1. By using the hint, we have to use limit equals 0 to find the maximum:

$$\lim_{\varepsilon\to 0}\frac{J(f(x,z)+\varepsilon g(x,z))-J(f(x,z))}{\varepsilon}=0$$

$$\to \lim_{\varepsilon\to 0}\frac{\int[f(x,z)+\varepsilon g(x,z)]P_{XZ}(x,z)dxdz-\int[f(x,z)]P_{XZ}(x,z)dxdz}{\varepsilon}-$$

$$\lim_{\varepsilon\to 0}\frac{\log\int exp[f(x,z)+\varepsilon g(x,z)]P_X(x)P_Z(z)dxdz-\log\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}{\varepsilon}$$
$$=0$$

$$\to \lim_{\varepsilon\to 0}\frac{\int[\varepsilon g(x,z)]P_{XZ}(x,z)dxdz}{\varepsilon}-$$

$$\lim_{\varepsilon\to 0}\frac{\log\frac{\int exp[f(x,z)+\varepsilon g(x,z)]P_X(x)P_Z(z)dxdz}{\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}}{\varepsilon}=0$$

Now we need to make use of **Hopital's** theorem because we have $\frac{0}{0}$ in limits. So, we have to take the derivative from the numerator and denominator and divide them.

$$\to \lim_{\varepsilon\to 0}\left(\frac{\frac{d}{d\varepsilon}(\int[\varepsilon g(x,z)]P_{XZ}(x,z)dxdz)}{1}-\frac{\frac{d}{d\varepsilon}\left(\log\frac{\int exp[f(x,z)+\varepsilon g(x,z)]P_X(x)P_Z(z)dxdz}{\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}\right)}{1}\right)=0$$

interval [a, b] of the integrals is not a function of ε so we can move the derivative into the first integral and take the derivative of the second integral with chain rule.

$$\to \lim_{\varepsilon\to 0}\left(\int\left[\frac{d}{d\varepsilon}[\varepsilon]g(x,z)\right]P_{XZ}(x,z)dxdz\right)-$$

$$\lim_{\varepsilon\to 0}\left(\frac{\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}{\int exp[f(x,z)+\varepsilon g(x,z)]P_X(x)P_Z(z)dxdz}\times\frac{d}{d\varepsilon}\left(\frac{\int exp[f(x,z)+\varepsilon g(x,z)]P_X(x)P_Z(z)dxdz}{\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}\right)\right)=0$$

It is simplified to:

$$\to \lim_{\varepsilon\to 0}\left(\int[g(x,z)]P_{XZ}(x,z)dxdz\right)-$$

$$\lim_{\varepsilon\to 0}\left(\frac{\frac{d}{d\varepsilon}(\int exp[f(x,z)+\varepsilon g(x,z)]P_X(x)P_Z(z)dxdz)}{\int exp[f(x,z)+\varepsilon g(x,z)]P_X(x)P_Z(z)dxdz}\right)=0$$

We can separate the power of exponential, move the derivative inside and what we have is:

$$\rightarrow \left(\int [g(x,z)]P_{XZ}(x,z)dxdz\right) -$$

$$\lim_{\varepsilon\to 0}\left(\frac{\left(\int \left[\frac{d}{d\varepsilon}\exp[\varepsilon g(x,z)]\right]\exp\big(f(x,z)\big)\,P_X(x)P_Z(z)dxdz\right)}{\int \exp\big(\varepsilon g(x,z)\big)\exp\big(f(x,z)\big)\,P_X(x)P_Z(z)dxdz}\right) = 0$$

Due to the fact that only variable function is $g(x,z)$ as the perturbation function, we can move $\int \exp\,(\varepsilon g(x,z))\exp\,(f(x,z))P_X(x)P_Z(z)dxdz$ inside the as the dominator of the second integral:

$$\rightarrow \left(\int [g(x,z)]P_{XZ}(x,z)dxdz\right) -$$

$$\lim_{\varepsilon\to 0}\left(\int \frac{g(x,z)\exp[\varepsilon g(x,z)]\exp\big(f(x,z)\big)\,P_X(x)P_Z(z)dxdz}{\int \exp\big(\varepsilon g(x,z)\big)\exp\big(f(x,z)\big)\,P_X(x)P_Z(z)dxdz}\right) = 0$$

By using $\exp\big(\varepsilon g(x,z)\big) = 1$ for $\varepsilon \to 0$ we can simplify it to:

$$\rightarrow \left(\int [g(x,z)]P_{XZ}(x,z)dxdz\right) -$$

$$\left(\int \frac{g(x,z)\exp\big(f(x,z)\big)\,P_X(x)P_Z(z)dxdz}{\int \exp\big(f(x,z)\big)\,P_X(x)P_Z(z)dxdz}\right) = 0$$

