PS, weekend session, 21/11/2020

① 16 Nov '20

$$A = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}_{3\times 4}$$

ⓐ Define the null space of the matrix A.

$A = [a_{ij}]_{m\times n}$ $\quad Null(A) = \{ x \in \mathbb{R}^n \mid \underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0} \}$

$A = [a_{ij}]_{3\times 4}$ $\quad Null(A) = \{ \underset{\sim}{x} \in \mathbb{R}^4 \mid \underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0} \}$

ⓑ Find a basis for the null space of A and determine the dimension of Null (A).

$$A = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{0}$$

$$x_1 - 4x_2 - 3x_3 - 7x_4 = 0$$

$$2x_1 - x_2 + x_3 + 7x_4 = 0$$

$$x_1 + 2x_2 + 3x_3 + 11x_4 = 0$$

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

n

$x_1 \quad x_2$

echelon form

$$\begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 - 3x_3 - 7x_4 = 0$$

$$x_2 + x_3 + 3x_4 = 0$$

# of unknowns $= n = 4$

# of leading ... $= k = 2$

# of free parameters $= r = n - k = 2$

Let $x_3 = r$, $x_4 = s$ $\longrightarrow$ $x_2 = -r - 3s$

$x_1 = -r - 5s$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -r-5s \\ -r-3s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$s=0, r=1$ $\qquad$ $s=1, r=0$

$$\text{Null}(A) = \left\{ x \in \mathbb{R}^4 \;\middle|\; \underset{\sim}{x} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix},\ r, s \in \mathbb{R} \right\}$$

Null(A) is spanned by

A basis for Null (A) is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

dim Null (A) = 2

(c) Find a basis for the column space of A and its rank.

$$A = \begin{array}{cccc} \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \\ \begin{array}{cccc} \underset{\sim}{c_1} & \underset{\sim}{c_2} & \underset{\sim}{c_3} & \underset{\sim}{c_4} \end{array} \end{array} \sim \begin{bmatrix} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E$$

Columns of A corresponding to the columns of E that include the leading entries form a basis for col(A).

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}$$ is a basis for col(A)

rank A = column rank of A = dim Col(A) = 2 (= row rank of A)

$A = [a_{ij}]_{m \times n}$

$$\dim \text{Null}(A) + \text{Rank } A = n$$

$\dim \text{Null}(A) = 2$ $\quad\quad\quad$ $2+2=4$ ✓

$\text{Rank}(A) = 2$

$n = 4$

② Find a subset of the vectors $\underset{\sim}{v_1} = (1,-1,2,2)$, $\underset{\sim}{v_2} = (-3,4,1,2)$, $\underset{\sim}{v_3} = (0,1,7,4)$, $v_4 = (-5,7,6,-2)$ that forms a basis for the subspace of $\mathbb{R}^4$ spanned by those vectors.

* This question requires that you know the def. of column space of a matrix A.

* If the Q was " find a basis for the col(A) of $A = \begin{bmatrix} & & \\ - & -- & \\ & & \end{bmatrix}$, $\xrightarrow[\text{obvious}]{\text{the answer is}} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \end{bmatrix}$
"

(2) Find a subset of the vectors $\underset{\sim}{v_1} = (1, -1, 2, 2)$, $\underset{\sim}{v_2} = (-3, 4, 1, 2)$, $\underset{\sim}{v_3} = (0, 1, 7, 4)$, $v_4 = (-5, 7, 4, -2)$ that forms a basis for the subspace of $\mathbb{R}^4$ spanned by those vectors.

The set spanned by $\underset{\sim}{v_1}, \underset{\sim}{v_2}, v_3, v_4$ consists of vectors

$$k_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 1 \\ 7 \\ 4 \end{bmatrix} + k_4 \begin{bmatrix} -5 \\ 7 \\ 4 \\ -2 \end{bmatrix}$$

where $k_i \in \mathbb{R}$. $= \operatorname{col}(A)$

$$A = \begin{bmatrix} 1 & -3 & 0 & -5 \\ -1 & 4 & 1 & 7 \\ 2 & 1 & 7 & 4 \\ 2 & 2 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E$$

the mentioned basis is $= \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \end{bmatrix} \right\}$

3. Are $\underset{\sim}{u} = (5, -2, 4)$, $\underset{\sim}{v} = (2, -3, 5)$, $w = (4, 5, -7)$ linearly dependent / independent ??

