Introduction to Online Learning Algorithms

Yoav Freund

January 7, 2025

Outline

Class web site

- All of the class material is available from the github repository https://github.com/yoavfreund/2025-online-learning

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HW / Evaluation

- 5 HW assignments for 5*15 = 75 opints

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- A final for 25 points.

Example trace for Halving Algorithm

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.
outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	
outcome	

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expert6	0
expert7	1
expert8	1
alg.	1
outcome	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	
outcome	1	

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	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
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expert5	1	0
expert6	0	-
expert7	1	1
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alg.	1	0
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	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	
outcome	1	1	1	

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	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
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alg.	1	0	1	1
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	0

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	-
expert8	1	1	1	0	0
alg.	1	0	1	1	
outcome	1	1	1	0	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	-
expert8	1	1	1	0	0
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
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expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	-
expert8	1	1	1	0	0
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Mistake bound for Halving algorithm

- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).

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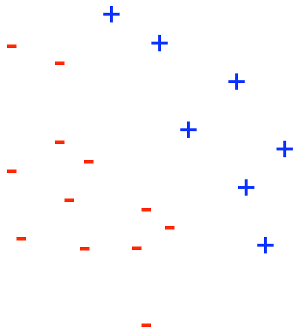
- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.
- No stochastic assumptions whatsoever.
- Proof is based on combining a lower and upper bounds on the number of perfect experts.

The Perceptron Problem

The Perceptron Problem



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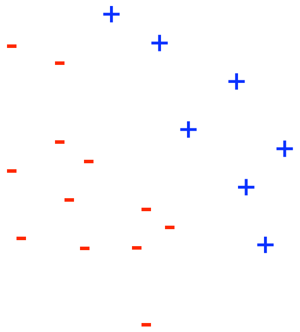

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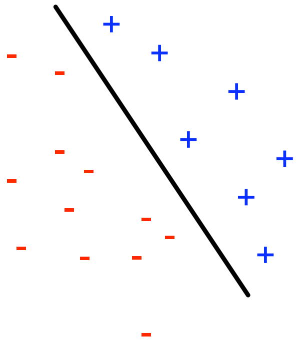

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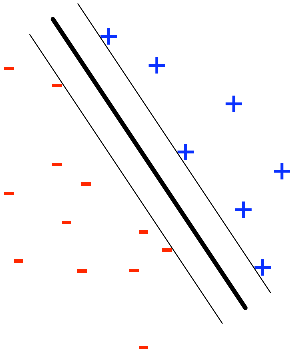

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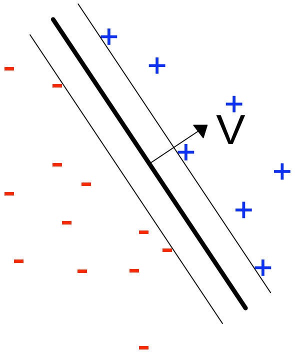

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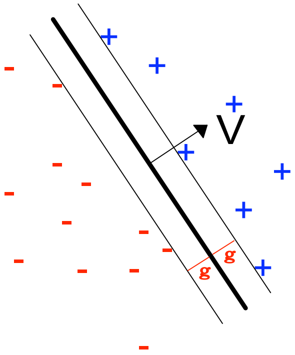

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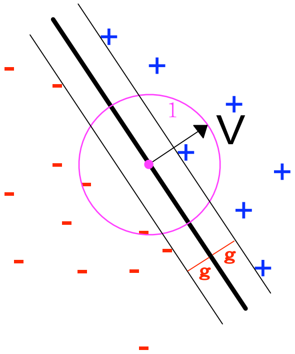

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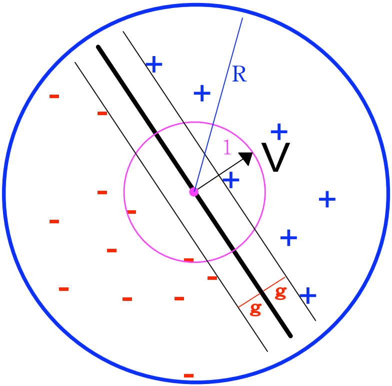

