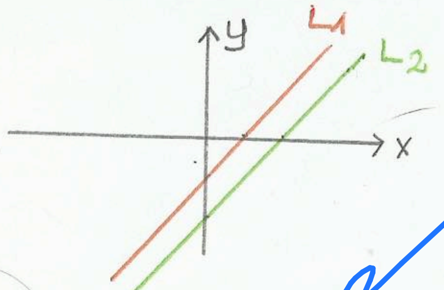
# Ch3 Linear Systems & Matrices

## 3.1 Intro. to Linear Systems

### LINEAR SYSTEM

A Linear system is said to be consistent if it has at least one solution and inconsistent if it has no solution.

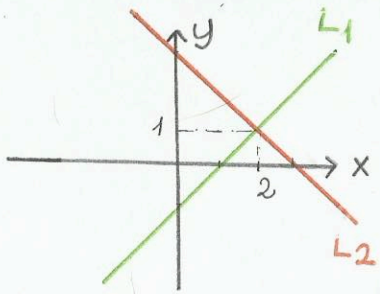
$L_1: x-y=1$ (a)
$L_2: 2x-2y=4$ (b)



$L_1 \parallel L_2 \Rightarrow$ no solution

Sol. set $= \phi$

$L_1: x-y=1$
$L_2: x+y=3$

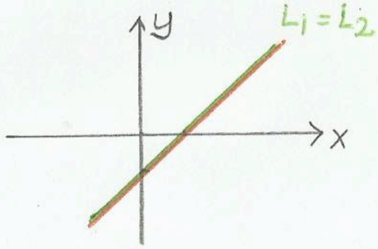


Intersecting lines
$\Rightarrow$ unique solution

sol. set $= \{(1,2)\}$

$L_1: x-y=1$
$L_2: 2x-2y=2$

$y=t$
$x=t+1$



coincident Lines
$\Rightarrow$ infinitely many solutions

sol set $= \{(t+1,\ t) \mid t \in \mathbb{R}\}$

**Ex**

$$\begin{array}{ll} (a) & x+2y+z=4 \\ (b) & 3x+8y+7z=20 \\ (c) & 2x+7y+9z=23 \end{array} \xrightarrow[-2(a)+(c)]{-3(a)+(b)} \begin{array}{l} x+2y+z=4 \\ 2y+4z=8 \\ 3y+7z=15 \end{array} \xrightarrow{(b)/2} \begin{array}{l} x+2y+z=4 \\ y+2z=4 \\ 3y+7z=15 \end{array} \xrightarrow{-3(b)+(c)} \begin{array}{l} x+2y+z=4 \\ y+2z=4 \\ z=3 \end{array}$$

$z=3 \Rightarrow y=4-2z=-2,\ x=4-2y-z=5$ (Back substitution) $\rightarrow$ a unique solution

**Ex**

$$\begin{array}{ll} (a) & 3x-8y+10z=22 \\ (b) & x-3y+2z=5 \\ (c) & 2x-9y-8z=-11 \end{array} \rightarrow \begin{array}{ll} (b) & x-3y+2z=5 \\ (a) & 3x-8y+10z=22 \\ (c) & 2x-9y-8z=-11 \end{array} \xrightarrow[-2(b)+(c)]{-3(b)+(a)} \begin{array}{l} x-3y+2z=5 \\ y+4z=7 \\ -3y-12z=-21 \end{array} \xrightarrow{-\frac{1}{3}(c)} \begin{array}{l} x-3y+2z=5 \\ y+4z=7 \\ y+4z=7 \end{array}$$

$z=t \Rightarrow y=7-4t,\ x=26-14t \Rightarrow$ infinitely many solutions

$$ax + by = c$$
$$dx + ey = f$$

$x, y$ : unknowns

~~to~~ two eqs. in two unknowns

No terms like

$x_1^2$

$x_1 x_3$

$x_3^{2/3}$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = d_1$$
$$b_1 x_1 + b_2 x_2 + b_3 x_3 = d_2$$
$$c_1 x_1 + c_2 x_2 + c_3 x_3 = d_3$$

Linear system

three unknowns, three eqs.

