EHB 211E
Basics of Electrical Circuits

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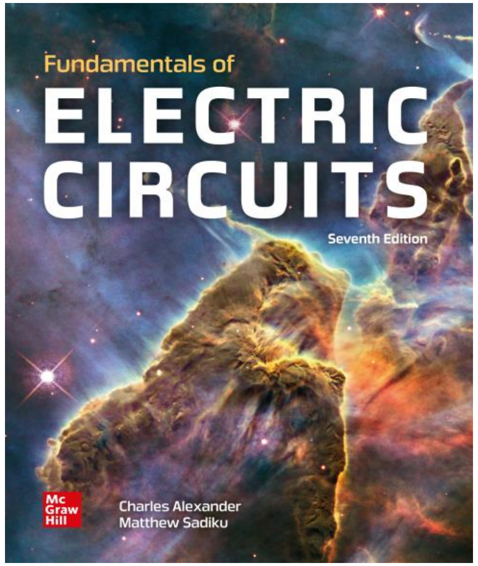
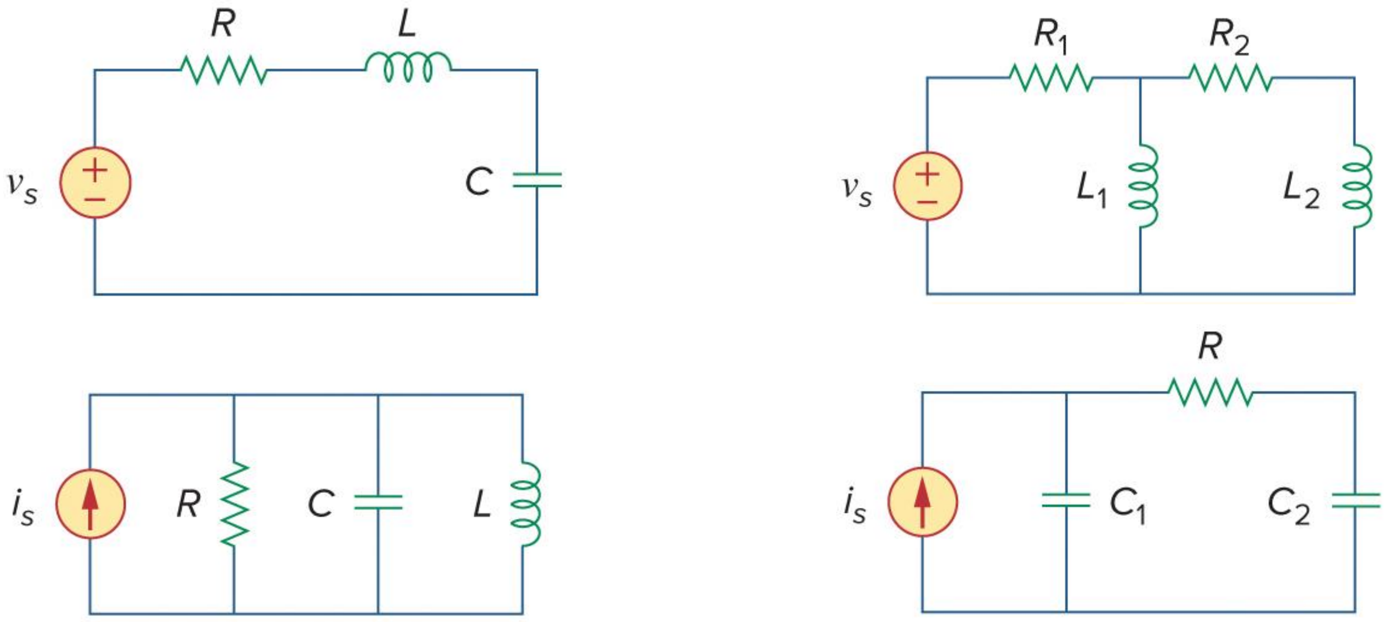
Second-Order Circuits



Fundamentals of Electric Circuits by C. Alexander and M. Sadiku, Seventh Edition (McGraw-Hill Education)

Introduction

- Circuit with single storage element such as capacitor or inductor: first-order circuit.
  - response is first-order differential equation
- Circuit containing two storage elements together with capacitor and inductor: second-order circuit.
  - Response is second order differential equation.
- Typical example of second-order circuits: RLC circuits
  - Two storage elements of different type
  - Two storage elements of same type (cannot represented by an equivalent single element)



Finding Initial and Final Values

- To solve second order differential equation of a second-order circuit, the following parameters must be found:

$$v(0)\ ,\ \ i(0)\ \ ,\ \ \frac{dv(0)}{dt}\ ,\ \frac{di(0)}{dt}\ \ ,\ v(\infty)\ \ ,\ i(\infty)$$

- Recall:
  - Capacitor voltage is continuous

$$v(0^+) = v(0^-)$$

  - Inductor current is continuous

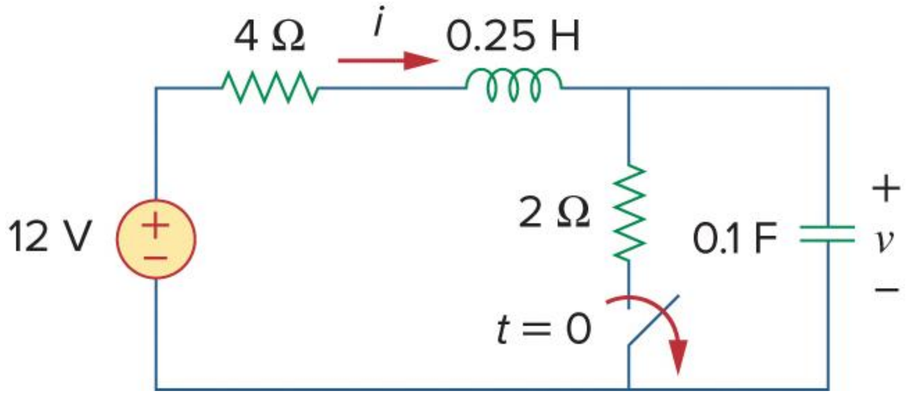
$$i(0^+) = i(0^-)$$

0^-: time just before switching event
0^+: time just after switching event

Example 1

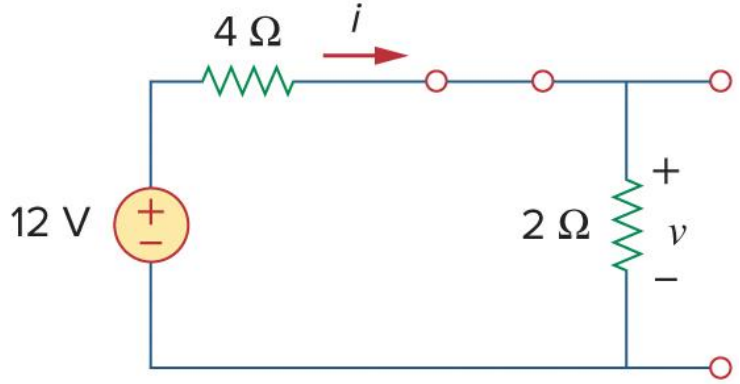
The switch in the circuit shown below has been closed for a long time. It is open at $t = 0$. Find a-) $i(0^+)$, $v(0^+)$, b-) $di(0^+)/dt$, $dv(0^+)/dt$, c-) $i(\infty)$, $v(\infty)$.

Solution:



- At $t < 0$, the switch is closed.
- At $t = 0$, the circuit has reached dc steady-state.
- Under dc condition, the inductor acts as short circuit and the capacitor acts as open circuit.

a-) At $t < 0$,



$$i(0^-) = \frac{12}{4+2} = 2\text{ A}, \qquad v(0^-) = 2i(0^-) = 4\text{ V}$$

Inductor current and capacitor voltage cannot change suddenly,

$$i(0^+) = i(0^-) = 2\text{ A}, \qquad v(0^+) = v(0^-) = 4\text{ V}$$

Solution

b-) At $t = 0^+$, the switch is open

$$i_C(0^+) = i(0^+) = 2\ \text{A}$$

Since $C\,dv/dt = i_C$, $dv/dt = i_C/C$,

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\ \text{V/s}$$

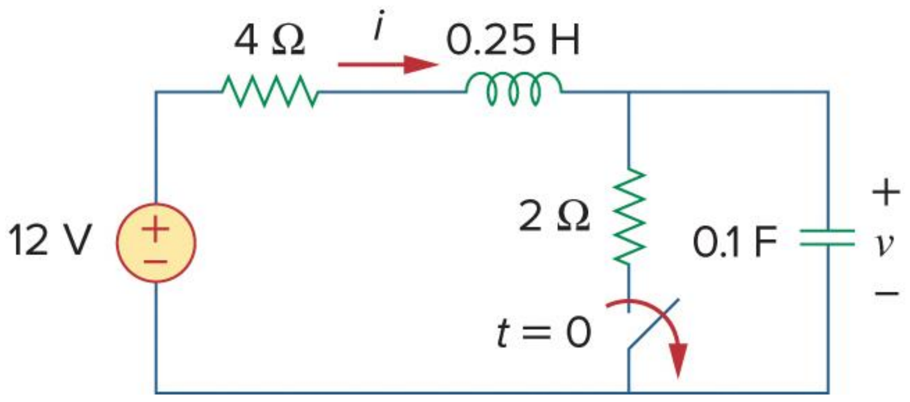
Similarly, since $L\,di/dt = v_L$, $di/dt = v_L/L$.

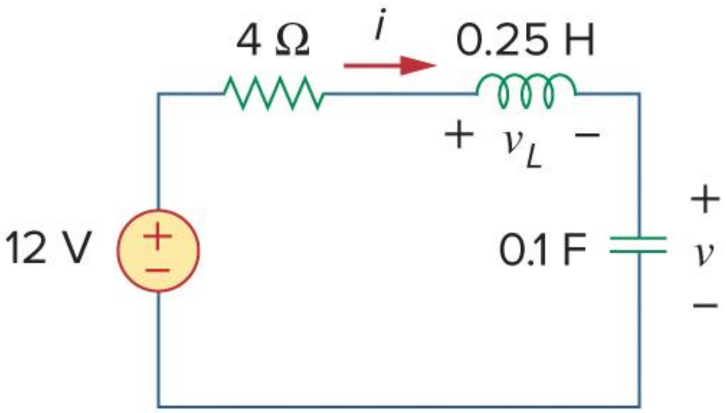
Obtain v_L by applying KVL (going clockwise):

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

$$v_L(0^+) = 12 - 8 - 4 = 0$$

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\ \text{A/s}$$



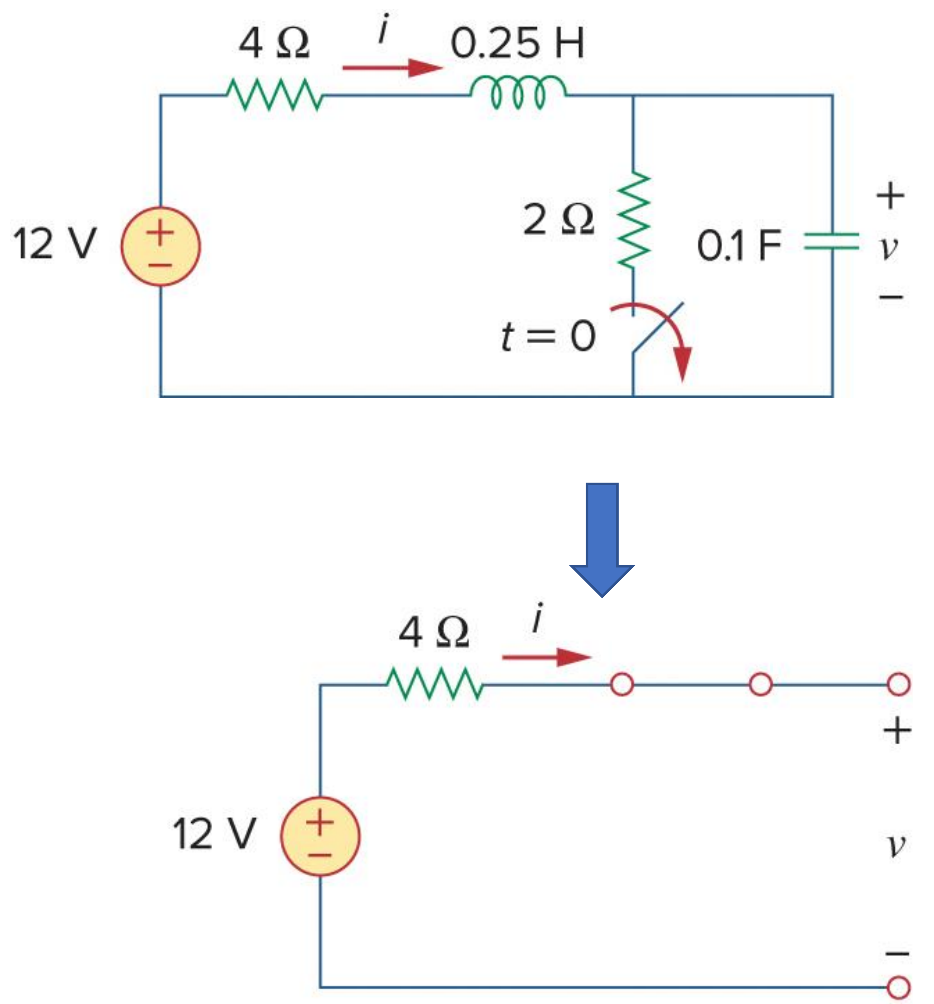


Solution

c-) At $t > 0$: as $t \to \infty$, the circuit reaches steady-state again. Under dc condition, inductor acts as short circuit and capacitor acts as open circuit.