Given n vectors $v_1, \dots, v_n$ in $\mathbb{R}^n$

$$\begin{vmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{vmatrix} \begin{array}{l} = 0 \rightarrow \text{lin dependent} \\ \neq 0 \Rightarrow \text{lin. independent} \end{array}$$

$$\begin{vmatrix} 5 & 2 & 4 \\ -2 & -3 & 5 \\ 4 & 5 & -7 \end{vmatrix} = 0 \Rightarrow \text{linearly dependent.}$$

④ Express $\underset{\sim}{r}$ as a linear comb. of $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$

where $\underset{\sim}{r} = (0, 0, 19)$, $\underset{\sim}{u} = (1, 4, 3)$, $\underset{\sim}{v} = (-1, -2, 2)$

$\underset{\sim}{w} = (4, 4, 1)$.

$$\underset{\sim}{r} = c_1 \underset{\sim}{u} + c_2 \underset{\sim}{v} + c_3 \underset{\sim}{w} \qquad c_1 = ? \quad c_2 = ? \quad c_3 = ?$$

$$\begin{bmatrix} 0 \\ 0 \\ 19 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{aligned} c_1 - c_2 + 4c_3 &= 0 \\ 4c_1 - 2c_2 + 4c_3 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 19 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & -1 & 4 & 0 \\ 4 & -2 & 4 & 0 \\ 3 & 2 & 1 & 19 \end{array}\right]$$

when we solve this system (DIY) we find

$c_1 = 2, \quad c_2 = 6, \quad c_3 = 1.$

Observation If we can find c_1, c_2, c_3 from the system, the answer to the question is affirmative.

$$\begin{aligned} c_1 - c_2 + 4c_3 &= 0 \\ 4c_1 - 2c_2 + 4c_3 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 19 \end{aligned} \Longrightarrow \begin{bmatrix} 1 & -1 & 4 \\ 4 & -2 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 19 \end{bmatrix}$$

$\underset{\sim}{A}\,\underset{\sim}{C} = \underset{\sim}{b}$

$\det A = 0$
- the system may have inf. many sols
- the system may have no sols.

$\det A \neq 0 \Rightarrow A^{-1}$ exists $\quad A^{-1}AC = A^{-1}b$

$\underset{\sim}{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1}\underset{\sim}{b}$ is obtained uniquely

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

det $= 0$

inf. many sol.

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 3$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

det $= 0$

no sols.

5. W is a subset of vectors in $\mathbb{R}^4$ such that $x_1 = 3x_3$. Is W a subspace of $\mathbb{R}^4$?

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;\middle|\; x_1 = 3x_3 \right\}$$

(i) $u, v \in W \overset{?}{\Rightarrow} u+v \in W$

(ii) $c \in \mathbb{R},\ u \in W \Rightarrow c\,\underset{\sim}{u} \overset{?}{\in} W$

(i) $\underset{\sim}{u} = \begin{bmatrix} 3x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \underset{\sim}{v} = \begin{bmatrix} 3y_3 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in W$

