July 20, 2013

Chapter 9: Nonlinear Systems and Optimization

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Slides for the book
A First Course in Numerical Methods (published by SIAM, 2011)
`http://www.ec-securehost.com/SIAM/CS07.html`

Goals of this chapter

- To devise and assess algorithms for (continuous) unconstrained optimization, a problem setting which arises frequently in applications;
- to solve systems of nonlinear equations, thus extending Chapter 3;
- to consider more advanced conditions and techniques for constrained optimization.

Outline

- Systems of nonlinear equations
- Unconstrained optimization
- *Constrained optimization

*advanced

Systems of equations

- Equipped with knowledge on how to solve scalar nonlinear equations as well as linear systems, it is time to combine them and consider systems of nonlinear equations

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0, \\ f_2(x_1, x_2, \ldots, x_n) &= 0, \\ \vdots \quad &= \quad \vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0. \end{aligned}$$

- In vector notation, write this as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.
- Chapter 3 promises that Newton's method extends directly to the present problem. But Newton's method requires the derivative of f, so here we have to extend the concept of derivative first.

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Multivariate Taylor expansion

- Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, function $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$, direction $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$.
  Assume that $\mathbf{f}$ is sufficiently smooth (at least two bounded derivatives).
- Then Taylor expansion gives

$$\mathbf{f}(\mathbf{x}+\mathbf{p}) = \mathbf{f}(\mathbf{x}) + J(\mathbf{x})\mathbf{p} + \mathcal{O}(\|\mathbf{p}\|^2),$$

where $J(\mathbf{x})$ is the Jacobian matrix of first derivatives of $\mathbf{f}$ at $\mathbf{x}$,

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Newton's method

- Derivation: By Taylor series,
$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_k) + J(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k) + \mathcal{O}(\|\mathbf{x} - \mathbf{x}_k\|^2).$$
- For $\mathbf{x} = \mathbf{x}^*$, also $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.
- Neglect nonlinear term and define method by
$$\mathbf{0} = \mathbf{f}(\mathbf{x}_k) + J(\mathbf{x}_k)(\mathbf{x}_{k+1} - \mathbf{x}_k).$$
- This is conceptually identical to the procedure (Chapter 3) for one function in one variable.
- Algorithm: Given an initial guess $\mathbf{x}_0$;

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for $k = 0, 1, \ldots,$ until convergence
        solve $J(\mathbf{x}_k)\mathbf{p} = -\mathbf{f}(\mathbf{x}_k)$,
        set   $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{p}$.
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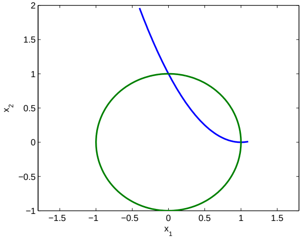
Example: a parabola meets a circle

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^2 - 2x_1 - x_2 + 1 \\ x_1^2 + x_2^2 - 1 \end{pmatrix}.$$



Two solutions: $(0,1)^T$ and $(1,0)^T$.

$$J(\mathbf{x}) = \begin{pmatrix} 2x_1 - 2 & -1 \\ 2x_1 & 2x_2 \end{pmatrix}$$

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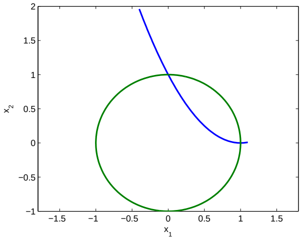

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Example (cont.)

Using Newton's method for finding the two roots, varying the starting point $\mathbf{x}_0$.

Stopping tolerance `tol` $=$ 1.e-7.

1. Starting at $\mathbf{x}_0 = (0,0)^T$ is bad because $J(\mathbf{x}_0)$ is singular!
2. Starting at $\mathbf{x}_0 = (1,1)^T$ obtain root $(0,1)^T$ in 5 iterations. Observe quadratic convergence.
3. Starting at $\mathbf{x}_0 = (-1,1)^T$ obtain root $(1,0)^T$ in 5 iterations. Observe quadratic convergence.

