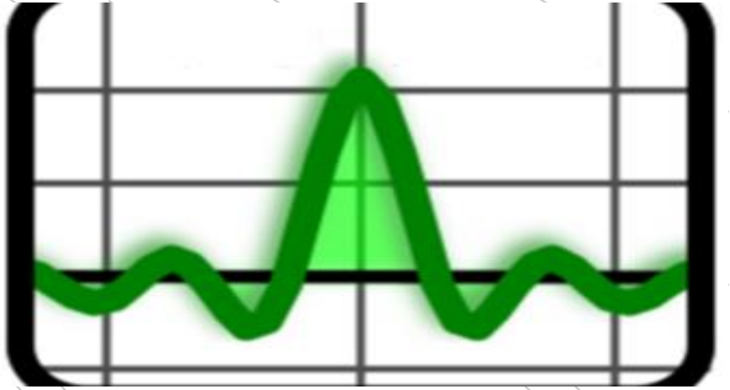

Signals & Systems for Computer Engineering

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BLG354E / CRN: 21560

5th Week Lecture

Response of DT LTI Systems to an Arbitrary Input:

Since the system is linear, the response y[n] of the system to an arbitrary input x[n] can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$y[n] = \mathbf{T}\{x[n]\} = \mathbf{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]\mathbf{T}\{\delta[n-k]\}$$

Since the system is time-invariant, $h[n-k] = \mathbf{T}\{\delta[n-k]\}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Therefore a discrete-time LTI system is completely characterized by its impulse response h[n]

(Lecture Book1: Schaum Outline Series, Signals and Systems)

Example:

Impulse response of an LTI DT system is given as h[m]={5, 0, 3, 2}

Find its output pattern if the signal x[m]={4, 1, 3, 2} is applied as the input.

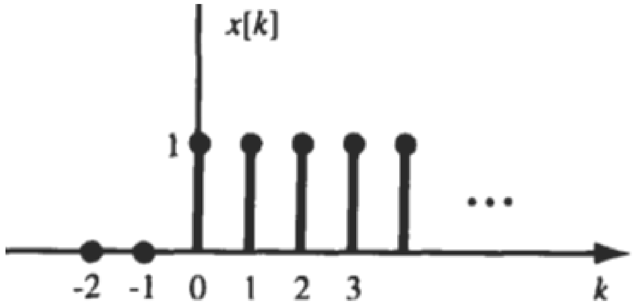
Output: y[m]=x[m]*h[m]

h(0-m)={2, 3, 0, 5} for m=3, 2, 1, 0

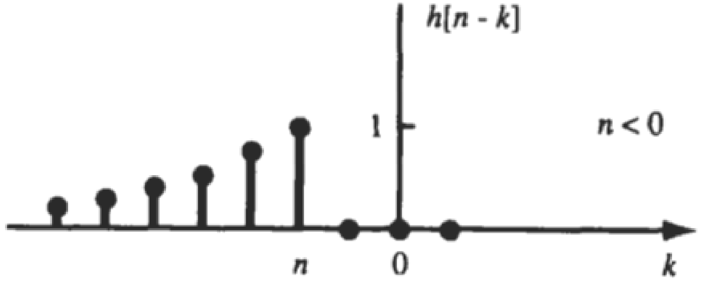
m				0	1	2	3			
$h(m)$				5	0	3	2			
$x(m)$				4	1	3	2			
$h(0-m)$	2	3	0	5						
$h(1-m)$		2	3	0	5					
$h(2-m)$			2	3	0	5				
$h(3-m)$				2	3	0	5			
$h(4-m)$					2	3	0	5		
$h(5-m)$						2	3	0	5	
$h(6-m)$							2	3	0	5
n				0	1	2	3	4	5	6
$y(n)$				20	5	27	21	11	12	4

Example:

The input x[n] and the impulse response h[n] of a discrete-time LTI system are given by
$$\begin{cases} x[n] = u[n] \\ h[n] = \alpha^n u[n] \qquad 0 < \alpha < 1 \end{cases}$$