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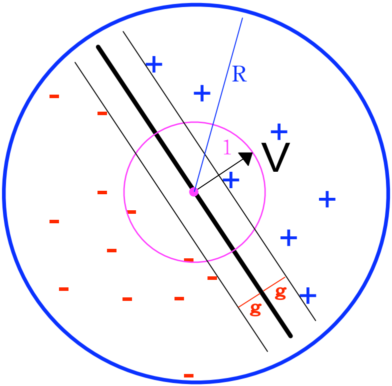

- $\|\vec{V}\| = 1$

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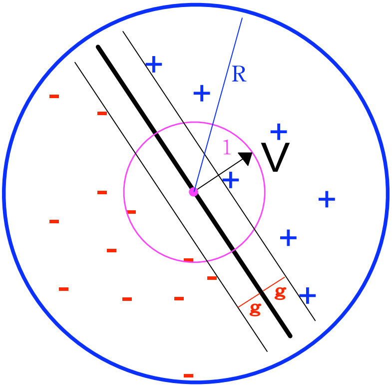

- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.

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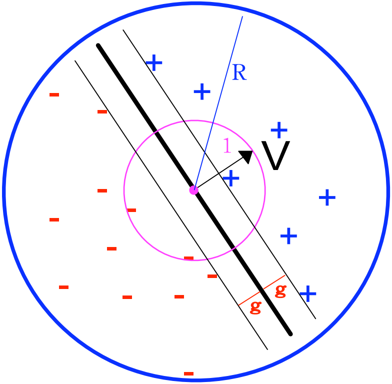

- $\|\vec{V}\| = 1$
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- $\forall \vec{X}, \;\; \|\vec{X}\| \le R$.

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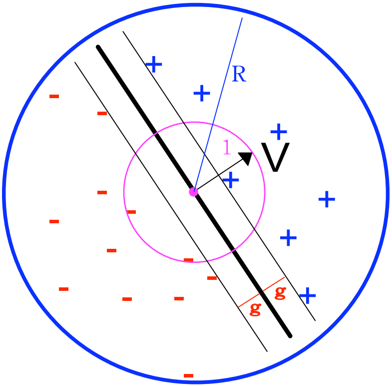

- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.
- $\forall \vec{X},\ \ \|\vec{X}\| \leq R$.
- $\forall(\vec{X}, y)$, $y(\vec{X} \cdot \vec{V}) \geq g$

The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.

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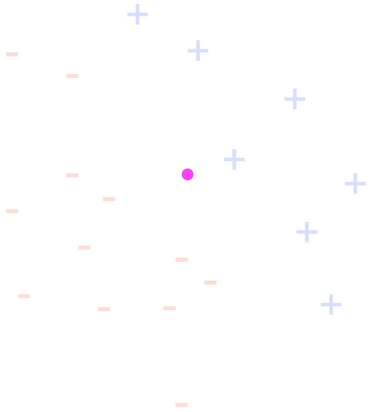
- An online algorithm. Examples presented one by one.
- start with $\vec{W}_0 = \vec{0}$.
- If mistake: $(\vec{W}_i \cdot \vec{X}_i) y_i \leq 0$
  - Update $\vec{W}_{i+1} = \vec{W}_i + y_i X_i$.

Example trace for the perceptron algorithm



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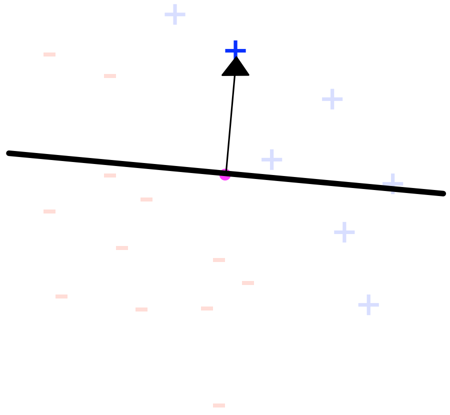