$\downarrow$

$x_1, x_2, x_3$

"linear"

the powers of the unknowns are "1"

none of them are multiplied by each other

Ex 6 (a) $x+2y+z=4$

(b) $3x+8y+7z=20$

(c) $2x+7y+9z=23$

$$\xrightarrow[(-2)(a)+(c)]{(-3)(a)+(b)}$$

$$\begin{aligned} x+2y+z&=4\\ 2y+4z&=8\\ 3y+7z&=15 \end{aligned}$$

$$\xrightarrow{\frac{1}{2}(b)}$$

$$\begin{aligned} x+2y+z&=4\\ y+2z&=4\\ 3y+7z&=15 \end{aligned}$$

$$\xrightarrow{(-3)(b)+(c)}$$

$$\begin{aligned} x+2y+z&=4\\ y+2z&=4\\ z&=3 \end{aligned}$$

triangular form

backward substitution

- $z=3$
- $y+2.3=4 \longrightarrow y=-2$
- $x+2.(-2)+3=4 \longrightarrow x=5$

$x=5$
$y=-2$
$z=3$

solution set $=\{(5,-2,3)\}$

Ex 7 (a) $3x - 8y + 10z = 22$

(b) $x - 3y + 2z = 5$

(c) $2x - 9y - 8z = -11$

$$\left.\begin{aligned} 3x - 8y + 10z &= 22 \\ x - 3y + 2z &= 5 \\ 2x - 9y - 8z &= -11 \end{aligned}\right\} \xrightarrow{(a) \leftrightarrow (b)} \begin{aligned} &(b)\quad 1x - 3y + 2z = 5 \\ &(a)\quad 3x - 8y + 10z = 22 \\ &(c)\quad 2x - 9y - 8z = -11 \end{aligned}$$

$$\xrightarrow[(-2)(b)+(c)]{(-3)(b)+(a)} \left.\begin{aligned} x - 3y + 2z &= 5 \quad (b) \\ 0 + y + 4z &= 7 \quad (a) \\ 0 - 3y - 12z &= -21 \quad (c) \end{aligned}\right\} \xrightarrow[3(a)+(c)]{}$$

$$\left.\begin{aligned} x - 3y + 2z &= 5 \\ y + 4z &= 7 \\ 0 &= 0 \end{aligned}\right\}$$

in the beginning: three unknows, three eqs.

we see afterwards: three unknows, two equations!!!

Let us say $z = t \in \mathbb{R}$. $y = 7 - 4t$, $x = 26 - 14t$

Solution set $= \{(26 - 14t, 7 - 4t, t) \mid t \in \mathbb{R}\}$ inf. many solution

In the examples above, we performed one of the following three operations:

1. Multiplying one equation by a nonzero constant
2. Interchanging two equations
3. Add a constant multiple of one equation to another equation

These we call "elementary operations".

# 3.2 Matrices & Gaussian Elimination

$$
\begin{aligned}
x + 2y + z &= 4 \\
3x + 8y + 7z &= 20 \\
2x + 7y + 9z &= 23
\end{aligned}
$$

Coefficient matrix:

$$
\begin{array}{c}
\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{bmatrix}_{3\times 3} \\
\begin{matrix} \;x\; & \;\;y\;\; & \;\;z\; \end{matrix}
\end{array}
$$

Augmented coefficient matrix: (Artırılmış matris)

$$
\left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]_{3\times 4}
$$

$$
\underbrace{\hphantom{1 \quad 2 \quad 1}}_{LHS} \qquad \underbrace{\hphantom{23}}_{RHS}
$$

# LINEAR EQUATIONS

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\}$$

General system of $m$ linear equations

$a_{ij}$ : coefficients

$x_j$ : variable

$b_i$ : constants

\# of unknowns : $n$

\# of eqs : $m$

$b_1 = b_2 = \cdots = b_m = 0 \Rightarrow$ homogeneous

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad \begin{array}{l} \to \text{1st row} \\ \to \text{2nd row} \end{array}$$

1st column, 2nd col.

