ISTANBUL TECHNICAL UNIVERSITY

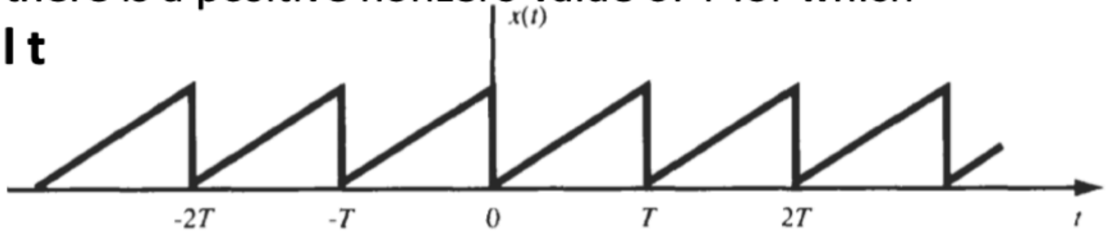
BLG354E - Recitation 2

11.06.2022

Periodic Signals

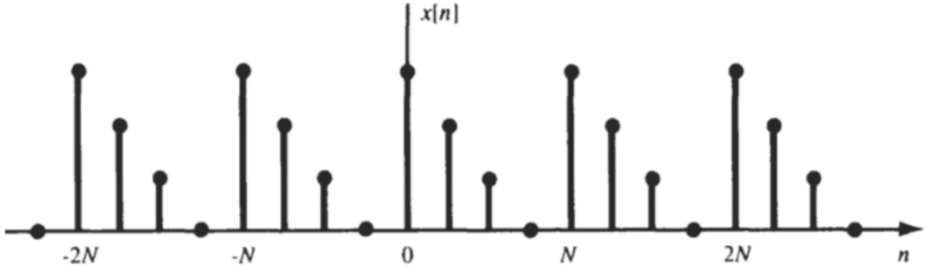
A continuous-time signal x(t) is said to be periodic with period T if there is a positive nonzero value of T for which

$$x(t+T)=x(t) \quad \text{for all } t$$



A discrete-time signal x[n] is said to be periodic with period N if there is a positive nonzero value of N for which

$$x[n+N]=x[n] \quad \text{for all } n$$



Smallest value of T or N that satisfies the above condition is called fundamental period

Periodic Signals

Example:

x(t) is a CT signal given as x(t)=cos(15t). Find the fundamental period of the DT signal x[n] if x[n] is discretized by sampling x(t) at the sampling frequency $f_s = \frac{10}{\pi}$ Hz

x[n] = x(nT_s) T_s=1/f_s=0.1π seconds   x(t)=cos(ωt)=cos(15t)→ ω=15 rad/s

The fundamental period of x(t) : $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{15}$

x[n] is periodic if $\frac{T_s}{T_0} = \frac{T_s}{2\pi/15} = \frac{m}{N_0}$

Since m and N_0 are positive integers → $T_s = \frac{m}{N_0} T_0 = \frac{m}{N_0} \frac{2\pi}{15}$ → $\frac{T_s}{T_0} = \frac{\pi/10}{2\pi/15} = \frac{15}{20} = \frac{3}{4}$

If x[n] is periodic then $N_0 = m\frac{T_0}{T_s} = m\frac{4}{3}$

m=3 provides N_0 to be the smallest positive integer. Fundamental period of x[n] is N_0=4

Periodic Signals

Example: (Case study for the signal having multiple periodic components)

Find the fundamental frequency of the signal x[n] $= e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$

$$\frac{\omega_0}{2\pi} = \frac{k}{N}$$

$$\frac{\frac{2\pi}{3}}{2\pi} = \frac{1}{3} \qquad \frac{\frac{3\pi}{4}}{2\pi} = \frac{3}{8}$$

Fundamental period of the first exponential is 3

Fundamental period of the second exponential is 8

Since the least common multiple of the periods of the two signals is 24,
x[n] is periodic with N_0=24

Periodic Signals

Example: Find the fundamental period of the DT signal $x[n] = sin(\frac{5\pi}{6}n) + cos(\frac{3\pi}{4}n) + sin(\frac{\pi}{3}n)$

The least common multiple of the denominators is 12 → $x[n] = sin(\frac{10\pi}{12}n) + cos(\frac{9\pi}{12}n) + sin(\frac{4\pi}{12}n)$

Fundamental frequency is $\omega_0 = \pi/12$ → The fundamental period is $T = 2\pi/\omega_0 = 24$ and the three terms are the 4th, 9th and 10th harmonic of ω_0

Example: Find the fundamental frequency of the CT signal $x(t) = \underbrace{sin(\frac{5\pi}{6}t)}_{x_1(t)} + \underbrace{cos(\frac{3\pi}{4}t)}_{x_2(t)} + \underbrace{sin(\frac{\pi}{3}t)}_{x_3(t)}$

The frequencies and periods of $x_1(t)$, $x_2(t)$ and the $x_3(t)$ are:

$$\left.\begin{aligned} \omega_1 &= \frac{5\pi}{6},\ f_1 = \frac{5}{12},\ T_1 = \frac{12}{5} \\ \omega_2 &= \frac{3\pi}{4},\ f_2 = \frac{3}{8},\ T_2 = \frac{8}{3} \\ \omega_3 &= \frac{\pi}{3},\ f_3 = \frac{1}{6},\ T_3 = 6 \end{aligned}\right\} f_0 = GCD(\frac{5}{12}, \frac{3}{8}, \frac{1}{6}) = GCD(\frac{10}{24}, \frac{9}{24}, \frac{4}{24}) = \frac{1}{24}$$

