From our book:

3.1 Intro. to linear systems ✓

3.2 Matrices & Gaussian Elimination ✓

3.3 Reduced Row-Echelon Matrices

} · Gaussian elimination
· Gauss-Jordan elimination

Today

3.4 Matrix Operations

3.5 Inverses of Matrices

3.6 Determinants

Subsection 3.4 MATRIX OPERATIONS

$A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$

① Equality: $a_{ij}=b_{ij} \Rightarrow A=B$

② Sum and Difference: $A \mp B = [a_{ij} \mp b_{ij}]_{m\times n}$

③ Scalar Multiplication: $cA=[ca_{ij}]_{m\times n}$, c: scalar

④ Matrix Multiplication

$A=[a_{ij}]_{m\times p}$, $B=[b_{ij}]_{p\times n}$, $C=[c_{ij}]_{m\times n}$

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ip}b_{pj}=\sum_{k=1}^{p} a_{ik}b_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} \bigcirc & \bigcirc & & \bigcirc \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

⑤ Vector Multiplication

$$A=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1} = (a_1, a_2, \ldots, a_n)$$ column vector (n vector)

$B=[b_1\ b_2 \cdots b_n]$ row vector

$$A\cdot B = a_1b_1+a_2b_2+\cdots+a_nb_n$$

$A = [a_{ij}]_{m \times n}$ m: number of rows

n: " of columns

$B = [b_{ij}]_{m \times n}$

$C = A + B = [c_{ij}] = [a_{ij} + b_{ij}]_{m \times n}$

Ex. $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 & 6 \\ 9 & 0 & -2 \end{bmatrix}$

$A + B = \begin{bmatrix} 3+4 & 0-3 & -1+6 \\ 2+9 & -7+0 & 5-2 \end{bmatrix}$

$D = \begin{bmatrix} 3 & -2 \\ -1 & 6 \end{bmatrix}$ $A + D = ??$ is not defined as the dimensions are not the same!!

$$A = [a_{ij}]_{m \times n}, \qquad c \in \mathbb{R}$$

$$c\underset{\sim}{A} = [c\,a_{ij}]_{m \times n}$$

↑ just to stress that this quantity/object is a MATRIX

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad 2\underset{\sim}{A} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

↑ Multiplication of a matrix by a scalar

↑ NOT matrix multiplication!!!

"Vectors"

A vector is an $n \times 1$ matrix, in the form

$$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n \times 1} = (a_1, a_2, \ldots, a_n)$$

$n \times 1$ matrix

Please do not confuse $\underset{\sim}{a} = (a_1, \ldots, a_n)$

with the matrix $\underset{\sim}{a} = [a_1 \quad a_2 \quad \ldots \quad a_n]$

They're different objects!

$1 \times n$ matrix

Example Consider the hom. system

$$x_1 + 3x_2 - 15x_3 + 7x_4 = 0$$
$$x_1 + 4x_2 - 19x_3 + 10x_4 = 0$$
$$2x_1 + 5x_2 - 26x_3 + 11x_4 = 0$$

The reduced echelon form corresponding to this hom. system is

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \text{RHS} \end{array} \end{array}$$

x_1, x_2 : are leading variables

x_3, x_4 : free variables

$x_1 = 3s + 2t$ $\qquad x_3 = s$

$x_2 = 4s - 3t$ $\qquad x_4 = t$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix}$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3s \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 2t \\ -3t \\ 0 \\ t \end{bmatrix}$$

$$= s \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\underset{\sim}{x}_1} + t \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\underset{\sim}{x}_2}$$

$$= s\,(3, 4, 1, 0) + t(2, -3, 0, 1)$$

$$= s\,\underset{\sim}{x}_1 + t\,\underset{\sim}{x}_2$$

Matrix Multiplication

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}, \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix}_{2\times 3}$$

$$\begin{array}{c|c} & \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} -1 & 6 & -2 \\ -1 & 12 & -2 \end{bmatrix}_{2\times 3} \end{array}$$

$\uparrow$ AB

$1.1 + 2.(-1) = -1$

$1.0 + 2.3 = 6$

$1.2 + 2.(-2) = -$

$3.1 + 4.(-1) = -1$

$3.0 + 4.3 = 12$

$3.2 + 4.(-2) = -2$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}, \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix}_{2\times 3}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} -1 & 6 & -2 \\ -1 & 12 & -2 \end{bmatrix}_{2\times 3}$$

$$1.1 + 2.(-1) = -1$$

$$1.0 + 2.3 = 6$$

$$1.2 + 2.(-2) = -2$$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$, $\quad B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix}_{2\times3}$

<u>Question</u> Can we calculate BA ??

