Drifting games, Boosting & Online Learning

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UCSD

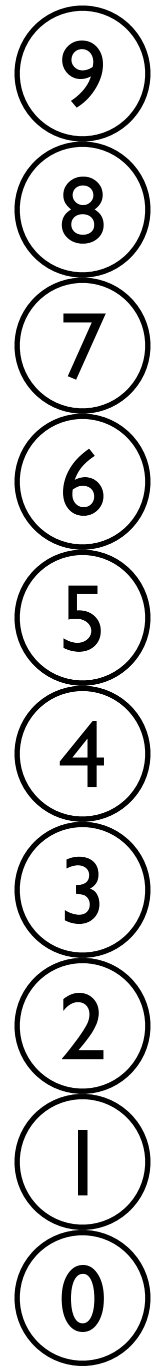
20 questions

called "Ulam's game" by mathematicians

- Alice's goal: find Bob's secret
  (a number in 1..n)
- In each iteration Alice asks:
  is the number in the set S ?
- Alice wants to minimize number of iterations, Bob wants to maximize it.

Strategies for Bob

- Naive Bob: Choose a number, hope for the best.
- Sneaky Bob: Keep the set of consistent secrets as large as possible.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8
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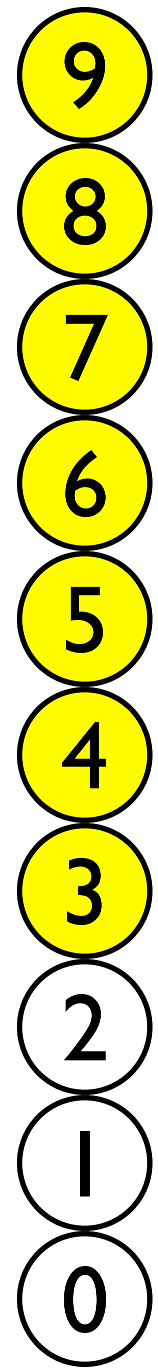
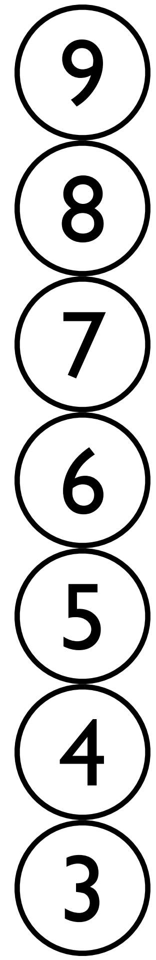
0

Alice: asking about a large set or a small set is a bad idea

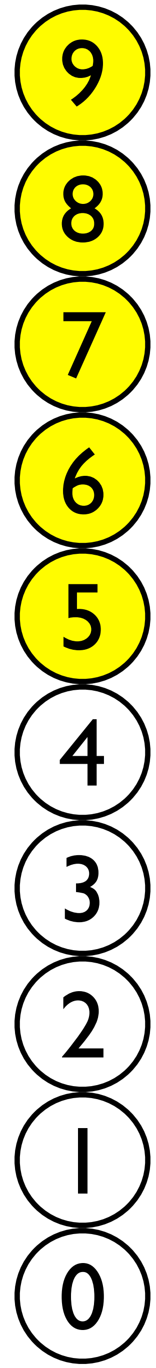


Alice: asking about a large set or a small set is a bad idea



Bob: answer to keep larger set consistent.

Alice: best to ask about half of the currently consistent answers



Bob: no advantage to answering yes vs. no

Secret revealed after

$$\lceil \log_2 10 \rceil = 4$$

questions

Alice: If the size of the set is odd, it cannot be split it into two equal parts.

Assume n is a power of 2

4

3

2

1

0

secret identified after $\log_2 n$ questions

Strategies for Alice

- Deterministic: split set into two equal parts.
- Stochastic: choose each element to be in or our of the set independently at random with equal probabilities (1/2,1/2).

Min-Max strategies

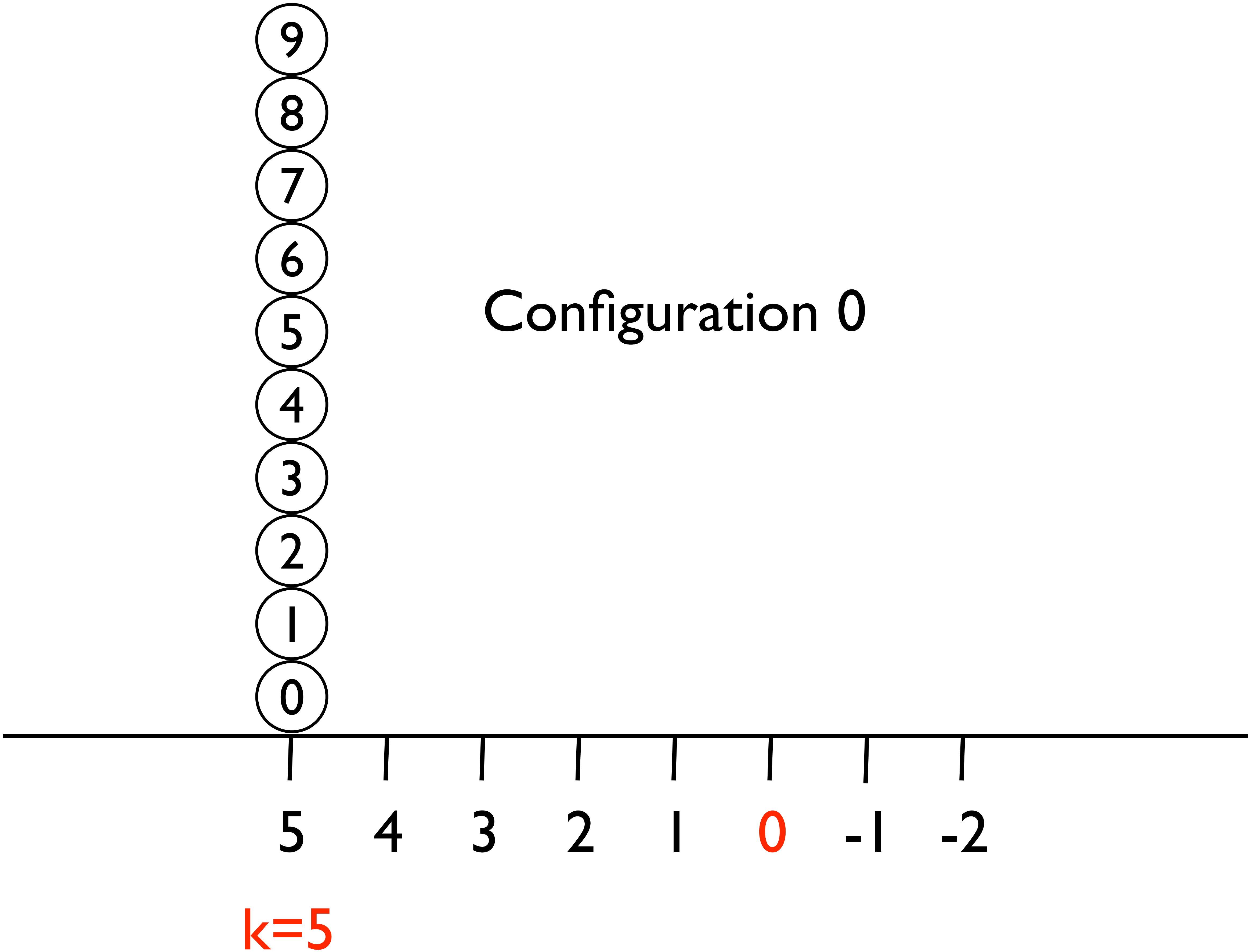
- Alice: each element is in S independently at random with probabilities 1/2, 1/2
- Bob: Answer "Yes" or "No" independently at random with probabilities 1/2, 1/2
- Guarantees that the expected number of steps from |S|=n to |S|<=1 is ceil(log_2(n))
- If one side is known to deviate, the other side would deviate too.

one step look-ahead

- One can derive the optimal response to a known strategy for the current step by assuming that all of the later steps will be done using min/max optimal strategies.

Ulam's game with k lies

- Bob can give an incorrect answer $\leq k$ times.
- Sneaky Bob: keep as many answers as possible consistent with $\leq k$ lies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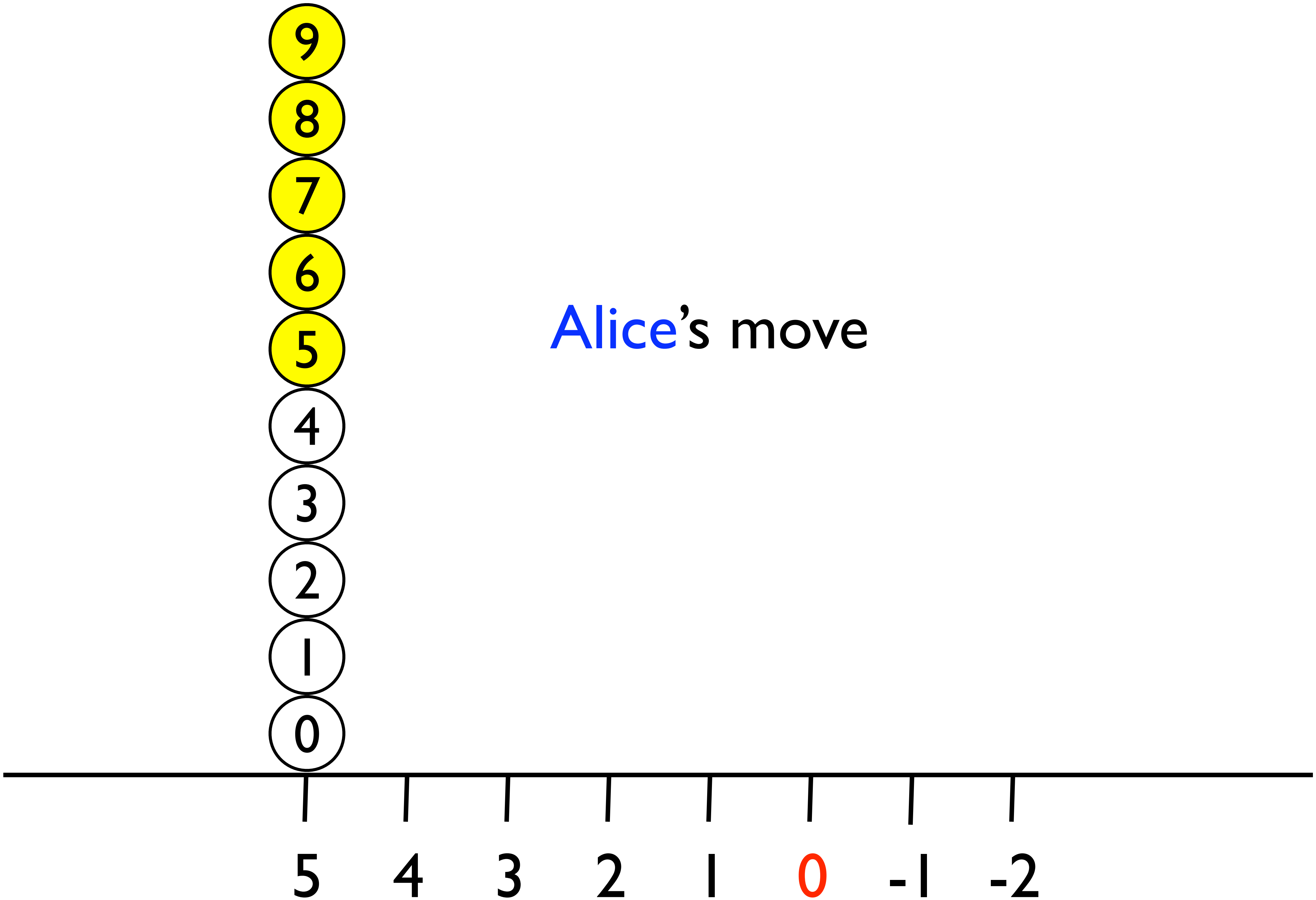
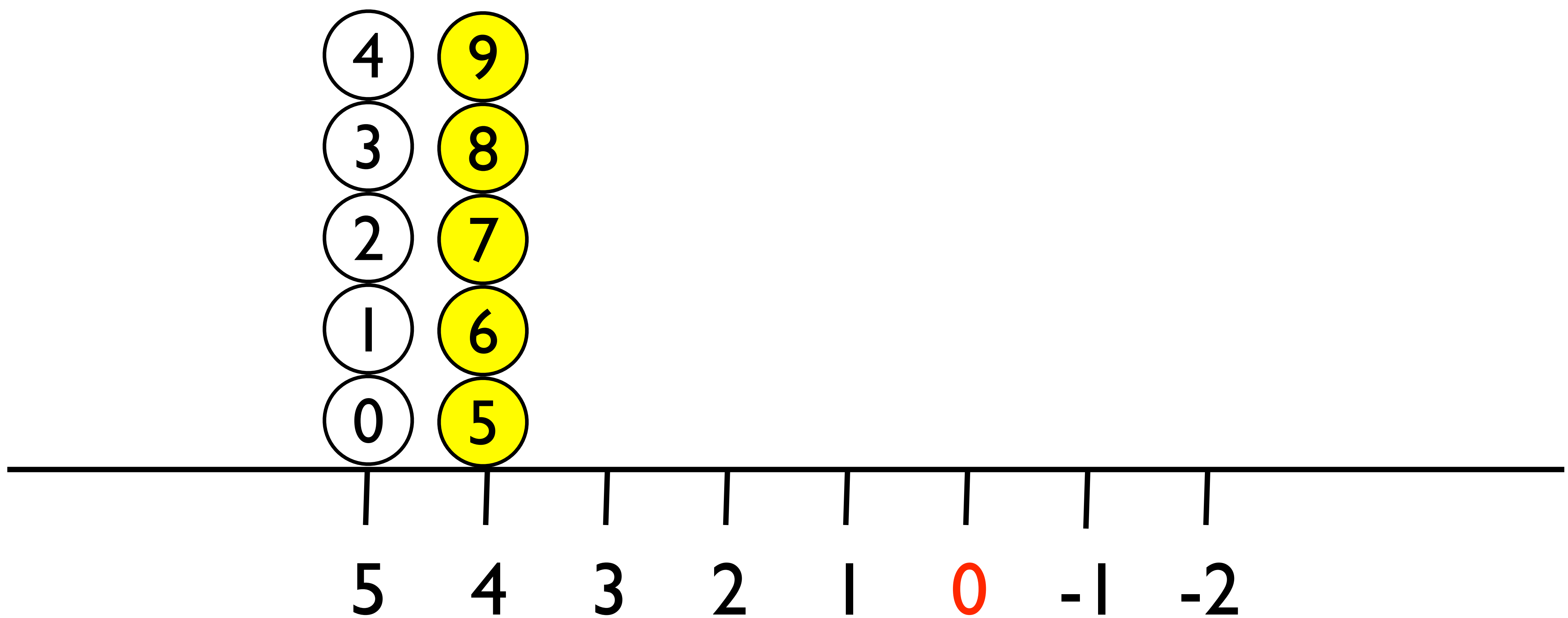
8
7
6
5
4
3
2
1
0
Configuration 0
5 4 3 2 1 0 -1 -2
k=5