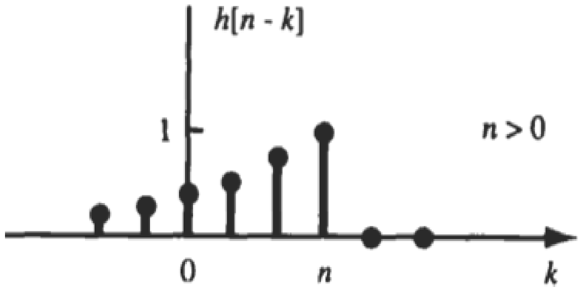


$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

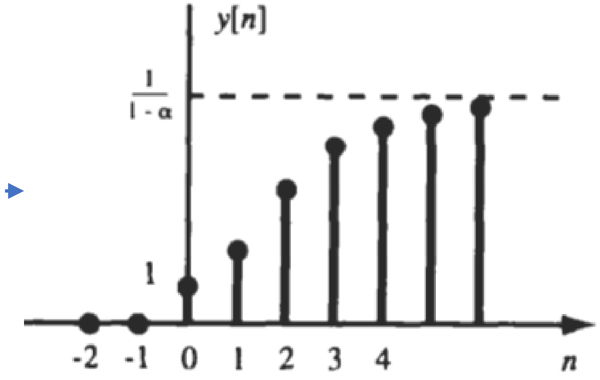


$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

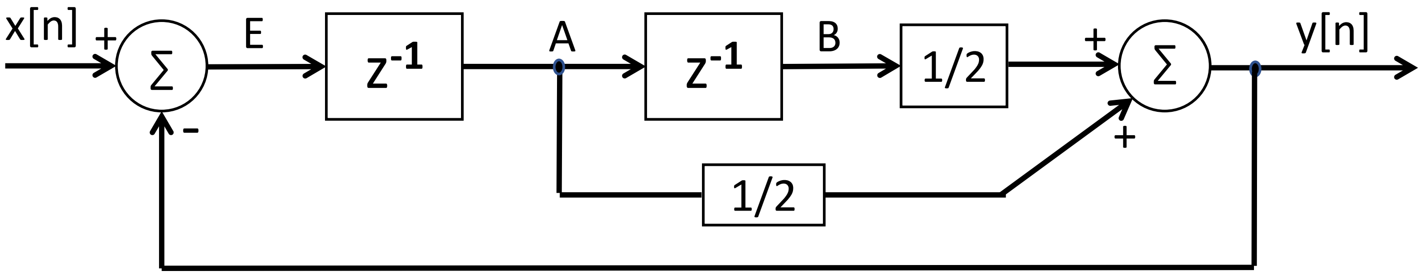
$$y[n] = \sum_{m=n}^{0} \alpha^m = \sum_{m=0}^{n} \alpha^m = \frac{1-\alpha^{n+1}}{1-\alpha}$$



$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$



Example:



a) Find the impulse response of the system via simulation

b) Find the system response for x[n]=n(u[n]-u[n-3]) by simulation

c) Find the system response for x[n]=n(u[n]-u[n-3]) by convolution for the first 5 values

d) Find the impulse response through analytical solution of the transfer function

Psuedo Code:

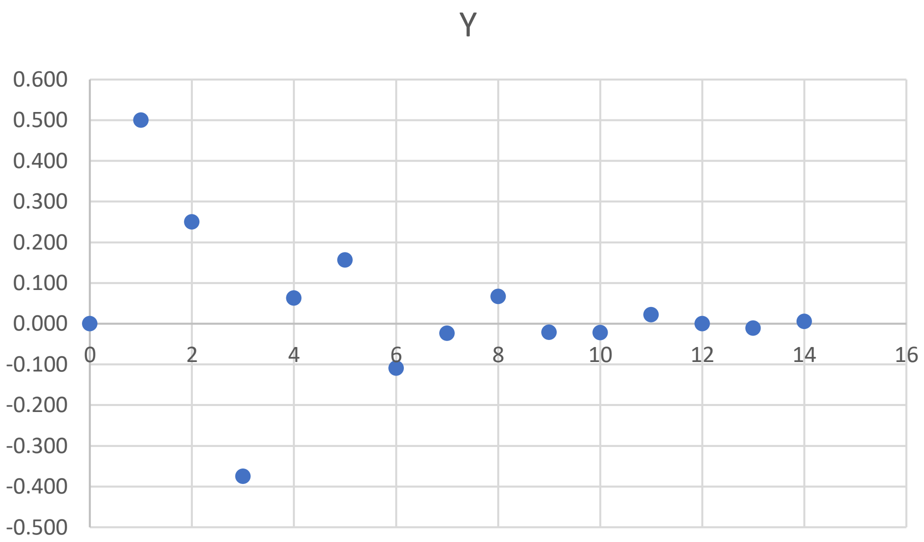
Y=0.5A+0.5B

E=X-Y

B=A

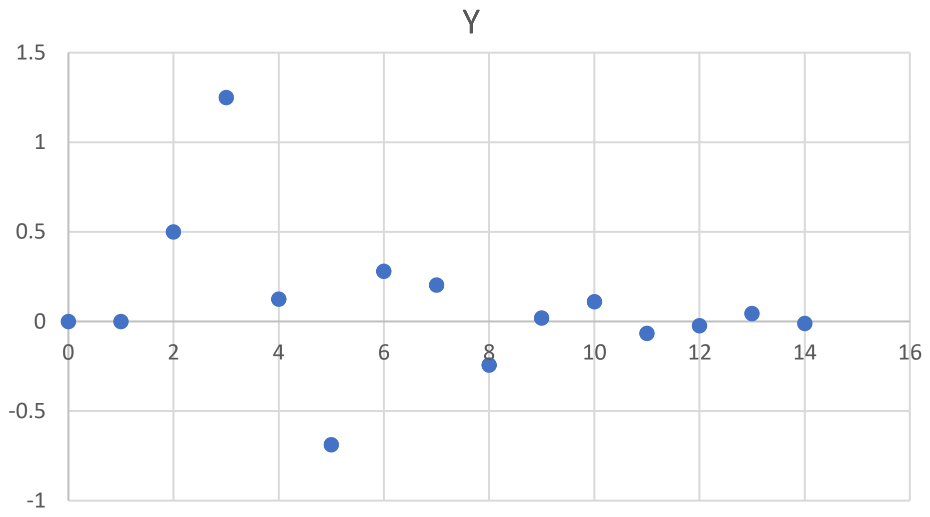
A=E

n	X	A	B	E	Y
0	1	0.000	0.000	1.000	0.000
1	0	1.000	0.000	-0.500	0.500
2	0	-0.500	1.000	-0.250	0.250
3	0	-0.250	-0.500	0.375	-0.375
4	0	0.375	-0.250	-0.063	0.063
5	0	-0.063	0.375	-0.156	0.156
6	0	-0.156	-0.063	0.109	-0.109
7	0	0.109	-0.156	0.023	-0.023
8	0	0.023	0.109	-0.066	0.066
9	0	-0.066	0.023	0.021	-0.021
10	0	0.021	-0.066	0.022	-0.022
11	0	0.022	0.021	-0.022	0.022
12	0	-0.022	0.022	0.000	0.000
13	0	0.000	-0.022	0.011	-0.011
14	0	0.011	0.000	-0.005	0.005



b) $x[n]=n(u[n]-u[n-3]) = \{0, 1, 2\}$

n	X	A	B	E	Y
0	0	0.000	0.000	0.000	0.000
1	1	0.000	0.000	1.000	0.000
2	2	1.000	0.000	1.500	0.500
3	0	1.500	1.000	-1.250	1.250
4	0	-1.250	1.500	-0.125	0.125
5	0	-0.125	-1.250	0.688	-0.688
6	0	0.688	-0.125	-0.281	0.281
7	0	-0.281	0.688	-0.203	0.203
8	0	-0.203	-0.281	0.242	-0.242
9	0	0.242	-0.203	-0.020	0.020
10	0	-0.020	0.242	-0.111	0.111
11	0	-0.111	-0.020	0.065	-0.065
12	0	0.065	-0.111	0.023	-0.023
13	0	0.023	0.065	-0.044	0.044
14	0	-0.044	0.023	0.011	-0.011