$B = [b_{ij}]_{2\times \underbrace{3}}$, $A = [a_{ij}]_{\underbrace{2}\times 2}$

$$3 \neq 2$$

$\Rightarrow$ BA is not defined !!

$A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{r \times s}$

In order that AB is defined $\Rightarrow$ $n = r$

In order that BA is defined $\Rightarrow$ $s = m$

For two matrices A and B both AB and BA are defined (calculable) if

$A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times m}$

Warning

$$C = A_{m\times p} \, B_{p\times n} = (AB)_{m\times n}$$

as if these "cancel".

Question Why to multiply two matrices in such an interesting way??

Answer: The motivation is again the linear system of eqs!!!

Example

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 2x_1 - 3x_2 + 5x_3 &= 0 \\ -3x_1 + 4x_2 - 2x_3 &= 7 \end{aligned}\right\}$$

Coef. matrix

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ -3 & 4 & -2 \end{bmatrix}_{\underline{\underline{3\times 3}}}$$

RHS

$$\begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}_{3\times 1}$$

and the unknowns can be collected in a vector $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{\underline{3\times 1}}$.

Now see the following:

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ -3 & 4 & -2 \end{bmatrix}_{3\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1.x_1 - 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 5x_3 \\ -3x_1 + 4x_2 - 2x \end{bmatrix}_{3\times 1} = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}$$

$\uparrow$ = $\qquad$ = $\uparrow$

Example

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 2x_1 - 3x_2 + 5x_3 &= 0 \\ -3x_1 + 4x_2 - 2x_3 &= 7 \end{aligned}\right\}$$

Based on the def. of matrix multiplication, this linear system can be expressed using matrices as

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ -3 & 4 & -2 \end{bmatrix}_{3\times 3}}_{\underset{\sim}{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}}_{\underset{\sim}{x}} = \begin{bmatrix} 1.x_1 - 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 5x_3 \\ -3x_1 + 4x_2 - 2x \end{bmatrix}_{3\times 1} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}}_{\underset{\sim}{b}}$$

$$\Rightarrow \underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

$$a_{11}\,\underline{\underline{x_1}} + \dots + a_{1\,10^6}\,\underline{\underline{x_{10^6}}} = b_1$$

$$\vdots$$

$$a_{10^6\,1}\,x_1 + \dots + a_{10^6\,10^6}\,x_{10^6} = b_{10^6}$$

10^6 unknowns, 10^6 eqs!!!

$$\Rightarrow \quad \underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b} \quad !!!$$

compact notation + useful for calculating in computers!!!

In the general case; a linear system of m eqs. in n unknowns;

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

is equivalent to the matrix equation

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}}_{\underset{\sim}{A}} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{\underset{\sim}{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{\underset{\sim}{b}}$$

Ex

$$
\begin{aligned}
3x_1 - 4x_2 + x_3 + 7x_4 &= 10 \\
4x_1 \qquad\quad - 5x_3 + 2x_4 &= 0 \\
x_1 + 9x_2 + 2x_3 - 6x_4 &= 5
\end{aligned}
$$

unknowns: $x_1, x_2, x_3, x_4 \Rightarrow n = 4$

# of eqs $= 3 \Rightarrow m = 3$

$$
\begin{bmatrix} 3 & -4 & 1 & 7 \\ 4 & 0 & -5 & 2 \\ 1 & 9 & 2 & -6 \end{bmatrix}_{3\times 4}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1}
=
\begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}_{3\times 1}
$$

RULES FOR MATRIX ALGEBRA (Rules for Matrix operations)

