LAPLACE TRANSFORM

Let $f(t)$ be defined for all $t \geqslant 0$. If the following improper integral converges, then it is called the Laplace transform of f.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

① $f(t) = 1, \quad t \geqslant 0$

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}\, dt = \lim_{b\to\infty} -\frac{1}{s} e^{-st} \Big|_0^b = \lim_{b\to\infty} -\frac{1}{s}\big(\overset{\to 0 \text{ if } s>0}{\cancel{e^{-sb}}} - 1\big)$$

$$= \frac{1}{s} \text{ for } s>0$$

② $f(t) = e^{at}, \quad t \geqslant 0$

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\, dt = \int_0^{\infty} e^{-(s-a)t}\, dt = \lim_{b\to\infty} -\frac{1}{s-a} e^{-(s-a)t} \Big|_0^b$$

$$= -\frac{1}{s-a} \lim_{b\to\infty} \big(\overset{\to 0 \text{ if } s>a}{\cancel{e^{-(s-a)b}}} - 1\big) = \frac{1}{s-a} \text{ for } s>a$$

If a is a complex number, the formula still holds.

$$a = \alpha + i\beta \Rightarrow e^{-(s-a)b} = e^{-(s-\alpha)b} \cdot e^{ib} = e^{-(s-a)b}(\cos\beta b + i\sin\beta b)$$

$$\Rightarrow \lim_{b\to\infty} e^{-(s-a)b}(\cos\beta b + i\sin\beta b) = 0$$

$$\left\{ \begin{array}{l} \text{Recall that } \quad -1 \leq \cos\beta b \leq 1 \\ \qquad -e^{-(s-a)b} \leq e^{-(s-a)b}\cos\beta b \leq e^{-(s-a)b} \\ \\ \text{Since } \lim\limits_{b\to\infty} -e^{-(s-a)b} = \lim\limits_{b\to\infty} e^{-(s-a)b} = 0 \text{ for } s>a. \\ \text{by the Sandwich Theorem.} \end{array} \right\}$$

Gamma Function: $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\,dt \quad , x>0$

$$\Gamma(1) = \int_0^{\infty} e^{-t} \cdot 1 \; dt = -e^{-t}\Big|_0^{\infty} = -(0-1) = 1$$

$$\Gamma(x+1) = \int_0^{\infty} e^{-t}\, t^{x}\, dt \quad \begin{Bmatrix} t^{x} = u \Rightarrow x t^{x-1} dt = du \\ e^{-t} dt = dv \Rightarrow v = -e^{-t} \end{Bmatrix}$$

$$= \cancel{-e^{-t}\, t^{x}\Big|_0^{\infty}}^{\,0} + x\int_0^{\infty} e^{-t}\, t^{x-1}\, dt = x\,\Gamma(x)$$

$$\Rightarrow \Gamma(x+1) = x\,\Gamma(x)$$

If n is a positive integer, then

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \dots = n(n-1)\cdots 2.1\,\cancel{\Gamma(1)}^{\,1} = n!$$

3 $f(t) = t^{a}$, $a>-1$ and a is real.

$$\mathcal{L}\{t^{a}\} = \int_0^{\infty} e^{-st}\, t^{a}\, dt \quad \begin{Bmatrix} u = st \\ du = s\,dt \end{Bmatrix} = \int_0^{\infty} e^{-u} \cdot \frac{u^{a}}{s^{a}}\, \frac{1}{s}\, du$$

$$= \frac{1}{s^{a+1}} \int_0^{\infty} e^{-u}\, u^{a}\, du = \frac{\Gamma(a+1)}{s^{a+1}}$$

$n \geq 0$ integer $\Rightarrow \mathcal{L}\{t^{n}\} = \dfrac{n!}{s^{n+1}}$, $s>0$ since $\Gamma(n+1) = n!$

LINEARITY OF LAPLACE TRANSFORM

a, b: constants , $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{g(t)\}$ exist

$$\mathcal{L}\{af(t) + bg(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$$

PROOF: $\mathcal{L}\{af(t)+bg(t)\} = \int_0^{\infty} e^{-st}\,(af(t) + bg(t))\, dt$

$$= a\int_0^{\infty} e^{-st} f(t)\, dt + b\int_0^{\infty} e^{-st} g(t)\, dt = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$$

① $\Gamma(1/2) = \sqrt{\pi}$ is known since $\Gamma(x+1) = x\Gamma(x)$, then

$$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2) = \frac{3}{4}\sqrt{\pi}$$

$$\mathcal{L}\{3t^2+4t^{3/2}\} = 3\mathcal{L}\{t^2\} + 4\mathcal{L}\{t^{3/2}\} \quad \left[\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ \mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}\right]$$

$$= 3\,\frac{2!}{s^3} + 4\,\frac{\Gamma(5/2)}{s^{5/2}} = \frac{6}{s^3} + 3\sqrt{\pi/s^5}$$