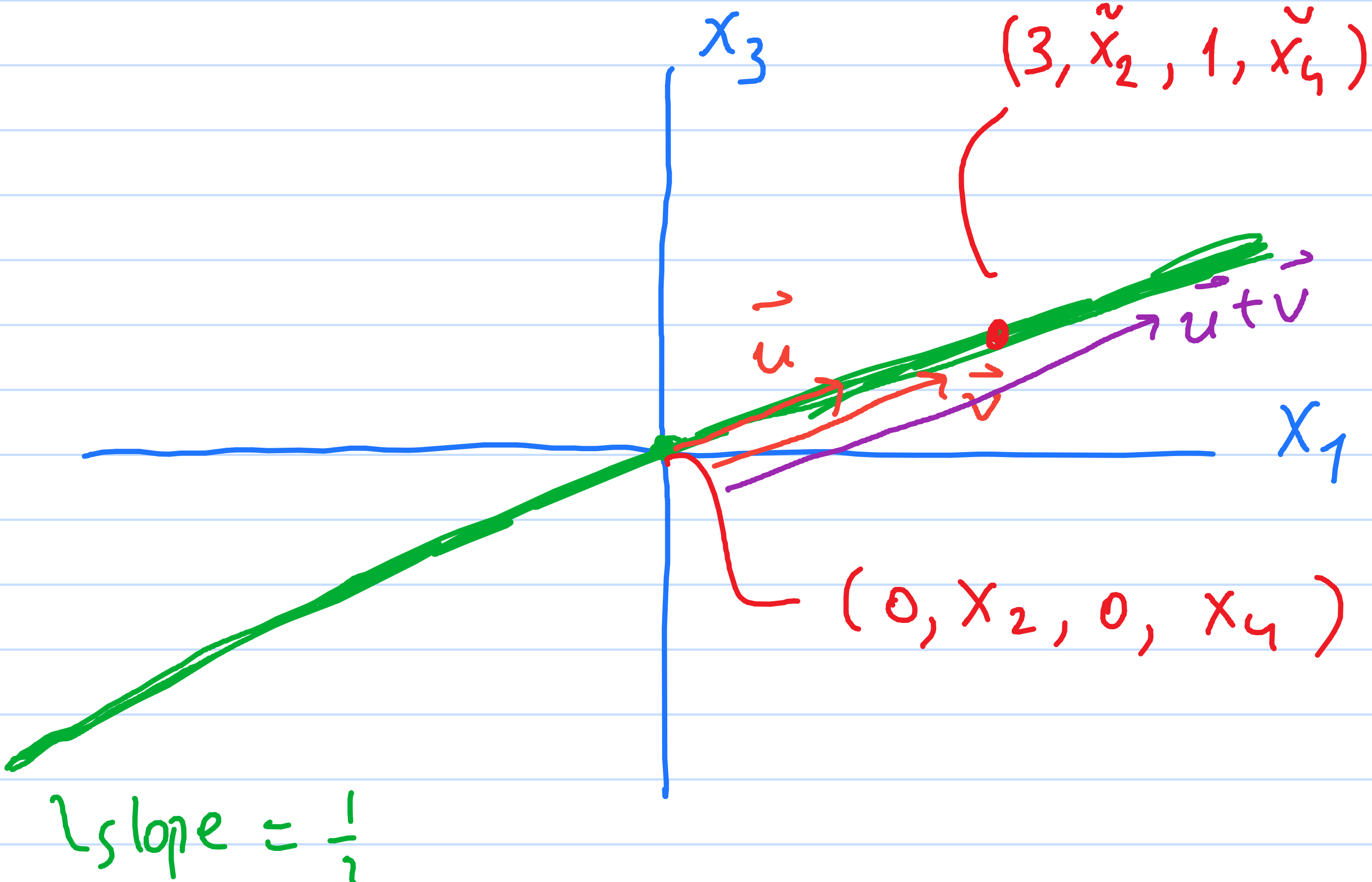
$$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} 3(x_3 + y_3) \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix} \in W \checkmark$$

(ii) $$c\underset{\sim}{u} = \begin{bmatrix} 3\,cx_3 \\ cx_2 \\ cx_3 \\ cx_4 \end{bmatrix} \in W \checkmark$$

$\Rightarrow W$ is a subspace of $\mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;\middle|\; x_1 = 3x_3 \right\}$$

Just for the moment, let's see what's happening in $\mathbb{R}^4$, but on the $x_1 x_3$-plane only!

