Introduction to Online Learning Algorithms

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January 2, 2025

Outline

Example trace for Halving Algorithm

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expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Halving Algorithm

expert1
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alg.
outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
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expert1	1
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expert7	1
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alg.	1
outcome	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	
outcome	1	

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	$t=1$	$t=2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
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	$t = 1$	$t = 2$
expert1	1	1
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expert6	0	-
expert7	1	1
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alg.	1	0
outcome	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	
outcome	1	1	

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	$t=1$	$t=2$	$t=3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
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Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
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expert4	1	0	-	-
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expert7	1	1	1	1
expert8	1	1	1	0
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Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	-
expert8	1	1	1	0	0
alg.	1	0	1	1	
outcome	1	1	1	0	

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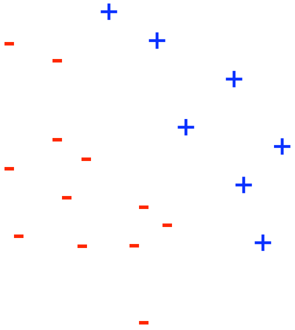
- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.
- No stochastic assumptions whatsoever.
- Proof is based on combining a lower and upper bounds on the number of perfect experts.

The Perceptron Problem

The Perceptron Problem



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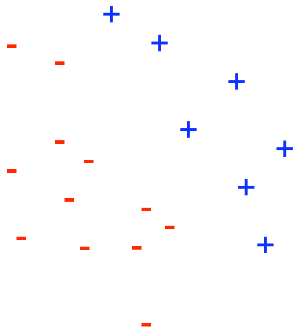

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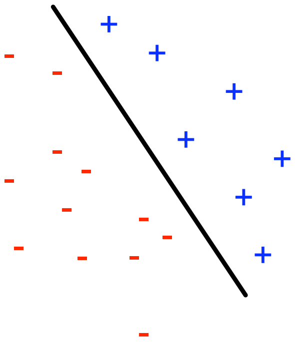

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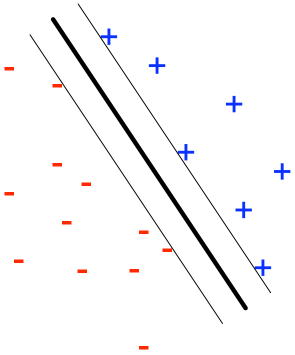

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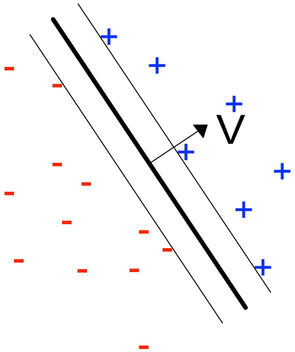

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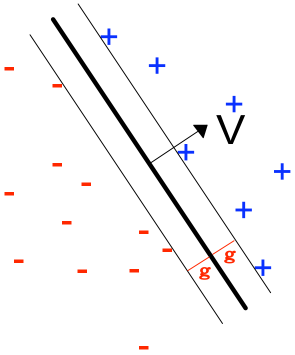

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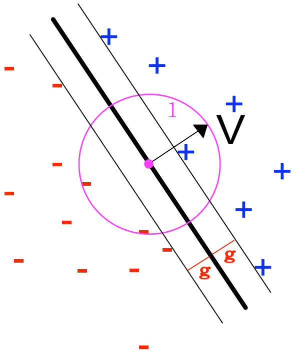

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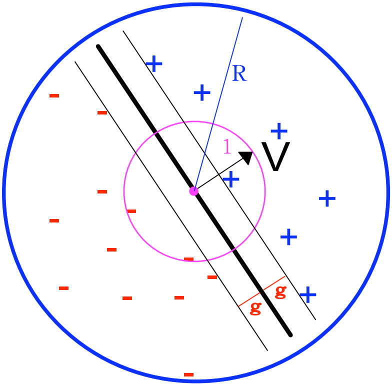

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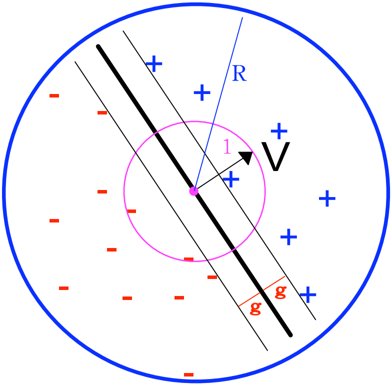

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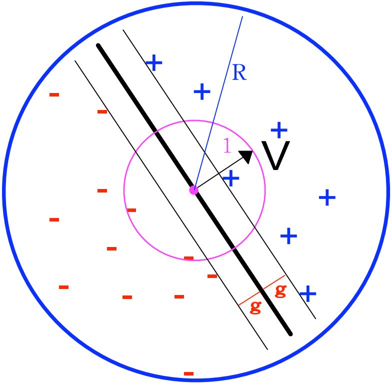

