Translation on the s-axis

$y = f(t)$ $\qquad \mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

✱

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

$$\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)$$

Ex $\quad \mathcal{L}^{-1}\left[\dfrac{s-2}{(s-2)^2+3^2}\right] = e^{2t}\cos(3t)$

Ex $\quad \mathcal{L}^{-1}\left[\underbrace{\dfrac{2}{(s-4)^3}}_{F(s-4)}\right] = \mathcal{L}^{-1}[F(s-4)] = e^{4t} f(t) = e^{4t} t^2$

$F(s-4) = \dfrac{2}{(s-4)^3} \xrightarrow[s \to s+4]{} F(s) = \dfrac{2}{s^3} \rightarrow f(t) = t^2$

Ex $\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-(-5)}\right] = e^{-5t}$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] = \mathcal{L}^{-1}\left[F(s+5)\right] = e^{-5t} f(t) = e^{-5t} \cdot 1$$

$$= e^{-5t}$$

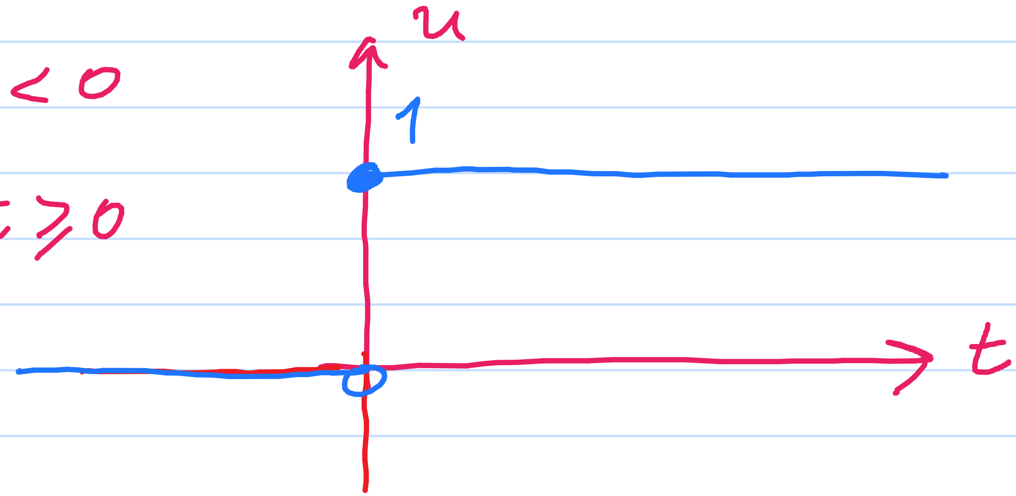
$$\parallel$$

$$F(s+5) = \frac{1}{s+5} \xrightarrow{s \to s-5} F(s) = \frac{1}{s} \to f(t) = 1$$

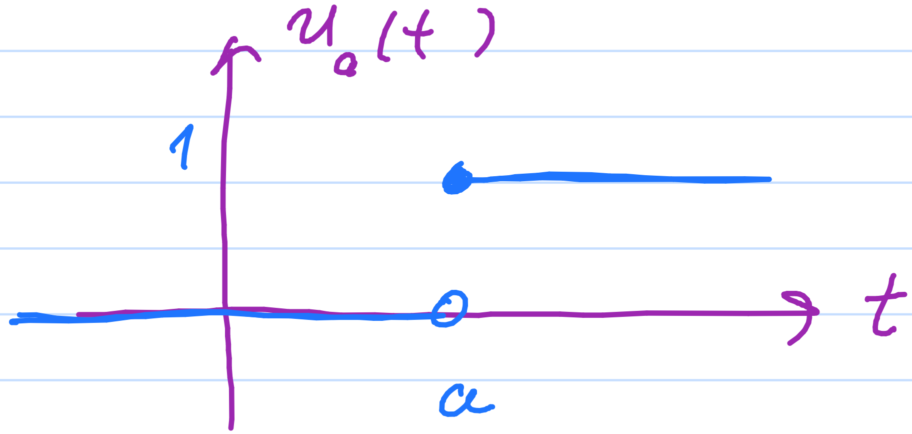
$$\mathcal{L}^{-1}\left[F(s-a)\right] = e^{at} f(t)$$

Ex

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & a \leq t \end{cases}$$



$$\mathcal{L}[u(t)] = \frac{1}{s}$$

$$\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$$

Ex $\mathcal{L}^{-1}\left[e^{-2s} \dfrac{s}{(s-3)^2} \right]$: Sorry, later; we need something more

Ex $x'' + 6x' + 34x = 0$; $x(0) = 3$, $x'(0) = 1$.

