03/12/2020 Review of Chs. 3 & 4

Test exam on 1 July 2020

(1)
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & -1 & 0 & 2 \\ 0 & b & 2b & 3 \end{array}\right]$$
inf. sols.? ?

if $b=0 \rightarrow$ last row: $0\ 0\ 0 \mid 3$ the system is inconsistent.

$$\overset{-2R_1+R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & -2 & 4 \\ 0 & b & 2b & 3 \end{array}\right]$$

$3+4b=0$

$$\boxed{b=-\frac{3}{4}}$$

$$\overset{bR_2+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & 0 & 3+4b \end{array}\right]$$

c

② A, B invertible 3×3 matrices

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \qquad \det A = 3, \quad \det(A^{-1}B^{T}) = ?$$

$$\det(AB) = \det A \det B \qquad |AB| = |A|\,|B|$$

$$|A^{T}| = |A| \quad , \quad |A^{-1}| = \frac{1}{|A|}$$

$$\det|\text{adj}\,A| = |A|^{n-1} \qquad A = [a_{ij}]_{n\times n}$$

$$\det(AB) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1.(-1)^{1+1} \begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} = 18$$

$$|A|\,|B| = 18 \Rightarrow 3.|B| = 18 \Rightarrow \underline{\underline{|B| = 6}}$$

$$|A^{-1}B^T| = |A^{-1}|\,|B^T| = \frac{1}{|A|}\cdot|B| = \frac{1}{3}\cdot 6$$
$$= 2$$

3. Which of the following is the value of k that makes $v_1 = (1,1,1)$ $v_2 = (1,0,-1)$ $v_3 = (1,-1,k)$ linearly dependent??

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{3\times 1}$$

When we're given # of n vector in $\mathbb{R}^n$

$$\begin{vmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{vmatrix} \begin{array}{l} = 0 \rightarrow \text{linearly dep.} \\ \\ \neq 0 \Rightarrow \text{linearly ind.} \end{array}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & k \end{vmatrix} = 0 \qquad k = \ldots$$

③ Which of the following is the value of k that makes $v_1 = (1, 1, 1)$ $v_2 = (1, 0, -1)$ $v_3 = (1, -1, k)$ linearly dependent??

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_{3\times 1}$$

When we're given # of n vector in $\mathbb{R}^n$

$$\begin{vmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{vmatrix} \begin{array}{l} = 0 \rightarrow \text{linearly dep.} \\ \\ \neq 0 \Rightarrow \text{linearly ind.} \end{array}$$

23 March 2019

$$A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 1 & 3 \\ 1 & 0 & +1 \end{bmatrix} \qquad A^{-1} = ?$$

(i) $$\left[\begin{array}{ccc|ccc} 3 & 1 & -2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & +1 & 0 & 0 & 1 \end{array}\right]$$

(ii) $$A^{-1} = \frac{1}{|A|} \operatorname{adj} A \qquad \operatorname{adj} A = [A_{ij}]^T$$

A_{ij}: cofactor of the element a_{ij}

$$|A| = \begin{vmatrix} 3 & 1 & -2 \\ -1 & 1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 1.(-1)^{3+1} . \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} + 1.(-1)^{3+3} \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 3 - (-2) + 3 - (-1) = 9 \;//$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}.$$

$$\text{adj}\, A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$A^{-1} = \frac{1}{|A|} \text{adj}\,(A)$$

Complete this yourself

3c

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 2 & -4 \\ 2 & 0 & 4 & -7 \\ 0 & 0 & 3 & -6 \\ 2 & -1 & -5 & -10 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \\ 7 \\ 10 \end{bmatrix}$$

By Cramer's rule, find x_3 in the eq $AX = B$.

$$x_3 = \frac{\begin{vmatrix} 1 & 0 & 3 & -4 \\ 2 & 0 & 4 & -7 \\ 0 & 0 & 7 & -6 \\ 2 & -1 & 10 & -10 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 & -4 \\ 2 & 0 & 4 & -7 \\ 0 & 0 & 3 & -6 \\ 2 & -1 & -5 & -10 \end{vmatrix}} = \frac{\cdots}{\cdots}$$

(4)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 4 \\ 2 & 5 & 11 & 12 \end{bmatrix}$$

(a) Find a basis for the null space of A and find the dimension of the null space.