② $$\mathcal{L}\{\cosh kt\} = \mathcal{L}\left\{\frac{1}{2}(e^{kt}+e^{-kt})\right\} = \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right] = \frac{s}{s^2-k^2}, \quad (s>k>0)$$

$$\mathcal{L}\{\sinh kt\} = \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} - \mathcal{L}\{e^{-kt}\}\right] = \frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right] = \frac{k}{s^2-k^2} \quad (s>k>0)$$

$$\left.\begin{aligned} e^{ikt} &= \cos kt + i\sin kt \\ e^{-ikt} &= \cos kt - i\sin kt \end{aligned}\right\} \quad \begin{aligned} \cos kt &= \frac{1}{2}(e^{ikt}+e^{-ikt}) = \cosh(ikt) \\ \sin kt &= \frac{1}{2}(e^{ikt}-e^{-ikt}) = \sinh(ikt) \end{aligned}$$

$$\mathcal{L}\{\cos kt\} = \mathcal{L}\{\cosh(ikt)\} = \frac{s}{s^2-(ik)^2} = \frac{s}{s^2+k^2} \quad (s>0)$$

$$\Rightarrow \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \quad (s>0)$$

③ $$\sin^2 kt = \frac{1}{2}(1-\cos 2kt)$$

$$\mathcal{L}\{\sin^2 kt\} = \frac{1}{2}\left[\mathcal{L}\{1\} - \mathcal{L}\{\cos 2kt\}\right] = \frac{1}{2}\left(\frac{1}{s} - \frac{s^2}{s^2+36k^2}\right)$$

INVERSE LAPLACE TRANSFORM

$$F(s) = \mathcal{L}\{f(t)\} \Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}$$

* $$\mathcal{L}\{1\} = \frac{1}{s} \Rightarrow 1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\mathcal{L}\{\cos kt\} = \frac{s^2}{s^2+k^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s^2}{s^2+k^2}\right\} = \cos kt$$

PIECEWISE CONTINUOUS FUNCTION

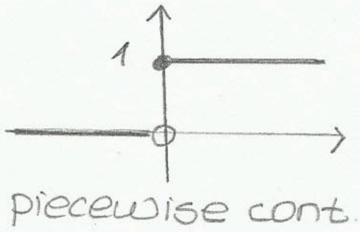
1- f is continuous in the interior of each subintervals of $[a,b]$

2- $f(t)$ has a finite limit as t approaches each endpoint of these subintervals from its interior

A function f which satisfies the properties above is called a piecewise continuous function on $[a,b]$

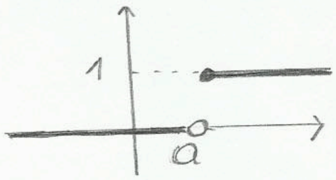
STEP FUNCTION

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$$ unit step function



$\Rightarrow$ for $t \geq 0$, $\mathcal{L}\{u(t)\} = \frac{1}{s}$, $s>0$.

$$u_a(t) = \begin{cases} 0, & t<a \\ 1, & t \geq a \end{cases} = u(t-a)$$



* Let $a>0$.

$$\mathcal{L}\{u_a(t)\} = \int_0^{\infty} e^{-st} u_a(t)\,dt = \int_a^{\infty} e^{-st} \cdot 1\,dt = -\frac{1}{s} e^{-st} \Big|_a^b$$

$$= \frac{e^{-as}}{s} \quad (s>0,\ a>0)$$

EXISTENCE OF LAPLACE TRANSFORMS

If f is a piecewise continuous function and satisfies

$|f(t)| \leq Me^{ct}$ for $t \geq T$, (M, c, T are nonneg. const)
(f is of exponential order as $t \to \infty$)

then $F(s)$ exists for all $s>c$ and $\lim_{s \to \infty} F(s) = 0$.

UNIQUENESS OF INVERSE LAPLACE TR.

Suppose that $F(s)$ and $G(s)$ which are the Laplace transforms of $f(t)$ and $g(t)$ both exist.
If $F(s) = G(s)$ for all $s > c$ for some c, then $f(t) = g(t)$ wherever on $[0, \infty)$ both f and g are continuous.