Let $\mathcal{L}[x(t)] = X(s)$

$$\mathcal{L}[x'' + 6x' + 34x] = \mathcal{L}[0]$$

$$\mathcal{L}[x''] + 6\mathcal{L}[x'] + 34\mathcal{L}[x] = 0$$

$$s^2 \underline{X(s)} - s\,x(0) - x'(0) + 6\left[s\,\underline{X(s)} - x(0)\right] + 34\,\underline{X(s)} = 0$$

$$(s^2 + 6s + 34)X(s) = 3s + 19 \Rightarrow X(s) = \frac{3s + 19}{s^2 + 6s + 34}$$

$$\frac{3s+19}{s^2+6s+34} = \frac{3s+19}{s^2+2.s.3+3^2+25} = \frac{3s+19}{(s+3)^2+5^2}$$

$$X(s) = \frac{3s+19}{(s+3)^2+5^2} = \frac{3s+9+10}{(s+3)^2+5^2}$$

$$X(s) = 3\,\frac{s+3}{(s+3)^2+5^2} + 2.\,\frac{5}{(s+3)^2+5^2}$$

$$x(t) = 3.e^{-3t}.\cos(5t) + 2.e^{-3t}.\sin 5t$$

$$\mathcal{L}[\cos 5t] = \frac{s}{s^2+5^2} \qquad \mathcal{L}[\sin 5t] = \frac{5}{s^2+5^2}$$

Ex $\mathcal{L}^{-1}\left[\dfrac{s^2+1}{s^3-2s^2-8s}\right]$

$$s^3-2s^2-8s = s(s^2-2s-8) = s(s+2)(s-4)$$

-4

$+2$

$$\frac{s^2+1}{s(s+2)(s-4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-4}$$

$$A = \frac{0^2+1}{(0+2)(0-4)} = -\frac{1}{8}, \quad B = \frac{(-2)^2+1}{(-2)\cdot(-2-4)} = \frac{5}{12}$$

$$C = \frac{4^2+1}{4(4+2)} = \frac{17}{24}$$

$$\mathcal{L}^{-1}\left[\frac{s^2+1}{s(s+2)(s-4)}\right] = \mathcal{L}^{-1}\left[\frac{-\frac{1}{5}}{s} + \frac{\frac{5}{12}}{s+2} + \frac{\frac{17}{24}}{s-4}\right]$$

$$= -\frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{5}{12}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{17}{24}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]$$

$$= -\frac{1}{5}\cdot 1 + \frac{5}{12}\cdot e^{-2t} + \frac{17}{24}\cdot e^{4t}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

Convolution of Two Functions

Given two functions $f(t), g(t)$

$$f(t) * g(t) = \int_0^t f(\tau) . g(t-\tau) . d\tau$$

is called the convolution of $f(t)$ and $g(t)$.
Afte the transformation $u = t - \tau$ $f * g$ can also be evaluated as

$*$ is commutative

$$f(t) * g(t) = \int_0^t g(u) f(t-u) \, du$$

$$= \int_0^t g(\tau) f(t-\tau) \, d\tau = g * f$$

$$f(t) = \cos t \quad , \quad g(t) = \sin t$$

$$\cos t * \sin t = \int_0^t \cos \tau \sin(t-\tau)\, d\tau$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$- \quad \sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos \tau \sin(t-\tau) = \frac{1}{2}\left[\sin(\tau+t-\tau) - \sin(\tau-t+\tau)\right]$$

$$\cos t * \sin t = \int_0^t \frac{1}{2}\left[\sin t - \sin(2\tau - t)\right] d\tau$$

$$\cos t * \sin t = \frac{1}{2}\int_0^t \sin t \, d\tau - \frac{1}{2}\int_0^t \underbrace{\sin(2\tau - t)} \, d\tau$$

$$= \frac{1}{2} \cdot \sin t \cdot \tau \Big|_{\tau=0}^{\tau=t} - \frac{1}{2} \cdot \frac{1}{-2} \cos(2\tau - t) \Big|_{\tau=0}^{\tau=t}$$

$$= \frac{1}{2} \cdot \sin t \cdot t + \frac{1}{4}\left[\cos(2t - t) - \cos(0 - t)\right]$$

$$= \frac{t \sin t}{2}$$

$$(\cos t) * (\sin t) = \frac{1}{2} t \sin t$$

The Convolution Theorem

Let $\mathcal{L}[f(t)] = F(s)$, $\mathcal{L}[g(t)] = G(s)$.