↓ coefficient matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

↓ column vector

$$[A \mid b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & & \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

↓ augmented coefficient matrix

**Leading entry**: The first nonzero element in a row

$a_{ij}$: $i$ → row number, $j$ → column number

**Matrix** is a rectangular array of numbers in the form $A = [a_{ij}]_{m \times n}$, $a_{ij} \in \mathbb{R}$

$m$: number of rows of the matrix $A$

$n$: " " columns " " "

$a_{ij}$ : The element/entry of $A$ at row $i$, column $j$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad C = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}_{3\times 1}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \qquad D = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$$

$$E = [2]_{1\times 1}$$

<u>Warning</u> :

$$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

" matrix quantity"

In special, we call an $m\times 1$ matrix a " vector "

$\vec{u} \in \mathbb{R}^2$ : $\vec{u} = a\vec{i} + b\vec{j} = \begin{bmatrix} a \\ b \end{bmatrix}_{2\times 1}$

$\vec{u} \in \mathbb{R}^3$ $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1}$

$\vec{u} \in \underline{\mathbb{R}^n}$ $\vec{u} = a_1\vec{i}_1 + a_2\vec{i}_2 + \cdots + a_n\vec{i}_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1}$

**Def.** Elementary Row Operations for a Matrix

Given a matrix $A$;

1. Multiplying any row of $A$ by a nonzero constant
2. Interchanging two rows of $A$
3. Add a constant multiple of one row of $A$ to another row.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[3R_1+R_2]{3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 6 & 10 \end{bmatrix}$$

| Type | Op. | Notation |
|---|---|---|
| 1 | Multiply row $p$ by $c$ | $cR_p$ |
| 2 | Interchange row $p$ and row $q$ | $R_p \longleftrightarrow R_q$ |
| 3 | Add $c$ times row $p$ to row $q$ | $c.R_p + R_q$ |

Ex $\quad x_1 + 2x_2 + x_3 = 4$ $\quad$ Solve this system.

$$3x_1 + 8x_2 + 7x_3 = 20$$

$$2x_1 + 7x_2 + 9x_3 = 23$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_2 + 2x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

Bacward substitution:

$$x_3 = 3$$

$$x_2 = 4 - 2 \cdot 3 = -2$$

$$x_1 = 4 - 2x_2 - x_3 = 5 \quad //$$

$$\left.\begin{array}{l} x_1, \ \ldots\ldots \ x_{1000000} \ \text{unknowns} \\ \sim 1\,000\,000 \ \text{eqs.} \end{array}\right\} \Rightarrow \left[ \qquad \qquad \right]_{10^6 \times 10^6}$$

solve the system by computer algebra in the language of matrizes!!!

**ELEMENTARY ROW OPERATIONS** →

- Multiply any row by a nonzero constant
- Interchange two rows
- Add a constant multiple of one row to another row

**ROW EQUIVALENT MATRICES** →

2 matrices are called row equivalent if one can be obtained from the other by a finite sequence of elementary row operations. Row equivalent matrices correspond to same linear system of equations.

**THEOREM** →

If the augmented coefficient matrices of 2 linear systems are row equivalent, then the 2 systems have the same solution set.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[2R_1 + R_2 \to R_2]{2R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix} = B$$

## ECHELON MATRIX

① If there is a row that consists entirely of zeroes, it must be the last row.

② A leading entry of each row lies strictly to the right of the leading entry in the preceding row. For each row, below the first nonzero entry, we all have zeros

## REDUCED ECHELON MATRIX

① It is an echelon matrix

② Each leading entry is 1.