① Commutative law of addition $A+B=B+A$ ✓

② Associative law of addition $A+(B+C)=(A+B)+C$

③ Associative law of multiplication $A.(B.C)=(A.B)C$

④ Distributive laws $A(B+C)=AB+AC$

$(A+B)C=AC+BC$

★★ In general $A.B \neq B.A$

⑤ Zero matrix : All elements are all zero

$\underset{\sim}{0}+A=A+\underset{\sim}{0}=A$, $\underset{\sim}{A}.\underset{\sim}{0}=\underset{\sim}{0}.\underset{\sim}{A}=\underset{\sim}{0}$

$$\underset{\sim}{0}=\begin{bmatrix} 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdots & \end{bmatrix}_{m\times n}$$

⑥ Identity matrix: A square matrix with a principal diagonal of 1s and 0s elsewhere.

$$I=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ 0 & 0 & & \ddots & \\ 0 & \cdots & & 0 & 1 \end{bmatrix}_{n\times n}$$

$\underline{AI}=\underline{IA}=\underline{\underline{A}}$

A is a square matrix if # of rows = # of columns

Proof of 4: $A=[a_{ij}]_{m\times p}$, $B=[b_{je}]_{p\times n}$, $C=[c_{je}]_{p\times n}$

$i=1,2,\ldots,m$, $j=1,2,\ldots,p$, $e=1,2,\ldots,n$

$$A.(B+C)=A([b_{je}]+[c_{je}])=[a_{ij}].[b_{je}+c_{je}]$$

$$=\left[\sum_{k=1}^{p} a_{ik}(b_{ke}+c_{ke})\right]$$

$$=\left[\sum_{k=1}^{p} a_{ik}b_{ke}\right]+\left[\sum_{k=1}^{p} a_{ik}c_{ke}\right]$$

$$=A.B+A.C$$

DIAGONAL MATRIX

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}_{n \times n}$$

A square matrix such that every element off the main diagonal is zero.

* Identity matrix (I) is a diagonal matrix

TRACE OF A MATRIX

The sum of the elements on the principal diagonal of a square matrix

INVERSE OF A MATRIX

A: a nonzero square matrix

$AB = BA = I \Rightarrow A$ is called invertible and $B = A^{-1}$

THEOREM

If the matrix A is invertible, then there exists precisely one matrix B such that $AB = BA = I$.

Proof: $B = A^{-1} \Rightarrow AB = BA = I$

Let's assume that $C = A^{-1} \Rightarrow AC = CA = I$

$$C = CI = C(AB) = (CA).B = IB = B \Rightarrow \underline{\underline{C = B}}$$

$$A = [a_{ij}]_{n \times n} \qquad \text{Trace } A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} & \dots & a_{1n} \\ a_{21} & \boxed{a_{22}} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & \boxed{a_{nn}} \end{bmatrix}$$

3.5 Inverses of Matrices (***)

$n \times n$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n}$$

Def. The square matrix A is invertible if there exists a matrix B such that

$$AB = BA = I$$

A matrix B satisfying these two properties is called an inverse matrix of A, and denoted $B = A^{-1}$ (not $\frac{1}{A}$!!!)

See the next example:

* Inverse is defined for only square matrices!!! (with some extra cond. we'll introduce later)

Ex

$$A = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$4 \cdot \frac{3}{2} + 6 \cdot \left(-\frac{5}{6}\right) = 1$$

$$4 \cdot (-1) + 6 \cdot \frac{2}{3} = 0$$

$$5 \cdot \frac{3}{2} + 9 \cdot \left(-\frac{5}{6}\right) = 0$$

$$5 \cdot (-1) + 9 \cdot \frac{2}{3} = 1$$

$AB = I$

we need to check also $BA \overset{?}{=} I$

Ex

$$A = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{3}{2} & -1 \\ \frac{-5}{6} & \frac{2}{3} \end{bmatrix}$$

$$BA = \begin{bmatrix} 3/2 & -1 \\ -5/6 & 2/3 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$AB = I \quad \wedge \quad BA = I \quad \Rightarrow \quad B = A^{-1}$$

Theorem If A^{-1} exists, it is unique.

Proof Suppose A has two inverses, say B and C.

Then $AB = BA = I$ & $AC = CA = I$.

$$C = CI = C(AB) = (CA)B = (I)B = B$$

Theorem Inverses of 2×2 matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex $A = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}$

$$A^{-1} = \frac{1}{4.9-5.6}\begin{bmatrix} 9 & -6 \\ -5 & 4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 9 & -6 \\ -5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9/6 & -6/6 \\ -5/6 & 4/6 \end{bmatrix} = \begin{bmatrix} 3/2 & -1 \\ -5/6 & 2/3 \end{bmatrix}$$

Proof

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \begin{bmatrix} x & y \\ z & m \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} ax + bz & ay + bm \\ cx + dz & cy + dm \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$ax + bz = 1$ 4 eqs;

$ay + bm = 0$ 4 unknowns!!

