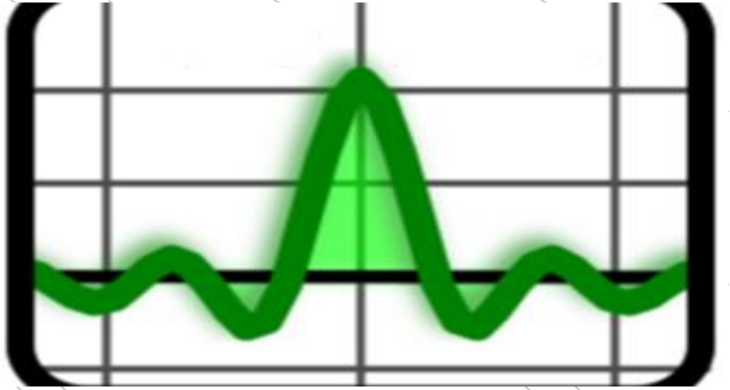

$$e^{i\pi}+1=0$$

Signals & Systems For Computer Engineering

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BLG354E / CRN: 21560

9th Week Lecture

Jean-Baptiste Joseph Fourier

1768-1830

What is Fourier Transform ?

the Fourier transform is a special case of the Laplace transform in which s=jω

$$X(s)|_{s=j\omega} = \mathscr{F}\{x(t)\}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \rightarrow X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Series Representation of Periodic Signals:

a continuous-time signal x(t) is periodic if there is a positive nonzero value of T for which x(t + T) =x(t) all t

complex exponential signal $x(t) = e^{j\omega_0 t}$ where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ fundamental angular frequency

Complex Exponential Fourier Series Representation:

The complex exponential Fourier series representation of a periodic signal x(t) with fundamental period To is given by

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}}$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

where c_k are known as the complex Fourier coefficients

If x(t) is a real signal then $c_{-k}=c_k^*$

for k=0 → $$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$ equals the average value of x(t) over a period

Trigonometric Fourier Series:

$$\boxed{x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)}$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t) \cos k\omega_0 t\, dt$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t) \sin k\omega_0 t\, dt$$

a_k, and b_k, are the Fourier coefficients

Relationship between complex Fourier Series and the Trigonometric Fourier Series:

$$\frac{a_0}{2} = c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k})$$

If x(t) is real then a_k and b_k are real: $a_k = 2\,\mathrm{Re}[c_k]$ $\qquad b_k = -2\,\mathrm{Im}[c_k]$

If the periodic signal x(t) is even then b_k=0 and its Fourier series contains only cosine terms:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t$$

If the periodic signal x(t) is odd then a_k=0 and its Fourier series contains only sine terms:

$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega_0 t$$

Example: Determine the complex exponential Fourier series representation for each of the following signals

a) $x(t)=\sin(\omega_0 t)$

b) $x(t)=\cos(4t)+\sin(6t)$

a) $$\sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) = \boxed{-\frac{1}{2j}} e^{-j\omega_0 t} + \boxed{\frac{1}{2j}} e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_1 = \frac{1}{2j} \qquad c_{-1} = -\frac{1}{2j} \qquad c_k = 0,\ |k| \neq 1$$

b) $x(t)=\cos(4t)+\sin(6t)$ → $x(t) = x_1(t+mT_1) + x_2(t+kT_2)$

If $x(t)$ is periodic then it must satisfy, $x(t+T) = x_1(t+T) + x_2(t+T) = x_1(t+mT_1) + x_2(t+kT_2)$

There must be positive integers m, k that $mT_1 = kT_2 = T \rightarrow \frac{T_1}{T_2} = \frac{k}{m} = \text{rational number}$

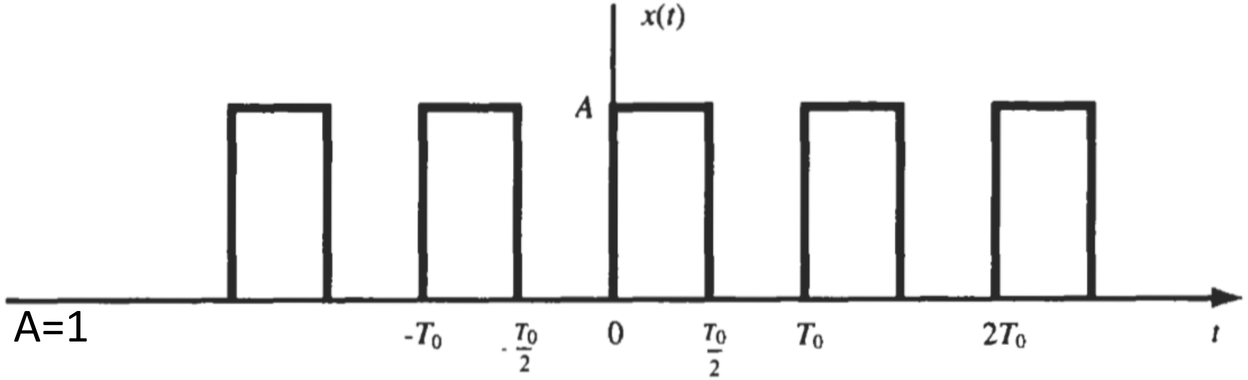
The sum of periodic two signals is periodic iff their respective periods can be expressed as a rational number.