$i(\infty) = 0\ A$ ⟶ No current flows

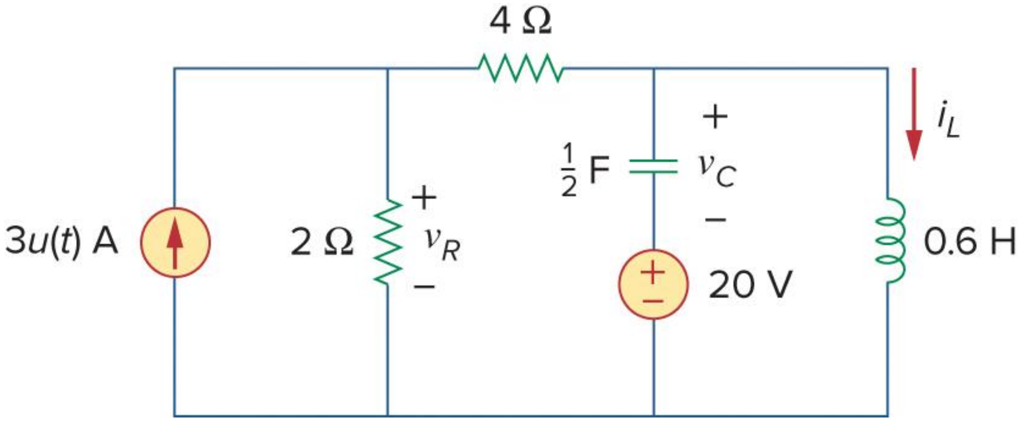
$v(\infty) = 12\ V$ ⟶ Same as applied voltage as they are parallel.



Example 2

In the circuit shown below, calculate a-) $i_L(0^+), v_C(0^+), v_R(0^+)$, b-) $\mathrm{d}i_L(0^+)/dt$, $\mathrm{d}v_C(0^+)/dt$, $\mathrm{d}v_R(0^+)/dt$, c-)$i_L(\infty), v_C(\infty), v_R(\infty)$.

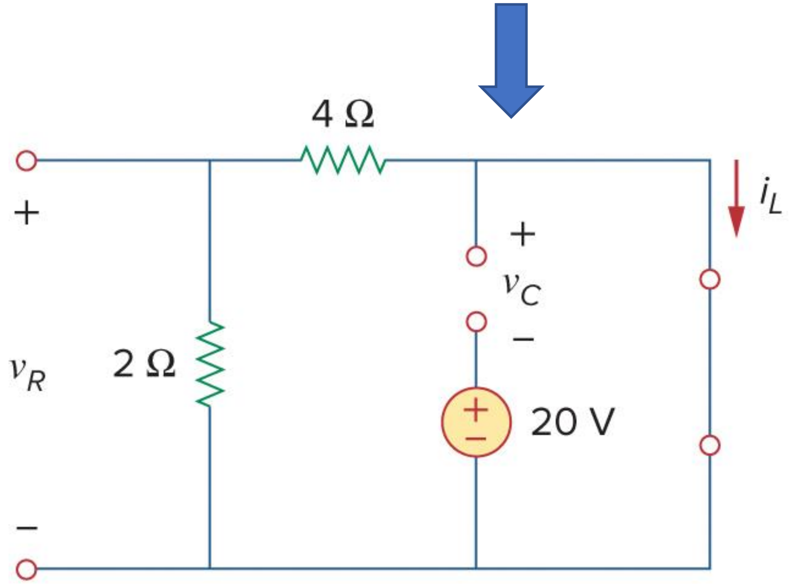
Solution:



a-) At $t < 0$, $3\mathrm{u}(t) = 0$ since $\mathrm{u}(t) = 0$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

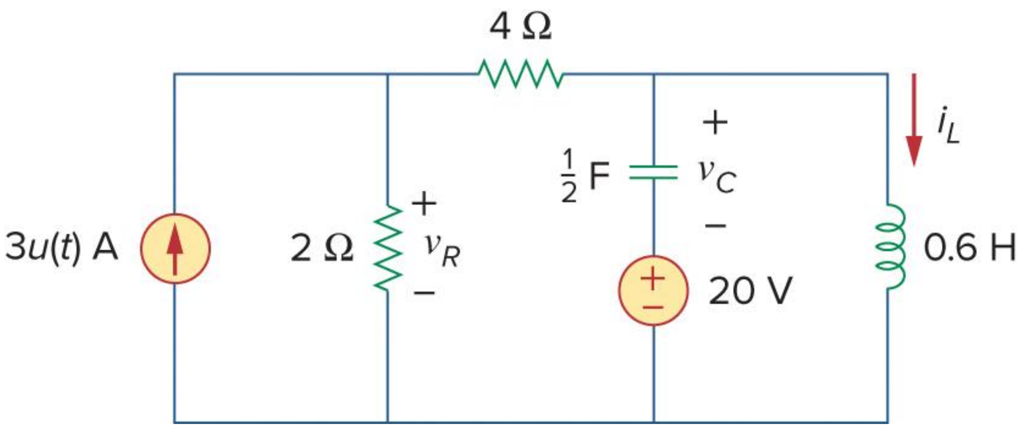
- At $t = 0^-$, the circuit reaches steady-state.
- The inductor acts as short circuit and the capacitor acts as open circuit.



$$i_L(0^-) = 0, \qquad v_R(0^-) = 0, \qquad v_C(0^-) = -20 \text{ V}$$

Solution

- At $t > 0$, $3\mathrm{u}(t) = 3$ since $\mathrm{u}(t) = 1$



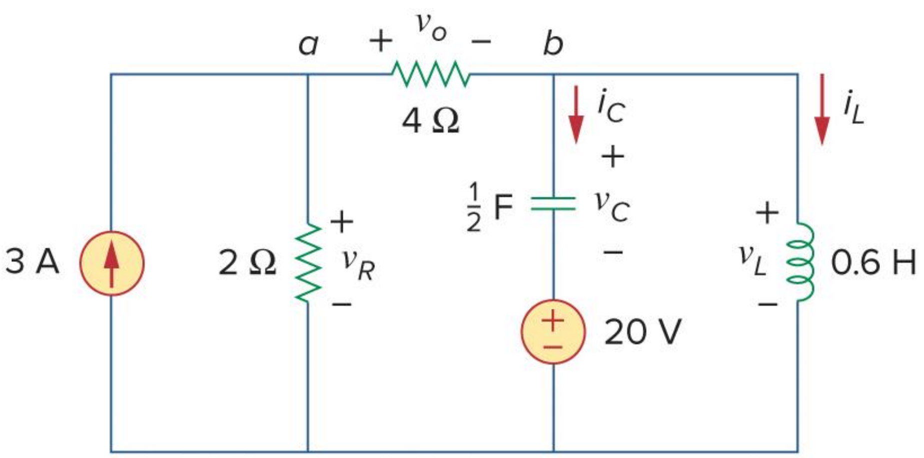
$$i_L(0^+) = i_L(0^-) = 0, \qquad v_C(0^+) = v_C(0^-) = -20 \text{ V}$$

- Apply KCL at node a:

$$\sum i_{in} = \sum i_{out}$$

$$3 = \frac{v_R(0^+)}{2} + \frac{v_o(0^+)}{4}$$

- Apply KVL to the middle mesh: $\sum_{m=1}^{n} v_m = 0$



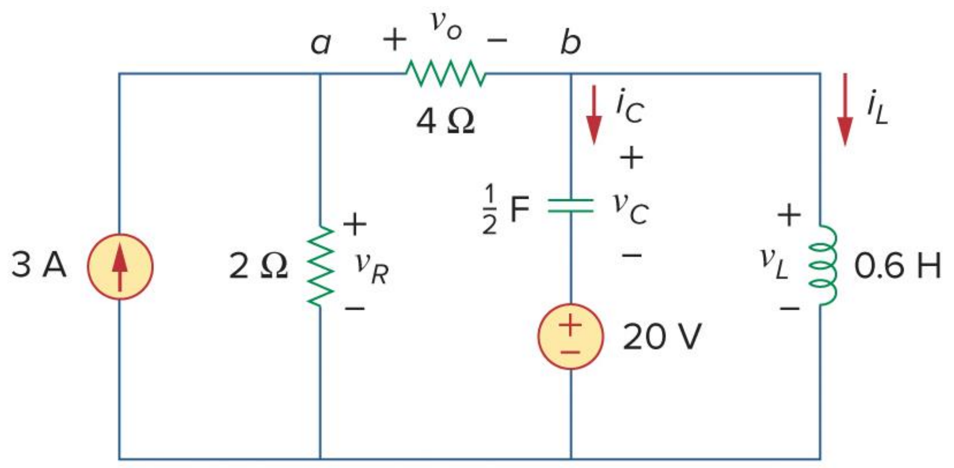
$$-v_R(0^+) + v_o(0^+) + v_C(0^+) + 20 = 0$$

$$v_C(0^+) = -20 \text{ V} \qquad v_R(0^+) = v_o(0^+)$$

$$v_R(0^+) = v_o(0^+) = 4 \text{ V}$$

Solution

b-) Since $L\, di_L/dt = v_L$, $\dfrac{di_L(0^+)}{dt} = \dfrac{v_L(0^+)}{L}$



Apply KVL to the right mesh:

$$v_L(0^+) = v_C(0^+) + 20 \qquad v_C(0^+) = -20 \text{ V}$$

$$\frac{di_L(0^+)}{dt} = 0$$

since $C\, dv_C/dt = i_C$, then $dv_C/dt = i_C/C$.

Apply KCL at node b: $\dfrac{v_o(0^+)}{4} = i_C(0^+) + i_L(0^+)$

Since $v_o(0^+) = 4$ and $i_L(0^+) = 0$, $i_C(0^+) = 4/4 = 1$ A.

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{1}{0.5} = 2 \text{ V/s}$$

Apply KCL at node a:

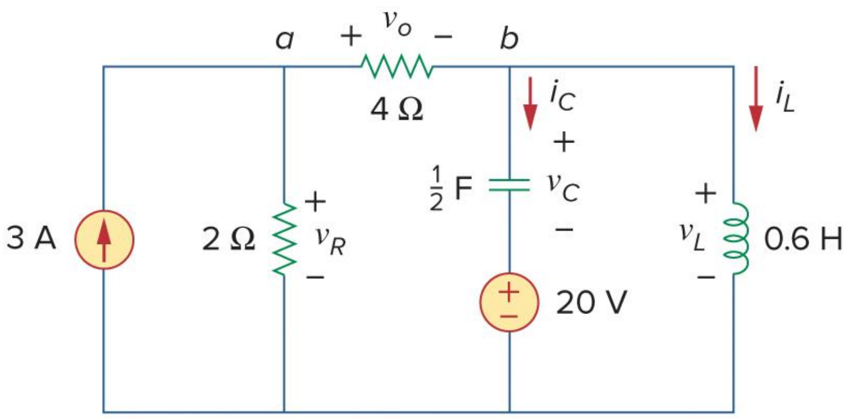
$$3 = \frac{v_R}{2} + \frac{v_o}{4}$$

Take derivative of both sides

$$0 = 2\frac{dv_R(0^+)}{dt} + \frac{dv_o(0^+)}{dt}$$

Solution

Apply KVL to the middle mesh:



$$-v_R + v_C + 20 + v_o = 0$$ Take derivative of both sides

$$-\frac{dv_R(0^+)}{dt} + \frac{dv_C(0^+)}{dt} + \frac{dv_o(0^+)}{dt} = 0 \qquad dv_C(0^+)/dt = 2$$

$$\frac{dv_R(0^+)}{dt} = 2 + \frac{dv_o(0^+)}{dt} \qquad \frac{dv_R(0^+)}{dt} = \frac{2}{3}\text{ V/s}$$

c-) As $t \rightarrow \infty$ ($t = \infty$), the circuit reaches steady-state. (inductor: short circuit and capacitor: open circuit).