TRANSFORMS OF DERIVATIVES

Suppose that $f(t)$ is continuous, piecewise smooth for $t \geqslant 0$ and is of exponential order as $t \to \infty$. Thet $\mathcal{L}\{f'(t)\}$ exists for $s > c$ and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \quad \left\{\begin{array}{l} f'(t)\,dt = dv \Rightarrow v = f(t) \\ e^{-st} = u \Rightarrow -se^{-st}\,dt = du \end{array}\right\}$$

$$= e^{-st} f(t) \Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + s\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{g'(t)\} \quad \text{where } g(t) = f'(t)$$

$$= s\,\mathcal{L}\{g(t)\} - g(0) = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = \mathcal{L}\{g''(t)\} \quad \text{where } g(t) = f'(t)$$

$$= s^2\mathcal{L}\{g(t)\} - sg(0) - g'(0) = s^2\mathcal{L}\{f'(t)\} - sf'(0) - f''(0)$$

$$= s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$$

$$\cdots$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

PARTIAL FRACTIONS

$$\frac{P(s)}{(s-a)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

$$\frac{P(s)}{(s-a)^2+b^2} = \frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{(s-a)^2+b^2} + \cdots + \frac{A_n s + B_n}{(s-a)^2+b^2}$$

SOLUTION OF INITIAL VALUE PROBLEM

We apply Laplace transform to the Lin. diff eq and find $X(s)$. Then apply inverse Laplace transform and find $x(t)$.

Ex $x'' - x' - 6x = 0$, $x(0) = 2$, $x'(0) = -1$ $\quad (\mathcal{L}\{0\} = 0)$

$\mathcal{L}\{x'' - x' - 6x\} = \cancel{\mathcal{L}\{0\}}^{\,0} \Rightarrow s^2\mathcal{L}\{x\} - s\,x(0) - x'(0) - (s\mathcal{L}\{x\} - x(0)) - 6\mathcal{L}\{x\} = 0$

$(s^2 - s - 6)\mathcal{L}\{x\} = s\,2 - 1 - 2 \Rightarrow \mathcal{L}\{x(t)\} = \dfrac{2s-3}{(s-3)(s+2)} = X(s)$

$\dfrac{2s-3}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2} \Rightarrow A(s+2) + B(s-3) = 2s-3$

$s = -2 \Rightarrow -5B = -7 \Rightarrow B = 7/5$, $s = 3 \Rightarrow 5A = 3 \Rightarrow A = 3/5$

$X(s) = \dfrac{3}{5}\dfrac{1}{s-3} + \dfrac{7}{5}\dfrac{1}{s+2} \Rightarrow \mathcal{L}^{-1}\{X(s)\} = \dfrac{3}{5}\mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} + \dfrac{7}{5}\mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\}$

Recall that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a} \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$

$\Rightarrow \mathcal{L}^{-1}\{X(s)\} = x(t) = \dfrac{3}{5}e^{3t} + \dfrac{7}{5}e^{-2t}$

Ex $x'' + 4x = \sin 3t$, $x(0) = x'(0) = 0$ $\quad \left(\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}\right)$

$s^2\mathcal{L}\{x\} - s\,x(0) - x'(0) + 4\mathcal{L}\{x\} = \mathcal{L}\{\sin 3t\}$

$\Rightarrow (s^2+4)\mathcal{L}\{x\} = \dfrac{3}{s^2+9} \Rightarrow \mathcal{L}\{x(t)\} = \dfrac{3}{(s^2+4)(s^2+9)} = X(s)$

$$\frac{3}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$\Rightarrow (As+B)(s^2+9) + (Cs+D)(s^2+4) = 3$

$s^2=-9 \Rightarrow (Cs+D)(-5) = 3 \Rightarrow C=0 \Rightarrow D=-3/5$

$s^2=-4 \Rightarrow (As+B)\,5 = 3 \Rightarrow A=0,\ B=3/5$

$$X(s) = \frac{3}{5}\frac{1}{s^2+4} - \frac{3}{5}\frac{1}{s^2+9} \Rightarrow \mathcal{L}^{-1}\{X(s)\} = \frac{3}{5}\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{3}{5}\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\Rightarrow x(t) = \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t$$

Ex Show that $\mathcal{L}\{te^{at}\} = \dfrac{1}{(s-a)^2}$. $\left\{\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}\right\}$

$f(t) = te^{at} \Rightarrow f(0)=0,\ f'(t) = (1+at)e^{at}$

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - \cancel{f(0)}^{\,0}$

$\mathcal{L}\{(1+at)e^{at}\} = \mathcal{L}\{e^{at}\} + a\mathcal{L}\{te^{at}\} = s\mathcal{L}\{te^{at}\}$

$$(s-a)\mathcal{L}\{te^{at}\} = \frac{1}{s-a} \Rightarrow \mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

Ex Use $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$ to find $\mathcal{L}\{t\sin kt\}$.

