(1)
$$x_1 - x_2 + x_3 - x_4 = 2$$
$$x_1 - x_2 + x_3 + x_4 = 0$$
$$4x_1 - 4x_2 + 4x_3 = 4$$
$$-2x_1 + 2x_2 - 2x_3 + x_4 = -3$$

Solve this system.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & +2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{array}\right] \begin{array}{c} -R_1+R_2 \\ \sim \\ -4R_1+R_3 \\ 2R_1+R_4 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{2}R_2+R_4 \\ \sim \\ -2R_2+R_3 \end{array} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3 \quad x_4$

$$\boxed{x_1 - x_2 + x_3 - x_4 = 2}$$

$x_4 \to -1$; $x_2 \to r$; $x_3 \to s$

$$2x_4 = -2 \checkmark$$
$$0 = 0 \checkmark$$
$$0 = 0 \checkmark$$

$$\begin{cases} x_4 = -1 \\ x_3 = r \\ x_2 = s \\ x_1 = 1 - r + s \end{cases} \qquad t, s \in \mathbb{R}$$

Warning

Suppose we have $x_3 - x_4 = 1$ a single eq. (leading variable)

we'll chose one of x_3 and x_4 as arbitrary

choose $x_4 = r$ as arbitrary and express the leading variable in terms of the other one

$$x_3 = 1 + x_4 = 1 + r.$$

It's possible that we say $x_3 = r$ arbitrary

$$x_4 = x_3 - 1 = r - 1$$

is not wrong, but we don't do that.

$$x_1 - x_2 + x_3 - x_4 = 2$$

$x_1 = r$

$x_2 = s$

$x_3 = 1 + s - r$

correct, but DON'T DO!!

(2) $x+4y-7z=8$ For which values of α

$-x-3y+5z=-6$ does the system has (a) unique

$2x+5y+(\alpha^2-17)z=\alpha+7$ (b) no (c) inf. many sols?

For (a) and (c) find those sols!

$$\left[\begin{array}{ccc|c} 1 & 4 & -7 & 8 \\ -1 & -3 & 5 & -6 \\ 2 & 5 & \alpha^2-17 & \alpha+7 \end{array}\right] \underset{-2R_1+R_3}{\overset{R_1+R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 4 & -7 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & \alpha^2-3 & \alpha-9 \end{array}\right]$$

$$\overset{3R_2+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & -7 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & \alpha^2-9 & \alpha-3 \end{array}\right]$$

Consider first the last line

$$\xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 4 & -7 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & \alpha^2-9 & \alpha-3 \end{array}\right]$$

Consider first the last line

$$0.x_1 + 0.x_2 + (\alpha^2-9).x_3 = \alpha-3$$

$$(\alpha-3)(\alpha+3)\,x_3 = \alpha-3$$

$\alpha = 3$ $\quad 0 = 0$

$\alpha = -3$ $\quad 0 = -6$

$\alpha = 3 \Rightarrow$ (c) inf. many sols.

$\alpha = -3 \Rightarrow$ (b) the system is inconsistent, NO SOLUTION

$\alpha^2 \neq 9 \Rightarrow$ (a) unique solution. ✓

(a) $\alpha^2 - 9 \neq 0$ $\qquad \dfrac{(\alpha^2-9)\,x_3}{\alpha^2-9} = \dfrac{\alpha-3}{\alpha^2-9} \Rightarrow x_3 = \dfrac{1}{a+3}$

$$\left[\begin{array}{ccc|c} 1 & 4 & -7 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & \alpha^2-9 & \alpha-3 \end{array}\right]$$

The second eq. gives

$$x_2 - 2x_3 = 2 \qquad x_3 = \frac{1}{\alpha+3}$$

$$x_2 - 2 \cdot \frac{1}{\alpha+3} = 2$$

$$x_2 = 2\,\frac{\alpha+4}{\alpha+3}, \qquad x_1 = \dots$$

x_1, x_2, x_3 are found uniquely!!