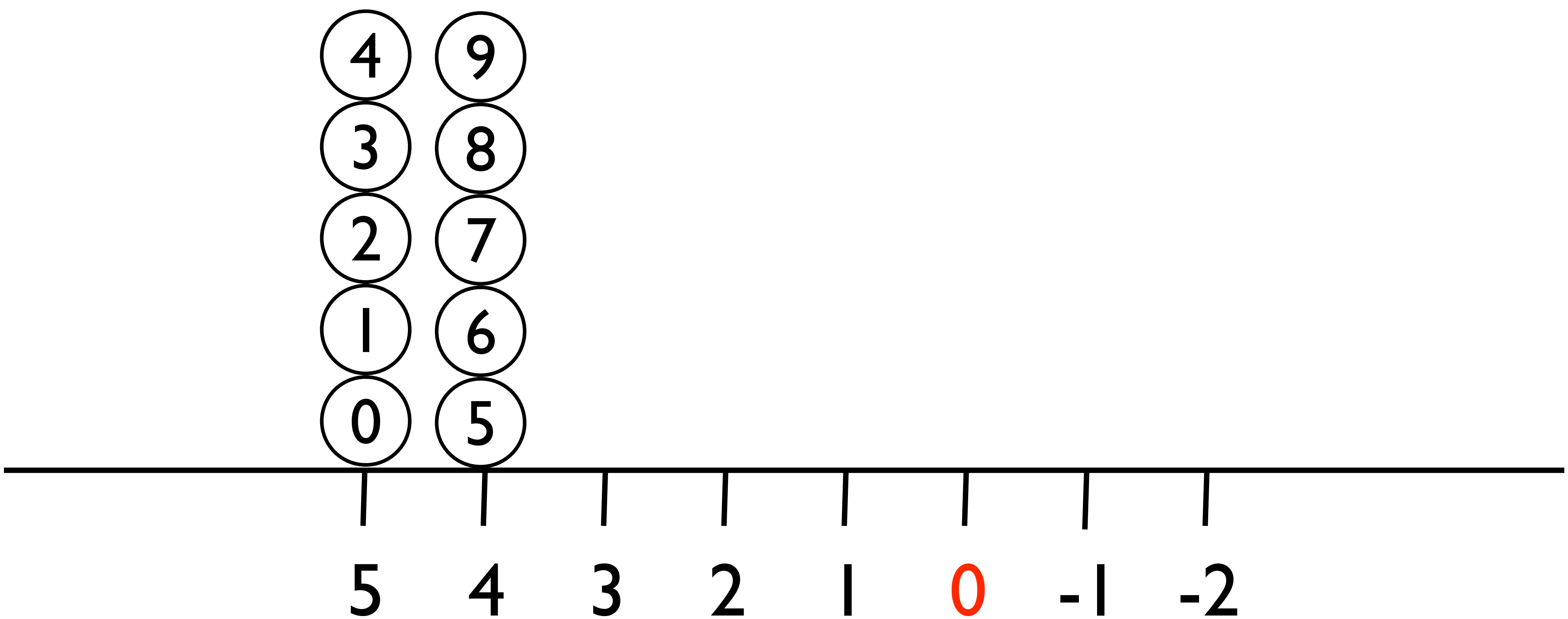


Alice's move

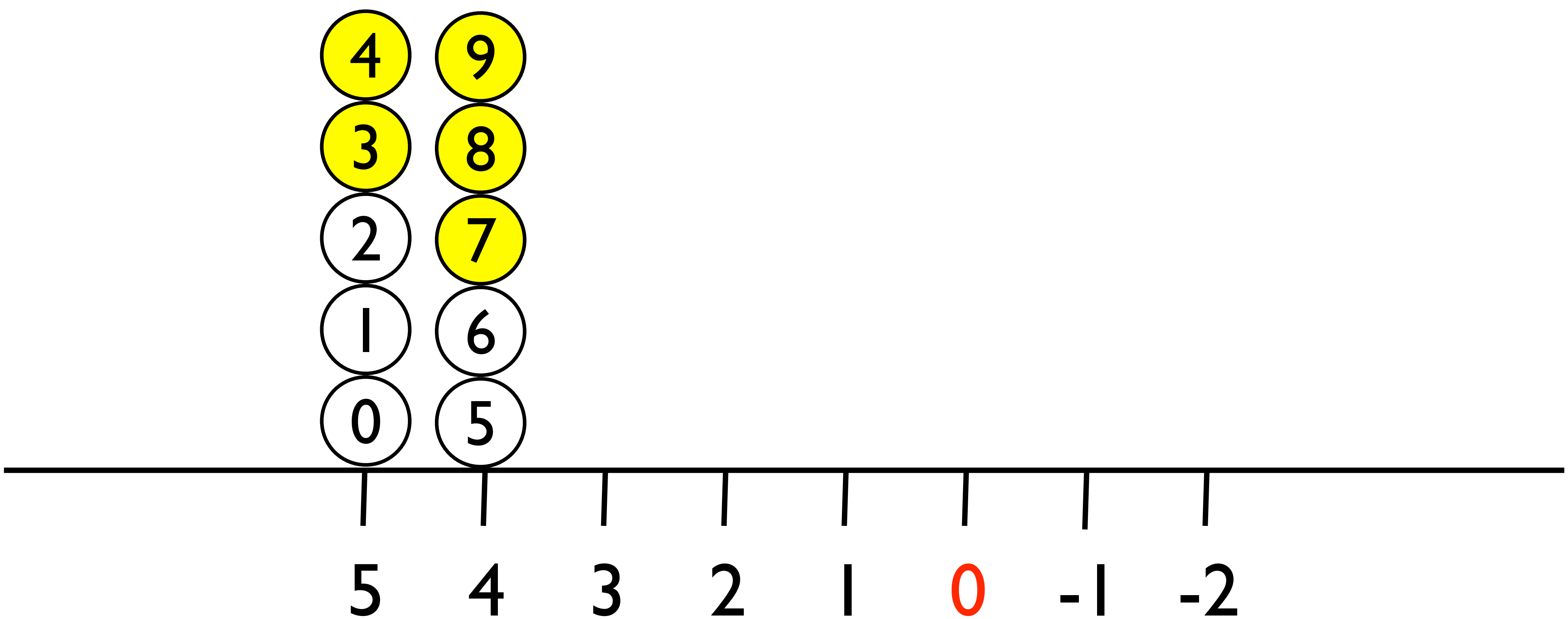
Bob's move



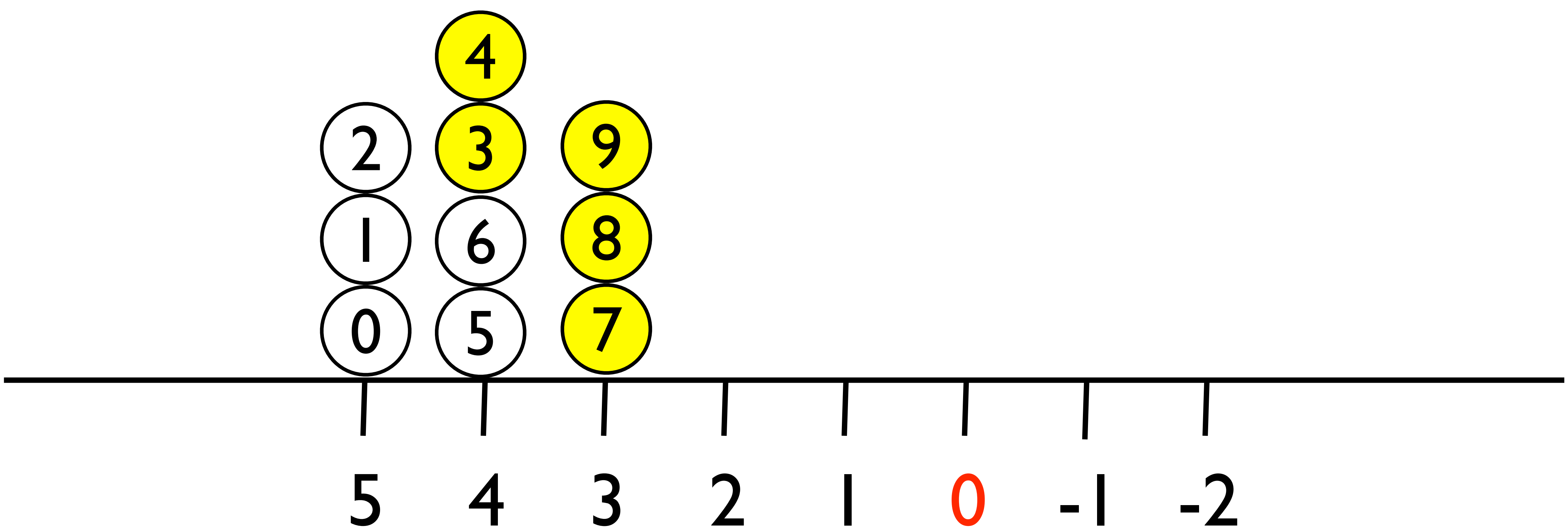
Configuration 1



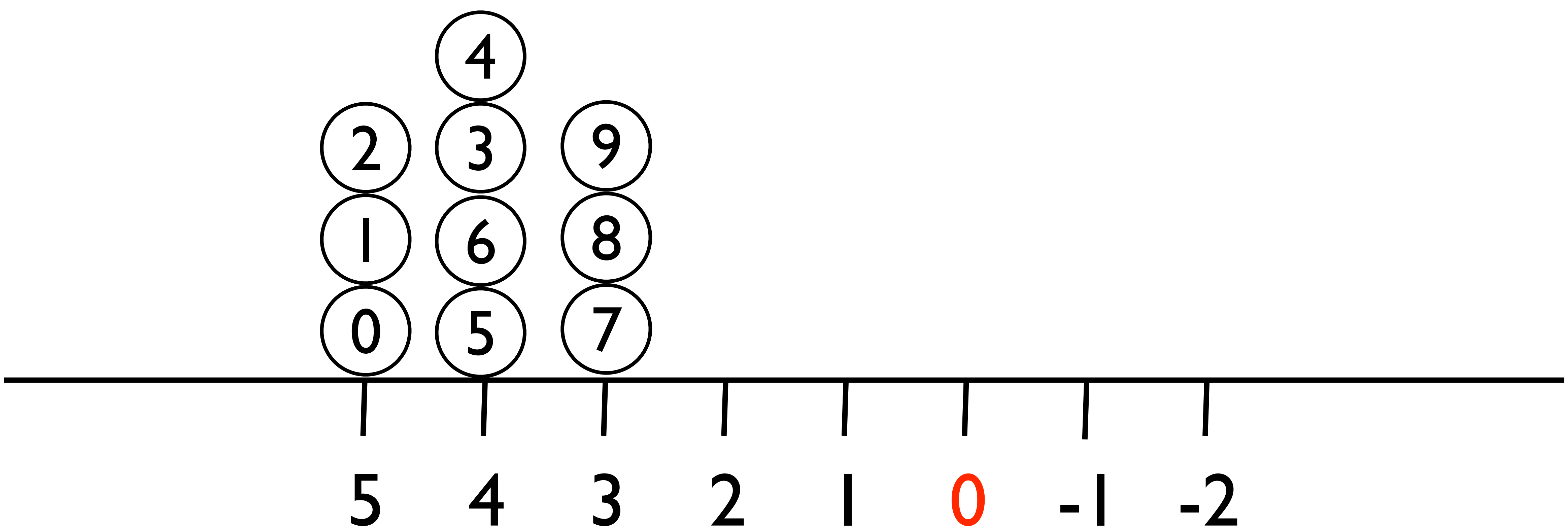
Alice's move



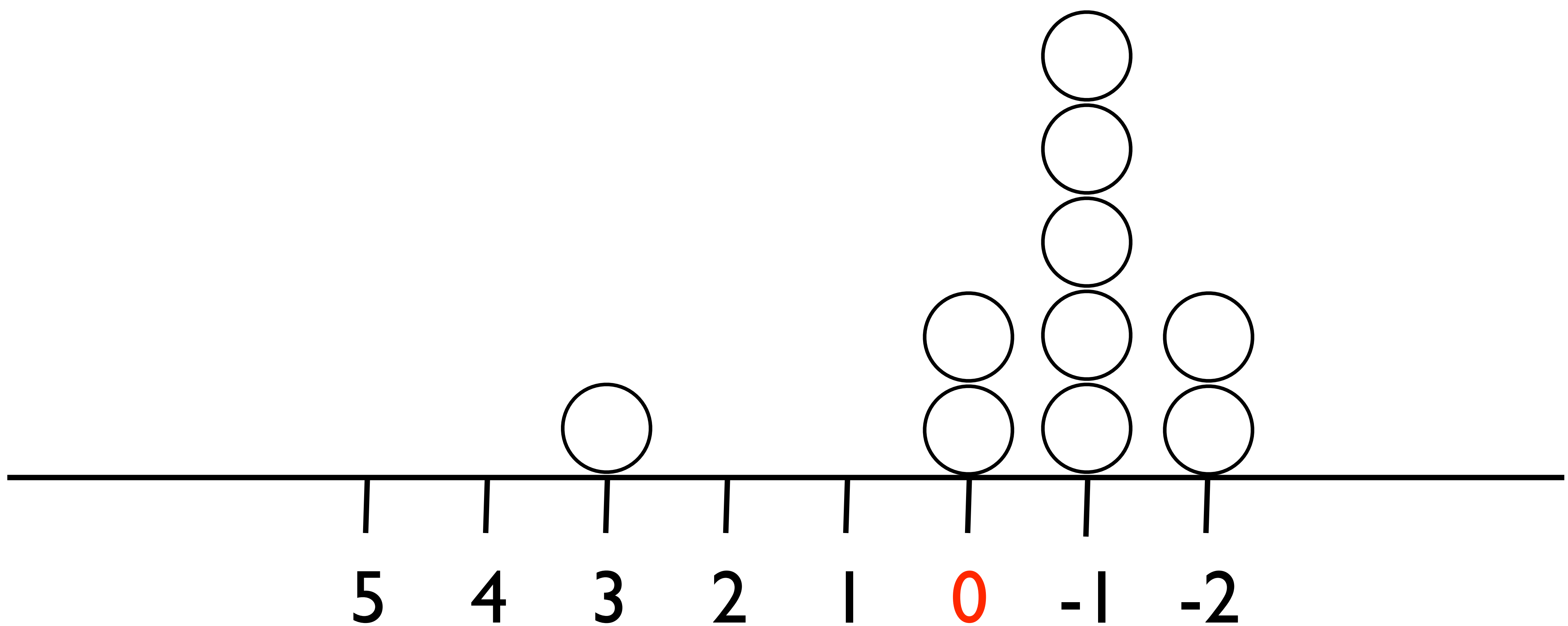
Bob's move



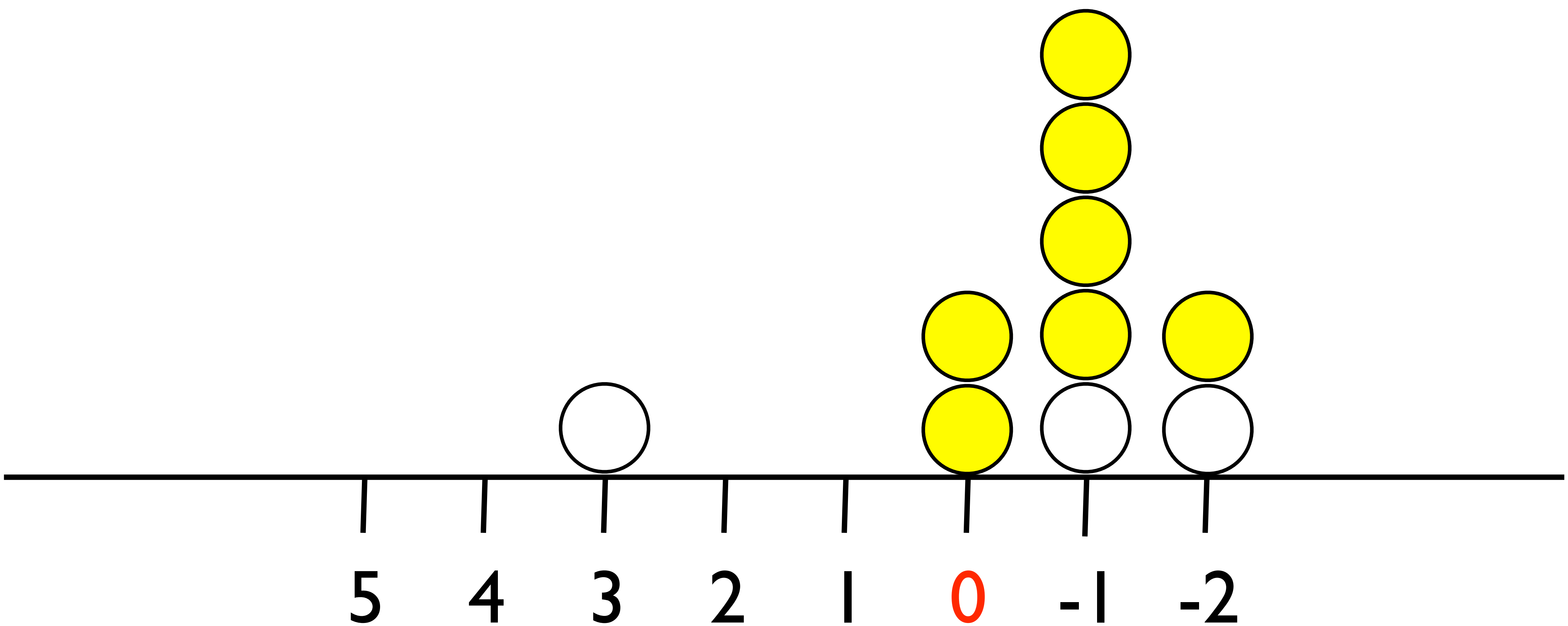
Configuration 2



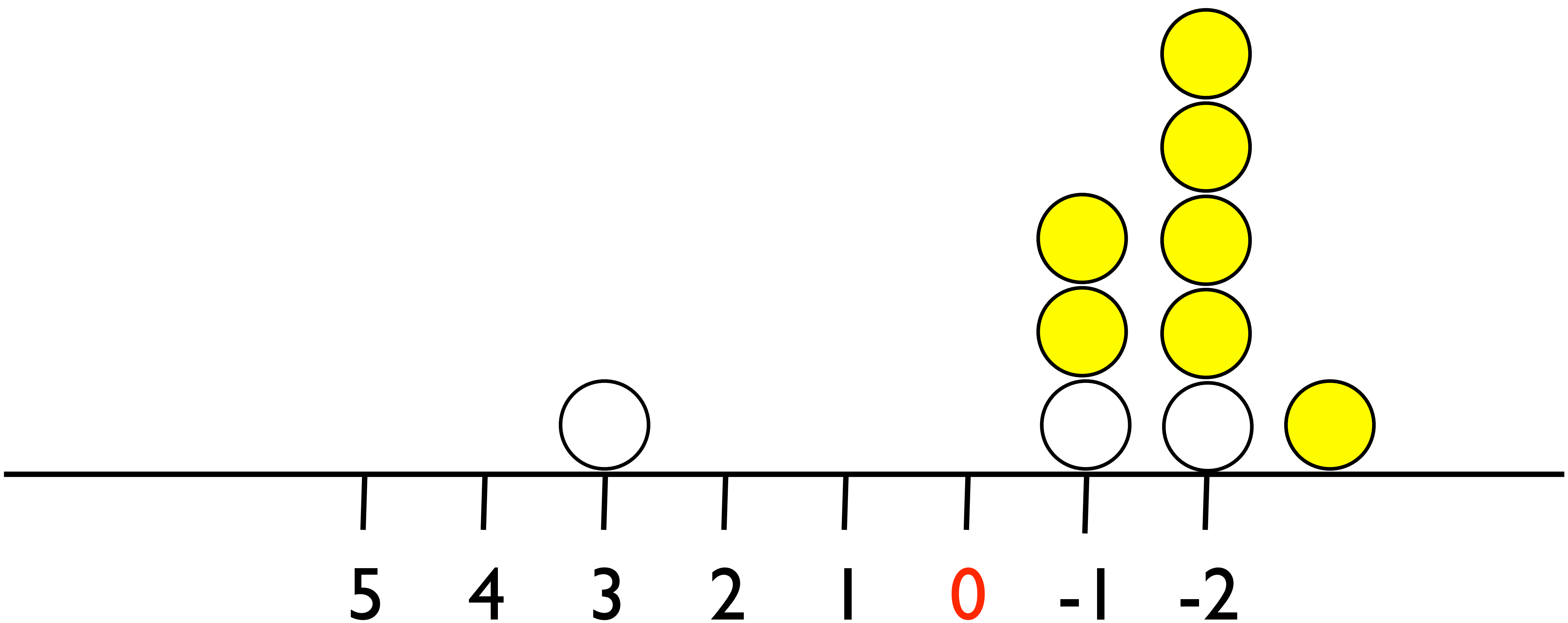
Configuration j



Alice's move

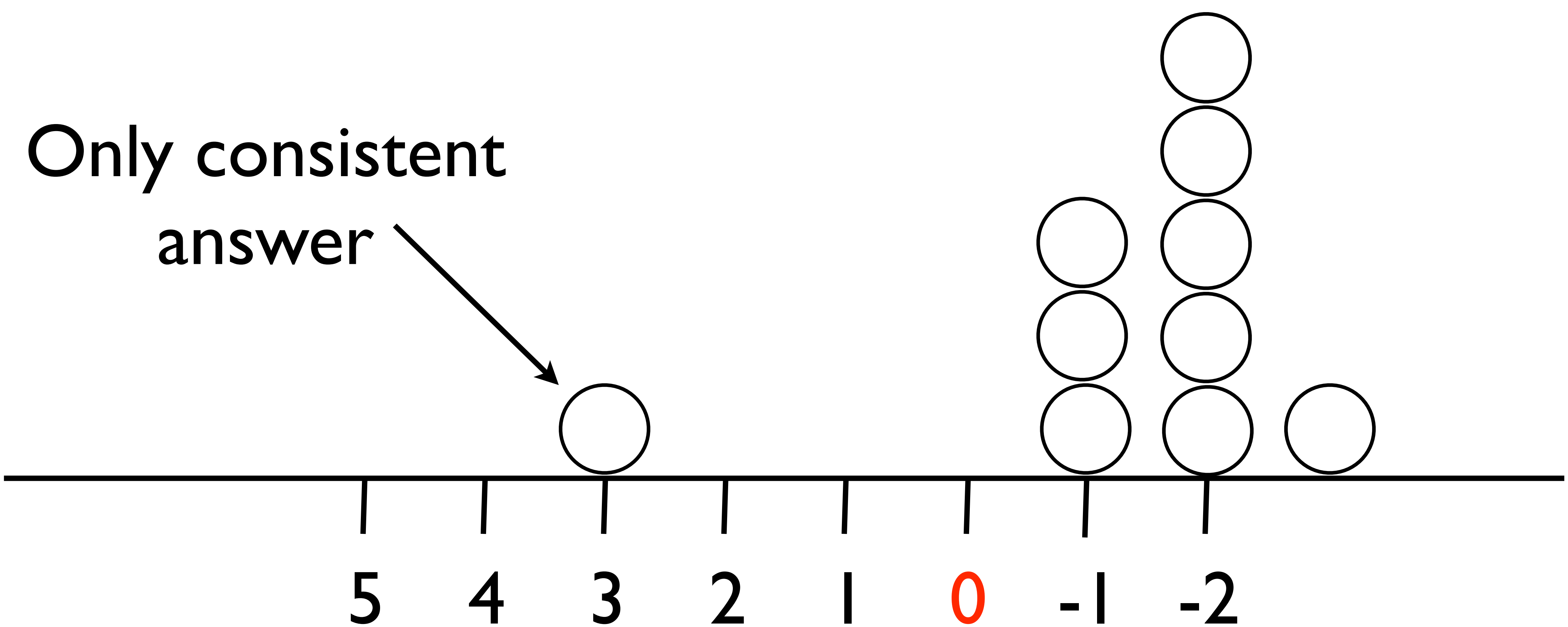


Bob's move



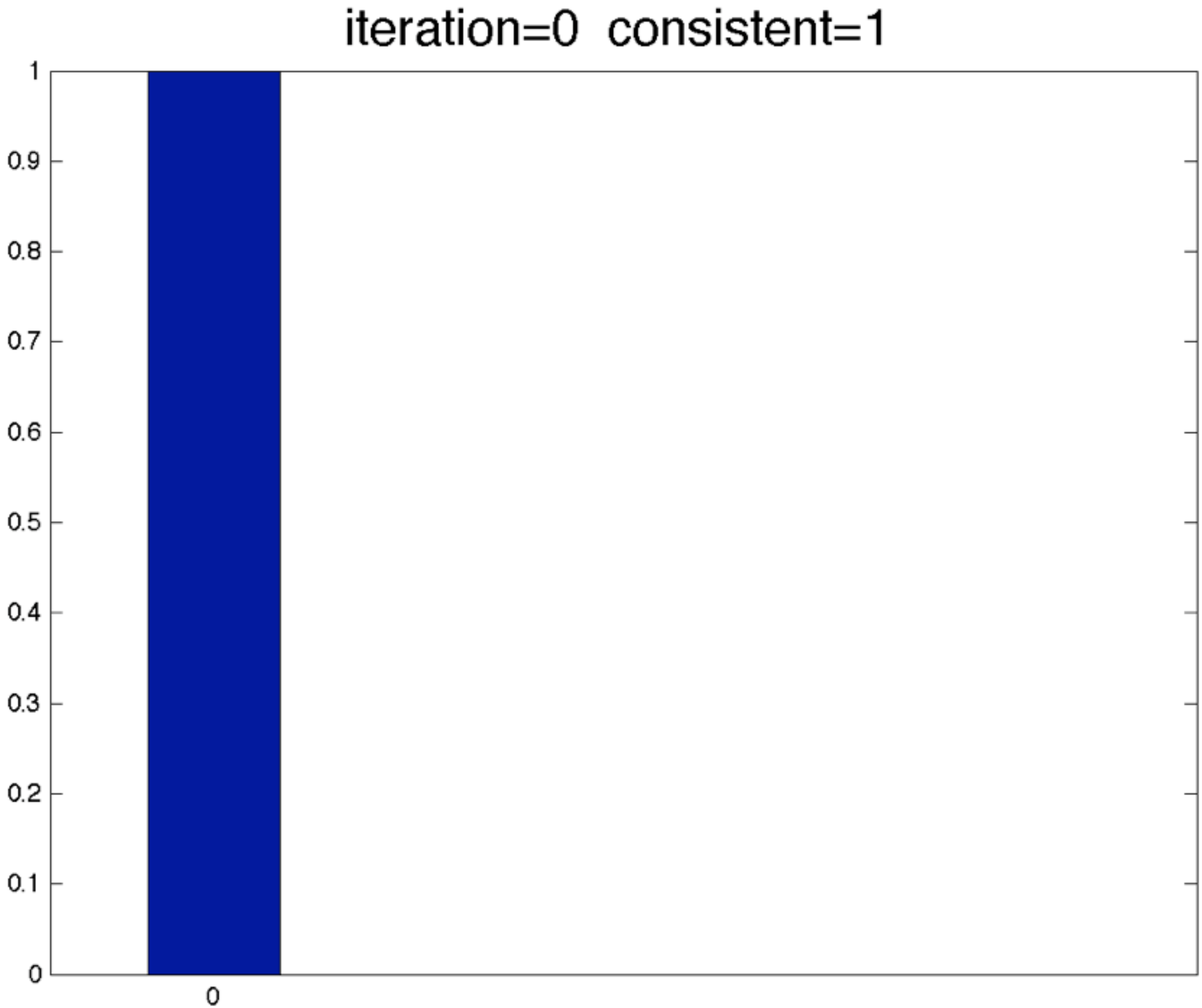
Final Configuration

GAME ENDS



Alice's strategy

- Alice: split each bin into two equal size parts.
- Assume that n is a high enough power of 2 so that equal size splits are always possible.
- Better yet: assume that the answers are a continuous set of volume (measure) 1 and set the goal to reduce volume of consistent set to $1/n$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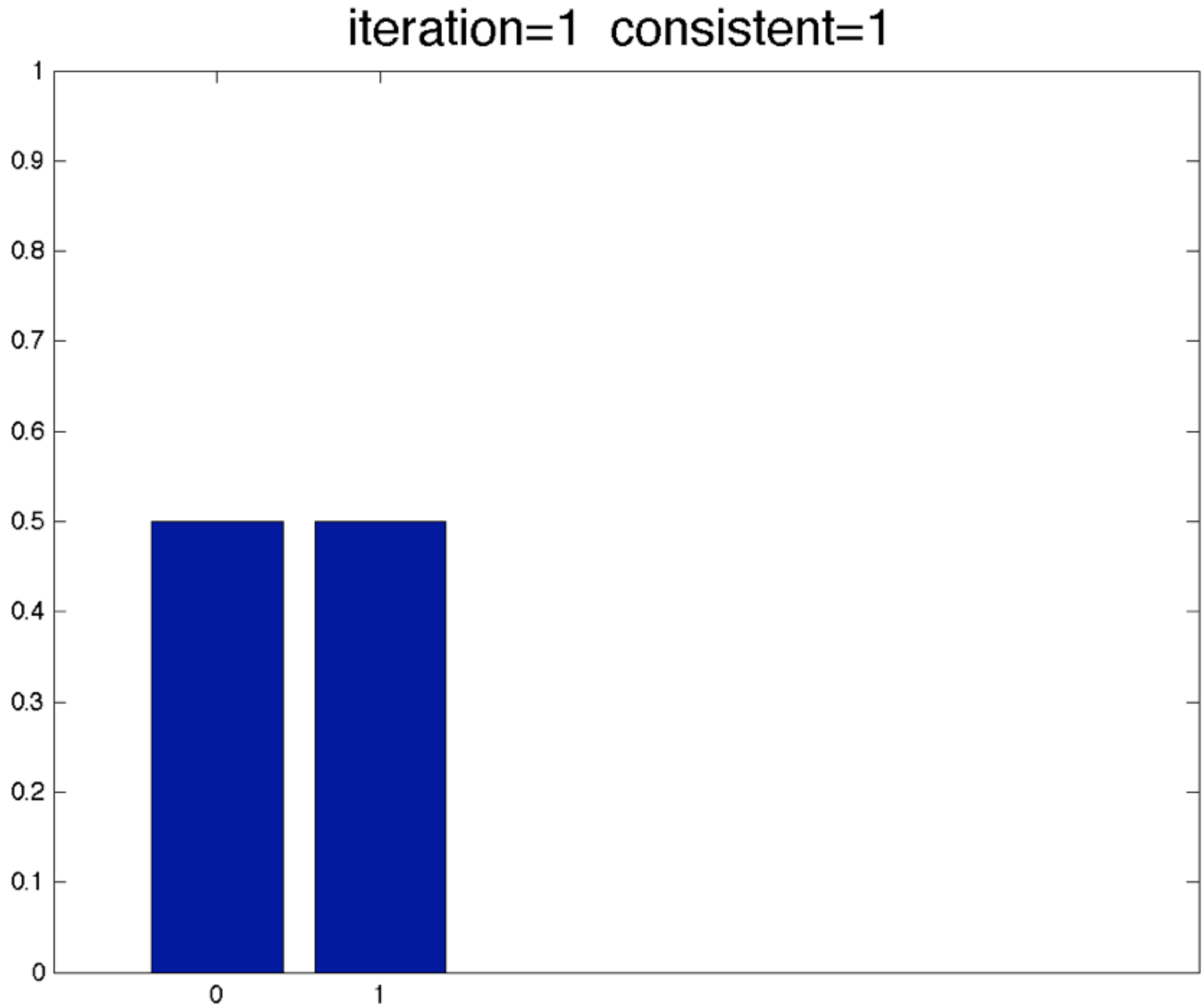
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0



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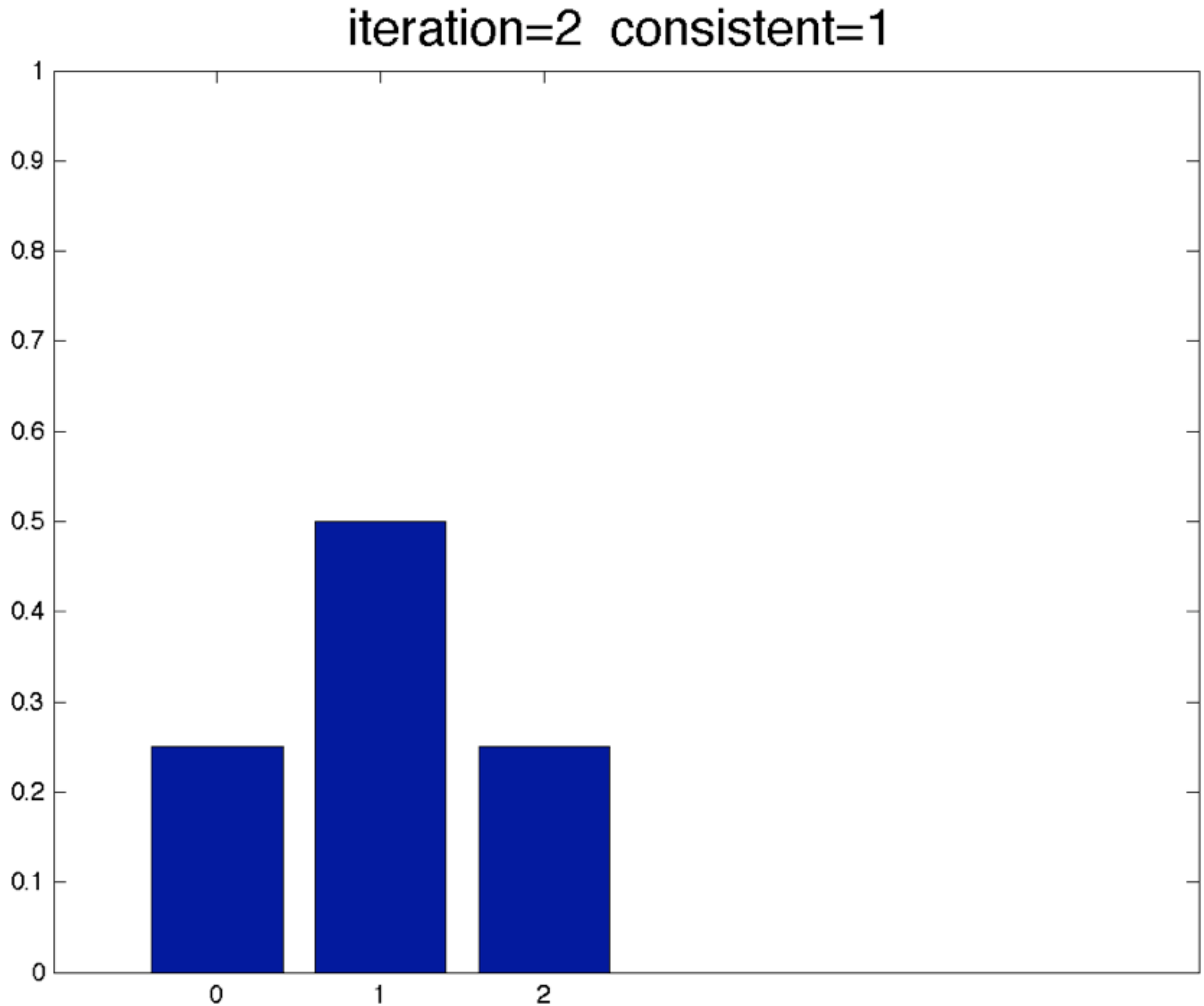
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1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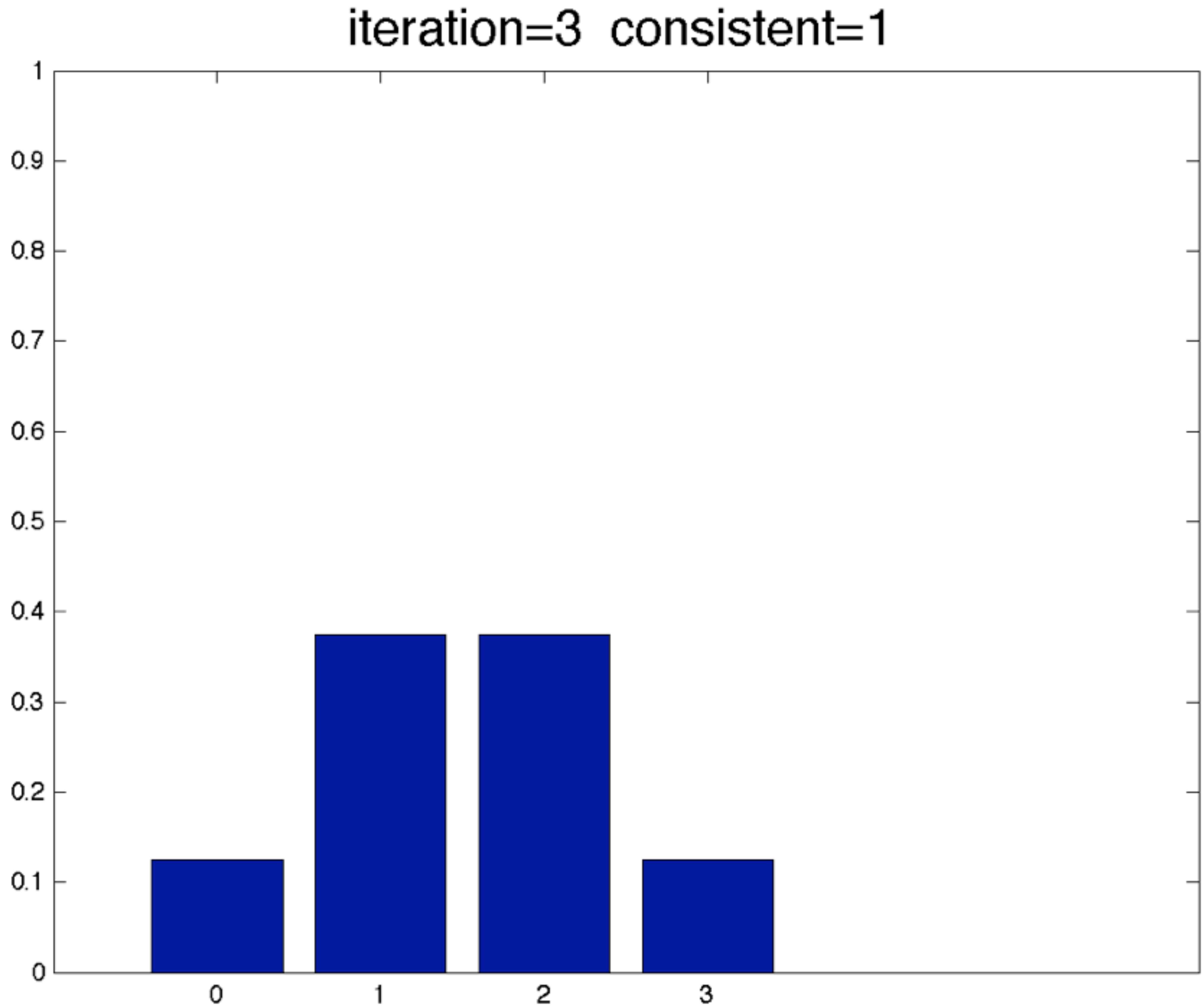
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2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0.6
0.5
0.4
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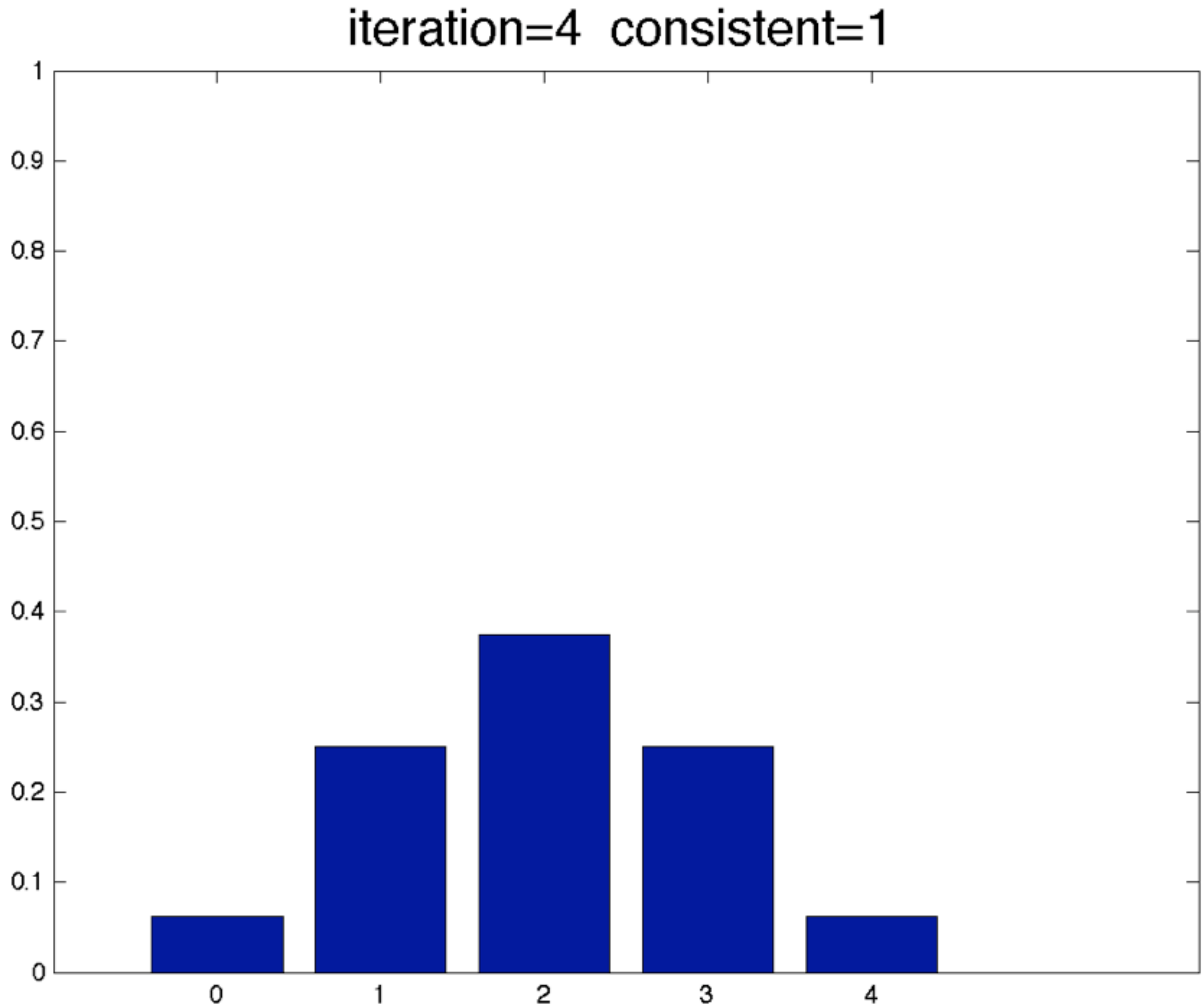
0.2
0.1
0
0
1
2
3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0.6
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0.4
0.3
0.2
0.1
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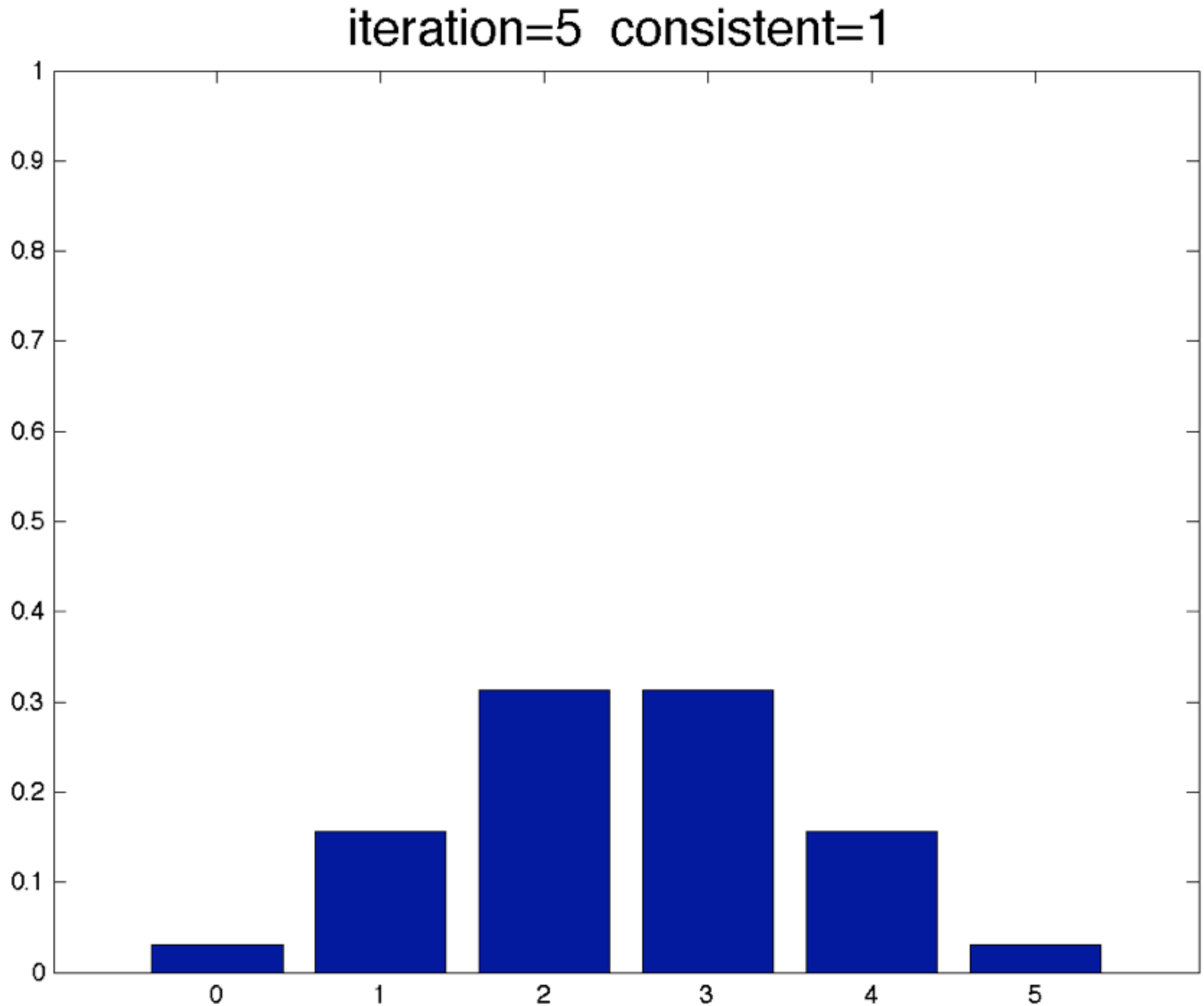
0
1
2
3
4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0.6
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0.4
0.3
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0.1
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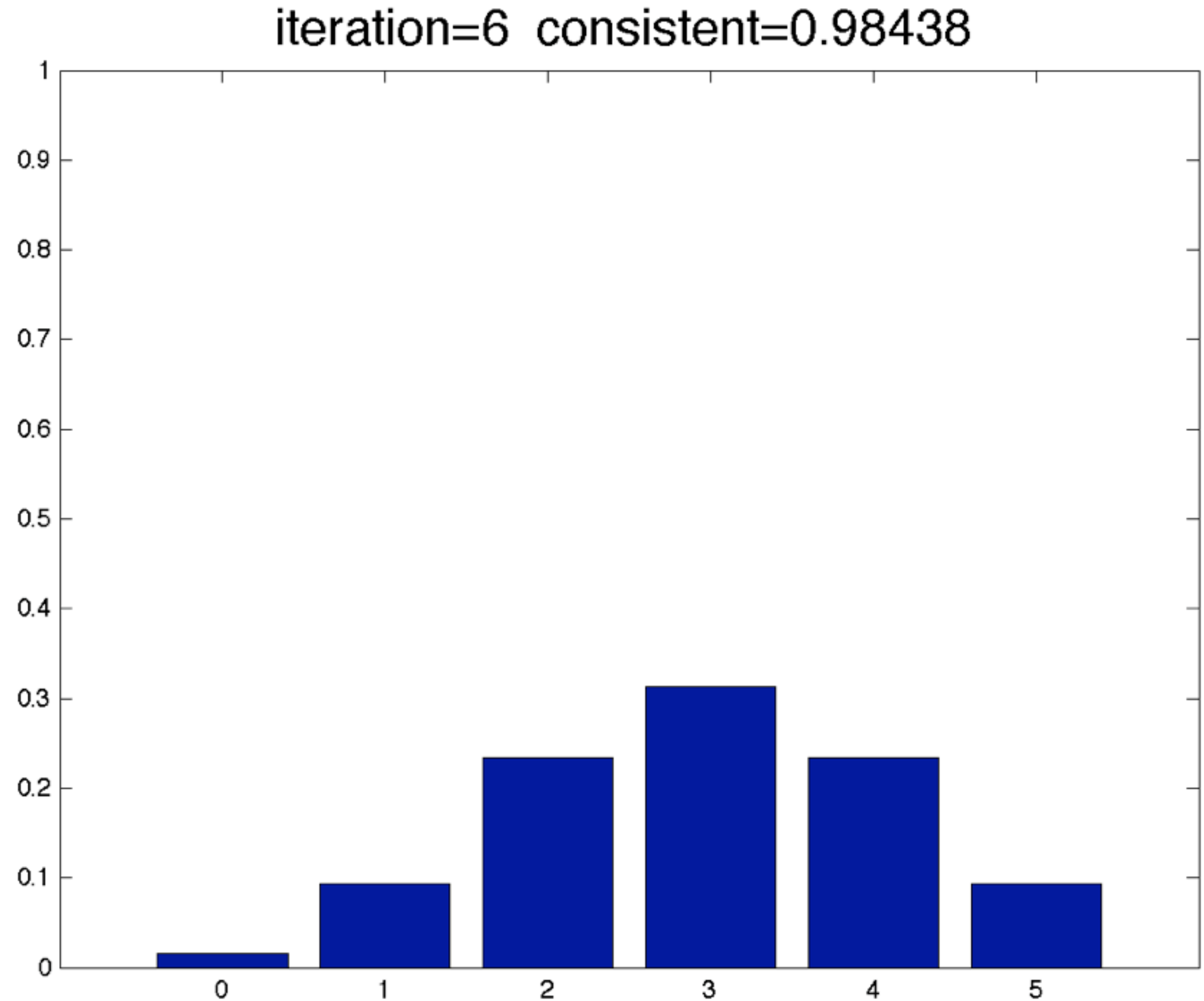
1
2
3
4
5