$f(t) = t\sin kt \Rightarrow f(0)=0,\ f'(t) = \sin kt + kt\cos kt \Rightarrow f'(0)=0$

$f''(t) = k\cos kt + k\cos kt - k^2 t\sin kt$

$\quad = 2k\cos kt - k^2 t\sin kt$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s\cancel{f(0)}^{\,0} - \cancel{f'(0)}^{\,0}$

$2k\,\mathcal{L}\{\cos kt\} - k^2\mathcal{L}\{t\sin kt\} = s^2\mathcal{L}\{t\sin kt\}$

$$(s^2+k^2)\mathcal{L}\{t\sin kt\} = 2k\cdot\frac{s}{s^2+k^2}$$

$$\Rightarrow \mathcal{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}$$

TRANSFORM OF INTEGRALS

Suppose that $f(t)$ is a piecewise cont. function for $t \geq 0$ and satisfies the condition of exponential order $|f(t)| \leq Me^{ct}$ for $t \geq T$. Then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \quad \text{for } s > c.$$

$$\left(\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau\right)$$

Proof: Let $g'(t) = f(t)$

Then $g(t) = \int_0^t f(\tau)\,d\tau$ $\left[\begin{array}{l}\text{Fund. Th. of calc:} \\ \dfrac{d}{dt}\displaystyle\int_{u(t)}^{v(t)} f(\tau)\,d\tau = f(v)\,v' - f(u)\,u'\end{array}\right]$

$$g(t) \leq \int_0^t |f(\tau)|\,d\tau \leq \int_0^t Me^{c\tau}\,d\tau = \frac{M}{c}(e^{ct}-1) < \frac{M}{c}e^{ct}$$

$\Rightarrow$ g is of exponential order as $t \to \infty$.

$\Rightarrow$ g is cont. and piecewise smooth. for $t \geq 0$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\,\mathcal{L}\{g(t)\} - \cancel{g(0)}^{\,0}$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

Ex Find the inverse Laplace transform of $G(s) = \dfrac{1}{s^2(s-a)}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}d\tau = \int_0^t e^{a\tau}\,d\tau$$

$$= \frac{1}{a}e^{a\tau}\Big|_0^t = \frac{1}{a}(e^{at}-1)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}d\tau = \int_0^t \frac{1}{a}(e^{a\tau}-1)\,d\tau$$

$$= \frac{1}{a^2}e^{a\tau} - \frac{1}{a}\tau\Big|_0^t = \frac{1}{a^2}e^{at} - \frac{1}{a}t - \frac{1}{a^2} = \frac{1}{a^2}(e^{at}-1-at)$$

TRANSLATION ON THE S-AXIS

If $F(s)=\mathcal{L}\{f(t)\}$ exists for $s>c$, then $\mathcal{L}\{e^{at}f(t)\}$ exists for $s>a+c$ and

$$\mathcal{L}\{e^{at}f(t)\}=F(s-a).$$

$$\left(\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)\right)$$

Proof: $F(s)=\int_0^{\infty}e^{-st}f(t)\,dt$

$$\Rightarrow F(s-a)=\int_0^{\infty}e^{-(s-a)t}f(t)\,dt=\int_0^{\infty}e^{-st}\left(e^{at}f(t)\right)dt=\mathcal{L}\{e^{at}f(t)\}.$$

Ex: $x''+6x'+34x=0$, $x(0)=3$, $x'(0)=1$

$$\mathcal{L}\{x''\}+6\mathcal{L}\{x'\}+34\mathcal{L}\{x\}=\mathcal{L}\{0\}$$

$$s^2\mathcal{L}\{x(t)\}-s\,x(0)-x'(0)+6\left(s\mathcal{L}\{x(t)\}-x(0)\right)+34\mathcal{L}\{x\}=0$$

$$(s^2+6s+34)\mathcal{L}\{x(t)\}=3s+1+6.3=3s+19$$

$$X(s)=\mathcal{L}\{x(t)\}=\frac{3s+19}{s^2+6s+34}=\frac{3s+19}{(s+3)^2+25}=3\,\frac{s+3}{(s+3)^2+25}+2\,\frac{5}{(s+3)^2+25}$$

$$3\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+5^2}\right\}+2\mathcal{L}^{-1}\left\{\frac{5}{(s+3)^2+5^2}\right\}=\mathcal{L}^{-1}\{X(s)\}$$

$$\Rightarrow x(t)=3e^{-3t}\cos 5t+2e^{-3t}\sin 5t$$

Recall that $\mathcal{L}\{\sin 5t\}=\dfrac{5}{s^2+5^2}=F(s)\Rightarrow F(s+3)=\dfrac{5}{(s+3)^2+5^2}$, $f(t)=\sin 5t$

