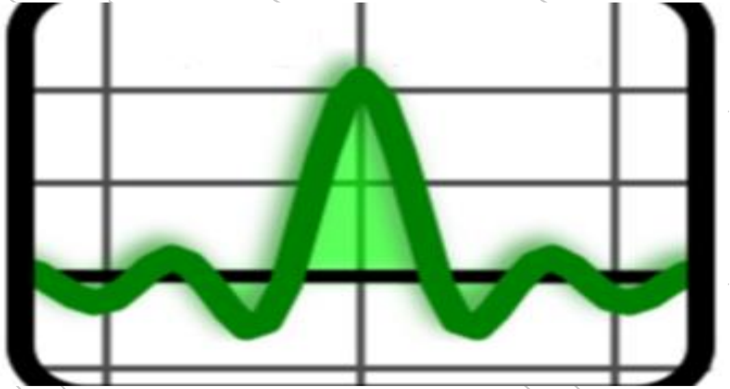


Signals & Systems for Computer Engineering

Prof.Dr. B.Berk ÜSTÜNDAĞ
Istanbul Technical University
Faculty of Computer Engineering and Informatics
bustundag@itu.edu.tr

BLG354E / CRN: 21560
4th Week Lecture

Step Response of an LTI System:

Step response s(t) of a CT LTI system is defined as the response of the system when the input is u(t)

$$s(t) = \mathbf{T}\{u(t)\}$$

The step response s(t) can be easily determined by convolution to impulse response $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$

$$\begin{aligned} s(t) = h(t) * u(t) &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau \\ &= \int_{-\infty}^{t} h(\tau)\,d\tau \end{aligned}$$

Therefore s(t) can be obtained by integrating the impulse response h(t).
By differentiating s(t) with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Step response s(t) of a system is one of the characterization methods

The impulse response h(t) can be determined by differentiating the step response s(t).

Eigenfunctions of Continuous-Time LTI systems

Eigenfunctions of continuous-time LTI systems represented by T are the complex exponentials e^{ST}, with s a complex variable

$$\mathbf{T}\{e^{st}\} = \lambda e^{st}$$

$$y(t) = \mathbf{T}\{e^{st}\} = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau = \left[\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau\right] e^{st} = H(s)\, e^{st} = \lambda e^{st}$$

where $\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$

Eigenvalue of a continuous-time LTI system associated with the eigenfunction e^{ST} is given by H(s) which is a complex constant whose value is determined by the value of s

If λ is a complex constant and s is a complex variable then $\mathbf{T}\{e^{st}\} = \lambda e^{st}$ in LTI systems

Proof:

Let y(t) be the output of the system with input $x(t) = e^{st}$ $\rightarrow$ $\mathbf{T}\{e^{st}\} = y(t)$

Since the system is time-invariant, $\mathbf{T}\{e^{s(t+t_0)}\} = y(t+t_0)$ for arbitrary real t_0.

Since the system is linear $\mathbf{T}\{e^{s(t+t_0)}\} = \mathbf{T}\{e^{st} e^{st_0}\} = e^{st_0}\mathbf{T}\{e^{st}\} = e^{st_0} y(t)$

$$y(t+t_0) = e^{st_0} y(t)$$

Setting t = 0, we obtain $y(t_0) = y(0)e^{st_0}$

Since t_0 is arbitrary, by changing t_0 to t, we can rewrite $y(t) = y(0)e^{st} = \lambda e^{st}$

or $\mathbf{T}\{e^{st}\} = \lambda e^{st}$

where $\lambda = y(0)$

Systems with or without Memory:

Since the output $y(t)$ of a memoryless system depends on only the present input $x(t)$, then, if the system is also linear and time-invariant, this relationship can only be of the form $y(t) = Kx(t)$

where K is a (gain) constant. Thus, the corresponding impulse response $h(t)$ is simply

$$h(t) = K\delta(t)$$

Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, the continuous-time LTI system has memory.

Casual LTI systems:

A causal system does not respond to an input event until that event actually occurs.
Therefore, for a causal continuous-time LTI system,

$$h(t) = 0 \qquad t < 0$$

Stability:

Bounded-input/Bounded-output (BIBO) stability of an LTI system can be determined from its impulse response.

A continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Example:

Find the impulse response of the casual LTI DT system described by the difference equation:

$$y[n] + 2y[n-1] = x[n] + x[n-1]$$

$x[n]=\delta[n] \rightarrow y[n]=h[n]$ $\qquad$ $h[n] = -2h[n-1]+\delta[n]+\delta[n-1]$

Since the system is casual, h[-1] = 0. Then

k=0 : $h[0] = -2h[-1]+\delta[0]+\delta[-1]= \delta[0]=1$

k=1 : $h[1] = -2h[0]+\delta[1]+\delta[0]=-2+1=-1$

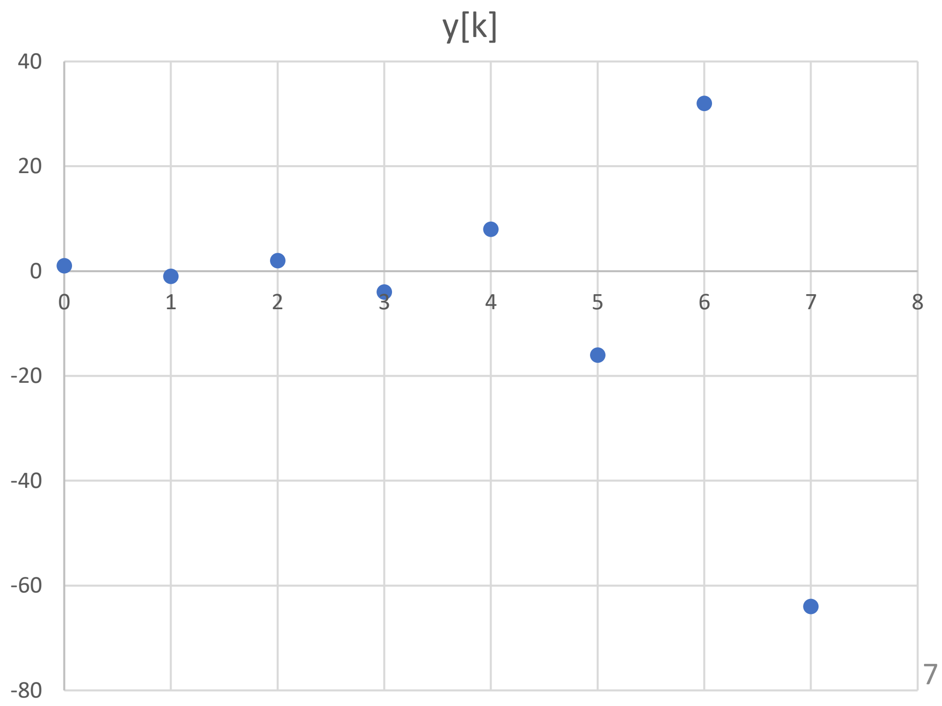
k=2 : $h[2] = -2h[1]+ \delta[2]+\delta[1] = -2(-1)=2$

k=3 : $h[3] = -2h[2]+\delta[3]+\delta[2] = -2(2) =-4$

k=n : $h[n] = -2h[n-1]+\delta[n]+\delta[n-1] = (-1)^n 2^{n-1}$

$h[n] = \delta[n] + (-1)^n 2^{n-1} u[n-1] \rightarrow$ IIR system

k	u[k-1]	δ[k]	y[k]
0	0	1	1
1	1	0	-1
2	1	0	2
3	1	0	-4
4	1	0	8
5	1	0	-16
6	1	0	32
7	1	0	-64
8	1	0	128
9	1	0	-256
10	1	0	512
11	1	0	-1024
12	1	0	2048



Solution by using the transfer function:

$$y[n] + 2y[n-1] = x[n] + x[n-1]$$

$$Y(z)+2z^{-1}Y(z)=X(z)+z^{-1}X(z)$$

$$T(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1+2z^{-1}}$$

$$Y(z)=T(z)X(z)$$

$$x[n]= \delta[n]=\{1\} \rightarrow X(z)=1$$

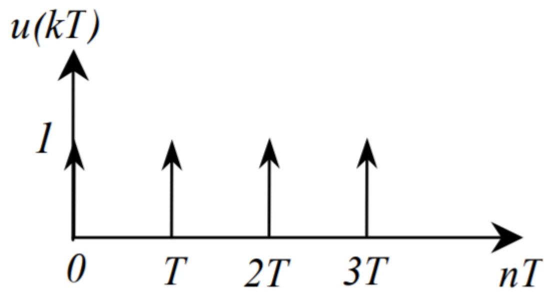
$$Y(z) = \frac{1}{1+z^{-1}} + \frac{z^{-1}}{1+2z^{-1}}$$

$$Y(z) = \frac{z}{z+2} + \frac{1}{z+2} \qquad y(k)=\mathcal{Z}^{-1}\{Y(z)\}$$

$$y(k) = (-2)^k u[k] + (-2)^{k-1} u[k-1]$$

$$y(k) = (-2)^k (u[k] - 0.5u[k-1])$$

10th week:



$$X(z) = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \leftrightarrow u[n] \leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| > 1$$