c) $h[n]=\{0, 0.5, 0.25, -0.375, 0.063, 0.156\}$

$h[-n]=\{0.156, 0.0625, -0.375, 0.25, 0.5, 0\}$

$Y=X*h$

						x0	x1	x2						
						0	1	2						
8									0.15625	0.0625	-0.375	0.25	0.5	0
7								0.15625	0.0625	-0.375	0.25	0.5	0	
6							0.15625	0.0625	-0.375	0.25	0.5	0		
5						0.15625	0.0625	-0.375	0.25	0.5	0			
4					0.15625	0.0625	-0.375	0.25	0.5	0				
3				0.15625	0.0625	-0.375	0.25	0.5	0					
2			0.15625	0.0625	-0.375	0.25	0.5	0						
1		0.15625	0.0625	-0.375	0.25	0.5	0							
0	0.15625	0.0625	-0.375	0.25	0.5	0								
	0	0	0.5	1.25	0.125	-0.6875	0.28125	0.3125						
	y0	y1	y2	y3	y4	y5								

$y[n]=\{0, 0, 0.5, 1.25, 0.125, -0.6875, \ldots\}$

Convolution of periodic DT signals:

If $x_1[n]$ and $x_2[n]$ are both periodic sequences with common period N, the convolution of $x_1[n]$ and $x_2[n]$ does not converge. In this case, we define the periodic convolution of $x_1[n]$ and $x_2[n]$ as

$$f[n] = x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k]x_2[n-k]$$

Proof:

Show that f[n] as convolution two periodic sequences is periodic with period N.

$$x_2[(n-k)+N] = x_2[n-k]$$

$$f[n+N] = \sum_{k=0}^{N-1} x_1[k]x_2[n+N-k] = \sum_{k=0}^{N-1} x_1[k]x_2[(n-k)+N] = \sum_{k=0}^{N-1} x_1[k]x_2[(n-k)] = f[n]$$

f[n] is periodic

DT Systems with or without Memory:

Since the output y[n] of a memoryless system depends on only the present input x[n], then, if the system is also linear and time-invariant, this relationship can only be of the form

$$y[n] = Kx[n]$$

where K is a (gain) constant. Thus, the corresponding impulse response h[n] is simply

$$h[n] = K\delta[n]$$

Therefore, if $h[n_0] \neq 0$ for $n_0 \neq 0$, the discrete-time LTI system has memory.

Stability of DT LTI systems:

A discrete-time LTI system is BIB0 stable if its impulse response is absolutely summable $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

Proof:

Let’s prove the BIBO stability condition for discrete-time LTI systems.

Assume that the input x[n] of a discrete-time LTI system is bounded → $|x[n]| \le k_1 \quad \forall n$

Due to convolution of impulse response and the input, the output will be:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \le k_1 \sum_{k=-\infty}^{\infty} |h[k]|$$

$|x[n-k)| \le k_1$ if the impulse response is absolutely summable → $\sum_{k=-\infty}^{\infty} |h[k]| = K < \infty$