③ Each leading entry is the only nonzero element in its column.

## MAIN DIFFERENCES

a nonzero number ← Leading entry → 1

below consists of 0s ← column of leading entry → below and above consists of 0s

## GAUSSIAN ELIMINATION — STEPS TO FOLLOW — GAUSS-JORDAN ELIMINATION

① Locate the 1st column of A that contains a nonzero element. If the 1st entry in this column is zero, interchange the 1st row of A with a row underneath in which the corresponding entry is nonzero.

(1*) Divide each element in that row by its leading entry.

② Replace the entries below it in the same column with zeros by adding appropriate multiples of the 1st row of A to lower rows.

(2*) Repeat 2, to upper rows in the same column.

Cover the row and the rows above it. Repeat for the remaining submatrix.

THEOREM → Every matrix is row equivalent to a unique reduced echelon matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ In echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ Not in echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$ In echelon form

✓ ✓

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$ not in echelon

✓ not zero form

$$\begin{bmatrix} 2 & -1 & 0 & 4 & 7 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in echelon form

← zero row at the end.

**Ex 4**

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\
x_3 \qquad\qquad + 2x_5 &= -3 \\
x_4 - 4x_5 &= 7
\end{aligned}
$$

The augmented matrix of this linear system is

$$
\begin{bmatrix}
1 & -2 & 3 & 2 & 1 & 10 \\
0 & 0 & 1 & 0 & 2 & -3 \\
0 & 0 & 0 & 1 & -4 & 7
\end{bmatrix}
$$

Already in echelon form

$x_1 = 5 + 2s - 3t$

$x_2 = s \in \mathbb{R}$

$x_3 = -3 - 2t$ $\quad x_4 = 7 + 4t$ $\quad x_5 = t \in \mathbb{R}$

# of unknowns = $n$

# of eqs. in echelon form = $m$

# of free parameters = $r = n - m$

In the previous example;

$$n = 5 \quad , \quad m = 3$$

$$r = n - m = 5 - 3 = 2 \text{ free parameters}$$

$x_2 = s \in \mathbb{R}$

$x_5 = t \in \mathbb{R}$

**Ex5** Solve

$$x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10$$
$$2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 = 7$$
$$3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 = 27$$

by Gaussian elimination (- Write the augmented matrix
- convert to echelon form

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & -1 & 8 & -13 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{array}\right]$$

in echelon form

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{array}\right] \xrightarrow[\text{in echelon form}]{-R_3}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right]$$ in echelon form

solve this

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ x_3 + 2x_5 &= -3 \\ x_4 - 4x_5 &= 7 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 &= \\ x_2 &= \\ x_3 &= \end{aligned}\right\},$$

$x_4 = 4t + 7$

$x_5 = t \in \mathbb{R}$

## 3.3 Reduced Row-Echelon Matrices

E is in reduced echelon form if

(i) E is in echelon form

(ii) Each leading nonzero entry (of each row) is 1

(iii) Each leading entry is the only nonzero element in its column

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

reduced echelon form.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

in echelon form? YES

in reduced echelon form NO

Can we convert this to reduced echelon form? YES

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

in echelon form? ✓

in reduced echelon?

NO

$$\xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

reduced echelon form ✓

Another example

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

echelon form ✓

each leading entry is 1 ✓

leading entries are the only nonzero elements in their columns

reduced echelon form

**Ex2** Find the reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix}_{3\times 4}$$

See yourself that the echelon form of A is

$$\begin{bmatrix} \boxed{1} & \boxed{2} & \boxed{1} & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

✓ X X

$$\xrightarrow[-3R_3+R_1]{-2R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] ✓$$

$1.x_1 + 0.x_2 + 0.x_3 = 5 \Rightarrow x_1 = 5$

$x_2 = -2$

$x_3 = 3$

Gaussian Elimination

Convert the (augmented) matrix to row-echelon form

leading entries are not necessarily 1

Gauss-Jordan Elimination

Convert the (augmented) matrix to reduced row-echelon form

leading entries are "1".