$$A = [a_{ij}]_{m \times n}$$

$$\text{Null}(A) = \{ x \in \mathbb{R}^n \mid \underset{\sim}{A} \underset{\sim}{x} = \underset{\sim}{0} \}$$

$$[\; A \;]_{m \times n} \begin{bmatrix} x \end{bmatrix}_{n \times 1} = \begin{bmatrix} \; \end{bmatrix}_{m \times 1}$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 3x_1 + x_2 - 3x_3 + 4x_4 &= 0 \\ 2x_1 + 5x_2 + 11x_3 + 12x_4 &= 0 \end{aligned}$$

4

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 4 \\ 2 & 5 & 11 & 12 \end{bmatrix}_{3\times 4}$$

$$\mathrm{Null}(A) = \{ x \in \mathbb{R}^4 \mid \underset{\sim}{A}\underset{\sim}{x} = 0 \}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 4 \\ 2 & 5 & 11 & 12 \end{bmatrix} \underset{-2R_1+R_3}{\overset{-3R_1+R_2}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 1 \\ 0 & 3 & 9 & 10 \end{bmatrix}$$

$$\overset{R_2+R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -6 & 1 \\ 0 & 1 & 3 & 11 \end{bmatrix} \overset{2R_3+R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 23 \\ 0 & 1 & 3 & 11 \end{bmatrix}$$

$$\begin{array}{c} \\ R_2 \Leftrightarrow R_3 \\ \sim \end{array} \quad \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 11 & 0 \\ 0 & 0 & 0 & 23 & 0 \end{array}\right]$$

$\dim \mathrm{Null}(A) = 1$

$$x_1 + x_2 + x_3 + x_4 = 0$$

$$x_2 + 3x_3 + 11x_4 = 0$$

$$23x_4 = 0$$

$x_4 = 0$

$x_3 = \text{arbit.} = \alpha$

$x_2 = -3\alpha$

$$x_1 - 3\alpha + \alpha + 0 = 0 \rightarrow x_1 = 2\alpha$$

$$\underset{\sim}{x} = \begin{bmatrix} 2\alpha \\ -3\alpha \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) Find a basis for column space of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -3 & 4 \\ 2 & 5 & 11 & 12 \end{bmatrix}_{3\times 4} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

A basis for $Col(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 12 \end{bmatrix} \right\}$

$\dim Col(A) = 3 = \dim Row\, A = Row\ rank(A) = \underline{Rank(A)}$

$\quad\,\parallel$

$col\ rank(A)$

Rank-Nullity th.

$A = [a_{ij}]_{m\times n}$

$\dim Null(A) + Rank\, A = n$

$1 + 3 = 4$

When we're given $A = [a_{ij}]$, to find

Null(A), Row(A), Col(A)

$$A \rightsquigarrow \bar{E}$$

echelon

16 Nov. 2019

36 A and B are 3x3 invertible matrices with $\det A = 2$ and $\det B = -3$. Calculate the following.

i) $\det(3A^T B^{-1}) = ?$ ii) $\det(4 \operatorname{adj} A + 2A^{-1})$

$A = [a_{ij}]_{n \times n}$ $|kA| = k^n |A|$

$|3A^T B^{-1}| = 3^3 . |A^T| . |B^{-1}| = 3^3 . |A| \frac{1}{|B|} = \frac{27.2}{3} = 18$ //

$\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(ii) $\det A = 2, \quad \det B = -3$

$\det(4\,\mathrm{Adj}\,A + 2A^{-1}) = ?$

$A^{-1} = \frac{1}{|A|}(\mathrm{adj}\,A) \qquad \mathrm{adj}\,A = |A|\,A^{-1}$

$|4\,\mathrm{adj}\,A + 2A^{-1}| = |4.\underbrace{|A|}_{2}A^{-1} + 2\,A^{-1}|$

$= |10\,A^{-1}| = 10^3\,|A^{-1}|$

$= 10^3\,\frac{1}{|A|} = 1000.\,\frac{1}{2} = \underline{\underline{500}}$

24 Mar '18

(3c) Two matrices A and B are similar provided there's a matrix P such that

$$A = P^{-1} B P$$

Show that similar matrices have the same determinant.

$$|A| = |P^{-1} B P|$$

$$|A| = \underline{|P^{-1}|}\ |B|\ |P| \qquad |P^{-1}| = \frac{1}{|P|}$$

$$|A| = |B|$$

* Prove that $|A^{-1}| = \dfrac{1}{|A|}$

$$A A^{-1} = I = A^{-1} A$$

$$|A A^{-1}| = |I| \rightarrow |A||A^{-1}| = 1$$

$$|A^{-1}| = \frac{1}{|A|}$$

$*$ $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + 2x_2 + x_3 = 0 \right\}$

Is S a subspace of $\mathbb{R}^3$?