$cx + dz = 0$ and $cy + dm = 1$ (x, y, z, m)

INVERSE OF A 2X2 MATRIX

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc \neq 0$ and then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

POWERS OF A SQUARE MATRIX

$A^0 = I$ (def.)

$A^1 = A$ (def.)

$A^{n+1} = A^n A$ for $n \geqslant 1$

$A^{-n} = (A^{-1})^n$

$A^r . A^s = A^{r+s}$

$(A^r)^s = A^{rs}$

ALGEBRA OF INVERSE MATRICES

A, B: Invertible matrices of the same size

* A^{-1} is invertible and $(A^{-1})^{-1} = A$
* A^n is invertible where n is a nonnegative integer and $(A^n)^{-1} = (A^{-1})^n$
* $A.B$ is invertible and $(A.B)^{-1} = B^{-1}A^{-1}$

Proof: $(AB).(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1}).(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Theorem If A & B are invertible (that is, A^{-1} and B^{-1} exist)

(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

$$AA^{-1} = A^{-1}A = I$$

(b) A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$

$$A^n (A^{-1})^n = (A \cdot A \cdot \ldots \cdot A)(A^{-1} A^{-1} \ldots A^{-1}) = I$$

$$(A^{-1})^n A^n = (A^{-1} A^{-1} \ldots A^{-1})(A A \ldots A) =$$

(c) The product AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I$$

$$\Rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$$

Th 4 Inverse Matrix Solution of $Ax = b$

If A is an $n \times n$ invertible matrix, then for any vector $\underset{\sim}{b}$ the system

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

has the unique solution

$$\underset{\sim}{x} = \underset{\sim}{A}^{-1}\,\underset{\sim}{b}.$$

Proof $A x = b \xrightarrow[\text{exists}]{A^{-1}} A^{-1} A x = A^{-1} b$

$$\underset{\sim}{I}\,\underset{\sim}{x} = A^{-1}\,\underset{\sim}{b}$$

$$\underset{\sim}{x} = \underset{\sim}{A}^{-1}\,\underset{\sim}{b} \quad //$$

$A = [a_{ij}]_{n \times n}$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

☆☆

Example

Ex $\left.\begin{aligned} 4x_1+6x_2&=6\\ 5x_1+9x_2&=18 \end{aligned}\right\}$ To find the solution of the lin. system

$$\underbrace{\begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 18 \end{bmatrix}}_{b} \Rightarrow A^{-1} = \frac{1}{6}\begin{bmatrix} 9 & -6 \\ -5 & 4 \end{bmatrix} \quad \text{since } 4\cdot 9-6\cdot 5=6\neq 0$$

$$X = A^{-1}\cdot b = \begin{bmatrix} 3/2 & -1 \\ -5/6 & 2/3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\cdot 6-1\cdot 18 \\ -\frac{5}{6}\cdot 6+\frac{2}{3}\cdot 18 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}$$

$\Rightarrow x_1=-9,\ x_2=7$ is the unique solution.

In what follows, we'll develop a method for finding inverses of (larger) square matrices.

Def. Elementary Matrix The $n \times n$ matrix E is called an elementary matrix if it can be obtained from the $n \times n$ identity matrix I by a single elementary row operation.

Elementary row ops : cR_i : multiplying R_i by c

$kR_i + R_j \rightarrow R_j$:

$R_i \leftrightarrow R_j$: switch/swap R_i R_j

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = E_1$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E_2$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Theorem Elementary Matrices & Row Ops.

If an elementary row operation is performed on the $m \times n$ matrix A, then the result is the product matrix EA, where E is the elementary matrix obtained by performing the same op on $I_{m \times m}$.

Ex

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 6 & 9 \\ 4 & 1 \end{bmatrix} = A^*$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = E$$

$$EA = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 4 & 1 \end{bmatrix} = A^*$$

Ex.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 5 & 7 \end{bmatrix} = A^*$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

$$= A^*$$

Performing elementary row ops $=$ multiplying by the corres-ponding E from the left.