$$\mathcal{L}^{-1}\{F(s+3)\}=e^{-3t}f(t)=e^{-3t}\sin 5t$$

and

$\mathcal{L}\{\cos 5t\}=\dfrac{s}{s^2+5^2}=F(s)\Rightarrow F(s+3)=\dfrac{s+3}{(s+3)^2+5^2}$, $f(t)=\cos 5t$

$$\mathcal{L}^{-1}\{F(s+3)\}=e^{-3t}f(t)=e^{-3t}\cos 5t$$

Ex: Find the inverse Laplace transform of $R(s)=\dfrac{s^2+1}{s^3-2s^2-8s}$

$$\frac{s^2+1}{s(s+2)(s-4)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s-4}\Rightarrow A(s+2)(s-4)+Bs(s-4)+Cs(s+2)=s^2+1$$

$s=0 \Rightarrow -8A=1\Rightarrow A=-1/8$

$s=-2\Rightarrow -2.(-6).B=4+1\Rightarrow B=5/12$

$s=4\Rightarrow 4.6.C=16+1\Rightarrow C=17/24$

$$\mathcal{L}^{-1}\{R(s)\}=-\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{5}{12}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{17}{24}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$=-\frac{1}{8}+\frac{5}{12}e^{-2t}+\frac{17}{24}e^{4t}$$

Recall that $\mathcal{L}\{1\}=\frac{1}{s}$ and $\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$.

Ex: $y''+4y'+4y=t^2,\ y(0)=y'(0)=0$

$s^2\mathcal{L}\{y\}-sy(0)-y'(0)+4(s\mathcal{L}\{y\}-s(0))+4\mathcal{L}\{y\}=\mathcal{L}\{t^2\}$

$(s^2+4s+4)\mathcal{L}\{y\}=\dfrac{2!}{s^3}\Rightarrow \mathcal{L}\{y\}=\dfrac{2}{s^3(s+2)^2}$

$$\frac{2}{s^3(s+2)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+2}+\frac{E}{(s+2)^2}\quad (*)$$

$\Rightarrow As^2(s+2)^2+Bs(s+2)^2+C(s+2)^2+Ds^3(s+2)+Es^3=2$

$s=0\Rightarrow 4C=2\Rightarrow C=1/2$

$s=-2\Rightarrow E(-8)=2\Rightarrow E=-1/4$

$(As^2+Bs+C)(s+2)^2+(D(s+2)+E)s^3=2\quad (**)$

Differentiate $(**)$, then

$(2As+B)(s+2)^2+(As^2+Bs+C)2(s+2)+Ds^3+(D(s+2)+E)3s^2=0$

$s=0\Rightarrow 4B+C.2.2=0\Rightarrow B=-C=-1/2$

$s=-2\Rightarrow -8D+E.3.4=0\Rightarrow D=3E/2=-3/8$

Multiply $(*)$ by s and take the Limit as $s\to\infty$, then

$A+D=0\Rightarrow A=+3/8$

$$Y(S)=\frac{3}{8}\frac{1}{S}-\frac{1}{2}\frac{1}{S^2}+\frac{1}{2}\frac{1}{S^3}-\frac{3}{8}\frac{1}{S+2}-\frac{1}{4}\frac{1}{(S+2)^2}$$

$$\Rightarrow y=\mathcal{L}^{-1}\{Y(S)\}=\frac{3}{8}-\frac{1}{2}t+\frac{1}{4}t^2-\frac{3}{8}e^{-2t}-\frac{1}{4}te^{-2t}$$

Recall that $\mathcal{L}\{t^n\}=\frac{n!}{S^{n+1}}$, $\mathcal{L}\{e^{at}\}=\frac{1}{S-a}$, $\mathcal{L}\{te^{at}\}=\frac{1}{(S-a)^2}$

Ex: $x''+6x'+34x=30\sin 2t$, $x(0)=x'(0)=0$

$$S^2\mathcal{L}\{x\}-Sx(0)-x'(0)+6(S\mathcal{L}\{x\}-x(0))+34\mathcal{L}\{x\}=30\mathcal{L}\{\sin 2t\}$$

$$(S^2+6S+34)\mathcal{L}\{x\}=\frac{60}{S^2+4}\Rightarrow\mathcal{L}\{x\}=\frac{60}{(S^2+4)(S^2+6S+34)}$$

$$\frac{60}{(S^2+4)(S^2+6S+34)}=\frac{AS+B}{S^2+4}+\frac{CS+D}{S^2+6S+34}$$

$$(AS+B)(S^2+6S+34)+(CS+D)(S^2+4)=60 \quad (*)$$

$$S^2=-4\Rightarrow(AS+B)(-4+6S+34)=60\Rightarrow 6A\overset{=-4}{S^2}+6BS+30AS+30B=60$$