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0.4
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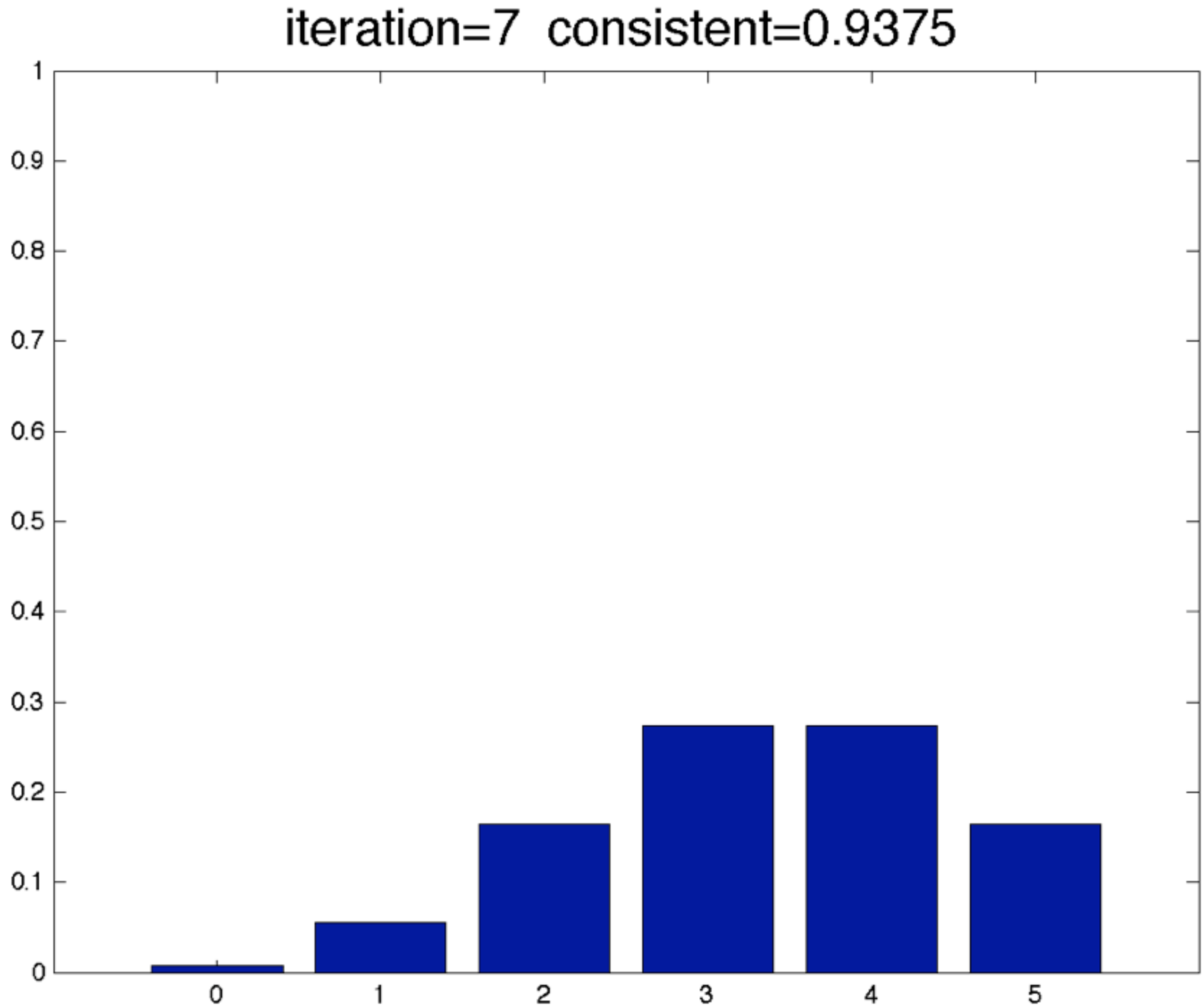
0.2
0.1
0
0
1
2
3
4
5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0.2
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0.8
0.9
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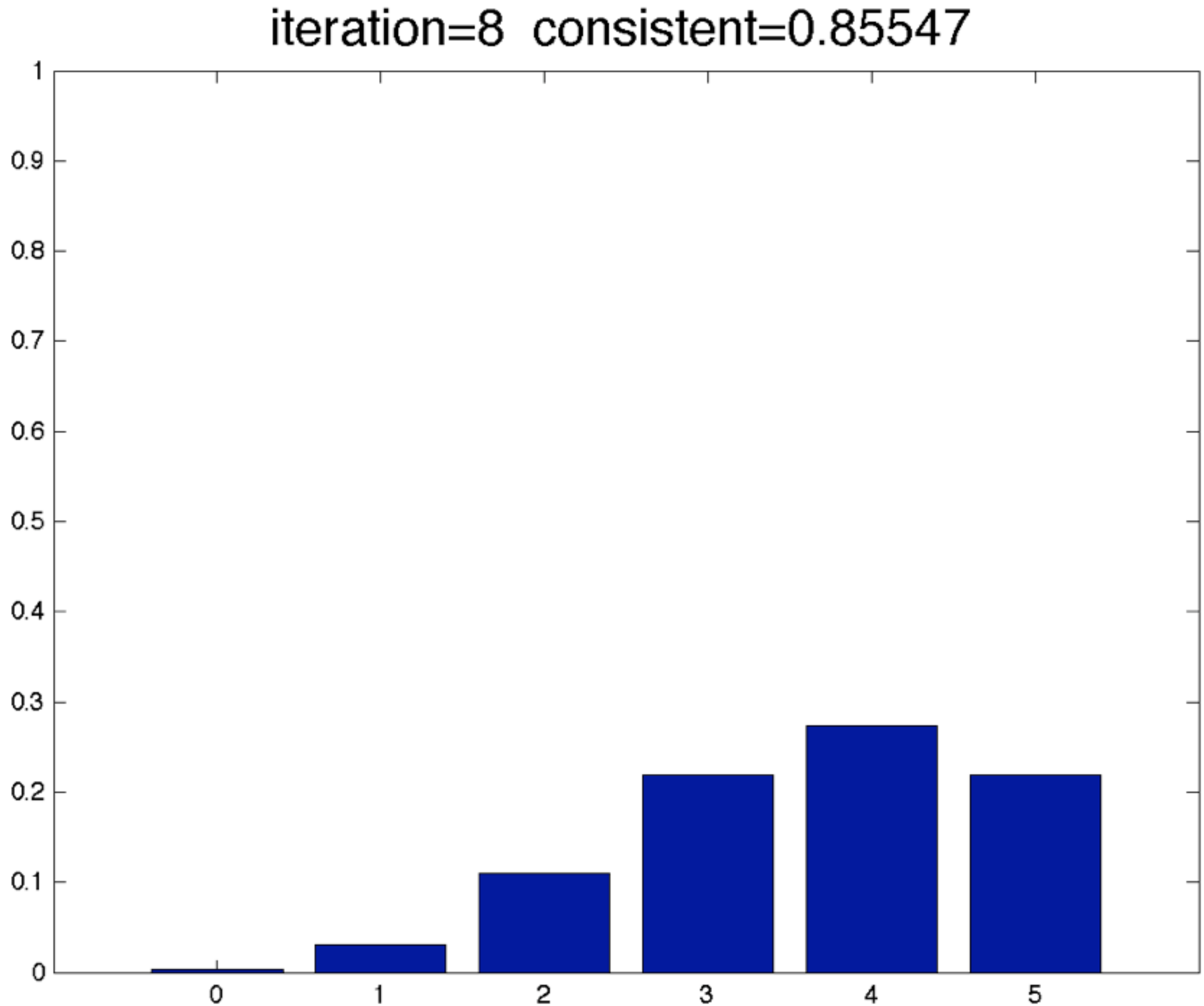
1
2
3
4
5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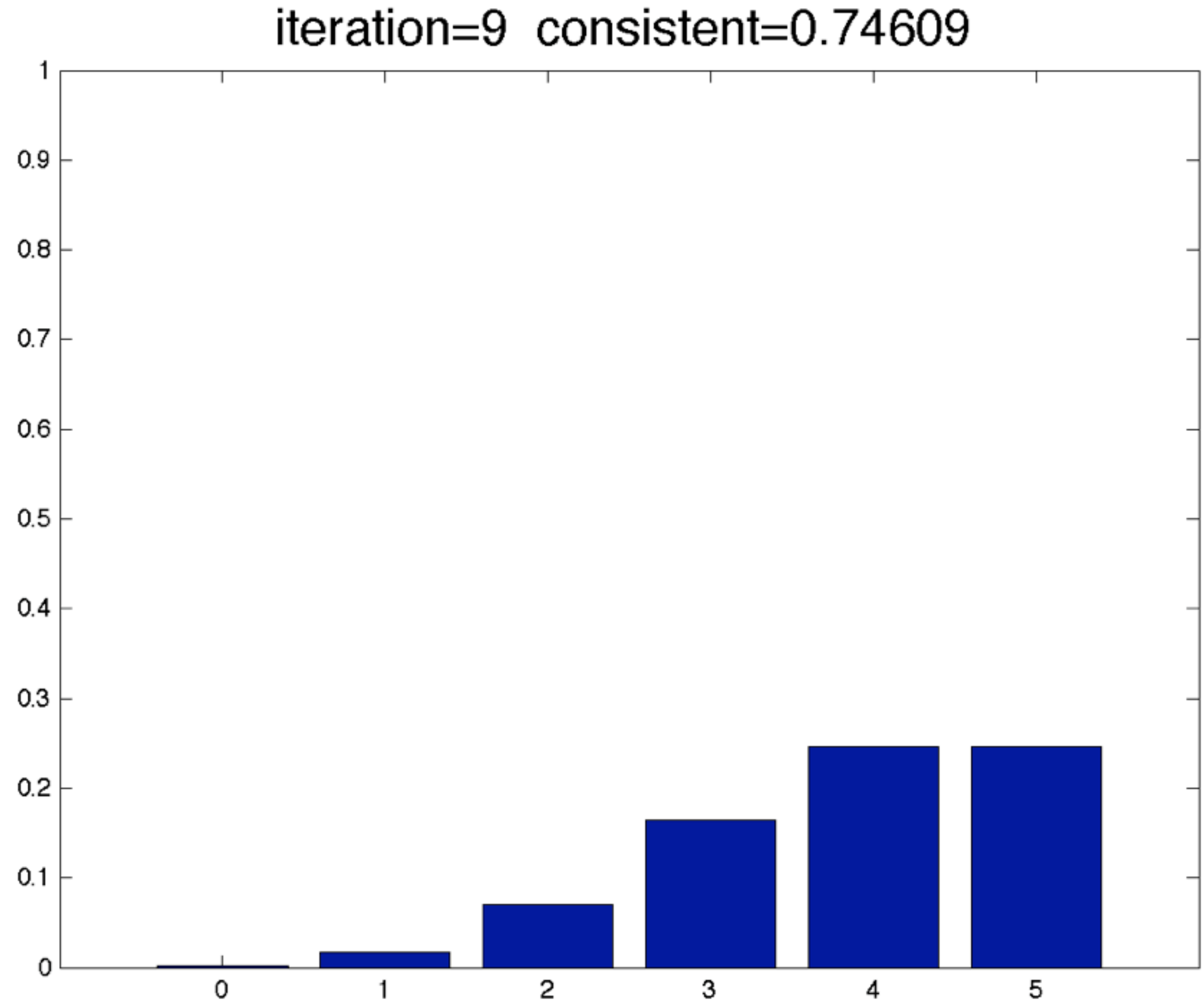
1
2
3
4
5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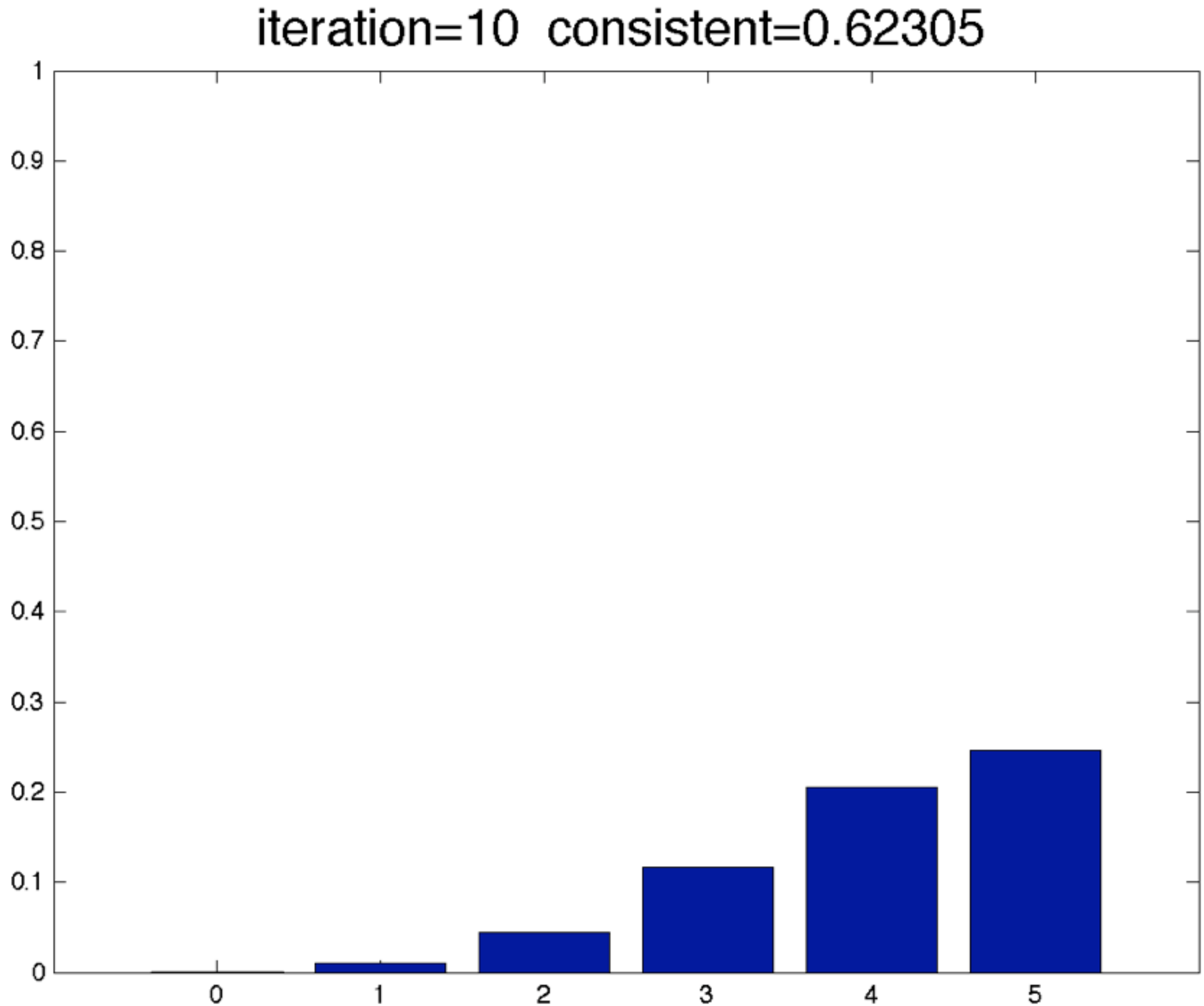
0.1
0
0
1
2
3
4
5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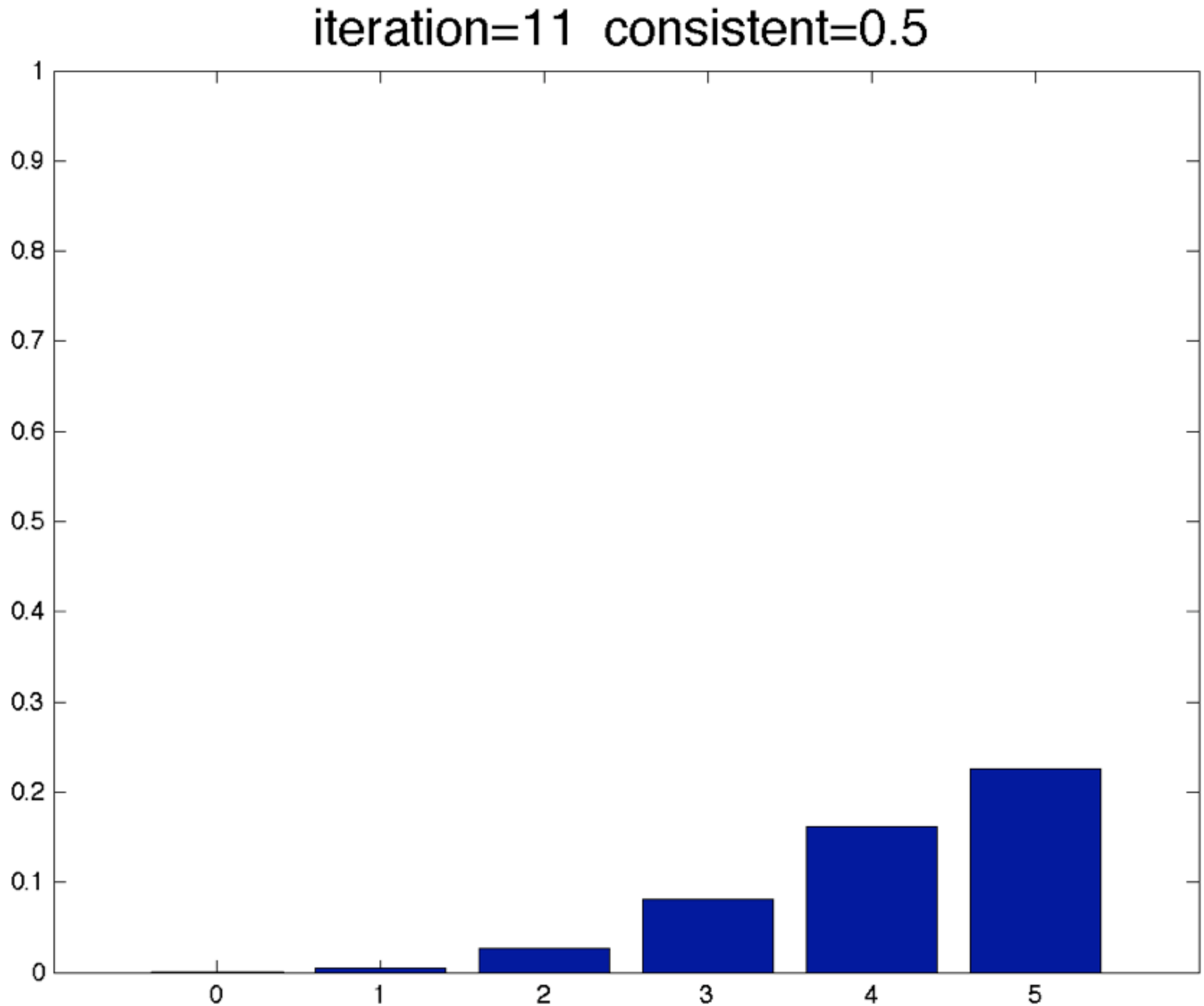
2
3
4
5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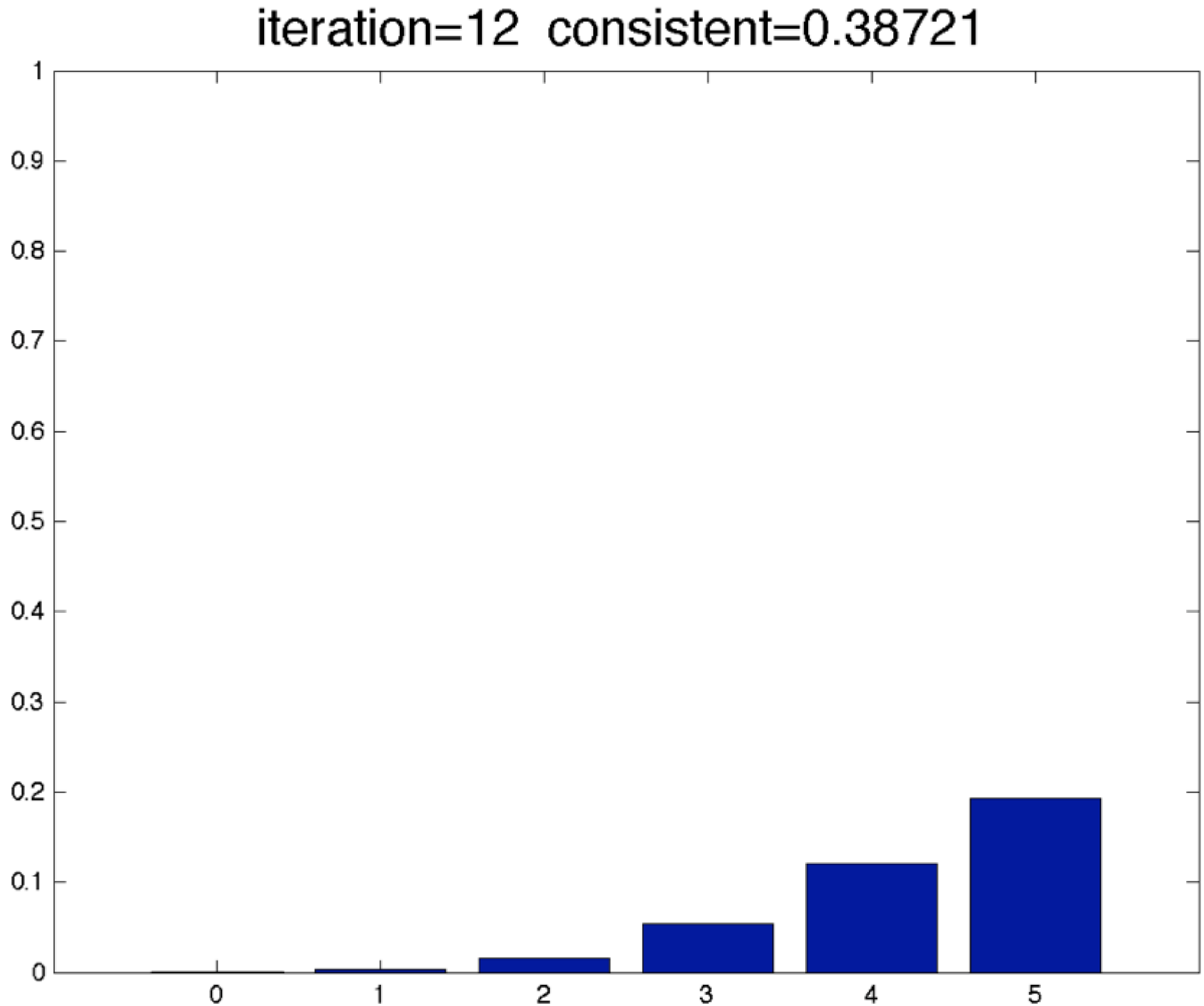
1
2
3
4
5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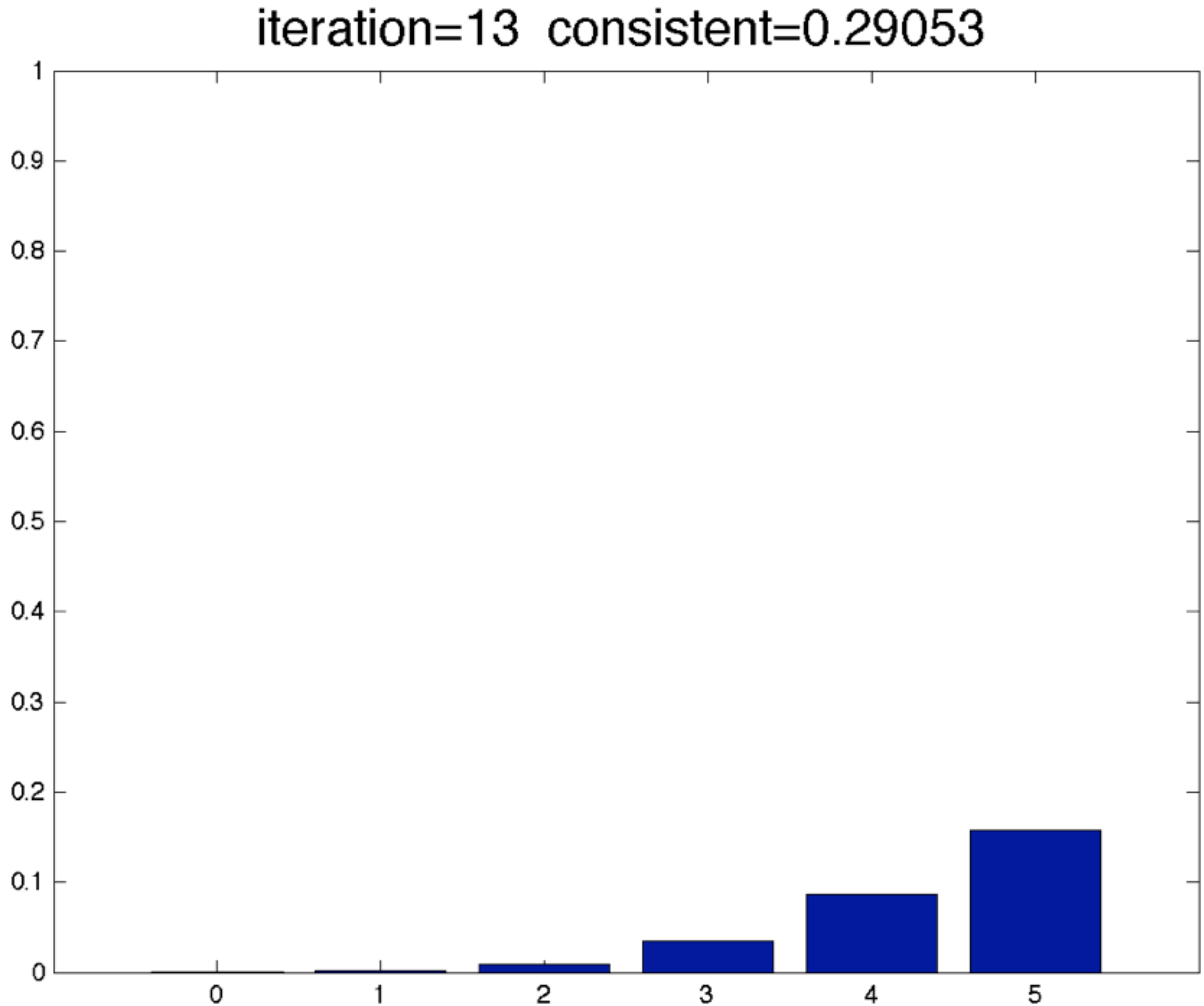
1
2
3
4
5



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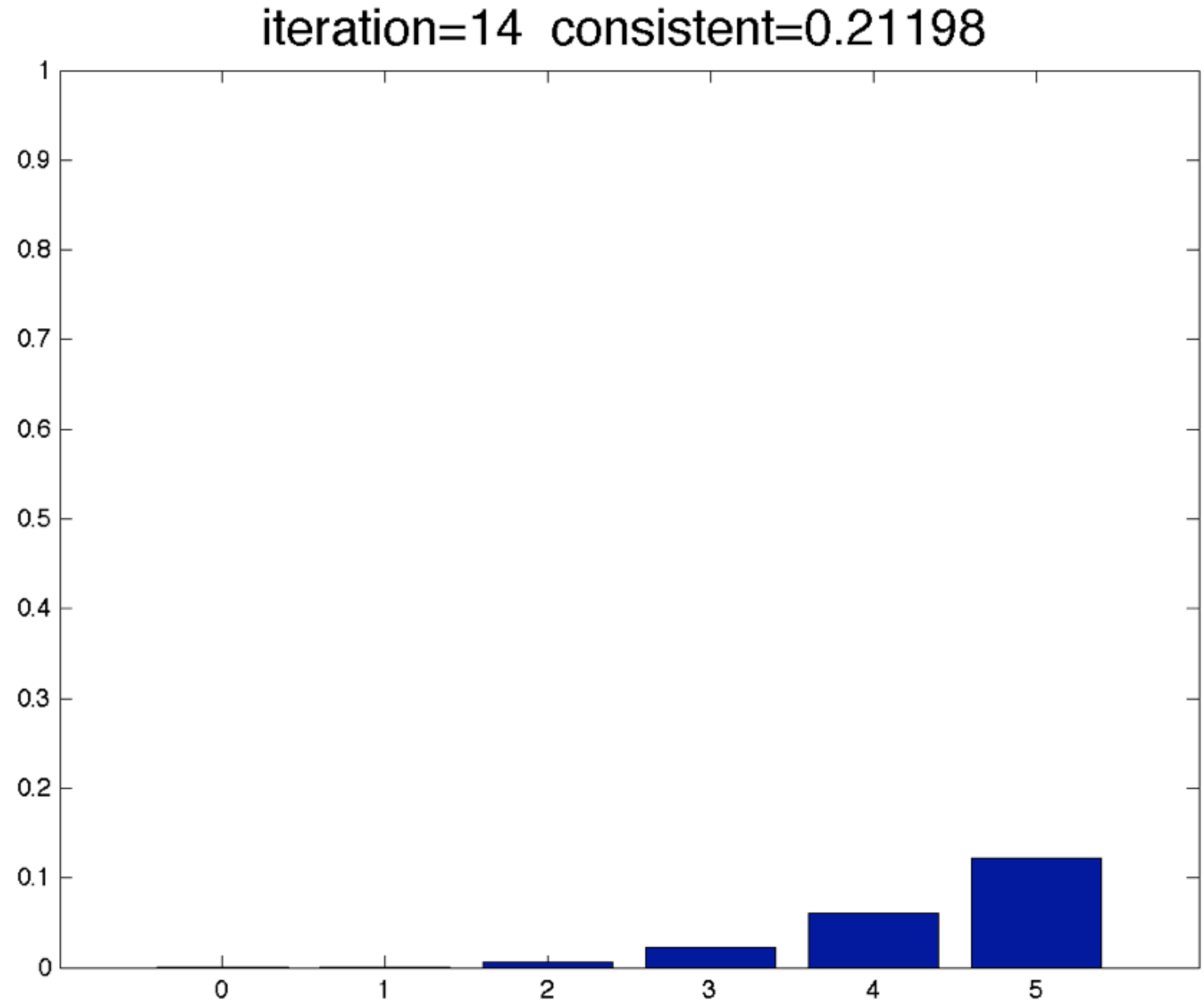
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5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1
0.9
0.8
0.7
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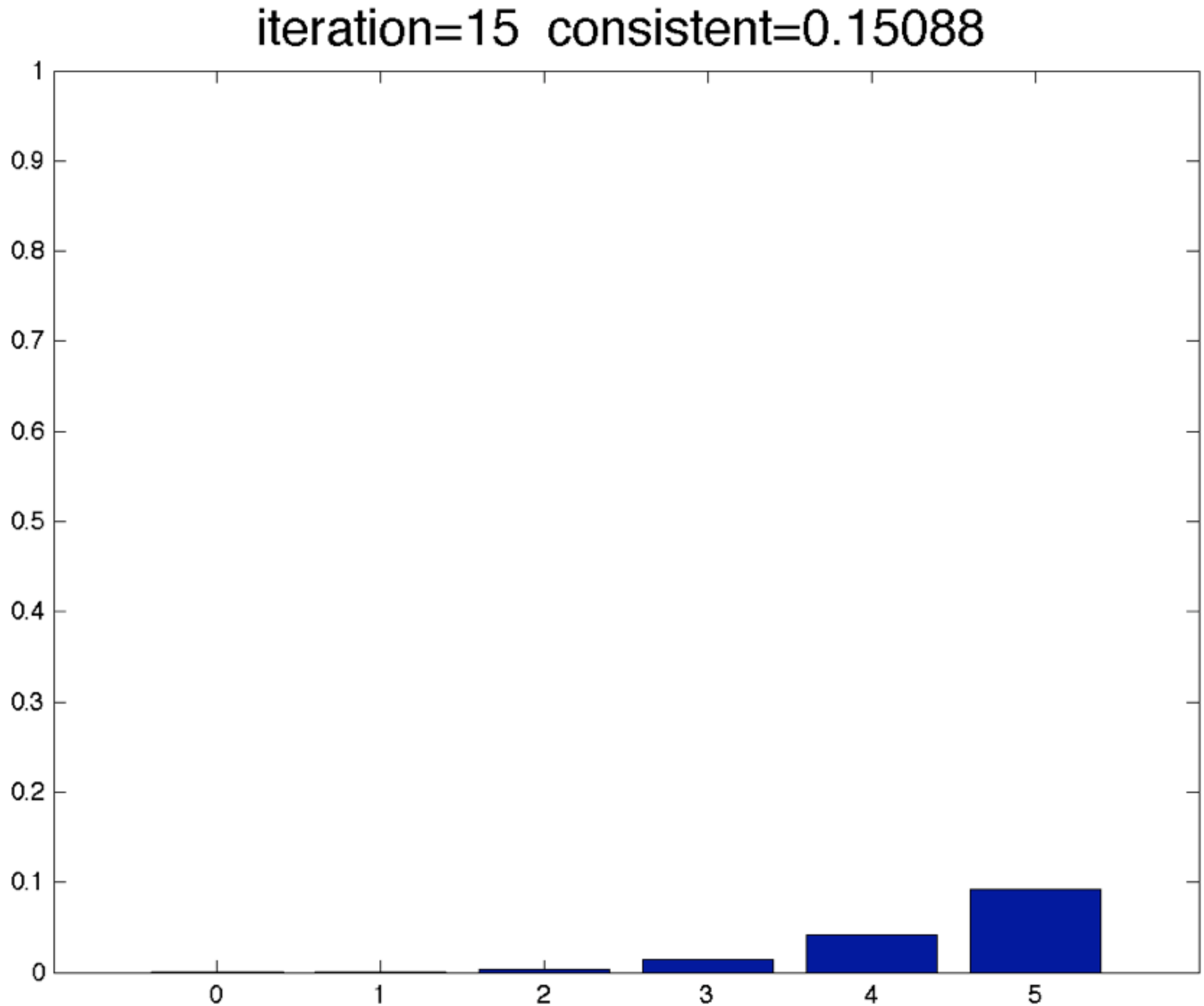
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5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0.5
0.4
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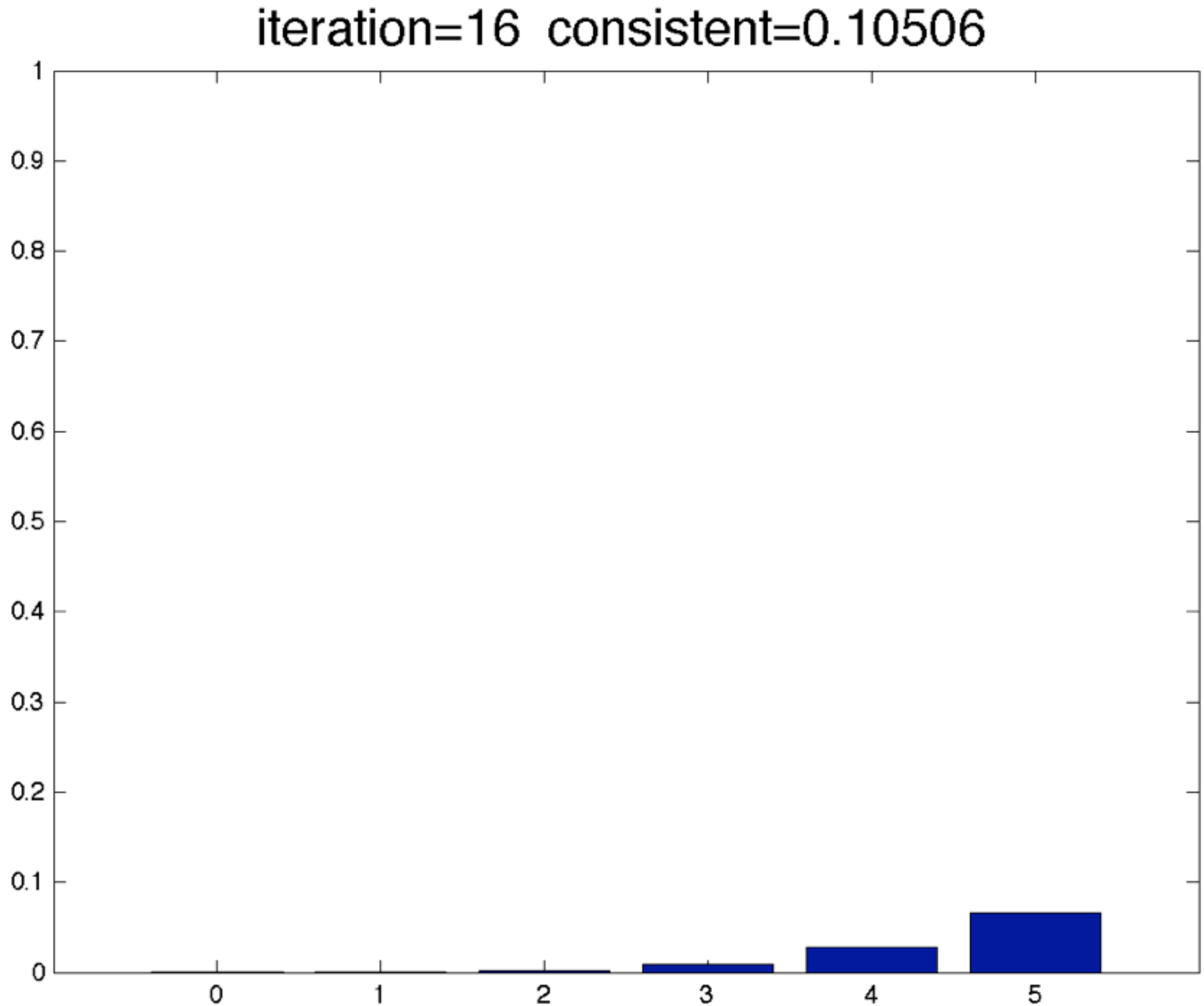
0.2
0.1
0
0
1
2
3
4
5