$\boxed{\text{Theorem}}$ Every matrix is row equivalent to a unique reduced echelon matrix.

Ex Use Gauss-Jordan elimination to solve the linear system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 12 \\ x_1 + 2x_2 \phantom{+ 4x_3} + 5x_4 &= 17 \\ 3x_1 + 2x_2 + 4x_3 - x_4 &= 31 \end{aligned}$$

$\Rightarrow$ we must write and transform the augmented matrix to the reduced echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 1 & 2 & 0 & 5 & 17 \\ 3 & 2 & 4 & -1 & 31 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & -1 & 1 & -4 & -5 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in echelon form ✓

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 7 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

in reduced echelon form

$x_1 \quad x_2 \quad x_3 \quad x_4$

$$x_1 + 2x_3 - 3x_4 = 7$$

$$x_2 - x_3 + 4x_4 = 5$$

$$0 = 0$$

$n = 4$

$m = 2$

$r = n - m = 2$ free parameters

$$\left.\begin{aligned} &x_3 = t \in \mathbb{R}, \quad x_4 = s \in \mathbb{R} \\ &x_2 = 5 + t - 4s \\ &x_1 = 7 - 2t + 3s \end{aligned}\right\}$$

infinite set of solutions depending on two parameters $t, s \in \mathbb{R}$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7-2t+3s \\ 5+t-4s \\ t \\ s \end{bmatrix} \in \mathbb{R}^4$$

$$\underset{\sim}{x} = \begin{bmatrix} 7 \\ 5 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$$\in \mathbb{R}^4 \qquad \in \mathbb{R}^4 \qquad \in \mathbb{R}^4$$

$$\underset{\sim}{x} = \vec{a} + t\,\vec{b} + s\,\vec{c}$$

Ex $x_2 + 3x_3 = 5$

$x_1 + 2x_2 + x_3 = 1$

$2x_1 - 3x_2 + x_3 = 7$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & -3 & 1 \end{bmatrix}_{3\times 3}$$

1,1,2: leading entries

→ coefficient matrix

DIY

$$[A|b] = \left[\begin{array}{ccc|c} 0 & 1 & 3 & 5 \\ 1 & 2 & 1 & 1 \\ 2 & -3 & 1 & 7 \end{array}\right]_{3\times 4}$$

→ augmented coefficient matrix

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 2 & -3 & 1 & 7 \end{bmatrix} \xrightarrow[\to R_3]{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & -7 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow[\to R_3]{7R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ \boxed{0} & 1 & 3 & 5 \\ \boxed{0} & \boxed{0} & 20 & 40 \end{bmatrix}$$

echelon matrix (Gaussian elimination)

$20x_3 = 40 \Rightarrow x_3 = 2$, $x_2 + 3x_3 = 5 \Rightarrow x_2 = 5 - 6 = -1$, $x_1 + 2x_2 + x_3 = 1 \Rightarrow x_1 = 1$

unique solution

$$\xrightarrow[\to R_1]{-2R_2+R_1} \begin{bmatrix} 1 & 0 & -5 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 20 & 40 \end{bmatrix} \xrightarrow[\to R_3]{\frac{1}{20}R_3} \begin{bmatrix} 1 & 0 & -5 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[\to R_1]{5R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[\to R_2]{-3R_3+R_2} \begin{bmatrix} 1 & \boxed{0} & \boxed{0} & 1 \\ \boxed{0} & 1 & \boxed{0} & -1 \\ \boxed{0} & \boxed{0} & 1 & 2 \end{bmatrix}$$

reduced echelon matrix (Gauss-Jordan elimination)

$x_1 = 1$, $x_2 = -1$, $x_3 = 2$

**Ex** $x_1 - x_2 + x_3 - x_4 - x_5 = 2$

$2x_3 + x_5 = -1$

$x_3 + x_4 - 2x_5 = 3$

DIY

$$\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 2 \\ 0 & 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -2 & 3 \end{bmatrix} \xrightarrow[\to R_3]{-\frac{R_2}{2}+R_3} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array} \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 2 \\ 0 & 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -\frac{5}{2} & \frac{7}{2} \end{bmatrix}$$

echelon matrix (Gaussian elim.)