Since the fundamental period is the least common multiple of T_1 and T_2 → T_0 of $x(t)$ is π and $\omega_0=2\pi/T_0=2$

$$x(t)=\cos(4t)+\sin(6t) = \frac{1}{2}\left(e^{j4t} + e^{-j4t}\right) + \frac{1}{2j}\left(e^{j6t} - e^{-j6t}\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt}$$

$$= -\frac{1}{2j}e^{-j6t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j4t} + \frac{1}{2j}e^{j6t} = \sum_{k=-\infty}^{\infty} c_k e^{j2kt} \rightarrow c_{-3} = -\frac{1}{2j} \qquad c_{-2} = \frac{1}{2} \qquad c_2 = \frac{1}{2} \qquad c_3 = \frac{1}{2j}$$

Example:



Consider the periodic square wave x(t) shown in the figure.

a) Determine the complex exponential Fourier series of x(t)
b) Determine the trigonometric Fourier series of x(t)
c) Apply the determined Fourier series for the case that T=2, A=1

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_0^{T_0/2} A\, e^{-jk\omega_0 t}\, dt = \frac{A}{-jk\omega_0 T_0} e^{-jk\omega_0 t}\Big|_0^{T_0/2} = \frac{A}{-jk\omega_0 T_0}\left(e^{-jk\omega_0 T_0/2} - 1\right)$$

$$= \frac{A}{jk2\pi}\left(1 - e^{-jk\pi}\right) = \frac{A}{jk2\pi}\left[1 - (-1)^k\right]$$

$$\omega_0 T_0 = 2\pi \text{ and } e^{-jk\pi} = (-1)^k$$

$$c_k = 0 \qquad k = 2m \neq 0$$

$$c_k = \frac{A}{jk\pi} \qquad k = 2m+1$$

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt = \frac{1}{T_0}\int_0^{T_0/2} A\, dt = \frac{A}{2}$$

$$x(t) = \frac{A}{2} + \frac{A}{j\pi}\sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

b) Trigonometric Fourier Series of x(t):

$$a_{2m} = b_{2m} = 0, \; m \neq 0 \qquad \frac{a_0}{2} = c_0 = \frac{A}{2}$$

Since x(t) is an odd function $\quad a_{2m+1} = 2\,\mathrm{Re}[c_{2m+1}] = 0$

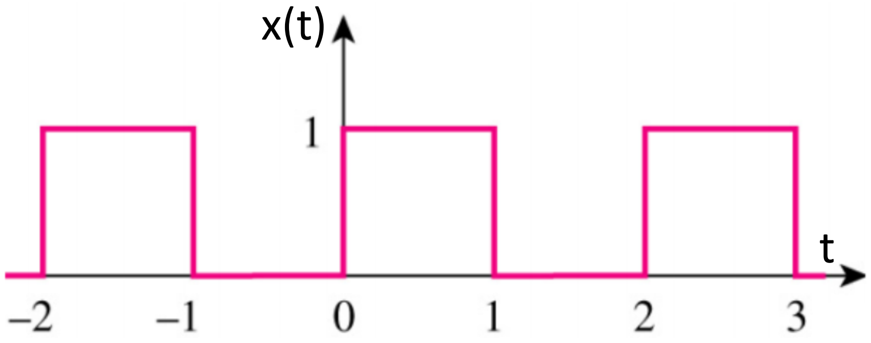
$$b_{2m+1} = -2\,\mathrm{Im}[c_{2m+1}] = \frac{2A}{(2m+1)\pi}$$

Trigonometric Fourier Series is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} \sin(2m+1)\omega_0 t = \frac{A}{2} + \frac{2A}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$$

c) When $T=2$, $T_0/2=1$ and $A=1$ signal $x(t)$ is drawn as:



$$x(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$$x(t+2)=x(t)$$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$$