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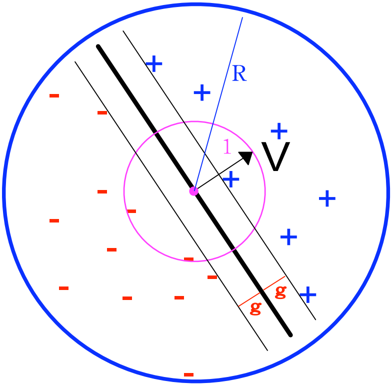

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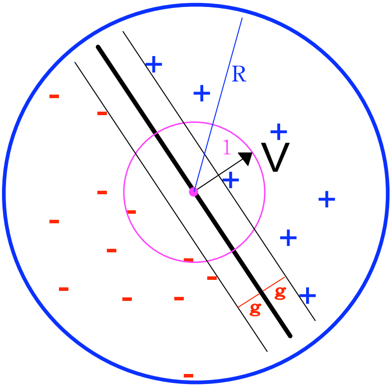

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- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.
- $\forall \vec{X}, \;\; \|\vec{X}\| \leq R$.
- $\forall (\vec{X}, y)$, $y(\vec{X} \cdot \vec{V}) \geq g$

The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.

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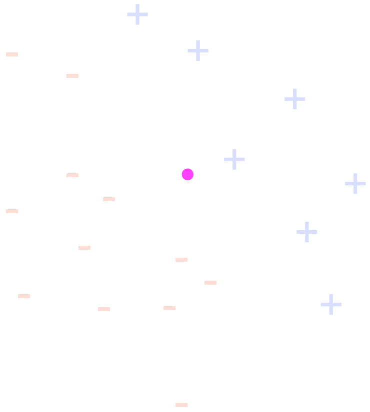
- An online algorithm. Examples presented one by one.
- start with $\vec{W}_0 = \vec{0}$.
- If mistake: $(\vec{W}_i \cdot \vec{X}_i)y_i \leq 0$
  - Update $\vec{W}_{i+1} = \vec{W}_i + y_i X_i$.

Example trace for the perceptron algorithm



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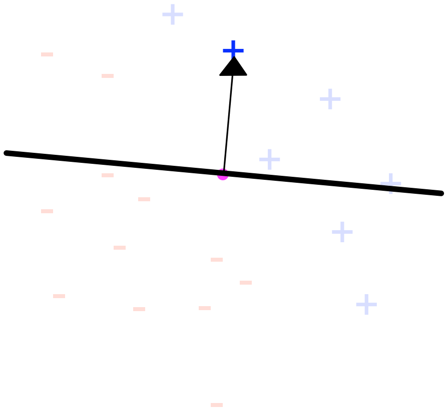

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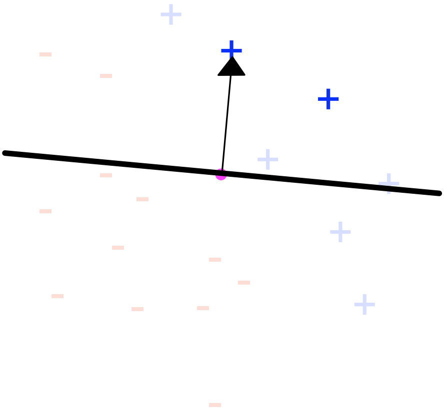

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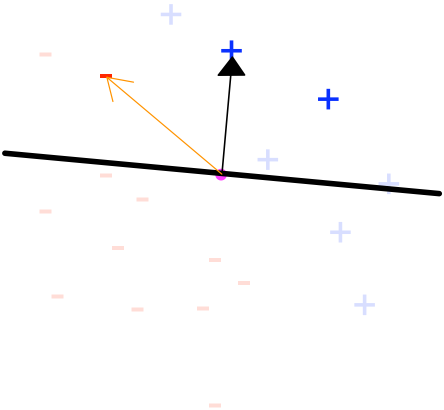

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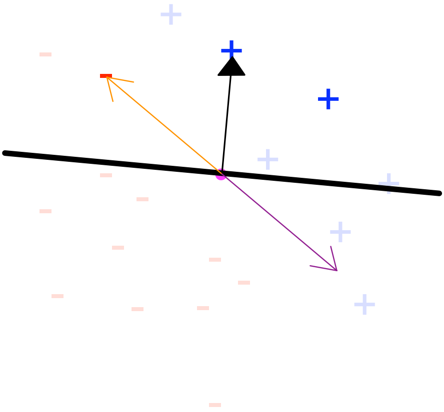

