26.12.2020, Weekend session

PS on Ch 5, Higher-Order DEs; Problems from Exercises 5.5, page 351.

(1) Find a particular solution to the following DEs.

Pr. 6 $\quad 2y'' + 4y' + 7y = x^2$

(1) $\quad 2y'' + 4y' + 7y = 0 \qquad y = e^{rx}$

$$2r^2 + 4r + 7 = 0$$

$$r_{1,2} = \frac{-4 \mp \sqrt{4^2 - 4.2.7}}{2.2} = \frac{-4 \mp \sqrt{-40}}{4}$$

$$= \frac{-4 \mp \sqrt{40 i^2}}{4} = \frac{-4 \mp 2\sqrt{10}\, i}{4} = -1 \mp \frac{\sqrt{10}}{2} i$$

$$y = e^{\alpha x}\left[ C_1 \cos \beta x + C_2 \sin \beta x \right] \qquad r = \alpha \mp \beta i$$

$$\alpha = -1, \quad \beta = \frac{\sqrt{10}}{2} \qquad y_c = e^{-1.x}\left[ C_1 \cos\left(\frac{\sqrt{10}}{2} x\right) + C_2 \sin\left(\frac{\sqrt{10}}{2} x\right) \right]$$

ii $\quad 2y'' + 4y' + 7y = x^2$

$$\boxed{y_p(x) = Ax^2 + Bx + C}, \quad y_p' = 2Ax + B, \quad y_p'' = 2A$$

$$2.(2A) + 4(2Ax + B) + 7(Ax^2 + Bx + C) = x^2$$

$$7Ax^2 + (8A + 7B)x + 4A + 4B + 7C = x^2$$

$$7A = 1 \longrightarrow A = \frac{1}{7}$$

$$8A + 7B = 0 \longrightarrow B = \frac{-8}{49}$$

$$4A + 4B + 7C = 0$$

$$\hookrightarrow C = -\frac{4}{7}A - \frac{4}{7}B = -\frac{4}{7}\cdot\frac{1}{7} - \frac{4}{7}\cdot\frac{(-8)}{49}$$

$$\boxed{\Rightarrow y_p = \frac{1}{7}x^2 - \frac{8}{49}x - \frac{4}{49.7}}$$

(*) $y'' + 9y' = 2x^2 e^{3x} + 5$ Find the <u>form</u> of the particular solution.

(i) $y'' + 9y' = 0$ $y = e^{rx}$ gives

$r^2 + 9r = 0$ $r_1 = 0,$ $r_2 = -9$

$y_c = c_1 \cdot e^{0 \cdot x} + c_2 \cdot e^{-9x} = c_1 + c_2 e^{-9x}$

(ii) $y'' + 9y' = 2x^2 e^{3x} + 5$

$y'' + 9y' = 2x^2 e^{3x}$ $\longrightarrow$ $Y_1 = (A_2 x^2 + A_1 x + A_0) e^{3x}$

$y'' + 9y' = 5$ $\longrightarrow$ $Y_2 = B \cdot x$

$y_p = Y_1 + Y_2$

* $y^{(4)} + 9y^{(3)} = \underbrace{2x^2e^{3x}} + \underbrace{xe^{-9x}} \underbrace{+5x+3}$

Find the form of the particular sol.

- $y^{(4)} + 9y^{(3)} = 0 \xrightarrow{y=e^{rx}} r^4 + 9r^3 = 0$

$r^3(r+9) = 0 \qquad r_1 = r_2 = r_3 = 0, \qquad r_4 = -9$

$y_c(x) = c_1 \cdot e^{0.x} + c_2 x e^{0.x} + c_3 x^2 e^{0.x} + c_4 e^{-9x}$

$y_1 = 1 \quad , \quad y_2 = x \quad , \quad y_3 = x^2 \quad , \quad y_4 = e^{-9x}$

$y_c(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-9x}$

- $y^{(4)} + 9y^{(3)} = 2x^2 e^{3x}$ $\quad Y_1 = (A_2 x^2 + A_1 x + A_0) e^{3x}$

$y^{(4)} + 9y^{(3)} = x e^{-9x}$ $\quad Y_2 = x (B_1 x + B_0) e^{-9x}$

$y^{(4)} + 9y^{(3)} = 5x + 3$ $\quad Y_3 = x^3 (D_1 x + D_0)$

$y_c(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-9x}$

$y_p = Y_1 + Y_2 + Y_3$ $\qquad s = 3$

$y = y_p + y_c$

The power $s$ in the term $x^s$ is nothing but the multiplicity of the root $r = 0$, from which we obtained the part of the comp. sol $c_1 + c_2 x + c_3 x^2$ that has a common part with the nonhom. term!!