Leading entries: $x_1, x_3, x_4$ free variables: $x_2, x_5$

$x_2 = s,\ x_5 = t \Rightarrow x_4 - \frac{5}{2}x_5 = \frac{7}{2} \Rightarrow x_4 = \frac{7+5t}{2}$, $2x_3 + x_5 = -1 \Rightarrow x_3 = \frac{-1-t}{2}$

$x_1 - x_2 + x_3 - x_4 - x_5 = 2 \Rightarrow x_1 = s + \frac{1+t}{2} + \frac{7+5t}{2} + t + 2 \Rightarrow x_1 = s + 4t + 6$

$\Rightarrow$ infinitely many solutions

$$\xrightarrow[\to R_2]{\frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 & -5/2 & 7/2 \end{bmatrix} \xrightarrow[\to R_1]{-R_2+R_1} \begin{bmatrix} 1 & -1 & 0 & -1 & -3/2 & 5/2 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 & -5/2 & 7/2 \end{bmatrix}$$

$$\xrightarrow[\to R_1]{R_3+R_1} \begin{bmatrix} 1 & -1 & 0 & 0 & -4 & 6 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 & -5/2 & 7/2 \end{bmatrix}$$

reduced echelon matrix (Gauss-Jordan elim.)

$x_2 = s,\ x_5 = t \Rightarrow x_4 - \frac{5}{2}x_5 = +\frac{7}{2} \Rightarrow x_4 = \frac{7+5t}{2}$, $x_3 + \frac{1}{2}x_5 = -\frac{1}{2} \Rightarrow x_3 = \frac{-1-t}{2}$

$x_1 - x_2 - 4x_5 = 6 \Rightarrow x_1 = 6 + s + 4t$

Ex For what values of a, b and c does the system

$$2x-y+3z=a$$
$$x+2y+z=b$$
$$7x+4y+9z=c$$

have a unique solution, no solution, infinitely many solutions?



$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 2 & -1 & 3 & a \\ 7 & 4 & 9 & c \end{array}\right] \xrightarrow[\to R_2]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & b \\ 0 & -5 & 1 & -2b+a \\ 7 & 4 & 9 & c \end{bmatrix} \xrightarrow[\to R_3]{-7R_1+R_3} \begin{bmatrix} 1 & 2 & 1 & b \\ 0 & -5 & 1 & -2b+a \\ 0 & -10 & 2 & -7b+c \end{bmatrix}$$

$$\xrightarrow[\to R_3]{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & b \\ 0 & -5 & 1 & -2b+a \\ 0 & 0 & 0 & -3b-2a+c \end{bmatrix}$$

$x_1 + 2x_2 + x_3 = b$

$-5x_2 + x_3 = -2b+a$

~~echelon matrix~~ (Gaussian elim.)

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = c-2a-3b$

$0 = c-2a-3b$

$0x+0y+0z = -3b-2a+c = 0 \Rightarrow -3b-2a+c \neq 0 \Rightarrow$ no solution

$c = 3b+2a \Rightarrow$ 3 unknowns, 2 equations $\Rightarrow$ infinetely many solutions

Ex
$$\left.\begin{array}{l} x+2y-z=1 \\ 2x+y+5z=2 \\ 3x+3y+4z=1 \end{array}\right\}$$ Solve the Lin. system by using Gaussian elimination

DIY

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 5 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix} \xrightarrow[\to R_2]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 7 & 0 \\ 3 & 3 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow[\to R_3]{-3R_1+R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 7 & 0 \\ 0 & -3 & 7 & -2 \end{bmatrix} \xrightarrow[\to R_3]{-R_2+R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 7 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$0x+0y+0z = 0 \neq -2 \Rightarrow$ no solution

# HOMOGENEOUS SYSTEMS

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underset{\sim}{0}$$

trivial

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\}$$

A homog. system has at least one solution (zero sol.)