Then $\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$

$$\mathcal{L}[f(t) * g(t)] = F(s)\,G(s)$$

Remark

$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] = F(s) + G(s)$$

$$\cancel{\mathcal{L}[f(t)\,g(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]}$$

Nothing like this!!

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$$

The Convolution Theorem

Let $\mathcal{L}[f(t)] = F(s)$, $\mathcal{L}[g(t)] = G(s)$.

Then $\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\, \mathcal{L}[g(t)]$

$$\mathcal{L}[f(t) * g(t)] = F(s)\, G(s)$$

$$\Rightarrow \mathcal{L}^{-1}[F(s)\, G(s)] = f(t) * g(t)$$

Example

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] = ?$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left[F(s)\, G(s) \right] = f(t) * g(t)$$

Example

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2} \right] = ?$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2} \right] = \mathcal{L}^{-1}\left[\underbrace{\frac{s}{s^2+1}}_{F(s)} \cdot \underbrace{\frac{1}{s^2+1}}_{G(s)} \right]$$

$$= \mathcal{L}^{-1}\left[F(s) \, . \, G(s) \right] = f(t) * g(t)$$

$$= \cos t * \sin t = \frac{1}{2} t \sin t$$

$$F(s) = \frac{s}{s^2+1} \Rightarrow f(t) = \cos t \, ; \quad G(s) = \frac{1}{s^2+1} \rightarrow g(t) = \sin t$$

Ex $\mathcal{L}^{-1}\left[\frac{1}{s^4(s^2+5)}\right] = ?$

Solution 1 $$\frac{1}{s^4(s^2+5)} = \frac{As+B}{s^2+5} + \frac{C}{s} + \frac{D}{s^2} + \frac{E}{s^3} + \frac{F}{s^4}$$

Find A, B, C, D, E, F, and find $\mathcal{L}^{-1}$.

$\Rightarrow$ Let's complete this TOMORROW

Ex $\mathcal{L}^{-1}\left[\frac{1}{s^4(s^2+5)}\right] = ?$

$$\int_0^t \tau^3 \sin(t-\tau)\, d\tau$$

$$\int \underbrace{x^3}_{u} \underbrace{\sin x\, dx}_{dv}$$

Solution 2 $= \mathcal{L}^{-1}\left[\underbrace{\frac{1}{s^4}}_{F(s)} \cdot \underbrace{\frac{1}{s^2+5}}_{G(s)}\right]$

$$F(s) = \frac{1}{s^4} = \frac{1}{6}\frac{3!}{s^4} \rightarrow f(t) = \frac{1}{6}t^3 \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$G(s) = \frac{1}{s^2+5} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2+(\sqrt{5})^2} \Rightarrow g(t) = \frac{1}{\sqrt{5}}\sin(\sqrt{5}t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^4}\frac{1}{s^2+5}\right] = \mathcal{L}^{-1}[F(s)G(s)] = f(t) * g(t)$$

DIY

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t \frac{1}{6}\tau^3 \frac{1}{\sqrt{5}}\sin(\sqrt{5}(t-\tau))\, d\tau$$

Ex

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] = ?$$

$$= \mathcal{L}^{-1}\left[\underbrace{\frac{1}{s-1}}_{F} \cdot \underbrace{\frac{2}{s^2+2^2}}_{G}\right] = \mathcal{L}^{-1}\left[F(s)\,G(s)\right]$$

$$= f(t) * g(t)$$

$$F(s) = \frac{1}{s-1} \Rightarrow f(t) = e^{t}$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$G(s) = \frac{2}{s^2+2^2} \Rightarrow g(t) = \sin 2t$$

$$= \int_0^t e^{\tau} \sin 2(t-\tau)\, d\tau$$

$$\vdots$$

$$= \frac{2}{5}e^{t} - \frac{\sin 2t}{5} - \frac{2}{5}\cos 2t$$

Ex

$$\mathcal{L}^{-1}\left[\frac{2}{(s-1)(s^2+4)}\right] = ?$$

OR

$$= \mathcal{L}^{-1}\left[\underbrace{\frac{1}{s-1}}_{F} \cdot \underbrace{\frac{2}{s^2+2^2}}_{G}\right] = \mathcal{L}^{-1}\left[F(s)\, G(s)\right]$$