(i) $u, v \in S$ $\quad u + v \in S$

$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad x_1 + 2x_2 + x_3 = 0$

$v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\quad + \; y_1 + 2y_2 + y_3 = 0$

$x_1 + y_1 + 2(x_2 + y_2) + x_3 + y_3 = 0$

$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} \in S$ ✓

(ii) $u \in S, c \in \mathbb{R}$ $\quad c\underset{\sim}{u} \in S$

$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in S$ $\quad x_1 + 2x_2 + x_3 = 0$

$cu = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} \overset{?}{\in} S$ ✓

$(cx_1) + 2(cx_2) + cx_3 = 0$

$\Rightarrow S$ is a subspace of $\mathbb{R}$

$\circledast$ $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2} \;\middle|\; 3a+d=b \right\}$

(i) Is S a subspace of $M_{2\times 2}$? ?

(ii) If so, find a basis for this subspace.

$u, v \in S$: $u = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $v = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$3a_1 + d_1 = b_1$ $\qquad$ $3a_2 + d_2 = b_2$

$u+v \overset{?}{\in} S$ $\qquad$ $u+v = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

(*) $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2} \;\middle|\; 3a+d=b \right\}$

$3a+d-b=0$

$3(a_1+a_2)+d_1+d_2-(b_1+b_2)$

$= 3a_1 + 3a_2 + d_1 + d_2 - b_1 - b_2$

$= \underbrace{3a_1 + d_1 - b_1}_{0} + \underbrace{3a_2 + d_2 - b_2}_{0} = 0+0$

$=0$

(ii) $\rightarrow$ check it yourself

$cu \in S$ ✓

(ii) $u \in S$ $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $3a + d = b$

$$u = \begin{bmatrix} a & 3a+d \\ c & d \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}}_{B_1} + c \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{B_2} + d \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{B_3}$$

(i) any $u \in S$ can be written as a linear. comb. of B_1, B_2, B_1 : $\underset{\sim}{u} = a B_1 + c B_2 + d B_3$

$\Rightarrow$ $\text{span}\{B_1, B_2, B_3\} = S$

(ii) <u>B_1, B_2, B_7 are linearly independent</u> $\Rightarrow$ $\{B_1, B_2, B_2\}$ is a basis for S.

12 July 2019

3 c

$$W = \left\{ v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x + y = 0 \right\}$$

(i) W is a subspace of $\mathbb{R}^3$ and $\dim W = 3$ F

(ii) W is not a subspace of $\mathbb{R}^3$ F

(iii) W is a subspace of $\mathbb{R}^2$ and $\dim W = 1$ F

(iv) W is a subspace of $\mathbb{R}^3$ and $\dim W = 2$.

T

12 July 2019

3 c

$$W = \left\{ v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x + y = 0 \right\}$$

i $\underset{\sim}{u}, \underset{\sim}{v} \in W$:

$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad x_1 + y_1 = 0$$

$$\underset{\sim}{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad x_2 + y_2 = 0$$

$$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \in W \; ? \; \checkmark$$

$$x_1 + x_2 + y_1 + y_2$$

$$= \underbrace{x_1 + y_1}_{0} + \underbrace{x_2 + y_2}_{0} = 0$$

(ii) $u = \begin{bmatrix} x \\ -x \\ z \end{bmatrix} \in W \quad \{x+y=0 \rightarrow y=-x\}$