$T=2 \rightarrow \omega_0=2\pi/T=\pi$

$A=1$

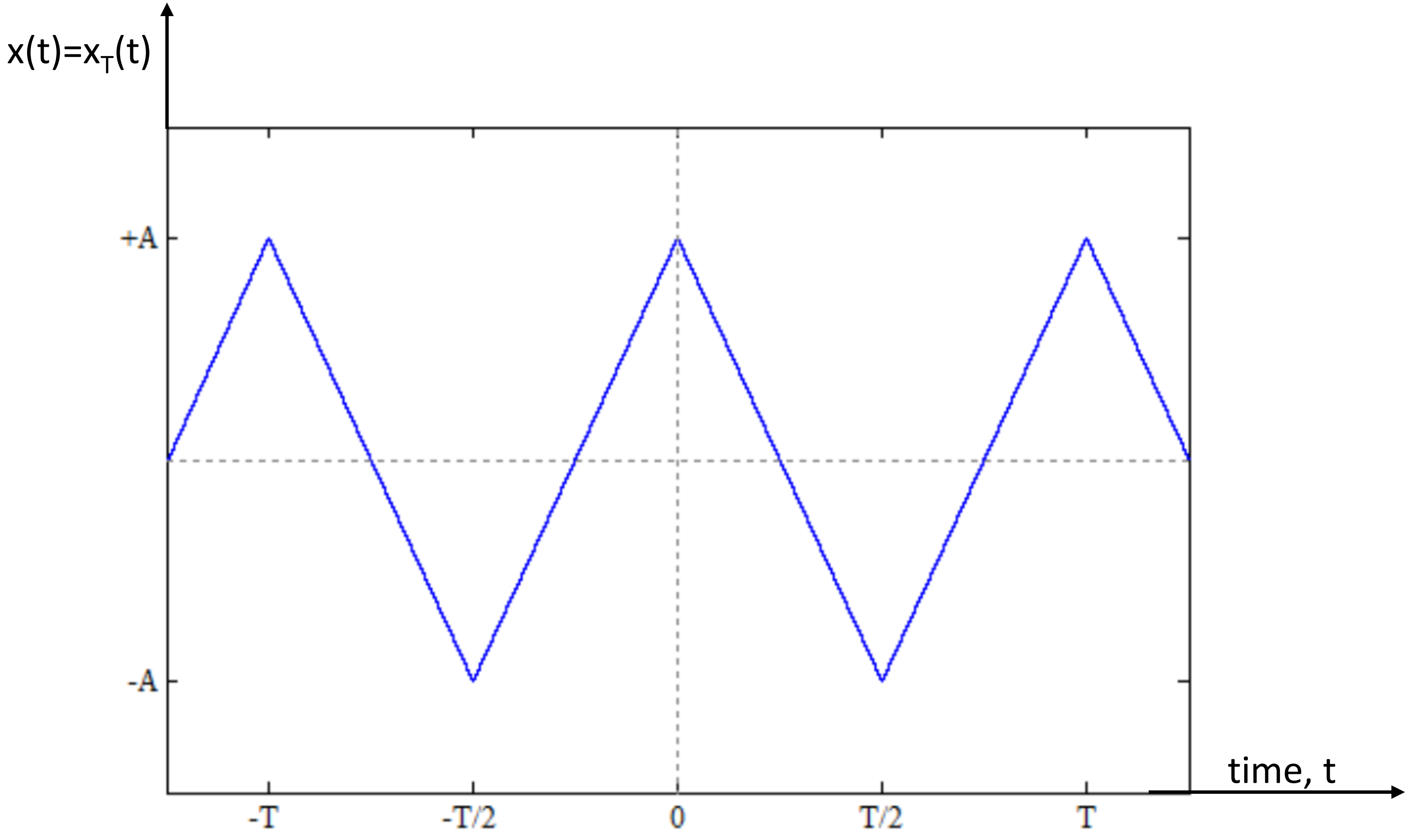
$2A/n\pi$

$n\omega_0$

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \ldots$$

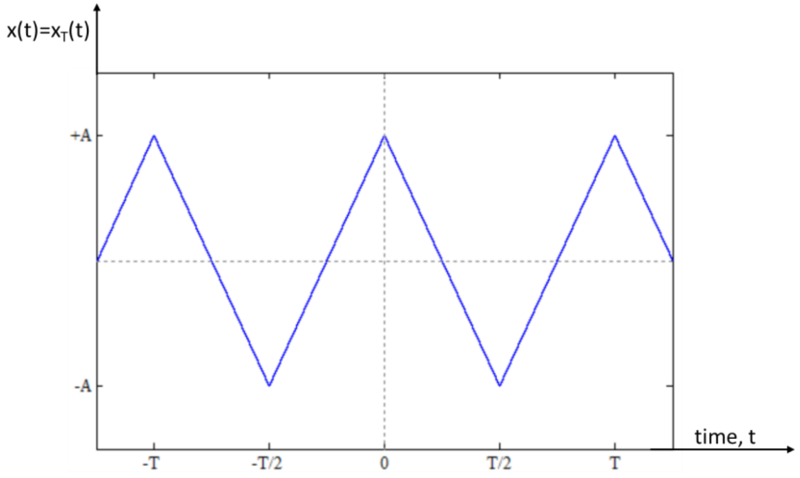
Example:

Find the Fourier Series representation of the Even Triangle Wave



Since a) average over the period –T/2 to +T/2=0→ a_0=0

b) $x_T(t)$ is an even signal → b_n=0



Between *t=0* and *t=T/2* the function is defined by $x_T(t) = A - \frac{4At}{T}$

$$a_n = \frac{2}{T}\int_{T} x_T(t)\cos(n\omega_0 t)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt = \frac{4}{T}\int_{0}^{+\frac{T}{2}} x_T(t)\cos(n\omega_0 t)dt$$

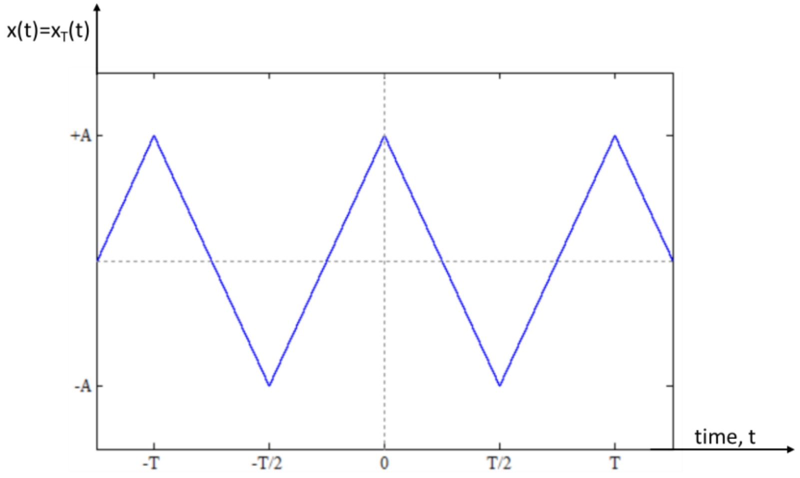
$$a_n = \frac{4}{T}\int_{0}^{+\frac{T}{2}}\left(A - \frac{4A}{T}t\right)\cos(n\omega_0 t)dt = \frac{4A}{T}\left(\int_{0}^{+\frac{T}{2}}\cos(n\omega_0 t)dt - \frac{4}{T}\int_{0}^{+\frac{T}{2}} t\cos(n\omega_0 t)dt\right)$$

$\omega_0 \cdot T = 2 \cdot \pi$ *and p*erform the integration by parts, or with a table of integrals:

$$a_n = \frac{4A}{T}\left(\frac{T\sin(\pi n)}{2\pi n} + \frac{4}{T}\frac{T^2\left(2\sin\left(\frac{\pi n}{2}\right)^2 - \pi n\sin(\pi n)\right)}{4\pi^2 n^2}\right)$$

this simplifies since $sin(\pi \cdot n)=0$:

$$a_n = \frac{4A}{T}\frac{4}{T}\frac{T^2 2\sin\left(\frac{\pi n}{2}\right)^2}{4\pi^2 n^2} = \frac{8A\sin\left(\frac{\pi n}{2}\right)^2}{\pi^2 n^2}$$



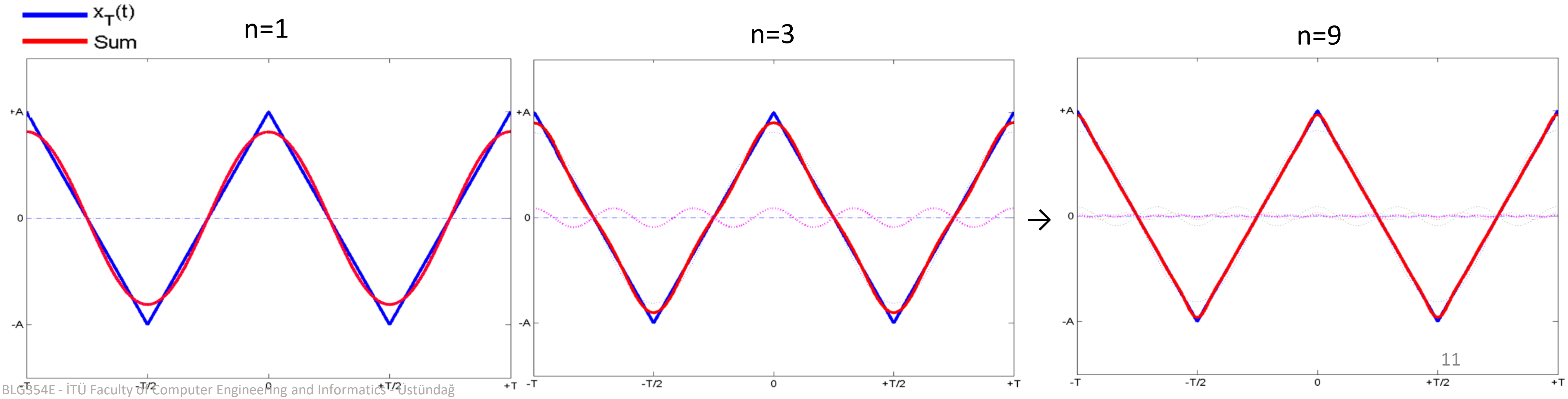
$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \quad \rightarrow \quad x_T(t) = \sum_{k=1}^{\infty} a_k \cos k\omega_0 t$$

