Vector Spaces

The n-dimensional space $\mathbb{R}^n$ is the set of all n-tuples $(x_1, \dots, x_n)$ of real numbers.

$n=2$ $\quad (x,y) \in \mathbb{R}^2$ Euclidean

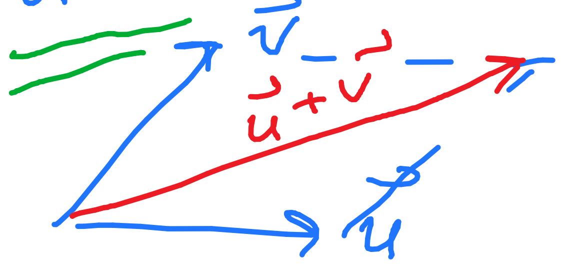
$n=3$ $\quad (x,y,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ Euclidean

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$ Euclidean

$$\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$$

$$\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$$

$$\vec{u} + \vec{v} = (u_1+v_1, \dots, u_n+v_n)$$



(Ilter) ⊕ (Yunus) = Cihangir

(ilter) ⊕ (Deniz) = Selin

(Cihangir) ⊕ (Yunus) = Serkan

2 (ilter) = Ece

3 (selin) = Esmanur

⊕ ⊙ V

- ilter
- Yunus
- Ece
- Selin
- Cihangir
- Deniz
- Serkan
- Esmanur

Definition of a Vector Space

Let V be any set for which two ops. addition & multiplication by a scalar are defined.

0. $\forall u, v \in V \Rightarrow u \oplus v \in V$

00. $c \in \mathbb{R},\ u \in V \Rightarrow cu \in V$

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = (-u) + u = 0$
5. $a(u + v) = au + av$
6. $(a + b)u = au + bu$
7. $a(bu) = (ab)u$
8. $1 \cdot u = u$

$\mathbb{R}$

$\mathbb{R}^2$

$\mathbb{R}^3$

$\mathbb{R}^n$

Two examples of vector spaces, other than $\mathbb{R}^n$

① $V = C[0,1]$: the space/set of cont. functions on $[0,1]$.

0. $f(x), g(x) \in V$ $\quad f(x)+g(x) \in V$?

If f is cont, and g is cont., so is $f(x)+g(x)$ // needs some proof, doable using your knowledge in Calc. I. ✓

00. $c \in \mathbb{R}$, $f(x) \in V \to$ is $c.f(x) \in V$?

Yes, indeed; f is cont., so is $c.f(x)$.

$C[0,1]$

- $f(x)$
- $g(x)$
- $f(x)+g(x)$
- $cf(x)$
- $2f(x)$
- $5f(x)$

$V = C[0,1]$ is indeed a vector space, that all the remaining properties 1-8 are satisfied.

So what??

② $M_{2\times 3} = \{$ Matrices of dimension 2×3 with real entries $\}$

$$M_{2\times 3} = \left\{ \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \;\middle|\; a, b, c, d, e, f \in \mathbb{R} \right\}$$

0. $u_1, u_2 \in M_{2\times 3} \rightarrow u_1 + u_2 \overset{?}{\in} M_{2\times 3}$??

$$u_1 = \begin{bmatrix} a_1 & c_1 & e_1 \\ b_1 & d_1 & f_1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} a_2 & c_2 & e_2 \\ b_2 & d_2 & f_2 \end{bmatrix}$$

$u_1 + u_2 \in M_{2\times 3}$

00. $\underset{\sim}{u} = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \Rightarrow k\underset{\sim}{u} = \begin{bmatrix} ka & kc & ke \\ kb & kd & kf \end{bmatrix}$

$k \in \mathbb{R}$ $\qquad\qquad$ $k\underset{\sim}{u} \in M_{2\times 3}$

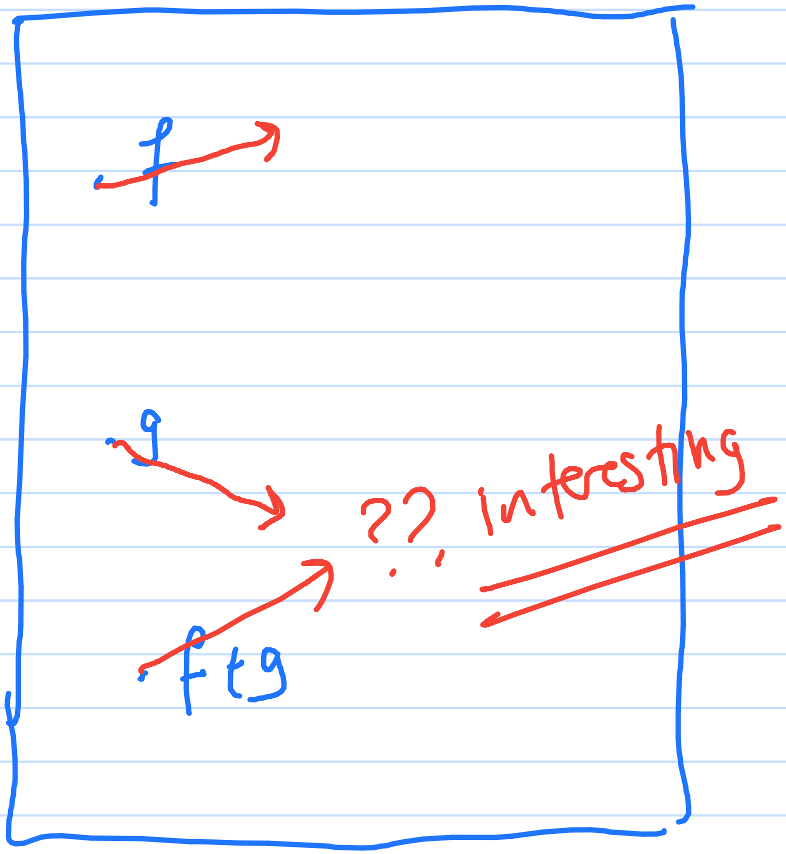
$M_{2\times 3}$

$$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

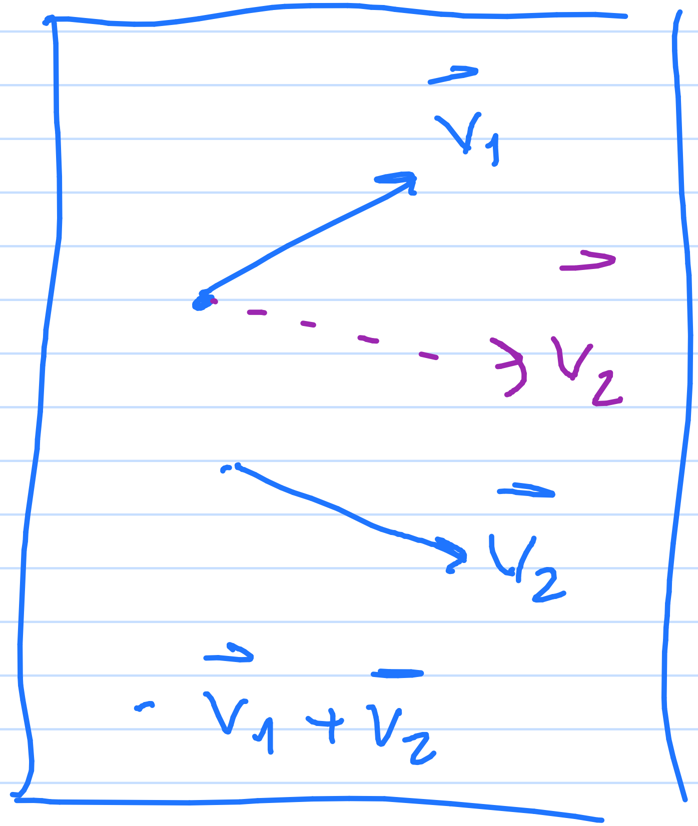
is also a vector space (#1 – #8 are also satisfied)

Moral: Vector spaces are not necesserally $\mathbb{R}^n$; there can be other interesting sets that can be treated as vector spaces, of which elements behave like VECTORS of $\mathbb{R}^n$!!!

$C[0,1]$



$\mathbb{R}^2$



$$f \cdot g = \langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$$

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1|\, |\vec{v}_2| \cos\theta$$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1|\,|\vec{v}_2|}\right)}$$

It is indeed possible to define the angle between two functions!!

$$\theta = \cos^{-1}\left(\frac{\langle f, g \rangle}{\|f\|\,\|g\|}\right)$$

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SUBSPACE (Sub vector space of a vector space)

Let $\emptyset \neq W$ be a subset of the vector space V.
If W is also a vector space with the operations of addition and multiplication by a scalar as defined in V, then W is called a subspace of V.

THEOREM $\emptyset \neq W \subset V$ is a subspace of the vector space V iff the following conditions are all satisfied:

$\underline{0} \leftarrow$ 1- $u, v \in W \Rightarrow u+v \in W$

$\underline{00} \leftarrow$ 2- $u \in W$, c: scalar $\Rightarrow cu \in W$

$W \subset V$

Example W: set of all vectors $(x_1, x_2, x_3, x_4) \in R^2$ such that $x_1 \cdot x_4 = 0$

$u, v \in W \Rightarrow u = (u_1, u_2, u_3, u_4)$ with $u_1 \cdot u_4 = 0$
$v = (v_1, v_2, v_3, v_4)$ with $v_1 \cdot v_4 = 0$

$u+v = (u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4) = (u_1+v_1, u_2+v_2, u_3+v_3, u_4+v_4)$

$(u_1+v_1)(u_4+v_4) = \underbrace{u_1 u_4}_{=0} + \underbrace{v_1 v_4}_{=0} + u_1 v_4 + v_1 u_4 = u_1 v_4 + v_1 u_4 \neq 0$ (not always)

$\Rightarrow u+v \notin W \Rightarrow W$: not a subspace

Example: W: subset of R^n such that if $(x_1, x_2, \ldots, x_n) \in W$, then $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$ where $a_1, a_2, \ldots, a_n$ are not all zero

* $x, y \in W \Rightarrow$
$$\left.\begin{array}{l} a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0 \\ a_1 y_1 + a_2 y_2 + \cdots + a_n y_n = 0 \end{array}\right\} a_1, \ldots, a_n: \text{not all zero}$$

$x+y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

$a_1(x_1+y_1) + a_2(x_2+y_2) + \cdots + a_n(x_n+y_n) = \underbrace{a_1 x_1 + \cdots + a_n x_n + a_1 y_1 + \cdots + a_n y_n}_{=0}$

Since $a_1, \ldots, a_n$ are not all zero, then $x+y \in W$

* $cx = (cx_1, cx_2, \ldots, cx_n)$

$a_1(cx_1) + a_2(cx_2) + \cdots + a_n(cx_n) = c(a_1 x_1 + \cdots + a_n x_n) = c.0 = 0$

Since $a_1, \ldots, a_n$ are not all zero, then $cx \in W$

$\Rightarrow W$: subspace of R^n

Examples of subspaces

(1) $V = \mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$ is a vector space

$W = \mathbb{R}^2 = \{(x,y,0) \mid x,y \in \mathbb{R}\}$ is a vector space

$W \subset V$ and $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

(2) $V = P_n(x) = \{$ polynomials of degree at most $n\}$

is a vector space. You can yourself show that axioms are satisfied.

e.g. $\underline{\underline{0}}$ $\quad u = a_0 + a_1 x + \cdots + a_n x^n \in V$

$v = b_0 + b_1 x + \cdots + b_n x^n \in V$

$u + v = a_0 + b_0 + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n \in V$

$c \in \mathbb{R}$, $\quad u = a_0 + a_1 x + \cdots + a_n x^n \in V = P_n(x)$

$$c.u = ca_0 + ca_1 x + \cdots + ca_n x^n \in V = P_n(x)$$

and, #1 – #8 are also satisfied.