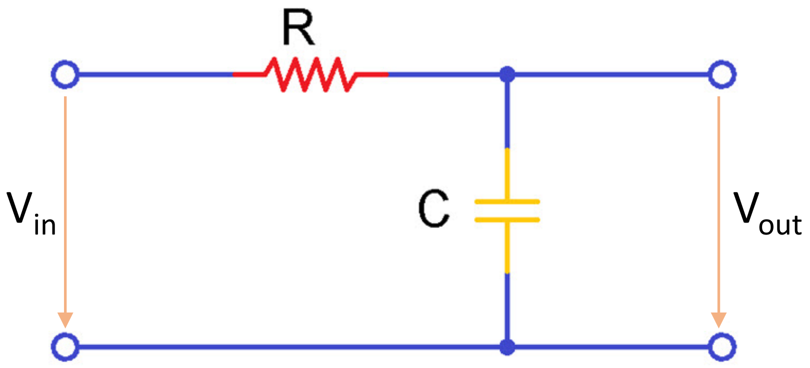
k	u[k]	u[k-1]	y[k]
0	1	0	1
1	1	1	-1
2	1	1	2
3	1	1	-4
4	1	1	8
5	1	1	-16
6	1	1	32
7	1	1	-64
8	1	1	128
9	1	1	-256
10	1	1	512
11	1	1	-1024
12	1	1	2048

Systems Described by Differential Equations

Differential equations that describe physical behavior CT LTI systems can be converted into equivalent Transfer Functions.

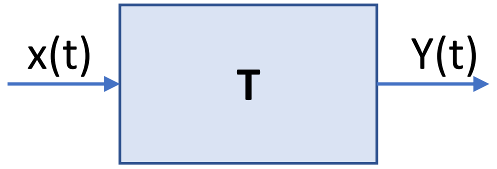
Their transfer functions can be used to estimate their output for a given input signal.
They also enable characteristic analysis including the BIBO stability.

Remember the RC Filter Example:



$$\left.\begin{aligned} v_s(t) &= Ri(t) + v_c(t) \\ i(t) &= C\frac{dv_c(t)}{dt} \end{aligned}\right\} \quad \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

Transfer function



$x(t)=V_{in}$
$y(t)=V_c=V_{out}$

A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

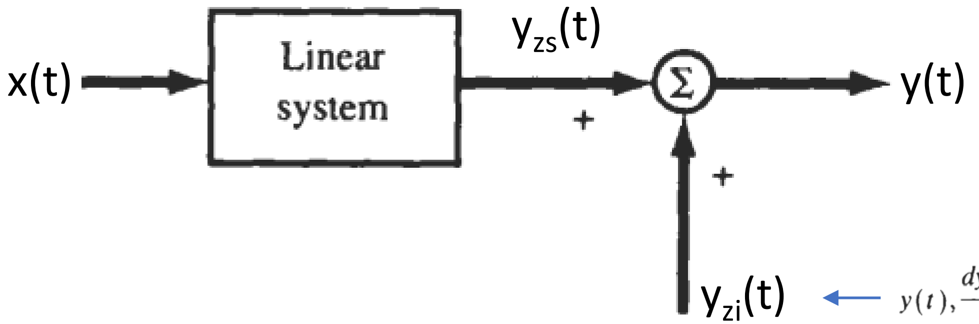
Its general solution for a particular input x(t) is given by $y(t) = y_p(t) + y_h(t)$

where $y_p(t)$ is a particular solution $y_h(t)$ is a homogeneous solution satisfying the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

Linearity of LTI systems

The exact form of homogenous solution $y_h(t)$ is determined by N auxiliary conditions $y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1}y(t)}{dt^{N-1}}$



$\leftarrow y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1}y(t)}{dt^{N-1}}$

$y_{zs}(t)$: Zero state response of the system

$y_{zi}(t)$: Zero input response of the system

The system specified by set of differential equations will be linear only if all of the auxiliary conditions are zero.