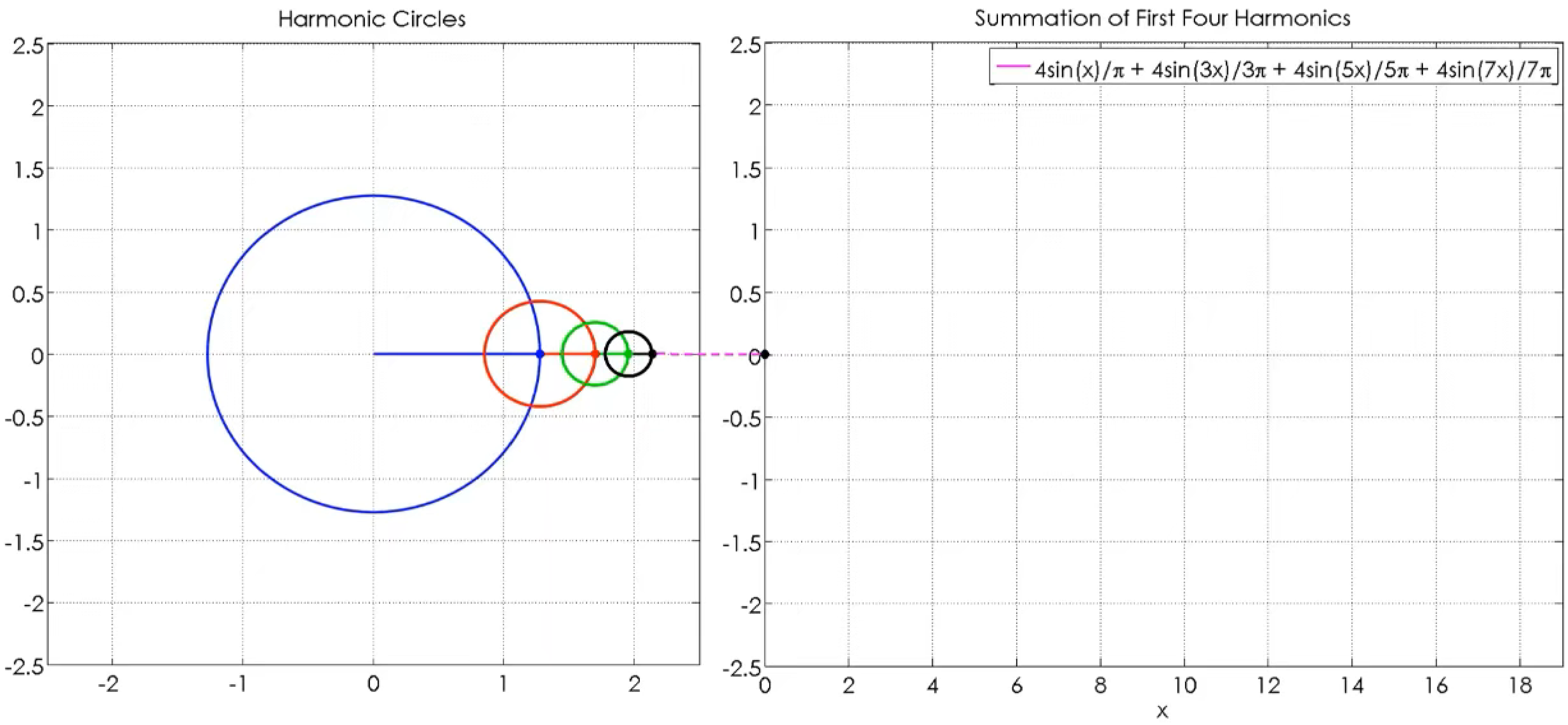
For n=0, 1, 2, 3, 4, …… → $\sin\left(\frac{\pi n}{2}\right)^2 = 0, 1, 0, 1, 0, 1, 0, \ldots = \frac{1-(-1)^n}{2}$

$$x_T(t) = a_1 \cos\frac{2\pi}{T}t + a_3 \cos\frac{6\pi}{T}t + a_5 \cos\frac{10\pi}{T}t + \ldots.$$