$$F(s) = \frac{1}{s-1} \Rightarrow f(t) = e^t$$

$$G(s) = \frac{2}{s^2+2^2} \Rightarrow g(t) = \sin 2t$$

$$= f(t) * g(t)$$

$$= g(t) * f(t)$$

$$= \int_0^t g(\tau) * f(t-\tau)\, d\tau$$

$$= \int_0^t \sin(2\tau)\, e^{t-\tau}\, d\tau$$

$$= \int_0^t e^{t-\tau} \sin(2\tau)\, d\tau$$

This form is available in the **book**

<u>Tomorrow</u>

- Differentiation & Integration of Transforms

- $\mathcal{L}[u_a(t)]$ Piecewise cont. fncts.

<u>Weekend session (PS)</u> Saturday, $13^{\underline{00}}$

$$f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$\mathcal{L}[f(t)] = F(s) \qquad \mathcal{L}[g(t)] = G(s)$$

$$\cancel{\mathcal{L}[f(t)\, g(t)] = \mathcal{L}[f(t)]\, \mathcal{L}[g(t)]}$$

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\, \mathcal{L}[g(t)]$$

$$\mathcal{L}[f(t) * g(t)] = F(s)\, G(s)$$

$$\mathcal{L}^{-1}[F(s)\, G(s)] = f(t) * g(t)$$

21/01/2021

Differentiation of Transforms

Theorem Let $\mathcal{L}[f(t)] = F(s)$

* $\mathcal{L}[-t f(t)] = F'(s)$

$\Rightarrow \mathcal{L}^{-1}[F'(s)] = -t f(t)$

$\Rightarrow f(t) = -\frac{1}{t} \mathcal{L}^{-1}[F'(s)]$

$\mathcal{L}[t f(t)] = -F'(s)$

$\mathcal{L}[t^2 f(t)] = F''(s)$

$\vdots$

* $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$.

* <u>Ex</u> $\mathcal{L}[t^2 \sin(kt)]$

$f(t) = \sin kt \implies F(s) = \frac{k}{s^2 + k^2}$

$\mathcal{L}[t^n f(t)] = (-1)^n$

$n=2 \implies \mathcal{L}[t^2 f(t)] = (-1)^2 F''(s)$

$$= \frac{d^2}{ds^2} \frac{k}{s^2+k^2}$$

$$= \frac{6ks^2 - 2k^3}{(s^2+k^2)^3}$$

Ex Find $\mathcal{L}^{-1}\left[\underbrace{\tan^{-1}\frac{1}{s}}_{F(s)}\right] = f(t) = ?$

$$\boxed{\mathcal{L}[f(t)] = F(s) \Rightarrow f(t) = -\frac{1}{t}\mathcal{L}^{-1}[F'(s)]}$$

$$F(s) = \tan^{-1}\frac{1}{s} \Rightarrow F'(s) = \frac{-\frac{1}{s^2}}{1+\left(\frac{1}{s}\right)^2} = \frac{-1}{1+s^2}$$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{-1}{1+s^2}\right] = \frac{1}{t}\cdot\mathcal{L}^{-1}\left[\frac{1}{1+s^2}\right]$$

$$= \frac{1}{t}\sin t$$

Ex $\quad tx'' + x' + tx = 0 \; ; \quad x(0) = 1, \; x'(0) = 0$

(Bessel's eq.)

$$\mathcal{L}[\, tx'' + x' + tx \,] = \mathcal{L}[0]$$

$$\mathcal{L}[tx''] + \mathcal{L}[x'] + \mathcal{L}[tx] = 0$$

Let $\mathcal{L}[x(t)] = X(s)$

$$\mathcal{L}[\, t \underbrace{x''}_{f(t)} \,] = -F'(s) = -\left[s^2 X(s) - s x(0) - x'(0) \right]'$$