So, $P_n(x)$ is a vector space.

Consider $W = P_m(x)$, $\quad m < n$.

definitley, • $W = P_m(x)$ is also a vector space.

• $W = P_m(x) \subset V = P_n(x)$

$\Rightarrow$ $P_m(x)$ is a subspace of $P_n(x)$
(sub vector space)

Example $V = \mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$

$W = \{(x_1, x_2, x_3, x_4) \mid x_1 \cdot x_4 = 0\}$

Question: Is W a subspace of V?

(i) we must have $u, v \in W \rightarrow u + v \in W$

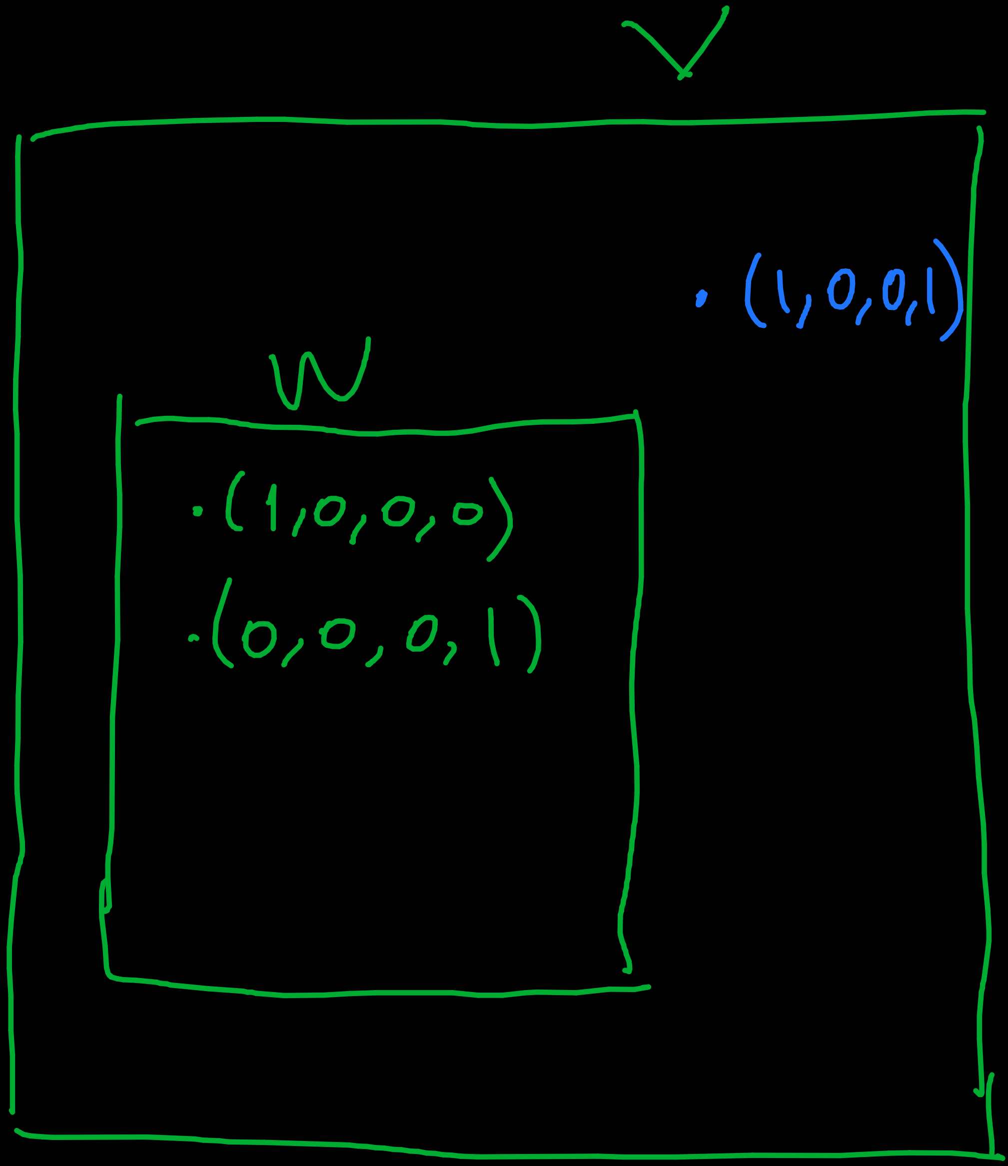
$$\left.\begin{aligned} u &= (1, 0, 0, 0) \in W \\ v &= (0, 0, 0, 1) \in W \end{aligned}\right\} \Rightarrow u + v = (1, 0, 0, 1) \notin W$$

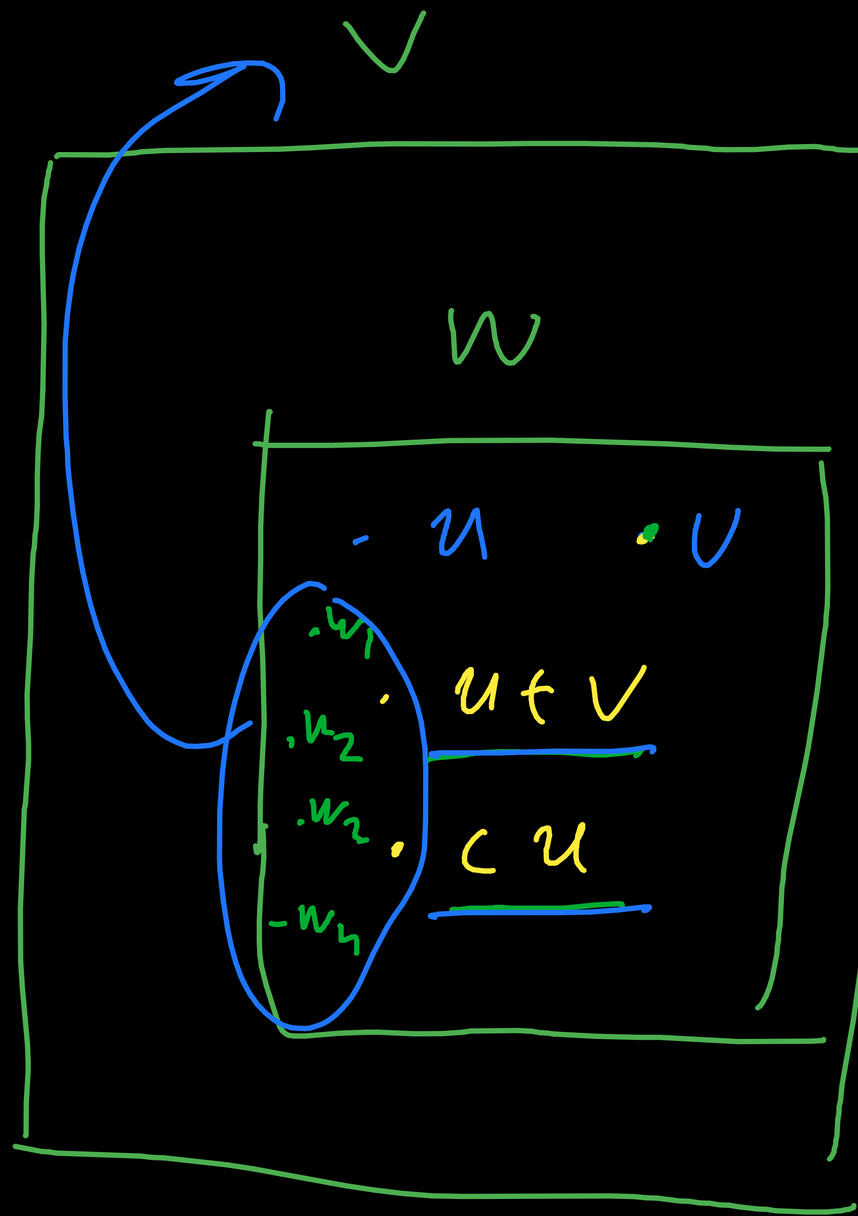
$x_1 \cdot x_4 = 1 \cdot 1 = 1 \neq 0$

(ii) we must have $c \in \mathbb{R},\ u \in W \Rightarrow cu \in W$

no need to check that

$\Rightarrow W$ is not a subspace of V.


V
W
(1,0,0,1)
(1,0,0,0)
(0,0,0,1)


V
W
u
v
w1
w2
w3
w4
u+v
cu

Ex $V = \mathbb{R}^3 = \{ (x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}$

$W = \{ (x_1, x_2, 0) \mid x_2 = 2x_1 \quad ; \quad x_1, x_2 \in \mathbb{R} \}$

Is W a subspace of V ??