Theorem Invertible matrices and elementary row ops.

The $n \times n$ matrix A is invertible if and only if it is row equivalent to the $n \times n$ identity matrix I.

Proof Let's see an example first.

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}$$

Q: Is A row equivalent to $I_{3\times3}$??

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow[E_1]{-R_3+R_1} \begin{bmatrix} 1 & -2 & 0 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow[E_2]{-5R_1+R_2}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow[E_2]{-5R_1+R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 16 & 3 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow[E_3]{-3R_1+R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 16 & 3 \\ 0 & 11 & 2 \end{bmatrix}$$

$$\xrightarrow[E_4]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 11 & 2 \\ 0 & 16 & 3 \end{bmatrix} \xrightarrow[E_5]{-R_2+R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 11 & 2 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\xrightarrow[E_6]{2R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 11 & 2 \\ 0 & 10 & 2 \end{bmatrix} \xrightarrow[E_7]{-R_3+R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 2 \end{bmatrix}$$

$$\xrightarrow[E_8]{-10R_2+R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[E_9]{\frac{1}{2}R_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{10}]{2R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Question: SO WHAT ??

$$E_{10} E_9 \ldots E_3 E_2 E_1 \quad A = I$$

$$A^{-1} = E_{10} E_9 \cdots E_3 E_2 E_1$$

or, equivalently,

$$A^{-1} = E_{10} E_9 \cdots E_3 E_2 E_1 I \quad //$$

Example Find the inverse of $A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}$.

Solution

$$\left[\begin{array}{ccc|ccc} \overbrace{\begin{matrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{matrix}}^{A} & \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}^{I} \end{array}\right]$$

convert this to I by elementary row ops

perform the same ops here; at the end, A^{-1} will appear here. why??

$$E_{10} E_9 \cdots E_2 E_1 A = I \longrightarrow \text{this will be done on the left}$$

$$E_{10} E_9 \cdots E_2 E_1 I = A^{-1} \longrightarrow \text{this will appear on the right}$$

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3]{-5R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 16 & 3 & -5 & 1 & 5 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 11 & 2 & -3 & 0 & 4 \\ 0 & 16 & 3 & -5 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 11 & 2 & -3 & 0 & 4 \\ 0 & 5 & 1 & -2 & 1 & 1 \end{array}\right] \xrightarrow{-2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 5 & 1 & -2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[2R_2+R_1]{-5R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right] \Rightarrow A^{-1} = \left[\begin{array}{ccc} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{array}\right] //$$

Matrix Equations

Suppose we have a number of linear systems in the form

$$A x_1 = b_1 , \quad A X_2 = b_2 , \quad \ldots , \quad A X_n = b_n$$

For all of them: the coeff. matrix is the same: A

the RHS b is differing !!

so, each time the unknown will be a different matrix $\underset{\sim}{x}$.

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} \\ X \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & | & & | \\ \underset{\sim}{b_1} & b_2 & b_3 & \cdots & b_n \\ | & | & | & & | \end{bmatrix}}_{\to B}$$

$$\underline{\underset{\sim}{X} = A^{-1} B}$$

THEOREM

If the $n \times n$ matrix A is invertible, then for any n vector b the system $Ax=b$ has the unique solution $x = A^{-1}.b$.

ELEMENTARY MATRIX

The $n \times n$ matrix E is called an elementary matrix if it can be obtained by performing a single elementary row operation on the $n \times n$ identity matrix I.

THEOREM

If an elementary row operation is performed on the $m \times n$ matrix A, then the result is the product matrix EA where E is the elementary matrix obtained by performing the same row operation on the $m \times m$ identity matrix

Elementary row operations are reversible. That is, to every elementary row operation there corresponds an inverse elementary row operation that cancels the effects

Elementary Row Operation	Inverse Operation
$(c).R_i$	$\frac{1}{c}.R_i$
$R_i \leftrightarrow R_j$	$R_i \leftrightarrow R_j$
$(c).R_i + R_j$.	$(-c)R_i + R_j$

This means that every elementary matrix is invertible.