(27) $y^{(4)} + 5y'' + 4y = \sin x + \cos 2x$

(i) $y^{(4)} + 5y'' + 4y = 0$ $\quad y = e^{rx}$ gives

$$r^4 + 5r^2 + 4 = 0 \rightarrow (r^2+1)(r^2+4) = 0$$

$$\begin{matrix} r^2 & +1 \\ r^2 & +4 \end{matrix}$$

$r^2 = -1 = i^2 \rightarrow r_{1,2} = 0 \mp i$ $\quad r^2 = -4 = 4i^2 \rightarrow r_{3,4} = 0 \mp 2i$

$\alpha = 0,\ \beta = 1$ $\quad \alpha = 0,\ \beta = 2$

$$y = e^{0.x}[c_1 \cos x + c_2 \sin x] + e^{0.x}[c_3 \cos 2x + c_4 \sin 2x]$$

$$y_c = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$$

$$y_p = x(A_1 \sin x + A_2 \cos x) + x(B_1 \cos 2x + B_2 \sin 2x)$$

* Write a linear, third order, constant coefficient, homogeneous DE of which general solution is $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-4x}$.

$r_1 = 1, \quad r_2 = 2, \quad r_3 = -4$

$(r-1)(r-2)(r+4) = 0$

$(D-1)(D-2)(D+4)\,y = 0 \qquad D = \frac{d}{dx}$

$(D^2 - 3D + 2)(D+4)\,y = 0$

$(D^3 + 4D^2 - 3D^2 - 12D + 2D + 8)\,y = 0$

$(D^3 + D^2 - 10D + 8)\,y = 0$

$y''' + y'' - 10y' + 8y = 0$

* The same question, for

$$y = e^{2x}\left[c_1 \cos 3x + c_2 \sin 3x\right] + c_3 e^{-2x} + c_4 x e^{-2x} + c_5 x^2 e^{-2x}$$

(This time the eq. will be $5^{th}$ order]

$$r_{1,2} = 2 \mp 3i \qquad r_3 = r_4 = r_5 = -2$$

$$(r-(2+3i))(r-(2-3i)).(r+2)^3 = 0$$

$$(r^2 - 4r + 13)(r+2)^3 = 0$$

$$\left(D^2 - 4D + 13\right)(D+2)^3 y = 0 \qquad D = \frac{d}{dx}$$

and expand this to get $y^{(5)} + \cdots = 0$

Ex Find the general solution to $y'' - 4y = \sinh 2x$.

* This can be solved by the method of undetermined coefficients, as $\sinh(2x) = \frac{1}{2}(e^{2x} - e^{-2x})$; i.e., the RHS is a combination of exponentials. Do this solution yourself.

* By the method of variation of parameters:

* $y'' - 4y = 0 \longrightarrow y = e^{rx}$; $r^2 - 4 = 0$

$r_1 = -2 \quad r_2 = 2 \longrightarrow \boxed{y_c = c_1 e^{-2x} + c_2 e^{2x}}$

$y_1 = e^{-2x} \qquad y_2 = e^{2x}$

$*\quad y'' - 4y = \sinh 2x \quad : \quad y = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = \sinh 2x$$

$$\begin{cases} u_1' e^{-2x} + u_2' e^{2x} = 0 \\ u_1' (-2) e^{-2x} + u_2' \, 2e^{2x} = \sinh(2x) \end{cases}$$

$$\begin{bmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh 2x \end{bmatrix}$$