The fundamental angular frequency is $\omega_0 = \pi/12$
and the fundamental period is $T_0 = 2\pi/\omega = 24$
$f_0 = 1/24$

→ $x(t) = sin(\frac{10\pi}{12}t) + cos(\frac{9\pi}{12}t) + sin(\frac{4\pi}{12}t)$

Question 1 (2019 Final Exam)

1- Transfer function H(z) of a digital filter is given as,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.1(z^2 + 2z + 1)}{z^2 - z + 0.5}$$

a) Find the difference equation where the output sequence is denoted by y[n] and input sequence is denoted by x[n].

b) Draw the block diagram of the system in terms of unit delays (z^{-1}).

c) Write a pseudo code for implementation of the given filter where the signal processing interrupt-subroutine is called at sampling period T_s. (Input and Output variables will be named as X and Y respectively. Internal variables will be assigned as A, B, C...).

d) Find the impulse response h[n] of the filter by using inverse Fourier transform of its frequency response H(Ω).

Question 2 (2019 Final Exam)

2- Transfer function of a first order low pass filter is given as $H(s) = H(j\omega) = \frac{1}{1+0.005 \cdot j\omega}$

a) Sketch the Bode plot for the frequency response.

b) Find a difference equation that is equivalent of the defined filter by using bilinear transformation $s = 2f_s \cdot \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ for the case that input signal is sampled at f_s=500Hz

Question 3 (2019 Final Exam)

3- Two continuous time signals x(t) and h(t) are given as,

$$x(t)=u(t-1),\ h(t)=e^{-2t}u(t)$$

Find their convolution $y(t)=x(t)*h(t)$.

Fourier Transform Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\Omega}}$

Question 4

Find the inverse z transform of $X(z) = \dfrac{z}{(z-1)(z-2)^2} \qquad |z| > 2$

Question 5

Calculate the convolution y[n] of the sequences

$v[n]=\{v_n\} = \{a^n\}$
$w[n]=\{w_n\}= \{b^n\}$

$a \neq b$

Question 6

$f(t) = 5 + 2\cos(2\pi t - 90^o) + 3\cos 4\pi t$ Find the 4 points DFT of this signal if it is sampled at 4Hz

Question 7 (from HW3)

1. Find 4-points DFT (Discrete Fourier Transform) of the periodic DT signal $x[n] = \{1, 2, 0, -1\}$ as $X[k] = DFT\{x[n]\}$ where

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \qquad W_N = e^{-j(2\pi/N)}$$

Question 8 (from HW3)

3. Transfer function of a discrete time system H(z) is given as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-1}}{(1 - 0.5z^{-1})^2}$$

where z^{-1} denotes the unit delay. Find the first 4 values of the output signal sequence $y[n] = \{y[0], y[1], y[2], y[3]\}$ if unit step signal $x[n] = u[n]$ is applied to this system.

(initial condition can be considered as zero)

Properties of Z Transform

Property	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1(z)+a_2X_2(z)$	$R' \supset R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	$R' \supset R \cap \{0<\lvert z\rvert<\infty\}$
Multiplication by z_0^n	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$R' = \lvert z_0\rvert R$
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$	$R' = R$
Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$R' = \frac{1}{R}$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$	$R' = R$
Accumulation	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1-z^{-1}}X(z)$	$R' \supset R \cap \{\lvert z\rvert > 1\}$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$R' \supset R_1 \cap R_2$

Common Z Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 if $(m>0)$ or ∞ if $(m<0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\|z < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\|z\| < \|a\|$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left[\frac{z}{z-a}\right]^2$	$\|z\| > \|a\|$
$(\cos \Omega_0 n)u[n]$	$\frac{z^2-(\cos\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\| > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin\Omega_0)z}{z^2-(2\cos\Omega_0)z+1}$	$\|z\| > 1$
$(r^n \cos \Omega_0 n)u[n]$	$\frac{z^2-(r\cos\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\| > r$
$(r^n \sin \Omega_0 n)u[n]$	$\frac{(r\sin\Omega_0)z}{z^2-(2r\cos\Omega_0)z+r^2}$	$\|z\| > r$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Fourier Transform Properties

Property	Sequence	Fourier Transform
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(\Omega) + a_2 X_2(\Omega)$
Time shifting	$x[n - n_0]$	$e^{-j\Omega n_0} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega) + \dfrac{1}{1 - e^{-j\Omega}} X(\Omega)$
		$\lvert\Omega\rvert \leq \pi$
Convolution	$x_1[n] * x_2[n]$	$X_1(\Omega) X_2(\Omega)$
Multiplication	$x_1[n] x_2[n]$	$\dfrac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$
Real sequence	$x[n] = x_e[n] + x_o[n]$	$X(\Omega) = A(\Omega) + jB(\Omega)$
		$X(-\Omega) = X^*(\Omega)$
Even component	$x_e[n]$	$\mathrm{Re}\{X(\Omega)\} = A(\Omega)$
Odd component	$x_o[n]$	$j\,\mathrm{Im}\{X(\Omega)\} = jB(\Omega)$

Common Fourier Transform Pairs

$x[n]$	$X[\Omega]$
$\delta[n]$	1
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$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
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$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{(1-ae^{-j\Omega})^2}$
$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\dfrac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \lvert n\rvert \le N_1\\ 0 & \lvert n\rvert > N_1\end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega)=\begin{cases}1 & 0 \le \lvert\Omega\rvert \le W\\ 0 & W < \lvert\Omega\rvert \le \pi\end{cases}$
$\sum_{k=-\infty}^{\infty} \delta[n-kN_0]$	$\Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega-k\Omega_0), \Omega_0=\dfrac{2\pi}{N_0}$