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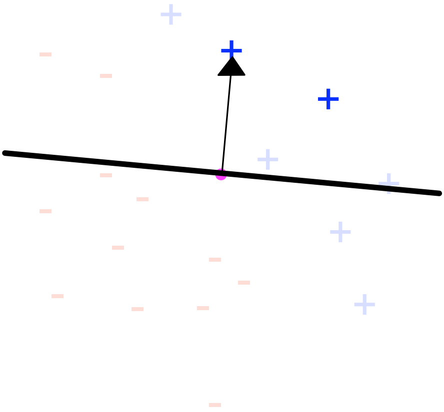

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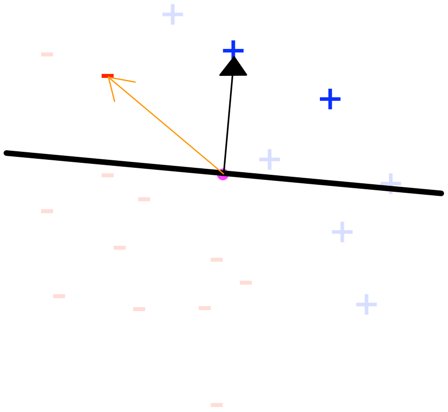

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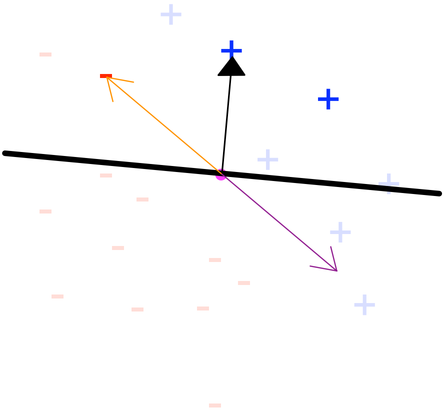

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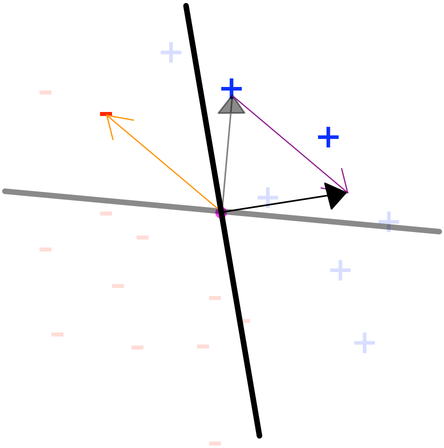

Bound on number of mistakes

- The number of mistakes that the perceptron algorithm can make is at most $\left(\frac{R}{g}\right)^2$.

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- Proof by combining upper and lower bounds on $\|\vec{W}\|$.

Pythagorian Lemma

If $(\vec{W}_i \cdot X_i)y < 0$ then

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If $(\vec{W}_i \cdot X_i)y < 0$ then

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i\vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$

Pythagorian Lemma

If $(\vec{W}_i \cdot X_i)y < 0$ then

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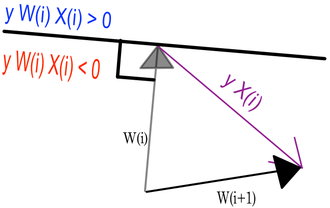

Upper bound on $\|\vec{W}_i\|$

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Proof by induction

- Claim: $\|\vec{W}_i\|^2 \leq iR^2$

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Proof by induction

- Claim: $\|\vec{W}_i\|^2 \leq iR^2$
- Base: $i = 0$, $\|\vec{W}_0\|^2 = 0$
- Induction step (assume for i and prove for $i + 1$):
  $\|\vec{W}_{i+1}\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$
  $\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2$

Lower bound on $\|\vec{W}_i\|$

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$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.

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We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ by induction over i

- Claim: $\vec{W}_i \cdot \vec{V} \geq ig$

Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.
Let i denote the number of mistakes made so far.