(c) $\alpha = 3$

$$\left[\begin{array}{ccc|c} 1 & 4 & -7 & 8 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 4x_2 - 7x_3 = 8$$

$$\boxed{x_2 - 2x_3 = 2}$$

$$0 = 0$$

$$\boxed{x_3 = s \in \mathbb{R}} \Rightarrow \boxed{x_2 = 2 + 2s} \quad \boxed{x_1 = 8 - 4(2+2s) + 7s}$$

③ $$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\downarrow \qquad\qquad \downarrow$$

$$I \qquad\qquad A^{-1}$$

will appear here

④ Find a 2×2 matrix A such that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\underset{\uparrow B}{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}} \quad \underset{\uparrow C}{\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}} \quad \underset{\uparrow D}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

$BAC = D \qquad A = ? \rightarrow B^{-1}BAC = B^{-1}D \rightarrow AC = B^{-1}D$

$ACC^{-1} = B^{-1}DC^{-1} \Rightarrow A = B^{-1}DC^{-1}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$B^{-1} = \frac{1}{1.1-2.0}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$C^{-1} = \frac{1}{1.1-3.0}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \text{- - - - -}$$

5

$$\begin{vmatrix}1&0&0&0\\2&0&5&0\\3&6&9&8\\4&0&10&7\end{vmatrix} = (-1)^{1+1}\cdot 1\cdot \begin{vmatrix}0&5&0\\6&9&8\\0&10&7\end{vmatrix}$$

$$= (-1)^{1+2}\cdot 5\cdot \begin{vmatrix}6&8\\0&7\end{vmatrix} = -5\cdot 42 = -210$$ //

(6)

Show that
$$\begin{array}{cccc} 11 & 12 & 13 & 14 \end{array}$$
$$\begin{vmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \\ a_3 & b_3 & 0 & 0 \\ a_4 & b_4 & 0 & 0 \end{vmatrix} = (a_1 b_2 - a_2 b_1)(a_3 b_4 - a_4 b_3)$$

$$= (-1)^{1+3} a_1 \begin{vmatrix} 0 & 0 & b_2 \\ a_3 & b_3 & 0 \\ a_4 & b_4 & 0 \end{vmatrix} + (-1)^{1+4} b_1 \begin{vmatrix} 0 & 0 & a_2 \\ a_3 & b_3 & 0 \\ a_4 & b_4 & 0 \end{vmatrix}$$

$$= a_1 . (-1)^{1+3} . b_2 . \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} - b_1 . (-1)^{1+3} . a_2 . \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \quad //$$

(7) Show that $\begin{vmatrix} -x^2 & xy & xz \\ xy & -y^2 & yz \\ xz & yz & -z^2 \end{vmatrix} = 4x^2y^2z^2$

Remember that $\begin{vmatrix} ka & d & g \\ kb & e & h \\ kc & f & i \end{vmatrix} = k \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

$$\begin{vmatrix} -x^2 & xy & xz \\ xy & -y^2 & yz \\ xz & yz & -z^2 \end{vmatrix} = x \begin{vmatrix} -x & xy & xz \\ y & -y^2 & yz \\ z & yz & -z^2 \end{vmatrix}$$

$$= xy \begin{vmatrix} -x & x & xz \\ y & -y & yz \\ z & z & -z^2 \end{vmatrix} = xyz \begin{vmatrix} -x & x & x \\ y & -y & y \\ z & z & -z \end{vmatrix}$$

$$= xyz \begin{vmatrix} -x & x & x \\ y & -y & y \\ z & z & -z \end{vmatrix} = xyz \cdot x \cdot y \cdot z \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$\xrightarrow{R_1 + R_2}$$

$$= x^2 y^2 z^2 \begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = x^2 y^2 z^2 \cdot (-1)^{1+3} \cdot 2 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= x^2 y^2 z^2 \cdot 2 \cdot 2$$

$$= 4 x^2 y^2 z^2$$

$$kR_1 + R_2$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ ka+d & kb+e & kc+f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & ka+b & c \\ d & kd+e & f \\ g & kg+h & i \end{vmatrix}$$

⑧ Show that $\begin{vmatrix} a+d & a-d & g \\ b+e & b-e & h \\ c+f & c-f & k \end{vmatrix} = -2 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix}$

$$\begin{vmatrix} a+d & a-d & g \\ b+e & b-e & h \\ c+f & c-f & k \end{vmatrix} = \begin{vmatrix} a+d & a-d-(a+d) & g \\ b+e & b-e-(b+e) & h \\ c+f & c-f-(c+f) & k \end{vmatrix}$$

$$= \begin{vmatrix} a+d & -2d & g \\ b+e & -2e & h \\ c+f & -2f & k \end{vmatrix} = -2 \begin{vmatrix} a+d & d & g \\ b+e & e & h \\ c+f & f & k \end{vmatrix}$$

$$= -2 \begin{vmatrix} a+d & d & g \\ b+e & e & h \\ c+f & f & k \end{vmatrix} = -2 \begin{vmatrix} a+d-d & d & g \\ b+e-e & e & h \\ c+f-f & f & k \end{vmatrix}$$

$$= -2 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix} \quad //$$

Column operations are allowed only for determinants !!!