Example: two-point boundary value ordinary differential equation

- Consider the differential problem

$$u''(t) + e^{u(t)} = 0, \quad 0 < t < 1,$$
$$u(0) = u(1) = 0.$$

- Discretize on a uniform mesh (grid) $t_i = ih, \ i = 0, 1, \ldots, n+1$, where $(n+1)h = 1$:

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + e^{v_i} = 0, \qquad i = 1, 2, \ldots, n.$$
$$v_0 = v_{n+1} = 0.$$

- This is a system of nonlinear equations, with $\mathbf{x} \leftarrow \mathbf{v}$ and $f_i(\mathbf{v}) = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + e^{v_i}$.

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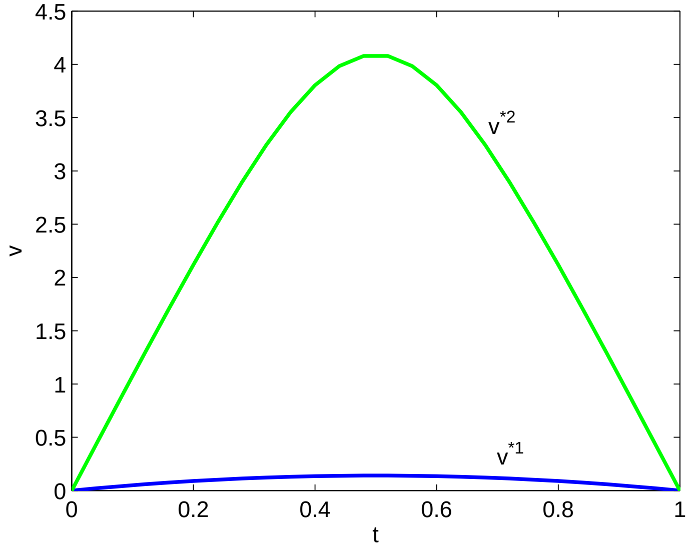
Newton for TPBVP

- Jacobian matrix is possibly large but tridiagonal

$$J = \frac{1}{h^2}\begin{pmatrix} -2+h^2e^{v_1} & 1 & & & \\ 1 & -2+h^2e^{v_2} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2+h^2e^{v_{n-1}} & 1 \\ & & & 1 & -2+h^2e^{v_n} \end{pmatrix}.$$

- Initial guess $\mathbf{v}_0 = \alpha\left(t_1(1-t_1),\ldots,t_n(1-t_n)\right)^T$.
- Take various values of α and see what happens, setting $\texttt{tol} = 1.\text{e-}8$, $n = 24$:
  1. $\alpha = 0 \;\Rightarrow$ converges in 4 iterations
  2. $\alpha = 10 \;\Rightarrow$ converges in 6 iterations
  3. $\alpha = 20 \;\Rightarrow$ converges in 6 iterations to another solution
  4. $\alpha = 50 \;\Rightarrow$ diverges

Two solutions



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Multivariate Taylor expansion

- Consider the problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \phi(\mathbf{x})$$

  Assume that $\phi(\mathbf{x})$ is smooth enough.

- Define gradient vector $\boldsymbol{\nabla}\phi(\mathbf{x})$ and Hessian matrix $\nabla^2\phi(\mathbf{x})$ by

$$\boldsymbol{\nabla}\phi(\mathbf{x}) = \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial x_2} \\ \vdots \\ \frac{\partial\phi}{\partial x_n} \end{pmatrix}, \quad \nabla^2\phi(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2\phi}{\partial x_1\partial x_n} \\ \frac{\partial^2\phi}{\partial x_2\partial x_1} & \frac{\partial^2\phi}{\partial x_2^2} & \cdots & \frac{\partial^2\phi}{\partial x_2\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\phi}{\partial x_n\partial x_1} & \frac{\partial^2\phi}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2\phi}{\partial x_n^2} \end{pmatrix}.$$

- Taylor expansion near a point $\mathbf{x}\in\mathbb{R}^n$:

$$\phi(\mathbf{x}+\mathbf{p}) = \phi(\mathbf{x}) + \boldsymbol{\nabla}\phi(\mathbf{x})^T\mathbf{p} + \frac{1}{2}\mathbf{p}^T\nabla^2\phi(\mathbf{x})\mathbf{p} + \mathcal{O}(\|\mathbf{p}\|^3).$$

