* Chapter 7 Linear Systems of Differential Eqs.

Two unknown functions $x_1 = x_1(t)$, $x_2 = x_2(t)$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}\, x_1(t) + a_{12}\, x_2(t) \\ \frac{dx_2}{dt} &= a_{21}\, x_1(t) + a_{22}\, x_2(t) \end{aligned}\right\} \quad \begin{aligned} x_1' &= a_{11}\, x_1 + a_{12}\, x_2 \\ x_2' &= a_{21}\, x_1 + a_{22}\, x_2 \end{aligned}$$

A coupled system of first order, linear, constant-coefficient, homogeneous differential eqs. for the unknown functions $x_1(t)$ & $x_2(t)$.

Solution of x_1 : requires knowledge of $x_2(t)$

" " x_2 : " " " $x_1(t)$

$$x_1' = a_{11} x_1 + a_{12} x_2$$

$$x_2' = a_{21} x_1 + a_{22} x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \end{bmatrix}$$

$$\underset{\sim}{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\underset{\sim}{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \boxed{\underset{\sim}{x}' = A \underset{\sim}{x}}$$

Question How do we obtain a system of DEs of the form $x' = Ax$. /

Ex $$u'' + 2u' + u = 0 \qquad u = u(t)$$

second-order, linear, constant-coefficient, hom. eq. for which you're able to find the general solution.

Let's call

$$u(t) = x_1 \quad , \quad u'(t) = x_2 \Rightarrow u'' = x_2'$$

$$u'' + 2u' + u = 0 \Rightarrow x_2' + 2x_2 + x_1 = 0 \rightarrow x_2' = -x_1 - 2x_2$$

$$x_1' = u' = x_2$$

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - 2x_2 \end{aligned}}$$

$$\boxed{\begin{aligned} x_1' &= \overset{a_{11}}{0} \cdot x_1 + \overset{a_{12}}{1} \cdot x_2 \\ x_2' &= \underset{a_{21}}{-} x_1 \underset{a_{22}}{- 2} x_2 \end{aligned}}$$

Ex $u^{(4)} + a u''' + b u'' + c u' + du = 0 \qquad u = u(t)$

Write this DE as a system of linear first order DEs.

$u = x_1$

$u' = x_2 \qquad \{ u' = x_1' \to x_1' = x_2 \}$

$u'' = x_3 \qquad \{ u'' = x_2' \to x_2' = x_3 \}$

$u''' = x_4 \qquad \{ u''' = x_3'$ and $x_4' + a x_4 + b x_3 + c x_2 + d x_1 = 0$

$x_1' = x_2$

$x_2' = x_3$

$x_3' = x_4$

$x_4' = -a x_4 - b x_3 - c x_2 - d x_1$

Then, we can say that, the linear, n-th order, constant-coefficient, homogeneous eq.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

can be recast as a linear, first-order constant coefficient system of DEs.

Vector Functions $(n=2)$

A 2×1 vector function is a vector of the form

$$\underset{\sim}{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

e.g. $\quad \underset{\sim}{x}(t) = \begin{bmatrix} \cos t \\ e^{t} \end{bmatrix}$

Def Two 2×1 vector functions

$$x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}, \quad \underset{\sim}{x_2}(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$$

are called linearly independent if

$$c_1 \underset{\sim}{x_1}(t) + c_2 \underset{\sim}{x_2}(t) = \underset{\sim}{0} \implies c_1 = c_2 = 0$$

Ex

$$x_1(t) = \begin{bmatrix} 3e^{t} \\ 3e^{2t} \end{bmatrix}, \qquad x_2(t) = \begin{bmatrix} 2e^{t} \\ 2e^{2t} \end{bmatrix}$$

$c_1 = (-2), \qquad c_2 = (3)$

$$c_1 \underset{\sim}{x_1}(t) + c_2 \underset{\sim}{x_2}(t) = -2 \begin{bmatrix} 3e^{t} \\ 3e^{2t} \end{bmatrix} + 3 \begin{bmatrix} 2e^{t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

x_1 and x_2 are <u>linearly dependent</u>.

Ex

$$x_1(t) = \begin{bmatrix} e^{t} \\ \cos t \end{bmatrix}, \qquad x_2(t) = \begin{bmatrix} e^{t} \\ 2\cos t \end{bmatrix}$$

Are x_1 & x_2 linearly dependent/independent?

$$c_1 \underset{\sim}{x_1}(t) + c_2 \underset{\sim}{x_2}(t) = \underset{\sim}{0} \Rightarrow c_1 = c_2 = 0$$

$$c_1 \begin{bmatrix} e^t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ 2\cos t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} c_1 e^t + c_2 e^t &= 0 \\ c_1 \cos t + 2 c_2 \cos t &= 0 \end{aligned} \qquad \begin{bmatrix} e^t & e^t \\ \cos t & 2\cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} e^t & e^t \\ \cos t & 2\cos t \end{vmatrix} = \cos t \, e^t \neq 0 \Rightarrow \text{The hom. system has a unique solution;}$$

$$\underset{\sim}{x_1}(t) \qquad \underset{\sim}{x_2}(t)$$

$$\boxed{c_1 = c_2 = 0}$$

$\Rightarrow$ $x_1(t)$ and $x_2(t)$ are linearly independent!!!

The actual reason that $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ are linearly independent is due to

$$\begin{vmatrix} e^t & e^t \\ \cos t & 2\cos t \end{vmatrix} \neq 0 \iff \begin{vmatrix} | & | \\ \underset{\sim}{x}_1(t) & x_2(t) \\ | & | \end{vmatrix} \neq 0$$

Now let's define for $\underset{\sim}{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}$, $\underset{\sim}{x}_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$

the Wronskian as

$$W\left(\underset{\sim}{x}_1(t), x_2(t)\right) = \begin{vmatrix} x_{11}(t) & x_{21}(t) \\ x_{12}(t) & x_{22}(t) \end{vmatrix}$$

$$\begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$$

Later!!

This will be modified according to the book, LATER

Question Under which cond. are $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$

linearly dependent / independent?

$$c_1 \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} + c_2 \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} c_1 x_{11} + c_2 x_{21} &= 0 \\ c_1 x_{12} + c_2 x_{22} &= 0 \end{aligned}\right\} \begin{bmatrix} x_{11}(t) & x_{21}(t) \\ x_{12}(t) & x_{22}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$W(\underset{\sim}{x}_1, \underset{\sim}{x}_2) \neq 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow \underset{\sim}{x}_1, x_2$ are linearly independent

$W(\underset{\sim}{x}_1, x_2) = 0 \Rightarrow \underset{\sim}{x}_1, \underset{\sim}{x}_2$ are linearly dependent

Solution of Linear Systems of Eqs.

A) Real, Distinct Eigenvalues (We'll see later why such a subtitle)

Ex Find the general sol. to the system

$$\underset{\sim}{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \underset{\sim}{x}$$

$$\underset{\sim}{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \underset{\sim}{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{aligned} x_1' &= 2x_1 \\ x_2 &= -3X \end{aligned}$$

$$x_1' = 2x_1 \qquad \frac{dx_1}{dt} = 2x_1 \Rightarrow \frac{dx_1}{x_1} = 2\,dt$$

$$x_2' = -3x_2$$

$$\ln x_1 = 2t + \ln c_1 \rightarrow \boxed{\begin{array}{l} x_1(t) = c_1 e^{2t} \\ \\ x_2(t) = c_2 e^{-3t} \end{array}}$$

$$\frac{dx_2}{dt} = -3x_2 \rightarrow \frac{dx_2}{x_2} = -3\,dt \longrightarrow$$

$$\underset{\sim}{x}(t) = \begin{bmatrix} x_1(t) \\ \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ \\ c_2 e^{-3t} \end{bmatrix} = c_1 \begin{bmatrix} e^{2t} \\ \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \\ e^{-3t} \end{bmatrix}$$

$$\underset{\sim}{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \underset{\sim}{x} \qquad \underset{\sim}{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}_{2\times 1}, \quad \underset{\sim}{x}_2(t) = \begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

↳ The general sol. is $\underset{\sim}{x}(t) = c_1\, \underset{\sim}{x}_1(t) + c_2\, \underset{\sim}{x}_2(t)$

See that $\underset{\sim}{x}_1(t)$ & $\underset{\sim}{x}_2(t)$ are linearly ind., as

$$W(x_1(t), x_2(t)) = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t} \neq 0$$

☆ There's a correspondence with the arguments of the previous section on higher-order DEs.

Ex Find the general sol. to $x' = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} x$

$\underset{\sim}{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ $\quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left.\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 - x_2 \end{aligned}\right\}$$

Motivated by
- the theory of the previous section
- the previous example

let's search for a solution of the form

$$\underset{\sim}{x}(t) = \underset{\sim}{v}\, e^{\lambda t} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$$

$x_1(t) = v_1 e^{\lambda t}, \quad x_2(t) = v_2 e^{\lambda t}$

$$\underset{\sim}{x}'(t) = \begin{bmatrix} v_1 \lambda e^{\lambda t} \\ v_2 \lambda e^{\lambda t} \end{bmatrix} = \lambda e^{\lambda t} \underset{\sim}{v} \qquad x' = Ax$$

$$\lambda e^{\lambda t} \underset{\sim}{v} = A . e^{\lambda t} \underset{\sim}{v} \Rightarrow e^{\lambda t} \underset{\sim}{A} \underset{\sim}{v} - \lambda e^{\lambda t} \underset{\sim}{v} = \underset{\sim}{0}$$

$$A \underset{\sim}{v} - \lambda \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow (A - \lambda I) \underset{\sim}{v} = \underset{\sim}{0}$$

{ Please be aware that this is eigenvalue-eigenvector equation for the matrix A! }

There are two cases.

Suppose $|A - \lambda I| \neq 0 \Rightarrow (A - \lambda I)^{-1}$ exists

$$(A - \lambda I)^{-1} (A - \lambda I) \underset{\sim}{v} = (A - \lambda I) \underset{\sim}{0} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0}$$

$$\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underset{\sim}{0} \implies v_1 = v_2 = 0$$

$$\underset{\sim}{x}(t) = \underset{\sim}{v}\, e^{\lambda t} = \underset{\sim}{0} \qquad \underline{x_1(t) = 0}, \quad \underline{x_2(t) = 0}$$

which makes sense:

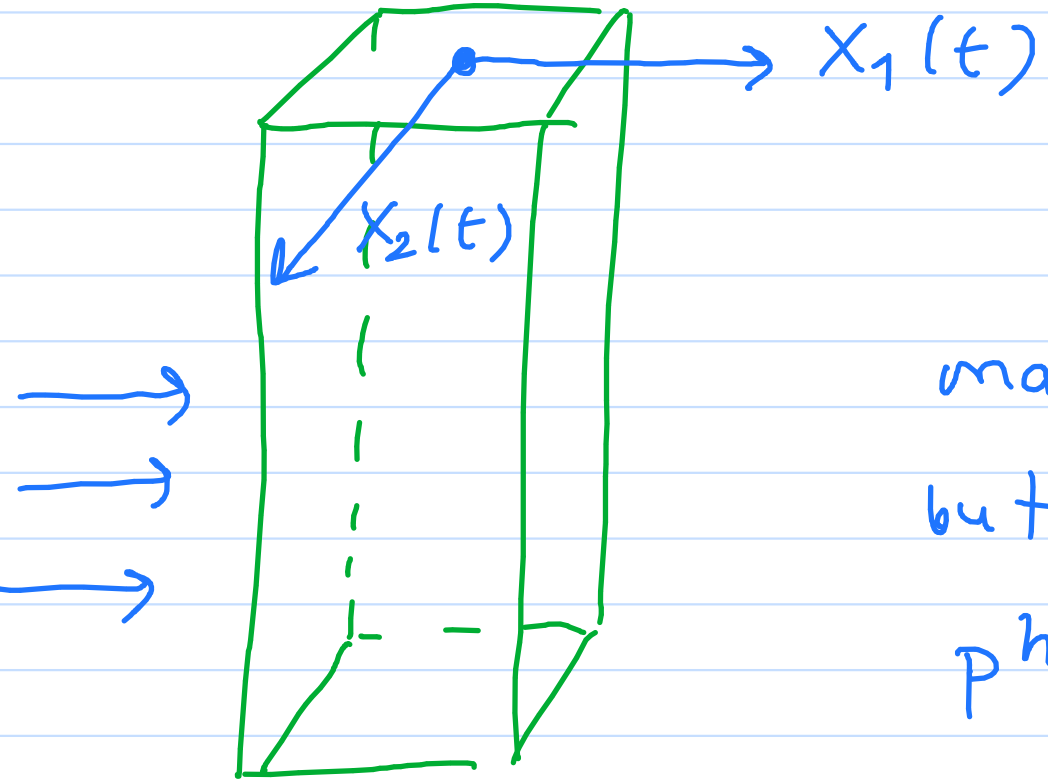
$$\left.\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}\right\} \text{ The system is satisfied!!}$$

$$\underset{\sim}{x}' = A\,\underset{\sim}{x} \longleftarrow \underset{\sim}{x}(t) = \underset{\sim}{0} \qquad \underset{\sim}{0}' = \underset{\sim}{A}\,\underset{\sim}{0}$$

But, there's no evolution in this system; it stays at rest, and at zero!! It does not give us any time evolution information, NOT USEFUL

Suppose in the system $\boxed{\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}}$

x_1 and x_2 represent displacements of a column in a building during an earthquake:



$x_1(t) \equiv 0$

$x_2(t) \equiv 0$

mathematically consistent, but no information on the physical world!!

The other case is $|A - \lambda I| = 0$ $\quad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

In this case we'll be able to find nonzero v_1 & v_2:

$\boxed{\lambda_1 = 3}$ $\quad (A - \lambda_1 I)\underset{\sim}{v} = \underset{\sim}{0}$

$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{array}{l} -2v_1 + v_2 = 0 \\ 4v_1 - 2v_2 = 0 \\ \text{Let } v_1 = 1 \rightarrow v_2 = 2 \\ \underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array} \right.$$

$$\underset{\sim}{x}_1(t) = v e^{\lambda t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$$

$\boxed{\lambda_2 = -1}$ $$(A - \lambda_2 I)\underset{\sim}{v} = 0$$

$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2v_1 + v_2 = 0$ let $v_2 = -2 \rightarrow v_1 = 1$

$$\underset{\sim}{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \underset{\sim}{x}_2(t) = \underset{\sim}{v} e^{\lambda t} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

$$\underset{\sim}{x}(t) = \underset{\sim}{v}\, e^{\lambda t} \Rightarrow \underset{\sim}{x}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}, \quad \underset{\sim}{x}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

$\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ are linearly independent as:

$$W\left(x_1(t), x_2(t)\right) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0$$

The general solution to $x' = Ax$ is

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{x}_1(t) + c_2 \underset{\sim}{x}_2(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

(to the matrix eq. $\underset{\sim}{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \underset{\sim}{x}$)

Explicitly, our system is $\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$

$$\underset{\sim}{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{bmatrix}$$

$$x_1(t) = c_1 e^{3t} + c_2 e^{-t}$$

$$x_2(t) = 2c_1 e^{3t} - 2c_2 e^{-t}$$

The methodology in solving $x' = Ax$:

- $x = \underset{\sim}{v}\, e^{\lambda t} \xrightarrow{\text{put in}} x' = Ax$
- This gives $(A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$
- Find the eigenvalues of A from $|A - \lambda I| = 0$
- For each eigenvalue λ_i, determine corresponding eigenvector v_i.
- This will give a solution $\underset{\sim}{x_i}(t) = \underset{\sim}{v_i}\, e^{\lambda_i t}$.

Example (p420) Find general solution of the system $x_1' = 4x_1 + 2x_2$, $x_2' = 3x_1 - x_2$.

$$\left.\begin{aligned} x_1' &= 4x_1 + 2x_2 \\ x_2' &= 3x_1 - x_2 \end{aligned}\right\} \Rightarrow X' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} X, \quad \underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underset{\sim}{x} = \underset{\sim}{v} e^{\lambda t} \Rightarrow (A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5) = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = 5$$

$$\boxed{\lambda_1 = -2} \qquad (A - \lambda_1 I)\underset{\sim}{v} = \underset{\sim}{0}$$

$$\begin{bmatrix} 4-(-2) & 2 \\ 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3v_1 + v_2 = 0$; Let $v_2 = -3 \rightarrow v_1 = 1$

$$v = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \underset{\sim}{x_1}(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

$\lambda_2 = 5 \qquad (A - \lambda_2 I)v = 0 \qquad -v_1 + 2v_2 = 0$

Let $v_2 = 1 \rightarrow v_1 =$

$$\begin{bmatrix} 4-5 & 2 \\ 3 & -1-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \underset{\sim}{x_2}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

The general solution is

$$\underset{\sim}{X}(t) = c_1 \underset{\sim}{X}_1(t) + c_2 \underset{\sim}{X}_2(t)$$

$$= c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

$$= \begin{bmatrix} c_1 e^{-2t} + 2 c_2 e^{5t} \\ -3 c_1 e^{-2t} + c_2 e^{5t} \end{bmatrix}$$

Example Find solution of the IVP $x_1' = 4x_1 + 2x_2$, $x_2' = 3x_1 - x_2$ with $x_1(0) = 2$, $x_2(0) = -3$.

$$\underset{\sim}{X}(t) = \begin{bmatrix} c_1 e^{-2t} + 2c_2 e^{5t} \\ -3c_1 e^{-2t} + c_2 e^{5t} \end{bmatrix} \begin{matrix} \rightarrow x_1(t) \\ \rightarrow x_2(t) \end{matrix}$$

$$\underset{\sim}{X}(0) = \begin{bmatrix} c_1 + 2c_2 \\ -3c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad \begin{aligned} c_1 + 2c_2 &= 2 \\ -3c_1 + c_2 &= -3 \end{aligned}$$

$$\left.\begin{aligned} c_1 + 2c_2 &= 2 \\ -6c_1 + 2c_2 &= -6 \end{aligned}\right\} \quad \begin{aligned} 7c_1 &= 8 \\ c_1 &= 8/7 \end{aligned} \qquad \begin{aligned} c_2 &= -3 + 3c_1 = -3 + 3 \cdot \frac{8}{7} \\ c_2 &= \frac{3}{7} \end{aligned}$$

The solution of the IVP is

$$\underset{\sim}{x}(t) = \frac{8}{7}\begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + \frac{3}{7}\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

$$= \begin{bmatrix} \frac{8}{7} e^{-2t} + \frac{6}{7} e^{5t} \\ -\frac{24}{7} e^{-2t} + \frac{3}{7} e^{5t} \end{bmatrix} \begin{matrix} \Rightarrow x_1(t) \\ \Rightarrow x_2(t) \end{matrix}$$