9 $\begin{vmatrix} x & y+z & y^2+z^2 \\ y & x+z & x^2+z^2 \\ z & x+y & x^2+y^2 \end{vmatrix} = (x+y+z)(x-y)(x-z)(x-y)$ Show that

$$\begin{array}{c} -R_1+R_2 \\ = \\ -R_1+R_3 \end{array} \begin{vmatrix} x & y+z & y^2+z^2 \\ y-x & x-y & x^2-y^2 \\ z-x & x-z & x^2-z^2 \end{vmatrix}$$

$$= (x-y)(x-z) \begin{vmatrix} x & y+z & y^2+z^2 \\ -1 & 1 & x+y \\ -1 & 1 & x+z \end{vmatrix}$$

$$\overset{-R_2+R_3}{=} \begin{vmatrix} x & y+z & y^2+z^2 \\ -1 & 1 & x+y \\ 0 & 0 & z-y \end{vmatrix} (x-y)(x-z)$$

$$= (x-y)(x-z) \cdot (-1)^{3+3} \cdot (z-y) \cdot \begin{vmatrix} x & y+z \\ -1 & 1 \end{vmatrix}$$

$$= (x-y)(x-z)(z-y)(x+y+z) \quad //$$

(10)

$$A = \begin{vmatrix} 2 & 5 & 3 & 4 \\ -1 & -2 & -2 & -3 \\ 2 & 6 & 4 & 4 \\ 1 & 3 & 8 & 9 \end{vmatrix}$$

(a) Evaluate $|A|$

(b) Find $|A^T|$, $|A^{-1}|$, $|A^4|$, $|2A|$, $|\text{adj}\, A|$.

$|kA| = k^n |A|$

$|AB| = |A||B|$

(a) $|A| = 2 \rightarrow$ Verify this yourself.

(b). $|A^T| = |A| = 2$

$\cdot\ |2A| = 2^4 |A| = 2^4 \cdot 2$

$\cdot\ |A^{-1}| = \dfrac{1}{|A|} = \dfrac{1}{2}$

$\cdot\ |\text{adj}\, A| = 2^{n-1} \cdot |A| = 2^{4-1} \cdot 2$

$A\,(\text{adj}\, A) = |A|\, I$

$\downarrow$

$|\text{adj}\, A| = |A|^{n-1}$

$|A^4| = |AAAA| = |A|^4$

$= 2^4$

(11) Show that the homogeneous system

$$Ax = 0$$

has only the trivial solution for $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

If A is invertible,

$$Ax = 0$$

$$\underset{\sim}{A}^{-1} \underset{\sim}{A} \underset{\sim}{x} = \underset{\sim}{A}^{-1} \underset{\sim}{0}$$

$$\underset{\sim}{I} \underset{\sim}{x} = \underset{\sim}{0}$$

$$\underset{\sim}{x} = \underset{\sim}{0}$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We need to show that A is invertible.

A is invertible $\Longleftrightarrow$ $\det A \neq 0$.

$$|A| = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} = 2 \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -2 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-2) \cdot 1 \cdot (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (-2) \cdot 1 \cdot 1 \cdot (-1) = 2$$

$|A| = 2 \neq 0 \Rightarrow A^{-1}$ exists

$\Rightarrow \underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}$ has only the trivial solution

$\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}$

$\underbrace{A^{-1}A}_{I} X = \underset{\sim}{A}^{-1}\underset{\sim}{0}$

$I\, x = \underset{\sim}{0}$

$\underset{\sim}{X} = \underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\underset{\sim}{x} = \underline{\underline{0}}$

$X_1 = X_2 = X_3 = X_4 = 0$ //