We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ by induction over i

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  $\geq ig + g = (i+1)g$

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Thus:

$$i \leq \left(\frac{R}{g}\right)^2$$

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- Impossible without additional constraints.

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- **Expected regret**: compare performance of algorithm to Regret $= \mathrm{E}_{Y^T} \left[(x_t - Y_t)^2\right] - \sigma^2$

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- **Goal:** sublinear regret $\lim_{T \to \infty} \frac{\text{Regret}_T}{T} = 0$

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- $x_{t+1} = \operatorname{argmin}_{x \in [0,1]} \sum_{i=1}^{t} (x - y_i)^2 = x_t^*$
- We will prove that the regret of this algorithm is upper bound by $2 + 2 \ln T$

Regret Bound

Theorem

Let $y_t \in [0,1]$ for $t = 1, \ldots T$ an arbitrary sequence of numbers. Let the algorithm output be $x_t = x^*_{t-1} = \frac{1}{t-1}\sum_{i=1}^{t-1} y_i$, then

$$Regret_T = \sum_{t=1}^{T}(x_t - y_t)^2 - \sum_{t=1}^{T}(x^*_T - y_t)^2 \leq 2\,(1 + \ln T)$$

Lemma

Let $x_1, x_2, \ldots$ be the squence of predictions produced by FTL. Then for all $u \in R$ (In particular, for $u = x_T^*$) :

$$\sum_{t=1}^{T} \left((x_t - y_t)^2 - (u - y_t)^2 \right) \leq \sum_{t=1}^{T} \left((x_t - y_t)^2 - (x_t^* - y_t)^2 \right)$$

Proof Sketch:

Subtract $\sum_{t=1}^{T} (x_t - y_t)^2$ from both sides to get an equivalent claim:

$$\sum_{t=1}^{T} (x_t^* - y_t)^2 \leq \sum_{t=1}^{T} (u - y_t)^2$$

The inequality is proven by induction on T.

Proof.

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- Base case ($T=1$): $(x_1^* - y_1)^2 = (y_1 - y_1)^2 = 0 \leq (u - y_1)^2$
- Induction hypothesis: $\sum_{t=1}^{T-1}(x_t^* - y_t)^2 \leq \sum_{t=1}^{T-1}(u - y_t)^2$
- Induction step:

$$\sum_{t=1}^{T-1}(x_t^* - y_t)^2 \leq \sum_{t=1}^{T-1}(x_{T-1}^* - y_t)^2 \leq \sum_{t=1}^{T-1}(x_T^* - y_t)^2$$

Adding $(x_T^* - y_T)^2$ to both sides gives:

$$\sum_{t=1}^{T}(x_t^* - y_t)^2 \leq \sum_{t=1}^{T}(x_T^* - y_t)^2 \leq \sum_{t=1}^{T}(u - y_t)^2$$

□

Proof of the theorem

First, note that in FTL we have:

$$x_t = x^*_{t-1} = \frac{1}{t-1}\sum_{i=1}^{t-1} y_i = \frac{t}{t-1}\cdot\left(\frac{1}{t}\sum_{i=1}^{t} y_i - \frac{y_t}{t}\right) = \frac{t}{t-1}\cdot\left(x^*_t - \frac{y_t}{t}\right)$$

Subtracting x^*_t from both sides, we get $x_t - x^*_t = \frac{x^*_t - y_t}{t-1}$. Then:

$$\begin{aligned}
\text{Regret}_T &= \sum_{t=1}^{T}(x_t - y_t)^2 - \sum_{t=1}^{T}(x^*_T - y_t)^2 \\
&\leq \sum_{t=1}^{T}(x_t - y_t)^2 - (x^*_t - y_t)^2 \qquad \text{(Lemma)} \\
&= \sum_{t=1}^{T}(x_t + x^*_t - 2y_t)(x_t - x^*_t) \leq \sum_{t=1}^{T}\frac{2}{t-1} \leq 2\,(1 + \ln T)
\end{aligned}$$