$$(6B+30A)S+30B-24A=0S+60$$

$$\left.\begin{array}{l}30B-24A=60\\6B+30A=0\end{array}\right\}B=-5A\Rightarrow-150A-24A=60\Rightarrow-174A=60\Rightarrow A=-10/29$$

$$\Rightarrow B=50/29$$

Differentiate $(*)$, then

$$A(S^2+6S+34)+(AS+B)(2S+6)+C(S^2+4)+(CS+D)2S=0$$

$$S=0\Rightarrow 34\cdot\frac{-10}{29}+6\cdot\frac{50}{29}+4C=0\Rightarrow C=10/29$$

$$S^2=-4\Rightarrow A(30+6S)+(2A(-4)+6AS+2BS+6B)+(2C(-4)+2DS)=0$$

$$\Rightarrow(6A+6A+2B+2D)S+(30A-8A+6B-8C)=0$$

$$\Rightarrow 6\frac{-10}{29}\cdot\cancel{2}+\cancel{2}\cdot\frac{50}{29}+\cancel{2}D=0\Rightarrow D=10/29$$

$$\Rightarrow X(S)=\mathcal{L}\{x\}=\left(-\frac{10}{29}\frac{S-5}{S^2+4}+\frac{10}{29}\frac{S+1}{(S+3)^2+25}\right)$$

$$\Rightarrow x=-\frac{10}{29}\left(\mathcal{L}^{-1}\left\{\frac{S}{S^2+4}\right\}-\frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{S^2+4}\right\}+\mathcal{L}^{-1}\left\{\frac{S+3}{(S+3)^2+5^2}\right\}-\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{5}{(S+3)^2+5^2}\right\}\right)$$

$$=-\frac{10}{29}\left(\cos 2t-\frac{5}{2}\sin 2t+e^{-3t}\cos 5t-\frac{2}{5}e^{-3t}\sin 5t\right)$$

CONVOLUTION OF TWO FUNCTIONS

f, g : piecewise cont func defined for $t \geq 0$

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$\downarrow$

the convolution of f and g $(f(t) * g(t))$

* $f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \quad \begin{Bmatrix} \tau = t-u \\ d\tau = -du \end{Bmatrix}$

$$= \int_t^0 f(t-u)\, g(u)\, (-du) = \int_0^t g(u)\, f(t-u)\, du = g * f$$

Ex: Convolution of $\cos t$ and $\sin t$

$$(\cos t) * (\sin t) = \int_0^t \cos\tau \cdot \sin(t-\tau)\, d\tau$$

$$= \int_0^t \frac{1}{2}\left[\sin t - \sin(2\tau - t)\right] d\tau \quad \text{since } \cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$= \frac{1}{2}\left[\tau \sin t + \frac{1}{2}\cos(2\tau - t)\right]\Bigg|_{\tau=0}^{t} = \frac{1}{2}\left[t\sin t + \cancel{\frac{1}{2}\cos t} - \cancel{\frac{1}{2}\underbrace{\cos(-t)}_{=\cos t}}\right]$$

$$= \frac{1}{2} t \sin t$$

THEOREM Suppose that $f(t)$ and $g(t)$ are piecewise cont for $t \geq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \to \infty$. Then, the Laplace transform of the convolution $f(t) * g(t)$ exists for $s > c$ and

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\},$$

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = f(t) * g(t)$$

Ex: $f(t)=\sin 2t$, $g(t)=e^t$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^2+4} = F(s), \quad \mathcal{L}\{g(t)\} = \frac{1}{s-1} = G(s)$$

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)(s-1)}\right\} = (\sin 2t) * e^t$$

$$= \int_0^t e^{t-\tau} \sin 2\tau \, d\tau = I$$

$$I = \int_0^t e^{t-\tau} \sin 2\tau \, d\tau \quad \left\{\begin{array}{l} u = e^{t-\tau} \Rightarrow du = -e^{t-\tau} d\tau \\ \sin 2\tau \, d\tau = dv \Rightarrow v = -\frac{1}{2}\cos 2\tau \end{array}\right\}$$

$$= -\frac{1}{2} e^{t-\tau} \cos 2\tau \Big|_0^t - \frac{1}{2}\int_0^t e^{t-\tau} \cos 2\tau \, d\tau \quad \left\{\cos 2\tau \, d\tau = dv \Rightarrow v = \frac{1}{2}\sin 2\tau\right\}$$

$$= -\frac{1}{2}\cos 2t + \frac{1}{2}e^t - \frac{1}{2}\Big[\frac{1}{2} e^{t-\tau} \sin 2\tau \Big|_0^t + \frac{1}{2}\underbrace{\int_0^t e^{t-\tau} \sin 2\tau \, d\tau}_{I}\Big]$$

$$\left(1+\frac{1}{4}\right) I = -\frac{1}{2}\cos 2t + \frac{1}{2}e^t - \frac{1}{4}\sin 2t$$

$$I = \frac{4}{5}\left(-\frac{1}{2}\cos 2t + \frac{1}{2}e^t - \frac{1}{4}\sin 2t\right)$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)(s-1)}\right\} = \frac{1}{5}\left(-2\cos 2t + 2e^t - \sin 2t\right)$$