Example

- Given the function

$$\phi(x_1, x_2) = x_1^4 - 2x_1^3 x_2^2 + 4x_1 x_2^3,$$

- the gradient at a point $\mathbf{x}$ is

$$\boldsymbol{\nabla}\phi(\mathbf{x}) = \begin{pmatrix} 4x_1^3 - 6x_1^2 x_2^2 + 4x_2^3 \\ -4x_1^3 x_2 + 12x_1 x_2^2 \end{pmatrix};$$

- the Hessian matrix is

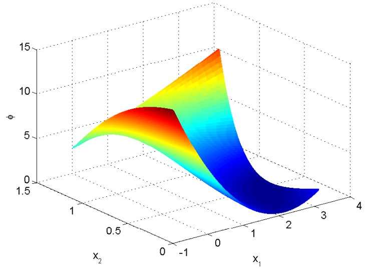
$$H(\mathbf{x}) = \nabla^2\phi(\mathbf{x}) = \begin{pmatrix} 12x_1^2 - 12x_1 x_2^2 & -12x_1^2 x_2 + 12x_2^2 \\ -12x_1^2 x_2 + 12x_2^2 & -4x_1^3 + 24x_1 x_2 \end{pmatrix}.$$

Critical points

- Taylor expansion near a suspected minimum point $\mathbf{x}^*$: in any direction $\mathbf{p}$ with magnitude $\|\mathbf{p}\|$ small enough, must have $\phi(\mathbf{x}^* + \mathbf{p}) \geq \phi(\mathbf{x}^*)$.
- Hence

$$\nabla\phi(\mathbf{x}^*) = \mathbf{0}.$$

  This defines a critical point.
- Similar condition also for a maximum or a saddle point.



Conditions for unconstrained minimum

$$\min_{\mathbf{x}\in\mathbb{R}^n} \phi(\mathbf{x}).$$

Assume that $\phi(\mathbf{x})$ is smooth enough.

- A necessary condition for having a local minimum at $\mathbf{x}^*$ is that $\mathbf{x}^*$ be a critical point

  $$\boldsymbol{\nabla}\phi(\mathbf{x}^*) = \mathbf{0},$$

  and that the symmetric Hessian matrix $\nabla^2\phi(\mathbf{x}^*)$ be positive semi-definite.
- A sufficient condition is that also $\nabla^2\phi(\mathbf{x}^*)$ be positive definite.

Descent direction

- At point $\mathbf{x}$ the vector $\mathbf{p}$ is a descent direction if

$$\nabla \phi(\mathbf{x})^T \mathbf{p} = \sum_{i=1}^{n} p_i \frac{\partial \phi}{\partial x_i} < 0.$$

- A small step in a descent direction gives reduction in the objective function:

$$\phi(\mathbf{x} + \alpha \mathbf{p}) < \phi(\mathbf{x})$$

  for scalar $0 < \alpha \ll 1$.

- Therefore, we can construct an iterative method that keeps reducing ϕ until convergence by using descent directions and controlled step sizes.

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Gradient-based methods

Consider

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{x}_k + \alpha_k \mathbf{p}_k, \quad \text{where} \\ \mathbf{p}_k &= -B_k^{-1} \boldsymbol{\nabla} \phi(\mathbf{x}_k). \end{aligned}$$

If B_k is symmetric positive definite then $\mathbf{p}_k$ is a descent direction. Note $\alpha_k > 0$.

1. Gradient descent: $B_k = I$. How to choose α_k?
2. Nonlinear conjugate gradients.
3. Newton: $B_k = \nabla^2 \phi(\mathbf{x}_k)$. Set $\alpha_k = 1$ or damped Newton: search for $\alpha_k \le 1$ guaranteeing descent.
4. Secant, or quasi-Newton.
5. Inexact Newton, Newton-CG.
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Newton's method

- Simply solve the nonlinear equations

$$\mathbf{f}(\mathbf{x}) \equiv \boldsymbol{\nabla}\phi(\mathbf{x}) = \mathbf{0}$$

  using Newton's method.
- At iteration k do

$$\text{solve } \nabla^2\phi(\mathbf{x}_k)\ \mathbf{p}_k = -\boldsymbol{\nabla}\phi(\mathbf{x}_k) \text{ for } \mathbf{p}_k$$
$$\text{set } \mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{p}_k.$$