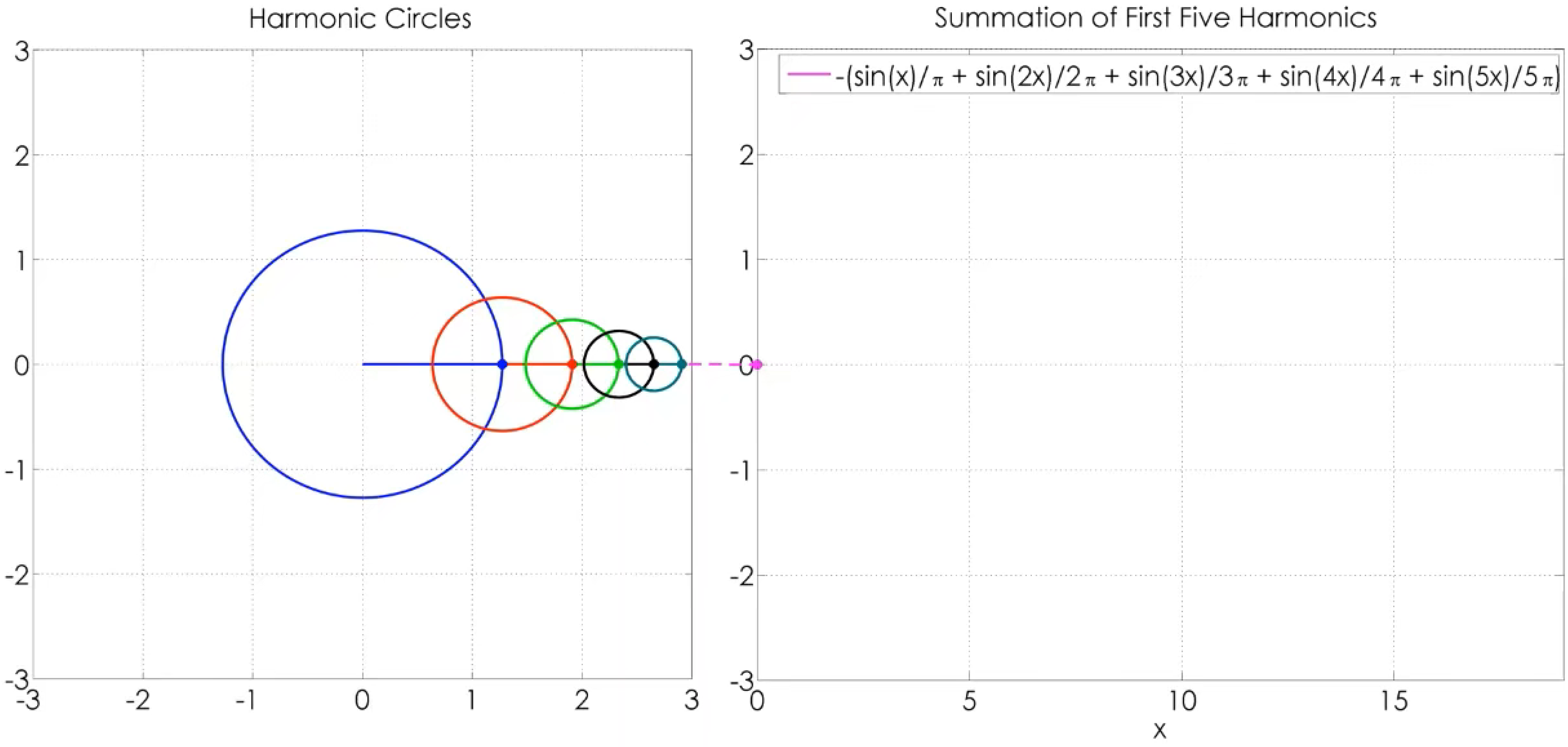
$$a_n = \frac{8A \sin\left(\frac{\pi n}{2}\right)^2}{\pi^2 n^2} \rightarrow a_n = \begin{cases} 4A\frac{1-(-1)^n}{\pi^2 n^2}, & \text{n odd} \\ 0, & \text{n even} \end{cases}$$

n	0	1	2	3	4	5	6	7
a_n	0	0.8106	0	0.0901	0	0.0324	0	0.0165





Harmonic Circles
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1
0
1
2
Summation of First Four Harmonics
4sin(x)/π + 4sin(3x)/3π + 4sin(5x)/5π + 4sin(7x)/7π
0
2
4
6
8
10
12
14
16
18
x


Harmonic Circles
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3
Summation of First Five Harmonics
-(sin(x)/π + sin(2x)/2π + sin(3x)/3π + sin(4x)/4π + sin(5x)/5π)
3
2
1
0
-1
-2
-3
0
5
10
15
x