THEOREM If $f(t)$ is piecewise cont for $t \geq 0$ and $|f(t)| \leq Me^{ct}$ as $t \to \infty$, then

$$\mathcal{L}\{-t f(t)\} = F'(s) \quad \text{for } s > c$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

. . . .

. . . .

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Ex: $\mathcal{L}\{\underbrace{t^2 \sin kt}_{=f(t)}\} = (-1)^2 F''(s) = [k(s^2+k^2)^{-1}]'' = [-k(s^2+k^2)^{-2} 2s]'$

$\Rightarrow F(s) = \dfrac{k}{s^2+k^2}.$

$\Rightarrow \mathcal{L}\{t^2 \sin kt\} = -2k\,[-2(s^2+k^2)^{-3}\, 2s\, s + (s^2+k^2)^{-2}]$

$= \dfrac{8ks^2}{(s^2+k^2)^3} - \dfrac{2k}{(s^2+k^2)^2} = \dfrac{6ks^2 - 2k^3}{(s^2+k^2)^3}$

Ex: $\mathcal{L}^{-1}\{\underbrace{\tan^{-1}\frac{1}{s}}_{=F(s)}\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = +\frac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{-1/s^2}{1+\frac{1}{s^2}}\right\}$

$= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\dfrac{1}{s^2+1}\right\} = -\frac{1}{t}(-\sin t) = \dfrac{\sin t}{t}$

Ex: $tx'' + x' + tx = 0,\ x(0) = 1,\ x'(0) = 0$ (Bessel's equation)

$\mathcal{L}\{\underbrace{tx''}_{=f(t)}\} = -F'(s) = -[s^2 X(s) - s\,\overset{1}{\cancel{x(0)}} - \overset{0}{\cancel{x'(0)}}]'$

$= -[2s\,X(s) + s^2 X'(s) - 1]$

$\mathcal{L}\{\underbrace{tx}_{=f(t)}\} = -F'(s) = -(X(s))' = -X'(s)$

$\Rightarrow -2s\,X(s) - s^2 X'(s) + 1 + s\,X(s) - \overset{1}{\cancel{x(0)}} - X'(s) = 0$

$(1+s^2)\,X'(s) = -s\,X(s) \Rightarrow \dfrac{X'(s)}{X(s)} = \dfrac{-s}{s^2+1}$

$\Rightarrow \ln X(s) = -\displaystyle\int \frac{s\,ds}{s^2+1} = -\frac{1}{2}\ln(s^2+1) + \ln c = \ln\frac{c}{\sqrt{s^2+1}} \Rightarrow X(s) = \frac{c}{\sqrt{s^2+1}}$

THEOREM Suppose that f is piecewise cont for $t \geq 0$, $\lim\limits_{t\to 0^+} \dfrac{f(t)}{t}$ exists and finite, and that $|f(t)| \leq Me^{ct}$ as $t \to \infty$

Then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(\tau)\,d\tau \quad \text{for } s > c$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^{\infty} F(\tau)\,d\tau\right\}$$

Ex: $\lim_{t\to 0^+} \frac{\sinh t}{t} = \lim_{t\to 0^+} \frac{e^t - e^{-t}}{2t} = \lim_{t\to 0^+} \frac{e^t+e^{-t}}{2} = 1$, $f(t) = \sinh t$

$\Rightarrow F(s) = \frac{1}{s^2-1}$

$$\mathcal{L}\left\{\frac{\sinh t}{t}\right\} = \int_s^\infty F(\tau)\,d\tau = \int_s^\infty \frac{d\tau}{\tau^2-1} = \frac{1}{2}\int_s^\infty \left(\frac{1}{\tau-1} - \frac{1}{\tau+1}\right) d\tau$$

$$= \frac{1}{2}\ln \frac{\tau-1}{\tau+1}\Bigg|_s^\infty = \frac{1}{2}\left(\cancel{\ln 1}^{\,0} - \ln\frac{s-1}{s+1}\right) = \frac{1}{2}\ln\frac{s+1}{s-1}$$