If the auxiliary conditions are not zero, then the response y(t) of a system can be expressed as

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

Example:

The following differential equation describes the input-output relationship of a physical system.

$$\frac{d^2y(t)}{dt^2}+a_1\frac{dy(t)}{dt}+a_2y(t)=x(t)$$

a) Find its CT LTI system block diagram

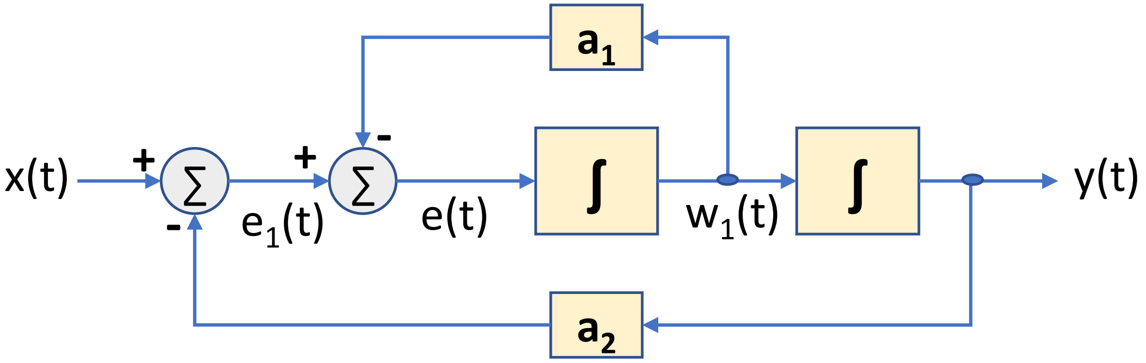
b) Find its Transfer Function (impulse response)

c) Find its output if the CT signal x(t)=2u(t) is applied to its input

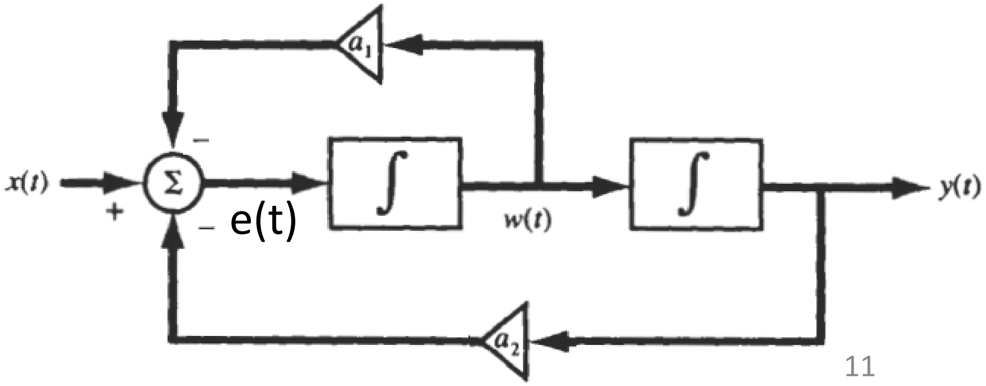
d) Simulate its output y(kT) if for x(t)=2u(t) if the sampling period T is 0.1seconds $a_1=a_2=0.5$

$$\frac{d^2y(t)}{dt^2}=-a_1\frac{dy(t)}{dt}-a_2y(t)+x(t) \qquad w(t)=\frac{dy(t)}{dt}$$

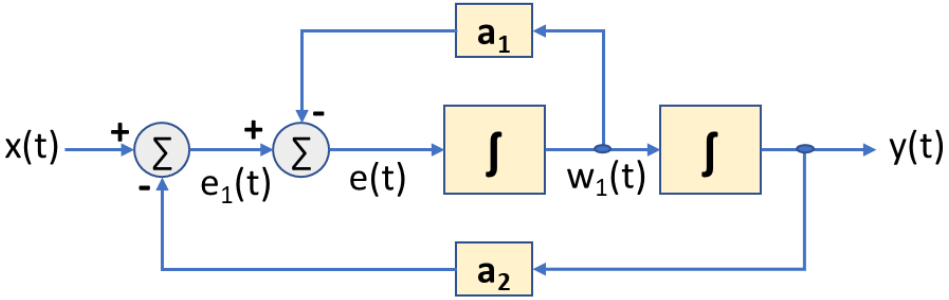
$$\frac{dw(t)}{dt}=-a_1w(t)-a_2y(t)+x(t)=e(t) \quad \begin{cases} e_1(t)=x(t)-a_2y(t) \\ e(t)=e_1(t)-a_1y(t) \end{cases}$$