- Note that if the Hessian matrix $\nabla^2\phi(\mathbf{x}_k)$ is positive definite then $\mathbf{p}_k$ minimizes a quadratic approximation to ϕ at $\mathbf{x}_k$:

$$\phi(\mathbf{x}_k + \mathbf{p}) \approx \phi(\mathbf{x}_k) + \boldsymbol{\nabla}\phi(\mathbf{x}_k)^T\mathbf{p} + \tfrac{1}{2}\mathbf{p}^T\nabla^2\phi(\mathbf{x}_k)\mathbf{p}.$$

- Furthermore, Newton's method has the following advantages:
  1. locally, it converges quadratically;
  2. (tautologically, it) retains Hessian sparsity.

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Newton's method deficiencies

1. Requires existence of the Hessian.
2. Requires evaluation of the Hessian.
3. Requires solving a linear system at each iteration.
4. $B_k = \nabla^2\phi(\mathbf{x}_k)$ is symmetric but may not be positive definite.
5. No control over convergence (does it converge? to a minimum point?)

Advanced methods are based on capitalizing on the strengths of Newton's method, while trying to weaken or eliminate its deficiencies.

(Weak) line search

- Suppose that $\mathbf{p}_k$ is a descent direction at $\mathbf{x}_k$. Then for $\alpha_k > 0$ small enough, $\phi(\mathbf{x}_k + \alpha_k \mathbf{p}_k) < \phi(\mathbf{x}_k)$.
- Here is a simple algorithm (Armijo) for determining step size α_k, given descent direction $\mathbf{p}_k$:
  Starting from $\alpha = \alpha_{max}$, repeat until sufficient decrease in ϕ is obtained,

$$\alpha \leftarrow \alpha/2.$$

- The result is $\alpha_k = \alpha$, and set $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k$.
- For Newton's method, set $\alpha_{max} = 1$ (and ensure that $\mathbf{p}_k$ is a descent direction).
- For gradient descent, $B_k = I$ so $\mathbf{p}_k$ is always a descent direction, but there is no obvious default value α_{max}.

Combining Newton and gradient descent

- Want the Newton efficiency. But what if $\nabla^2\phi(\mathbf{x}_k)$ is not positive definite?! – Need to ensure a descent search direction.
- So, consider mixing Newton and gradient descent:

$$B_k = \nabla^2\phi(\mathbf{x}_k) + \mu_k I.$$

For $\mu_k > 0$ large enough, this shifts the eigenvalues of B_k into positivity.

- Big question: choose scalar $\mu_k \geq 0$ adaptively, not too small and not too large; this leads to trust region methods.

Secant (quasi-Newton)

- Do not want to form or evaluate Hessian matrix explicitly; and *also*,
- want a positive definite B_k that is *easy to invert*.
- Note by Taylor's expansion

$$\nabla\phi(\mathbf{x}_k) \approx \nabla\phi(\mathbf{x}_{k+1}) - \nabla^2\phi(\mathbf{x}_{k+1})(\mathbf{x}_{k+1} - \mathbf{x}_k).$$

The essential action of the Hessian is therefore in the direction of $\mathbf{w}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$.

- Thus, equire

$$B_{k+1}\mathbf{w}_k = \mathbf{y}_k, \quad \mathbf{y}_k = \nabla\phi(\mathbf{x}_{k+1}) - \nabla\phi(\mathbf{x}_k).$$

- Obtain B_{k+1} as a positive definite rank-2 update of B_k, thus satisfying the above requirements.

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BFGS method

Update $G_k = B_k^{-1}$ directly.