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0.8
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0.5
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0
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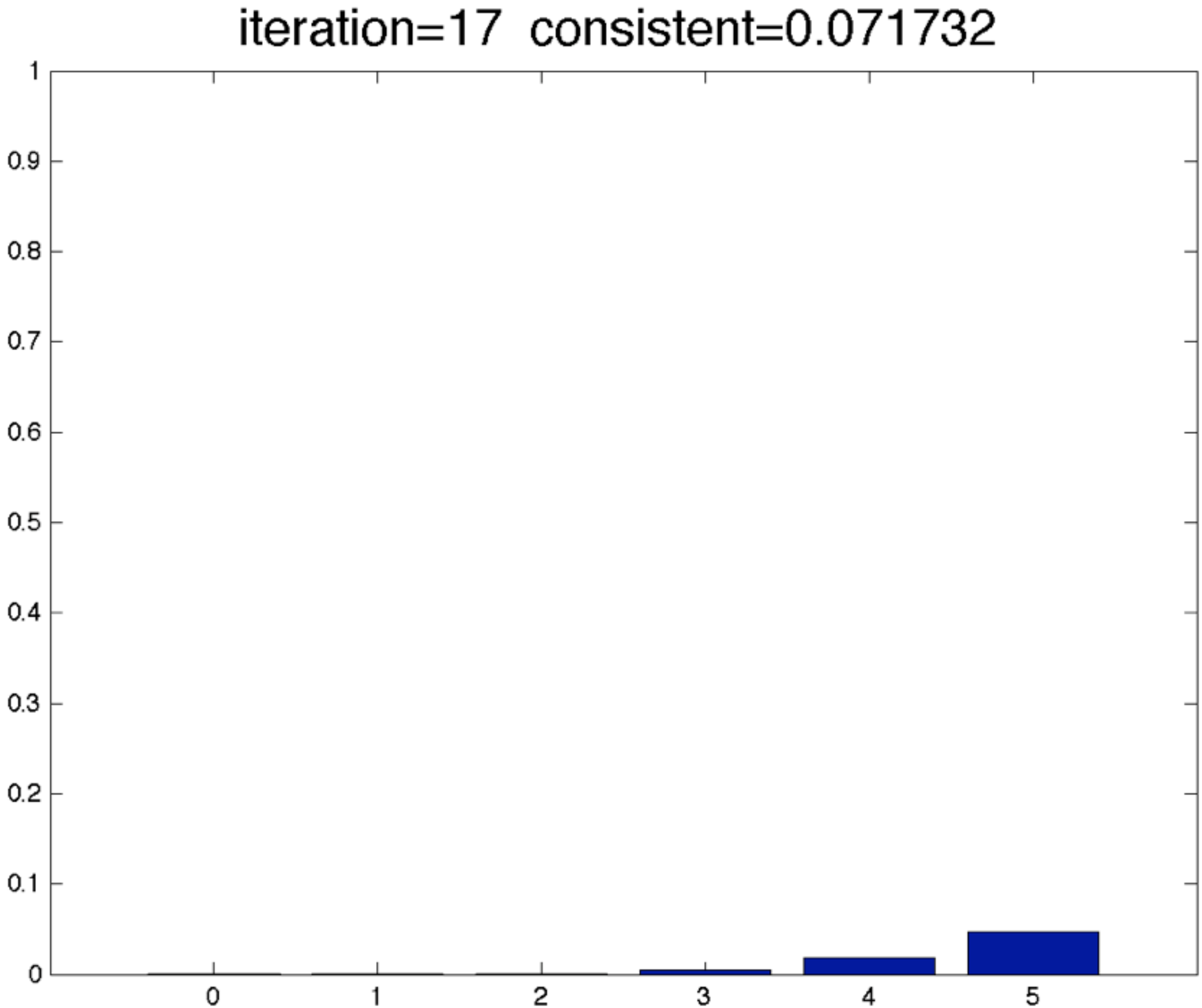
1
2
3
4
5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0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5

The length of the game

- n = size of the set S
- k = maximal number of lies
- m = length of game
  (number of questions until secret is found).

$$m = \max\left\{ q : \frac{1}{n} \le 2^{-q} \binom{q}{\le k} \right\}$$

$$m \le 2k + 2\sqrt{k \ln n} + \log_2 n$$

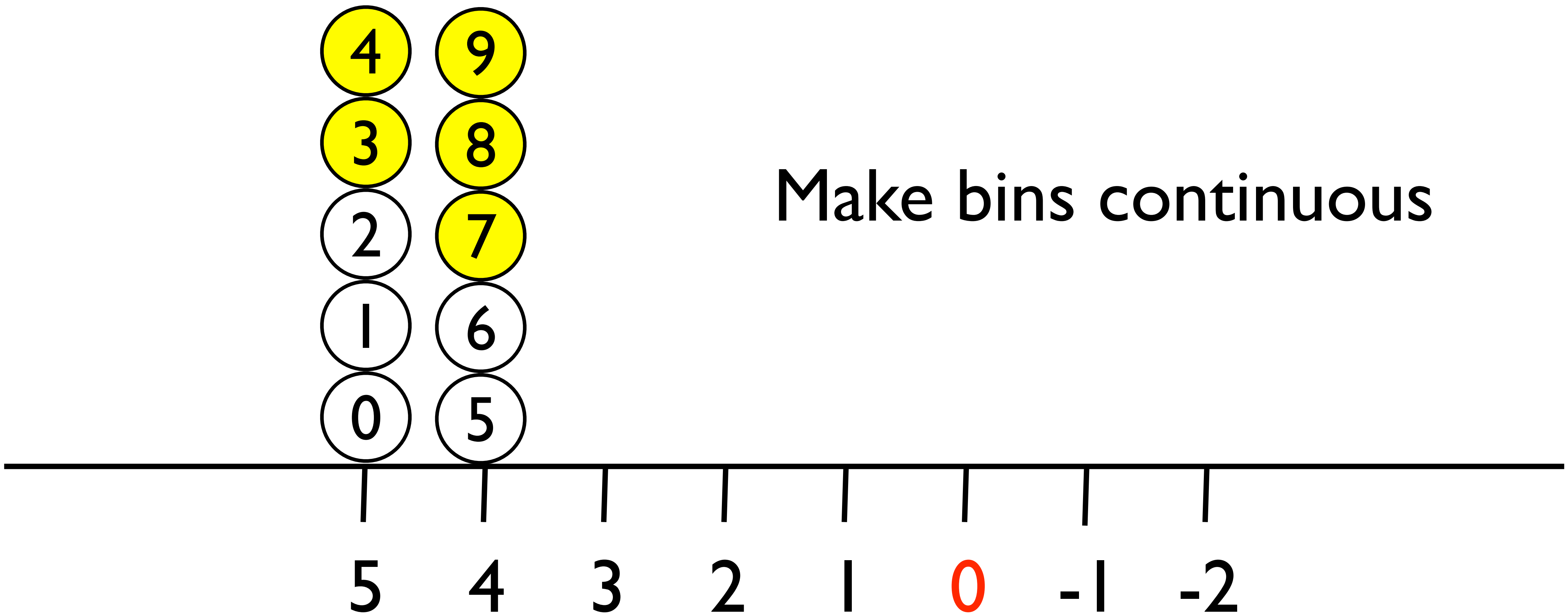
Taking advantage of Alice's mistakes

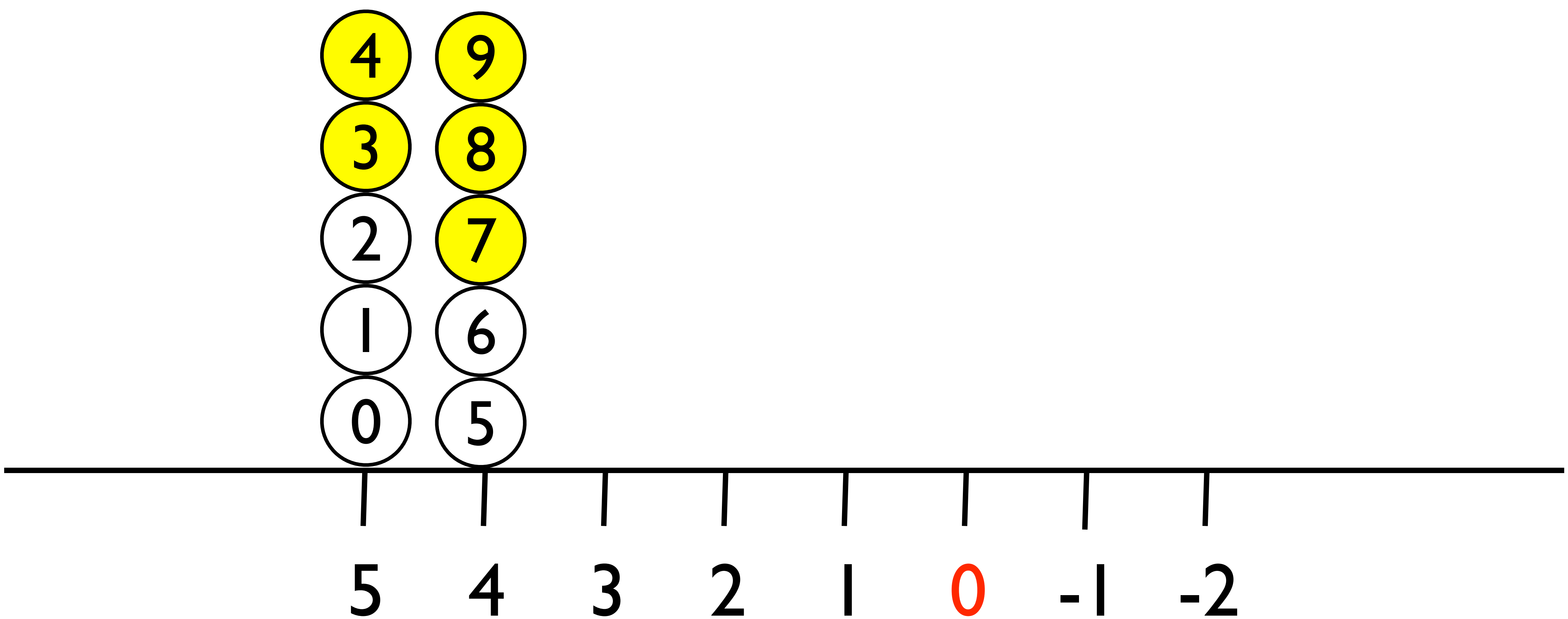
- If Alice splits each bin into two equal parts
  - Bob's choice does not matter
  - does not impact the next configuration
  - can be random with 1/2 1/2 probability.
- If splits are not equal - next configuration <u>depends</u> on Bob's choice - which one should he choose?

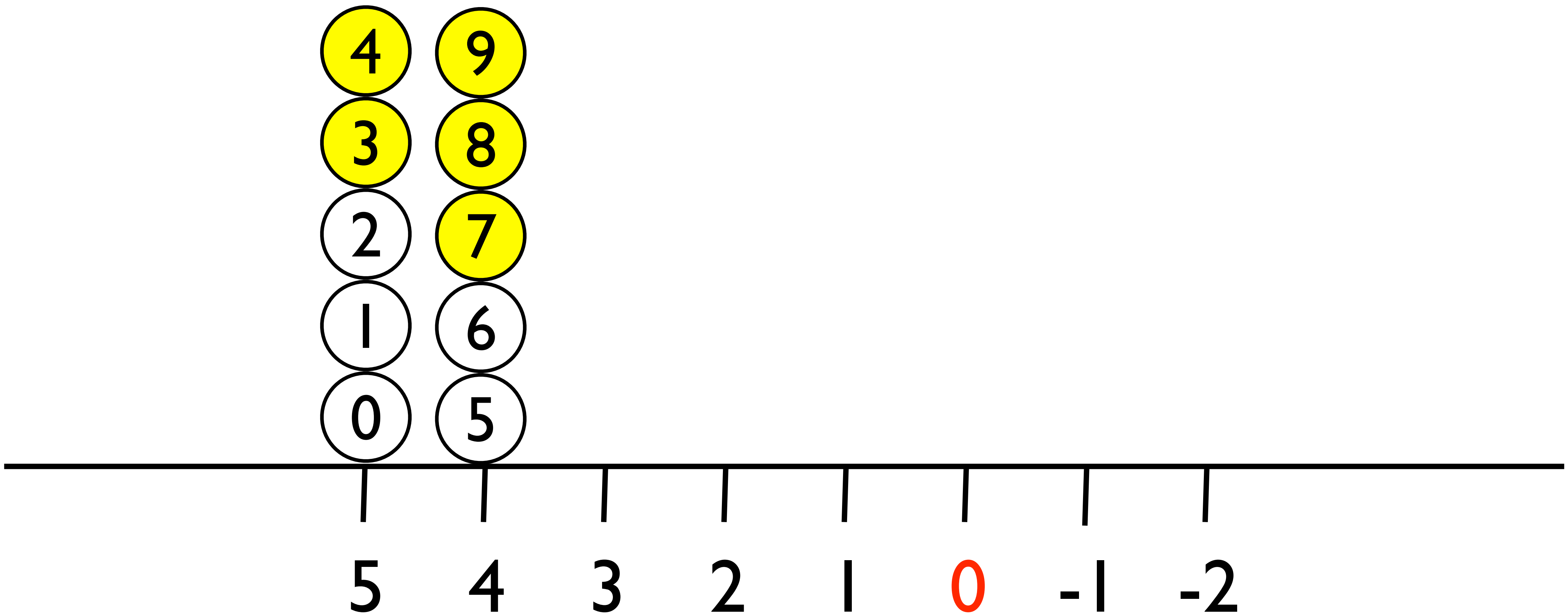
One step look-ahead

- Assume that Alice will play optimally from the next step and onward (worst case assumption).
- Compute the number of steps till game ends.
- Choose configuration that makes game longer.

Make bins continuous



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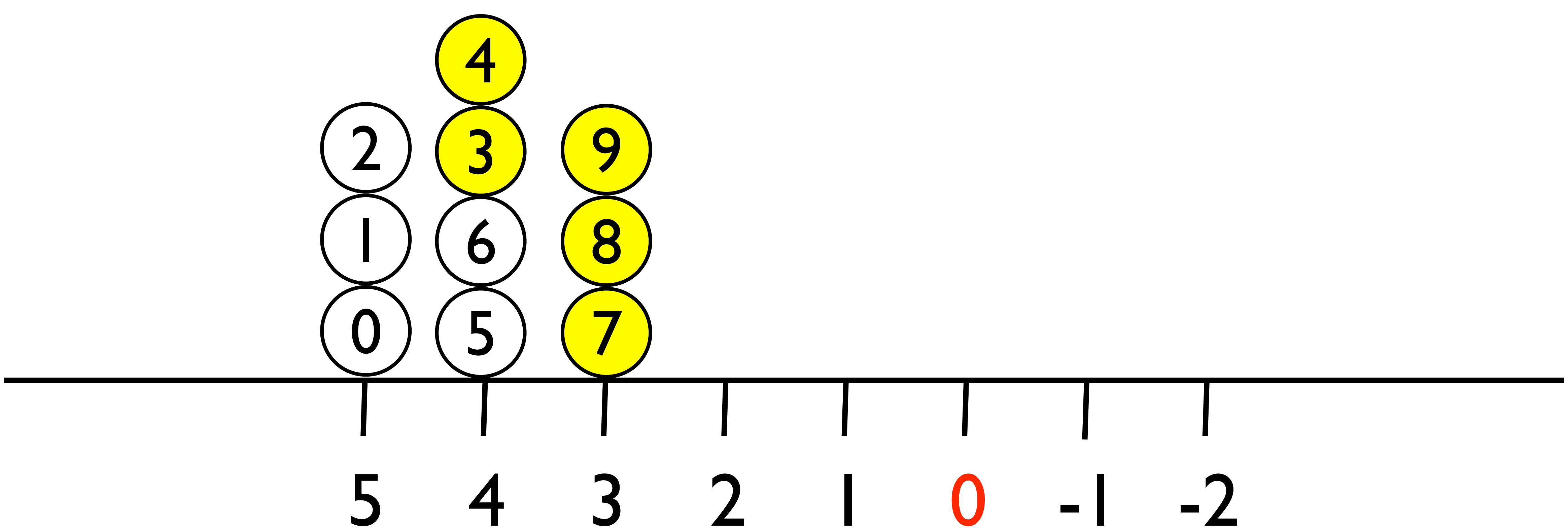
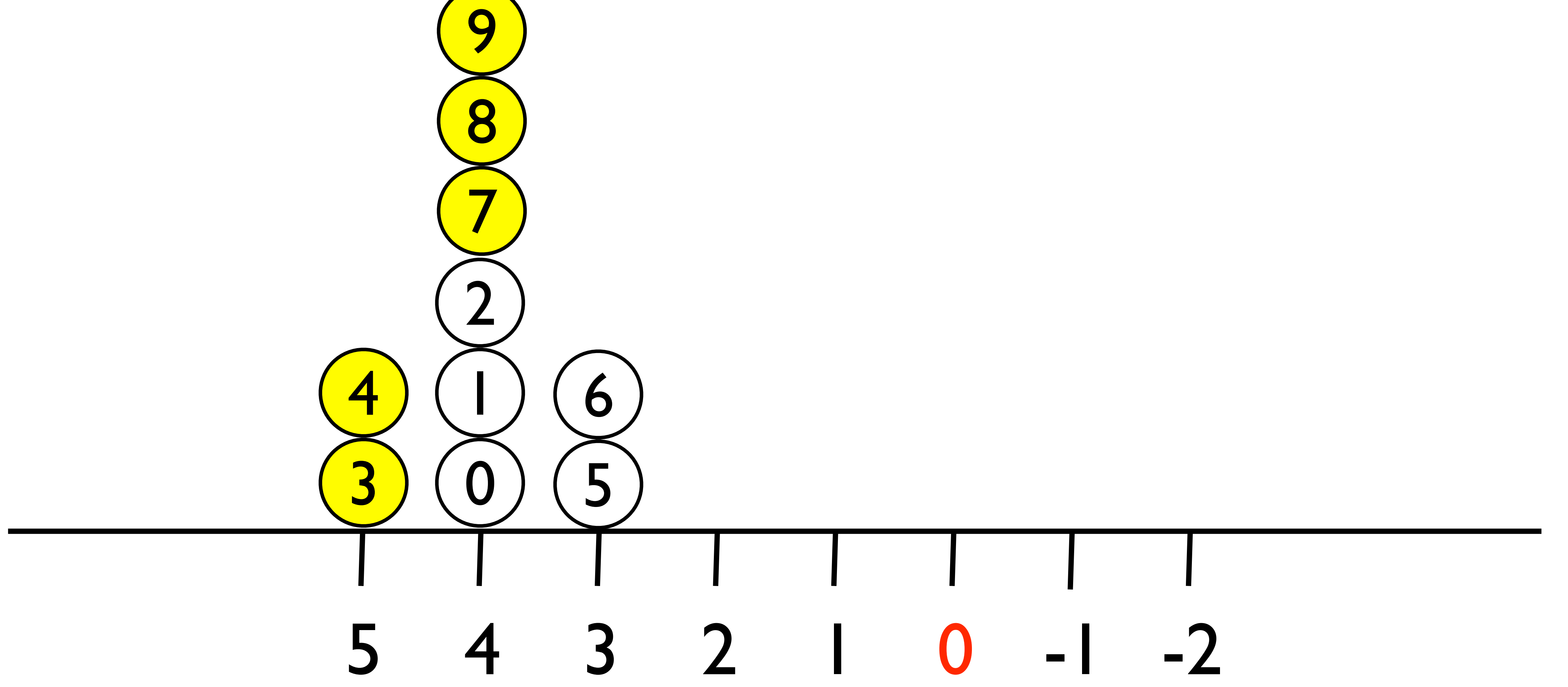
2 7
1 6
0 5
5 4 3 2 1 0 -1 -2


4
2 3 9
1 6 8
0 5 7
5 4 3 2 1 0 -1 -2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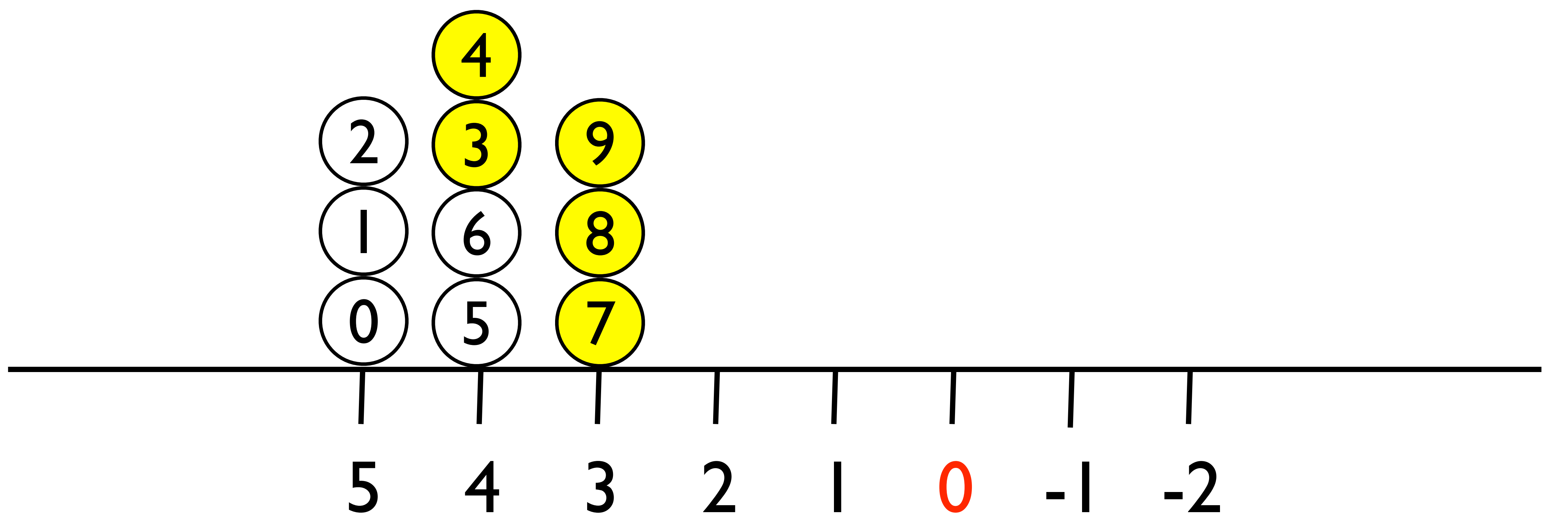
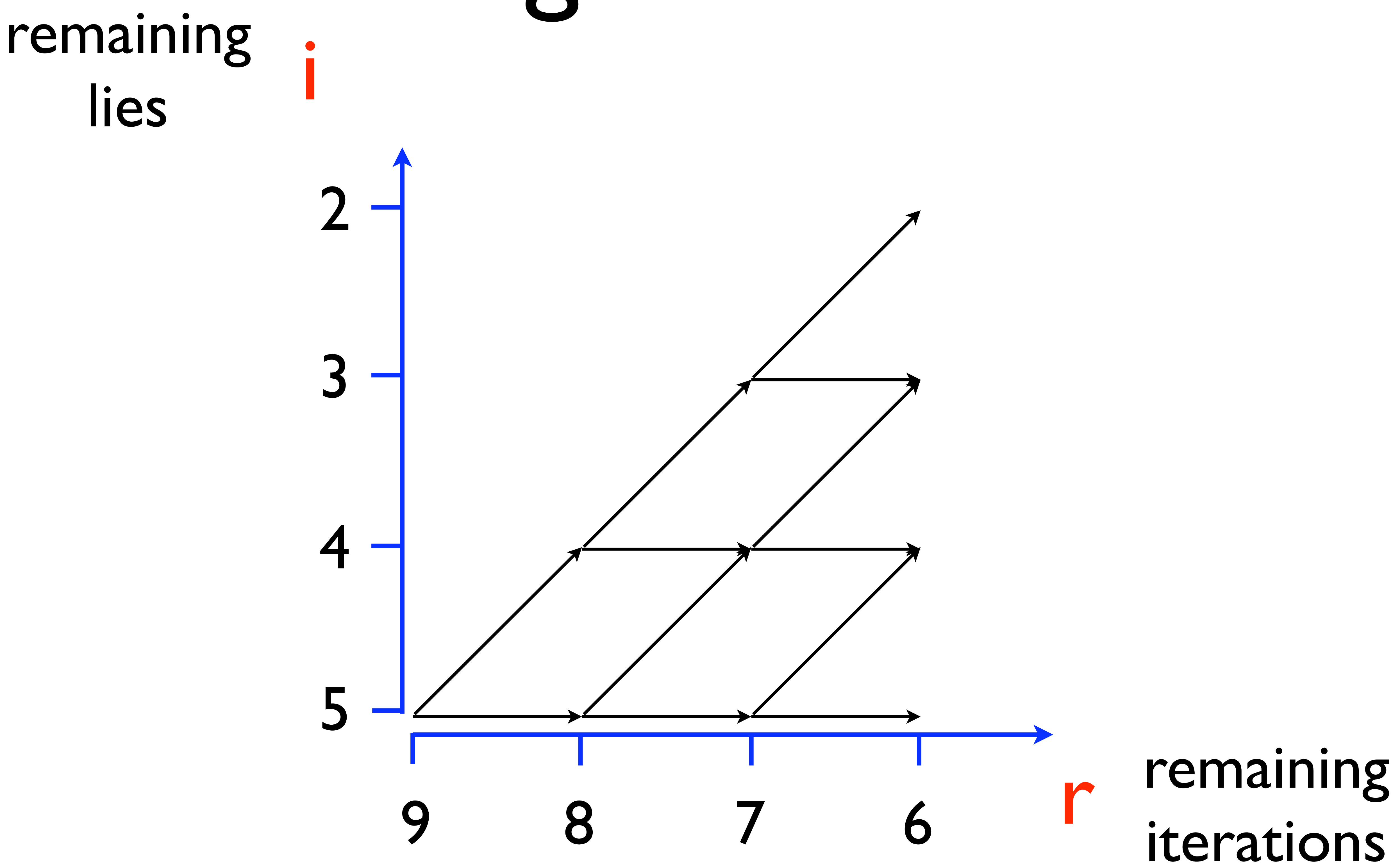
7
2
4
1
6
3
0
5
5 4 3 2 1 0 -1 -2


4
2
3
9
1
6
8
0
5
7
5 4 3 2 1 0 -1 -2

A more refined strategy for bob.

- In some cases, the length of the game will not differ, so bob's choice does not matter.
- Let r be the number of remaining steps before Alice made her suboptimal move.
- Let V_1, V_2 be the volume of the consistent set after $r-1$ optimal steps starting at each of the two configurations.
- Bob chooses the configuration with the larger volume.
- If $V_{max} \geq 1/n$, then number of remaining steps ($r-1$) is decreased by (an additional) 1.

The game lattice



Potential

$\psi(i,r)$ the fraction elements with i remaining lies that will be consistent after r iterations of optimal play.