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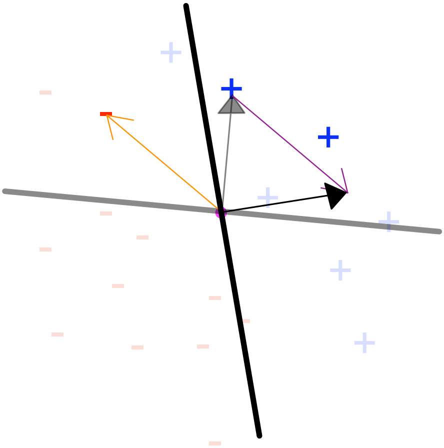

Bound on number of mistakes

- The number of mistakes that the perceptron algorithm can make is at most $\left(\frac{R}{g}\right)^2$.

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- Proof by combining upper and lower bounds on $\|\vec{W}\|$.

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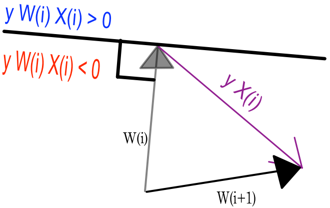

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  $\|\vec{W}_{i+1}\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$
  $\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2$

Lower bound on $\|\vec{W}_i\|$

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$$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V} \text{ because } \|\vec{V}\| = 1.$$

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  $\geq ig + g = (i+1)g$

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Thus:

$$i \leq \left(\frac{R}{g}\right)^2$$

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- Impossible without additional constraints.

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- Online prediction: predict x_{t+1} from $Y^t = \langle Y_1, Y_2, \ldots, Y_t \rangle$.
- **Expected regret**: compare performance of algorithm to Regret $= \mathrm{E}_{Y^T} \left[(x_t - Y_t)^2 \right] - \sigma^2$

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  $\text{Regret}_T = \sum_{t=1}^{T} (x_t - y_t)^2 - \sum_{t=1}^{T} (x_T^* - y_t)^2$

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  $\text{Regret}_T = \sum_{t=1}^T (x_t - y_t)^2 - \sum_{t=1}^T (x_T^* - y_t)^2$
- **Goal:** sublinear regret $\lim_{T \to \infty} \frac{\text{Regret}_T}{T} = 0$

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Follow the Leader

- Idea: set x_{t+1} to be the best constant prediction on $y_1, \ldots, y_t$
- $x_{t+1} = \text{argmin}_{x \in [0,1]} \sum_{i=1}^{t} (x - y_i)^2$
- We will prove that the regret of this algorithm is upper bound by $4 + 4 \ln T$

regret bound

Theorem

Let $y_t \in [0,1]$ for $t = 1, \ldots T$ an arbitrary sequence of numbers. Let the algorithm output be $x_t = x^*_{t-1} = \frac{1}{t-1}\sum_{i=1}^{t-1} y_i$, then

$$Regret_T = \sum_{t=1}^{T}(x_t - y_t)^2 - \sum_{t=1}^{T}(x^*_T - y_t)^2 \leq 1 + \ln T$$

Lemma

Let $x_1^*, x_2^*, \ldots$ be the squence of predictions produced by FTL. Then for all $u \in R$ (In particular, for $u = x_{T+1}^*$) :

$$\begin{aligned} Regret_T(u) &= \sum_{t=1}^{T} \left((x_t^* - y_t)^2 - (u - y_t)^2 \right) \\ &\leq \sum_{t=1}^{T} \left((x_t^* - y_t)^2 - (x_{t+1}^* - y_t)^2 \right) \end{aligned}$$

proof sketch:
Subtract $\sum_{t=1}^{T} (x_t^* - y_t)^2$ from both sides to get an equivalent claim:

$$\sum_{t=1}^{T} (x_{t+1}^* - y_t^*)^2 \leq \sum_{t=1}^{T} (u - y_t)^2$$

The inequality is proven by induction on T.

Sketch of proof of theorem

Using the fact that FTL is $x_t^* = \frac{1}{t-1} \sum_{i=1}^{t-1} y_i$
one can show that $x_{t+1}^* - y_t = \frac{t-1}{t}(x_t^* - y_t)$
and therefor that $(x_t^* - y_t)^2 - (x_{t+1}^* - y_t)^2 = \frac{1}{t}(x_t^* - y_t)^2$
From the fact that $0 \leq x_t^*, y_t \leq 1$ we get that $(x_t^* - y_t)^2 \leq 1$.
From which we obtain $\sum_{t=1}^{T}((x_t^* - y_t)^2 - (x_{t+1}^* - y_t)^2) \leq \sum_{t=1}^{T} \frac{1}{t}$
Combing the last statement with the Lemma concludes the proof of the theorem.