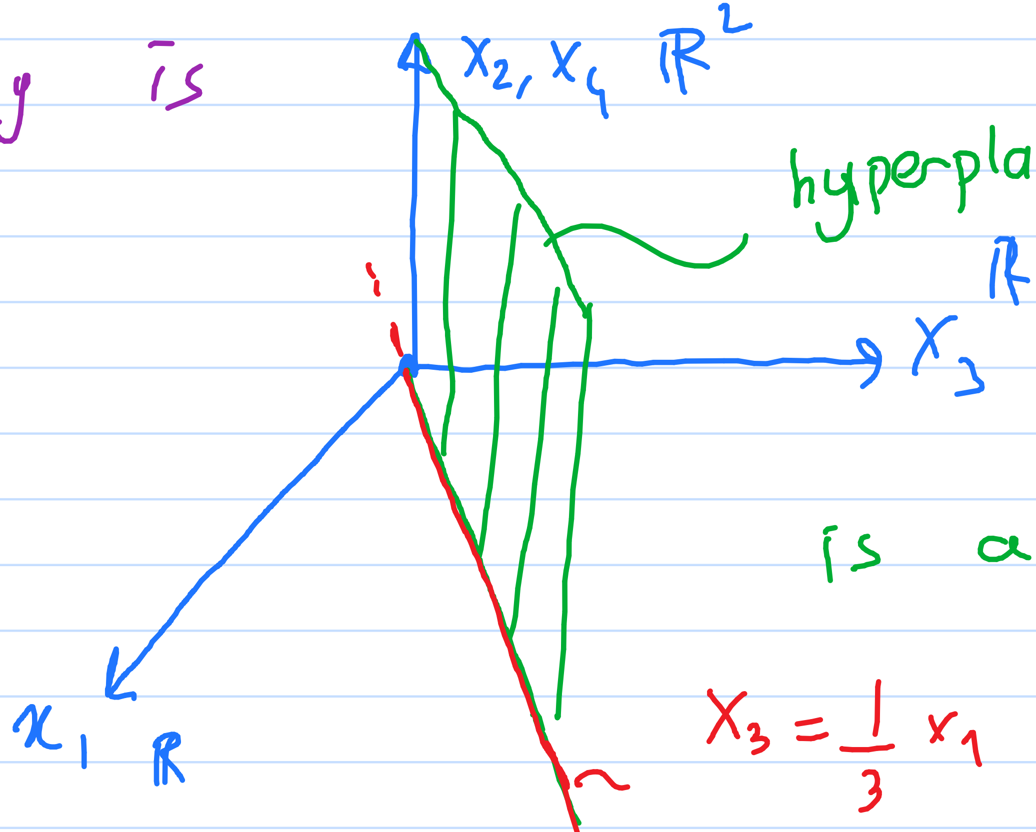


$$x_3 = \frac{1}{3} x_1$$

$$y = \frac{1}{3} x$$

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;\middle|\; x_1 = 3x_3 \right\}$$

If we don't ignore x_2, and x_4; but embed them on a single axis (as we don't have two more axes in addition to x_1 and x_3), the thing happening geometrically is



hyperplane in $\mathbb{R}^4$ with eq $x_3 = \frac{1}{3} x_1$ is a subspace of $\mathbb{R}^4$.

6. W is a subset of all vectors in $\mathbb{R}^2$ such that $x_1 + x_2 = 1$.

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, x_1 + x_2 = 1 \right\}$$

(i) $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 + x_2 = 1$; $\underset{\sim}{v} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $y_1 + y_2 = 1$

$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$ is not in W as

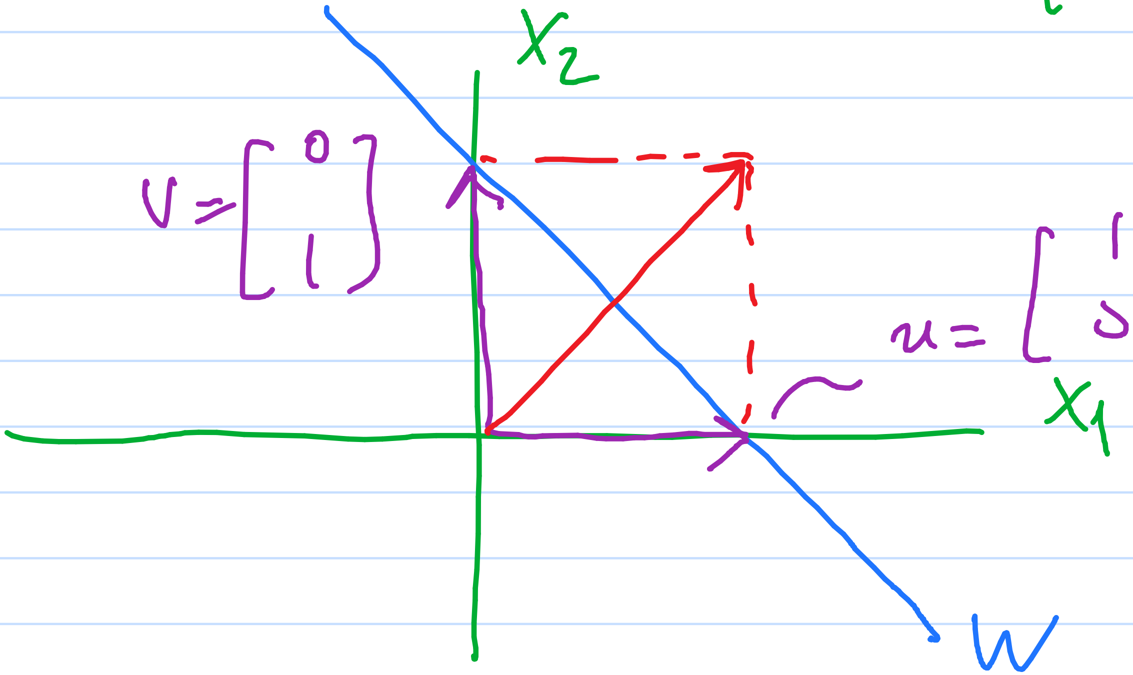
$(x_1 + y_1) + (x_2 + y_2) = 1 + 1 = 2 \neq$

$u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W$, $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W \Rightarrow u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W$

geometrically :