$$|y[n]| \le k_1 K = k_2 < \infty$$

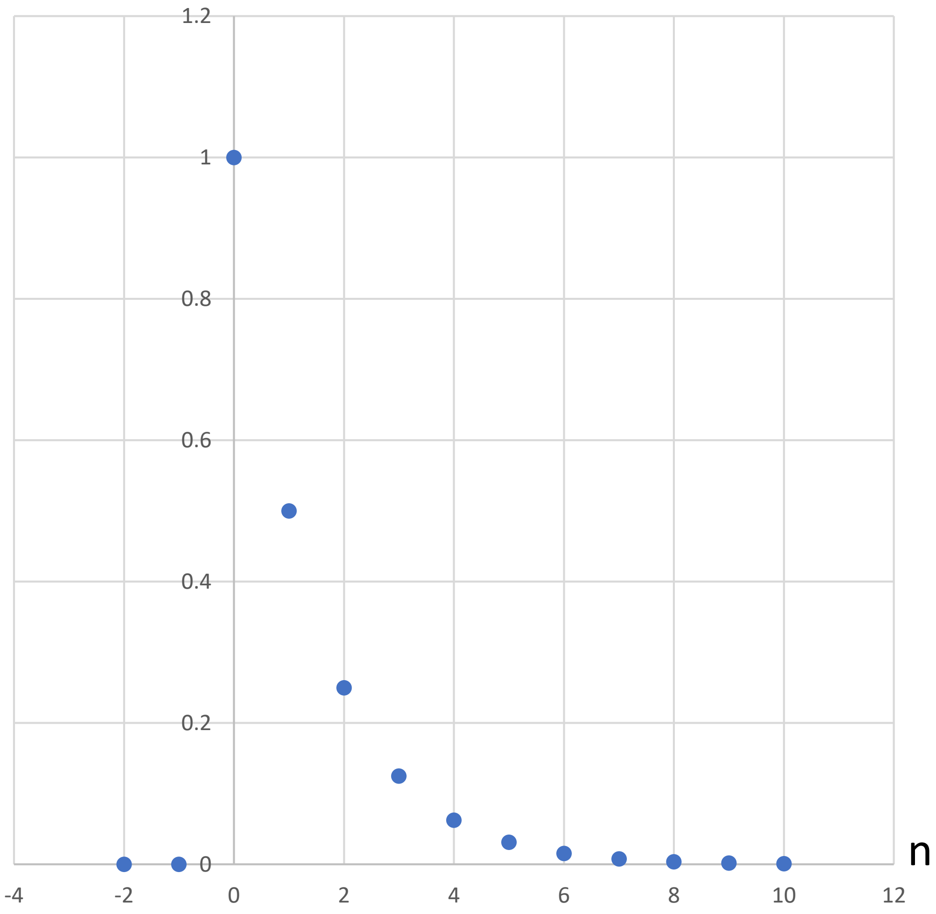
Hence the system is BIB0 stable

Example:

Impulse response of a discrete-time LTI system is given by $h[n]=0.5^n u[n]$

a) Is this system causal? b) Is this system BIBO stable?

h[n]



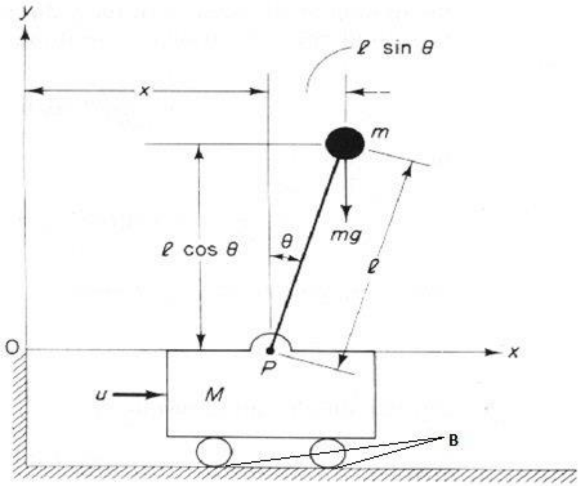
a) The system is causal because $h[n]=0$ for $n<0$

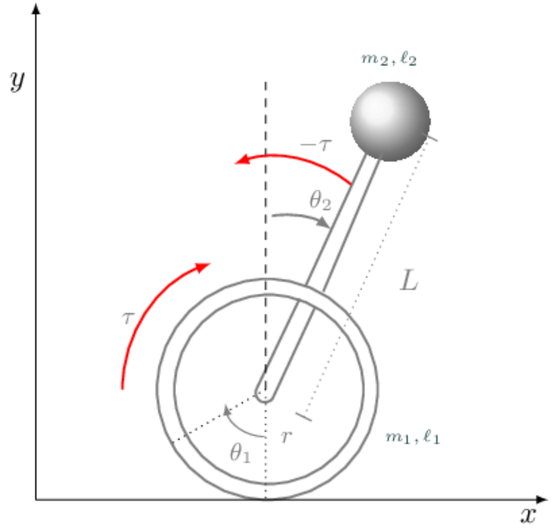
b)