$$= -2s\, X(s) + s^2 X'(s) - 1)$$

$$\mathcal{L}[\, t x \,] = -F'(s) = -\left[X(s) \right]' = -X'(s)$$

$n = 1$

$$\mathcal{L}[\, t^n f(t) \,] = (-1)^n F^{(n)}(s).$$

$$-2s\,X(s) - s^2 X'(s) + 1 + s\,X(s) - x(0) - X'(s) = 0$$

$$(1+s^2)\,X'(s) = -s\,X(s)$$

$$(1+s^2)\frac{dX}{ds} = -s\,X \rightarrow \frac{dX}{X} = \frac{-s}{1+s^2}\,ds$$

$$\ln X = -\frac{1}{2}\ln 1+s^2 + \ln C$$

$$X(s) = \frac{C}{\sqrt{s^2+1}} \quad ; \quad X(s=0) = \frac{C}{\sqrt{0^2+1}}$$

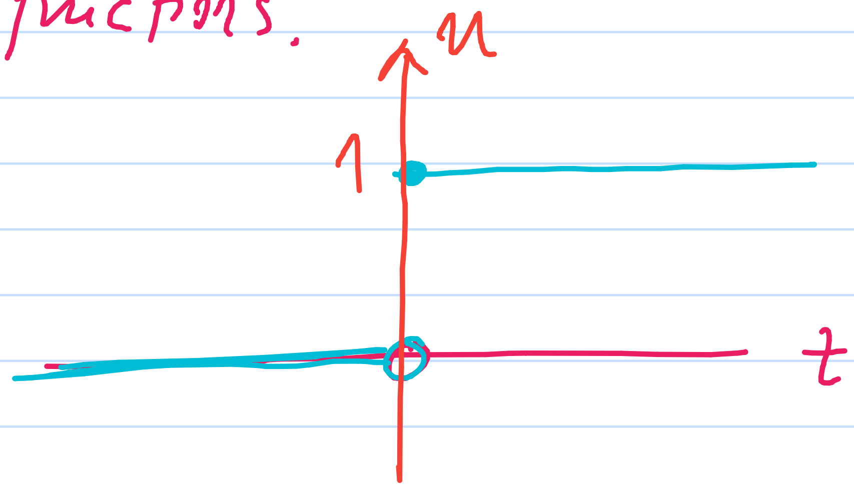
$$C = X(0)$$

$$X(0) = \int_0^\infty x(t)\,dt$$

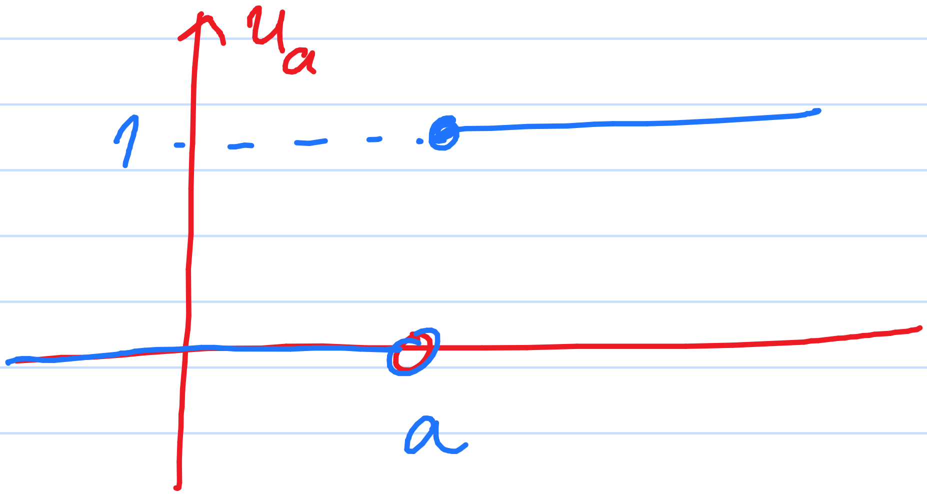
$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty e^{-st} x(t)\,dt \qquad = C$$

Solution of DEs involving unit step functions / piecewise cont. functions.