(1) $u, v \in W \overset{?}{\Rightarrow} u+v \in W$

$u = (x_1, x_2, 0) \in W \quad x_2 = 2x_1$

$v = (y_1, y_2, 0) \in W \quad y_2 = 2y_1$

$\rightarrow x_2 + y_2 = 2(x_1+y_1)$

$u+v = (x_1+y_1, x_2+y_2, 0) \overset{?}{\in} W$ since $u+v \in W$ ✓

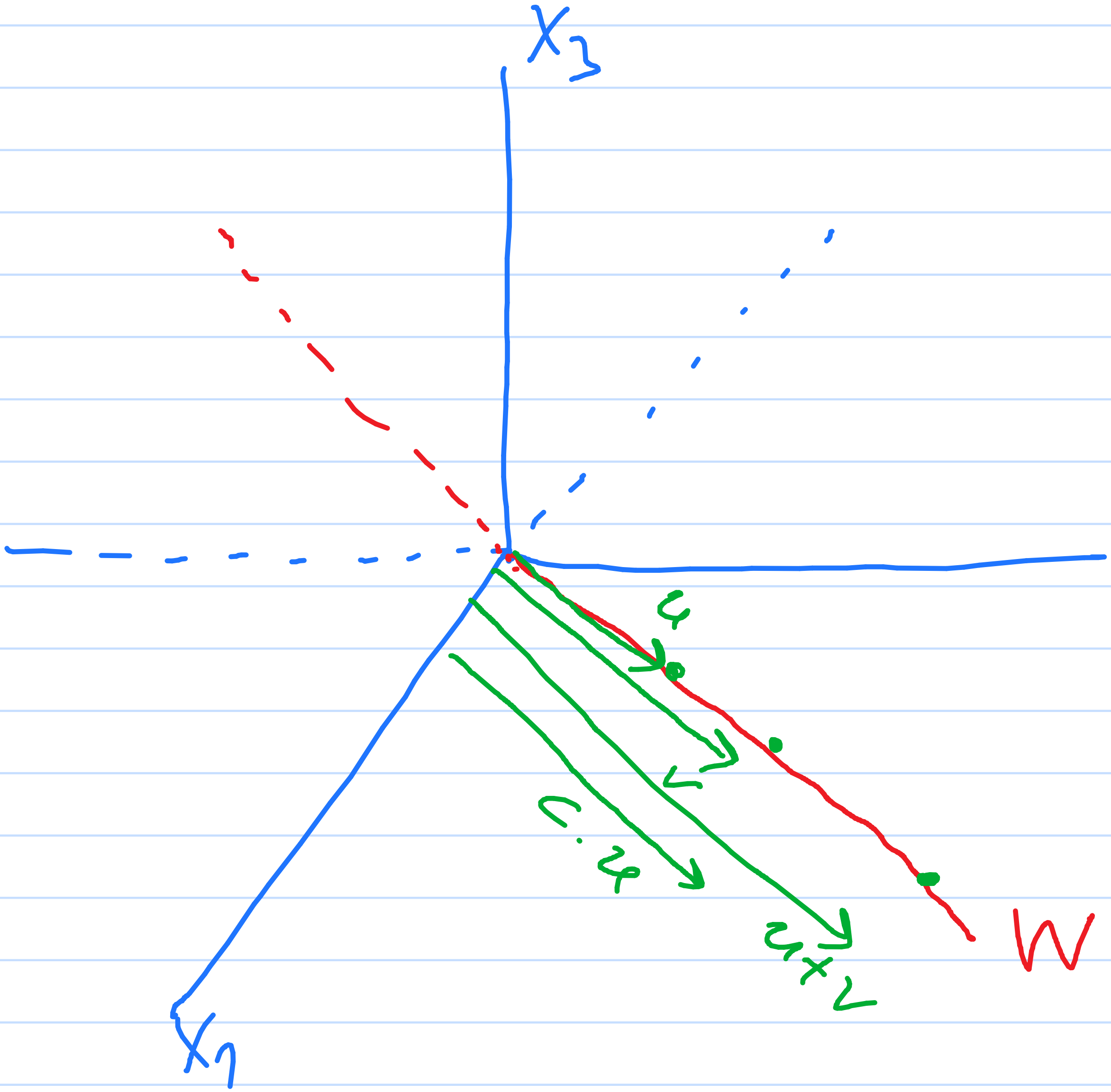
(2) $u = (x_1, x_2, 0) \in W \quad x_2 = 2x_1 \; ; \quad c \in \mathbb{R}$

$cu = (cx_1, cx_2, 0) \overset{?}{\in} W \quad cx_2 = 2(cx_1) \rightarrow cu \in W$

W is a subspace of V.

$V = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\} = \mathbb{R}^3$

$W = \{(x_1, x_2, 0) \mid x_2 = 2x_1 \ ; x_1, x_2 \in \mathbb{R}\}$



$V = \mathbb{R}^3$

$(x_1, x_2, 0)$

$x_2 = 2x_1$

$y = 2x$

Another example: Is $W = \{(x, y, z) \mid y = 2x;\ x, z \in \mathbb{R}\}$ a subspace of $V = \mathbb{R}^3$?

(i) $u = (x_1, 2x_1, z_1) \in W$

$v = (x_2, 2x_2, z_2) \in W$

$u + v = (\underbrace{x_1 + x_2}_{x}, \underbrace{2(x_1 + x_2)}_{y}, z_1 + z_2)$

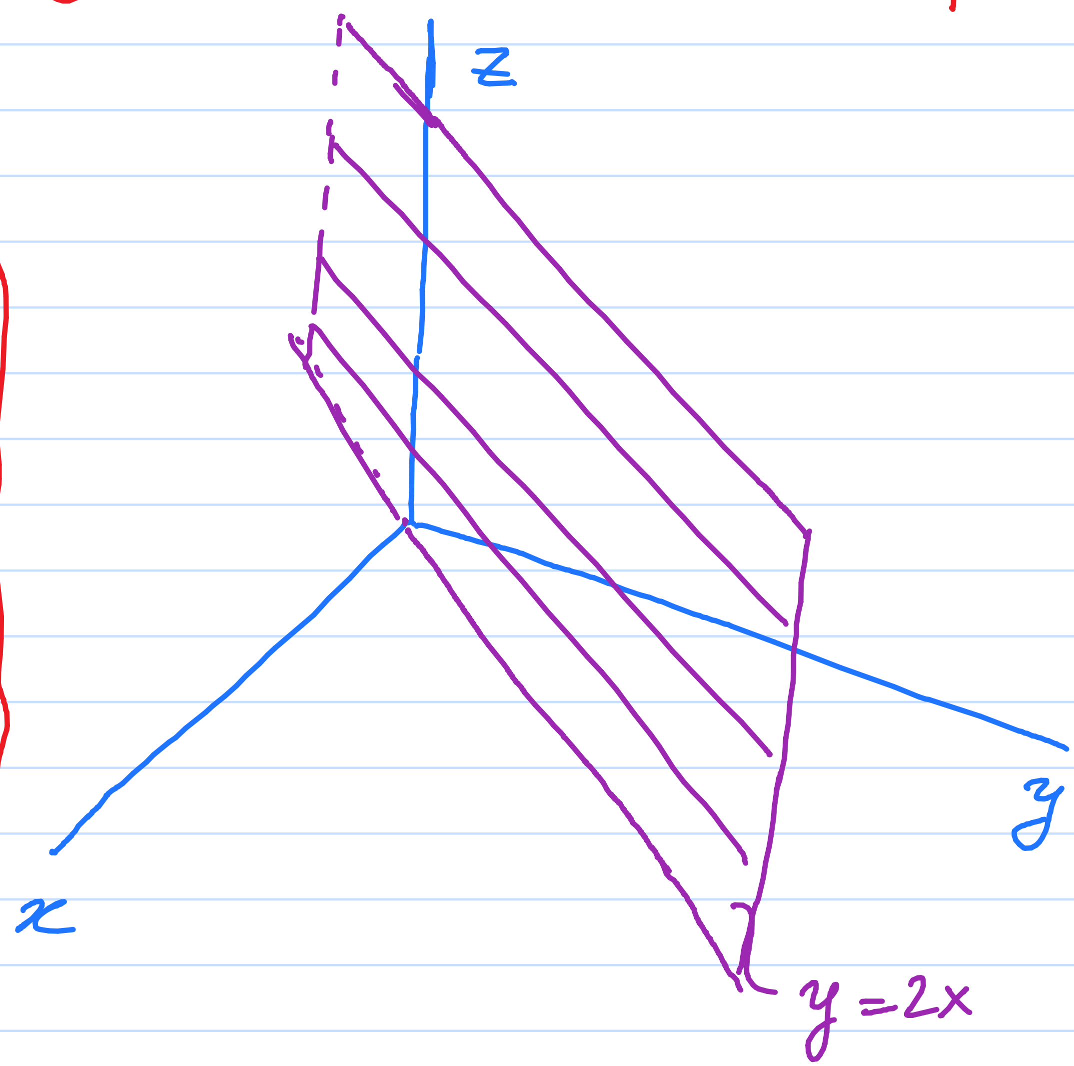
Since $y = 2x \Rightarrow u + v \in W$ ✓

(ii) $u = (x, 2x, z) \in W$, $c \in \mathbb{R}$;

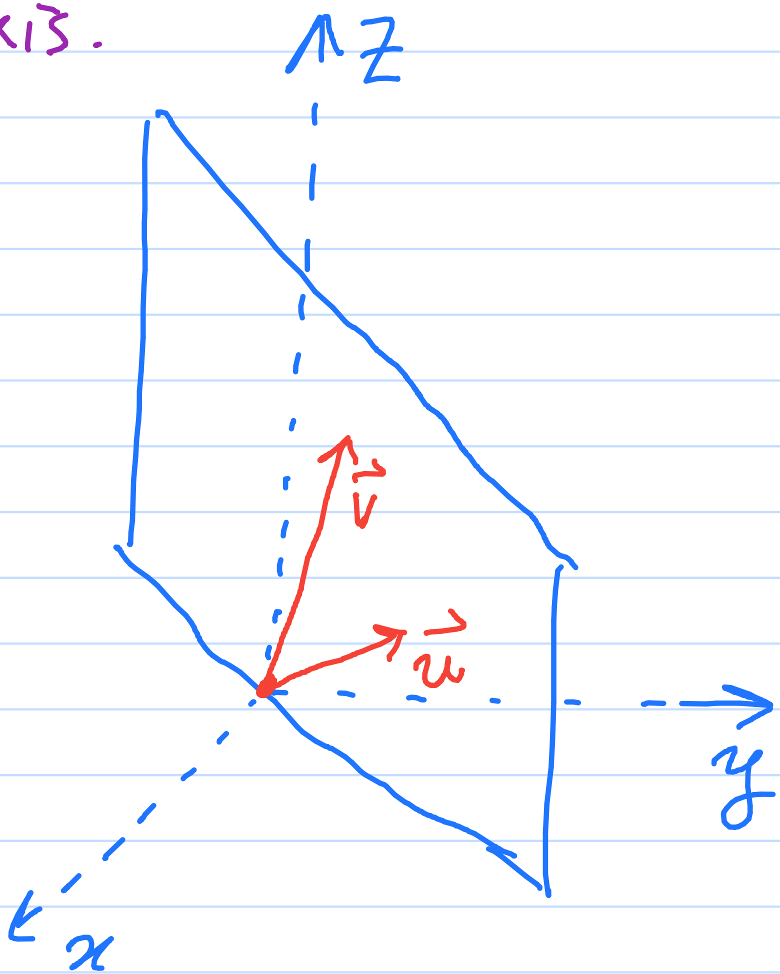
$cu = (\underbrace{cx}_{x}, \underbrace{2c\,x}_{y}, z)$

Since $y = 2x \Rightarrow c\underset{\sim}{u} \in W$

(i) $\wedge$ (ii) $\Rightarrow W$ is a subspace of $\mathbb{R}^3$.



when we plot the points (x, y, z) that belong to W, we see there's no restriction on z, it's arbitrary, and the coords. x, y has the cond. $y = 2x$. For $z = 0$, we draw the line $y = 2x$ on the xy-plane; for $z = 1$, we again draw the line $y = 2x$ on the plane $z = 1$. when we repeat this for every $z \in \mathbb{R}$, we obtain the collection of lines $y = 2x$ along z-direction; namely, the plane $y = 2x$; like a door that includes the z-axis.