≡

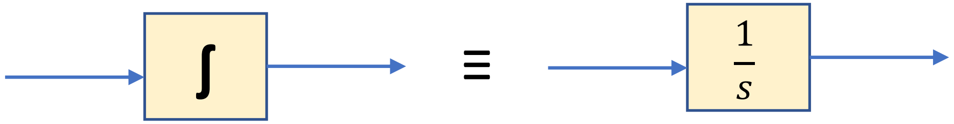


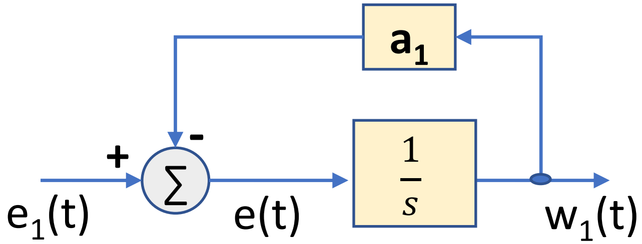
b)



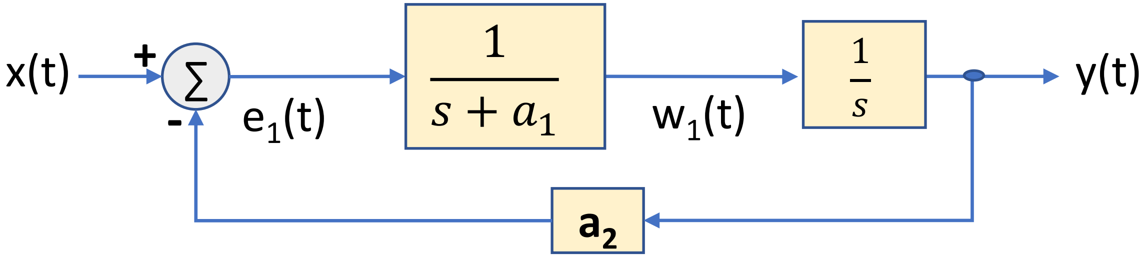
$$\frac{d}{dt} \triangleq s \rightarrow \int \equiv \frac{1}{s}$$

For more detailed explanations:
Week7: Laplace Transform





$$w_1(t) = \frac{\frac{1}{s}}{1+\frac{a_1}{s}} e_1(t) \quad \rightarrow \quad \frac{w_1(t)}{e_1(t)} = \frac{1}{s+a_1}$$



$$\mathrm{Y}(s) = \frac{\frac{1}{s(s+a_1)}}{1+\frac{a_2}{s(s+a_1)}} X(s) \qquad T(s) = \frac{Y(s)}{X(s)} = \frac{1}{s(s+a_1)+a_2} = \frac{1}{s^2+a_1 s+a_2}$$

c) $$y(t) = L^{-1}\left\{ \overbrace{\frac{1}{s^2+a_1 s+a_2}}^{\mathrm{T(s)}} \cdot \overbrace{\frac{2}{s}}^{\mathrm{X(s)\ for\ x(t)=2u(t)}} \right\}$$