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \end{cases}$$



$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & a \le t \end{cases}$$



$$\mathcal{L}[u(t)] = \frac{1}{s} \qquad \mathcal{L}[u_a(t)] = e^{-as}\frac{1}{s}$$

$$\mathcal{L}[u_3(t)] = \frac{e^{-3s}}{s}$$

$\boxed{\text{Theorem}}$ Let $\mathcal{L}[f(t)] = F(s)$

$$\mathcal{L}[u_a(t)\, f(t-a)] = \mathcal{L}[u(t-a)\, f(t-a)] = e^{-as} F(s)$$

$$\mathcal{L}^{-1}[e^{-as} F(s)] = u_a(t)\, f(t-a) = u(t-a)\, f(t-a)$$

$$u_a(t)\, f(t-a) = u(t-a)\, f(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

$$\mathcal{L}^{-1}\left[e^{-as} F(s) \right] = u_a(t)\, f(t-a) = u(t-a)\, f(t-a)$$

$$\mathcal{L}^{-1}\left[e^{-3s} \frac{s}{s^2+2^2} \right] = u_3(t) \cos\left[2(t-3)\right]$$

$$= u(t-3) \cos\left[2(t-3)\right]$$

$$\mathcal{L}^{-1}\left[e^{-2s} \frac{1}{s^4} \right] = u_2(t)(t-2)^3$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^4} \right] = \mathcal{L}^{-1}\left[\frac{1}{3!} \frac{3!}{s^4} \right] = \frac{1}{6} \mathcal{L}^{-1}\left[\frac{3!}{s^4} \right]$$

$$\mathcal{L}\left[t^n \right] = \frac{n!}{s^{n+1}}$$

$$= \frac{1}{6} \cdot t^3$$

$$* \quad g(t) = \begin{cases} 0 & t < 3 \\ t^2 & t \geq 3 \end{cases} \Rightarrow \mathcal{L}[g(t)] = ?$$

$$g(t) = t^2 \begin{Bmatrix} 0 & t < 3 \\ 1 & t \geq 3 \end{Bmatrix} = t^2 u_3(t)$$

$$\boxed{\cancel{\mathcal{L}[u_3(t)\, t^2] = e^{-3s} \cdot \frac{2}{s^3}} \quad \cancel{= u_3(t)(t-3)^2}}$$

$$\mathcal{L}[u_a(t)\, f(t-a)] = e^{-as} F(s)$$

$$\mathcal{L}[u_3(t)\, \underbrace{t^2}_{f(t-3)}] = \mathcal{L}[u_3(t)\, f(t-3)] = e^{-as} F(s)$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

$$f(t-3) = t^2 \Rightarrow f(t) = (t+3)^2 = t^2 + 6t + 9$$

$$t \to t+3 \qquad F(s) = \frac{2!}{s^3} + 6 \cdot \frac{1}{s^2} + 9 \cdot \frac{1}{s}$$

$$f(t) = \begin{cases} \cos 2t & 0 \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases} \qquad \mathcal{L}[f(t)] = ?$$

Remark

$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & a \leq t \end{cases} \checkmark$$

$$1 - u_a(t) = 1 - u(t-a) = \begin{cases} 1 & t < a \\ 0 & a \leq t \end{cases} \checkmark$$

$$f(t) = \cos 2t \left\{ \begin{matrix} 1 & t < 2\pi \\ 0 & 2\pi \leq t \end{matrix} \right\} = \cos 2t \cdot \left(1 - u_{2\pi}(t)\right)$$

$$f(t) = \left[1 - u_{2\pi}(t)\right] \cos 2t$$

$$= \cos 2t - u_{2\pi}(t) \cos 2t.$$

$$\mathcal{L}[f(t)] = \mathcal{L}[\cos 2t] - \mathcal{L}[u_{2\pi}(t) \cos 2t]$$

$$= \frac{s}{s^2 + 2^2} - \mathcal{L}[u_{2\pi}(t) \cos 2t]$$

$$\mathcal{L}[u_{2\pi}(t) \underbrace{\cos 2t}_{f(t-2\pi)}] = e^{-2\pi s} \frac{s}{s^2 + 2^2}$$

$$f(t-2\pi) = \cos 2t \Rightarrow f(t) = \cos 2t \rightarrow F(s) = \frac{s}{s^2 + 2^2}$$

Ex Solve $x'' + 4x = f(t)$, $\quad x(0) = x'(0) = 0$

where $f(t)$ is the function in the previous example.