Choose $\mathbf{x}_0$ and G_0 (e.g., $G_0 = I$)

for $k = 0, 1, \ldots,$ until convergence

```
    $\mathbf{p}_k = -G_k \nabla_\phi(\mathbf{x}_k)$
    find a suitable step size $\alpha_k$
    $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k$
    $\mathbf{w}_k = \alpha_k \mathbf{p}_k$
    $\mathbf{y}_k = \nabla_\phi(\mathbf{x}_{k+1}) - \nabla_\phi(\mathbf{x}_k)$
    $G_{k+1} = (I - \frac{\mathbf{w}_k \mathbf{y}_k^T}{\mathbf{y}_k^T \mathbf{w}_k}) G_k (I - \frac{\mathbf{y}_k \mathbf{w}_k^T}{\mathbf{y}_k^T \mathbf{w}_k}) + \frac{\mathbf{w}_k \mathbf{w}_k^T}{\mathbf{y}_k^T \mathbf{w}_k}.$
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$$
\begin{aligned}
&\mathbf{p}_k = -G_k \nabla_\phi(\mathbf{x}_k) \\
&\text{find a suitable step size } \alpha_k \\
&\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k \\
&\mathbf{w}_k = \alpha_k \mathbf{p}_k \\
&\mathbf{y}_k = \nabla_\phi(\mathbf{x}_{k+1}) - \nabla_\phi(\mathbf{x}_k) \\
&G_{k+1} = (I - \frac{\mathbf{w}_k \mathbf{y}_k^T}{\mathbf{y}_k^T \mathbf{w}_k}) G_k (I - \frac{\mathbf{y}_k \mathbf{w}_k^T}{\mathbf{y}_k^T \mathbf{w}_k}) + \frac{\mathbf{w}_k \mathbf{w}_k^T}{\mathbf{y}_k^T \mathbf{w}_k}.
\end{aligned}
$$

Secant (quasi-Newton) pros and cons

- Cheap update of $G_k = B_k^{-1}$ using rank-2 updates.
- Local superlinear convergence.
- Only descent directions because $G_k = B_k^{-1}$ symmetric positive definite.
- The method of choice for most problems.
- May lose Hessian sparsity, though.
- Limited memory versions L-BFGS exist for large, sparse matrices. They have their pros and cons...

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Inexact Newton, Newton-CG

- If the problem is large and has a sparse Hessian, may want to use Newton. But how to solve the linear system at each iteration?
- Use iterative method (Chapter 7) for the linear system at each iteration:

  apply preconditioned conjugate gradient iterations towards solving
  $\nabla^2\phi(\mathbf{x}_k)\ \mathbf{p}_k = -\nabla_\phi(\mathbf{x}_k)$ for $\mathbf{p}_k$
  set $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{p}_k$.

- Newton - outer iteration
  PCG - inner iteration
- How many inner iterations?! (intuitively, fewer when far, more when close) – leads to inexact Newton.

Nonlinear least squares

$$\min_{\mathbf{x}} \phi(\mathbf{x}) = \frac{1}{2}\|\mathbf{g}(\mathbf{x}) - \mathbf{b}\|^2,$$

where $\mathbf{b}$ is data (m values) and $\mathbf{g}$ a nonlinear function of n parameters $\mathbf{x}$.

$$A(\mathbf{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{m-1}}{\partial x_1} & \frac{\partial g_{m-1}}{\partial x_2} & \cdots & \frac{\partial g_{m-1}}{\partial x_n} \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix},$$

Then

$$\boldsymbol{\nabla}\phi(\mathbf{x}) = A(\mathbf{x})^T(\mathbf{g}(\mathbf{x}) - \mathbf{b}).$$

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Gauss-Newton

Hessian matrix

$$\nabla^2\phi(\mathbf{x}) = A(\mathbf{x})^T A(\mathbf{x}) + L(\mathbf{x}),$$

where L is $n \times n$

$$L_{i,j} = \sum_{l=1}^{m} \frac{\partial^2 g_l}{\partial x_i \partial x_j}(g_l - b_l).$$

L can be ugly. Appears in Newton's method!

Gauss-Newton: drop the ugly term. Define iteration by

$$\begin{aligned}\left[A(\mathbf{x}_k)^T A(\mathbf{x}_k)\right]\mathbf{p}_k &= -\boldsymbol{\nabla}\phi(\mathbf{x}_k)\\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \mathbf{p}_k.\end{aligned}$$

These are normal equations for

$$\min_{\mathbf{p}} \|A(\mathbf{x}_k)\mathbf{p} - (\mathbf{b} - \mathbf{g}(\mathbf{x}_k))\|,$$

so at each iteration solve a linear least squares problem (Chapter 6).