E: el. matrix obtained by el. row op. from I

E^*: el. matrix obtained by inverse el. row op. from I

$$\Rightarrow E E^* = E^* E = I$$

$$\Rightarrow E^* = E^{-1}$$

THEOREM The $n \times n$ matrix A is invertible iff it is row equivalent to the $n \times n$ identity matrix I.

A is invertible $\Longleftrightarrow$ $A \underset{row}{\sim} I$

Proof: $* \Rightarrow$ Let A be an invertible matrix.

We showed before that if A is invertible, then the homog. system $AX=0$ has only the trivial solution $x=0$.

$$AX=0 \Rightarrow A^{-1}AX = A^{-1}0 \Rightarrow X=0$$

We also know that a homog system with coeff. matrix A has only the trivial solution iff A is row equivalent to I.

$* \Leftarrow$ Let A be row eq. to I.

This means that there is a sequence of elementary row operations that transforms A into I.

$$\Rightarrow E_k E_{k-1} \dots E_2 E_1 A = I. \qquad (E_k\text{s have inverse matrices})$$

$$\Rightarrow A = (E_1)^{-1}(E_2)^{-1} \dots (E_k)^{-1}$$

Since A is a product of invertible matrices, then A is invertible.
(If A and B is invertible, then $A.B$ is invertible)

FINDING A^{-1} $\quad [A \mid I] \xrightarrow[\text{op.}]{\text{Elem. row}} [I \mid A^{-1}]$

Pay attention that when applying elem. row operations if a row appears with zeros of all entries, then A can't be row eq. to I. Thus A is not invertible.

MATRIX EQUATIONS $\quad AX=B$

If A is invertible, then $A^{-1}AX = A^{-1}.B \Rightarrow X = A^{-1}B$

NONSINGULAR MATRIX If A is invertible, then A is called a nonsingular matrix. Otherwise, it is called singular.

A^{-1} does not exist : A is called singular

Properties of nonsingular matrices

THEOREM Let A be an $n \times n$ matrix. Then, the following properties are equivalent.

$1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$

Think about these!!!

1. A is invertible.
2. A is row equivalent to the $n \times n$ identity matrix I.
3. $Ax = 0$ has only the trivial solution.
4. For every n-vector b, $Ax = b$ has a unique solution.
5. For every n-vector b, $Ax = b$ is consistent.

Proof: We already know that if A is invertible, then A is row equivalent to I; and if so, $Ax = 0$ has only the trivial solution. $(1 \to 2 \to 3)$

If $Ax = 0$ has only the trivial solution, then A is invertible and if so, then $Ax = b$ has a unique solution. $(3 \to 4)$

If $Ax = b$ has a unique solution, then the system is consistent. $(4 \to 5)$

Now, Let's assume that $Ax = b$ is consistent.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix} = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} e_{1j} \\ e_{2j} \\ \vdots \\ e_{nj} \end{bmatrix}$$

Let's choose the jth column of b as the jth column of I

jth column of x and b

Then $a_{11}x_{1j} + a_{12}x_{2j} + \cdots + a_{1n}x_{nj} = e_{1j}$

$\cdots$

$a_{n1}x_{1j} + a_{n2}x_{2j} + \cdots + a_{nn}x_{nj} = e_{nj}$

$\Rightarrow$ Let's define $B = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$

$$AB = \begin{bmatrix} a_{11}x_{11} + a_{12}x_{21} + \cdots + a_{1n}x_{n1} & \cdots & a_{11}x_{1n} + a_{12}x_{2n} + \cdots + a_{1n}x_{nn} \\ \vdots & & \\ a_{n1}x_{11} + a_{n2}x_{21} + \cdots + a_{nn}x_{n1} & & a_{n1}x_{1n} + a_{n2}x_{2n} + \cdots + a_{nn}x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} & \cdots & e_{1n} \\ e_{21} & \cdots & e_{2n} \\ \vdots & & \\ e_{n1} & \cdots & e_{nn} \end{bmatrix} = I \Rightarrow$$

$AB = I \Rightarrow (AB)x = A(Bx) = x$

$Bx = 0 \Rightarrow x = 0 \Rightarrow$ only a trivial sol.