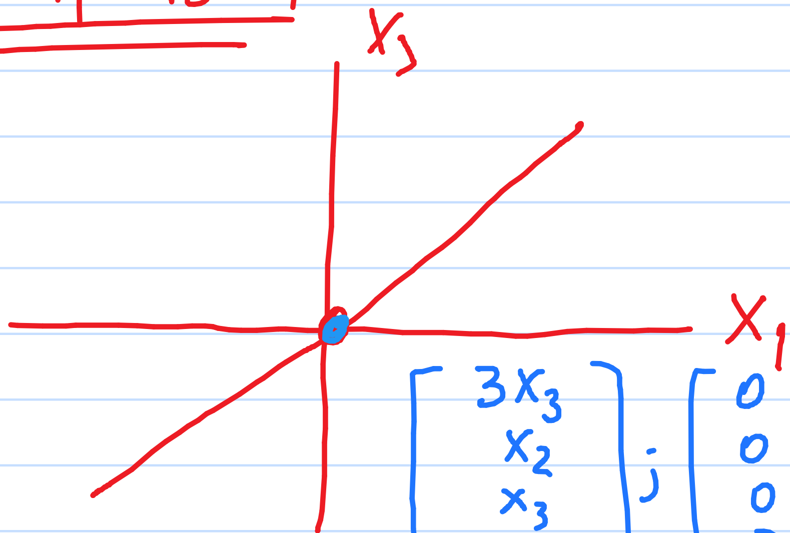
$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \; x_1 + x_2 = 1 \right\}$$

$$x + y = 1$$



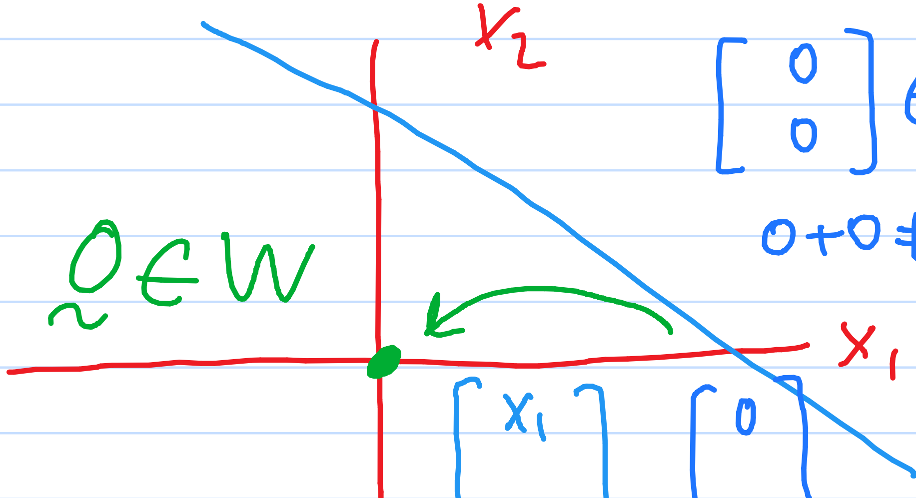
$$\underset{\sim}{u} + \underset{\sim}{v} \notin W$$

Comparison



$$\begin{bmatrix} 3x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} ; \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in W$$

$$\underset{\sim}{0} \in W$$



$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin W$$

$0 + 0 \neq$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(6) $A = [a_{ij}]_{n \times n}$, $k \in \mathbb{R}$; show that the set of all vectors $\underset{\sim}{x}$ such that

$$\underset{\sim}{A}\,\underset{\sim}{x} = k.\,\underset{\sim}{x}$$

is a subspace of $\mathbb{R}^n$.

$$W = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \;\middle|\; \underset{\sim}{A}\,\underset{\sim}{x} = k\,\underset{\sim}{x} \;,\; k \in \mathbb{R} \right\}$$

DIY

Think

The sol. space of $A\underset{\sim}{x} = 0$ is a subspace of $\mathbb{R}^n$

" " " " $A\underset{\sim}{x} = \underset{\sim}{b}$ is not a subspace of $\mathbb{R}^n$

⑦ $A = \begin{bmatrix} 1 & 1 & 3 & 3 & 1 \\ 2 & 3 & 7 & 8 & 2 \\ 2 & 3 & 7 & 8 & 3 \\ 3 & 1 & 7 & 5 & 4 \end{bmatrix}_{m \times n}$ Find bases for row and column spaces of A.

$E = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis of the row space

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \leftarrow$

Basis of the col space

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\} \leftarrow$

Rank(A) = Row rank of A = dim Row(A) $=$ Col Rank of A = dim Col A

For any matrix $A = [a_{ij}]_{m \times n}$, which of the followings are always True / False?

I) # of vectors in the basis of Row A
$\parallel$
# of vectors in the basis of Col A

True

II) Row A and Col A are both subspaces of $\mathbb{R}^n$? False

III) Row A and Col A are both subspaces of $\mathbb{R}^m$? False