See that, for any vectors $\vec{u}, \vec{v}$ in this plane;

$\vec{u}+\vec{v}$ is again in this plane;

$c\vec{u}$ is also in this plane.

Example Is $W = \{(x,2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

(i) $u = (x_1, 2) \in W$

$v = (x_2, 2) \in W$

$u + v = (x_1 + x_2, 4) \notin W$

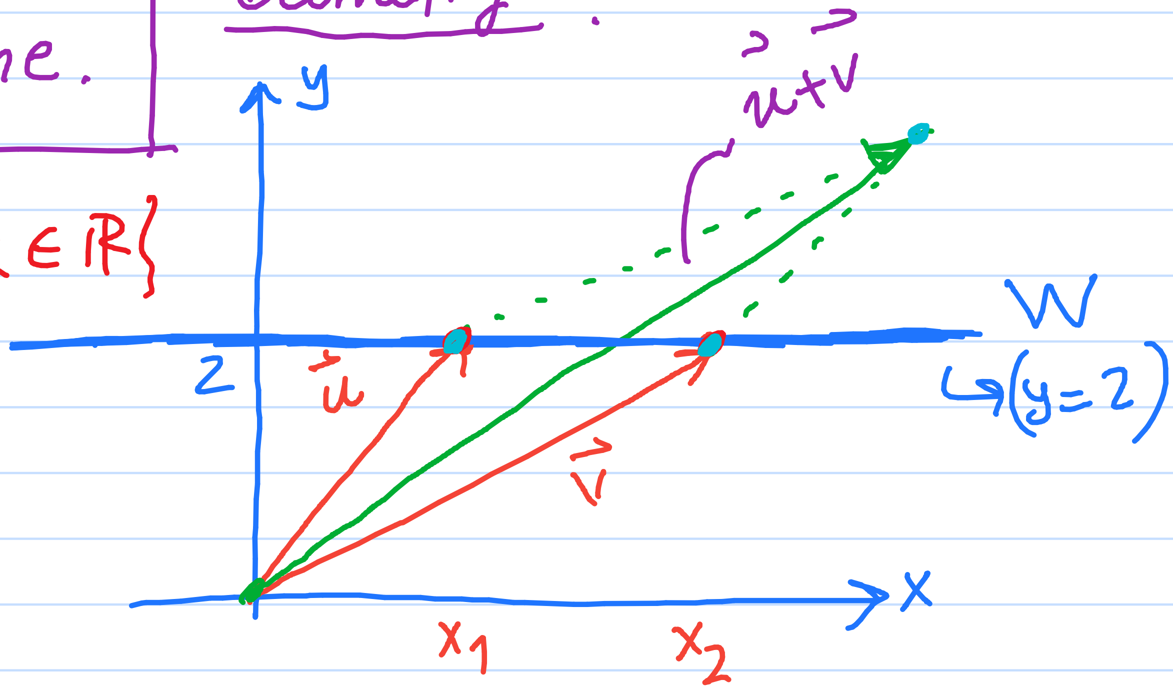
$\Rightarrow$ W is not a subspace of $\mathbb{R}^2$.

(ii) observe that

$cu = (cx_1, 2c) \notin W$

if $c \neq 1$.

Geometry :



$\underset{\sim}{u}, \underset{\sim}{v} \in W \Rightarrow \underset{\sim}{u} + \underset{\sim}{v} \notin W$

and $c\underset{\sim}{u} \notin W$

$(c \neq 1)$

Warning When checking whether $W \subset V$ is a sub vector space of V, we only check closedness under addition and multiplication by a scalar is true on W. The other eight properties are already satisfied on W as V is a vector space and any element of W is also an element of V.

Remarks

- Observe that, in the previous question, W does not include the additive identity element $\vec{0} = \langle 0, 0 \rangle = 0\vec{i} + 0\vec{j}$ of $V = \mathbb{R}^2$. W itself cannot be a vector space, due to lack of identity element of the addition.
- Can you identify subspaces W of $\mathbb{R}^2$?

{ The set W you will write must include the additive identity (0,0), therefore the geometric plot of the points of W must pass through the origin)

Example $W = \{(x_1, \dots, x_n) \mid a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0\}$

Is W a subspace of $\mathbb{R}^n$?

Definitely $W \subset \mathbb{R}^n = V$

$*$ $x = (x_1, \dots, x_n) \in W$: $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$

$y = (y_1, \dots, y_n) \in W$: $a_1 y_1 + a_2 y_2 + \dots + a_n y_n = 0$

$+$

$x, y \in W \Rightarrow x + y \overset{?}{\in} W$ $\quad a_1(x_1+y_1) + \dots + a_n(x_n+y_n) = 0$

$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in W$

$*$ $c \in \mathbb{R}$, $x = (x_1, \dots, x_n) \in W$ $\quad a_1 x_1 + \dots + a_n x_n = 0$

$c(a_1 x_1 + \dots + a_n x_n) = 0$

$c\underset{\sim}{x} = (cx_1, \dots, cx_n) \overset{?}{\in} W$ ✓ $\longleftrightarrow a_1(cx_1) + \dots + a_n(cx_n) = 0$

W is a subspace of $\mathbb{R}^n$

THEOREM Let A be a $m\times n$ matrix. Then the solution set of the homogeneous linear system $AX=0$ is a subspace of R^n.

Proof: W: solution set of $AX=0$ (also called solution space)

$u,v\in W \Rightarrow Au=Av=0$

$$\left.\begin{array}{l} * \ A(u+v)=Au+Av=0+0=0 \Rightarrow u+v\in W \\ * \ A(cu)=c(Au)=c0=0 \Rightarrow cu\in W \end{array}\right\} W: \text{subspace}$$

Important Note The solution set of the nonhom. system $AX=b$ where $b\neq 0$ is never a subspace.

$u\in W \Rightarrow Au=b$ but $A(cu)=c(Au)=cb\neq b$ for $c\neq 1 \Rightarrow cu\notin W$

Example
$$\left.\begin{array}{r} x_1+3x_2-15x_3+7x_4=0 \\ x_1+4x_2-19x_3+10x_4=0 \\ 2x_1+5x_2-26x_3+11x_4=0 \end{array}\right\} \text{solution space?}$$

$$\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -15 & 7 \\ 0 & 1 & -4 & 3 \\ 0 & -1 & 4 & -3 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

free variables: x_3, x_4

$x_3=t$, $x_4=s \Rightarrow x_2-4x_3+3x_4=0 \Rightarrow x_2=4t-3s$

$x_1-3x_3-2x_4=0 \Rightarrow x_1=3t+2s$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 3t+2s \\ 4t-3s \\ t \\ s \end{bmatrix}=t\underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{=u}+s\underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{=v} \Rightarrow X=tu+sv$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

$A \underset{\sim}{x} = \underset{\sim}{0} \qquad A = [a_{ij}]_{m\times n}$

The solution set of the hom. system $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0}$ is a subspace of $\mathbb{R}^n$.

$W = \{\text{Solution space of } \underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}\} = \left\{ \underset{\sim}{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \;\middle|\; \underset{\sim}{A}\,\underset{\sim}{X} = \underset{\sim}{0} \right\}$

(i) $x, y \in W$: $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{A}\,\underset{\sim}{y} = 0$

$\underset{\sim}{x} + \underset{\sim}{y} \overset{?}{\in} W \qquad \underset{\sim}{A}(\underset{\sim}{x} + \underset{\sim}{y}) = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{A}\,\underset{\sim}{y} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$

$\therefore$ $\underset{\sim}{x} + \underset{\sim}{y}$ also solves the hom. system $\Rightarrow \underset{\sim}{x} + \underset{\sim}{y} \in W$

(ii) $c \in \mathbb{R}$, $\underset{\sim}{x} \in W$ $\quad \underset{\sim}{A}\,\underset{\sim}{x} = 0$

Is $c\,\underset{\sim}{x} \in W$??

$$\underset{\sim}{A}(c\,\underset{\sim}{x}) = c(\underset{\sim}{A}\,\underset{\sim}{x}) = c\,\underset{\sim}{0} = \underset{\sim}{0}$$

$\Rightarrow c\,\underset{\sim}{x} \in W$

$\Rightarrow$ Solution space of $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0}$ is a subspace of $\mathbb{R}^n$; $(A = [a_{ij}]_{m \times n})$.

Addition to Example on page 16

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \quad \underset{\sim}{X} = t \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

The sol. space of this linear system is the set

$$W = \left\{ t \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t, s \in \mathbb{R} \text{ arbitrary} \right\}$$

Indeed W is a subspace of $\mathbb{R}^4$, as:

It is obvious that W is a subset of $\mathbb{R}^4$

(i) take $u, v \in W$. $u = t_1 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$,

$$v = t_2 \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow u + v = (t_1 + t_2) \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + (s_1 + s_2) \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$u + v \in W$!!!

(ii) $c \in \mathbb{R}$, $\underset{\sim}{u} = t \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \in W$

Is $c\underset{\sim}{u} \in W$??

$$c\underset{\sim}{u} = (ct) \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + (cs) \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \in W!$$

$\Rightarrow$ W is a subspace of $\mathbb{R}^4$

* Solution spaces of $\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}$ form subspaces / sub(vector) spaces of $\mathbb{R}^n$!!!

LINEAR COMBINATION

$\vec{u} = 3\vec{i} + 4\vec{j}$

$\vec{u}$ is a linear comb. of $\vec{i}, \vec{j}$.

If there exist scalars $c_1, c_2, \dots, c_k$ such that

$$w = c_1 v_1 + c_2 v_2 + \dots + c_k v_k,$$

then the vector w is called a linear combination of the vectors $v_1, v_2, \dots, v_k$. Furthermore, $\vec{u} = \frac{3}{2}(2\vec{i}) + \frac{4}{5}(5\vec{j})$

$\vec{u}$ is a linear comb. of $2\vec{i}$ and $5\vec{j}$.

Example Determine whether the vector $w = (2,5,8) \in \mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1,1,2)$ and $v_2 = (2,1,3)$.

$\underset{\sim}{w} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$ $\underset{\sim}{v_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\underset{\sim}{v_2} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$$w = c_1 v_1 + c_2 v_2 \Rightarrow (2,5,8) = (c_1, c_1, 2c_1) + (2c_2, c_2, 3c_2)$$

$$\left.\begin{aligned} c_1 + 2c_2 &= 2 \\ c_1 + c_2 &= 5 \\ 2c_1 + 3c_2 &= 8 \end{aligned}\right\} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 1 & 5 \\ 2 & 3 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -1 & 3 \\ 0 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

The third row tells us that the system is inconsistent. Thus, such c_1 and c_2 do not exist. Therefore, w is not a linear combination of v_1 and v_2.

Example Determine whether the vector $w = (1,7,3) \in \mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1,1,2)$, $v_2 = (2,1,0)$ and $v_3 = (3,2,1)$.

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\begin{array}{cccc} v_1 & v_2 & v_3 & w \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 7 \\ 2 & 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 6 \\ 0 & -4 & -5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 13 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & -1 & -23 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 1 & 23 \end{array}\right] \end{array}$$

$\Rightarrow c_1 = -10,\ c_2 = -29,\ c_3 = 23$ $\boxed{w = (-10)\underset{\sim}{v_1} + (-29)\underset{\sim}{v_2} + (23)\underset{\sim}{v_3}}$

$\Rightarrow w$ is a linear combination of v_1, v_2 and v_3 for specific valus of c_1, c_2 and c_3.

$w = (1, 7, 3)$ $\quad v_1 = (1, 1, 2)$, $\quad v_2 = (2, 1, 0)$ and

$v_3 = (3, 2, 1)$

$$\underset{\sim}{w} = \underset{(-10)}{c_1}\, \underset{\sim}{v}_1 + \underset{(-29)}{c_2}\, \underset{\sim}{v}_2 + \underset{(23)}{c_3}\, v_3$$

$$\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 1c_1 + 2c_2 + 3c_3 \\ c_1 + c_2 + 2c_3 \\ 2c_1 + c_3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 7 \\ 2 & 0 & 1 & 3 \end{array}\right]$$

$$\underline{\underline{c_1 = -10}} \qquad \underline{\underline{c_2 = -29}} \qquad \underline{\underline{c_3 = 23}}$$

Example Determine if $w=(-7,7,11)$ is a linear combination of the vectors $v_1=(1,2,1)$, $v_2=(-4,-1,2)$ and $v_3=(-3,1,3)$.

$$w=c_1v_1+c_2v_2+c_3v_3$$

$$\left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 6 & 6 & 18 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 \rightarrow \text{free var.} & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$c_3=t \Rightarrow c_2+c_3=3 \Rightarrow c_2=3-t \Rightarrow c_1+c_3=5 \Rightarrow c_1=5-t$

$t=0 \Rightarrow c_1=5,\ c_2=3,\ c_3=0 \Rightarrow w=5v_1+3v_2+0\cdot v_3$

$t=1 \Rightarrow c_1=4,\ c_2=2,\ c_3=1 \Rightarrow w=4v_1+2v_2+1\cdot v_3$

Thus, w can be written as a linear combination of v_1, v_2 and v_3 in many different ways.

SPAN

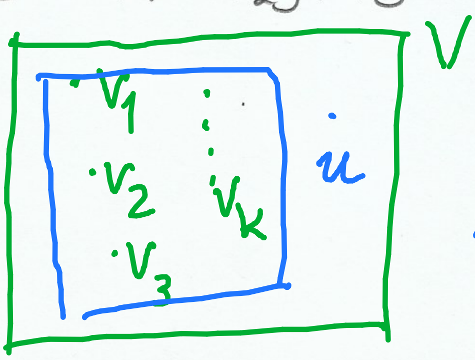
Let $v_1, v_2, \ldots, v_k$ are vectors in a vector space V. If every vector $v \in V$ is a linear combination of $v_1, v_2, \ldots, v_k$, then we say that $v_1, v_2, \ldots, v_k$ span the vector space V.

$S=\{v_1, v_2, \ldots, v_k\}$: spanning set of V

Example: $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$ and $i,j,k \in R^3$.
Every vector in R^3 can be written as a linear combination of i, j and k.

$$x=(x_1,x_2,x_3)=x_1 i+x_2 j+x_3 k$$

$\Rightarrow i,j,k$ span R^3



$u \in V$

$\underset{\sim}{u}=c_1\underset{\sim}{v_1}+c_2\underset{\sim}{v_2}+\ldots+c_k\underset{\sim}{v_k}$

$V = \mathbb{R}^3$

$u \in \mathbb{R}^3 \; : \quad u = (x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\underset{\sim}{u} = x_1 \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}^{\vec{i}} + x_2 \overbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}^{\vec{j}} + x_3 \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{\vec{k}}$$

Since $\forall u \in \mathbb{R}^3$ can be written as

$$\underset{\sim}{u} = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}$$

$\Rightarrow \{\vec{i}, \vec{j}, \vec{k}\}$ is a spanning set of $\mathbb{R}^3$.

and we write

$$\text{span}\{\vec{i}, \vec{j}, \vec{k}\} = \mathbb{R}^3$$

$\boxed{\text{Another example}}$

$V = \mathbb{R}^2 \qquad \underset{\sim}{u} = (x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

$$\underset{\sim}{u} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2.$$

Question: Are there any other spanning sets for $\mathbb{R}^2$??

For example, let's show that

$$\{v_1, v_2\} = \{\vec{i}+\vec{j}, \vec{i}-\vec{j}\}$$ is also

a spanning set for $\mathbb{R}^2$.

This is indeed true if any $\underset{\sim}{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as

$$\underset{\sim}{u} = c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2}$$

Let the vector $\underset{\sim}{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}$ be given to us.

We need to find c_1, c_2 that satisfy

$$x_1 \vec{i} + x_2 \vec{j} = c_1 (\vec{i}+\vec{j}) + c_2 (\vec{i}-\vec{j})$$

$$x_1 \vec{i} + x_2 \vec{j} = (c_1 + c_2)\vec{i} + (c_1 - c_2)\vec{j}$$

$$\left.\begin{aligned} c_1 + c_2 &= x_1 \\ c_1 - c_2 &= x_2 \end{aligned}\right\}$$ if this system is consistent, we can determine c_1 & c_2,

and the answer to the Q is affirmative

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

unknows: x_1, x_2

- unique sol.
- no sols.
- inf. many. sols.

$\underset{\sim}{A}\,\underset{\sim}{c} = \underset{\sim}{x}$ $\quad$ $\det A = 1.(-1) - 1.1 = -2 \neq 0$

A^{-1} exists $\quad A^{-1}Ac = A^{-1}x \quad \underset{\sim}{c} = A^{-1}x$

uniquely.

$$\left.\begin{aligned} c_1 + c_2 &= x_1 \\ c_1 - c_2 &= x_2 \end{aligned}\right\} \quad c_1 = \frac{x_1+x_2}{2}, \; c_2 = \frac{x_1 - x_2}{2}$$

Then, any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{i} + x_2\vec{j} \in \mathbb{R}^2$

can be written as

$$\underset{\sim}{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\frac{x_1+x_2}{2}}^{c_1}(\vec{i}+\vec{j}) + \overbrace{\frac{x_1-x_2}{2}}^{c_2}(\vec{i}-\vec{j})$$

Since we can do it for any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

$$\text{span}\{\vec{i}+\vec{j}, \vec{i}-\vec{j}\} = \mathbb{R}^2.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

unknowns : x_1, x_2

- unique sol.
- no sols.
- inf. many. sols.

$$\underset{\sim}{A}\ \underset{\sim}{c} = \underset{\sim}{x} \qquad \det A = 1.(-1) - 1.1 = -2 \neq$$

A^{-1} exists $\quad A^{-1}Ac = A^{-1}x \quad \underset{\sim}{c} = A^{-1}x$

uniquely.

$$\left.\begin{aligned} c_1 + c_2 &= x_1 \\ c_1 - c_2 &= x_2 \end{aligned}\right\} \quad c_1 = \frac{x_1+x_2}{2}, \quad c_2 = \frac{x_1-x_2}{2}$$

Then, any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{i} + x_2\vec{j} \in \mathbb{R}^2$

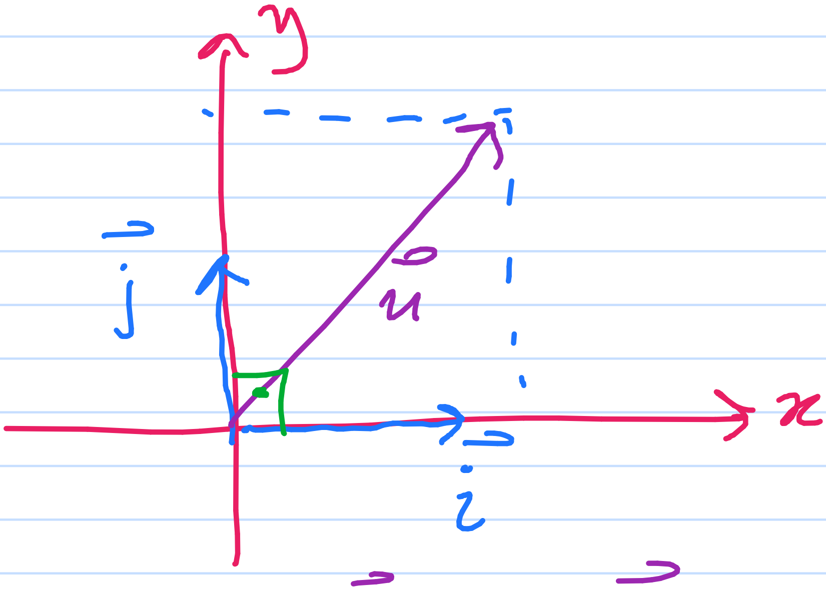
can be written as

$$\underset{\sim}{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\frac{x_1+x_2}{2}}^{c_1}(\vec{i}+\vec{j}) + \overbrace{\frac{x_1-x_2}{2}}^{c_2}(\vec{i}-\vec{j})$$

Since we can do it for any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

$$\operatorname{span}\{\vec{i}+\vec{j}, \vec{i}-\vec{j}\} = \mathbb{R}^2.$$

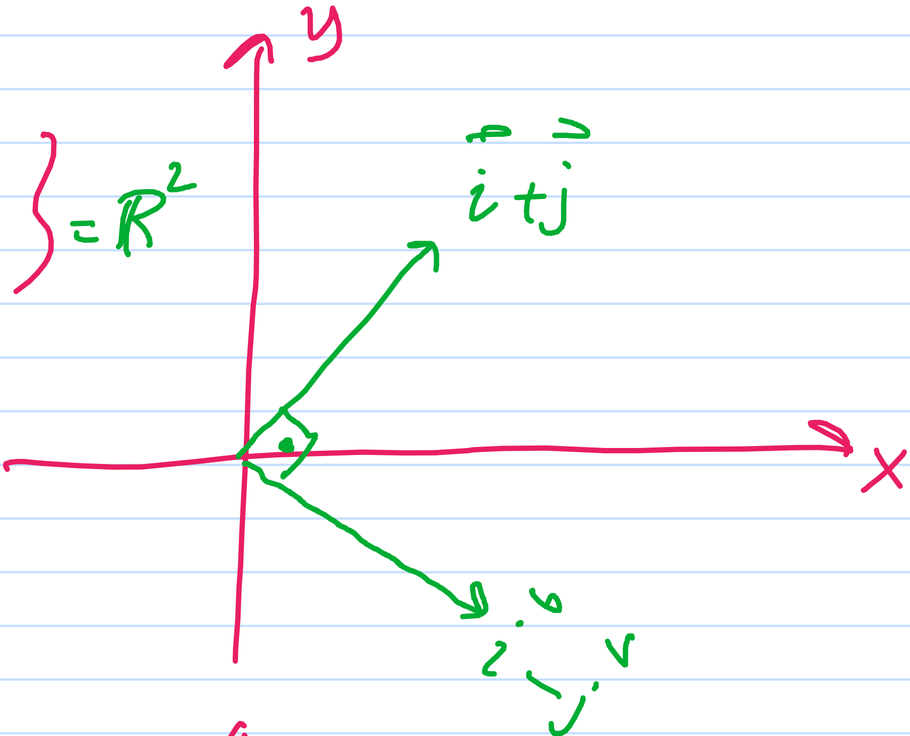
$\text{span}\{\vec{i}, \vec{j}\} = \mathbb{R}^2$



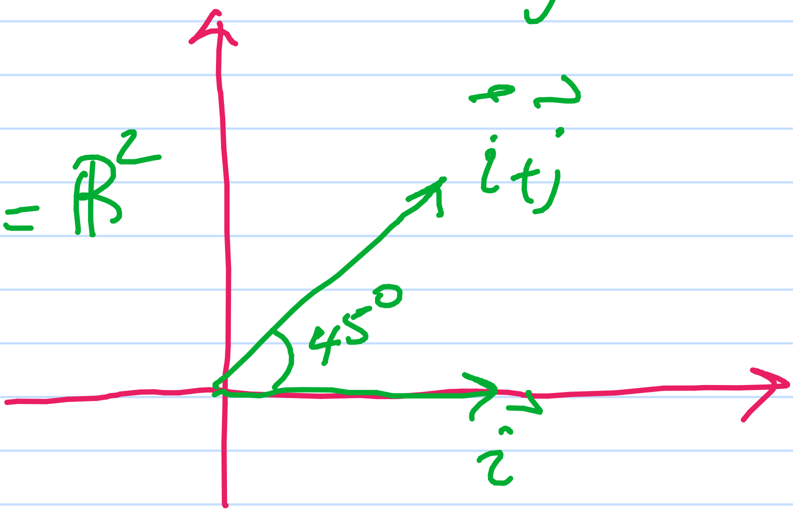
{any non-parallel two vectors span $\mathbb{R}^2$}

$\vec{u} = x_1 \vec{i} + x_2 \vec{j}$

$\text{span}\{\vec{i}+\vec{j}, \vec{i}-\vec{j}\} = \mathbb{R}^2$



$\text{span}\{\vec{i}, \vec{i}+\vec{j}\} = \mathbb{R}^2$



THEOREM Let $v_1, v_2, \dots, v_k$ be vectors in a vector space V. Then the set W of all linear combinations of $v_1, v_2, \dots, v_k$ is a subspace of V. $W = \{ c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \dots + c_k \underset{\sim}{v_k} \mid c_i \in \mathbb{R} \}$

Proof $u, v \in W \Rightarrow u = a_1 v_1 + a_2 v_2 + \dots + a_k v_k$

$$v = b_1 v_1 + b_2 v_2 + \dots + b_k v_k$$

$W \subset V$

$$u + v = (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \dots + (a_k + b_k) v_k$$
$$= c_1 v_1 + c_2 v_2 + \dots + c_k v_k \Rightarrow u + v \in W$$

· $\underset{\sim}{u}, \underset{\sim}{v} \in W$
$\rightarrow \underset{\sim}{u} + \underset{\sim}{v} \in W$

· $c\, \underset{\sim}{u} \in W$

$$cu = (ca_1) v_1 + (ca_2) v_2 + \dots + (ca_k) v_k = d_1 v_1 + d_2 v_2 + \dots + d_k v_k$$

$\Rightarrow cu \in W$

$\Rightarrow W$: subspace of V

$$\Rightarrow W = \operatorname{span} S = \operatorname{span} \{ v_1, v_2, \dots, v_k \}$$

LINEAR INDEPENDENCE of $\{ \underset{\sim}{v_1}, \dots, \underset{\sim}{v_k} \}$

If $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$ has only the trivial solution $c_1 = c_2 = \dots = c_k = 0$, then the vectors $v_1, v_2, \dots, v_k$ in a vector space V are called to be linearly independent
($c_1, c_2, \dots, c_k$: not all zero $\Rightarrow v_1, v_2, \dots, v_k$: Lin. dependent)

Example Let's determine if the vectors $v_1 = (1,2,2,0)$, $v_2 = (0,1,1,1)$, $v_3 = (2,1,0,1) \in \mathbb{R}^4$ are linearly independent

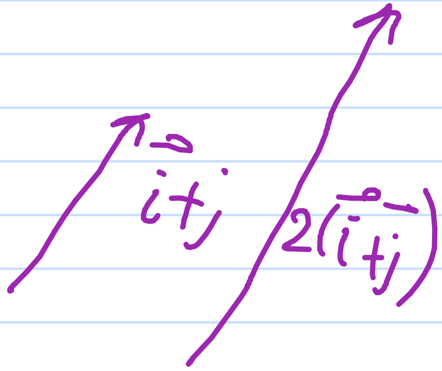
$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow (c_1, 2c_1, 2c_1, 0) + (0, c_2, c_2, c_2) + (2c_3, c_3, 0, c_3) = (0,0,0,0)$$

$$c_1 + 2c_3 = 0$$
$$2c_1 + c_2 + c_3 = 0$$
$$2c_1 + c_2 = 0$$
$$c_2 + c_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Ex $\quad \underset{\sim}{u} = \vec{i} + \vec{j} \qquad \underset{\sim}{v} = 2(\vec{i} + \vec{j})$

$\tilde{u}$ and $\tilde{v}$ are linearly dependent because



$c_1 = (-2)$, $\quad c_2 = 1$ and calculate

$$c_1 \underset{\sim}{u} + c_2 \underset{\sim}{v} = (-2)(\vec{i} + \vec{j}) + 1.2.(\vec{i} + \vec{j}) = 0\vec{i} + 0\vec{j}$$
$$= \underset{\sim}{0}$$

If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly dependent; then

$\exists\, c_1, c_2$ not both zero such that

$$c_1 \underset{\sim}{u} + c_2 \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \underset{\sim}{v} = -\frac{c_1}{c_2} \underset{\sim}{u}$$

$\underset{\sim}{v}$ is a constant multiple of $\underset{\sim}{u}$.

Ex Are $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{i} + \vec{j}$, $\underset{\sim}{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{i} - \vec{j}$

linearly dependent / independent ?

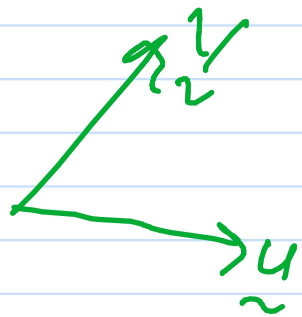
Def. $c_1 \underset{\sim}{u} + c_2 \underset{\sim}{v} = \underset{\sim}{0} \Longrightarrow c_1 = c_2 = 0$

then $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly independent.

$$c_1 \underset{\sim}{u} + c_2 \underset{\sim}{v} = 0 \Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{array}{l} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$$

$\Rightarrow$ $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly independent.



Observation

If $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k$ are linearly dependent, one of these vectors can be expressed as a linear combination of the others.

If $v_1, \ldots, v_k$ are linearly dependent, then there are constants $c_1, \ldots, c_k$, not all zero, such that

$$c_1 \underset{\sim}{v}_1 + \cdots + c_k \underset{\sim}{v}_k = \underset{\sim}{0}.$$

Suppose $c_i \neq 0$:

$$c_i \underset{\sim}{v}_i = -\left(c_1 \underset{\sim}{v}_1 + \cdots + c_{i-1} \underset{\sim}{v}_{i-1} + c_{i+1} v_{i+1} + \cdots + c_k v_k\right)$$

$$\underset{\sim}{v}_i = -\frac{c_1}{c_i} \underset{\sim}{v}_1 - \cdots - \frac{c_{i-1}}{c_i} \underset{\sim}{v}_{i-1} - \frac{c_{i+1}}{c_i} \underset{\sim}{v}_{i+1} - \cdots - \frac{c_k}{c_i} \underset{\sim}{v}_k$$

$\Rightarrow c_3 = c_2 = c_1 = 0 \Rightarrow v_1, v_2, v_3$: Lin. independent

PROPERTIES OF LINEAR INDEPENCE

1. Any set of more than n vectors in $\mathbb{R}^n$ is linearly dependent.

2. $v_1, v_2, \dots, v_n$ in $\mathbb{R}^n$ are linearly independent if and only if the $n \times n$ matrix

$$A = [v_1 \; v_2 \; \cdots \; v_n]$$

having them as its column vectors has nonzero determinant.

3. Let $k < n$. $v_1, v_2, \dots, v_k$ in $\mathbb{R}^n$ are linearly independent if and only if some $k \times k$ submatrix of A has nonzero determinant.

Example DIY

1– $v_1 = (1,2,3)$, $v_2 = (1,0,4)$, $v_3 = (0,2,-1)$, $v_4 = (7,6,1) \in \mathbb{R}^3$ are linearly dependent $(k = 4 > n = 3)$

$(v_1 - v_2 - v_3 + 0v_4 = 0)$

2– $v_1 = (1,2,3)$, $v_3 = (0,2,-1)$, $v_4 = (7,6,1) \in \mathbb{R}^3$

$$\begin{vmatrix} 1 & 0 & 7 \\ 2 & 2 & 6 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 7 \\ 0 & 2 & -8 \\ 0 & -1 & -20 \end{vmatrix} = 1(-40-8) = -48 \neq 0 \Rightarrow \text{Lin. ind.}$$

3– $v_1 = (9,1,3)$, $v_2 = (3,0,1) \in \mathbb{R}^3$

$$\begin{bmatrix} 9 & 3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 9 & 3 \\ 3 & 1 \end{vmatrix} = 9 - 9 = 0 \Rightarrow \text{Lin. dep.}$$

Properties of linear independence

#1 In $\mathbb{R}^3$: $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3$

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \right\}$ is linearly dependent!!

#2 In $\mathbb{R}^3$, suppose we're given $\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. How to determine $\underset{\sim}{u}, \underset{\sim}{v}, w$ are linearly dependent/independent?

Linearly ind. if $\underline{c_1 \underset{\sim}{u} + c_2 \underset{\sim}{v} + c_3 \underset{\sim}{w} = \underset{\sim}{0}}$ $\Rightarrow$ $c_1 = c_2 = c_3 = 0$

$$c_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + c_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + c_3 \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 u_1 + c_2 v_1 + c_3 w_1 \\ c_1 u_2 + c_2 v_2 + c_3 w_2 \\ c_1 u_3 + c_2 v_3 + c_3 w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is a hom. linear system of eqs.

- a unique sol.
- inf. many sols.
- ~~no solution~~

as at least we have the trivial solution $\underset{\sim}{c} = 0$.

$$\underbrace{\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}}_{\underset{\sim}{A}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\underset{\sim}{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{A} \; \underset{\sim}{C} = \underset{\sim}{0}$$

Remember $\left\{\begin{array}{l} \underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0} \text{ has only} \\ \text{the trivial solution} \\ \underset{\sim}{x} = 0 \end{array}\right\} \iff \det A \neq 0$

$$\left\{ \det A \neq 0 \Rightarrow A^{-1} \text{ exists} \Rightarrow \underset{\sim}{A}^{-1}\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{0} \Rightarrow \underset{\sim}{x} = \underset{\sim}{0} \right\}$$

$$\Rightarrow \left\{\begin{array}{l} \underset{\sim}{A}\ \underset{\sim}{c} = \underset{\sim}{0} \text{ has only the} \\ \text{trivial sol. } \underset{\sim}{c} = 0 \end{array}\right\} \iff \det A \neq 0$$

$$\underset{\sim}{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underset{\sim}{0} \iff \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0$$

e.g.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

$\neq 0$ linearly independent

$= 0$ linearly dependent

BASIS

A finite set S of vectors in a vector space V is called a basis for V if

- the vectors in S are linearly independent and
- the vectors in S span V.

$\text{span}\{e_1, e_2, \dots, e_n\} = \mathbb{R}^n$

Example $e_1 = (1,0,0,\dots,0)$, $e_2 = (0,1,0,\dots,0)$, $\dots$, $e_n = (0,0,0,\dots,1)$.

$e_1, e_2, \dots, e_n$ are lin. independent. $c_1 e_1 + c_2 e_2 + \dots + c_n e_n = 0$

$(c_1, c_2, \dots, c_n) = (0,0,\dots,0)$

$c_1 = c_2 = \dots c_n = 0$

If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

$\Rightarrow \{e_1, e_2, \dots, e_n\}$ spans $\mathbb{R}^n \Rightarrow$ standard basis

Example $\boxed{v_1 = (1,-1,-2,-3),\ v_2 = (1,-1,2,3),\ v_3 = (1,-1,-3,-2),}$ $v_4 = (0,3,-1,2)$.

Is $S = \{v_1, v_2, v_3, v_4\}$ a basis for $\mathbb{R}^4$??

Remember that $v_1 = (1,-1,-2,-3) = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -3 \end{bmatrix}$

$$\begin{array}{c} \begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 3 \\ -2 & 2 & -3 & -1 \\ -3 & 3 & -2 & 2 \end{vmatrix} \\ \ v_1 \quad v_2 \quad v_3 \quad v_4 \end{array} = 30 \neq 0 \Rightarrow \{v_1, v_2, v_3, v_4\} : \text{Lin. indep}$$

(i) $\{v_1, v_2, v_3, v_4\}$ must be linearly independent

$\Downarrow$ a basis for $\mathbb{R}^4$

(ii) $\text{span}\{v_1, v_2, v_3, v_4\} = \mathbb{R}^n$

THEOREM: Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis for the vector space V. Then any set of more than n vectors in V is linearly dependent. *Think about this*

THEOREM: Any two bases for a vector space consist of the same number of vectors.

$\mathbb{R}^n \Rightarrow \dim \mathbb{R}^n = n$

$\Rightarrow$ standard basis $\{e_1, e_2, \dots, e_n\}$

$\Rightarrow$ any basis of $\mathbb{R}^n$ has n vectors.

Ex $\underset{\sim}{e}_1 = (1, 0, \ldots, 0)$, $\underset{\sim}{e}_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$

is a basis for $\mathbb{R}^n$, since:

(i) $\{ \underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n \}$ are linearly independent as

$$c_1 \underset{\sim}{e}_1 + c_2 \underset{\sim}{e}_2 + \cdots + c_n \underset{\sim}{e}_n = (c_1, c_2, \ldots, c_n) = (0, \ldots, 0)$$

implies $c_1 = c_2 = \cdots = c_n = 0$

(ii) $\operatorname{span}\{ \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n \} = \mathbb{R}^n$ since any vector $\underset{\sim}{x} \in \mathbb{R}^n$ can be written as a linear comb. of $\underset{\sim}{e}_1, \ldots, e_n$:

$$\underset{\sim}{x} = (x_1, x_2, \ldots, x_n) = x_1 (1, 0, 0, \ldots, 0) + \cdots + x_n (0, 0, \ldots, 0, 1)$$

$$= x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + \cdots + x_n \underset{\sim}{e}_n$$

$\{ \underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n \}$ is called the STANDARD BASIS of $\mathbb{R}^n$.

dim of V := {# of vectors in its basis}

THEOREM Let V be an n-dimensional vector space and let S be a subset of V. Then

1- If S is lin. independent and consists of n vectors, then S is a basis for V.

2- If S spans V and consists of n vectors, then S is a basis for V.

3- If S is lin. independent, then S is contained in a basis for V.

4- If S spans V, then S contains a basis for V.

Read these

~~HOW~~ TO FIND A BASIS FOR THE SOLUTION SPACE

Basis for the solution space of $\underline{AX=0}$ (A: n x n matrix)

1. Reduce the coefficient matrix A to echelon form
2. r: leading entries $\Rightarrow k=n-r$: free variables
   $\Rightarrow k=0 \Rightarrow W=\{0\}$.
3. $k\neq 0 \Rightarrow k$: parameters

r : # of leading entries

$k=n-r$: # of free variables

Find $V_1, V_2, \dots, V_k$ by setting each parameter to be 1 and the rest to be 0.

$\Rightarrow W=\{V_1, V_2, \dots, V_k\}$: basis for W.

Example

$$\left.\begin{aligned} 3x_1+6x_2-x_3-5x_4+5x_5&=0\\ 2x_1+4x_2-x_3-3x_4+2x_5&=0\\ 3x_1+6x_2-2x_3-4x_4+x_5&=0 \end{aligned}\right\}$$

Basis for the solution space?

# of unknowns $= n = 5$

# of leading vars. $= r = 2$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \qquad\qquad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{bmatrix} 3 & 6 & -1 & -5 & 5 \\ 2 & 4 & -1 & -3 & 2 \\ 3 & 6 & -2 & -4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1, x_2

x_2, x_4, x_5 : free var.

$\boxed{x_2=t}$

$\boxed{x_4=s}$

$\boxed{x_5=r}$

$x_3-x_4+4x_5=0 \Rightarrow \underline{x_3}=s-4r$

$x_1+2x_2-2x_4+3x_5=0 \Rightarrow \underline{x_1}=-2t+2s-3r$

# of free parameters

$= k = n-r = 5-2 = 3$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t+2s-3r \\ t \\ s-4r \\ s \\ r \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2s \\ 0 \\ s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -3r \\ 0 \\ -4r \\ 0 \\ r \end{bmatrix}$$

$s, t, r \in \mathbb{R}$

$$\underset{\sim}{x} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$\underset{\sim}{x} \in \mathbb{R}^5$; $t, s, r \in \mathbb{R}$, arbitrary

Solution space of this system $\underset{\sim}{A}_{3\times5}\ \underset{\sim}{x}_{3\times1} = \underset{\sim}{0}_{3\times1}$

is

$$= \left\{ t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t, s, r \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \underset{\underset{\sim}{u}}{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{\underset{\sim}{v}}{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\underset{\sim}{w}}{\begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

$\{\underset{\sim}{u}, \underset{\sim}{v}, w\}$ is a BASIS of the solution space of $\underline{A}\,\underset{\sim}{x} = \underset{\sim}{0}$.

dim sol.space $= \underline{\underline{3}}$

As we saw before, the solution space of the hom. linear system

$$\underset{\sim}{A}_{m \times n} \underset{\sim}{X}_{n \times 1} = \underset{\sim}{0}_{n \times 1}$$

is a <u>vector space</u>.

$t=1,\ s=r=0 \Rightarrow v_1 = (-2,1,0,0,0)$

$s=1,\ t=r=0 \Rightarrow v_2 = (2,0,1,1,0)$ $\quad W = \{v_1, v_2, v_3\}$

$r=1,\ s=t=0 \Rightarrow v_3 = (-3,0,-4,0,1)$

ROW SPACE

$AX=0 \Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

$$\left.\begin{array}{l} r_1 = (a_{11}, a_{12}, \ldots, a_{1n}) \\ r_2 = (a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ r_m = (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{array}\right\} \quad r_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$$

The subspace of R^n spanned by m row vectors $r_1, r_2, \ldots, r_m$ is called the row space of A (Row (A))

The dimension of Row (A) is called the row rank of A.

PROPERTIES OF ROW SPACES

1) The nonzero row vectors of an echelon matrix are linearly independent and thus form a basis for its row space.

2) If the matrices A and B are equivalent, then they have the same row space.

↓

To find a basis for row (A), find the echelon matrix E of A. Then the nonzero row vectors of E form a basis for row (A).

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \\ 4 & 5 & 2 \end{bmatrix}$

$r_1 = (1, 2, 3) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $r_3 = (4, 5, 2) = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$

$r_2 = (-2, 1, 3) = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

The subspace of $\mathbb{R}^3$ spanned by those 3 vectors is

$$Row(A) = \{ k_1 \underset{\sim}{r_1} + k_2 \underset{\sim}{r_2} + k_3 \underset{\sim}{r_3} \mid k_1, k_2, k_3 \}$$

$$\mathrm{Row}(A) = \left\{ k_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \;\middle|\; k_1, k_2, k_3 \right\}$$

<u>Ex</u>

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix} \qquad r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix},$$

$$r_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}. \qquad \mathrm{Row}(A) = \left\{ k_1 \underset{\sim}{r_1} + k_2 \underset{\sim}{r_2} + k_3 \underset{\sim}{r_3} \;\middle|\; \ldots \right\}$$

$$= \left\{ k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; k_1, k_2, k_3 \in \mathbb{R} \right\}$$

$$= \left\{ (k_1 + 2k_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \;\middle|\; k_1, k_2, k_3 \right.$$

$$Row(A) = \left\{ a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \;\middle|\; a_1, a_2 \in \mathbb{R} \right\}$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ are linearly independent.

$$\ell_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \ell_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \ell_1 \\ 2\ell_2 \\ 3\ell_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\ell_1 = \ell_2 = 0 \Rightarrow$ linearly ind.

- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ are linearly ind.
- $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \right\} = Row(A)$

Row space of A is a vector space with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \right\}$ of dim = 2.

Row space of a $A_{m \times n}$ matrix is an $\leq m$ dimensional subspace of $\mathbb{R}^n$.

!

Example $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$ → basis for the row space of A?

$r_1 = (1,2,1,3,2)$, $r_2 = (3,4,9,0,7)$

$r_3 = (2,3,5,1,8)$, $r_4 = (2,2,8,-3,5)$

- row (A) = span $\{r_1, r_2, r_3, r_4\}$

$$A \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} v_1 = (1,2,1,3,2) \\ v_2 = (0,1,-3,5,-4) \\ v_3 = (0,0,0,1,-7) \end{array}$$

$\{v_1, v_2, v_3\}$: a basis for Row (A) ⇒ rank Row (A) = 3

COLUMN SPACE

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \Rightarrow c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The subspace of R^m spanned by the n column vectors $c_1, c_2, \ldots, c_n$ is called the colum space of A (COL(A))

The dimension of COL(A) is called the column rank

Example $c_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$, $c_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 2 \end{bmatrix}$, $c_3 = \begin{bmatrix} 1 \\ 9 \\ 5 \\ 8 \end{bmatrix}$, $c_4 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -3 \end{bmatrix}$, $c_5 = \begin{bmatrix} 2 \\ 7 \\ 8 \\ 5 \end{bmatrix}$

COL(A) = span $\{c_1, c_2, c_3, c_4, c_5\}$

$\{c_1, c_2, c_4\}$: a basis for COL (A) ⇒ rank COL (A) = 3

↳ colums of A which corresponds to the colums of Leading entries of E

↓ be careful

Rank (A) = Rank Row (A) = Rank COL (A)

→ AX=0

Rank (A) + dim (Null (A)) = n

To find a basis for row and column spaces of a matrix A, find echelon form E of the matrix.

* Rows of E including the leading entries form a basis for $Row(A)$.

* Columns of A corresponding to the columns of E that include the leading entries is a basis for $Col(A)$.

$A = [a_{ij}]_{m \times n}$:

$Row(A)$ is a $\leq m$ dimensional subspace of $\mathbb{R}^n$

$Col(A)$ is a $\leq n$ dimensional " " $\mathbb{R}^m$.

4.5 Row and Column Spaces

Th. 1 The nonzero row vectors of an echelon matrix are linearly independent and form a basis for the row space.

Algorithm 1 To find a basis for the row space of a matrix A, use elementary row ops. to reduce A to an echelon matrix E. Then, the nonzero row vectors of E form a basis for $\mathrm{Row}(A)$.

Example 2

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

Find a basis for Row(A).

$$E = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Def.

Row rank of A $= \dim \text{Row}(A)$

$$\left.\begin{aligned} \underset{\sim}{r_1} &= (1, 2, 1, 3, 2) \\ \underset{\sim}{r_2} &= (0, 1, -3, 5, -4) \\ \underset{\sim}{r_3} &= (0, 0, 0, 1, -7) \end{aligned}\right\}$$

are linearly independent and form a basis for Row(A)

$\dim \text{Row } A = 3$

Row rank of A is 3.

Column space

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\underset{\sim}{c_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \underset{\sim}{c_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$\underset{\sim}{c_1}, \ldots, \underset{\sim}{c_n} \in \mathbb{R}^m$

The subspace of $\mathbb{R}^m$ spanned by the column vectors $\underset{\sim}{c_1}, \ldots, \underset{\sim}{c_n}$ is called the column space $col(A)$ of A. The dimension of $col(A)$ is called column rank of of A.

Ex

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{c_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underset{\sim}{c_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underset{\sim}{c_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underset{\sim}{c_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathrm{Col}(A) = \left\{ k_1 \underset{\sim}{c_1} + k_2 \underset{\sim}{c_2} + k_3 \underset{\sim}{c_3} + k_4 \underset{\sim}{c_4} \;\middle|\; k_i \in \mathbb{R} \right\}$$

$$= \left\{ k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right.$$

$\underset{\sim}{c_1}, \underset{\sim}{c_2}, c_3, \underset{\sim}{c_4}$ are linearly ind. $\therefore$ a basis for $\mathrm{col}(A) = \{ \underset{\sim}{c_1}, \underset{\sim}{c_2}, \underset{\sim}{c_3}, \underset{\sim}{c_4} \}$; $\dim \mathrm{Col}\, A = \mathrm{col.rank}(A) = 4$

Ex

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

$$\mathrm{Col}(A) = \left\{ k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + k_4 \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} \;\middle|\; k_i \in \mathbb{R} \right\}$$

$$= \left\{ (k_1 + 2k_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (k_3 + 2k_4) \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ are linearly independent (verify this yourself)

A basis for $\mathrm{Col}(A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Column rank of $A = \dim \mathrm{Col}(A) = 2$

Ex 3

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

Find a basis for $\text{Col}(A)$

$$\begin{array}{c} \begin{array}{ccccc} \#1 & \#2 & \#3 & \#4 & \#5 \end{array} \\ E = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\underset{\sim}{c_1} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 2 \end{bmatrix}$$

$$c_4 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

Columns of A corresponding to the leading entries of E are $\underset{\sim}{c_1}$, $\underset{\sim}{c_2}$, and c_4. $\Rightarrow$ $\{ \underset{\sim}{c_1}, \underset{\sim}{c_2}, c_4 \}$ form a basis for $\text{col}(A)$. Column rank of A which is $\dim \text{Col}(A) = 3$

Algorithm 2   A basis for the column space

- To find a basis for the column space of a matrix A, reduce it to an echelon matrix E.

- The column vectors of A that correspond to the pivot columns of E (columns carrying the leading entries) form a basis for $Col(A)$.

Example Find bases for the row and column spaces of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 5 \\ 3 & 2 & 4 & -1 \end{bmatrix}_{3\times 4}.$$

Also find the row and column ranks.

Row(A) is a subspace of $\mathbb{R}^4$

Col(A) is a subspace of $\mathbb{R}^3$

$$E = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Row A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 4 \end{bmatrix} \right\}$

Basis for Row A $= \{ (1, 0, 2, -3), (0, 1, -1, 4) \}$

Row Rank of $A = 2$

Example Find bases for the row and column spaces of the matrix

$$A = \begin{array}{cccc} c_1 & c_2 & & \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 5 \\ 3 & 2 & 4 & -1 \end{bmatrix}_{3\times 4}.$$

Also find the row and column ranks.

Row(A) is a subspace of $\mathbb{R}^4$

Col(A) is a subspace of $\mathbb{R}^3$

$$E = \begin{array}{cccc} \#1 & \#2 & & \\ \end{array}$$
$$E = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Col A $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$

Column Rank of A = dim Col A = 2

From the book, p 266, Example 5 ⇒ DIY

In order to cover up the material we did so far, a PS on Saturday,

at 12.00 – 13.30

Please STUDY the notes, before this.

Warning $\boxed{\text{Theorem}}$ Equality of Row and Column Ranks

The row rank and column rank of any matrix are equal.

Row rank of A = Column rank of A := Rank of A.

Remark : (Row) rank of a matrix is the number of the nonzero columns in echelon form.

Null Space of A

$A = [a_{ij}]_{m \times n}$

$$\mathrm{Null}(A) = \left\{ x \in \mathbb{R}^n \;\middle|\; \underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0} \right\}$$

"A'nın sıfır uzayı, sıfırlığı"

Example : in PS time

Theorem $A = [a_{ij}]_{m \times n} \Rightarrow \mathrm{rank}(A) + \dim \mathrm{Null}(A) = n$