Equivalently: probability of getting at most i heads in r flips of a fair coin

$$\psi(i,0) = \begin{cases} 1 & i \geq 0 \\ 0 & i < 0 \end{cases}$$

$$\psi(i,r) = 2^{-r} \binom{r}{\leq i}$$

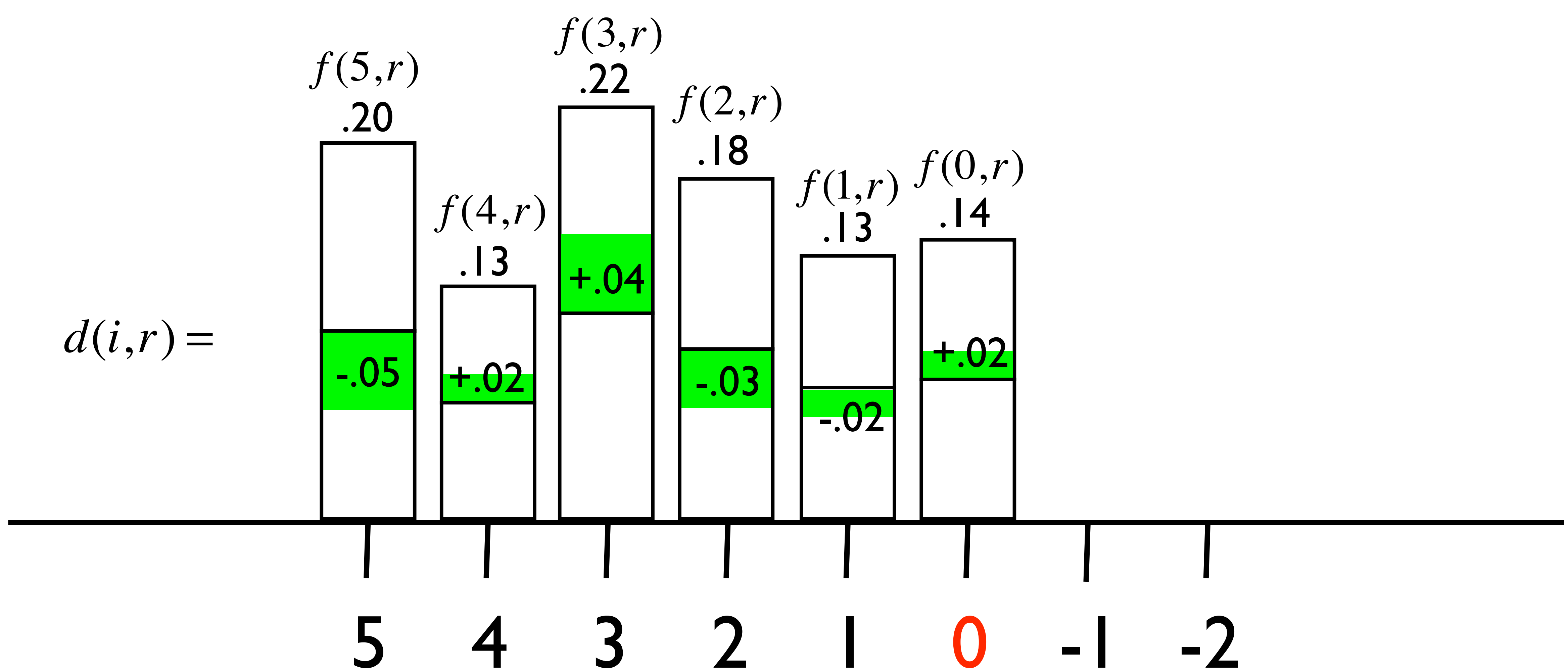
$$\psi(i,r+1) = \frac{\psi(i,r) + \psi(i+1,r)}{2}$$

notation

$f(i,r)$ The fraction of answers in bin i when r iterations remain

$d(i,r)$ The difference between the two parts chosen by Alice in bin i when r steps remain

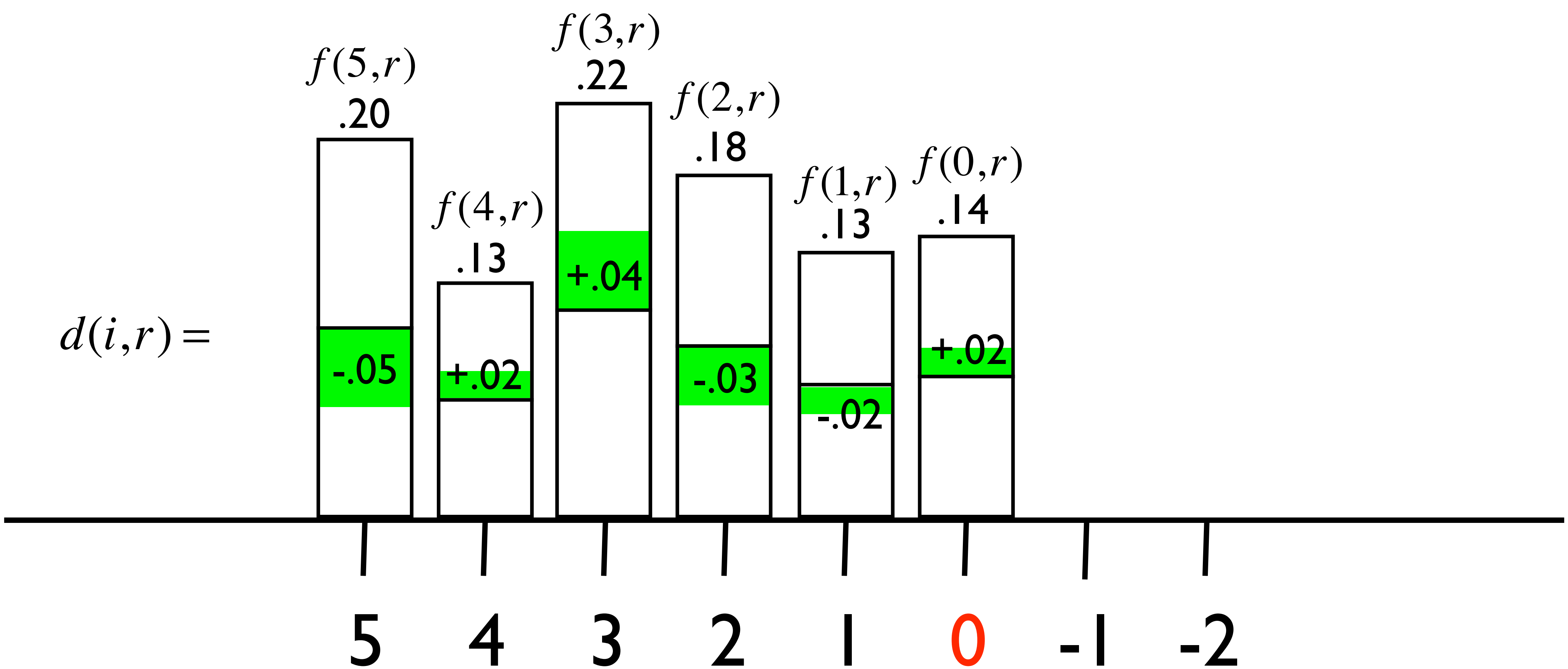
$b(r)$ Bob's choice (+1 or -1)



$$f(i,r-1)=\frac{f(i-1,r)+f(i,r)}{2}+b(r)\big(d(i-1,r)-d(i,r)\big)$$

Total potential $V=\Psi(r)=\sum_i f(i,r)\psi(i,r)$

$$\Psi(r-1)=\sum_i\left[\frac{f(i-1,r)+f(i,r)}{2}+b(r)\big(d(i-1,r)-d(i,r)\big)\right]\psi(i,r-1)$$



$$f(i,r-1)=\frac{f(i-1,r)+f(i,r)}{2}+b(r)\big(d(i-1,r)-d(i,r)\big)$$

Total potential $V=\Psi(r)=\sum_i f(i,r)\psi(i,r)$

$$\Psi(r-1)=\sum_i\left[\frac{f(i-1,r)+f(i,r)}{2}+b(r)\big(d(i-1,r)-d(i,r)\big)\right]\psi(i,r-1)$$

$$\Psi(r-1)=\sum_i\left[f(i,r)\frac{\psi(i,r-1)+\psi(i+1,r-1)}{2}+b(r)d(i,r)\big(\psi(i,r-1)-\psi(i+1,r-1)\big)\right]$$

$$w(i,r)\doteq\psi(i,r-1)-\psi(i+1,r-1)$$

$$\Psi(r-1)=\sum_i\big[f(i,r)\psi(i,r)+b(r)d(i,r)w(i,r)\big]$$

$$f(i,r-1)=\frac{f(i-1,r)+f(i,r)}{2}+b(r)\big(d(i-1,r)-d(i,r)\big)$$

Total potential $V=\Psi(r)=\sum_i f(i,r)\psi(i,r)$

$$\Psi(r-1)=\sum_i\left[\frac{f(i-1,r)+f(i,r)}{2}+b(r)\big(d(i-1,r)-d(i,r)\big)\right]\psi(i,r-1)$$

$$\Psi(r-1)=\sum_i\left[f(i,r)\frac{\psi(i,r-1)+\psi(i+1,r-1)}{2}+b(r)d(i,r)\big(\psi(i,r-1)-\psi(i+1,r-1)\big)\right]$$

$$w(i,r)\doteq\psi(i,r-1)-\psi(i+1,r-1)$$

$$\Psi(r-1)=\sum_i\left[f(i,r)\psi(i,r)+b(r)d(i,r)w(i,r)\right]$$

$$\Psi(r-1)=\Psi(r)+b(r)\sum_i d(i,r)w(i,r)$$

$$\Psi(r-1) = \Psi(r) + b(r)\sum_i d(i,r)w(i,r)$$

$b(r)$ Bob's choice (+1 or -1)

$d(i.r)$ Alice's choice

$$w(i,r) \doteq \psi(i,r-1) - \psi(i+1,r-1) = 2^{-(r-1)}\binom{r-1}{i}$$

Bob wants to lengthen the game
= increase the potential

$$b(r) = sign\left(\sum_i d(i,r)w(i,r)\right)$$

Application to online learning with expert advice

[Cesa-Bianchi, Freund, Helmbold, Warmuth 1996]

- n experts
- at each iteration:
  - Each expert makes a +1/-1 prediction
  - Master makes +1/-1 prediction
  - Nature generates +1/-1 output
- One of the experts makes $\leq k$ mistakes
- Goal: minimize number of mistakes of the master.

The halving algorithm

- k=0 - one of the experts is perfect
- Master's strategy: predict according to the majority of consistent experts
- If Master makes a mistake - consistent experts halved.
- Master makes at most $\log_2 n$ mistakes
- optimal when n is a power of two or continuous
- Equivalent to 20 questions with no lies.

Expert making ≤k mistakes

Expert = Chip

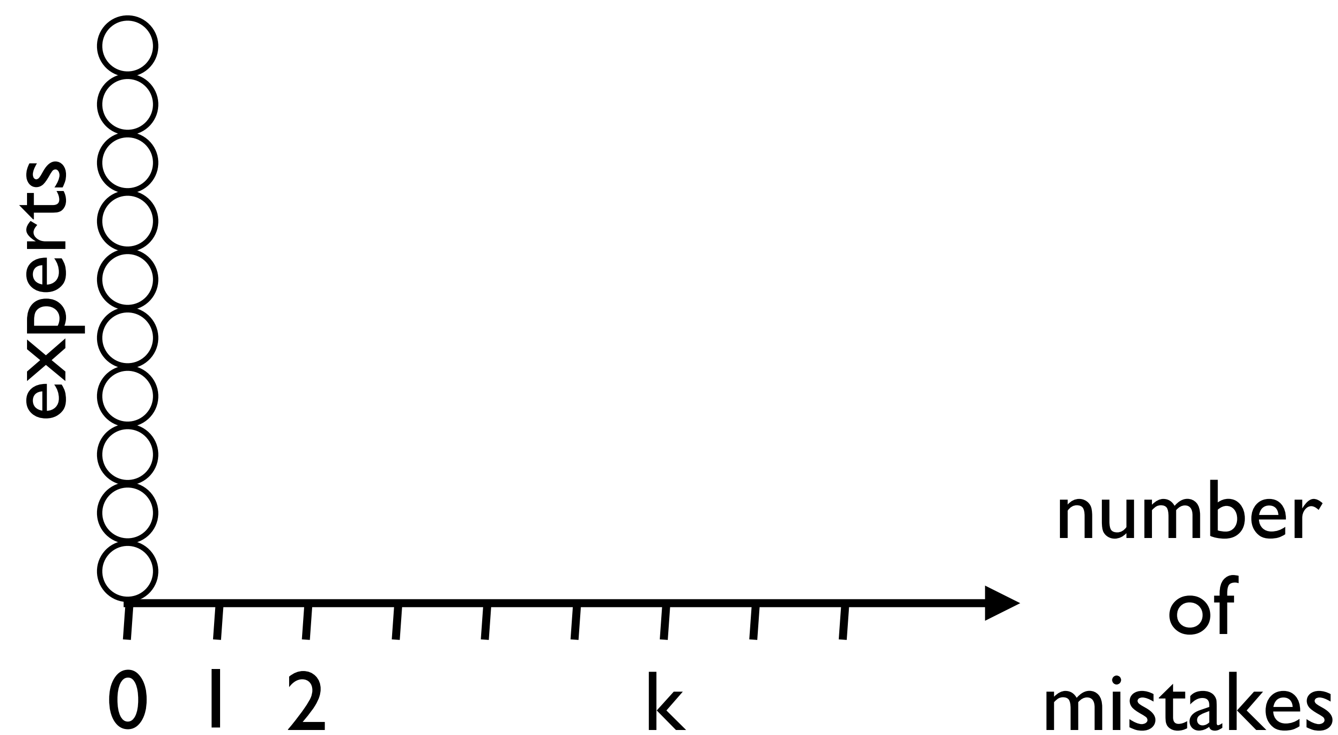
Bin = number of mistakes made by expert

Alice chooses predictions for all experts

Bob chooses master's predictions

wlog master makes a mistake on every round

Bob's goal: make the game as short as possible



Expert making ≤k mistakes

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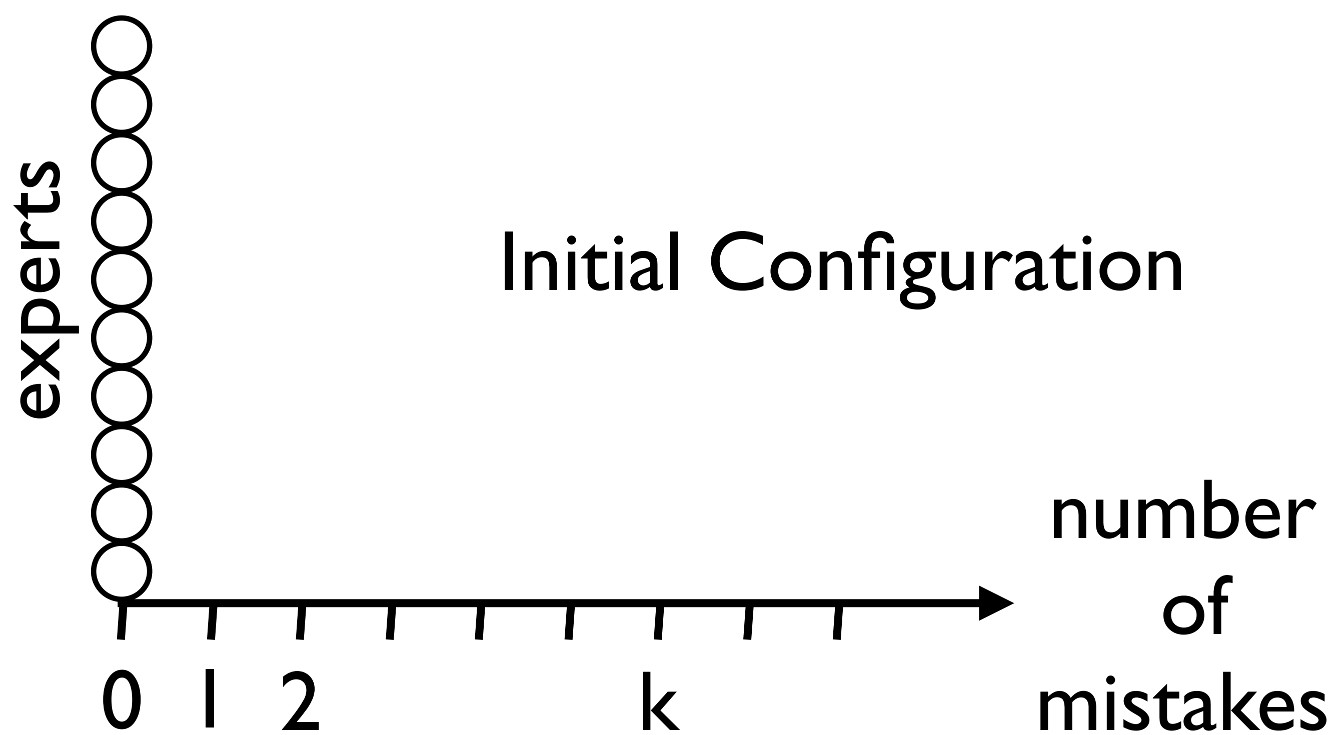
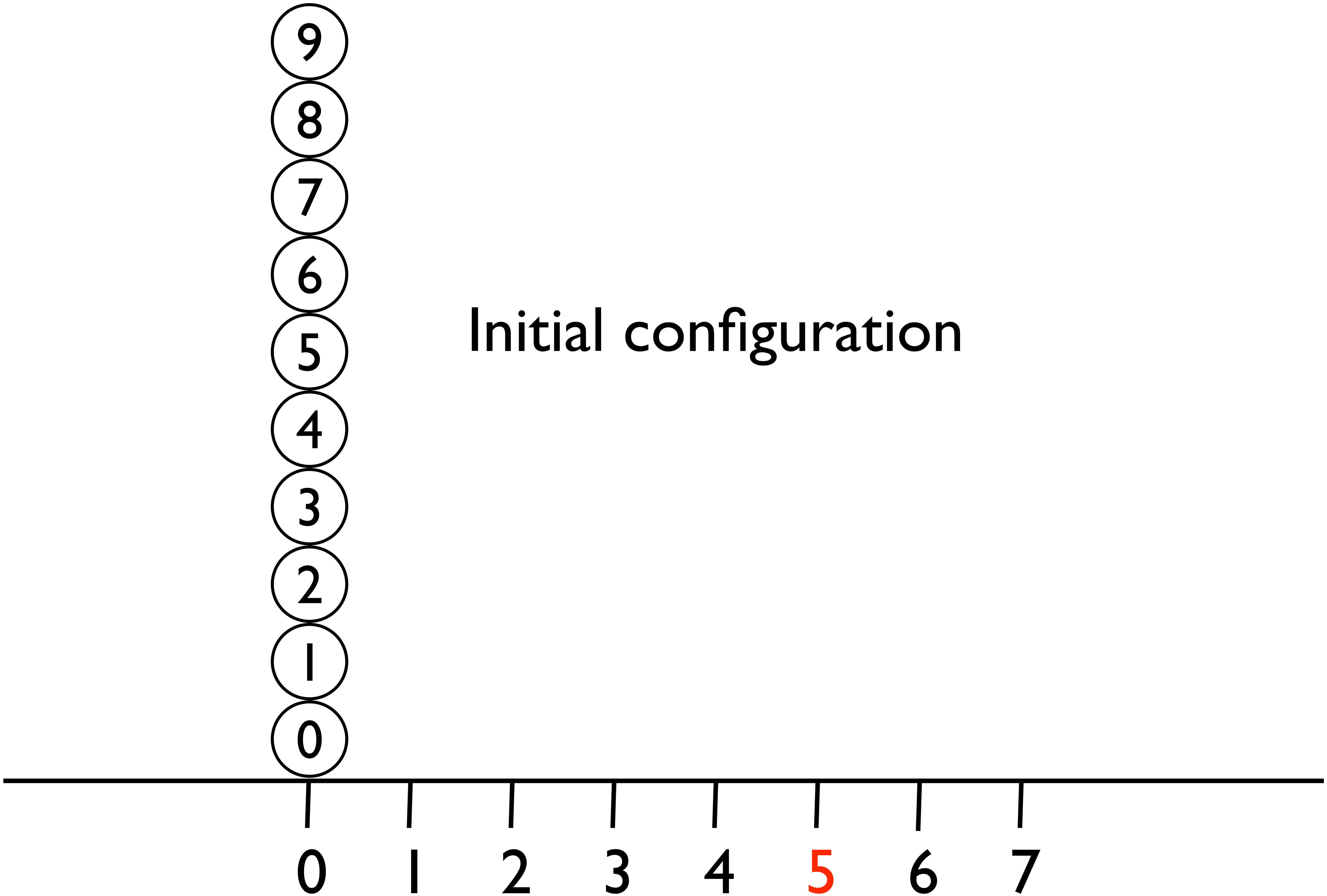
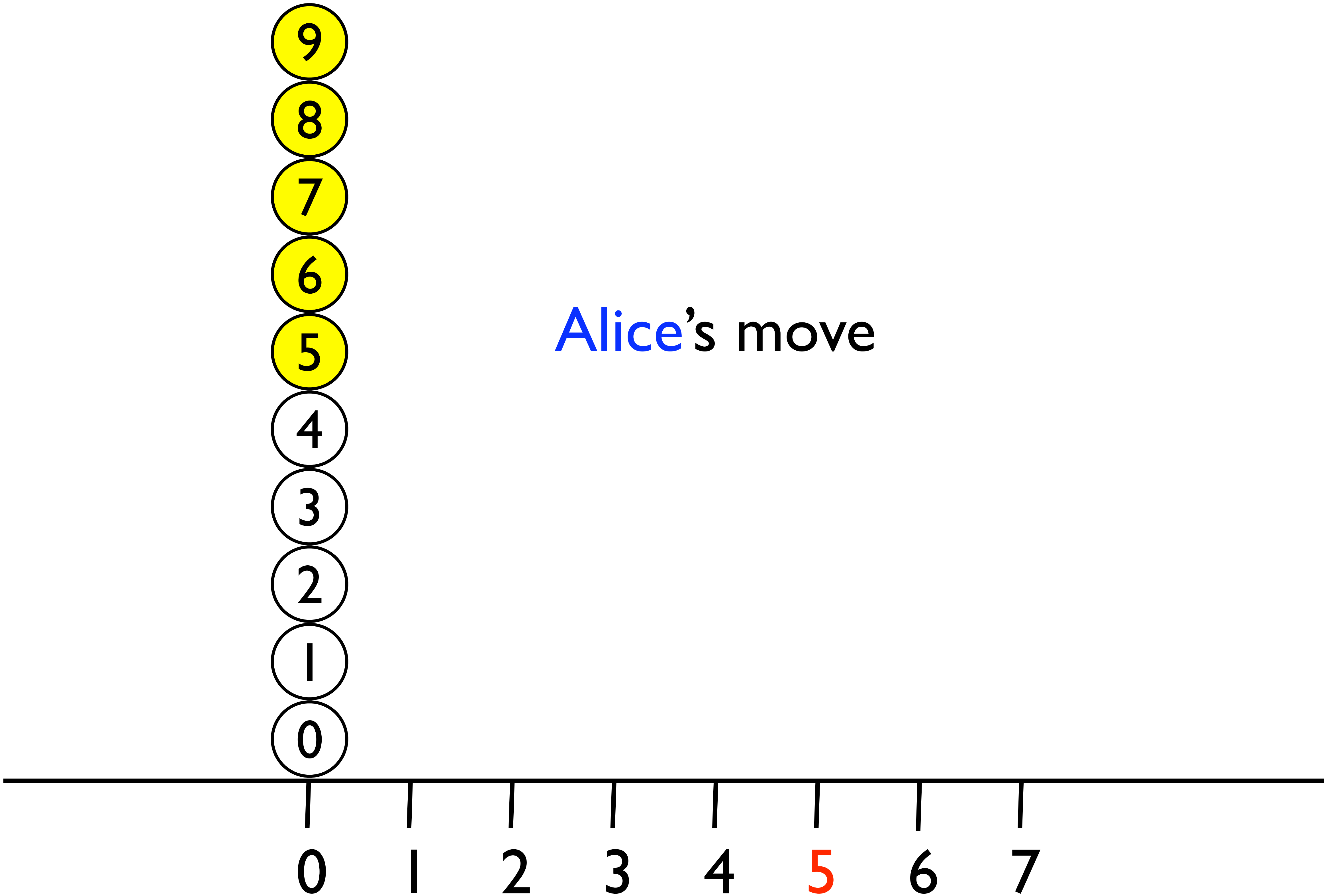