Analytic Fourier Series Expansion in Python

$$\mathcal{L}\{f\} = \int_0^{\infty} f(t)\exp(-st)\mathrm{d}t$$

$$\mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\mathrm{d}t$$

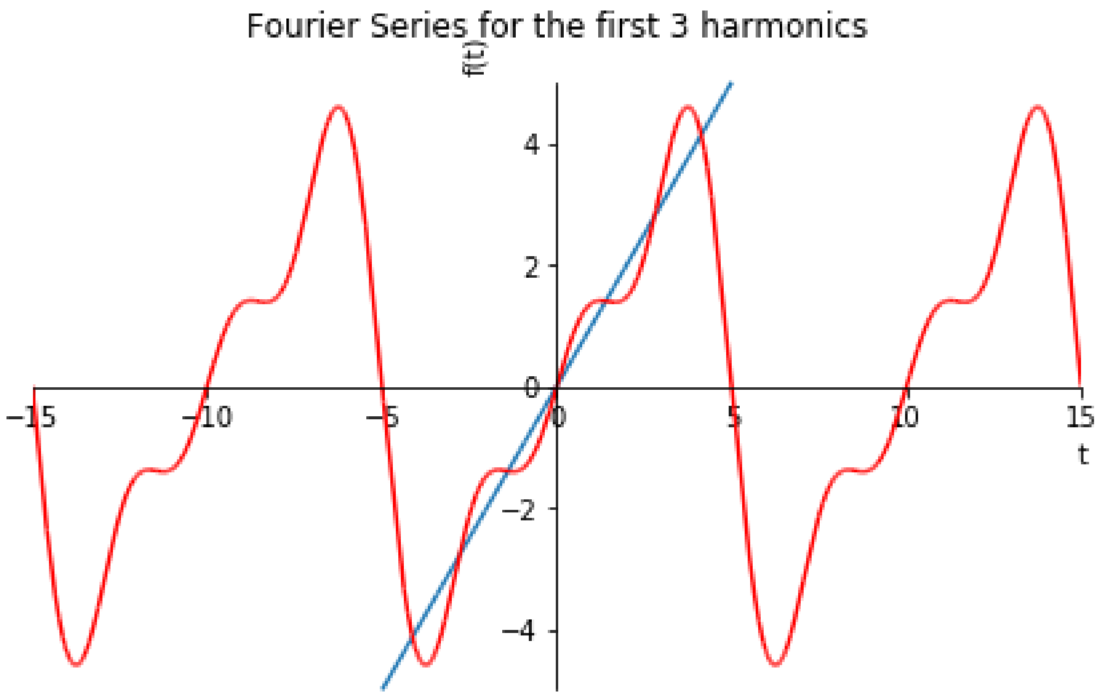
$$S_N(t) = \sum_{n=-N}^{N} c_n \exp\left(\frac{i2\pi n t}{T}\right)$$

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\exp\left(\frac{-i2\pi n t}{T}\right)\mathrm{d}t$$

