Let the power series for $f(x)$ be

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \quad (1)$$

where $a_0, a_1, a_2, \ldots$ are constants.

When $x=0$, $\boldsymbol{f(0) = a_0}$

Differentiating equation (1) with respect to x gives:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \quad (2)$$

When $x=0$, $\boldsymbol{f'(0) = a_1}$

Differentiating equation (2) with respect to x gives:

$$f''(x) = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2 + (5)(4)a_5x^3 + \cdots \quad (3)$$

When $x=0$, $f''(0)=2a_2=2!\,a_2$, i.e. $\boldsymbol{a_2 = \frac{f''(0)}{2!}}$

Differentiating equation (3) with respect to x gives:

$$f'''(x) = (3)(2)a_3 + (4)(3)(2)a_4x + (5)(4)(3)a_5x^2 + \cdots \quad (4)$$

When $x=0$, $f'''(0)=(3)(2)a_3=3!\,a_3$, i.e. $\boldsymbol{a_3 = \frac{f'''(0)}{3!}}$

Continuing the same procedure gives $\boldsymbol{a_4 = \frac{f^{\text{iv}}(0)}{4!}}$, $\boldsymbol{a_5 = \frac{f^{\text{v}}(0)}{5!}}$, and so on.

Substituting for $a_0, a_1, a_2, \ldots$ in equation (1) gives:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

i.e. $$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \quad (5)$$

Equation (5) is a mathematical statement called **Maclaurin's theorem** or **Maclaurin's series**.

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

Hence at some point $f(h)$ in Fig. 52.1:

$$f(h) = f(0) + h\, f'(0) + \frac{h^2}{2!} f''(0) + \cdots$$

If the y-axis and origin are moved a units to the left, as shown in Fig. 52.2, the equation of the same curve relative to the new axis becomes $y = f(a+x)$ and the function value at P is $f(a)$.
At point Q in Fig. 52.2:

$$f(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} f''(a) + \cdots \qquad (1)$$

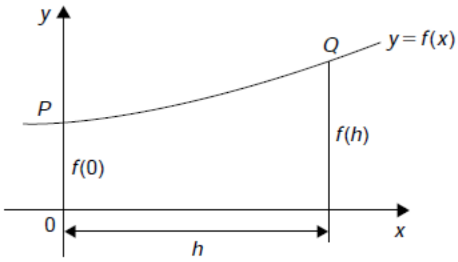

Figure 52.1

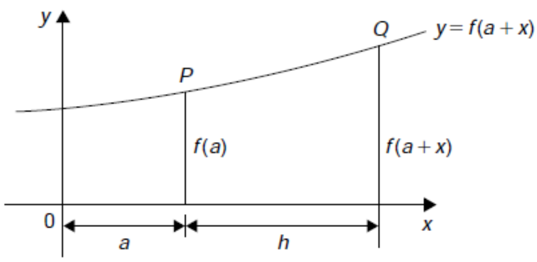

Figure 52.2

which is a statement called **Taylor's*** **series**.

Reference: Higher Engineering Mathematics, John Bird, Routledge, 2010.