Current division

$$i_L(\infty) = \frac{2}{2+4}3\text{ A} = 1\text{ A} \qquad v_R(\infty) = \frac{4}{2+4}3\text{ A} \times 2 = 4\text{ V}, \qquad v_C(\infty) = -20\text{ V}$$

Source-Free Series RLC Circuit

- The circuit is being excited by the energy initially stored in the capacitor and inductor.

  V_0 = initial capacitor voltage I_0 = initial inductor current

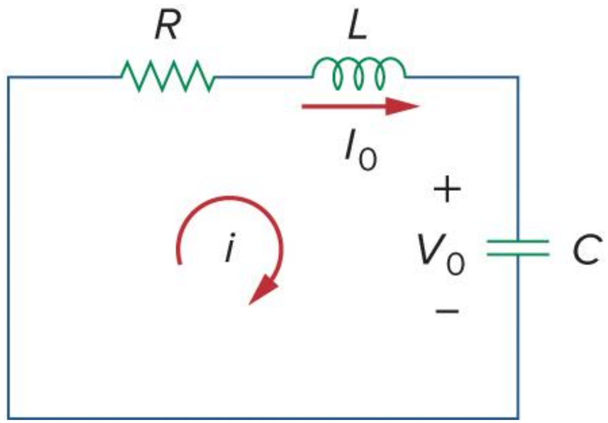
- At $t = 0$, $v(0) = V_0$ and $i(0) = I_0$
- At $t > 0$, excitation of the circuit is due to the energy stored in the capacitor and the inductor.
- Apply KVL: $\sum_{m=1}^{n} v_m = 0$

$$Ri + L\frac{di}{dt} + \underbrace{\boxed{\frac{1}{C}\int_0^t i(t)dt}}_{v(t)} = 0 \longrightarrow$$

To eliminate the integral, differentiate w.r.t t

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0 \Rightarrow \underbrace{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0}$$

Second-order differential equation. This is why RLC circuit is called the second-order circuit



Source-Free Series RLC Circuit

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

- To solve second-order differential equation, initial conditions should be obtained:
  - Initial value of i
  - Initial value of v
  - Initial value of the first derivative of i

Using KVL: $Ri + L\frac{di}{dt} + \boxed{\frac{1}{C}\int_0^t i(t)dt} = 0$ $\quad Ri(0) + L\frac{di(0)}{dt} + V_0 = 0 \quad i(0) = I_0$

$$v(0) = V_0$$

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

- With these initial conditions, the 2nd order differential equation can be solved.

Source-Free Series RLC Circuit

- Based on our experience from previous lecture, 1st order circuit suggests that the solution is of exponential form.

$i = Ae^{st}$ A and S are constants to be determined

- Substitute above current equation into 2nd order differential equation:

$$\frac{d^2(Ae^{st})}{dt^2} + \frac{R}{L}\frac{d(Ae^{st})}{dt} + \frac{Ae^{st}}{LC} = 0 \Rightarrow As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \Rightarrow$$ Since $i = Ae^{st}$, Ae^{st} cannot be zero

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \longrightarrow$$ Characteristic equation of the 2nd order differential equation

Source-Free Series RLC Circuit

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Recall:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\left.\begin{aligned} s_1 &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \\ s_2 &= -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \end{aligned}\right\} \text{Two roots}$$

$$\left.\begin{aligned} s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}\right\}$$

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

Ratio of $\frac{\alpha}{\omega_0}$: damping ratio

- s_1 and s_2: natural frequency, measured in nepers per sec (Np/s)
- α: neper frequency or damping factor, measured in nepers per sec (Np/s)
- ω_0: resonant frequency or undamped natural frequency (rad/sec)

Damping: decrease in amplitude of oscillation

Source-Free Series RLC Circuit

- Characteristic equation can be written as:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

- Two possible solution due to two roots (s_1 & s_2)

$$i_1 = Ae^{s_1 t} \qquad\qquad i_2 = Ae^{s_2 t}$$

- Complete or total solution of 2nd order differential equation is linear combination of i_1 & i_2

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Constant will be determined from the initial values $i(0)$ and $\frac{di(0)}{dt}$

Source-Free Series RLC Circuit

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

- Based on these equation, following three types of solutions:
  1. If $\alpha > \omega_0$, we have the overdamped case
  2. If $\alpha = \omega_0$, we have the critically damped case
  3. If $\alpha < \omega_0$, we have the underdamped case.

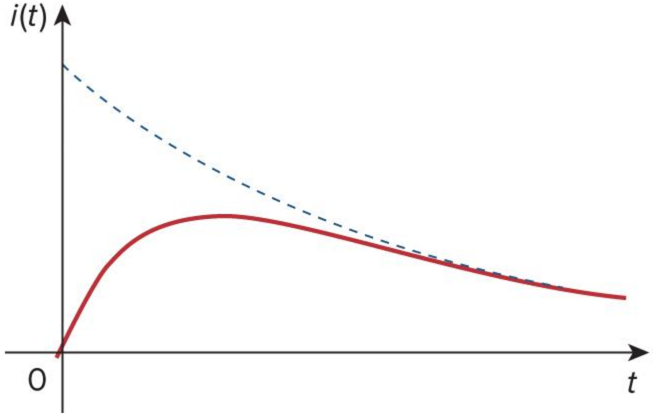
Source-Free Series RLC Circuit

➢ Overdamped case ($\alpha > \omega_0$):

- Both roots (s_1 and s_2) are negative and real.
- The response is:

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \longrightarrow$$

As t increases ($t \to \infty$), $i(t)$ decays and approaches zero



System returns to equilibrium position very slowly without any oscillation

➢ Critically damped case ($\alpha = \omega_0$):

- $s_1 = s_2 = -\alpha$ (Both roots are equal)

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

Where $A_3 = A_1 + A_2$

- This cannot be solution as two initial conditions cannot be satisfied with the single constant A_3. This means our assumption of an exponential solution is incorrect for the critical case.

Source-Free Series RLC Circuit

$\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\alpha = \omega_0$ for critically damped case

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \Rightarrow \frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0 \Rightarrow \frac{d^2i}{dt^2} + \alpha\frac{di}{dt} + \alpha\frac{di}{dt} + \alpha^2 i = 0$$

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\underbrace{\left(\frac{di}{dt} + \alpha i\right)}_{f} = 0$$

Let $\frac{di}{dt} + \alpha i = f$

$\frac{df}{dt} + \alpha f = 0 \longrightarrow$ 1st order differential equation

Solution of 1st order differential equation $f = A_1 e^{-\alpha t}$ where A_1 is constant

Product rule

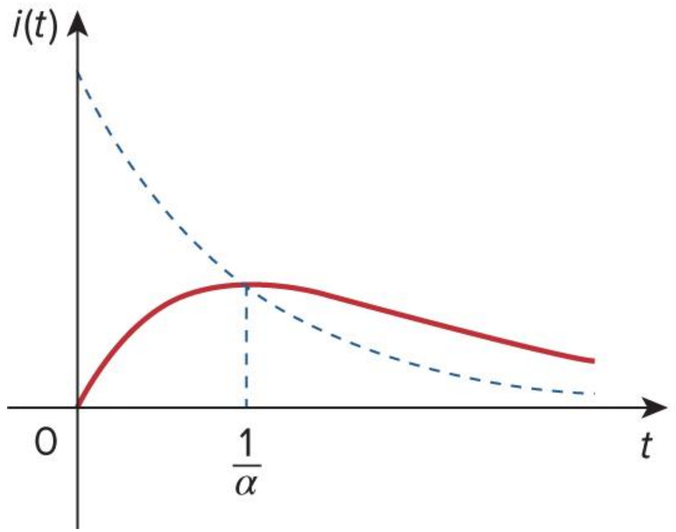
$$\frac{df}{dt} + \alpha f = 0 \Rightarrow \frac{di}{dt} + \alpha i = A_1 e^{-\alpha t} \Rightarrow e^{\alpha t}\frac{di}{dt} + (e^{\alpha t}\alpha i) = A_1 \Rightarrow \frac{d}{dt}(e^{\alpha t} i) = A_1$$

Source-Free Series RLC Circuit

$$\frac{d}{dt}(e^{\alpha t}i) = A_1$$ Integrate both sides $\Rightarrow$ $$e^{\alpha t}i = A_1 t + A_2$$ $\Rightarrow$ $$i = (A_1 t + A_2)e^{-\alpha t}$$

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}}$$ $\longrightarrow$ The natural response of the critically damped circuit

System returns to its equilibrium position in the shortest possible time without any oscillation



It reaches its max value of $\frac{e^{-1}}{\alpha}$ at $t = \frac{1}{\alpha}$ and then decays all the way zero.

Source-Free Series RLC Circuit

- Underdamped case ($\alpha < \omega_0$):
- Roots may be written as:

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} == -\alpha - j\omega_d$$

where j $= \sqrt{-1}$ and $\omega_d = \sqrt{(\omega_0^2 - \alpha^2)}$

Damping frequency or damped natural frequency

- Natural response is given by:

$$i(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \Rightarrow i(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$$

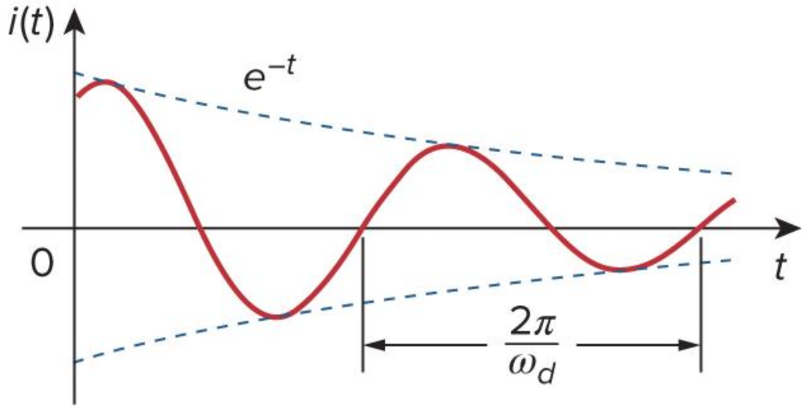
- Using Euler's identities:

$$e^{j\theta} = cos\theta + jsin\theta \qquad e^{-j\theta} = cos\theta - jsin\theta$$

Source-Free Series RLC Circuit

$$i(t) = e^{-\alpha t}(B_1 cos\omega_d t + B_2 sin\omega_d t)$$

- Natural response is exponentially damped and oscillatory due to the presence of sin and cos



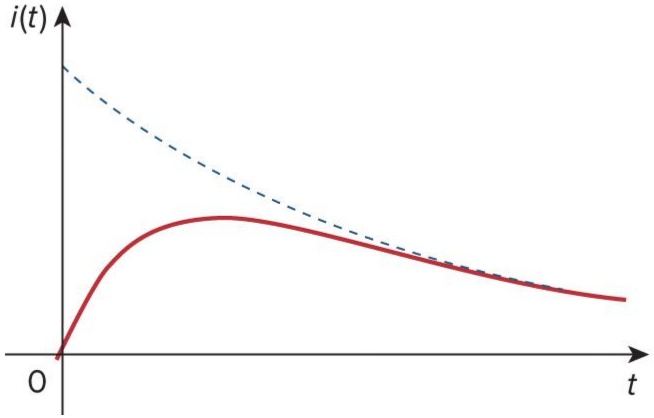
- Once the current $i(t)$ is found for RLC series circuit, other quantities can easily be found.
  - Resistor voltage as $v(t) = Ri(t)$
  - Inductor voltage as $v(t) = L\frac{di(t)}{dt}$