Ex: $$\mathcal{L}^{-1}\Bigg\{\underbrace{\frac{2s}{(s^2-1)^2}}_{F(s)}\Bigg\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\tau)\,d\tau\right\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2\tau}{(\tau^2-1)^2}\,d\tau\right\}$$

$$= t\,\mathcal{L}^{-1}\left\{-\frac{1}{\tau^2-1}\Bigg|_s^\infty\right\} = t\,\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = t\sinh t$$

THEOREM If $\mathcal{L}\{f(t)\}$ exists for $s>c$, then

$$\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a) \quad \text{for } s > c+a$$

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t\geq a \end{cases} \Rightarrow u(t-a)\,f(t-a) = \begin{cases} 0, & t<a \\ f(t-a), & t\geq a \end{cases}$$

We can also formulate the following:

$$\begin{cases} 1, & t<a \\ 0, & t\geq a \end{cases} = \begin{cases} 1-0, & t<a \\ 1-1, & t\geq a \end{cases} = 1 - \begin{cases} 0, & t<a \\ 1, & t\geq a \end{cases}$$

$$= 1 - u(t-a)$$

Ex: Find $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s^3}\right\}$.

$F(s)=\frac{1}{s^3} \Rightarrow f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\frac{1}{2!}t^2=\frac{t^2}{2}$ since $\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$

$\mathcal{L}^{-1}\left\{e^{-as}\frac{1}{s^3}\right\}=u(t-a)f(t-a)=\begin{cases}0, & t<a\\ \frac{1}{2}(t-a)^2, & t\geq a\end{cases}$

Ex: Find $\mathcal{L}\{g(t)\}$ if $g(t)=\begin{cases}0, & t<3\\ t^2, & t\geq 3\end{cases}$

$g(t)=\begin{cases}0\cdot t^2, & t<3\\ 1\cdot t^2, & t\geq 3\end{cases}=t^2\begin{cases}0, & t<3\\ 1, & t\geq 3\end{cases}=t^2u(t-3)$

$\mathcal{L}\{g(t)\}=\mathcal{L}\{\underbrace{t^2}_{=f(t-3)}u(t-3)\}=e^{-3s}F(s)$

$f(t-3)=t^2 \Rightarrow f(t)=(t+3)^2 \Rightarrow F(s)=\mathcal{L}\{f(t)\}=\mathcal{L}\{t^2+6t+9\}=\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}$

$\mathcal{L}\{g(t)\}=e^{-3s}\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right)$

Ex: Find $L\{f(t)\}$ if $f(t)=\begin{cases}\cos 2t, & 0\leq t<2\pi\\ 0, & t\geq 2\pi\end{cases}$

$f(t)=\cos 2t\begin{cases}1, & 0\leq t<2\pi\\ 0, & t\geq 2\pi\end{cases}=\cos 2t\begin{cases}1-0, & 0\leq t<2\pi\\ 1-1, & t\geq 2\pi\end{cases}$

$=\cos 2t\left[1-\begin{cases}0, & 0\leq t<2\pi\\ 1, & t\geq 2\pi\end{cases}\right]=\cos 2t\,(1-u(t-2\pi))$

$L\{f(t)\}=\mathcal{L}\{\cos 2t\}-\mathcal{L}\{\underbrace{\cos 2t}_{=f(t-2\pi)}u(t-2\pi)\}=\frac{s}{s^2+4}-e^{-2\pi s}F(s)$

$f(t-2\pi)=\cos 2t \Rightarrow f(t)=\cos 2(t+2\pi)=\cos 2t \Rightarrow F(s)=\frac{s}{s^2+4}$

$\mathcal{L}\{f(t)\}=(1-e^{-2\pi s})\frac{s}{s^2+4}$

Ex: $x'' + 4x = f(t)$, $x(0) = x'(0) = 0$, $f(t)$: as in the previous example

$$s^2 \mathcal{L}\{x\} - s\cancel{x(0)}^{\to 0} - \cancel{x'(0)}^{\to 0} + 4\mathcal{L}\{x\} = (1-e^{-2\pi s}) \frac{s}{s^2+4}$$

$$\mathcal{L}\{x\} = (1-e^{-2\pi s}) \frac{s}{(s^2+4)^2}.$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s} \frac{s}{(s^2+4)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4} t \sin 2t = f(t)$$

$$\Rightarrow x(t) = \frac{1}{4} t \sin 2t - \mathcal{L}^{-1}\{e^{-2\pi s} F(s)\}$$

$$= \frac{1}{4} t \sin 2t - u(t-2\pi) \cdot f(t-2\pi)$$

$$= \frac{1}{4} t \sin 2t - u(t-2\pi) \frac{1}{4}(t-2\pi) \sin 2t$$