let $\mathcal{L}[x(t)] = X(s)$

$$\mathcal{L}[x'' + 4x] = \mathcal{L}[f(t)]$$

$$s^2 X(s) - s\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=0} + 4X(s) = \frac{s}{s^2+2^2} - e^{-2\pi s}\frac{s}{s^2+2^2}$$

$$X(s) = \left(1 - e^{2\pi s}\right)\frac{s}{(s^2+4)^2}$$

$$x(t) = \mathcal{L}^{-1}\left\{\left(1-e^{-2\pi s}\right)\frac{s}{(s^2+4)^2}\right\}$$

$$H(s) = \frac{s}{(s^2+4)^2} = \frac{s}{s^2+4}\cdot\frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{1}{2}\,\mathcal{L}^{-1}\left[\underbrace{\frac{s}{s^2+2^2}}_{F(s)}\cdot\underbrace{\frac{2}{s^2+2^2}}_{G(s)}\right]$$

$f(t) = \cos 2t \qquad g(t) = \sin 2t$

$$= \frac{1}{2}\,\mathcal{L}^{-1}\left[F(s)\,G(s)\right] = \frac{1}{2}\,f(t) * g(t)$$

$$= \frac{1}{2}\,(\cos 2t) * (\sin 2t) = \frac{1}{4}\,t\sin 2t$$

$$x(t) = \mathcal{L}^{-1}\left\{ \left(1 - e^{-2\pi s}\right) \frac{s}{(s^2+4)^2} \right\}$$

$$= \mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2} \right] - \mathcal{L}^{-1}\left[e^{-2\pi s} \frac{s}{(s^2+4)^2} \right]$$

$$= \frac{1}{4} t \sin 2t - u_{2\pi}(t) \frac{1}{4} (t-2\pi) \sin 2(t-2\pi)$$

Remark: There's another way of evaluating

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2} \right].$$

$$\mathcal{L}^{-1}[F'(s)] = -t\,f(t)$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] = ? = (-t)\cdot\left(\frac{-1}{4}\right)\cdot \sin 2t$$

$$= \frac{1}{4}\,t\,\sin 2t \quad //$$

Let $F'(s) = \dfrac{s}{(s^2+4)^2} \Rightarrow F(s) = \displaystyle\int \frac{s\,ds}{(s^2+4)^2}$

$$F(s) = \int \frac{s\,ds}{(s^2+4)^2} = \frac{1}{2}\int \frac{2s\,ds}{(s^2+4)^2}$$

$u = s^2+4$

$du = 2s\,ds$

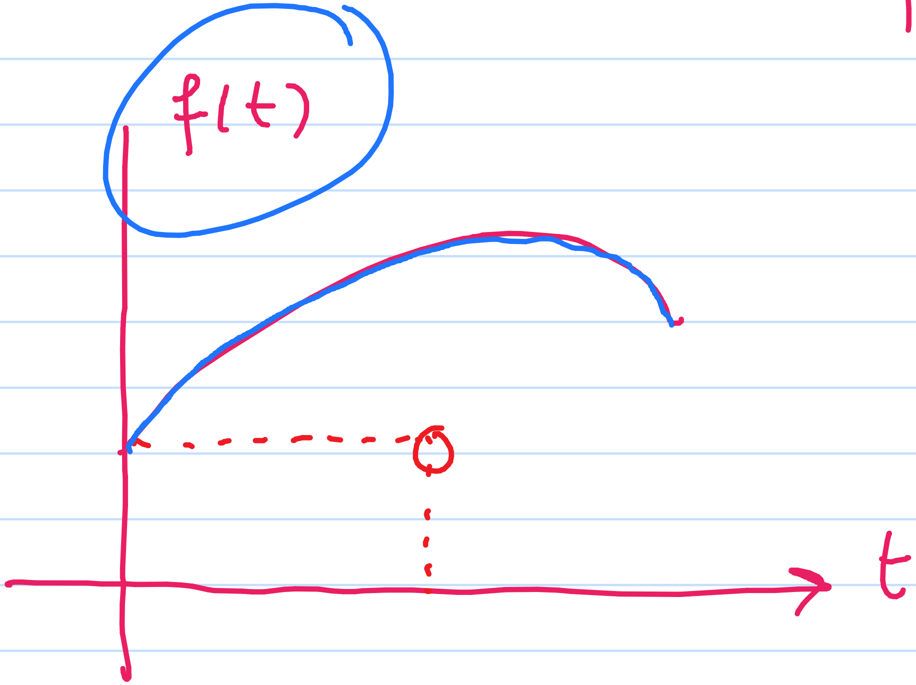
$$= \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} = -\frac{1}{2}\,\frac{1}{s^2+4} \Rightarrow f(t) = \frac{-1}{4}\sin 2t$$