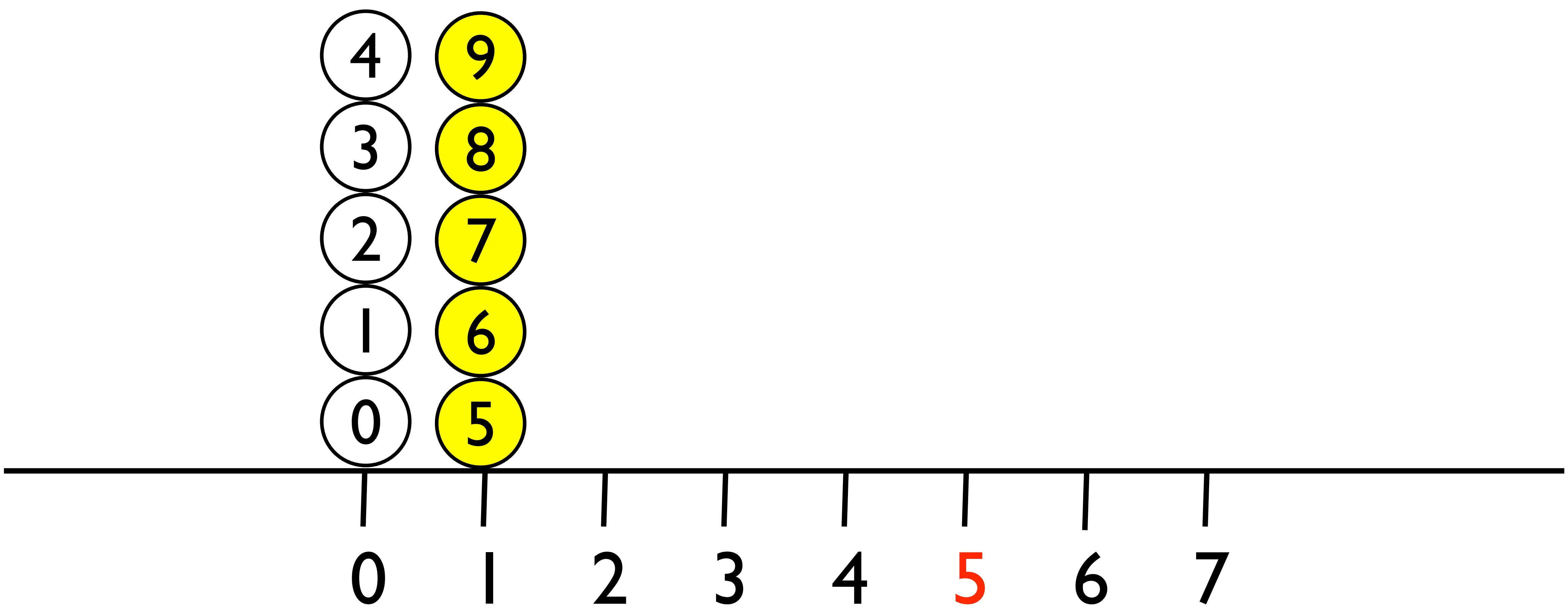
Initial configuration



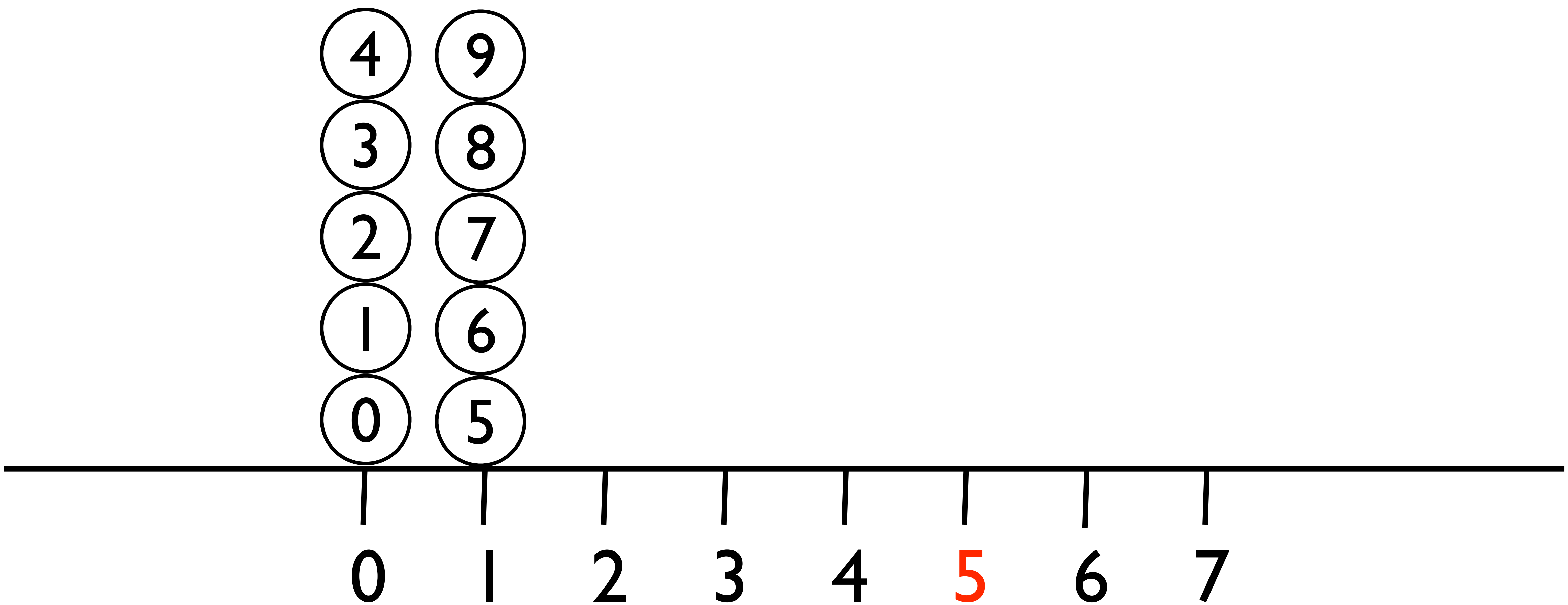


Alice's move

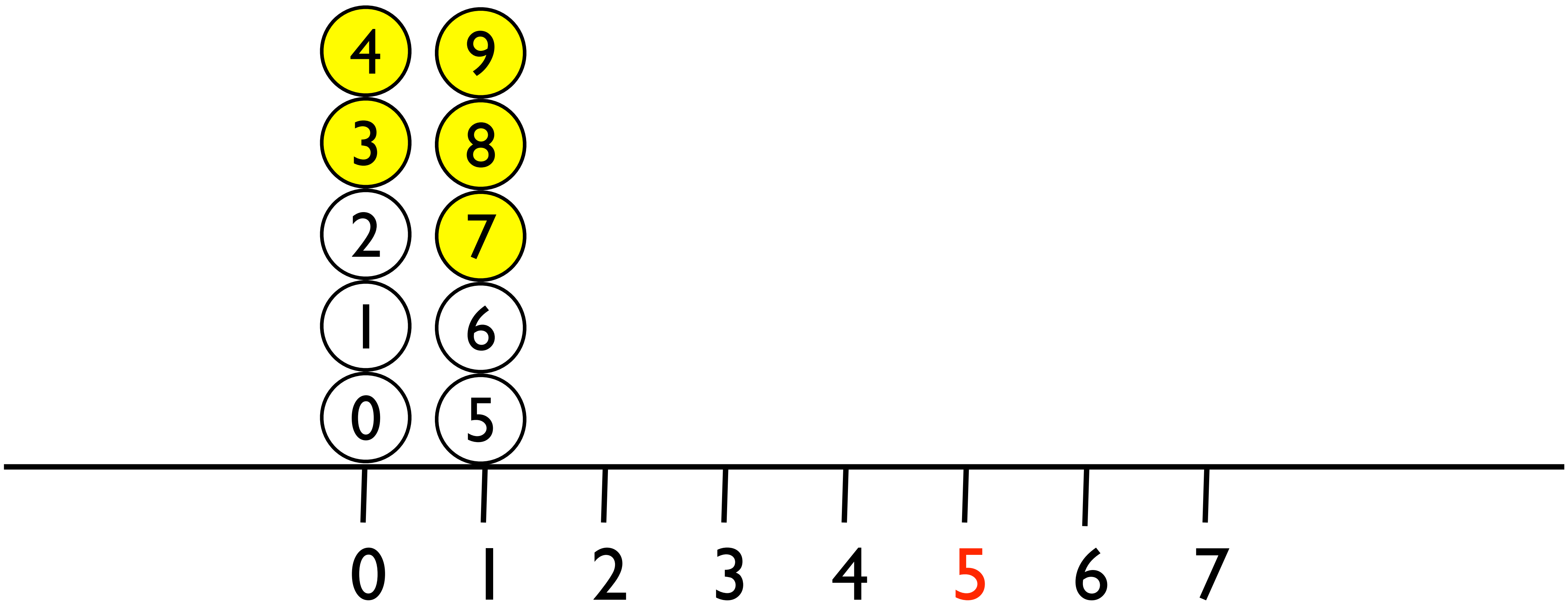
Bob's move



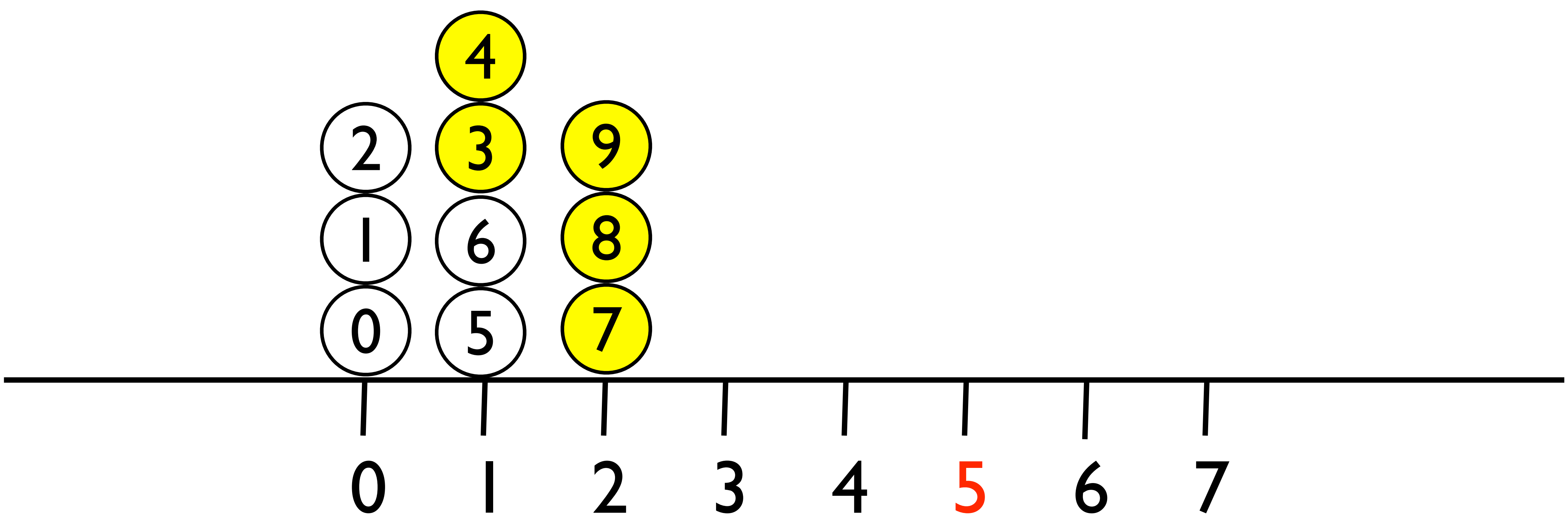
Configuration 1



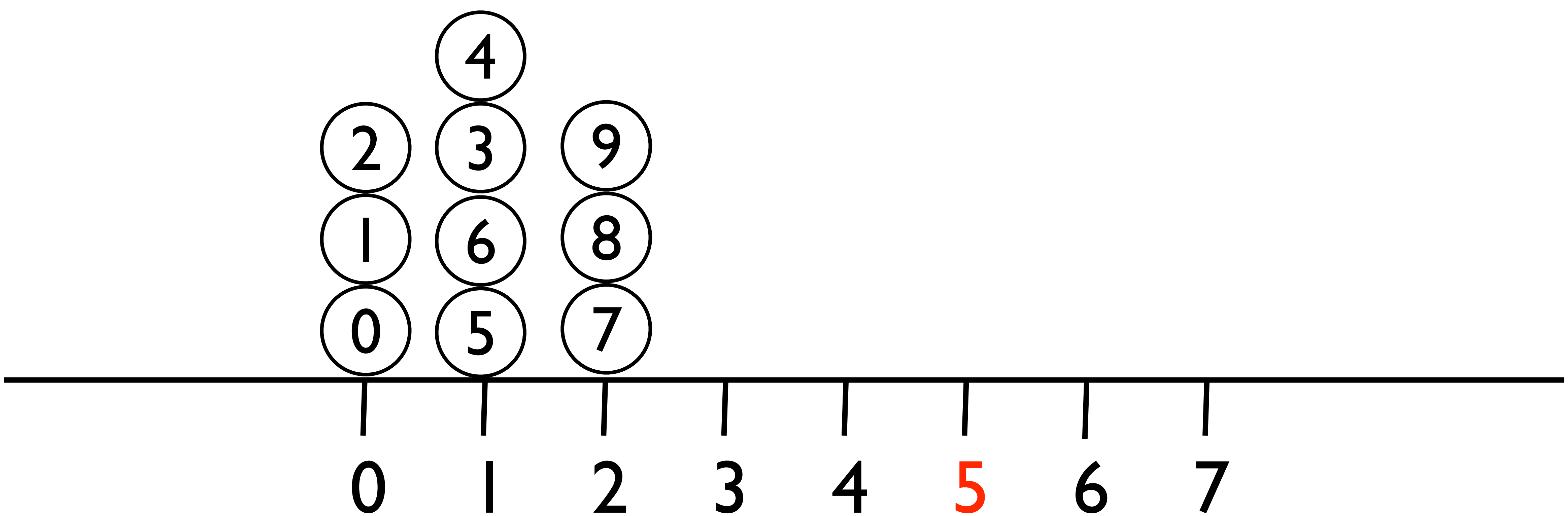
Alice's move



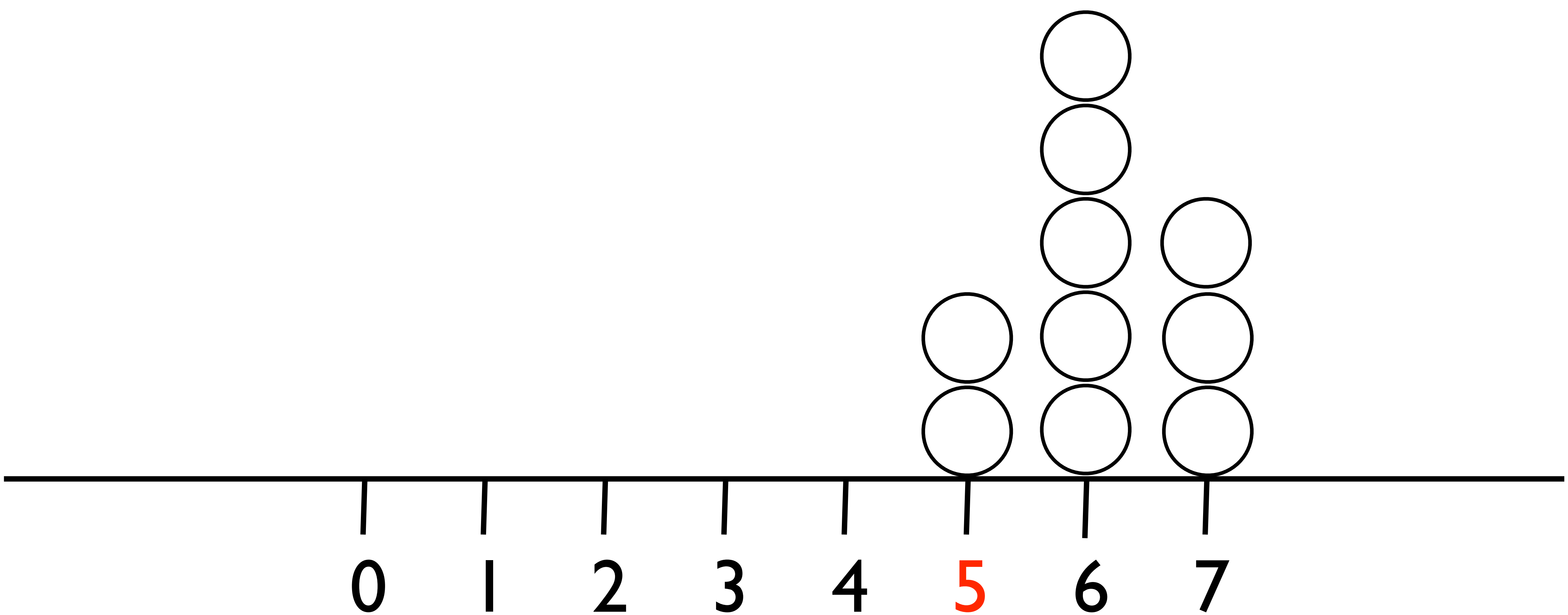
Bob's move



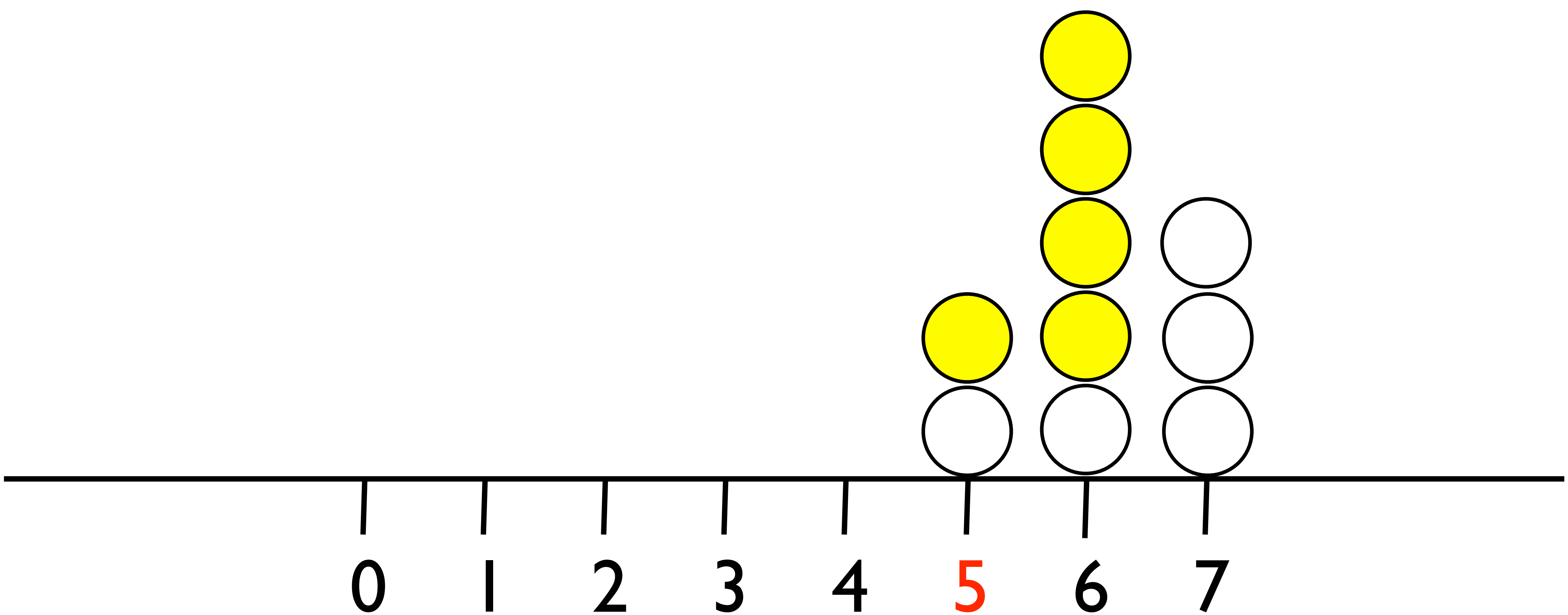
Configuration 2

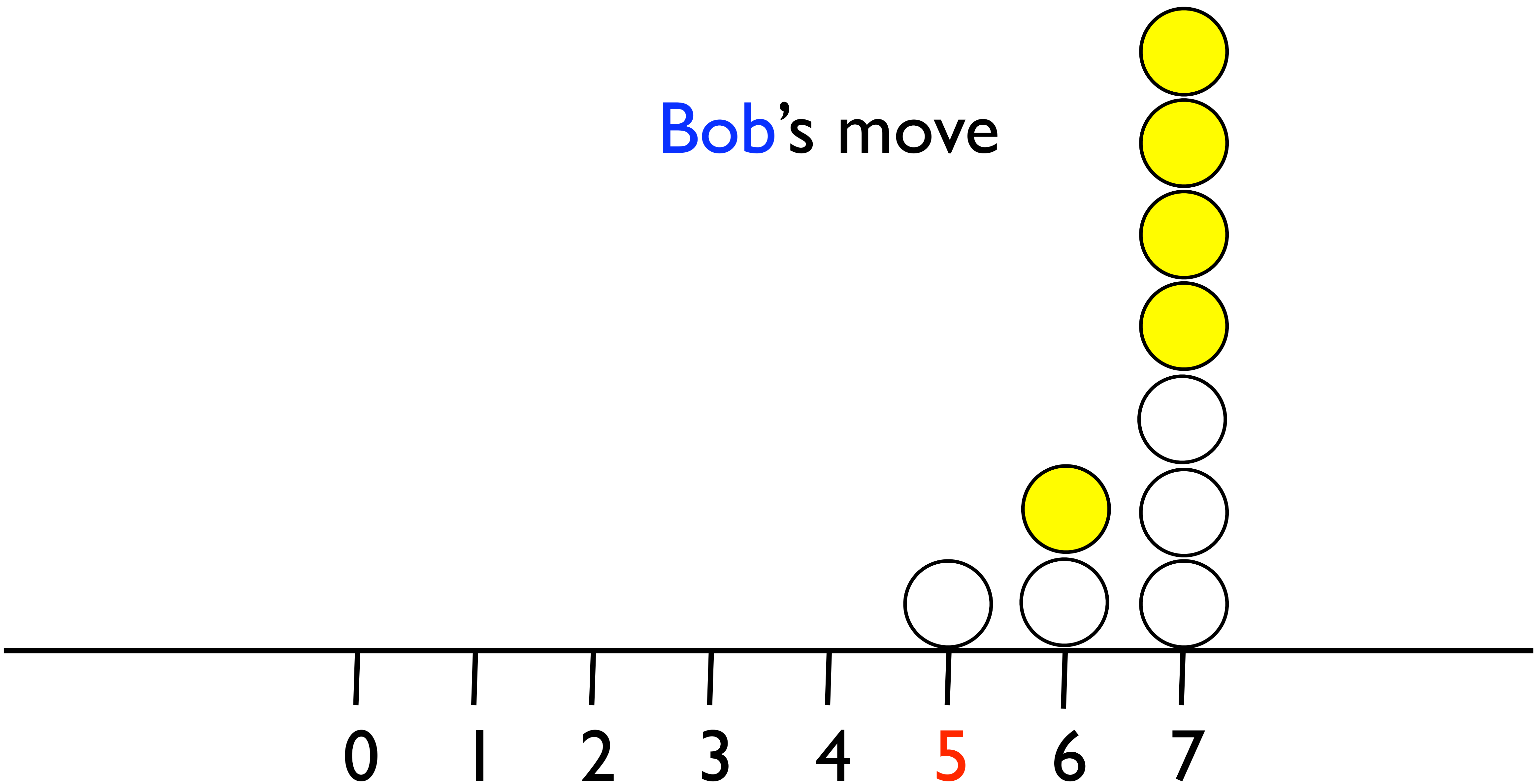


Configuration j



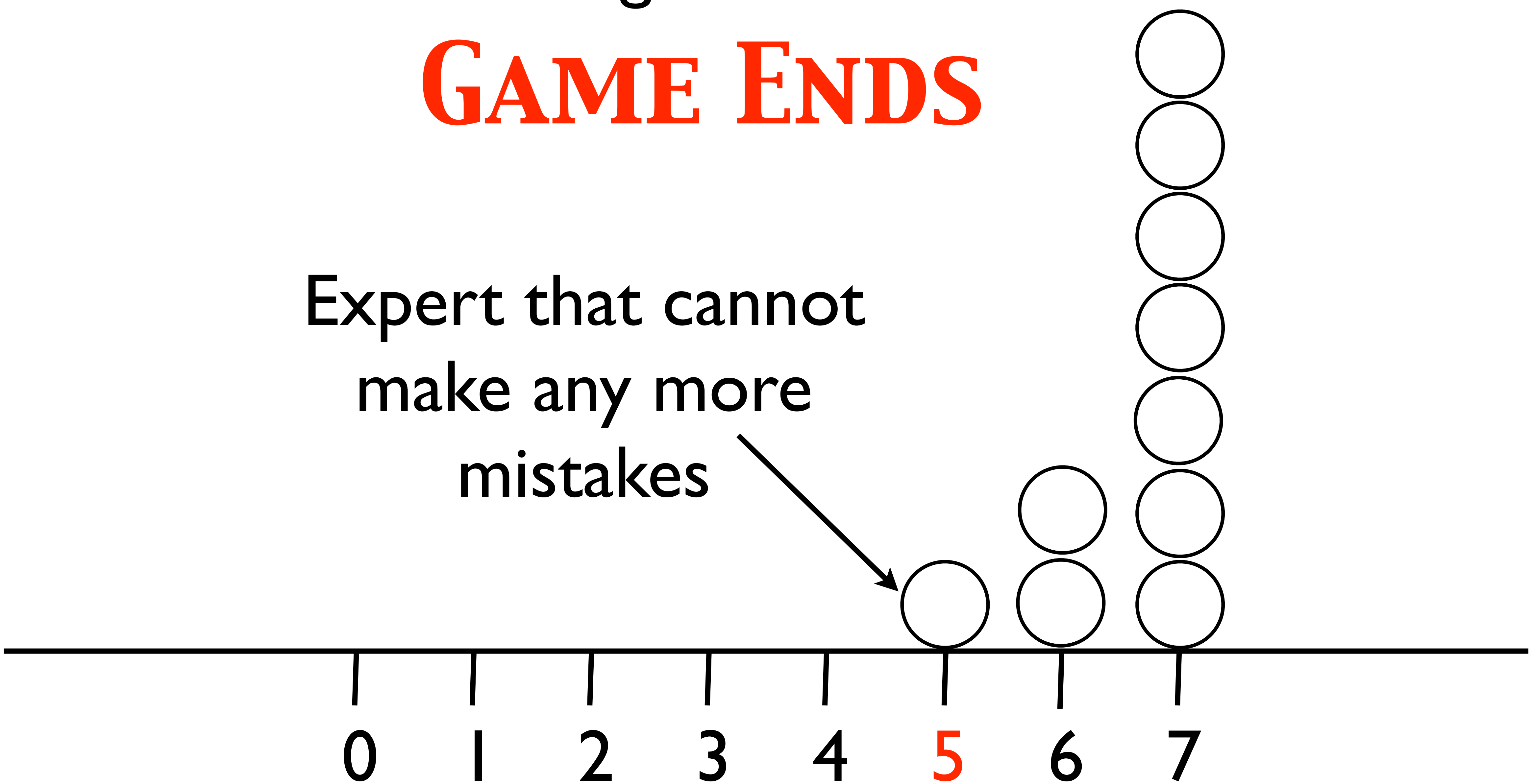
Alice's move




Bob's move
0
1
2
3
4
5
6
7

Final configuration

GAME ENDS



The Binomial Weights algorithm

- Main difference from Ulam's game with k lies - Bob wants a short game, not a long game.
- Using potentials and weights works the same way, only the sign is reversed.
- Better than exponential weights.
- Limited to +1/-1 predictions.

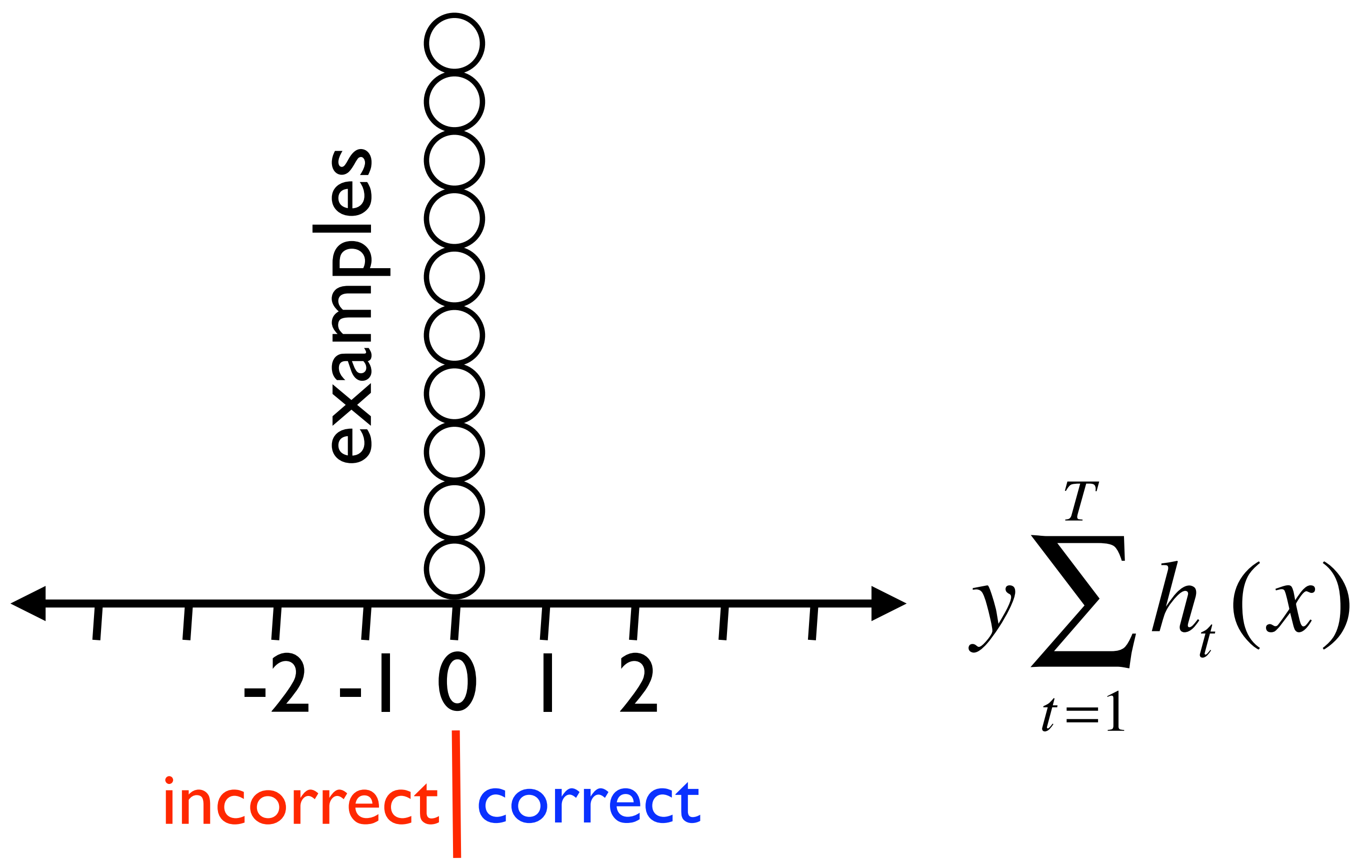
Application to Boosting Boost by Majority

[Freund 95]

- game between a booster and a weak learner.
- Boosting generates a simple (unweighted) majority rule over weak learners.
- T Number of iterations is **set in advance**
- On iteration $t=1..T$
  - booster assigns weights to the training examples.
  - learner chooses a rule whose error wrt the chosen weights is smaller than $1/2-\gamma$
  - Rule is added to majority rule
- Goal of booster is to minimize number of errors of final majority rule.

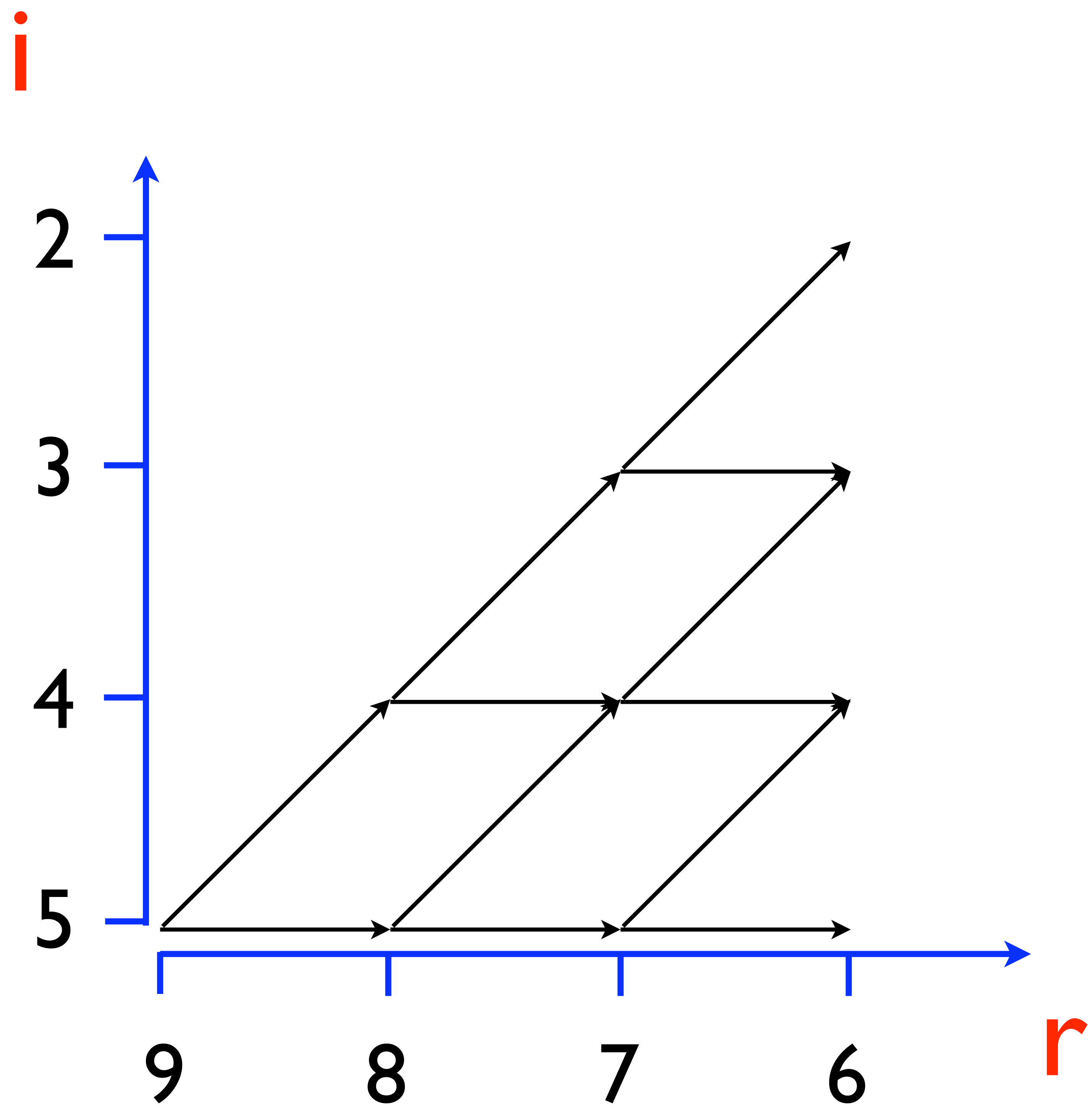
Boosting as a drifting game

- Chips = examples
- bin i contains the examples for which the difference between the number of correct and incorrect rules is i

