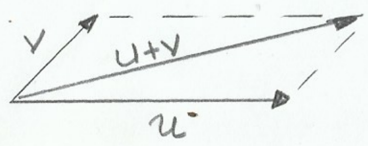
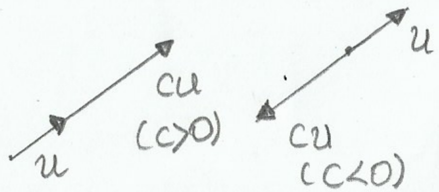
VECTOR SPACE R^n

The n dimensional space R^n is the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers.

$$(x,y) \in R^2, \quad (x,y,z) \in R^3, \quad \ldots, \quad (x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$$

* $u = (u_1, u_2, \cdots, u_n)$, $v = (v_1, v_2, \cdots, v_n) \Rightarrow u+v = (u_1+v_1, u_2+v_2, \cdots, u_n+v_n)$
* c : scalar $\Rightarrow cu = (cu_1, cu_2, \cdots, cu_n)$





VECTOR SPACE

Let V be a set of elements called vectors for which the operations of addition and multiplication by a scalar are defined. This means that if $u, v \in V$ and c is a scalar, then $u+v$, $cu \in V$ (closed under vector addition and multiplication by a scalar).

Then, if the following 8 conditions hold for $u, v, w \in V$ and scalars a, b, V is called a vector space

1. $u+v = v+u$ (commutative law)
2. $u+(v+w) = (u+v)+w$ (associative law)
3. $u+0 = 0+u = u$ (zero element)
4. $u+(-u) = (-u)+u = 0$ (inverse for addition)
5. $a(u+v) = au+av$ (distributive law)
6. $(a+b)u = au+bu$
7. $a(bu) = (ab)u$
8. $1.u = u$

<u>Example</u> $\mathcal{F}$: all real valued functions defined on $\mathbb{R}$.

$$\left.\begin{array}{l}(f+g)(x) = f(x)+g(x) \\ (cf)(x) = c(f(x))\end{array}\right\} \begin{array}{l}\text{8 conditions above are all satisfied} \\ \Rightarrow \text{a vector space}\end{array}$$

SUBSPACE

Let $\emptyset \neq W$ be a subset of the vector space V.
If W is also a vector space with the operations of addition and multiplication by a scalar as defined in V, then W is called a subspace of V.

THEOREM $\emptyset \neq W \subset V$ is a subspace of the vector space V iff the following conditions are all satisfied:

1- $u, v \in W \Rightarrow u+v \in W$

2- $u \in W$, c: scalar $\Rightarrow cu \in W$

Example W: set of all vectors $(x_1, x_2, x_3, x_4) \in R^2$ such that $x_1 \cdot x_4 = 0$

$u, v \in W \Rightarrow u = (u_1, u_2, u_3, u_4)$ with $u_1 \cdot u_4 = 0$
$\qquad v = (v_1, v_2, v_3, v_4)$ with $v_1 \cdot v_4 = 0$

$$u+v = (u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4) = (u_1+v_1, u_2+v_2, u_3+v_3, u_4+v_4)$$

$$(u_1+v_1)(u_4+v_4) = \underbrace{u_1 u_4}_{=0} + \underbrace{v_1 v_4}_{=0} + u_1 v_4 + v_1 u_4 = u_1 v_4 + v_1 u_4 \neq 0$$
(not always)

$\Rightarrow u+v \notin W \Rightarrow W$: not a subspace

Example: W: subset of R^n such that if $(x_1, x_2, \ldots, x_n) \in W$, then $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ where $a_1, a_2, \ldots, a_n$ are not all zero

* $x, y \in W \Rightarrow \left.\begin{array}{l} a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \\ a_1y_1 + a_2y_2 + \cdots + a_ny_n = 0 \end{array}\right\}$ $a_1, \ldots, a_n$: not all zero

$$x+y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$
$$a_1(x_1+y_1) + a_2(x_2+y_2) + \cdots + a_n(x_n+y_n) = \underbrace{a_1x_1 + \cdots + a_nx_n + a_1y_1 + \cdots + a_ny_n}_{=0}$$

Since $a_1, \ldots, a_n$ are not all zero, then $x+y \in W$

* $cx = (cx_1, cx_2, \ldots, cx_n)$
$$a_1(cx_1) + a_2(cx_2) + \cdots + a_n(cx_n) = c(a_1x_1 + \cdots + a_nx_n) = c \cdot 0 = 0$$
Since $a_1, \ldots, a_n$ are not all zero, then $cx \in W$

$\Rightarrow W$: subspace of R^n

THEOREM Let A be a $m\times n$ matrix. Then the solution set of the homogeneous linear system $AX=0$ is a subspace of R^n.

Proof: W: solution set of $AX=0$ (also called solution space)

$u, v \in W \Rightarrow Au = Av = 0$

$$\left.\begin{array}{l} * \ A(u+v) = Au + Av = 0+0 = 0 \Rightarrow u+v \in W \\ * \ A(cu) = c(Au) = c0 = 0 \Rightarrow cu \in W \end{array}\right\} W: \text{subspace}$$

Important Note The solution set of the nonhom. system $AX=b$ where $b\neq 0$ is never a subspace.

$u \in W \Rightarrow Au = b$ but $A(cu) = c(Au) = cb \neq b$ for $c\neq 1 \Rightarrow cu \notin W$

Example

$$\left.\begin{array}{r} x_1+3x_2-15x_3+7x_4=0 \\ x_1+4x_2-19x_3+10x_4=0 \\ 2x_1+5x_2-26x_3+11x_4=0 \end{array}\right\} \text{solution space ?}$$

→ free variables

$$\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -15 & 7 \\ 0 & 1 & -4 & 3 \\ 0 & -1 & 4 & -3 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_3 = t,\ x_4 = s \Rightarrow x_2 - 4x_3 + 3x_4 = 0 \Rightarrow x_2 = 4t - 3s$

$x_1 - 3x_3 - 2x_4 = 0 \Rightarrow x_1 = 3t + 2s$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3t+2s \\ 4t-3s \\ t \\ s \end{bmatrix} = t\underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{=u} + s\underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{=v} \Rightarrow X = tu + sv$$

LINEAR COMBINATION

If there exist scalars $c_1, c_2, \dots, c_k$ such that

$$w = c_1 v_1 + c_2 v_2 + \dots + c_k v_k,$$

then the vector w is called a linear combination of the vectors $v_1, v_2, \dots, v_k$.

Example Determine whether the vector $w = (2,5,8) \in \mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1,1,2)$ and $v_2 = (2,1,3)$.

$$w = c_1 v_1 + c_2 v_2 \Rightarrow (2,5,8) = (c_1, c_1, 2c_1) + (2c_2, c_2, 3c_2)$$

$$\left.\begin{aligned} c_1 + 2c_2 &= 2 \\ c_1 + c_2 &= 5 \\ 2c_1 + 3c_2 &= 8 \end{aligned}\right\} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 1 & 5 \\ 2 & 3 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -1 & 3 \\ 0 & -1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

The third row tells us that the system is inconsistent. Thus, such c_1 and c_2 do not exist. Therefore, w is not a linear combination of v_1 and v_2.

Example Determine whether the vector $w = (1,7,3) \in \mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1,1,2)$, $v_2 = (2,1,0)$ and $v_3 = (3,2,1)$.

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 7 \\ 2 & 0 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 6 \\ 0 & -4 & -5 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 13 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & -1 & -23 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 1 & 23 \end{array}\right] \end{array}$$

$\Rightarrow c_1 = -10,\ c_2 = -29,\ c_3 = 23$

$\Rightarrow w$ is a linear combination of v_1, v_2 and v_3 for specific valus of c_1, c_2 and c_3.

Example Determine if $w=(-7,7,11)$ is a linear combination of the vectors $v_1=(1,2,1)$, $v_2=(-4,-1,2)$ and $v_3=(-3,1,3)$.

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 6 & 6 & 18 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 \to \text{free var.} & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$c_3 = t \Rightarrow c_2 + c_3 = 3 \Rightarrow c_2 = 3-t \Rightarrow c_1 + c_3 = 5 \Rightarrow c_1 = 5-t$

$t=0 \Rightarrow c_1=5,\ c_2=3,\ c_3=0 \Rightarrow w = 5v_1 + 3v_2 + 0 \cdot v_3$

$t=1 \Rightarrow c_1=4,\ c_2=2,\ c_3=1 \Rightarrow w = 4v_1 + 2v_2 + 1 \cdot v_3$

Thus, w can be written as a linear combination of v_1, v_2 and v_3 in many different ways.

SPAN

Let $v_1, v_2, \dots, v_k$ are vectors in a vector space V. If every vector $v \in V$ is a linear combination of $v_1, v_2, \dots, v_k$, then we say that $v_1, v_2, \dots, v_k$ span the vector space V.

$$S = \{v_1, v_2, \dots, v_k\} \text{ : spanning set of } V$$

Example: $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$ and $i,j,k \in R^3$.
Every vector in R^3 can be written as a linear combination of i, j and k.

$$x = (x_1, x_2, x_3) = x_1 i + x_2 j + x_3 k$$

$\Rightarrow i, j, k$ span R^3.

THEOREM Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V. Then the set W of all linear combinations of $v_1, v_2, \ldots, v_k$ is a subspace of V.

Proof $u, v \in W \Rightarrow u = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$

$v = b_1 v_1 + b_2 v_2 + \cdots + b_k v_k$

$$u + v = (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \cdots + (a_k + b_k) v_k$$
$$= c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \Rightarrow u + v \in W$$

$$cu = (ca_1) v_1 + (ca_2) v_2 + \cdots + (ca_k) v_k = d_1 v_1 + d_2 v_2 + \cdots + d_k v_k$$
$\Rightarrow cu \in W$

$\Rightarrow W$: subspace of V

$\Rightarrow W = \text{span } S = \text{span } \{v_1, v_2, \ldots, v_k\}$

LINEAR INDEPENDENCE

If $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$, then the vectors $v_1, v_2, \ldots, v_k$ in a vector space V are called to be linearly independent
($c_1, c_2, \ldots, c_k$: not all zero $\Rightarrow v_1, v_2, \ldots, v_k$: Lin. dependent)

Example Let's determine if the vectors $v_1 = (1,2,2,0)$, $v_2 = (0,1,1,1)$, $v_3 = (2,1,0,1) \in \mathbb{R}^4$ are linearly independent

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow (c_1, 2c_1, 2c_1, 0) + (0, c_2, c_2, c_2) + (2c_3, c_3, 0, c_3) = (0,0,0,0)$$

$$c_1 + 2c_3 = 0$$
$$2c_1 + c_2 + c_3 = 0$$
$$2c_1 + c_2 = 0$$
$$c_2 + c_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\Rightarrow c_3 = c_2 = c_1 = 0 \Rightarrow v_1, v_2, v_3$: Lin. independent

PROPERTIES OF LINEAR INDEPENCE

1. Any set of more than n vectors in R^n is Linearly dependent.
2. $v_1, v_2, \dots, v_n$ in R^n are Linearly independent if and only if the $n \times n$ matrix

$$A = [v_1 \; v_2 \; \cdots \; v_n]$$

having them as its column vectors has nonzero determinant.

3. Let $k < n$. $v_1, v_2, \dots, v_k$ in R^n are Linearly independent if and only if some $k \times k$ submatrix of A has nonzero determinant.

Example

1- $v_1 = (1,2,3)$, $v_2 = (1,0,4)$, $v_3 = (0,2,-1)$, $v_4 = (7,6,1) \in R^3$ are Linearly dependent $(k = 4 > n = 3)$

$(v_1 - v_2 - v_3 + 0 v_4 = 0)$

2- $v_1 = (1,2,3)$, $v_3 = (0,2,-1)$, $v_4 = (7,6,1) \in R^3$

$$\begin{vmatrix} 1 & 0 & 7 \\ 2 & 2 & 6 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 7 \\ 0 & 2 & -8 \\ 0 & -1 & -20 \end{vmatrix} = 1(-40-8) = -48 \neq 0 \Rightarrow \text{Lin. ind.}$$

3- $v_1 = (9,1,3)$, $v_2 = (3,0,1) \in R^3$

$$\begin{bmatrix} 9 & 3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 9 & 3 \\ 3 & 1 \end{vmatrix} = 9 - 9 = 0 \Rightarrow \text{Lin. dep.}$$

BASIS

A finite set S of vectors in a vector space V is called a basis for V if

* the vectors in S are linearly independent and
* the vectors in S span V.

Example $e_1 = (1,0,0,\dots,0)$, $e_2 = (0,1,0,\dots,0), \dots, e_n = (0,0,0,\dots,1)$.
$e_1, e_2, \dots, e_n$ are Lin. independent.
If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

$\Rightarrow \{e_1, e_2, \dots, e_n\}$ spans $\mathbb{R}^n \Rightarrow$ standard basis

Example $v_1 = (1,-1,-2,-3)$, $v_2 = (1,-1,2,3)$, $v_3 = (1,-1,-3,-2)$,
$v_4 = (0,3,-1,2)$.

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 3 \\ -2 & 2 & -3 & -1 \\ -3 & 3 & -2 & 2 \end{vmatrix} = 30 \neq 0 \Rightarrow \{v_1, v_2, v_3, v_4\} : \text{Lin. indep}$$

$\Downarrow$

a basis for $\mathbb{R}^4$

THEOREM: Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis for the vector space V. Then any set of more than n vectors in V is linearly dependent

THEOREM: Any two bases for a vector space consist of the same number of vectors.

$\mathbb{R}^n \Rightarrow \dim \mathbb{R}^n = n$

$\Rightarrow$ standard basis $\{e_1, e_2, \dots, e_n\}$

$\Rightarrow$ any basis of $\mathbb{R}^n$ has n vectors.

THEOREM Let V be an n-dimensional vector space and let S be a subset of V. Then

1- If S is lin. independent and consists of n vectors, then S is a basis for V.

2- If S spans V and consists of n vectors, then S is a basis for V.

3- If S is lin. independent, then S is contained in a basis for V.

4- If S spans V, then S contains a basis for V.

HOW TO FIND A BASIS FOR THE SOLUTION SPACE

Basis for the solution space of $AX=0$ (A: $n\times n$ matrix)

1. Reduce the coefficient matrix A to echelon form

2. r: leading entries $\Rightarrow k=n-r$: free variables
$\Rightarrow k=0 \Rightarrow W=\{0\}$.

3. $k\neq 0 \Rightarrow k$: parameters
Find $V_1, V_2, \dots, V_k$ by setting each parameter to be 1 and the rest to be 0.
$\Rightarrow W=\{V_1, V_2, \dots, V_k\}$: basis for W.

Example

$$\left.\begin{aligned} 3x_1+6x_2-x_3-5x_4+5x_5&=0\\ 2x_1+4x_2-x_3-3x_4+2x_5&=0\\ 3x_1+6x_2-2x_3-4x_4+x_5&=0 \end{aligned}\right\}$$

Basis for the solution space?

$$\begin{bmatrix} 3 & 6 & -1 & -5 & 5\\ 2 & 4 & -1 & -3 & 2\\ 3 & 6 & -2 & -4 & 1 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\\ \begin{bmatrix} 1 & 2 & 0 & -2 & 3\\ 0 & 0 & 1 & -1 & 4\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

x_2, x_4, x_5 : free var.

$x_2=t$

$x_4=s$

$x_5=r$

$x_3-x_4+4x_5=0 \Rightarrow x_3=s-4r$

$x_1+2x_2-2x_4+3x_5=0 \Rightarrow x_1=-2t+2s-3r$

$t=1,\ s=r=0 \Rightarrow v_1=(-2,1,0,0,0)$

$s=1,\ t=r=0 \Rightarrow v_2=(2,0,1,1,0)$ $\qquad W=\{v_1,v_2,v_3\}$

$r=1,\ s=t=0 \Rightarrow v_3=(-3,0,-4,0,1)$

ROW SPACE

$$AX=0 \Rightarrow A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$\left.\begin{array}{l} r_1=(a_{11},a_{12},\dots,a_{1n}) \\ r_2=(a_{21},a_{22},\dots,a_{2n}) \\ \vdots \\ r_m=(a_{m1},a_{m2},\dots,a_{mn}) \end{array}\right\} \quad r_i=\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$$

The subspace of $\mathbb{R}^n$ spanned by m row vectors $r_1,r_2,\dots,r_m$ is called the row space of A (Row (A))

The dimension of Row (A) is called the row rank of A.

PROPERTIES OF ROW SPACES

1) The nonzero row vectors of an echelon matrix are linearly independent and thus form a basis for its row space.

2) If the matrices A and B are equivalent, then they have the same row space.

↓

To find a basis for row (A), find the echelon matrix E of A. Then the nonzero row vectors of E form a basis for row (A).

Example $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$ → basis for the row space of A?

$r_1 = (1,2,1,3,2)$, $r_2 = (3,4,9,0,7)$

$r_3 = (2,3,5,1,8)$, $r_4 = (2,2,8,-3,5)$

- row (A) = span $\{r_1, r_2, r_3, r_4\}$

$$A \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} V_1 = (1,2,1,3,2) \\ V_2 = (0,1,-3,5,-4) \\ V_3 = (0,0,0,1,-7) \end{array}$$

$\{V_1, V_2, V_3\}$: a basis for Row (A) ⇒ rank Row (A) = 3

COLUMN SPACE

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \Rightarrow c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The subspace of R^m spanned by the n column vectors $c_1, c_2, \ldots, c_n$ is called the colum space of A (COL(A))

The dimension of COL(A) is called the column rank

Example $c_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$, $c_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 2 \end{bmatrix}$, $c_3 = \begin{bmatrix} 1 \\ 9 \\ 5 \\ 8 \end{bmatrix}$, $c_4 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -3 \end{bmatrix}$, $c_5 = \begin{bmatrix} 2 \\ 7 \\ 8 \\ 5 \end{bmatrix}$

COL(A) = span $\{c_1, c_2, c_3, c_4, c_5\}$

$\{c_1, c_2, c_4\}$: a basis for COL (A) ⇒ rank COL(A) = 3

↳ colums of A which corresponds to the colums of Leading entries of E

↓ be careful

Rank (A) = Rank Row (A) = Rank CoL (A)

→ AX=0

Rank (A) + dim (Null (A)) = n