$x_1 = x_2 = \dots = x_n = 0$ (Trivial solution)

A homog. system either has only the trivial solution or has infinitely many solutions.

**THEOREM** A homogeneous Linear system with more variables than equations has infinitely many solutions.

- unique solution
- ~~no solution~~
- inf. many sols.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

→ square matrix (col. num = row num)

principal diagonal

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

→ identity matrix : principal diagonal consists of 1s and zeros elsewhere.

**THEOREM** Let A be an nxn matrix. Then the homogeneous system with coefficient matrix A has only the trivial solution if and only if A is row equivalent to the nxn identity matrix

- trivial solution
- inf. many sols. (includes the trivial)

**Ex** $x_1+2x_2+x_3=0$, $3x_1+8x_2+7x_3=0$, $2x_1+7x_2+9x_3=0$ } Solve the homogeneous system

$$\begin{bmatrix}1&2&1\\3&8&7\\2&7&9\end{bmatrix}\xrightarrow[-2R_1+R_3\to R_3]{-3R_1+R_2\to R_2}\begin{bmatrix}1&2&1\\0&2&4\\0&3&7\end{bmatrix}\xrightarrow{\frac{1}{2}R_2\to R_2}\begin{bmatrix}1&2&1\\0&1&2\\0&3&7\end{bmatrix}$$

echelon ✓

$$\xrightarrow[-3R_2+R_3\to R_3]{-2R_2+R_1\to R_1}\left[\begin{array}{ccc:c}1&0&3&0\\0&1&2&0\\0&0&1&0\end{array}\right]\xrightarrow[-2R_3+R_2\to R_2]{-3R_3+R_1\to R_1}\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$$

$x_1=x_2=x_3=0$

trivial solution

$$\left.\begin{aligned}x_1+3x_3&=0\\x_2+2x_3&=0\\x_3&=0\end{aligned}\right\}\ x_1=x_2=x_3=0$$

$$\xrightarrow{-R_3+R_1}\left[\begin{array}{ccc:c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]\begin{array}{l}\to x_1=0\\\to x_2=0\\\to x_3=0\end{array}$$

reduced echelon ✓

**Ex** $x_1+x_3+x_4=0$, $x_2-x_3-x_4=0$, $x_2+x_4=0$ } 4 unknowns, 3 equations

$\Rightarrow$ infinitely many solutions

$$\begin{bmatrix}1&0&1&1\\0&1&-1&-1\\0&1&0&1\end{bmatrix}\xrightarrow{-R_2+R_3\to R_3}\begin{bmatrix}1&0&1&1\\0&1&-1&-1\\0&0&1&2\end{bmatrix}\xrightarrow[R_3+R_2\to R_2]{-R_3+R_1\to R_1}\begin{array}{c}\begin{array}{cccc}x_1&x_2&x_3&x_4\end{array}\\\left[\begin{array}{cccc:c}1&0&0&-1&0\\0&1&0&1&0\\0&0&1&2&0\end{array}\right]\end{array}$$

reduced echelon form

$x_1-x_4=0$

$x_2+x_4=0$

$x_3+2x_4=0$

$x_4=t \Rightarrow x_3+2x_4=0 \Rightarrow x_3=-2t$, $x_2+x_4=0\Rightarrow x_2=-t$

$x_1-x_4=0\Rightarrow x_1=t$

$$\underset{\sim}{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}t\\-t\\-2t\\t\end{bmatrix}=t\begin{bmatrix}1\\-1\\-2\\1\end{bmatrix},\ t\in\mathbb{R}$$