If we combine the two terms above:

$$\left(\int g(x,z)\ \left[\frac{-\exp\big(f(x,z)\big)P_X(x)P_Z(z)}{\int \exp\big(f(x,z)\big)P_X(x)P_Z(z)dxdz} + P_{XZ}(x,z)\ \right]dxdz\right)=0$$

Now we can conclude that the following term is zero:

$$\left[\frac{-\exp\big(f(x,z)\big)\,P_X(x)P_Z(z)}{\int \exp\big(f(x,z)\big)\,P_X(x)P_Z(z)dxdz} + P_{XZ}(x,z)\ \right] = 0$$

$$\rightarrow\rightarrow\rightarrow \boldsymbol{P_{XZ}(x,z)} = \frac{\mathbf{exp}\big(\boldsymbol{f(x,z)}\big)\,\boldsymbol{P_X(x)P_Z(z)}}{\int \mathbf{exp}\big(\boldsymbol{f(x,z)}\big)\,\boldsymbol{P_X(x)P_Z(z)dxdz}}$$

Now, we need to prove that this extremum is a maximum. So, we take the second derivative respect to f.

$$\frac{d^2J}{df^2} = \frac{dJ}{df}\left(\int P_{XZ}(x,z)dxdz\right) -$$

$$\frac{dJ}{df}\left(\frac{exp[f(x,z)]P_X(x)P_Z(z)}{\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}\right)$$

$$\rightarrow -\frac{[exp[f(x,z)]P_X(x)P_Z(z)][(\int exp[f(x,z)]P_X(x)P_Z(z)dxdz\ )\ ] - (exp[f(x,z)]P_X(x)P_Z(z)dxdz)^2}{(\int exp[f(x,z)]P_X(x)P_Z(z)dxdz\ )^{\ 2}}$$

$$\rightarrow -\left(\frac{[exp[f(x,z)]P_X(x)P_Z(z)]}{\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}\right) + \left(\frac{[exp[f(x,z)]P_X(x)P_Z(z)]}{\int exp[f(x,z)]P_X(x)P_Z(z)dxdz}\right)^2$$

$$\rightarrow \frac{d^2J}{df^2} = -P_{XZ} + {P_{XZ}}^2 \ \ \&\&\ P_{XZ} < 1$$

$$\frac{d^2J}{df^2} < 0$$

Now, we see that the second derivative is negative. So, this function has a maximum.

2. We need to substitute the value that we concluded in part (A) in the original equation:

In order to simplify the equation, we can call:

$$\mathrm{A} = \exp\big(f(x,z)\big)\,P_X(x)P_Z(z)$$

So:

$$P_{XZ}(x,z) = \frac{\exp\big(f(x,z)\big)P_X(x)P_Z(z)}{\int \mathrm{A}\ dxdz} \rightarrow\ f(x,z) = \log\left(\frac{P_{XZ}(x,z)\int \mathrm{A}\ dxdz}{P_X(x)P_Z(z)}\right)$$

By substituting $f(x,z)$ in the original equation we would see:

$$MI = \int \log\left(\frac{P_{XZ}(x,z)\int \mathrm{A}\ dxdz}{P_X(x)P_Z(z)}\right) P_{XZ}(x,z)dxdz - \ \log\left(\int \mathrm{A}\ dxdz\ \right)$$

By separating the argument of the first logarithm:

$$MI = \int \left[\ P_{XZ}(x,z)\log\left(\frac{P_{XZ}(x,z)}{P_X(x)P_Z(z)}\right)\ \right] + \left[P_{XZ}(x,z)\log\int \mathrm{A}\ dxdz\right]dxdz -$$

$$- \log(\int \mathrm{A}\, dxdz)$$

By moving out $[\log \int \mathrm{A}\, dxdz\,]$ from the integral as a constant we have:

$$MI = \int [\, P_{XZ}(x,z) \log\left(\frac{P_{XZ}(x,z)}{P_X(x)P_Z(z)}\right)]dxdz +$$

$$\log(\int \mathrm{A}\, dxdz)\ [(\int P_{XZ}(x,z)\, dxdz) - 1]$$

Now we have to make use of the fact that the integral of marginal distribution over the whole domain is 1.

$$\int P_{XZ}(x,z)\, dxdz\ =\ 1$$

In conclusion the second term is zero and the result would be:

$$\rightarrow\rightarrow\rightarrow \boldsymbol{MI} = \int [\, \boldsymbol{P_{XZ}(x,z)} \,\mathbf{log}(\frac{\boldsymbol{P_{XZ}(x,z)}}{\boldsymbol{P_X(x)P_Z(z)}})\,]\ \boldsymbol{dxdz}$$