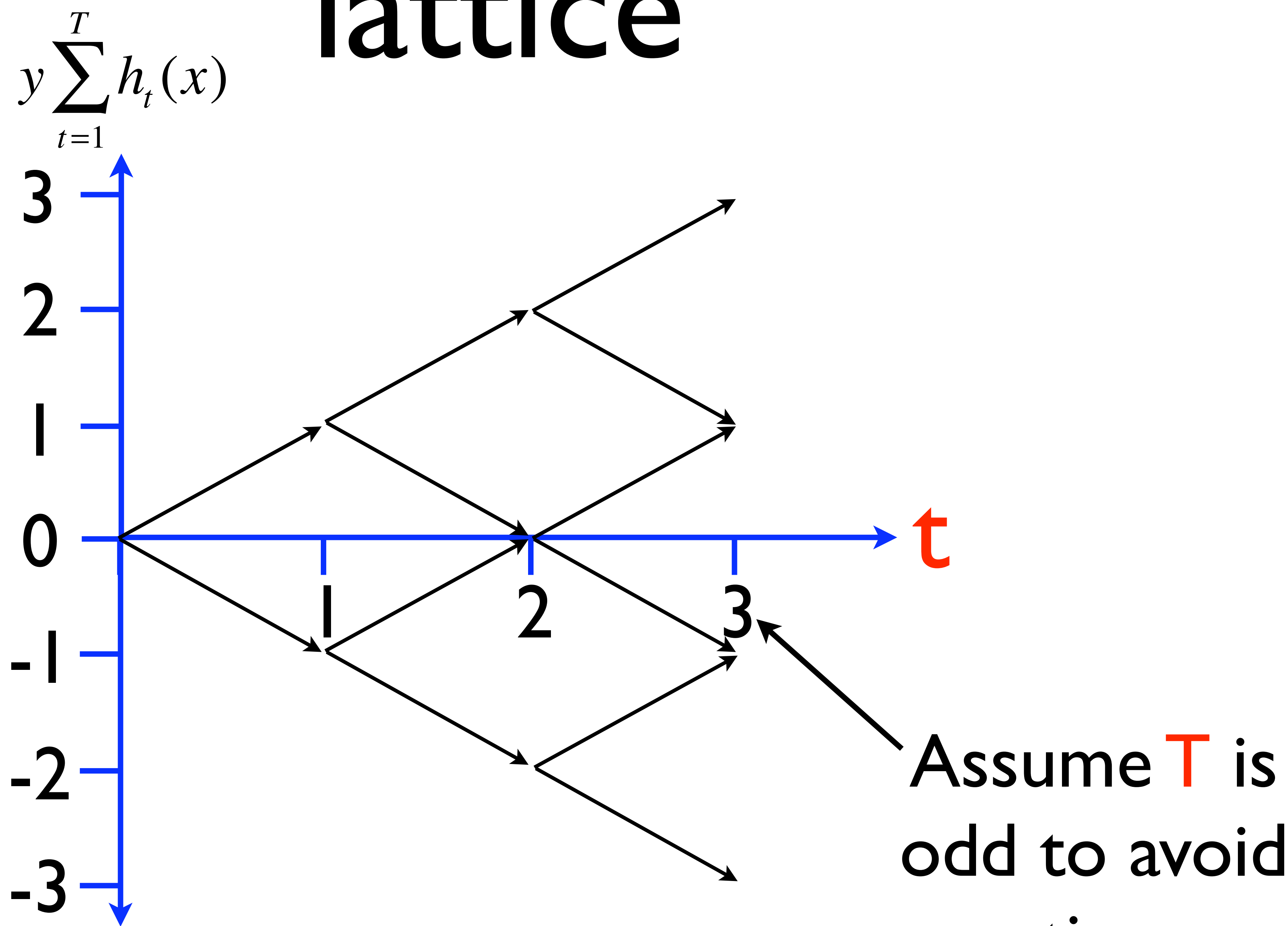


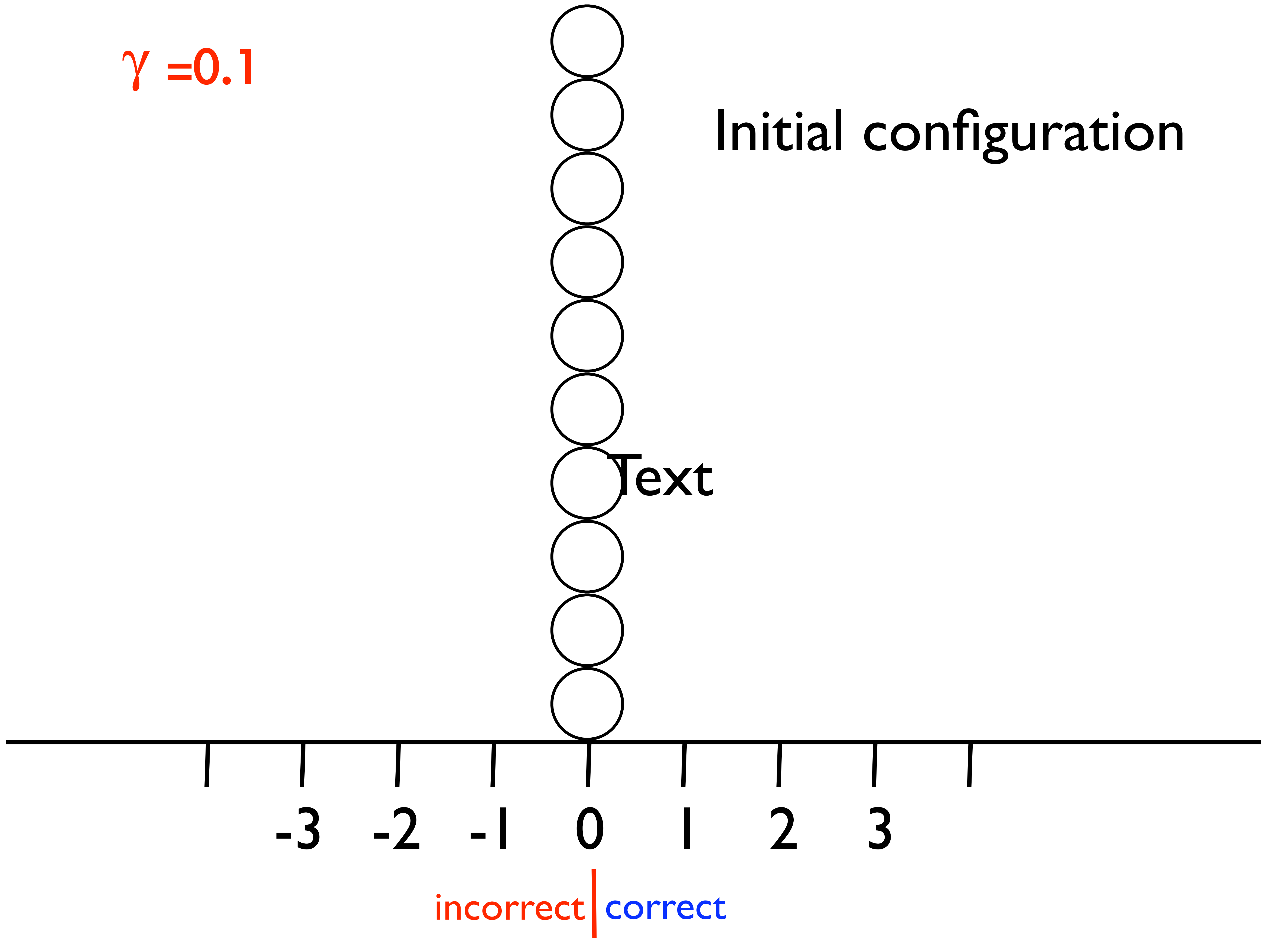
Initial Configuration

The Ulam game lattice

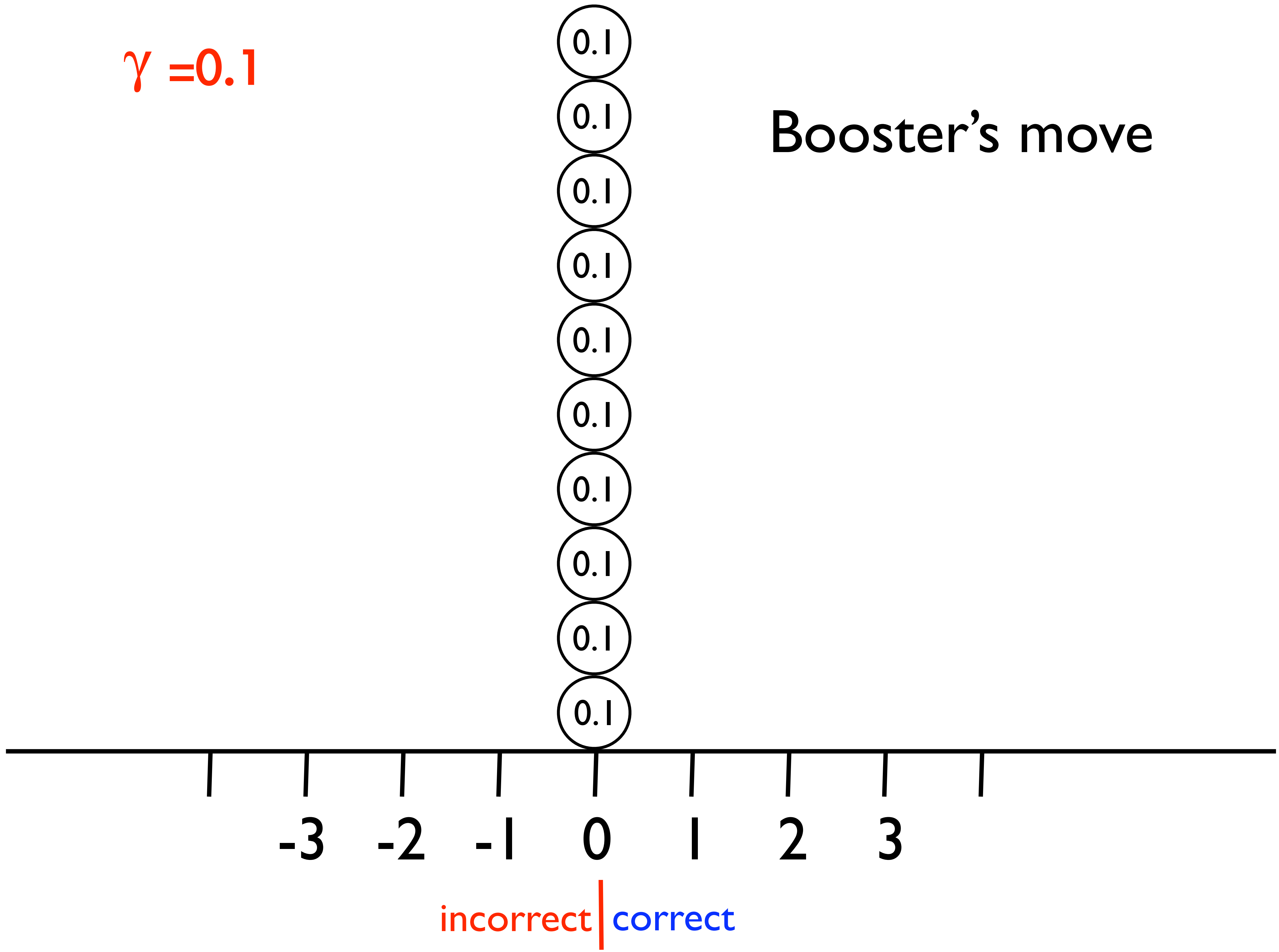


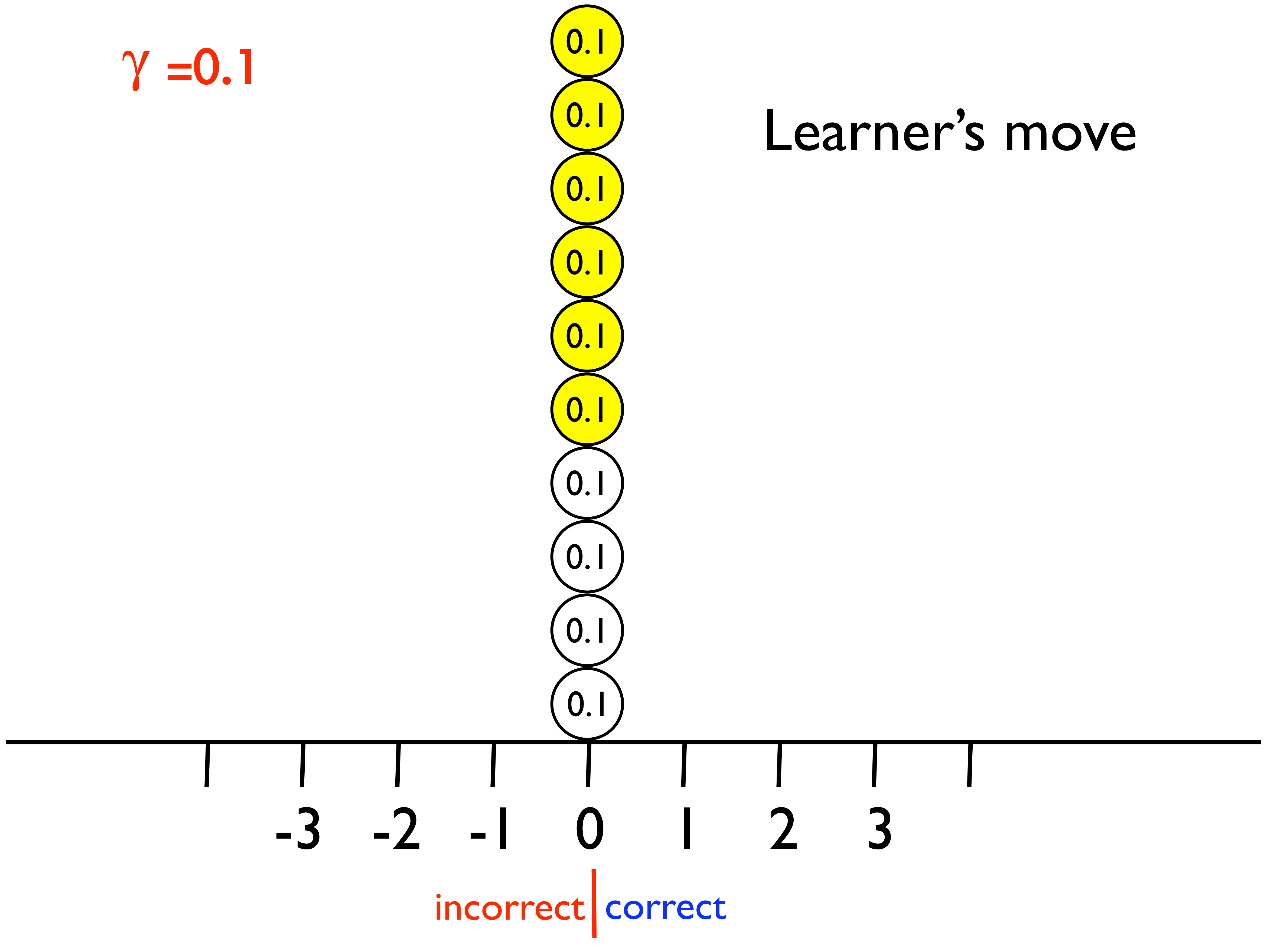
The boosting game lattice





γ =0.1
Initial configuration
Text
-3 -2 -1 0 1 2 3
incorrect
correct

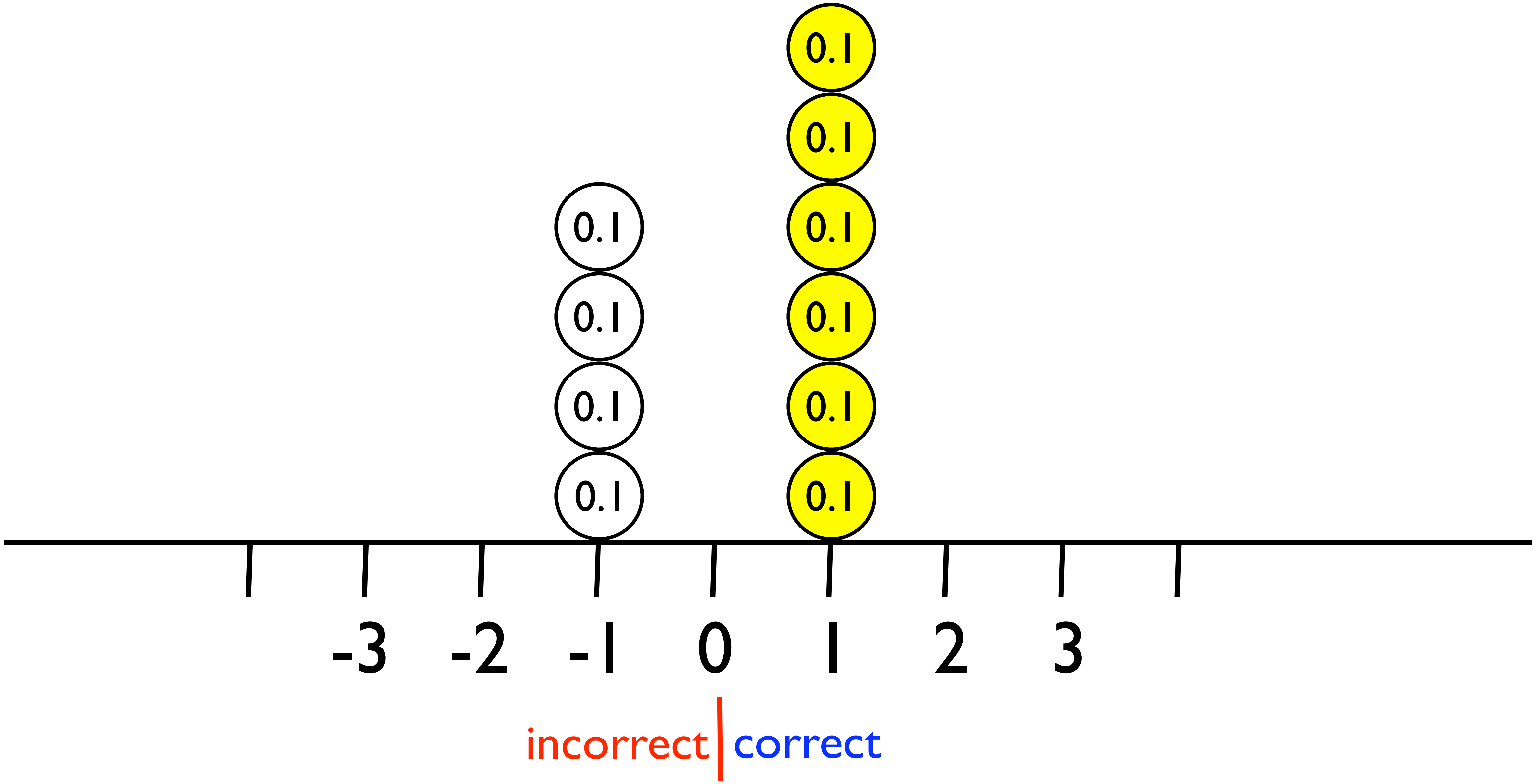

γ =0.1
Booster's move
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
-3
-2
-1
0
1
2
3
incorrect
correct



γ =0.1
Learner's move
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
0.1
-3 -2 -1 0 1 2 3
incorrect
correct