d) Set t=kT and find y(t) for the given values

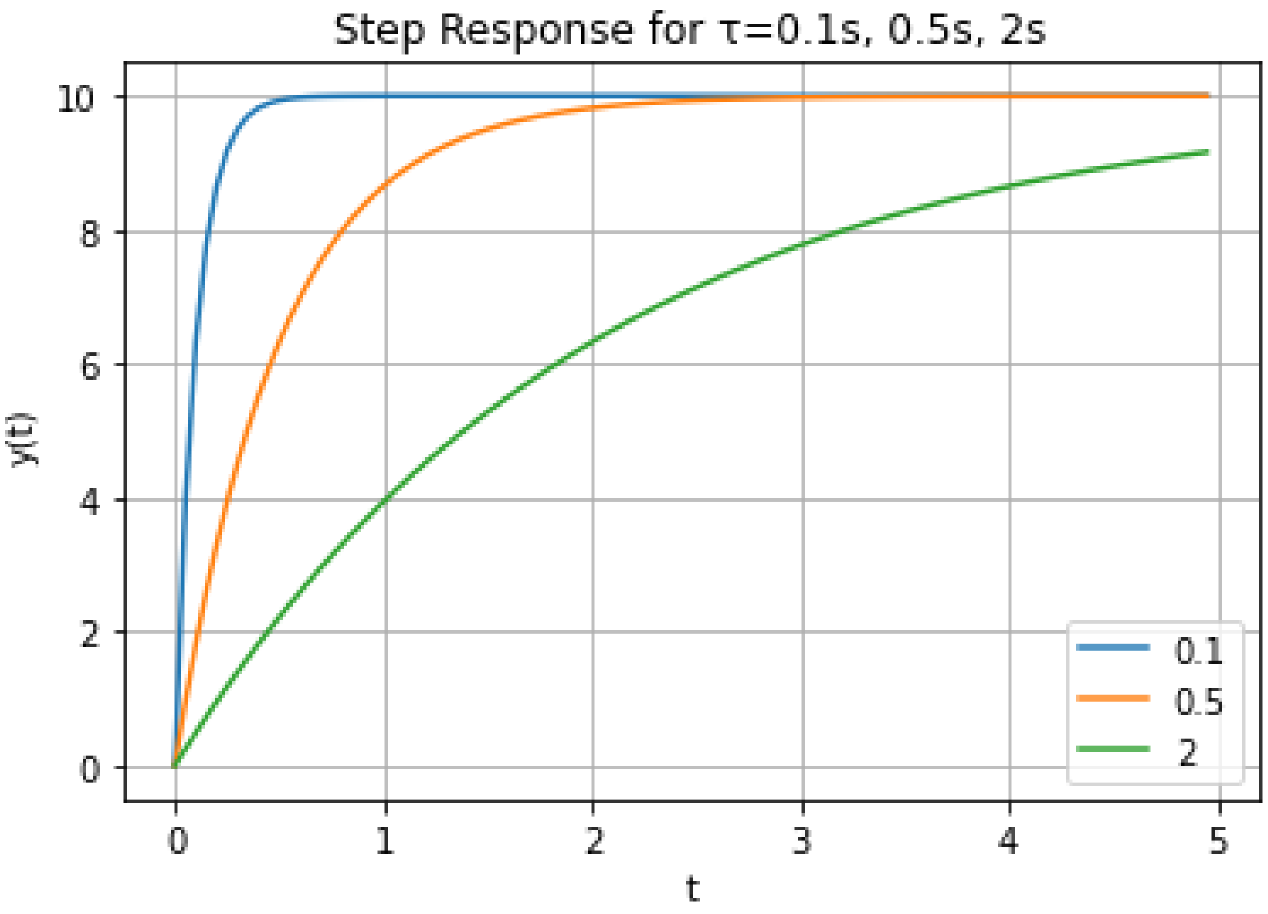
Step Response of 1st order Systems in Python

```
import control
import matplotlib.pyplot as plt
import numpy as np

Ts = [0.1, 0.5, 2] # time constants
K = 10
t = np.arange(0, 5, 0.05)
for T in Ts:
#Define the transfer function
    num = np.array ([K])
    den = np.array ([T , 1])
    H = control.tf(num , den)
    print ('H(s) =', H)
    t, y = control.step_response(H, t) #Step Response
    plt.plot(t, y) # Plot the output

plt.grid()
plt.xlabel("t")
plt.ylabel("y(t)")
plt.legend(Ts)
plt.title("Step Response for τ=0.1s, 0.5s, 2s")
plt.show()
```

$$H(s) = \frac{y(s)}{x(s)} = \frac{10}{\tau s + 1} \rightarrow y(t) = 10(1 - e^{-\frac{t}{\tau}})$$



Step Response of a 2nd order System in Python

```
import control
import matplotlib.pyplot as plt

s = control.TransferFunction.s
H1 = (20)/(s**2 + 5*s + 20)
H2 = (15*s + 20)/(s**2 + 5*s + 20)
print ('H1(s) =', H1)
print ('H2(s) =', H2)

t, y = control.step_response(H1)
plt.plot(t,y)
t, y = control.step_response(H2)
plt.plot(t,y)

plt.legend(["H1(s)","H2(s)"])
plt.xlabel("t")
plt.ylabel("y(t)")
plt.title("Step Response")
plt.grid()
```

$$H1(s) = \frac{20}{s^2 + 5s + 20}$$

$$H2(s) = \frac{15s + 20}{s^2 + 5s + 20}$$