Source-Free Series RLC Circuit

- Summarize:

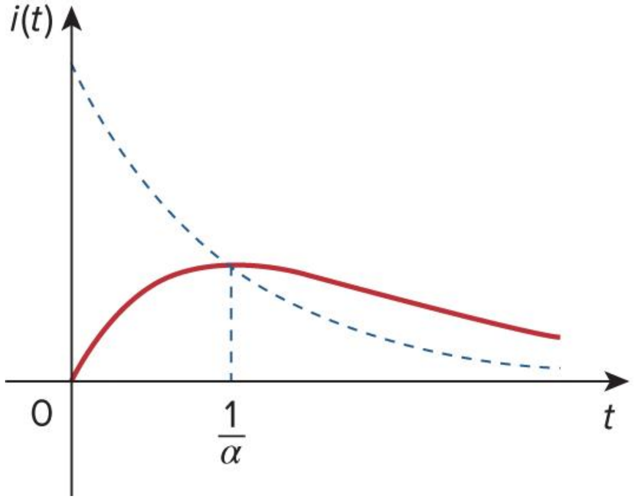
Overdamped case

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



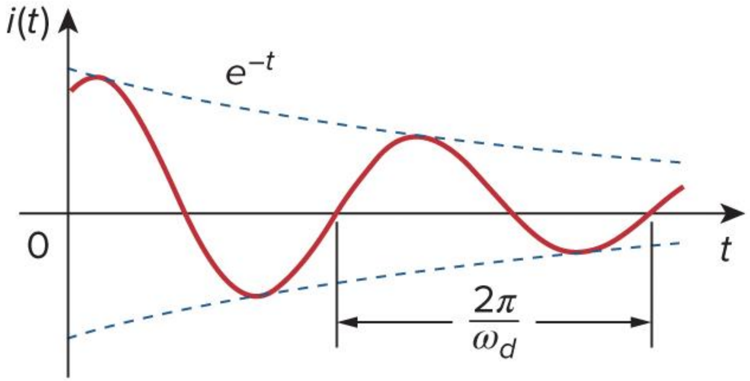
Critically damped case

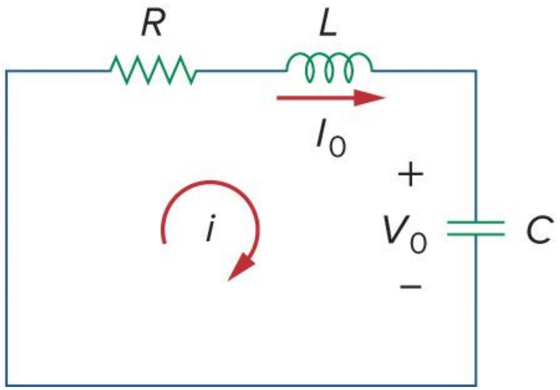
$$i(t) = (A_2 + A_1 t)e^{-\alpha t}$$



Underdamped case

$$i(t) = e^{-\alpha t}(B_1 cos\omega_d t + B_2 sin\omega_d t)$$



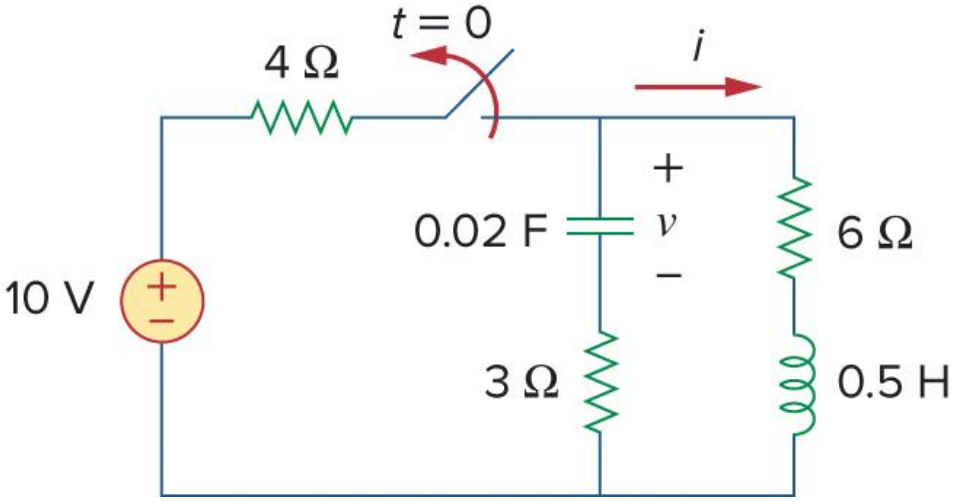


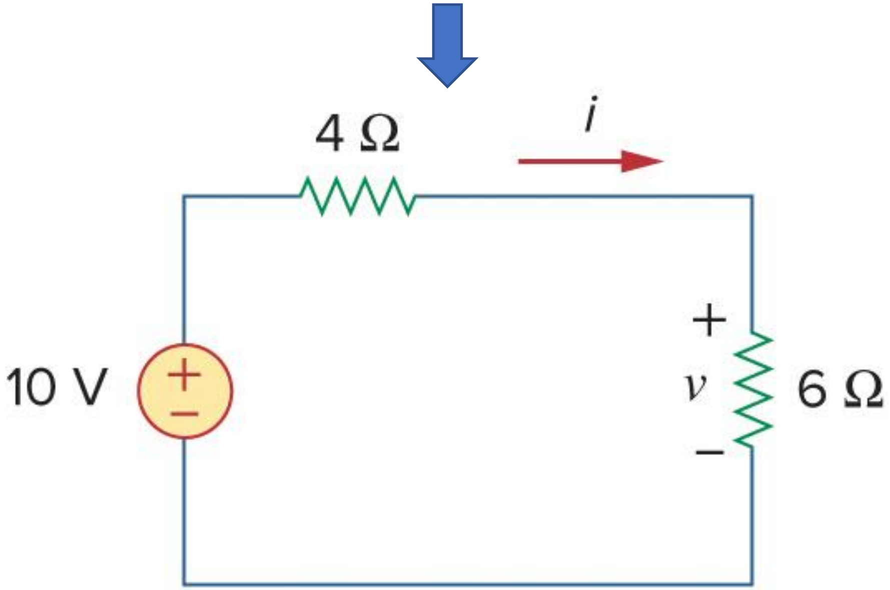
Example 3

Find $i(t)$ in the circuit shown below. Assume that the circuit has reached steady state at $t = 0^-$.

Solution:

- First, we need to calculate initial values of inductor and capacitor.
- For $t < 0$, the switch is closed.
- Under dc condition, the inductor acts as short circuit and the capacitor acts as open circuit.
- Capacitor voltage and inductor current cannot change instantly.



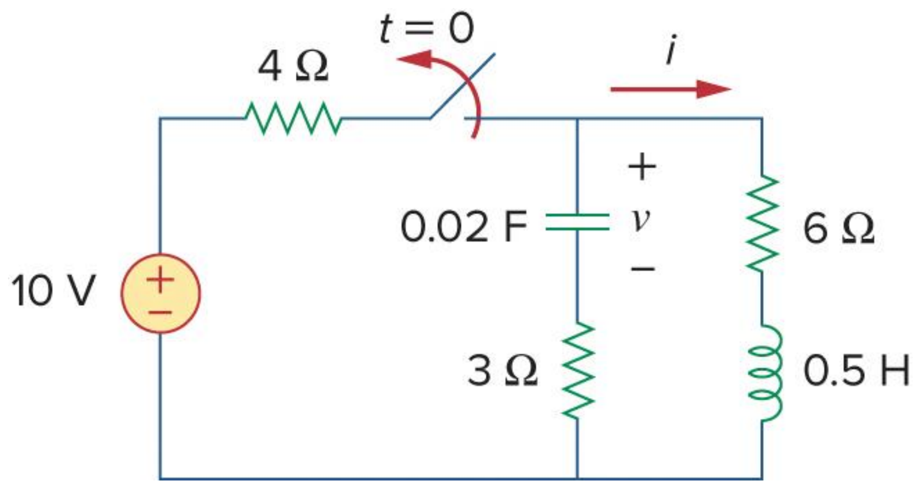


$$i(0) = \frac{10}{4+6} = 1 \text{ A}, \qquad v(0) = 6i(0) = 6 \text{ V}$$

- $i(0)$ *and* $v(0)$ are initial inductor current and initial capacitor voltage at $t = 0$, respectively.

Solution

- For $t > 0$, the switch is opened.
- Voltage source is disconnected.
- To determine which type of response or case the circuit has, characteristics roots of the circuit should be obtained.



$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100} \qquad s_{1,2} = -9 \pm j4.359$$

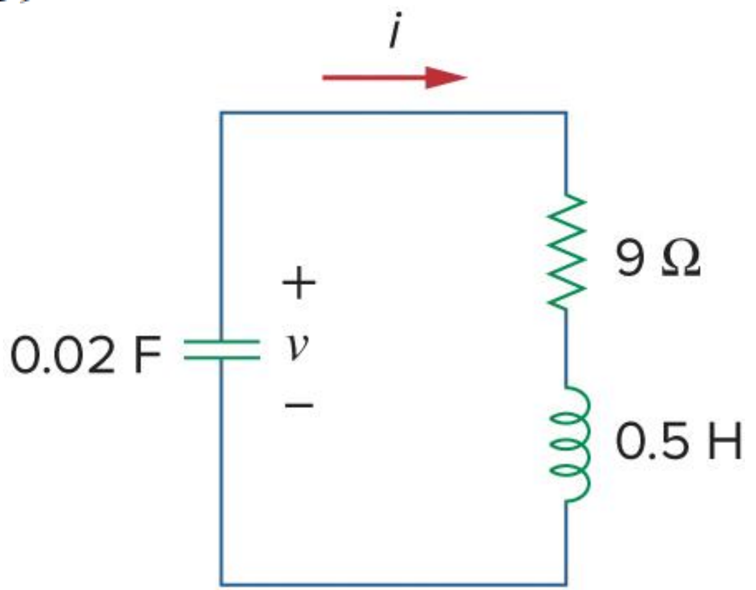
the response is underdamped ($\alpha < \omega$); that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$

obtain A_1 and A_2 using the initial conditions. At $t = 0$,

$$i(0) = 1 = A_1 \qquad v(0) = V_0 = -6 \text{ V}$$

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6 \text{ A/s}$$



Solution

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

at $t = 0$

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2) \qquad A_1 = 1$$

$$-6 = -9 + 4.359A_2 \quad \Rightarrow \quad A_2 = 0.6882$$

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

Source-Free Parallel RLC Circuit

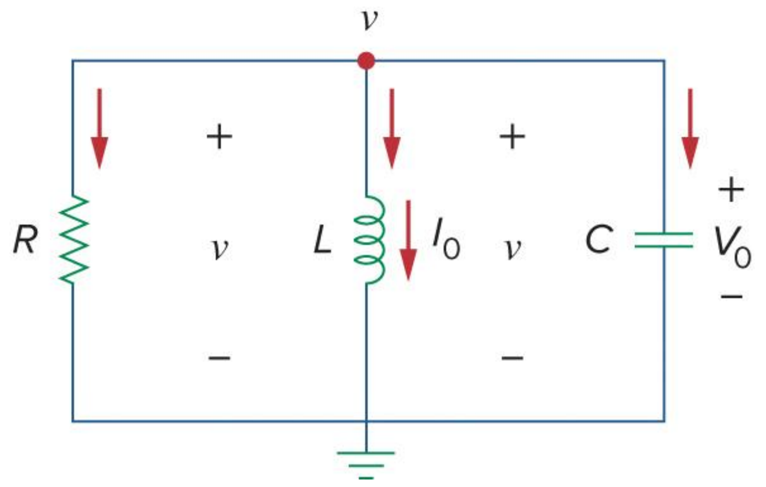
- Assume initial inductor current: I_0 and initial capacitor voltage V_0

$$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)dt \qquad\qquad v(0) = V_0$$

- Apply KCL at the top node: $\frac{v}{R} + \frac{1}{L}\int_{t}^{\infty} v(t)dt + C\frac{dv}{dt} = 0$ ➡ Take derivative and divide entire equation by C

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$ ⟶ 2nd order differential equation

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$ ⟶ Characteristic equation of the 2nd order differential equation



$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$ ➡ $\boxed{s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$

where $\alpha = \frac{1}{2RC}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

Source-Free Parallel RLC Circuit

- Three possible solutions depending on the value of α and ω_0
  1. Overdamped case ($\alpha > \omega_0$)
     - ❑ $\alpha > \omega_0$ when $L > 4R^2C$
     - ❑ Roots of the characteristics equations are real and negative. The response is:

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

  2. Critically damped case ($\alpha = \omega_0$)
     - ❑ $\alpha = \omega_0$ when $L = 4R^2C$
     - ❑ Roots of the characteristics equations are real and equal. The response is:

$$\boxed{\mathrm{v}(t) = (A_1 + A_2 t)e^{-\alpha t}}$$

  3. Underdamped case ($\alpha < \omega_0$)
     - ❑ $\alpha < \omega_0$ when $L < 4R^2C$
     - ❑ Roots of the characteristics equations are complex and expressed as:

$$s_{1,2} = -\alpha \pm \mathrm{j}\omega_d \qquad \text{where } \omega_d = \sqrt{\omega_0 - \alpha^2}$$

- ❑ The response: $\boxed{v(t) = e^{-\alpha t}(A_1 cos\omega_d t + A_2 sin\omega_d t)}$

Constant A_1 and A_2 in each case determined from initial conditions

Example 4

In the parallel circuit shown below, find $v(t)$ for $t > 0$, assuming $v(0) = 5\ V, i(0) = 0, L = 1\ H$, and C= $10\ mF$. Consider these cases: $R = 1.923\ \Omega, R = 5\ \Omega$, and R $= 6.25\ \Omega$.

