynomial
        - If a knot is the unknot, then its Conway polynomial is 1.
        - Is the converse true?

Slice Knots

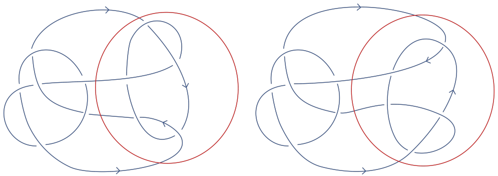
Definition

A knot K is **slice** if it bounds a smoothly embedded disk in B^4.

- A slice knot can be thought of as a knot that becomes "unknotted" in four dimensions.
- Various invariants prove that a knot is not slice.:
  - Slice genus
  - Knot signatures
  - Rasmussen's s–invariant

The Conway Knot

- John Horton Conway (1937–2020) introduced a knot whose Conway polynomial is 1, but it is not the unknot.
- It is a mutation of a slice knot.
- For decades, mathematicians did not know whether the Conway knot was slice.
- In 2020, Lisa Piccirillo resolved this question.



Conway knot (left) and a related slice knot (right).

The Main Result

Theorem (Piccirillo, 2020)

The Conway knot is not slice.

- Piccirillo's strategy:
  - Construct another knot K' such that $X(K') \cong X(K)$.
  - Compute Rasumussen's s-invariant for K'.
  - Conclude that K is not slice.

Knot Traces

Definition

The 0–**trace** of a knot K, denoted $X(K)$, is the 4–manifold obtained by attaching a 0–framed 2–handle to B^4 along $K \subset S^3$.

- Think of adding a thickened disk in B^4 whose boundary is the knot.

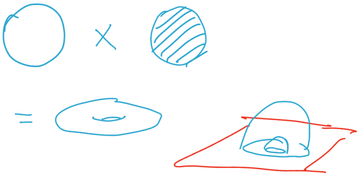
Knot Trace Visualization

n-dimensional
k-handle $= D^k \times D^{n-k}$

Dimension 4, 2-handle

$h^2 = D^2 \times D^2$

attaching region : $\partial_- h^2 = (\partial D^2) \times D^2$



Trace Embedding Lemma

Lemma

A knot K is slice if and only if $X(K)$ smoothly embeds in S^4.

- Thus, sliceness can be studied by analyzing the trace.

Construction of K'

- Piccirillo finds a knot K' such that $X(K') \cong X(K)$.
- $s(K) = 0$, so the s–invariant does not detect sliceness for the Conway knot.
- But $s(K') \neq 0$, so K' is not slice.

Conclusion

- If K' is not slice, then $X(K')$ does not embed in S^4.
- Since $X(K) \cong X(K')$, K is not slice either.
- Therefore, the Conway knot is not slice.