$\Rightarrow B$ is invertible

$AB = I \Rightarrow ABB^{-1} = B^{-1} \Rightarrow A = B^{-1} \Rightarrow A$: invertible

Ex $A=\begin{bmatrix}6 & -3 & -4\\ 5 & 2 & -1\\ 0 & 7 & 9\end{bmatrix}_{3x3}$, $B=\begin{bmatrix}-6 & 1 & 1\\ 0 & 2 & -2\\ 1 & -3 & 5\end{bmatrix}_{3x3}$, $C=\begin{bmatrix}-1 & 0\\ 3 & -2\\ 1 & 1\end{bmatrix}_{3x2}$

* $A+B=\begin{bmatrix}6-6 & -3+1 & -4+1\\ 5+0 & 2+2 & -1-2\\ 0+1 & 7-3 & 9+5\end{bmatrix}=\begin{bmatrix}0 & -2 & -3\\ 5 & 4 & -3\\ 1 & 4 & 14\end{bmatrix}$

* $A-B=\begin{bmatrix}6+6 & -3-1 & -4-1\\ 5-0 & 2-2 & -1+2\\ 0-1 & 7+3 & 9-5\end{bmatrix}=\begin{bmatrix}12 & -4 & -5\\ 5 & 0 & 1\\ -1 & 10 & 4\end{bmatrix}$

* $2A=\begin{bmatrix}2.6 & 2.(-3) & 2.(-4)\\ 2.5 & 2.2 & 2.(-1)\\ 2.0 & 2.7 & 2.9\end{bmatrix}=\begin{bmatrix}12 & -6 & -8\\ 10 & 4 & -2\\ 0 & 14 & 18\end{bmatrix}$

* $A.C=\begin{bmatrix}6.(-1)-3.3-4.1 & 6.0-3.(-2)-4.1\\ 5.(-1)+2.3-1.1 & 5.0+2(-2)-1.1\\ 0.(-1)+7.3+9.1 & 0.0+7(-2)+9.1\end{bmatrix}=\begin{bmatrix}-19 & 2\\ 0 & -5\\ 30 & -5\end{bmatrix}_{3x2}$

You can not calculate CA since the dimensions are inappropriate.

* $A.B=\begin{bmatrix}6(-6)-3.0-4.1 & 6.1-3.2-4(-3) & 6.1-3(-2)-4.5\\ 5(-6)+2.0-1.1 & 5.1+2.2-1(-3) & 5.1+2(-2)-1.5\\ 0(-6)+7.0+9.1 & 0.1+7.2+9(-3) & 0.1+7(-2)+9.5\end{bmatrix}$

$=\begin{bmatrix}-40 & 12 & -8\\ -31 & 12 & -4\\ 9 & -13 & 31\end{bmatrix}$

$B.A=\begin{bmatrix}-6.6+1.5+1.0 & -6(-3)+1.2+1.7 & \overbrace{-6(-4)+1(-1)+1.9}^{=32}\\ 0.6+2.5-2.0 & 0(-3)+2.2-2.7 & 0(-4)+2(-1)-2.9\\ 1.6-3.5+5.0 & 1(-3)-3.2+5.7 & 1(-4)-3(-1)+5.9\end{bmatrix}\neq AB$

* Trace $A = 6+2+9 = 17$

Ex $A=\begin{bmatrix}1 & 3\\ -2 & 2\end{bmatrix}$, $B=\begin{bmatrix}1/4 & -3/8\\ 1/4 & 1/8\end{bmatrix}$

$$AB=\begin{bmatrix}1(1/4)+3.(1/4) & 1(-3/8)+3(1/8)\\ -2(1/4)+2.(1/4) & -2.(-3/8)+2(1/8)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=B.A \Rightarrow B=A^{-1}$$

$$A^{-1}=\frac{1}{2-3(-2)}\begin{bmatrix}2 & -3\\ 2 & 1\end{bmatrix} \qquad (ad-bc=8\neq 0)$$

Ex: $A=\begin{bmatrix}1 & -3\\ -2 & 6\end{bmatrix}$, $B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$$A.B=I \Rightarrow \begin{bmatrix}a-3c & b-3d\\ -2a+6c & -2b+6d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow \begin{matrix}a-3c=1\\ -2a+6c=0 \Rightarrow a-3c=0\end{matrix}$$

$\Rightarrow$ inconsistent $\Rightarrow$ no solution $\Rightarrow$ inverse does not exist