Solution:

- Case 1: $R = 1.923\ \Omega$.

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

- $\alpha > \omega_0$, the response is overdamped

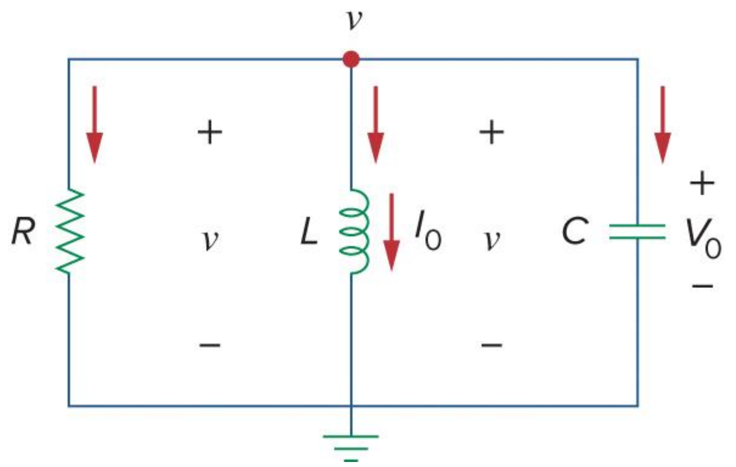
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

- Response for overdamped case:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{-2t} + A_2 e^{-50t}$$

- Apply initial condition to get A_1 and A_2

$$v(0) = 5 = A_1 + A_2 \longrightarrow 1$$



- Apply KCL at the top node:

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t = 0$, $\quad -260 = -2A_1 - 50A_2 \longrightarrow 2$

From equation 1 & 2: $A_1 = -0.2083$ and $A_2 = 5.208$.

$$v(t) = -0.2083e^{-2t} + 5.208e^{-50t}$$

Solution

- Case 2: $R = 5\ \Omega$.

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10 \qquad \omega_0 = 10$$

- $\alpha = \omega_0$, the response is critically damped: $s_1 = s_2 = -10$
- Response for critically damped case: $v(t) = (A_1 + A_2 t)e^{-10t}$
- Apply initial conditions: $v(0) = 5 = A_1$

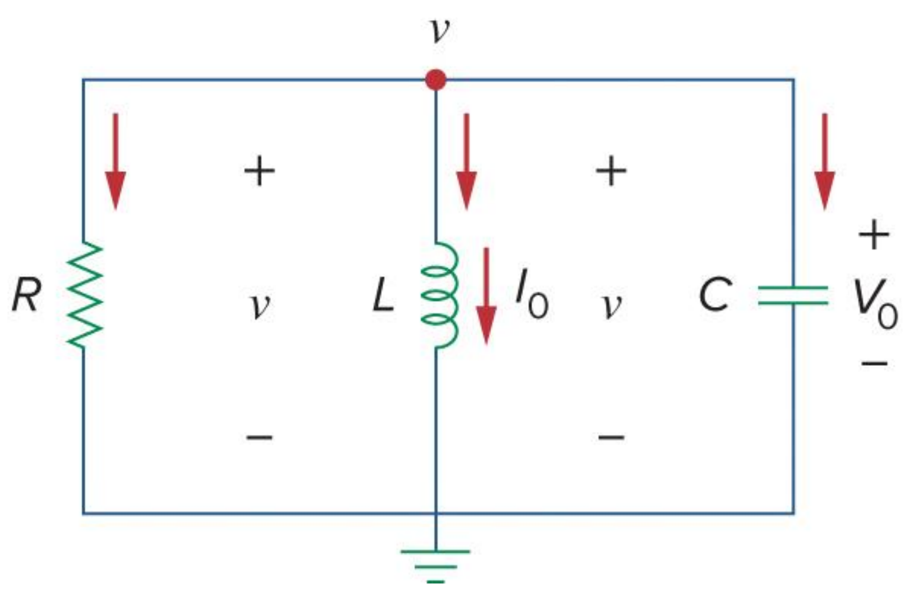
$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

At $t = 0$, $\dfrac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$

$$-100 = -10A_1 + A_2$$

$A_1 = 5$ and $A_2 = -50.$

$$v(t) = (5 - 50t)e^{-10t}\ \text{V}$$



Solution

- Case 3: $R = 6.25\ \Omega$.

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8 \qquad \omega_0 = 10$$

- $\alpha < \omega_0$, the response is underdamped: $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$

- Response for underdamped case: $v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$

- Apply initial conditions: $v(0) = 5 = A_1$

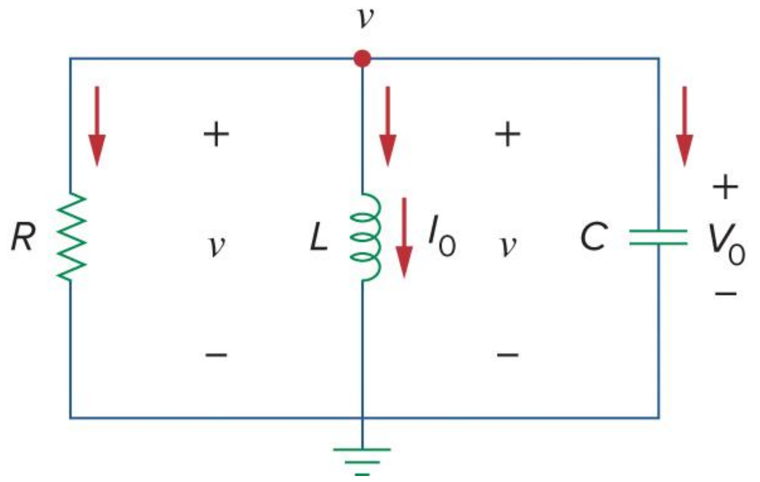
$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25 \times 10 \times 10^{-3}} = -80$$

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$, $\quad -80 = -8A_1 + 6A_2$

$A_1 = 5$ and $A_2 = -6.667$.

$$v(t) = (5 \cos 6t - 6.667 \sin 6t)e^{-8t}$$

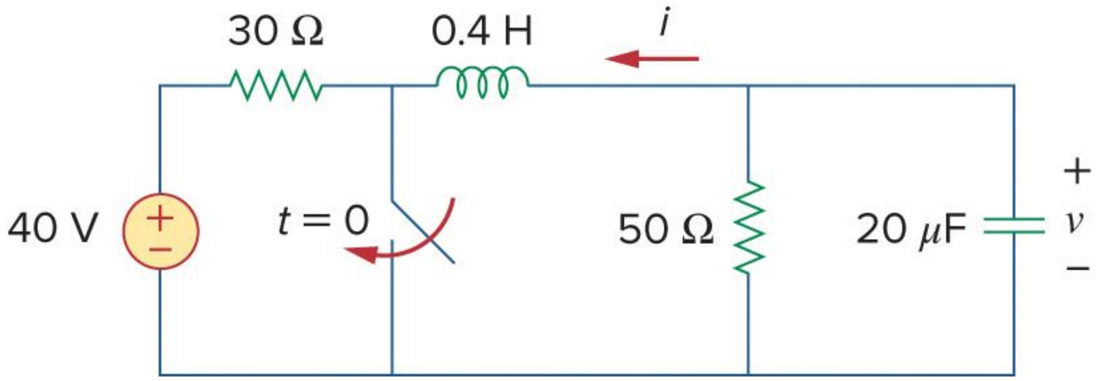


Example 5

Find $v(t)$ for $t > 0$ in the RLC circuit shown below.

Solution:

- First, find initial value of inductor current and capacitor voltage.
- At $t < 0$, switch is opened. Inductor acts like short circuit and capacitor acts like open circuit.



$$v(0) = \frac{50}{30 + 50}(40) = \frac{5}{8} \times 40 = 25 \text{ V}$$

$$i(0) = -\frac{40}{30 + 50} = -0.5 \text{ A}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{25 - 50 \times 0.5}{50 \times 20 \times 10^{-6}} = 0$$

At $t > 0$, switch is closed. The voltage source along with 3Ω is disconnected.

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 50 \times 20 \times 10^{-6}} = 500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.4 \times 20 \times 10^{-6}}} = 354$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$= -500 \pm \sqrt{250{,}000 - 124{,}997.6} = -500 \pm 354$$

$$s_1 = -854, \qquad s_2 = -146$$

Solution

- $\alpha > \omega_0$, the response is overdamped

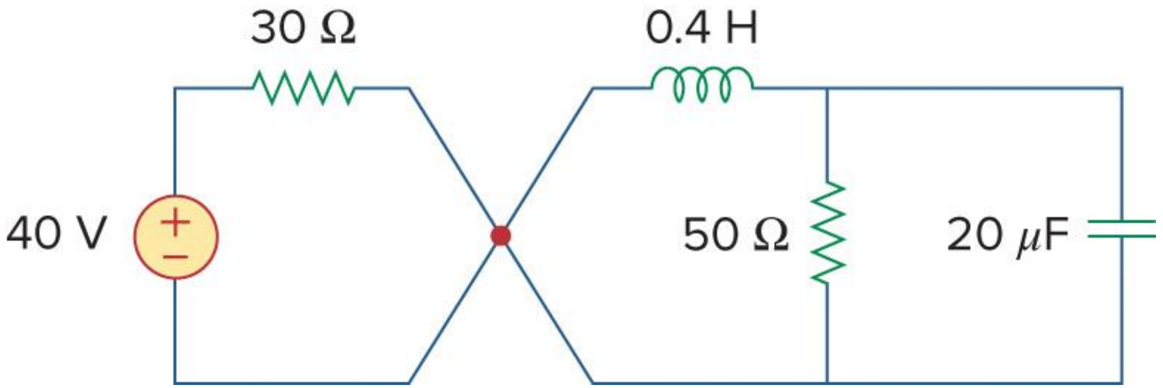
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad v(t) = A_1 e^{-854t} + A_2 e^{-146t}$$

At $t = 0$,

$$v(0) = 25 = A_1 + A_2 \quad \Rightarrow \quad A_2 = 25 - A_1$$

$$\frac{dv}{dt} = -854A_1e^{-854t} - 146A_2e^{-146t}$$

$t > 0$



$$\frac{dv(0)}{dt} = 0 = -854A_1 - 146A_2$$

$$0 = 854A_1 + 146A_2$$

$$A_1 = -5.156, \qquad A_2 = 30.16$$

- The complete solution:

$$v(t) = -5.156e^{-854t} + 30.16e^{-146t}\ \text{V}$$

Step Response of a Series RLC Circuit

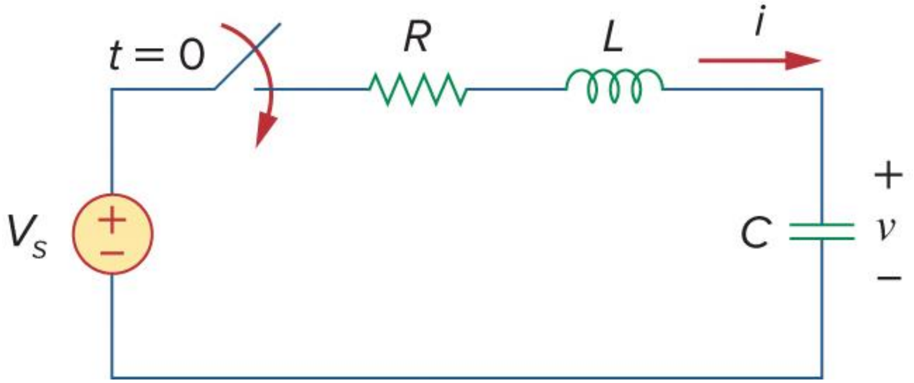
- Recall: Step response is obtained by the sudden application of a dc source.
- Apply KVL:

$$-V_S + Ri + L\frac{di}{dt} + v = 0 \Rightarrow V_S = L\frac{di}{dt} + Ri + v \qquad i = C\frac{dv}{dt}$$

$$V_S = L\frac{d}{dt}\left(C\frac{dv}{dt}\right) + R\left(C\frac{dv}{dt}\right) + v \Rightarrow V_S = LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_S}{LC}$$

$$\Downarrow$$



- The solution of the equation has two components:
- The transient response: $v_t(t)$
- The steady-state response $v_{ss}(t)$
- The complete response:

$$v(t) = v_t(t) + v_{ss}(t)$$

Step Response of a Series RLC Circuit

- The complete response: $v(t) = v_t(t) + v_{ss}(t)$

$v_t(t) = A_1e^{s_1t} + A_2e^{s_2t} \longrightarrow$ Overdamped

$v_t(t) = (A_1 + A_2t)e^{-\alpha t} \longrightarrow$ Critically damped

$v_t(t) = (A_1cos\omega_dt + A_2sin\omega_dt)e^{-\alpha t} \longrightarrow$ Underdamped

- The steady-state response: final value of $v_{ss}(t) = v(\infty) = V_s$
- The complete solution for all cases:

$v(t) = V_s + A_1e^{s_1t} + A_2e^{s_2t} \longrightarrow$ Overdamped

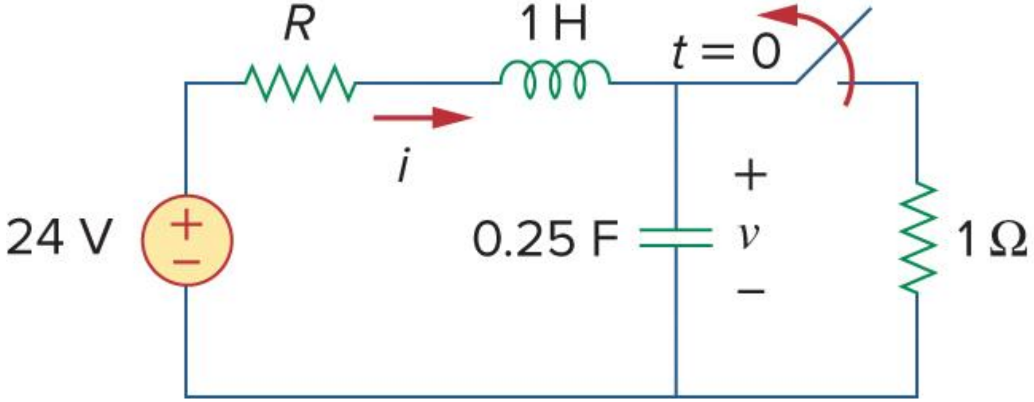
$v(t) = V_s + (A_1 + A_2t)e^{-\alpha t} \longrightarrow$ Critically damped

$v(t) = V_s + (A_1cos\omega_dt + A_2sin\omega_dt)e^{-\alpha t} \longrightarrow$ Underdamped

- Constants A_1 and A_2 are obtained from the initial conditions.

Example 6

For the circuit shown below, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega, R = 4\ \Omega$, and $\text{R} = 1\ \Omega$.



Solution:

- Case 1: $R = 5\ \Omega$
- First, find initial value of inductor current and capacitor voltage.
- At $t < 0$, switch is closed. Inductor acts like short circuit and capacitor acts like open circuit.

$$i(0) = \frac{24}{5 + 1} = 4\ \text{A} \qquad v(0) = 1i(0) = 4\ \text{V}$$

- At $t > 0$, switch is opened 1 Ω resistor is disconnected.

$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 1} = 2.5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

$\alpha > \omega_0$, we have the overdamped natural response.

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t})$$

v_{ss} is the steady-state response.

$v_f = 24$ V.

$$v(0) = 4 = 24 + A_1 + A_2$$

$$-20 = A_1 + A_2$$

$$i(0) = C\frac{dv(0)}{dt} = 4 \Rightarrow \frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16$$

Solution

$$\frac{dv}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t}$$

At $t = 0$,

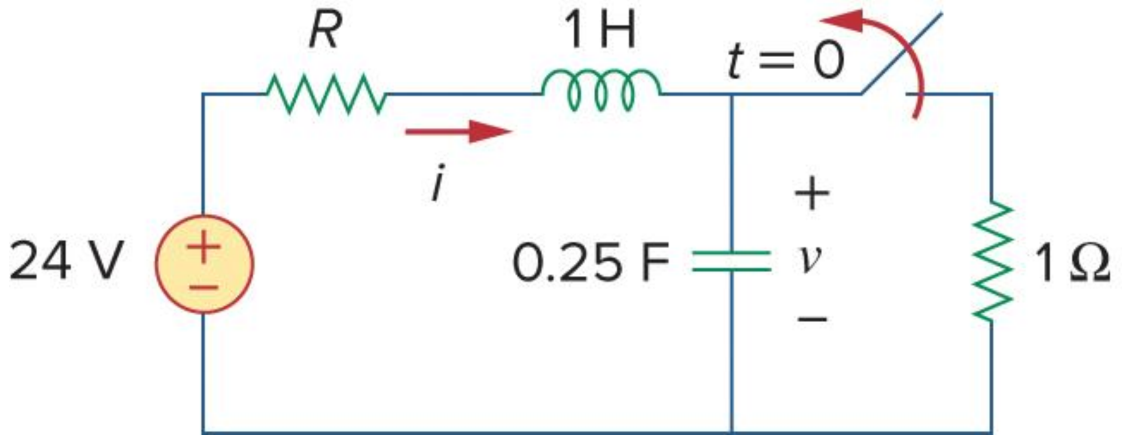
$$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2$$

$A_1 = -64/3$ and $A_2 = 4/3.$

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t}) \text{ V}$$

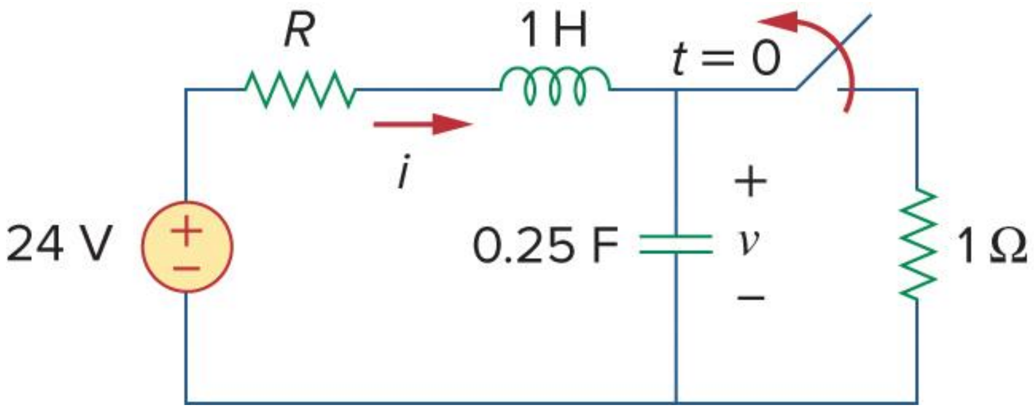
$$i(t) = C\frac{dv}{dt}$$

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t}) \text{ A}$$



Solution

- Case 2: $R = 4\ \Omega$
- First, find initial value of inductor current and capacitor voltage.
- At $t < 0$, switch is closed. Inductor acts like short circuit and capacitor acts like open circuit.



$$i(0) = \frac{24}{4 + 1} = 4.8\text{ A} \qquad v(0) = 1i(0) = 4.8\text{ V}$$

$$\alpha = \frac{R}{2L} = \frac{4}{2 \times 1} = 2 \qquad \omega_0 = 2$$

$s_1 = s_2 = -\alpha = -2$, the critically damped natural response.

$$v(t) = v_{ss} + (A_1 + A_2 t)e^{-2t} \qquad v_{ss} = 24\text{ V},$$

$$v(t) = 24 + (A_1 + A_2 t)e^{-2t}$$

$$v(0) = 4.8 = 24 + A_1 \quad \Rightarrow \quad A_1 = -19.2$$

Since $i(0) = C\,dv(0)/dt = 4.8$ $\quad \dfrac{dv(0)}{dt} = \dfrac{4.8}{C} = 19.2$

$$\frac{dv}{dt} = (-2A_1 - 2tA_2 + A_2)e^{-2t}$$

At $t = 0$, $\quad \dfrac{dv(0)}{dt} = 19.2 = -2A_1 + A_2$

$A_1 = -19.2$ and $A_2 = -19.2$.

$$v(t) = 24 - 19.2(1 + t)e^{-2t}\text{ V}$$

$$i(t) = C\frac{dv}{dt}$$

$$i(t) = (4.8 + 9.6t)e^{-2t}\text{ A}$$

Solution

- Case 3: $R = 1\ \Omega$

$$i(0) = \frac{24}{1+1} = 12 \text{ A} \qquad v(0) = 1i(0) = 12 \text{ V}$$

$$\alpha = \frac{R}{2L} = \frac{1}{2 \times 1} = 0.5$$

$\alpha = 0.5 < \omega_0 = 2$, we have the underdamped response

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936$$

$$v(t) = 24 + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t}$$

$$v(0) = 12 = 24 + A_1 \quad \Rightarrow \quad A_1 = -12$$

Since $i(0) = C\, dv(0)/dt = 12$,

$$\frac{dv(0)}{dt} = \frac{12}{C} = 48$$

$$\frac{dv}{dt} = e^{-0.5t}(-1.936A_1 \sin 1.936t + 1.936A_2 \cos 1.936t) - 0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t)$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 48 = (-0 + 1.936A_2) - 0.5(A_1 + 0)$$