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |\, 0.5^k u[n] \,|$$

$$= \sum_{k=0}^{\infty} |0.5|^k = \frac{1}{1-|0.5|} = 2 = C < \infty$$ → system is BIBO stable

Modelling and Simulation of the Systems





M	mass of the cart	0.5 kg
m	mass of the pendulum	0.5 kg
b	friction of the cart	0.1 N/m/sec
l	length to pendulum center of mass	0.3 m
I	inertia of the pendulum	0.006 kg*m^2
u	step force applied to the cart	
x	cart position coordinate	
phi	pendulum angle from vertical	

Physical system equation:

$$(M+m)\ddot{x}+b\dot{x}+ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$$

$$(I+ml^2)\ddot{\theta}+mgl\sin\theta = -ml\ddot{x}\cos\theta$$

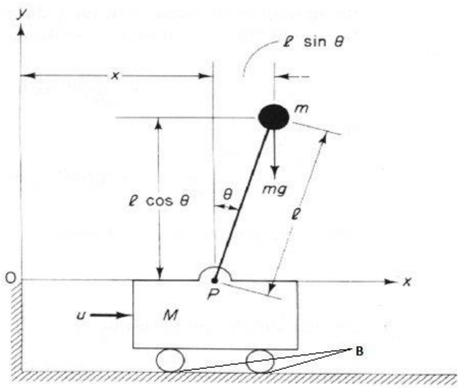
Linearization around very small Φ: Θ=π+ø ,
cos(Θ) = -1, sin(Θ) = -ø, and (d(Θ)/dt)^2 = 0

$$(I+ml^2)\Phi(s)s^2 - mgl\Phi(s) = mlX(s)s^2$$

$$(M+m)X(s)s^2 + bX(s)s - ml\Phi(s)s^2 = U(s)$$

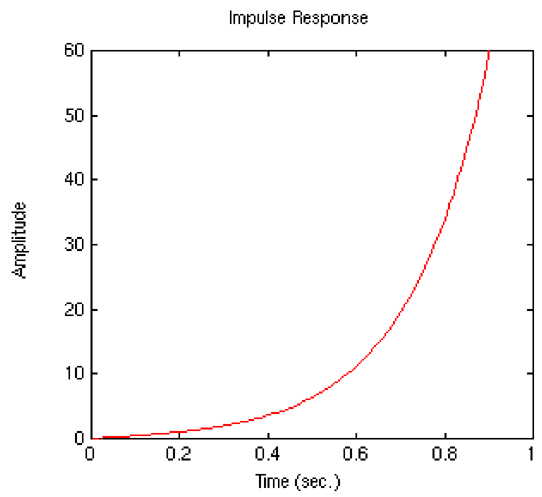
Transfer function:

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{b(I+ml^2)}{q}s^2 - \frac{(M+m)mgl}{q}s - \frac{bmgl}{q}}$$



$$\begin{bmatrix} x(k) \\ \dot{x}(k) \\ \phi(k) \\ \dot{\phi}(k) \end{bmatrix} = \begin{bmatrix} 1 & 0.01 & 0.0001 & 0 \\ 0 & 0.9982 & 0.0267 & 0.0001 \\ 0 & 0 & 1.0016 & 0.01 \\ 0 & -0.0045 & 0.3119 & 1.0016 \end{bmatrix} \begin{bmatrix} x(k-1) \\ \dot{x}(k-1) \\ \phi(k-1) \\ \dot{\phi}(k-1) \end{bmatrix} + \begin{bmatrix} 0.0001 \\ 0.0182 \\ 0.0002 \\ 0.0454 \end{bmatrix} [u(k-1)]$$