$cu = \begin{bmatrix} cx \\ -cx \\ cz \end{bmatrix} : \quad cx+(-cx)=0$ ✓

$cu \in W$

$\Rightarrow$ W is a subspace of $\mathbb{R}^3$

any $u \in W$ is of the form

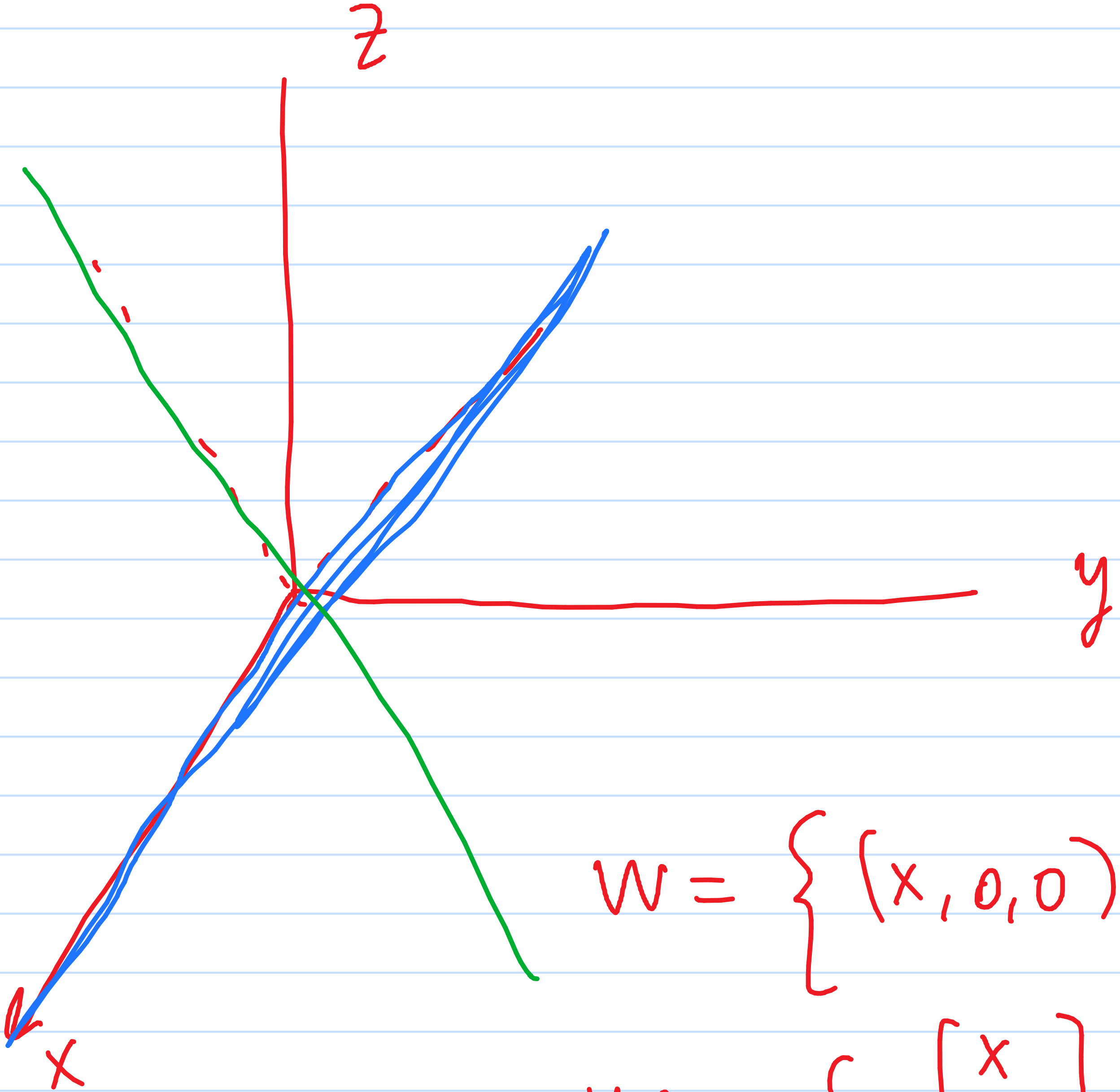
$$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x \\ z \end{bmatrix} = x \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{B_1} + z \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_2}$$

$x, z \in \mathbb{R}$

- Any $u \in W$ can be written as a linear comb. of B_1, B_2: $\quad u = x B_1 + z B_2$

- B_1 & B_2 are linearly ind.