$$A_1 = -12 \qquad A_2 = 21.694,$$

$$v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t} \text{ V}$$

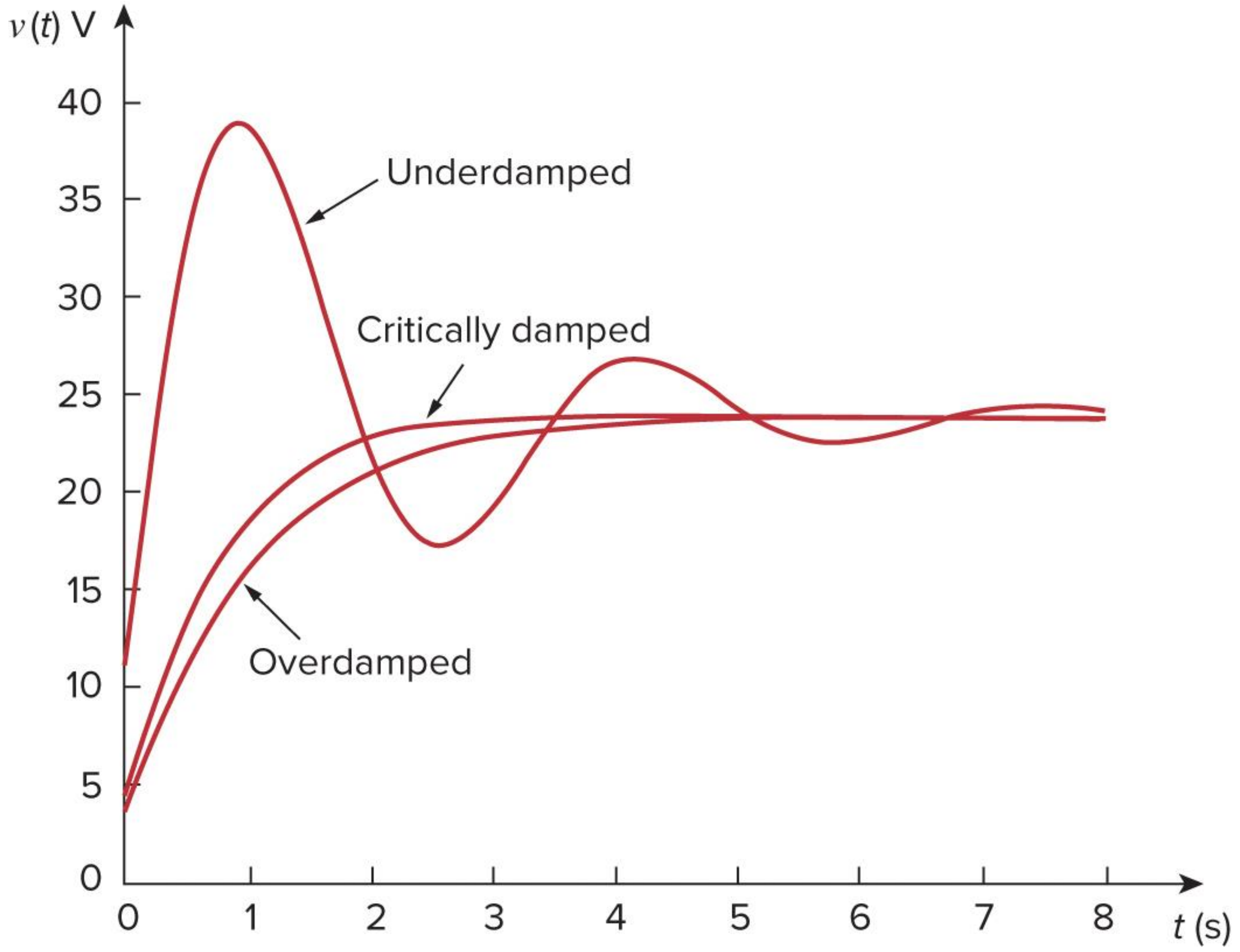
$$i(t) = C\frac{dv}{dt}$$

$$i(t) = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t} \text{ A}$$

Solution

- Graph shown below plots the responses for the three cases. From this graph, we observe that the critically damped response approaches the step input of 24 V the fastest.

- The system returns to equilibrium position without oscillation for both overdamped and critically damped responses.
- The system returns to equilibrium position faster for critically damped response as compared to overdamped response.
- For underdamped response, the system crosses the equilibrium or steady-state position very quickly but will continue to oscillate around the final value.



Step Response of a Parallel RLC Circuit

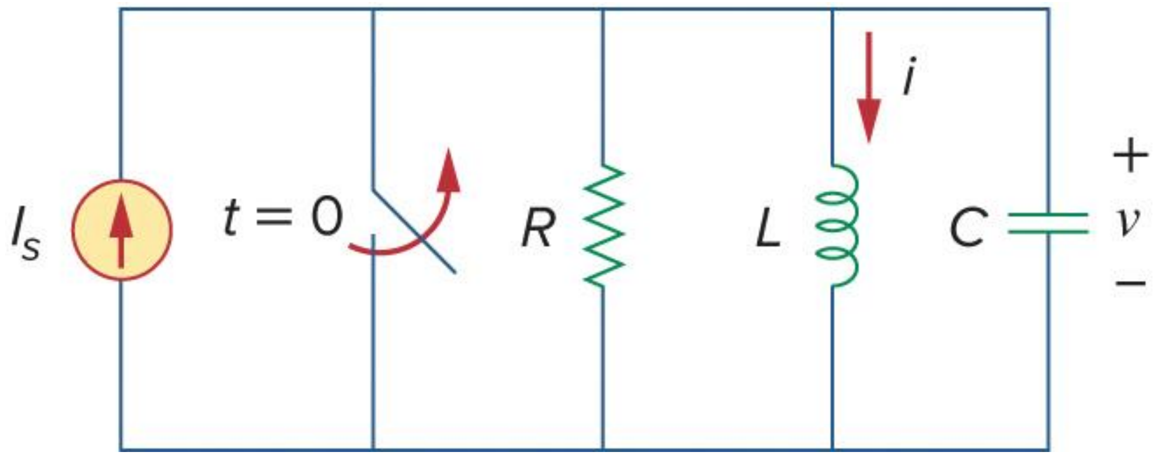
- Determine i due to a sudden application of a dc current.
- Apply KCL at the top node for $t > 0$.

$$I_s = \frac{v}{R} + i + C\frac{dv}{dt} \qquad v = L\frac{di}{dt} \qquad I_s = \frac{L\frac{di}{dt}}{R} + i + C\frac{d}{dt}\left(L\frac{di}{dt}\right)$$

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \longrightarrow$$ 2nd order differential equation

- This equation has same characteristic equation as the source free parallel RLC circuit equation.
- Complete response:

$$i(t) = i_t(t) + i_{ss}(t)$$



Step Response of a Parallel RLC Circuit

$i(t) = i_t(t) + i_{ss}(t)$

$i(t)$: complete response
$i_t(t)$: transient response
$i_{ss}(t)$: steady-state response

$i_{ss}(t) = i(\infty) = I_s$

$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \longrightarrow \text{Overdamped}$$

$$i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t} \longrightarrow \text{Critically damped}$$

$$i(t) = I_s + (A_1 cos\omega_d t + A_2 sin\omega_d t)e^{-\alpha t} \longrightarrow \text{Underdamped}$$

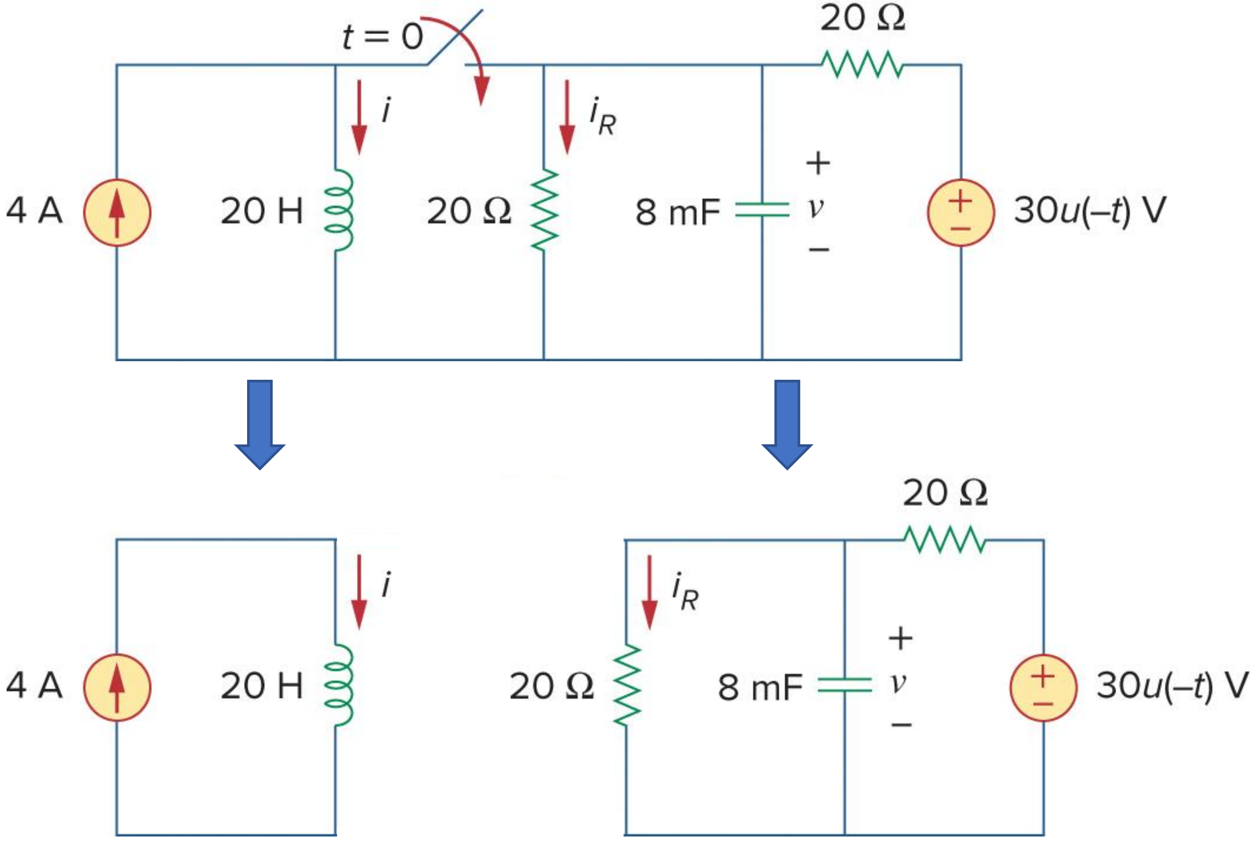
- Similarly, constants A_1 and A_2 can be determined from the initial conditions.

Example 7

In the circuit shown below, find $i(t)$ and $i_R(t)$ for $t > 0$.

Solution:

- For $t < 0$, switch is opened and circuit is partitioned into two independent subcircuits.



$$i(0) = 4\ A$$

Recall: Step function of $u(t)$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$u(-t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}$$

When $t < 0, 30u(-t) = 30$

When $t > 0, 30u(-t) = 0$

Solution

- For $t < 0$, switch is opened and capacitor acts like open circuit.

$$v(0) = \frac{20}{20 + 20}(30) = 15 \text{ V}$$

- For $t > 0$, switch is closed and we have a parallel RLC circuit with current source.
- For $t > 0$, voltage source is zero due to $u(-t) = 0$.

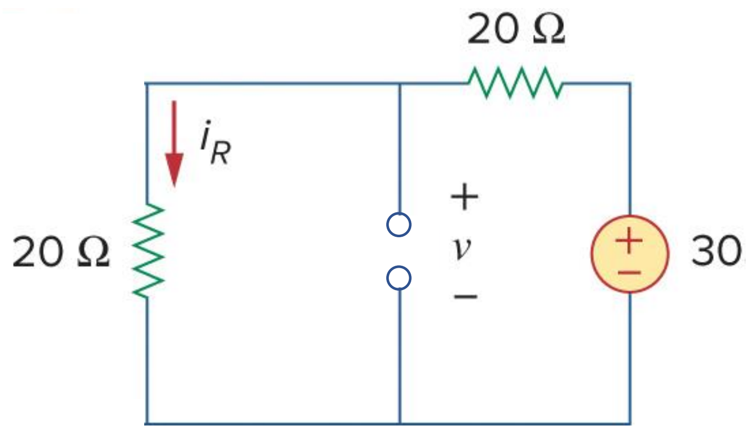
$$R = 20 \parallel 20 = 10 \ \Omega.$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 10 \times 8 \times 10^{-3}} = 6.25$$

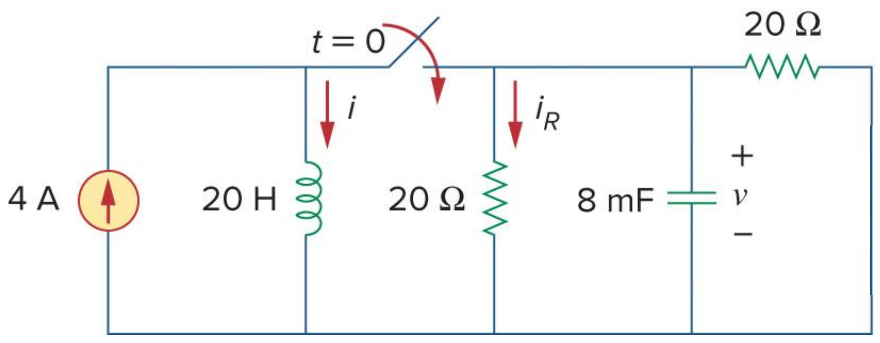
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20 \times 8 \times 10^{-3}}} = 2.5$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -6.25 \pm \sqrt{39.0625 - 6.25}$$
$$= -6.25 \pm 5.7282$$

$t < 0$



$t = 0$



$$s_1 = -11.978, \qquad s_2 = -0.5218$$

Solution

Since $\alpha > \omega_0$, we have the overdamped case.

$$i(t) = I_s + A_1 e^{-11.978t} + A_2 e^{-0.5218t}$$

where $I_s = 4$ is the final value of $i(t)$.

At $t = 0$, $\quad i(0) = 4 = 4 + A_1 + A_2 \quad \Rightarrow \quad A_2 = -A_1$

Taking the derivative of $i(t)$ $\Rightarrow$ $\dfrac{di}{dt} = -11.978A_1e^{-11.978t} - 0.5218A_2e^{-0.5218t}$

At $t = 0$, $\quad \dfrac{di(0)}{dt} = -11.978A_1 - 0.5218A_2$

$$L\frac{di(0)}{dt} = v(0) = 15 \quad \Rightarrow \quad \frac{di(0)}{dt} = \frac{15}{L} = \frac{15}{20} = 0.75$$

$0.75 = (11.978 - 0.5218)A_2 \quad \Rightarrow \quad A_2 = 0.0655 \qquad A_1 = -0.0655$

$$i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11.978t})\text{ A}$$

$$v(t) = L\,di/dt \qquad\qquad i_R(t) = \frac{v(t)}{20} = \frac{L}{20}\frac{di}{dt} = 0.785e^{-11.978t} - 0.0342e^{-0.5218t}\text{ A}$$

General Second-Order Circuits

- Other second-order circuits such as op amps
- For a given 2nd order circuit, determine its step response $x(t)$ (which may be voltage or current source) by taking the following steps:
  - Determine the initial conditions and final value:

$$x(0) \qquad \frac{dx(0)}{dt} \qquad x(\infty)$$

  - Turn off the independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL. When obtained 2nd order differential equation, determine its characteristics roots. Compare α and ω_0 to figure it out what kind of response the circuit has.
  - Obtain steady-state response as $x_{ss}(t) = x(\infty)$
  - Total response is sum of the transient response and steady-state response:

$$x(t) = x_t(t) + x_{ss}(t)$$

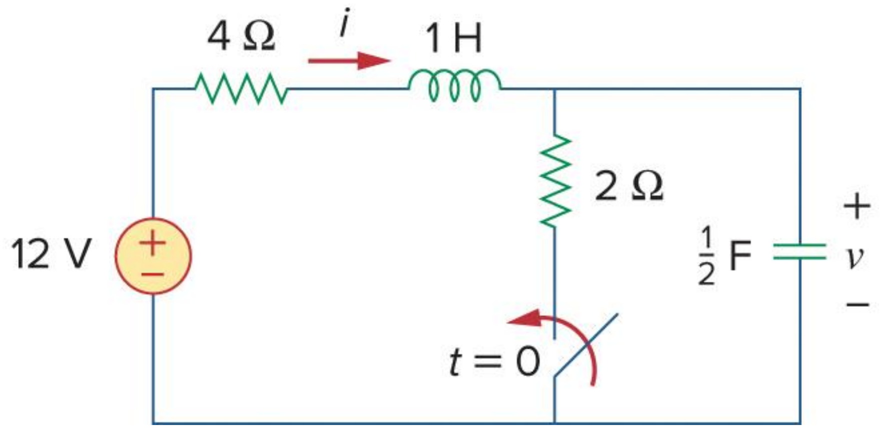
  - Finally, determine the constant of the transient response from the initial condition $x(0)$ and $\frac{dx(0)}{dt}$

Example 8

- Find the complete response v and i for $t > 0$ in the circuit shown below.

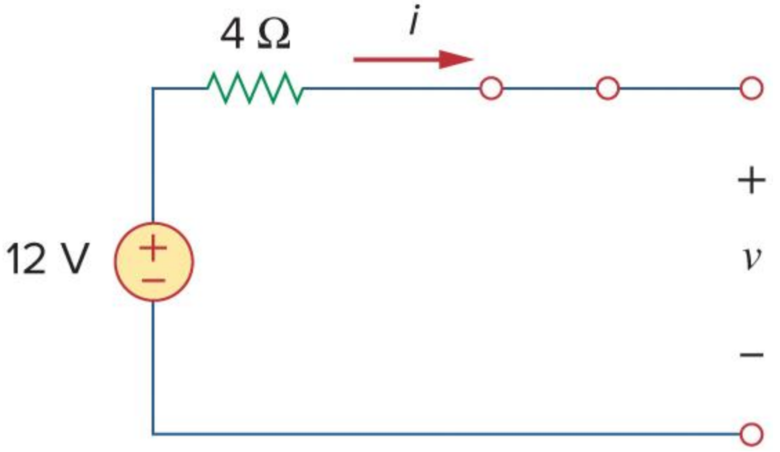
Solution:

- Find the initial and final values.



- $t = 0^-$, the circuit is at steady-state and switch is opened.

$$v(0^-) = 12 \text{ V}, \qquad i(0^-) = 0$$



Solution

- $t = 0^+$, the switch is closed.

$$v(0^+) = v(0^-) = 12 \text{ V}, \qquad i(0^+) = i(0^-) = 0$$

- Find initial value of the 1st derivative of voltage $\frac{dv}{dt}$, i.e., $\frac{dv(0^+)}{dt}$

$$C\, dv/dt = i_C \ \text{ or } \ dv/dt = i_C/C.$$

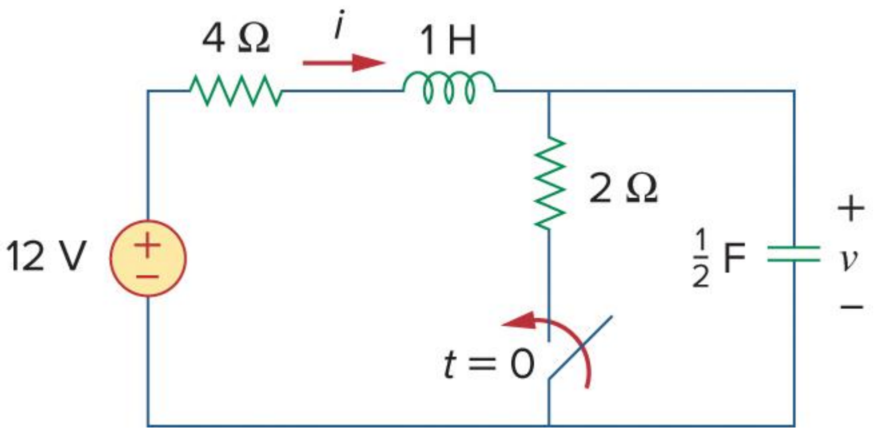
$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2} \qquad \text{(Apply KCL at node a)}$$

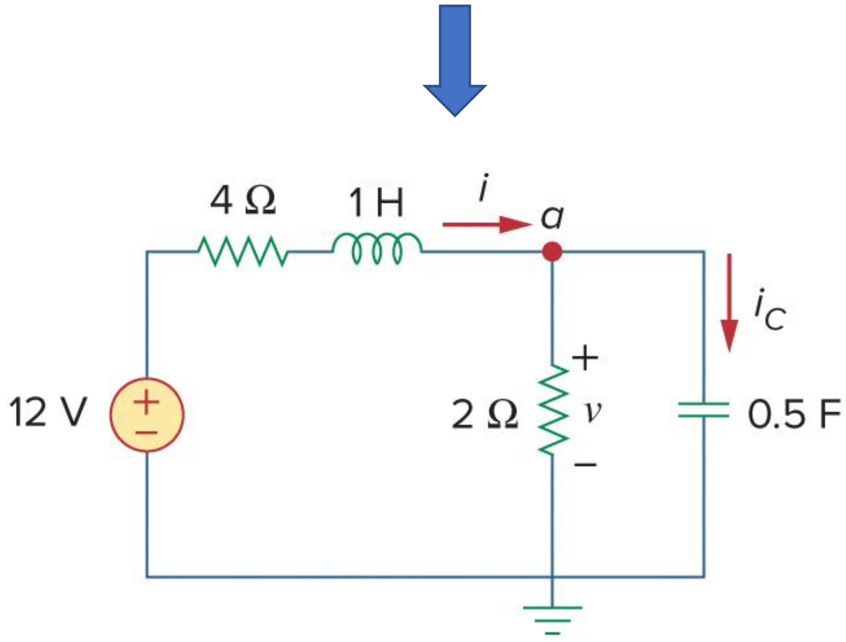
$$0 = i_C(0^+) + \frac{12}{2} \quad \Rightarrow \quad i_C(0^+) = -6 \text{ A}$$

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12 \text{ V/s}$$

- Final value is obtained as:

$$i(\infty) = \frac{12}{4 + 2} = 2 \text{ A}, \qquad v(\infty) = 2i(\infty) = 4 \text{ V}$$





Solution

- Obtain transient response for $t > 0$ by turning of independent source
- Apply KCL at node a:

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt}$$

- Apply KVL to the left mesh:

$$4i + 1\frac{di}{dt} + v = 0$$

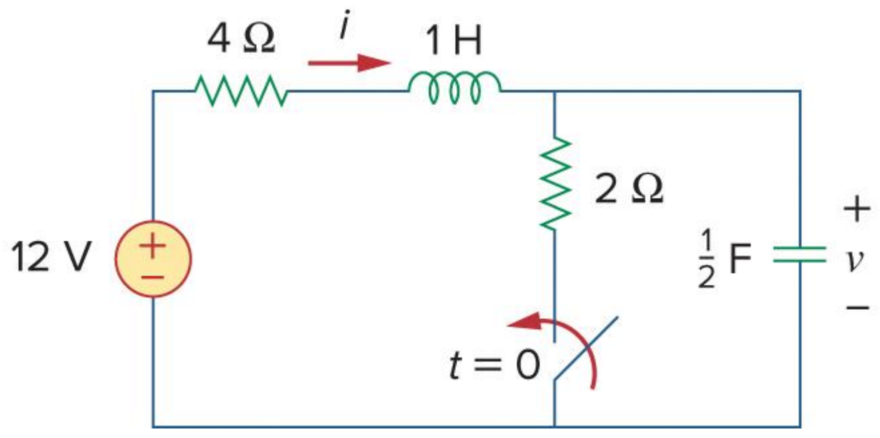
- Substitude i into KVL equation:

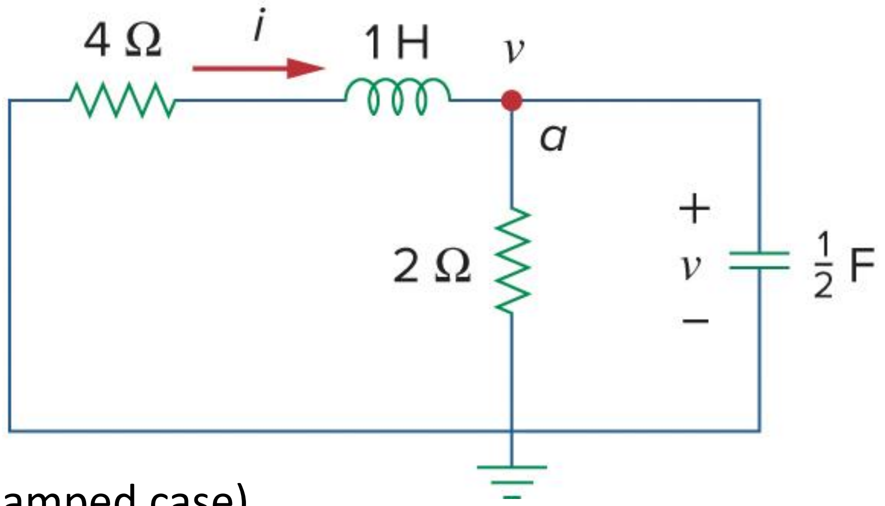
$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0$$

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0 \quad \Rightarrow \quad s^2 + 5s + 6 = 0$$

$$s = -2 \qquad s = -3$$

(Overdamped case)





Solution

- Natural response: $v_n(t) = Ae^{-2t} + Be^{-3t}$
- Steady-state response: $v_{ss}(t) = v(\infty) = 4$
- Complete response: $v(t) = v_t + v_{ss} = 4 + Ae^{-2t} + Be^{-3t}$

$v(0) = 12$ at $t = 0$ $\quad 12 = 4 + A + B \Rightarrow A + B = 8$

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t}$$

$-12 = -2A - 3B \Rightarrow 2A + 3B = 12 \qquad A = 12, \qquad B = -4$

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t} \text{ V},$$

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t}$$

$$= 2 - 6e^{-2t} + 4e^{-3t} \text{ A}, \qquad t > 0$$