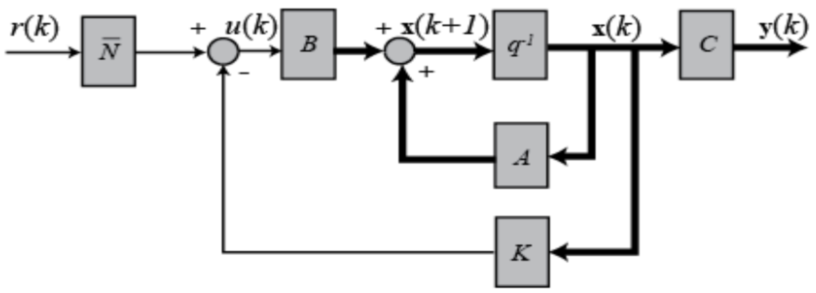
$$y(k-1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(k-1) \\ \dot{x}(k-1) \\ \phi(k-1) \\ \dot{\phi}(k-1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [u(k-1)]$$



Physical System ⟷ CT System Model ⟷ DT System Model ⟷ Simulation /Control

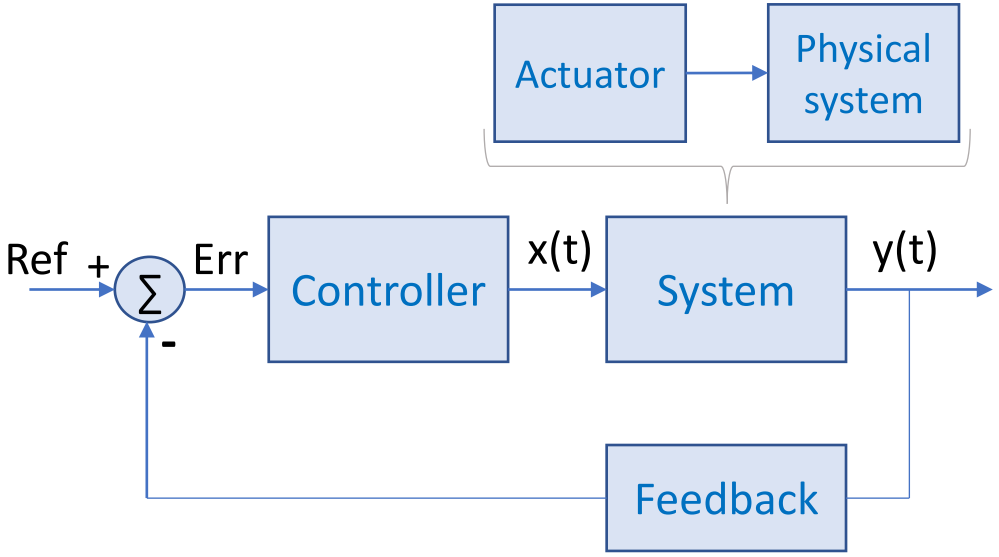
$$\begin{bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \\ \dot{\Phi} \\ \ddot{\Phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \Phi \\ \dot{\Phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \\ \Phi \\ \dot{\Phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mathbf{u}$$

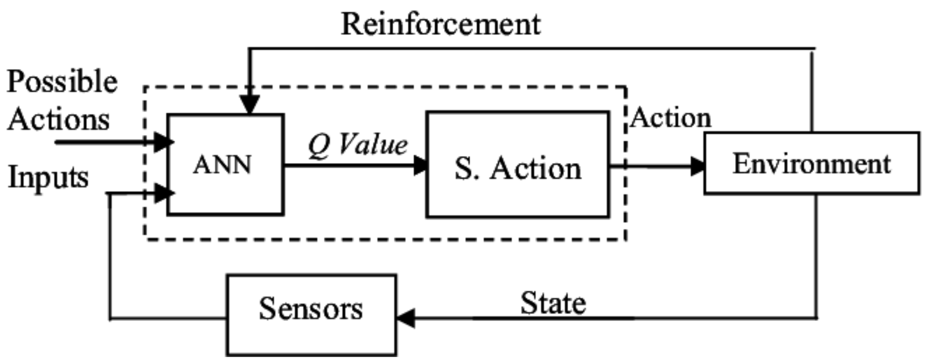


State space model

Feedback Control Systems



ML Approach to control systems:



PID controller example: