THEOREM Principle of Superposition for Homogeneous Eq.

Assume that $y_1, y_2, \dots, y_n$ are n solutions of the homog. Linear equation

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \dots + p_{n-1}(x)\, y' + p_n(x)\, y = 0$$

on the interval I. Then, the Linear combination

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \qquad (c_1, c_2, \dots, c_n: \text{constants})$$

is also a solution of the d.e. on I.

PROOF: Since $y_1, y_2, \dots, y_n$ are solutions, then

$$y_1^{(n)} + p_1 y_1^{(n-1)} + \dots + p_{n-1} y_1' + p_n y_1 = 0$$
$$y_2^{(n)} + p_1 y_2^{(n-1)} + \dots + p_{n-2} y_2' + p_n y_2 = 0$$
$$\text{-----}$$
$$\text{-----}$$
$$y_n^{(n)} + p_1 y_n^{(n-1)} + \dots + p_{n-2} y_n' + p_n y_n = 0$$

$$(c_1 y_1 + c_2 y_2 + \dots + c_n y_n)^{(n)} + p_1 (c_1 y_1 + c_2 y_2 + \dots + c_n y_n)^{(n-1)} + \dots$$
$$+ p_{n-1} (c_1 y_1 + c_2 y_2 + \dots + c_n y_n)' + p_n (c_1 y_1 + c_2 y_2 + \dots + c_n y_n)$$
$$= c_1 (y_1^{(n)} + p_1 y_1^{(n-1)} + \dots + p_{n-1} y_1' + p_n y_1)$$
$$+ c_2 (y_2^{(n)} + p_1 y_2^{(n-1)} + \dots + p_{n-1} y_2' + p_n y_2)$$
$$+ \dots\dots$$
$$+ c_n (y_n^{(n)} + p_1 y_n^{(n-1)} + \dots + p_{n-1} y_n' + p_n y_n) = 0 /\!/$$

Ex $y_1 = e^{-3x}$, $y_2 = \cos 2x$, $y_3 = \sin 2x$ are solutions of

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

$$y_1 = e^{-3x} \Rightarrow y_1' = -3e^{-3x}, \quad y_1'' = 9e^{-3x}, \quad y_1^{(3)} = -27e^{-3x}$$

$$\Rightarrow y_1^{(3)} + 3y_1'' + 4y_1' + 12y_1 = (-27 + 3.9 - 4.3 + 12)\, e^{-3x} = 0$$

$$y_2 = \cos 2x \Rightarrow y_2' = -2\sin 2x, \quad y_2'' = -4\cos 2x, \quad y_2^{(3)} = 8\sin 2x$$

$$\Rightarrow y_2^{(3)} + 3y_2'' + 4y_2' + 12y_2 = (-3.4 + 12)\cos 2x + (8 - 4.2)\sin 2x = 0$$

$y_3 = \sin 2x \Rightarrow y_3' = 2\cos 2x, \; y_3'' = -4\sin 2x, \; y_3^{(3)} = -8\cos 2x$

$\Rightarrow y_3^{(3)} + 3y_3'' + 4y_3' + 12y_3 = (-8+4.2)\cos 2x + (-3.4+12)\sin 2x = 0$

Thus, any Linear combination of y_1, y_2 and y_3 is also a solution of the d.e

$$y = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x.$$

THEOREM Existence and Uniqueness for Linear Eq.

Let $p_1, p_2, \dots, p_n$ and f be continuous functions on the open interval I containing the point a and let the n numbers $b_0, b_1, \dots, b_{n-1}$ be given. Then the nth order Linear equation

$$y^{(n)} + p_1(x)\,y^{(n-1)} + \dots + p_{n-1}(x)\,y' + p_n(x)\,y = f(x)$$

has a unique (one and only one) solution on I that satisfies the n initial conditions

$$y(a) = b_0, \; y'(a) = b_1, \; \dots, \; y^{(n-1)}(a) = b_{n-1}.$$

Ex $y^{(3)} + 3y'' + 4y' + 12y = 0$, $y = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x$

$y(0) = 0, \; y'(0) = 1, \; y''(0) = -13$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y' = -3c_1 e^{-3x} - 2c_2 \sin 2x + 2c_3 \cos 2x, \; y'(0) = 1 \Rightarrow -3c_1 + 2c_3 = 1$

$y'' = 9c_1 e^{-3x} - 4c_2 \cos 2x - 4c_3 \sin 2x, \; y''(0) = -13 \Rightarrow 9c_1 - 4c_2 = -13$

$$\left.\begin{array}{l} c_1 + c_2 = 0 \\ 9c_1 - 4c_2 = -13 \end{array}\right\} c_1 = -c_2 \Rightarrow -9c_2 - 4c_2 = -13 \Rightarrow c_2 = 1 \Rightarrow c_1 = -1$$

$-3c_1 + 2c_3 = 1 \Rightarrow +3 + 2c_3 = 1 \Rightarrow c_3 = -1$

$y = -e^{-3x} + \cos 2x - \sin 2x$

The theorem above implies that there is no other solution which satisfies the same initial values.

Ⓔⓧ Existence and uniqueness theorem for Linear eq. implies that $y=0$ is the only solution of the hom. eq.

$$y^{(n)}+p_1 y^{(n-1)}+\cdots+p_{n-1}y'+p_n y=0 \qquad (*)$$

that satisfies the trivial initial conditions

$$y(a)=y'(a)=\cdots=y^{(n-1)}(a)=0.$$

Now, Let's consider the d.e. $x^2y''-4xy'+6y=0$ with the initial conditions $y(0)=y'(0)=0$

As you can easily verify that the trivial solution $y=0$ satisfies the d.e with these initial conditions. But so do $y=x^2$ and $y=x^3$.

Does this contradict with the theorem?

the answer is no. This happens because when we write the d.e in the form of $(*)$ we get

$$y''-\frac{4}{x}y'+\frac{6}{x^2}y=0$$

for which $P_1(x)=-4/x$ and $P_2(x)=6/x^2$ are not continuous on an open interval containing the point $x=0$.

LINEAR INDEPENDENCE OF FUNCTIONS

$f_1, f_2, \ldots, f_n$ are Linearly independent on I if

$$c_1f_1+c_2f_2+\cdots+c_nf_n=0$$

only when $c_1=c_2=\cdots=c_n=0$. for all $x\in I$.

If you have 2 functions f and g, check if f/g or g/f is a constant valued func. on I. If so, they are Linearly dependent

$$\frac{f}{g}=c \Rightarrow f=cg.$$

Ex $f(x) = \sin x$, $g(x) = \cos x \Rightarrow \frac{f}{g} = \tan x \Rightarrow f, g$: Lin. indep.

$f(x) = \sin 2x$, $g(x) = \sin x \cdot \cos x \Rightarrow \frac{f}{g} = 2 \Rightarrow f, g$: Lin. dep.

WROSKIAN $f_1, f_2, \cdots, f_n$: $(n-1)$ times diff. func.

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \rightarrow \text{Wroskian}$$

THEOREM Wroskians of Solutions

Let $y_1, y_2, \cdots, y_n$ be the n solutions of the hom. nth order Linear equation

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_{n-1}(x)\, y' + p_n(x)\, y = 0$$

on an open int. I where $p_1, p_2, \cdots, p_n$ are all continuous.

* If $y_1, y_2, \cdots, y_n$ are Lin. dependent, then $w(y_1, \cdots, y_n) = 0$ on I.

* If $y_1, y_2, \cdots, y_n$ are Lin independent, then $w(y_1, \cdots, y_n) \neq 0$ for $\forall x \in I$.

PROOF: $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$

$c_1 y_1' + c_2 y_2' + \cdots + c_n y_n' = 0$

$\vdots$

$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)} = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}_{n\times n}}_{\text{coeff. matrix}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

We know that a hom. nxn Lin. system of equations has a nontrivial solution (at least one of c_k is nonzero) if and only if its coeff. matrix is not invertible. $\Rightarrow W=0$.

$\Rightarrow y_1, y_2, \dots, y_n$: Lin. dependent

** $w' = -p_1(x)\,w \Rightarrow w(x) = k e^{-\int p_1(x)\,dx}$ $\nearrow k=0 \Rightarrow w=0$ $\searrow k\neq 0 \Rightarrow w\neq 0$

Ex $w(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$

$\Rightarrow y_1 = \sin x$, $y_2 = \cos x$ Lin. indep.

$$w(\sin 2x, \sin x \cdot \cos x) = \begin{vmatrix} \sin 2x & \sin x \cos x \\ 2\cos 2x & \cos^2 x - \sin^2 x \end{vmatrix}$$

$$= \sin 2x\,(\cos^2 x - \sin^2 x) - \cos 2x \; 2\sin x \cos x$$

$$= \sin 2x \cdot \cos 2x - \cos 2x \cdot \sin 2x = 0$$

$\Rightarrow y_1 = \sin 2x$, $y_2 = \sin x \cdot \cos x$ Lin. dep.

Ex $y^{(3)} + p_1 y'' + p_2 y' + p_3 y = 0$, y_1, y_2, y_3

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1(y_2'y_3'' - y_2''y_3') - y_2(y_1'y_3'' - y_1''y_3') + y_3(y_1'y_2'' - y_1''y_2')$$

$$W' = y_1'(y_2'y_3'' - y_2''y_3') + y_1(\cancel{y_2''y_3''} + y_2'y_3^{(3)} - y_2^{(3)}y_3' - \cancel{y_2''y_3''})$$
$$-y_2'(\dots) - y_2(\dots)' + y_3'(\dots) + y_3(\dots)'$$
$$= y_1'(\cancel{y_2'y_3''} - \cancel{y_2''y_3'}) + y_1(y_2'y_3^{(3)} - y_2^{(3)}y_3')$$
$$-y_2'(\cancel{y_1'y_3''} - \cancel{y_1''y_3'}) - y_2(y_1'y_3^{(3)} - y_1^{(3)}y_3')$$
$$+y_3'(\cancel{y_1'y_2''} - \cancel{y_1''y_2'}) + y_3(y_1'y_2^{(3)} - y_1^{(3)}y_2') = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}$$

$$y_1^{(3)} = -P_1 y_1'' - P_2 y_1' - P_3 y_1$$

$$y_2^{(3)} = -P_1 y_2'' - P_2 y_2' - P_3 y_2$$

$$y_3^{(3)} = -P_1 y_3'' - P_2 y_3' - P_3 y_3 \ .$$

$$w' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -P_1 y_1'' - P_2 y_1' - P_3 y_1 & -P_1 y_2'' - P_2 y_2' - P_3 y_2 & -P_1 y_3'' - P_2 y_3' - P_3 y_3 \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -P_1 y_1'' & -P_1 y_2'' & -P_1 y_3'' \end{vmatrix} = -P_1 w \ .$$

$$w' = -P_1 w \Rightarrow \frac{dw}{w} = -P_1 \, dx \Rightarrow \ln w = -\int P_1 dx \Rightarrow w = e^{-\int P_1 dx}$$

THEOREM General Solutions of Homogeneous Eq.

Let $y_1, y_2, \dots, y_n$ be n Linearly indep. solutions of the hom. eq.

$$y^{(n)} + P_1(x) y^{(n-1)} + \dots + P_{n-1}(x) y' + P_n(x) y = 0$$

on an open interval I where $P_1, \dots, P_n$ are all continuous. If Y is any solution of the d.e, then there exist numbers $c_1, c_2, \dots, c_n$ such that

$$Y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

for all $x \in I$.

($y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$: general sol of the d.e).

Ex $y''-4y=0$

$y_1=e^{2x}$ and $y_2=e^{-2x}$ are solutions of the de.

$$w(y_1,y_2)=\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}=-2-2=-4\neq 0 \Rightarrow y_1,y_2 \text{: lin indep.}$$

$\Rightarrow y=c_1e^{2x}+c_2e^{-2x}$: general solution of the de.

$y_3=\cosh 2x$ and $y_4=\sinh 2x$ are also solutions of the de.

$$w(y_3,y_4)=\begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix}=2(\cosh^2 2x-\sinh^2 2x)=2\neq 0$$

$\Rightarrow y_3,y_4$: lin indep.

$\Rightarrow y_3,y_4$ can be written as a lin. combination of y_1 and y_2.

$$\cosh 2x=\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}, \quad \sinh 2x=\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}$$

This means that there are 2 different basis for the solution space of the d.e. $\{e^{2x},e^{-2x}\}$ and $\{\cosh 2x,\sinh 2x\}$.

$\Rightarrow$ Every particular solution Y can be written as a linear combination of both basis

THEOREM Solutions of Nonhomogeneous Eq.

Let y_p be a particular solution of the nonhom. eq

$$y^{(n)}+p_1(x)y^{(n-1)}+\dots+p_{n-1}(x)y'+p_n(x)y=f(x) \qquad (*)$$

on an open interval I where $p_1,p_2,\dots,p_n$ and f are all continuous. Let $y_1,y_2,\dots,y_n$ be n linearly independent solutions of the associated homog. eq.

$$y^{(n)}+p_1(x)y^{(n-1)}+\dots+p_{n-1}(x)y'+p_n(x)y=0.$$

If Y is any solution of $(*)$ on I, then

$$Y=\underbrace{c_1y_1(x)+c_2y_2(x)+\dots+c_ny_n(x)}_{y_c(x)\text{: complementary function}}+y_p(x)$$

for all $x\in I$.

HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0 \quad , \quad (a_n \neq 0)$$

↓

Offer a solution of the form $\boxed{y=e^{rx}}$

$$\underbrace{(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0)}_{=0} \underset{\neq 0}{\underset{\downarrow}{e^{rx}}} = 0$$

↓

Characteristic Equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

$r_1, r_2, \ldots, r_n$: roots of the char. eq.

Real and Distinct Roots

$r_1 \neq r_2 \neq \cdots \neq r_n$, all real

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$

$\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$

↓

basis

Repeated Real Roots

r: repeated real root of multiplicity k

$y = (c_1 + c_2 x + \cdots + c_k x^{k-1}) e^{rx}$

$\{e^{rx}, x e^{rx}, \ldots, x^{k-1} e^{rx}\}$

↓

basis

Complex Roots

$r = a \mp ib \quad (b \neq 0)$

$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$\{e^{ax} \cos bx, e^{ax} \sin bx\}$

↓

basis

Repeated Complex Roots

$r = a \mp ib$ of multiplicity of k

$$y = e^{ax}\left[(b_1 + b_2 x + \cdots + b_k x^{k-1}) \cos bx + (c_1 + c_2 x + \cdots + c_k x^{k-1}) \sin bx\right]$$

$\{x^p e^{ax} \cos bx, x^p e^{ax} \sin bx\}$

$p = 0, 1, \ldots, k-1$

↓

basis

EX $y'' + 5y' + 6y = 0, \; y(0) = 2, \; y'(0) = 3$

$r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0 \Rightarrow r_1 = -2, r_2 = -3$ (real dist.)

$\Rightarrow y = c_1 e^{-2t} + c_2 e^{-3t} \Rightarrow y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

$$\left.\begin{array}{l} y(0) = 2 \Rightarrow c_1 + c_2 = 2 \\ y'(0) = 3 \Rightarrow -2c_1 - 3c_2 = 3 \end{array}\right\} -c_2 = 7 \Rightarrow c_2 = -7 \Rightarrow c_1 = 9$$

$$\Rightarrow y = 9e^{-2t} - 7e^{-3t}$$

EX $y^{IV} - y = 0$

$r^4 - 1 = 0 \Rightarrow (r+1)(r-1)(r^2+1) = 0 \Rightarrow r_1 = -1, r_2 = 1, r_{3,4} = \pm i$

$\Rightarrow y = c_1 e^{-t} + c_2 e^{t} + c_3 \cos t + c_4 \sin t$

EX $9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$

$9r^5 - 6r^4 + r^3 = 0 \Rightarrow r^3(9r^2 - 6r + 1) = 0 \Rightarrow r^3(3r-1)^2 = 0$

$\Rightarrow r_{1,2,3} = 0 \quad , r_{4,5} = 1/3$

$\Rightarrow y = (c_1 + c_2 x + c_3 x^2) e^{0x} + (c_4 + c_5 x) e^{x/3}$

$\quad = c_1 + c_2 x + c_3 x^2 + (c_4 + c_5 x) e^{x/3}$

EX $Dy = y', D^2 y = y'', \ldots, D^n y = y^{(n)} \Rightarrow (D^2 + 6D + 13)^2 y = 0$

$\Rightarrow (r^2 + 6r + 13)^2 = 0 \Rightarrow \Delta = 36 - 4.1.13 = -16 = 16i^2$

$\Rightarrow r_{1,2} = \dfrac{-6 \mp 4i}{2} = -3 \mp 2i$, multiplicity $k = 2$

$\Rightarrow y = e^{-3x} [(c_1 + c_2 x) \cos 2x + (d_1 + d_2 x) \sin 2x]$

nth order nonhom. Lin. equations with const. coeff.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots\cdots + a_{n-1} y' + a_n y = f(x) \qquad (1)$$

Find $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

Is f(x) in the form of a polynomial $P_m(x)$, $P_m(x)\, e^{ax}$, $P_m(x)\, e^{ax} \cos bt$ or $P_m(x)\, e^{ax} \sin bt$?

Yes

No

Method of Undetermined Coeff.

f(x)	yp
$P_m(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$	$x^s (A_0 x^m + \cdots + A_m)$
$P_m(x)\, e^{ax}$	$x^s (A_0 x^m + \cdots + A_m)\, e^{ax}$
$P_m(x)\, e^{ax} \Big\langle \begin{matrix} \cos bx \\ \sin bx \end{matrix}$ or/and	$x^s \left[(A_0 x^m + \cdots + A_m) \cos bx + (B_0 x^m + \cdots + B_m) \sin bx \right] e^{ax}$

Here s is the smallest nonnegative integer for which every term in y_p differs from every term in y_c.

* Then, replace y_p in (1) to determine the values of the coeff in y_p.

Variation of Parameters

- Evaluate $W(y_1, y_2, \ldots, y_n)$
- Find $W_m(y_1, y_2, \ldots, y_n)$ for $m = 1, 2, \ldots, n$ where W_m is the determinant obtained by replacing the mth column of W with $(0, 0, \ldots, 0, 1)$.
- $u_m(x) = \int f(x) \dfrac{W_m}{W}\, dx, \quad m = 1, 2, \ldots, n$
- $y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$

Important Note:
You can use variation of parameters for nth order nonhom. Lin eq even if the coeff. are not constants.

$a_i \to P_i(x)$

$$y = y_c + y_p$$

Ex $y'''-4y'=e^{3x}$

$r^3-4r=0 \Rightarrow r(r^2-4)=0 \Rightarrow r_1=0,\ r_2=2,\ r_3=-2$

$\Rightarrow y_c=c_1+c_2e^{2x}+c_3e^{-2x}$

$\Rightarrow y_p=Ae^{3x},\ y_p'=3Ae^{3x},\ y_p''=9Ae^{3x},\ y_p'''=27Ae^{3x}$

$\Rightarrow (27A-4.3A)e^{3x}=e^{3x} \Rightarrow 15A=1 \Rightarrow A=1/15$

$y=y_c+y_p \Rightarrow y=c_1+c_2e^{2x}+c_3e^{-2x}+\frac{1}{15}e^{3x}$

* $y'''-4y'=12x+e^{-2x}+2\cos x$

For $12x$, $y_{p_1}=(Ax+B).x$
↳ because of c_1 in y_c.

$y_{p_1}'=2Ax+B,\ y_{p_1}''=2A,\ y_{p_1}'''=0$

$\Rightarrow 0-4(2Ax+B)=12x \Rightarrow -8Ax-4B=12x+0 \Rightarrow A=-12/8=-3/2,\ B=0$

$\Rightarrow y_{p_1}=-\frac{3}{2}x^2$

For e^{-2x}, $y_{p_2}=Ae^{-2x}.x$
↳ because of c_3e^{-2x} in y_c.

$y_{p_2}'=A(1-2x)e^{-2x},\ y_{p_2}''=A(-2-2+4x)e^{-2x}=A(-4+4x)e^{-2x}$

$y_p'''=A(4+8-8x)e^{-2x}=A(12-8x)e^{-2x}$

$\Rightarrow A[12-\cancel{8x}-4(1-\cancel{2x})]e^{-2x}=e^{-2x} \Rightarrow 8A=1 \Rightarrow A=1/8$

$\Rightarrow y_{p_2}=\frac{1}{8}e^{-2x}x$

For $2\cos x$, $y_{p_3}=A\cos x+B\sin x$

$y_{p_3}'=-A\sin x+B\cos x,\ y_{p_3}''=-A\cos x-B\sin x,\ y_{p_3}'''=A\sin x-B\cos x$

$A\sin x-B\cos x-4[-A\sin x+B\cos x]=2\cos x$

$\Rightarrow 5A\sin x-5B\cos x=2\cos x \Rightarrow A=0,\ B=-2/5$

$\Rightarrow y_{p_3}=-\frac{2}{5}\sin x$

$\Rightarrow y=y_c+y_{p_1}+y_{p_2}+y_{p_3}$

$y=c_1+c_2e^{2x}+c_3e^{-2x}-\frac{3}{2}x^2+\frac{1}{8}e^{-2x}x-\frac{2}{5}\sin x$

VARIATION OF PARAMETERS

$$y^{(n)}+P_1(x)y^{(n-1)}+\dots+P_ny=f(x)$$

$$y_c=c_1y_1+c_2y_2+\dots+c_ny_n$$

$$y_p=u_1y_1+u_2y_2+\dots+u_ny_n$$

$$y_p'=(u_1y_1'+u_2y_2'+\dots+u_ny_n')+\underbrace{(u_1'y_1+u_2'y_2+\dots+u_n'y_n)}_{=0}$$

We assume that the 2nd term on the right hand side is 0 to make the calculation as simple as possible.

$$y_p''=(u_1y_1''+u_2y_2''+\dots+u_ny_n'')+\underbrace{(u_1'y_1'+u_2'y_2'+\dots+u_n'y_n')}_{=0}$$

If we go on like this, we'll eventually get

$$y_p^{(n)}=(u_1y_1^{(n)}+u_2y_2^{(n)}+\dots+u_ny_n^{(n)})+(u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\dots+u_n'y_n^{(n-1)}).$$

Since $y^{(n)}+P_1(x)y^{(n-1)}+\dots+P_n(x)y=f(x)$, then

$$y_p^{(n)}+P_1y_p^{(n-1)}+\dots+P_ny_p=f(x).$$

$$(u_1y_1^{(n)}+u_2y_2^{(n)}+\dots+u_ny_n^{(n)})+(u_1'y_1^{(n-1)}+\dots+u_n'y_n^{(n-1)})$$

$$+P_1(u_1y_1^{(n-1)}+\dots+u_ny_n^{(n-1)})+\dots+P_n(u_1y_1+\dots+u_ny_n)=f(x)$$

$$\Rightarrow u_1(y_1^{(n)}+P_1y_1^{(n-1)}+\dots+P_ny_1)+u_2(y_2^{(n)}+P_1y_2^{(n-1)}+\dots+P_ny_2)$$

$$+\dots+u_n(y_n^{(n)}+P_1y_n^{(n-1)}+\dots+P_ny_n)+(u_1'y_1^{(n-1)}+\dots+u_n'y_n^{(n-1)})=f(x)$$

Each paranthesis except the last one is zero since $y_1, y_2, \dots, y_n$ are all solutions of the homog. equation $y^{(n)}+P_1y^{(n-1)}+\dots+P_ny=0$. Then $u_1'y_1^{(n-1)}+\dots+u_n'y_n^{(n-1)}=f(x)$

Now, we have n unknowns and n equations.

$$y_1u_1'+y_2u_2'+\dots+y_nu_n'=0$$

$$y_1'u_1'+y_2'u_2'+\dots+y_n'u_n'=0$$

$$\dots$$

$$\dots$$

$$y_1^{(n-1)}u_1'+y_2^{(n-1)}u_2'+\dots+y_n^{(n-1)}u_n'=f(x).$$

Since $w(y_1, y_2, \dots, y_n) \neq 0$ ($y_1, y_2, \dots, y_n$ are all Lin. indep), we can find $u_1', u_2', \dots, u_n'$ by using cramer's rule

$$\Rightarrow u_m' = \frac{f(x)\,w_m}{w} \Rightarrow u_m = \int f(x) \cdot \frac{w_m}{w}\,dx$$

$$y_p = \sum_{m=1}^{n} y_m u_m = \sum_{m=1}^{n} y_m \int f(x) \frac{w_m}{w}\,dx$$

Ex $y'' + y = \tan x$

$$\Rightarrow r^2+1=0 \Rightarrow r_{1,2} = \pm i \Rightarrow y_c = c_1 \underbrace{\cos x}_{=y_1} + c_2 \underbrace{\sin x}_{=y_2}$$

$$w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$w_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x \quad , \quad w_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$$

$$u_1 = \int \tan x \cdot \frac{-\sin x}{1}\,dx = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= \int \cos x\,dx - \int \sec x\,dx = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \tan x \frac{\cos x}{1}\,dx = \int \sin x\,dx = -\cos x$$

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2 = (\cancel{\sin x} - \ln|\sec x + \tan x|)\cos x + \cancel{(-\cos x)}\sin x$$

$$= -\ln|\sec x + \tan x| \cdot \cos x$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \cos x \cdot \ln|\sec x + \tan x|$$

EULER'S EQUATION

$$ax^2y''+bxy'+cy=0 \rightarrow \text{2nd order Euler eq.}$$

a, b, c: constants

Let's assume that $x>0$ and let's use the substitution $v=\ln x$

$$\frac{dy}{dx}=\frac{dy}{dv}\,\frac{dv}{dx}=\frac{1}{x}\,\frac{dy}{dv}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\,\frac{dy}{dv}\right)=-\frac{1}{x^2}\,\frac{dy}{dv}+\frac{1}{x}\,\frac{d}{dv}\left(\frac{dy}{dv}\right)\overset{\rightarrow\frac{1}{x}}{\boxed{\frac{dv}{dx}}}$$

$$=-\frac{1}{x^2}\,\frac{dy}{dv}+\frac{1}{x^2}\,\frac{d^2y}{dv^2}$$

$$\Rightarrow ax^2\left[-\frac{1}{x^2}\,\frac{dy}{dv}+\frac{1}{x^2}\,\frac{d^2y}{dv^2}\right]+bx\left[\frac{1}{x}\,\frac{dy}{dv}\right]+cy=0$$

$$\Rightarrow a\,\frac{d^2y}{dv^2}+(b-a)\,\frac{dy}{dv}+cy=0 \rightarrow \text{constant coefficient linear eq.}$$

Assume that r_1 and r_2 are two distinct real roots of the characteristic equation. Then

$$y=c_1e^{r_1v}+c_2e^{r_2v}$$

$$=c_1e^{r_1\ln x}+c_2e^{r_2\ln x}$$

$$=c_1e^{\ln x^{r_1}}+c_2e^{\ln x^{r_2}}$$

$$\Rightarrow \quad y=c_1x^{r_1}+c_2x^{r_2} : \text{gen. sol. of the Euler eq.}$$

Ex $x^2y''+xy'-y=0$ Eul eq. $a=1,\ b=1,\ c=-1 \Rightarrow \frac{d^2y}{dv^2}-y=0$

$\Rightarrow r^2-1=0 \Rightarrow r_{1,2}=\pm 1$

$\Rightarrow y=c_1x+c_2x^{-1}$

REDUCTION OF ORDER

$$y''+p(x)\,y'+q(x)\,y=0$$

Suppose that one solution $y_1(x)$ of the homogeneous 2nd order Linear diff eq. is known.

$\Rightarrow$ $y(x) = v(x)\,y_1(x)$. where y is the gen. sol. of the diff. eq.

$$y = v'y_1 + v\,y_1' \Rightarrow y'' = v''y_1 + 2v'y_1' + vy_1''$$

$$\Rightarrow v''y_1 + 2v'y_1' + vy_1'' + p\,(v'y_1 + vy_1') + q\,v\,y_1 = 0$$

$$v\,(\underbrace{y_1'' + py_1' + q\,y_1}_{=0 \text{ since } y_1 \text{ is the solution of diff. eq.}}) + v''y_1 + 2v'y_1' + pv'y_1 = 0$$

$$\Rightarrow y_1 v_1'' + (2y_1' + py_1)\,v' = 0$$

$$\Rightarrow \frac{dv'}{v'} = -\frac{2y_1' + py_1}{y_1}\,dx \rightarrow \text{separable diff eq.}$$

EX $x^2y'' + xy' - 9y = 0$, $x>0$, $y_1(x) = x^3$

$$y_1 = x^3 \Rightarrow y_1' = 3x^2,\ y_1'' = 6x \Rightarrow x^2\,6x + x\,.\,3x^2 - 9x^3 = 0$$

$$\Rightarrow x^2y_1'' + xy_1' - 9y_1 = 0$$

$$y'' + \frac{1}{x}y' - \frac{9}{x^2}y = 0 \Rightarrow p = \frac{1}{x}$$

$$\frac{dv'}{v'} = -\frac{2.3x^2 + \frac{1}{x}\,.x^3}{x^3}\,dx = -\frac{7}{x}\,dx \Rightarrow \ln v' = -7\ln x + \ln c_1$$

$$\ln v' = \ln(c_1x^{-7}) \Rightarrow v' = c_1x^{-7} \Rightarrow dv = c_1x^{-7}\,dx$$

$$\Rightarrow v = -c_1\frac{x^{-6}}{6} + c_2 \Rightarrow y = v\,y_1 = -c_1\underset{y_2}{\underbrace{\frac{x^{-3}}{6}}} + c_2\underset{y_1}{\underbrace{x^3}}$$

$$\Rightarrow y_2 = x^{-3}$$