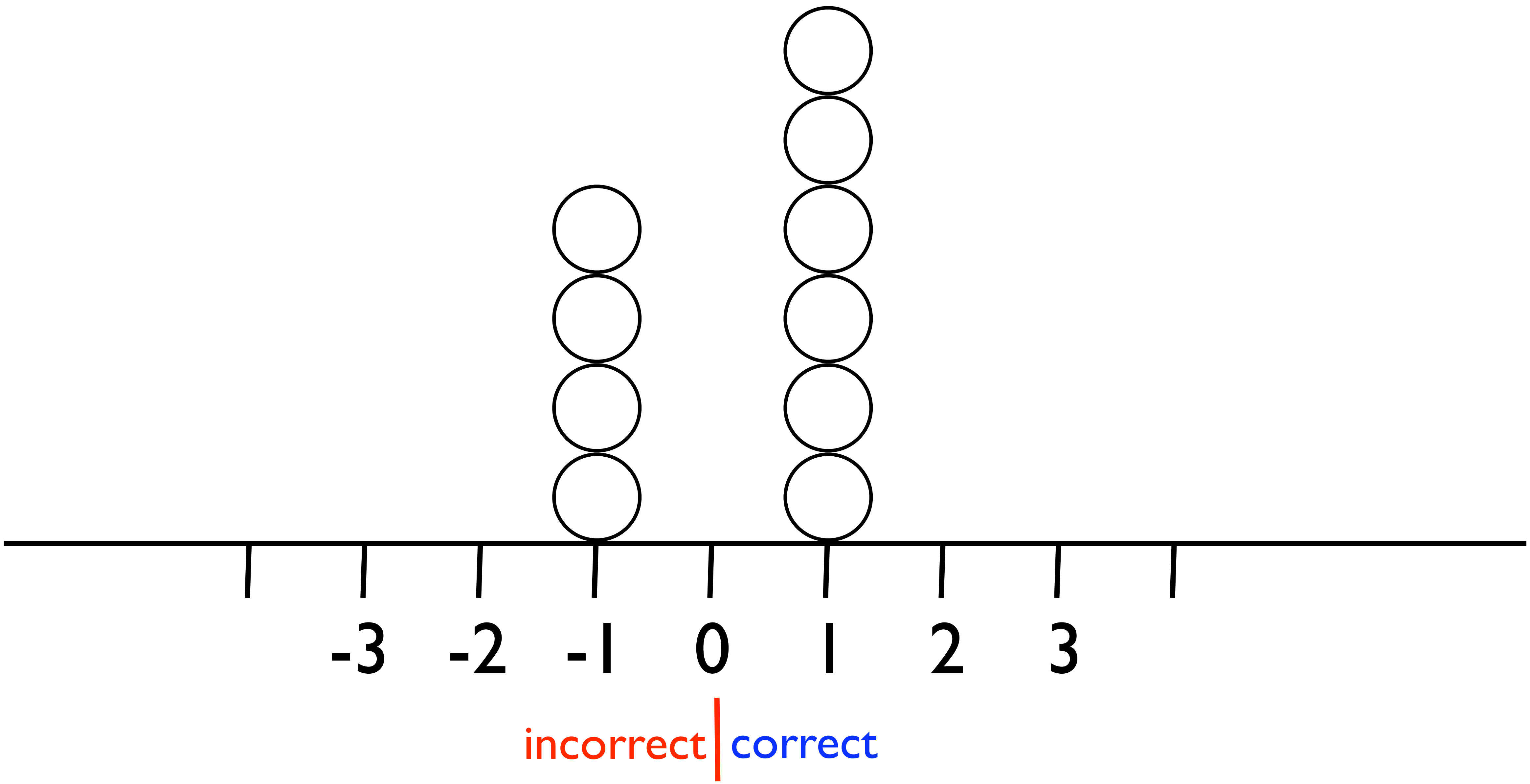
$\gamma = 0.1$

Update



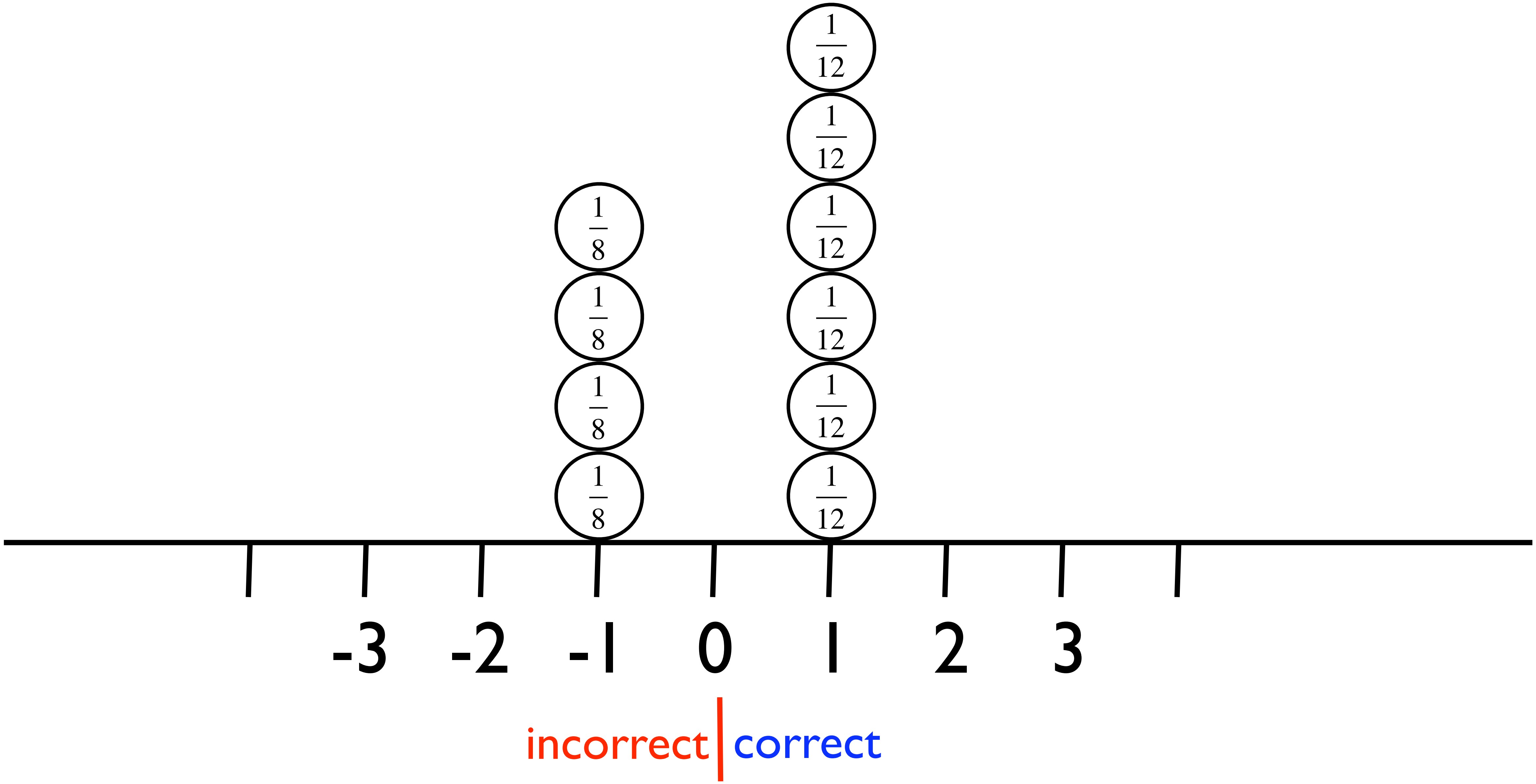
$\gamma = 0.1$

Configuration 1



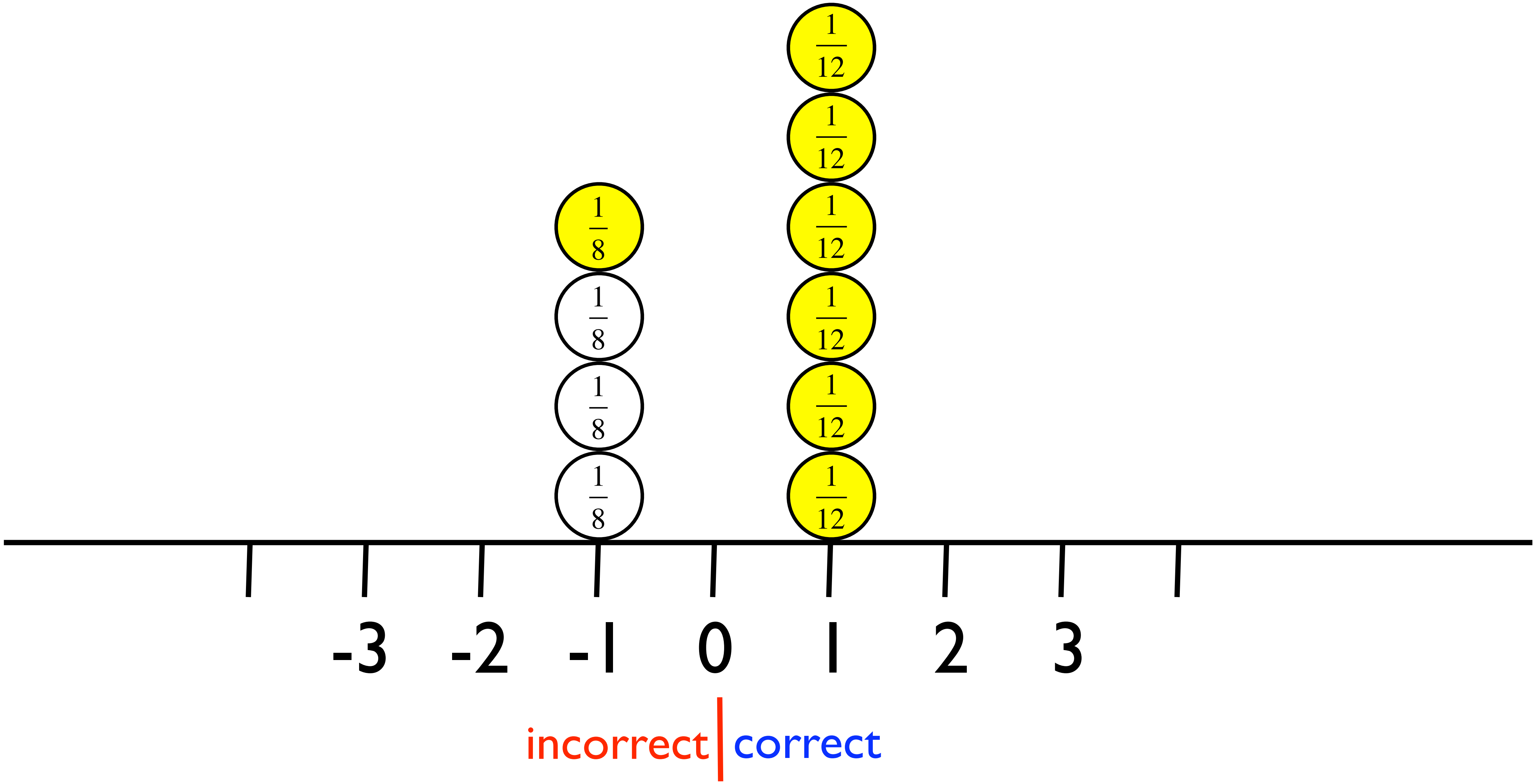
γ =0.1

Booster's move



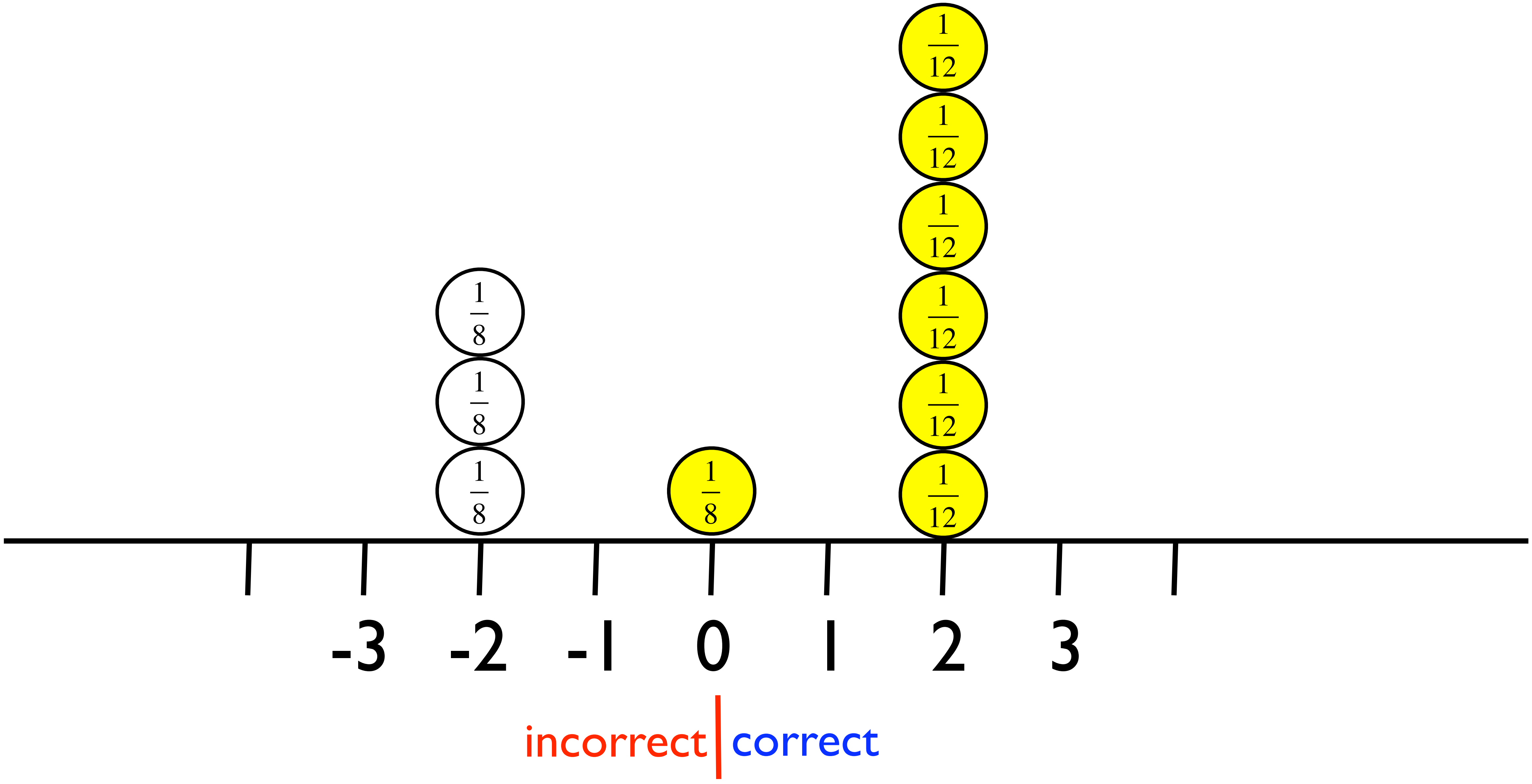
γ =0.1

Learner's move



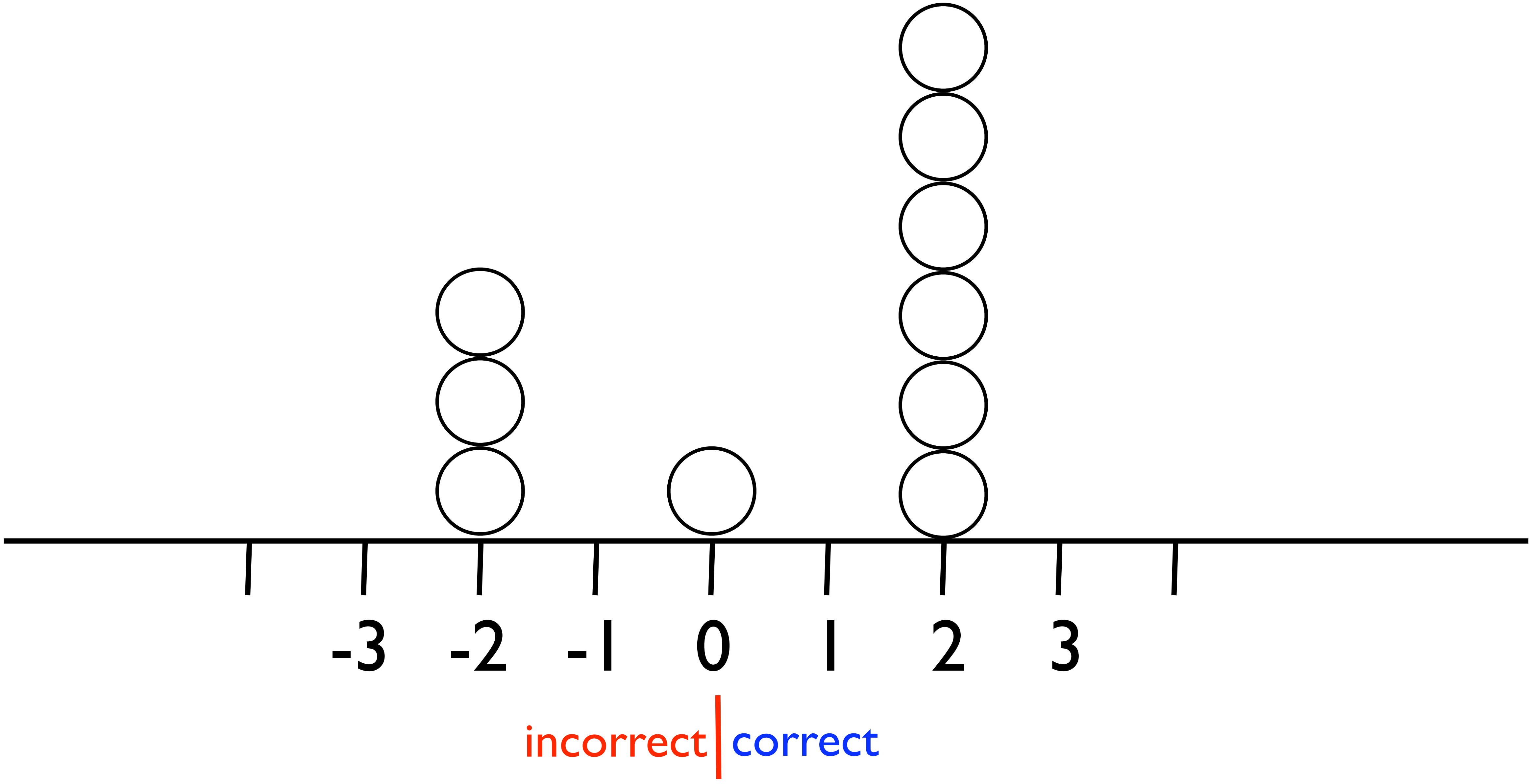
γ =0.1

update



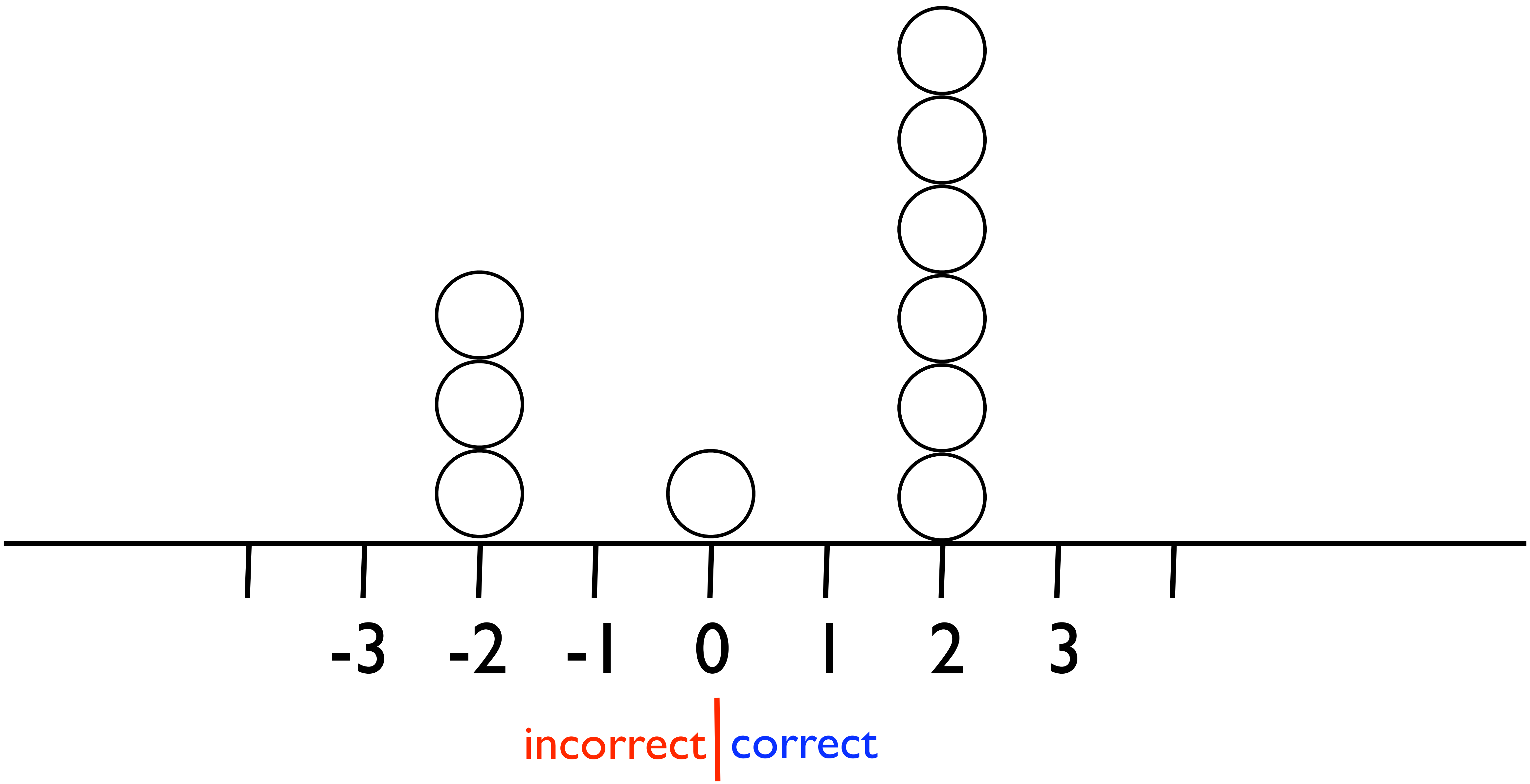
$\gamma = 0.1$

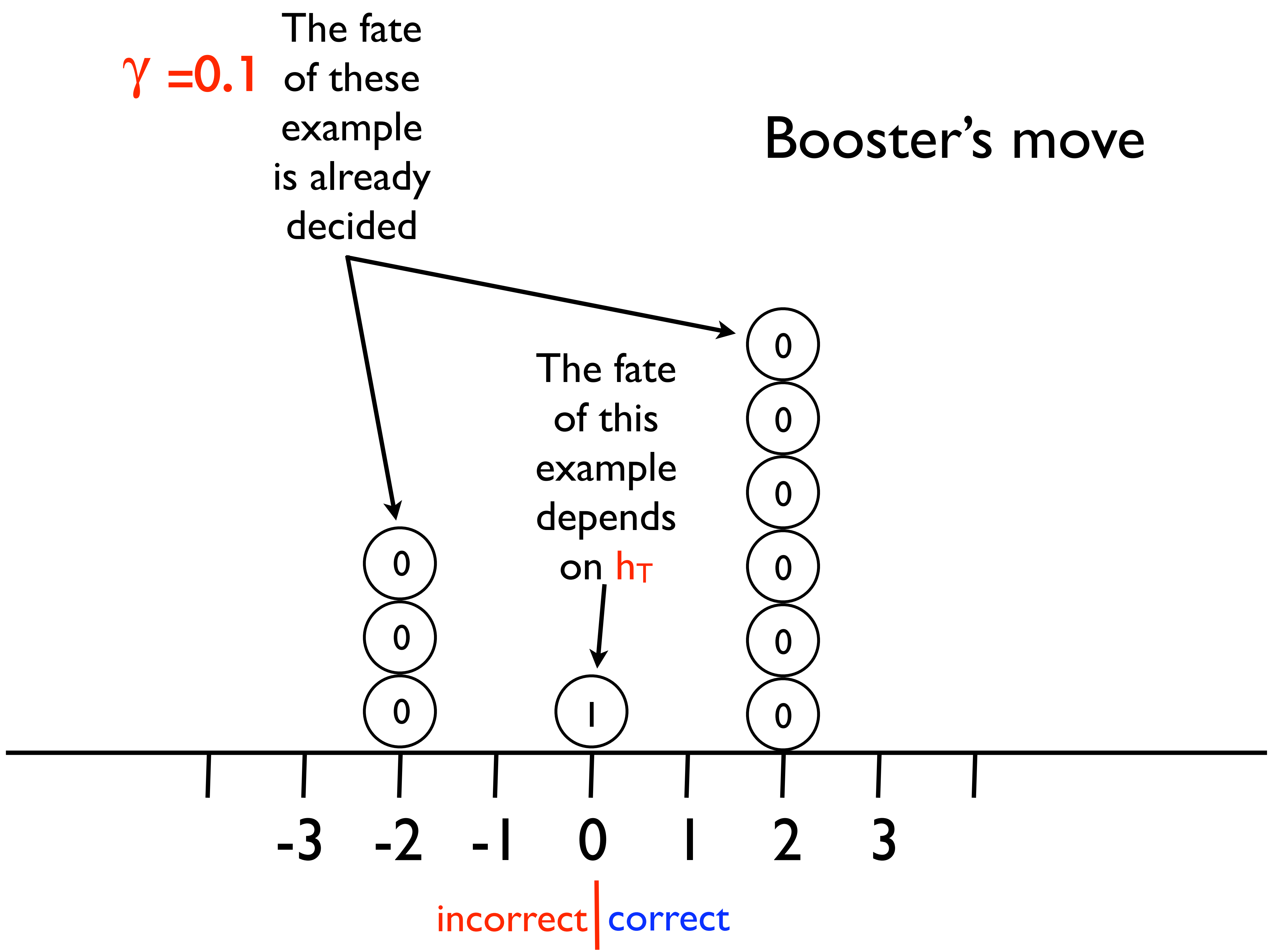
Configuration 2



$\gamma = 0.1$

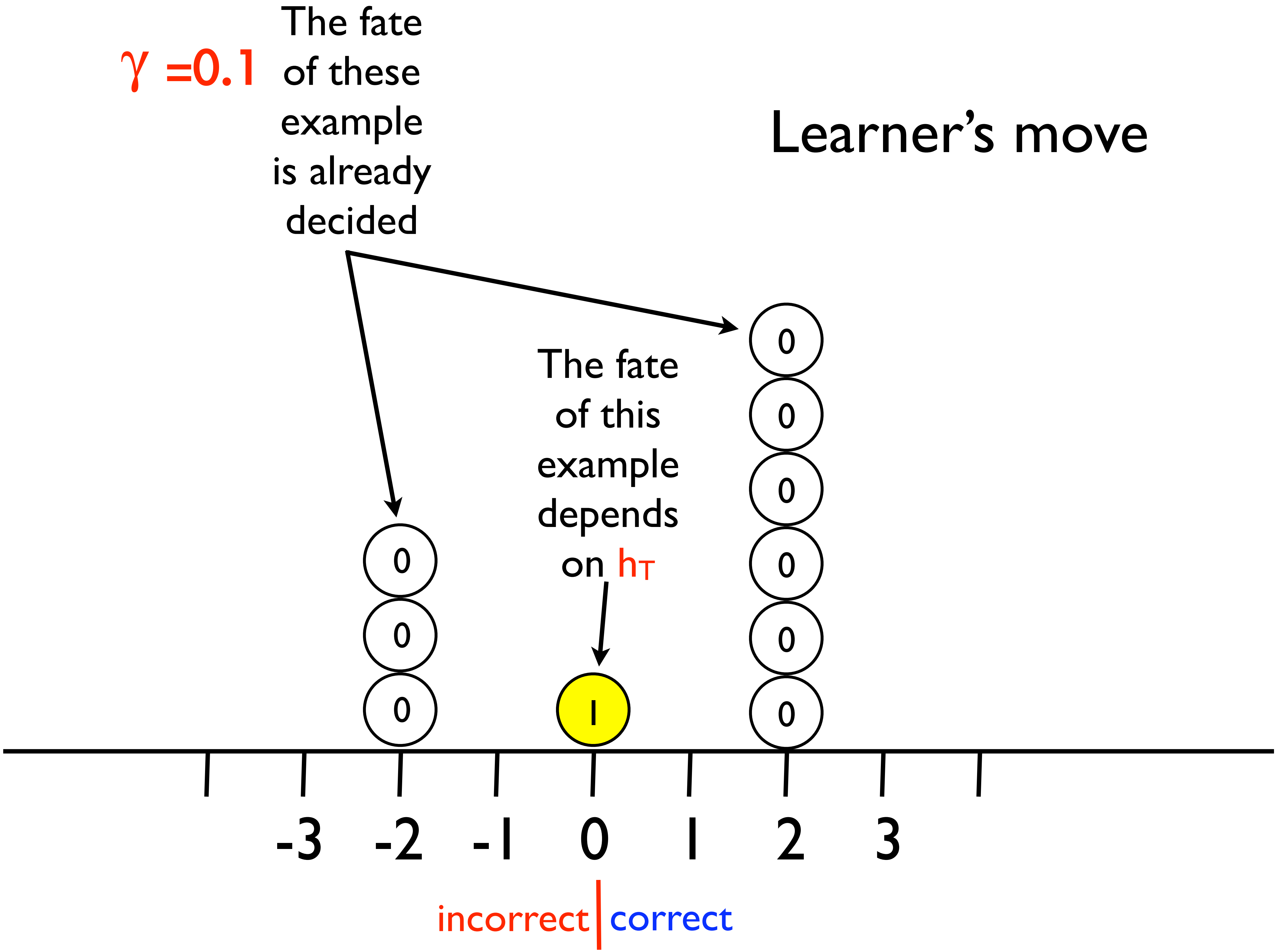
Configuration T-1





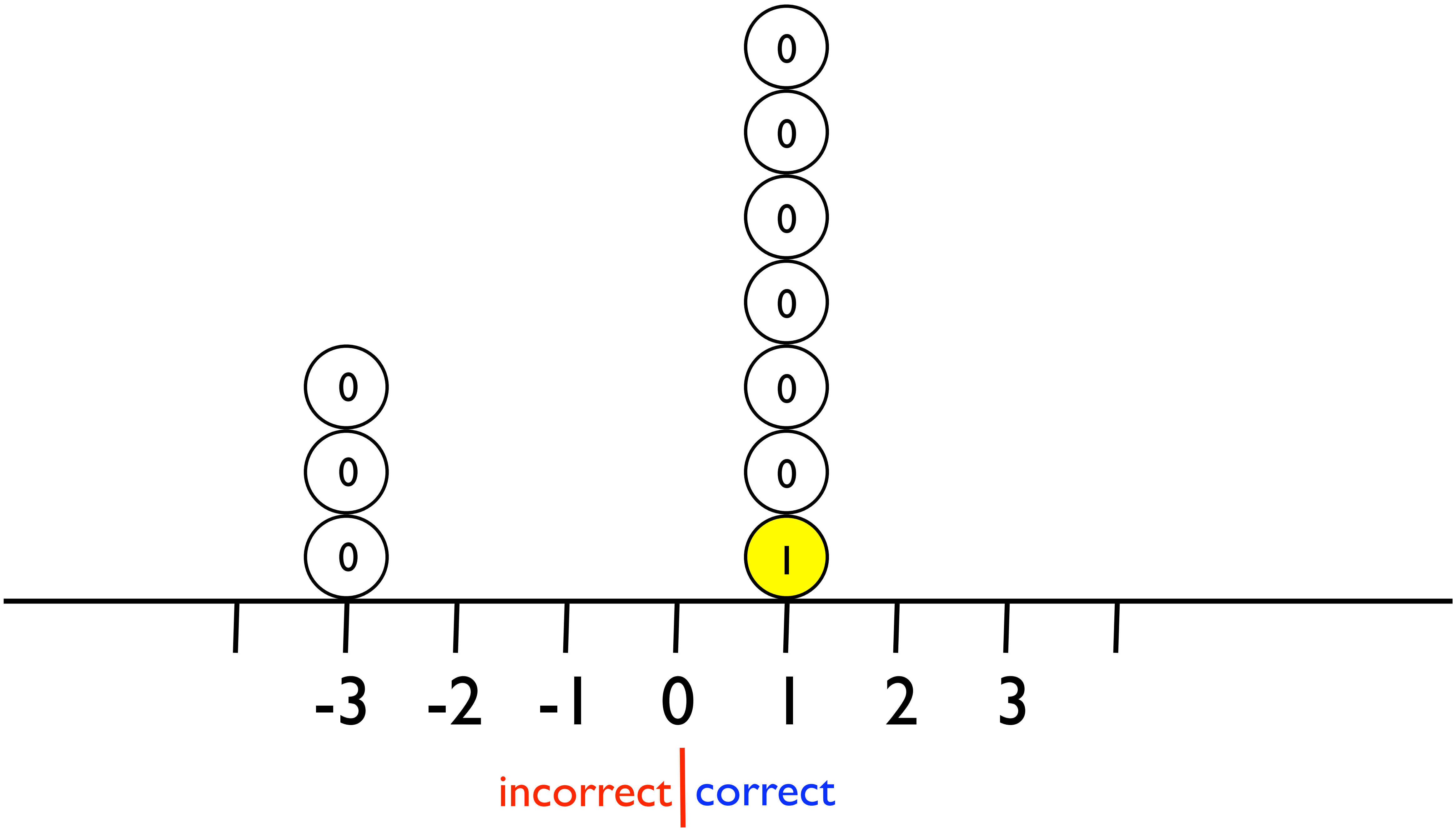
γ =0.1
The fate of these example is already decided
Booster's move
The fate of this example depends on h_T
0
0
0
1
0
0
0
0
0
0
-3
-2
-1
0
1
2
3
incorrect
correct

Learner’s move



$\gamma = 0.1$

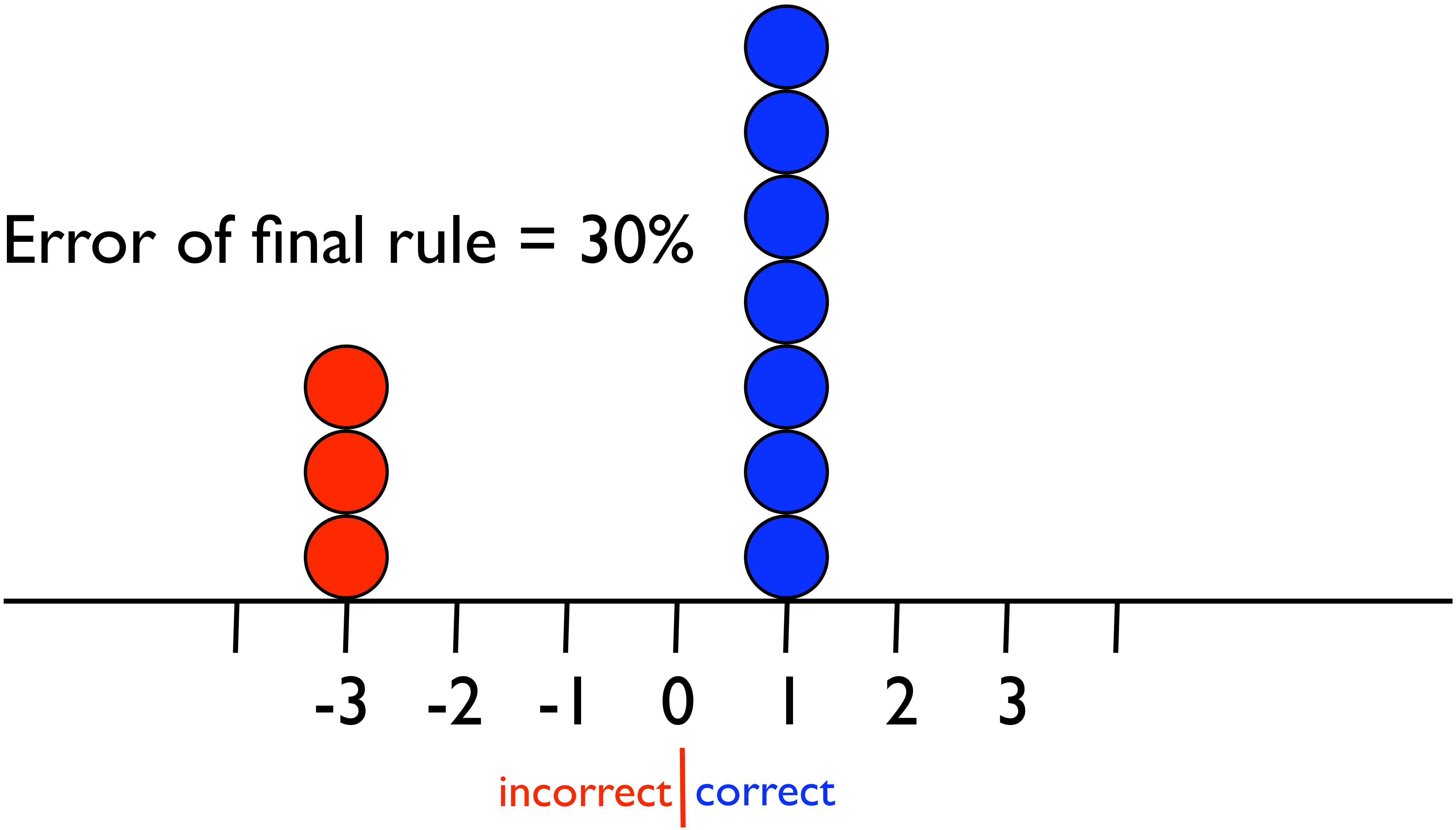
update



$\gamma = 0.1$

Final Configuration

Error of final rule = 30%



Learner's min/max strategy

- Choose $1/2+\gamma$ from each bin to be correct.
- Might not be possible if number of examples is finite.
- If set of examples is continuous - strategy is min/max optimal.
- Equivalent to producing the correct answer independently at random with

$$P(\text{correct}) = 1/2+\gamma$$

Potential and weight for boosting

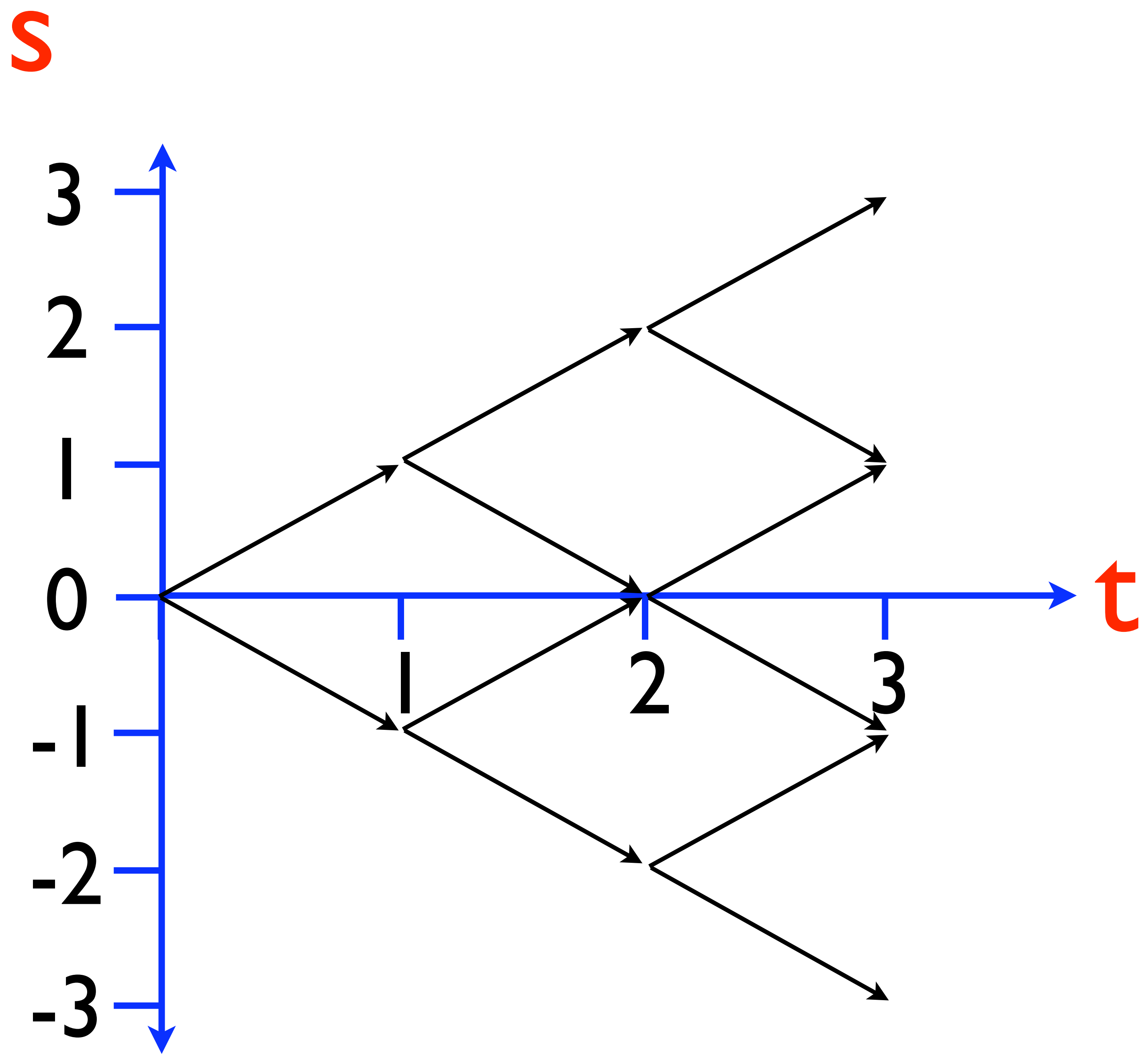
$$w(t,s) = \binom{T-t}{\left\lfloor \frac{T-t+s+1}{2} \right\rfloor} \left(\frac{1}{2}+\gamma\right)$$

From discrete to continuous time

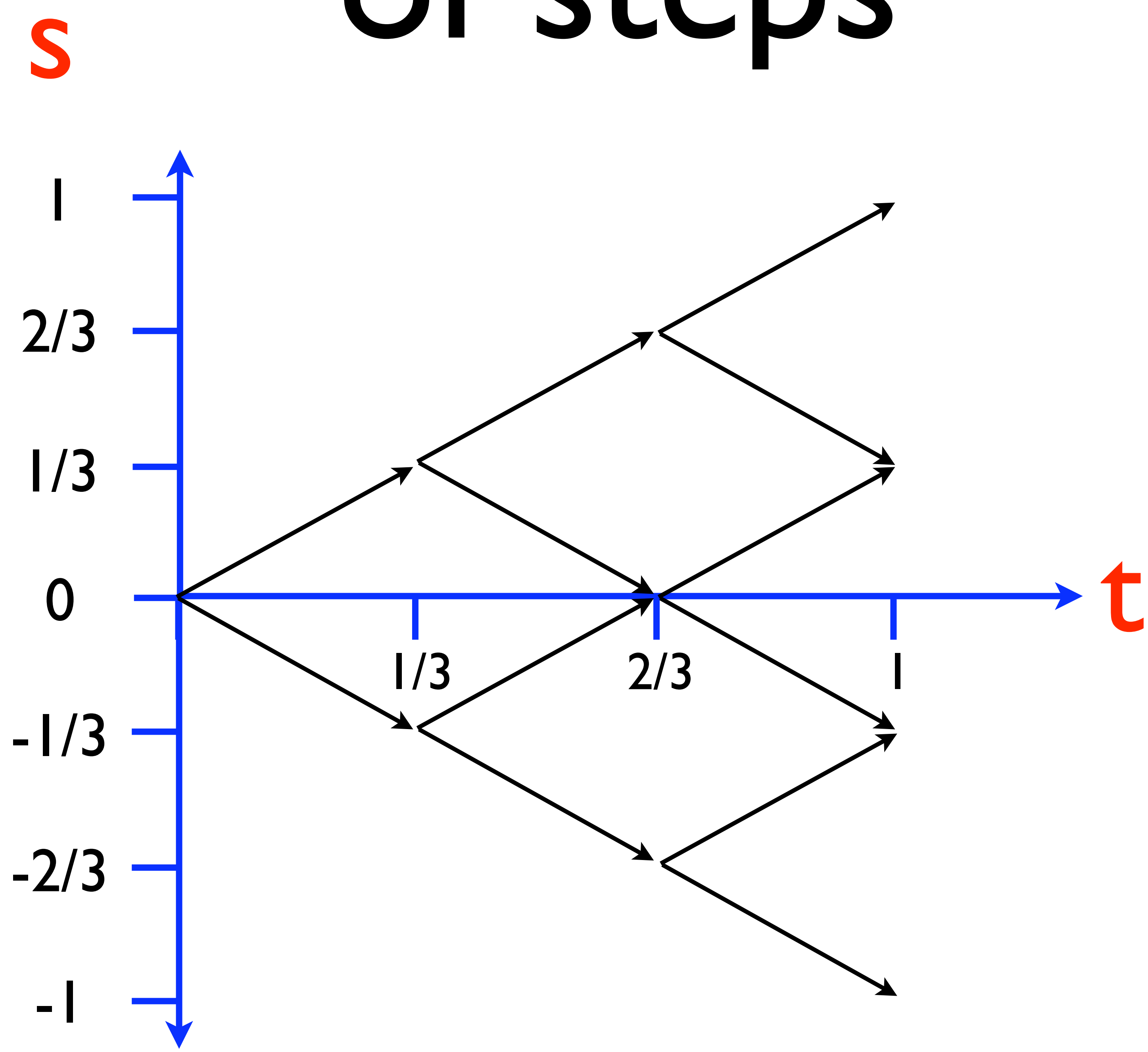
- BBM is not usable because it is not adaptive.
- BW is not usable because the predictions are restricted to be either 0 or 1.

Letting time step decrease to zero.

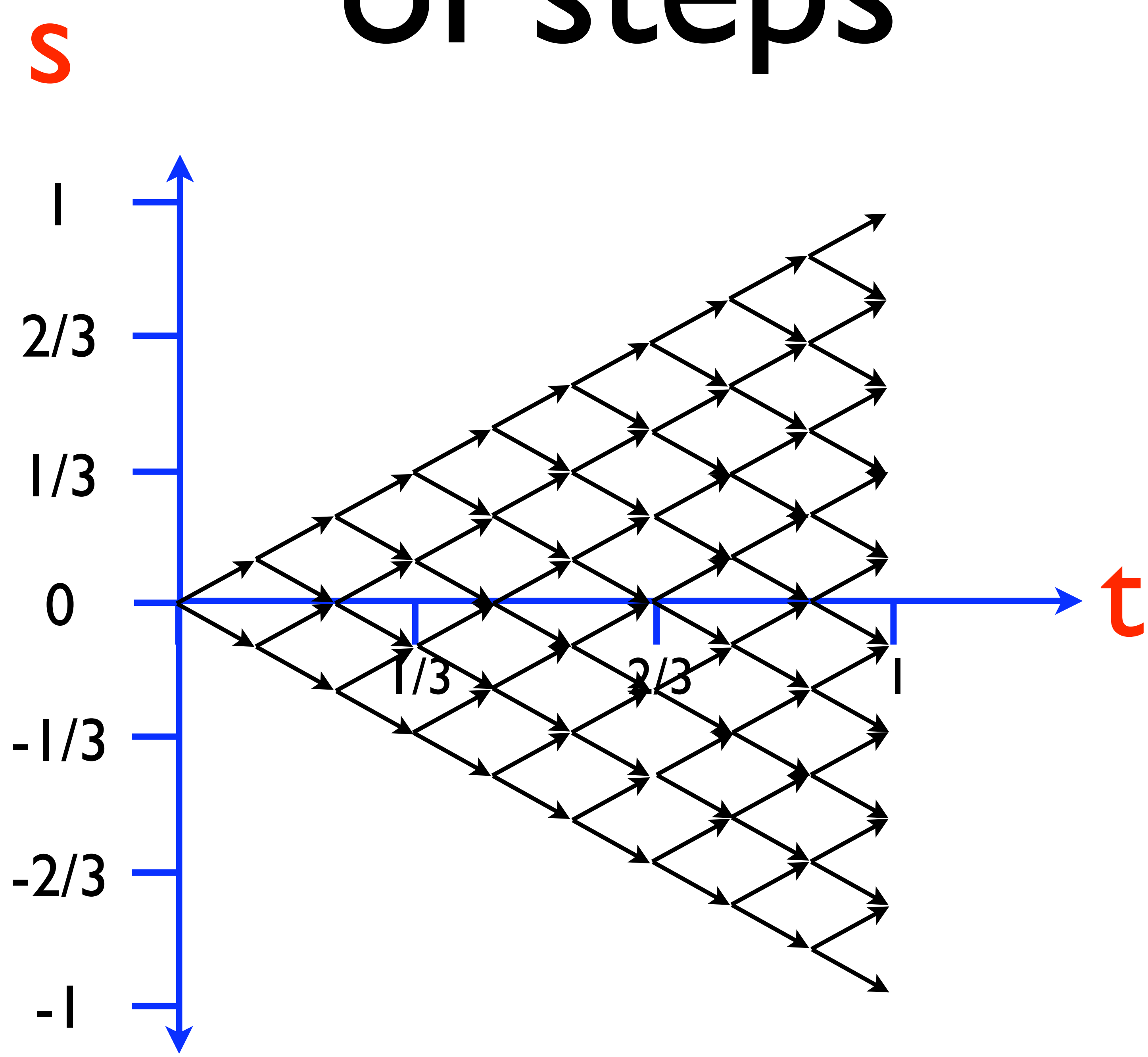
The game lattice



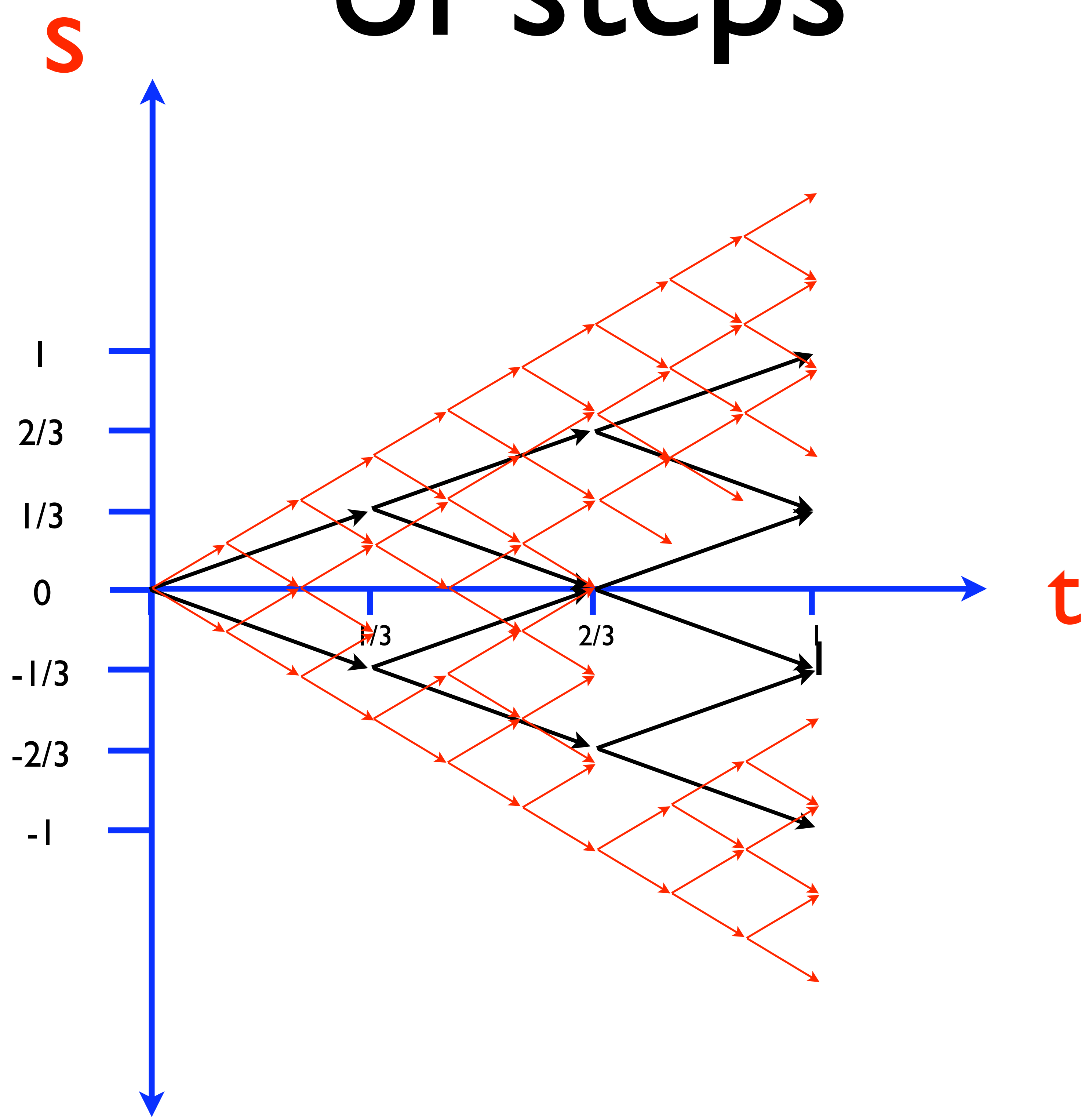
Increasing the number of steps



Increasing the number of steps



Increasing the number of steps



Boosting

- Boost by majority
- Brownboost
- Robustboost
- Experiments

Online Learning

- Multiplicative weights
- Binomial Weights
- Normalhedge
- Tracking faces