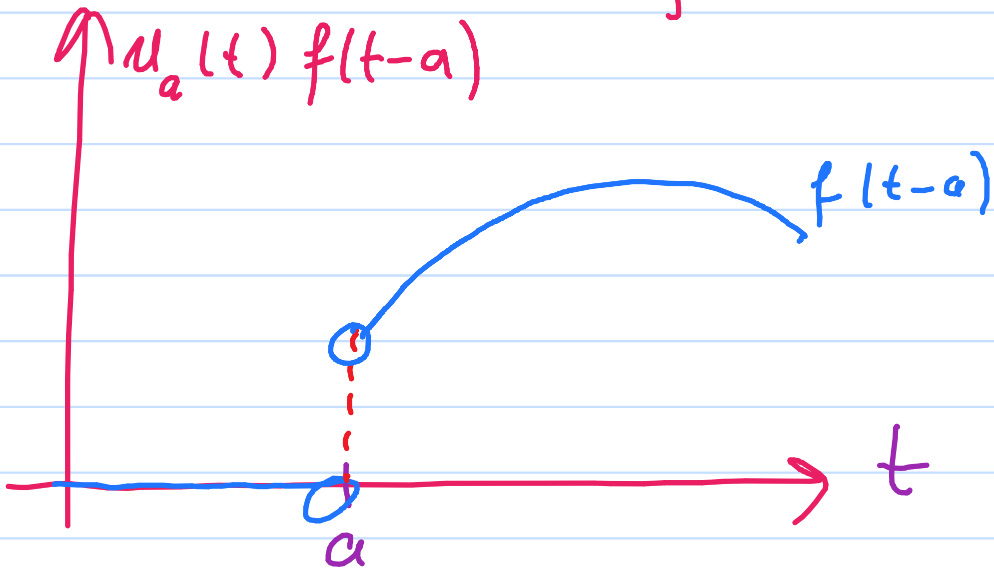
$\mathcal{L}[f(t)] = F(s) \Rightarrow \mathcal{L}[u_a(t)\, f(t-a)]$

$= \mathcal{L}[u(t-a)\, f(t-a)] = e^{-as} F(s)$

$m x'' + b x' + k x = f(t)$ → outer force

source fuction.





$f(t) : [0, \infty) \to \mathbb{R}$

$t \in [0, \infty)$

$$u_a(t)\, f(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$

retarded force

delayed fuctions

$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ t^2 & 1 \leq t < 3 \\ \sin 2t & 3 \leq t \end{cases} \qquad \mathcal{L}[f(t)] = ?$$

$$f(t) = \begin{cases} 0 & 0 < t < a \\ f_1(t) & a \leq t < b \\ f_2(t) & b \leq t \end{cases}$$

$$f(t) = u_a(t)\, g(t) + u_b(t)\, h(t)$$

$$f(t) = u_a(t)\, f_1(t) + u_b(t)\left[f_2(t) - f_1(t)\right]$$

$$f(t) = u_1(t)\, t^2 + u_3(t)\left[\sin 2t - t^2\right]$$

$$\mathcal{L}\left[u_1(t)\, t^2\right] = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

$$f(t-1) = t^2$$

$$f(t) = (t+1)^2 = t^2 + 2t + 1 \Rightarrow F(s) = \frac{2!}{s^3} + 2\cdot\frac{1}{s^2} + \frac{1}{s}$$

$$\mathcal{L}\left[u_3(t)\, \underbrace{t^2}_{f(t-3)}\right] = e^{-3s}\left(\frac{2}{s^3} + 6\cdot\frac{2}{s^2} + 9\cdot\frac{1}{s}\right)$$

$$f(t-3) = t^2$$

$$f(t) = (t+3)^2 = t^2 + 6t + 9$$

$$\mathcal{L}[\,u_3(t)\,\underbrace{\sin 2t}_{f(t-3)}\,] = ?$$

$$f(t-3) = \sin 2t \Rightarrow f(t+3-3) = \sin 2(t+3)$$

$$f(t) = \sin(2t+6) = \sin 2t \cos 6 + \sin 6 \cos 2t$$

$$F(s) = \cos 6 . \frac{2}{s^2+2^2} + \sin 6 . \frac{s}{s^2+2^2}$$

$$\rightarrow \; = e^{-3s}\left(\cos 6 . \frac{2}{s^2+4} + \sin 6 . \frac{s}{s^2+4}\right)$$