$(ad-bc=1.6-(-3).(-2)=0)$

Ex $\left.\begin{matrix}4x_1+6x_2=6\\ 5x_1+9x_2=18\end{matrix}\right\}$ To find the solution of the Lin. system

$$\underbrace{\begin{bmatrix}4 & 6\\ 5 & 9\end{bmatrix}}_{A}\cdot\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}6\\ 18\end{bmatrix}}_{b} \Rightarrow A^{-1}=\frac{1}{6}\begin{bmatrix}9 & -6\\ -5 & 4\end{bmatrix} \quad \text{since } 4.9-6.5=6\neq 0$$

$$x=A^{-1}.b=\begin{bmatrix}3/2 & -1\\ -5/6 & 2/3\end{bmatrix}\cdot\begin{bmatrix}6\\ 18\end{bmatrix}=\begin{bmatrix}\frac{3}{2}.6-1.18\\ -\frac{5}{6}.6+\frac{2}{3}.18\end{bmatrix}=\begin{bmatrix}-9\\ 7\end{bmatrix}$$

$\Rightarrow x_1=-9,\ x_2=7$ is the unique solution

Ex

$$I=\begin{bmatrix}1&0\\0&1\end{bmatrix}\xrightarrow{2R_1\to R_1}\begin{bmatrix}2&0\\0&1\end{bmatrix}=E_1\xrightarrow{\frac{1}{2}R_1\to R_1}\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$$

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow[\to R_3]{3R_1+R_3}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}=E_2\xrightarrow[\to R_3]{-3R_1+R_3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=E_3\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

E_1, E_2 and E_3 are elementary matrices since they are obtained from identity matrices by elementary row operations

Ex: $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}\xrightarrow{2R_1\to R_1}\begin{bmatrix}2&0\\0&1\end{bmatrix}=E_1$, $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}\xrightarrow{\frac{1}{2}R_1\to R_1}\begin{bmatrix}1/2&0\\0&1\end{bmatrix}=E_1^*$

$A=\begin{bmatrix}1&2&5\\-1&3&1\end{bmatrix}\xrightarrow{2R_1\to R_1}\begin{bmatrix}2&4&10\\-1&3&1\end{bmatrix}=E_1\cdot A$, $E_1\cdot E_1^*=E_1^*E=I\Rightarrow E_1^*=E_1^{-1}$

$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{3R_2+R_1\to R_1}\begin{bmatrix}1&3&0\\0&1&0\\0&0&1\end{bmatrix}=E_2$, $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow[\to R_1]{-3R_2+R_1}\begin{bmatrix}1&-3&0\\0&1&0\\0&0&1\end{bmatrix}=E_2^*$

$B=\begin{bmatrix}2&-1&0\\1&3&2\\-2&1&1\end{bmatrix}\xrightarrow{3R_2+R_1\to R_1}\begin{bmatrix}5&8&6\\1&3&2\\2&1&1\end{bmatrix}=E_2.B$, $E_2.E_2^*=E_2^*E_2=I\Rightarrow E_2^*=E_2^{-1}$

Ex: $A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \Rightarrow A^{-1} = ?$

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-3R_1+R_3]{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 & 1 & -1 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 2 \end{array}\right] \xrightarrow{R_2 \Leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 5 & 1 & -2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[-5R_2+R_3]{2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix}$$

$$\underbrace{\begin{matrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{matrix}}_{=A^{-1}}$$

Ex: $\underbrace{\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}}_{A} \cdot \underset{\sim}{X} = \underbrace{\begin{bmatrix} 3 & -1 & 2 & 6 \\ 7 & 4 & 1 & 5 \\ 5 & 2 & 4 & 1 \end{bmatrix}}_{\underset{\sim}{B}} \Rightarrow \underset{\sim}{X} = ?$

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} & \cdots\cdot & \end{bmatrix} \begin{bmatrix} \cdot\cdot \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} \cdot B = \begin{array}{cccc} & & & \\ \left[\begin{matrix} -4 \\ -1 \\ +11 \end{matrix}\right. & \begin{matrix} -13 \\ -5 \\ 33 \end{matrix} & \begin{matrix} 14 \\ 8 \\ -39 \end{matrix} & \left.\begin{matrix} 1 \\ -2 \\ 4 \end{matrix}\right] \\ \underset{\sim}{X_1} & \underset{\sim}{X_2} & & \end{array}$$