$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 4$$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{2x} \\ \sinh 2x & 2e^{2x} \end{vmatrix}}{4} = \frac{1}{4} \cdot (-1)\, e^{2x} \sinh(2x)$$

$$= -\frac{1}{4} e^{2x} \cdot \frac{1}{2} \left( e^{2x} - e^{-2x} \right) = -\frac{1}{8} \left( e^{4x} - 1 \right)$$

$$u_1' = -\frac{1}{8} e^{4x} + \frac{1}{8} \rightarrow \boxed{u_1 = -\frac{1}{32} e^{4x} + \frac{x}{8} + C_1}$$

$$u_2' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \sinh 2x \end{vmatrix}}{4} = \frac{1}{4} e^{-2x} \sinh 2x$$

$$= \frac{1}{4} e^{-2x} \left( \frac{e^{2x} - e^{-2x}}{2} \right) = \frac{1}{8} \left( 1 - e^{-4x} \right)$$

$$\boxed{u_2 = \frac{x}{8} + \frac{1}{32} e^{-4x} + C_2}$$

$$u_1 = -\frac{1}{32} e^{4x} + \frac{x}{8} + c_1$$

$$u_2 = \frac{x}{8} + \frac{1}{32} e^{-4x} + c_2$$

$$y = u_1 y_1 + u_2 y_2$$

$$= \left(c_1 - \frac{1}{32} e^{4x} + \frac{x}{8}\right) e^{-2x} + \left(c_2 + \frac{x}{8} + \frac{e^{-4x}}{32}\right) e^{2x}$$

$$= c_1 e^{-2x} + c_2 e^{2x} + \frac{x}{8} \left(e^{2x} + e^{-2x}\right) - \frac{1}{32} \left(e^{2x} - e^{-2x}\right)$$

$$= c_1 e^{-2x} + c_2 e^{2x} + \frac{x}{8} \cosh(2x)$$

* For the method of undetermined coefficients:

$$y'' - 4y = \sinh(2x) = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

$$y_c(x) = c_1 e^{-2x} + c_2 e^{2x}$$

$$y_p(x) = x \cdot A e^{-2x} + x \cdot B e^{2x}$$

DIY

# 6.2 Diagonalization of Matrices

<u>Example</u> $A = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$ has the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, with the eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Let's construct the matrix $P = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$

$$P = \begin{array}{c} \begin{array}{cc} \ \ v_1 & \ v_2 \end{array} \\ \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \end{array}$$ ; which has the inverse $P^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

<u>See that</u> $P^{-1}AP = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \overset{\lambda_1}{2} & 0 \\ 0 & \underset{\lambda_2}{1} \end{bmatrix}$

$$P^{-1} A P = D = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

<u>Def.</u> $n \times n$ matrices $A$ & $B$ are called similar matrices provided there's a nonsingular ($\det \neq 0$) matrix $P$ such that

$$B = P^{-1} A P.$$

This def. is symmetric in $A$ and $B$:

$$P B = A P$$

$$P B P^{-1} = A$$

$$A = P B P^{-1}$$

$$A = (P^{-1})^{-1} B P^{-1} \qquad Q = P^{-1}$$

$$A = Q^{-1} B Q$$

* Similar matrices have the same determinant:

$$B = P^{-1} A P \Rightarrow \det B = \det(P^{-1}AP)$$

$$\det B = \det(P^{-1}) \det A \det P$$

$$\det B = \det A$$

**Theorem 1** The $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

$$A = \begin{bmatrix} \cdot & - & \cdot & - \\ & - & - & - \\ & - & - & \\ & - & - & \end{bmatrix}_{n\times} : \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ \sim & \sim & & \sim \\ | & | & & | \end{bmatrix} = P$$

$$P^{-1} A P = D$$ : diagonalization of the matrix $A$.

Ex In the previous section, we found the eigenvalus and corresponding eigenvectors of

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -5 & -5 \end{bmatrix}$$ as

This example is illustrating Theorem 1 and Theorem 2.

$\lambda_1 = 3 \Rightarrow v_1 = (1, 0, 2)$

$\lambda_2 = 1 \Rightarrow v_2 = (0, 2, -5)$

$\lambda_3 = 0 \quad v_3 = (0, 1, -3)$

$$P = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & -5 & -3 \end{bmatrix} \end{array}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 1 \\ 4 & -5 & 2 \end{bmatrix} \Rightarrow D = P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 2 If the $n \times n$ matrix $A$ has $n$ distinct eigenvalues, then it's diagonalizable.

different

The following example illustrates the Theorem 1.

In the previous section, we saw that the matrix

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

has only two distinct eigenvalues,

I'll repeat this next Wednesday!!

$\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 3$

$\lambda_1 = \lambda_2 = 2 \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$\lambda_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$