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } W.$$

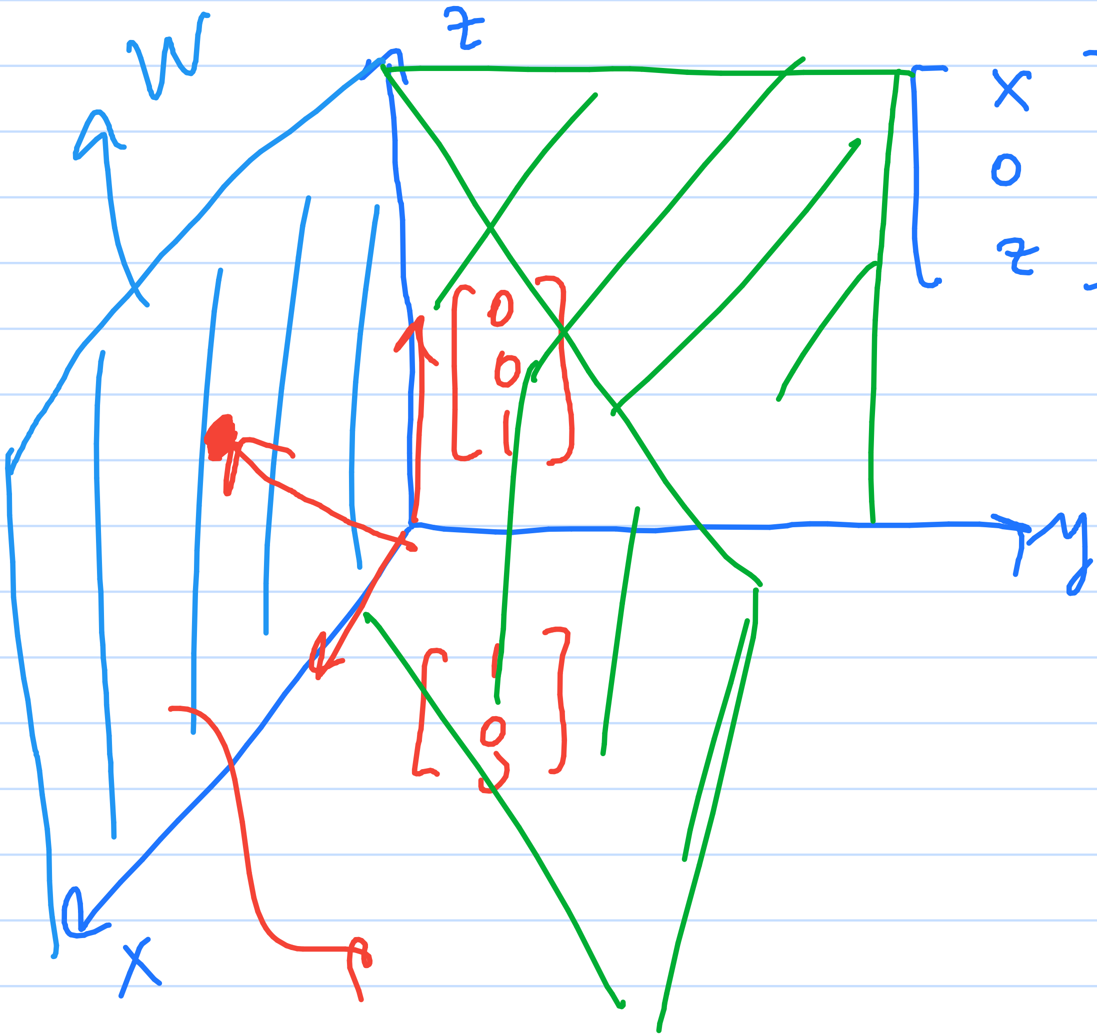


$$W = \{(x, 0, 0) \mid x \in \mathbb{R}\}$$

$$W = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \mid x \in \mathbb{R} \right\}$$

$$W = \left\{ \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x, z \in \mathbb{R} \right\}$$

$(y = 0)$



$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

this evening

$\boxed{\begin{matrix} 17.30 \\ 19.00 \end{matrix}}$

Review of DEs

$$\frac{dy}{dx} = y \longrightarrow \frac{dy}{y} = dx$$

$$\ln y = x + C \qquad\qquad \ln y = x + \ln C$$

$$y = e^{x+C} \qquad\qquad \ln y = \ln e^{x} + \ln C$$

$$y = e^{C} e^{x} \qquad\qquad y = C e^{x}$$

$$y = \hat{C} e^{x}$$

Ex

$$\frac{dy}{dx} = \frac{y^2}{x^2} \rightarrow \frac{dy}{y^2} = \frac{dx}{x^2}$$

$$-\frac{1}{y} = -\frac{1}{x} - C \qquad\qquad -\frac{1}{y} = -\frac{1}{x} - \frac{1}{\tilde{c}}$$

$$\frac{1}{y} = \frac{1}{x} + C \qquad\qquad \frac{1}{y} = \frac{1}{x} + \frac{1}{\tilde{c}}$$

$$y = \frac{x}{1 + Cx} \qquad \boxed{\tilde{c} = \frac{1}{C}} \qquad \frac{1}{y} = \frac{\tilde{c} + x}{\tilde{c}\,x}$$

$$y = \frac{x}{1 + \frac{1}{\tilde{c}}x} \leftarrow y = \frac{\frac{\tilde{c}x}{\tilde{c}}}{\frac{\tilde{c}}{\tilde{c}} + \frac{x}{\tilde{c}}} \leftarrow y = \frac{\tilde{c}x}{\tilde{c} + x}$$