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Gauss-Newton vs. Newton

- The Gauss-Newton direction, unlike Newton's, is guaranteed to be a descent direction. This is because $A^T A$ is positive definite even when $A^T A + L$ is not.
- The Gauss-Newton iteration is cheaper and can be better conditioned (more stable) than Newton's iteration.
- The convergence order of Gauss-Newton is only linear, as the difference between it and Newton's iteration does not vanish in the limit.
- Gauss-Newton converges faster for problems where the model fits the data well! This is because then $\|\mathbf{g}(\mathbf{x}) - \mathbf{b}\|$ is "small" near the solution, hence L is small and Gauss-Newton is closer to Newton.

Outline

- Systems of nonlinear equations
- Unconstrained optimization
- *Constrained optimization

Constrained problem

- General form

$$\min_{\mathbf{x}\in\Omega} \quad \phi(\mathbf{x}), \quad \text{where}$$
$$\Omega = \{\mathbf{x} \in \mathbb{R}^n | c_i(\mathbf{x}) = 0,\ i \in \mathcal{E}, \quad c_i(\mathbf{x}) \geq 0,\ i \in \mathcal{I}\}.$$

- Equality constraints: reducing space; algebraic; domain Ω has empty interior.
- Inequality constraints: combinatorial; if $\mathcal{E}$ empty then domain Ω can have nonempty interior.
- Active set

$$\mathcal{A}(\mathbf{x}) = \mathcal{E} \cup \{i \in \mathcal{I} | c_i(\mathbf{x}) = 0\}.$$

Consider problems where $\mathcal{A}(\mathbf{x}^*)$ is nonempty.

KKT conditions for a minimum

- Assume constraint qualification.
- Lagrangian
$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \phi(\mathbf{x}) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(\mathbf{x}).$$
- KKT conditions necessary for a minimum:
$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) &= \mathbf{0}, \\ c_i(\mathbf{x}^*) &= 0, \quad \forall i \in \mathcal{E}, \\ c_i(\mathbf{x}^*) &\geq 0, \quad \forall i \in \mathcal{I}, \\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, \\ \lambda_i^* c_i(\mathbf{x}^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}. \end{aligned}$$

Active set methods

- Assuming solution is on $\partial\Omega$, search for the optimum along the boundary.
- For inequality constraints, keep track of $\mathcal{A}(\mathbf{x}_k)$, shuffling constraints in and out of the active set.
- e.g. quadratic programming (QP): quadratic objective function subject to linear inequality constraints.
- Sequential quadratic programming (SQP): At each iteration solve QP for search direction $\mathbf{p}_k$ at $(\mathbf{x}_k, \boldsymbol{\lambda}_k)$

$$\min_{\mathbf{p}} \quad \frac{1}{2}\mathbf{p}^T W_k \mathbf{p} + \nabla_\phi(\mathbf{x}_k)^T \mathbf{p},$$

$$c_i(\mathbf{x}_k) + \nabla \mathbf{c}_i(\mathbf{x}_k)^T \mathbf{p} = 0, \ i \in \mathcal{E}, \quad c_i(\mathbf{x}_k) + \nabla \mathbf{c}_i(\mathbf{x}_k)^T \mathbf{p} \geq 0, \ i \in \mathcal{I}.$$

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Interior point and other methods

- Penalty methods

$$\min_{\mathbf{x}} \psi(\mathbf{x}, \mu) = \phi(\mathbf{x}) + \frac{1}{2\mu} \sum_{i \in \mathcal{E}} c_i^2(\mathbf{x}),$$

where $\mu \downarrow 0$.

- Barrier methods

$$\min_{\mathbf{x}} \psi(\mathbf{x}, \mu) = \phi(\mathbf{x}) - \mu \sum_{i \in \mathcal{I}} \log c_i(\mathbf{x}),$$

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- Augmented Lagrangian

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Given estimates $\boldsymbol{\lambda}_k, \ \mu_k$, solve the unconstrained minimization problem for $\mathbf{x} = \mathbf{x}_{k+1}$, then update the multipliers to $\boldsymbol{\lambda}_{k+1}, \ \mu_{k+1}$.

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