$f(t)=t \quad (t_0=-5 , T=10)$

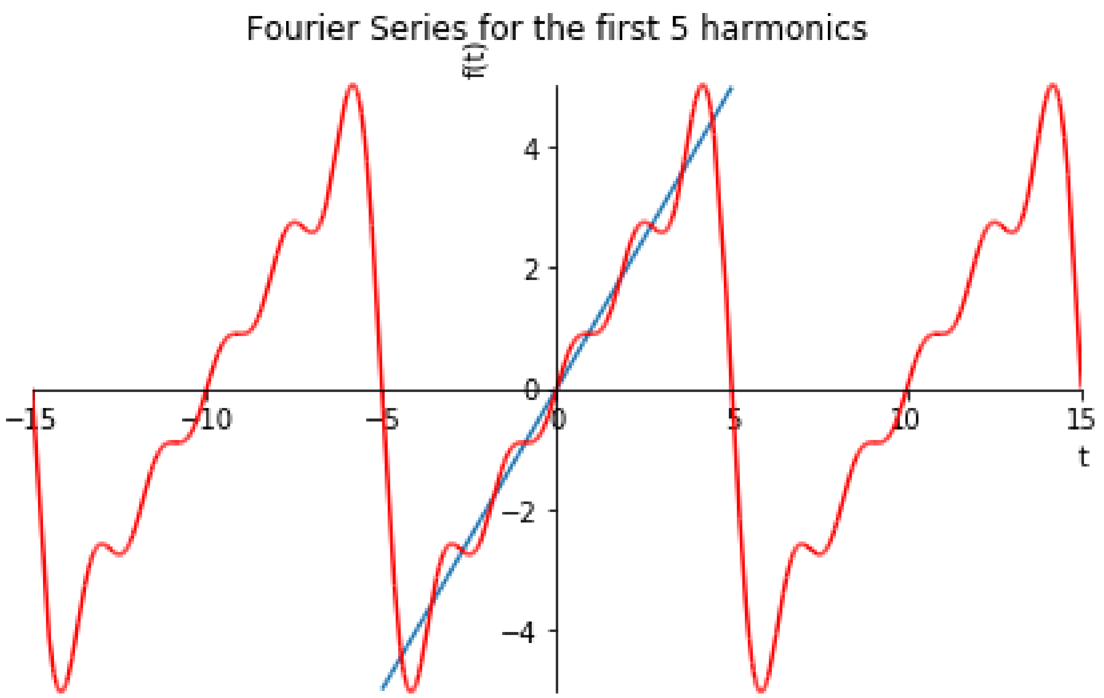
we can expand the mathematical expressions in the form of variables by using sympy.expand() method

```
import sympy
exp = sympy.exp
j2pi = sympy.I*2*sympy.pi

def S(N):
   return sum(c(n)*exp(j2pi*n*t/T) for n in range(-N, N+1)).expand(complex=True).simplify()
def c(n):
   return (sympy.integrate(
            f(t)*exp((-j2pi * n * t)/T),(t, t0, t0 + T))/T)

a = sympy.Symbol('a', positive=True)
def f(t):
   return t
T = 10
t0 = -5
t = sympy.Symbol('t', real=True)
N = 5
analytic_app = S(N).expand()
interval = (t, t0-T, t0+2*T)
p1 = sympy.plot(f(t), (t, t0, t0+T), title='Fourier Series for the first '+str(N)+' harmonics',
show=False)
p2 = sympy.plot(analytic_app, interval, show=False)
p2[0].line_color = 'red'
p1.extend(p2)
p1.show()
print(sympy.N(analytic_app))
```

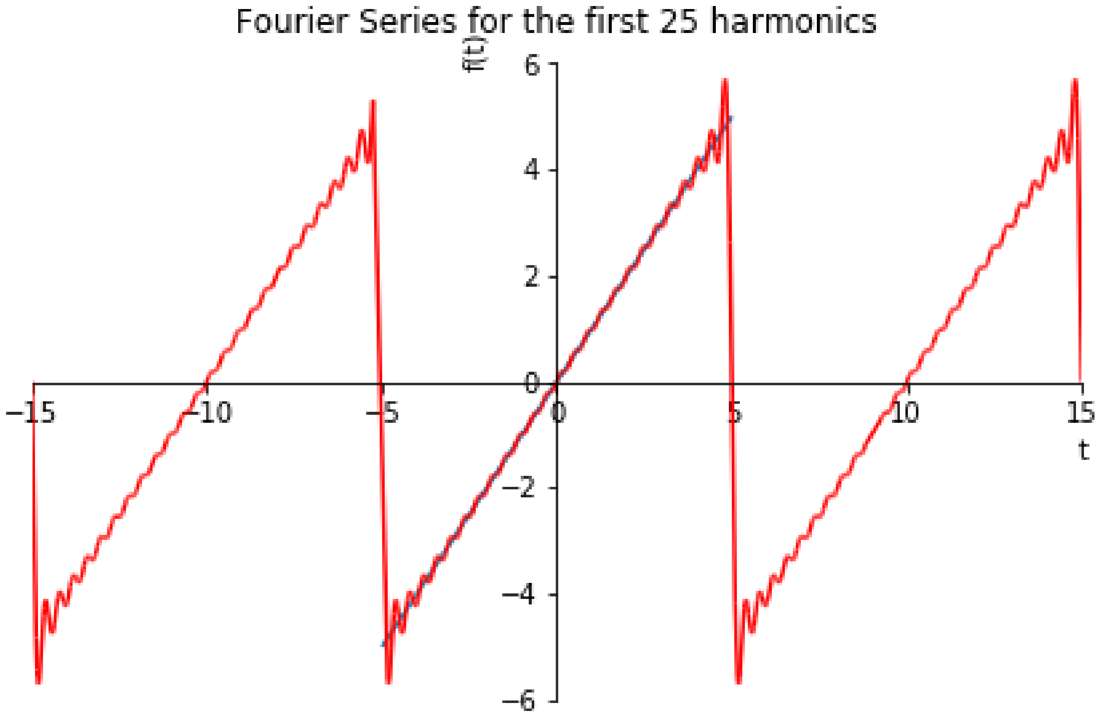


3.18309886183791*sin(pi*t/5) - 1.59154943091895*sin(2*pi*t/5) + 1.06103295394597*sin(3*pi*t/5

sympy.N(analytic_app) → $10\sin(\pi t/5)/\pi - 5\sin(2\pi t/5)/\pi + 10\sin(3\pi t/5)/3\pi$



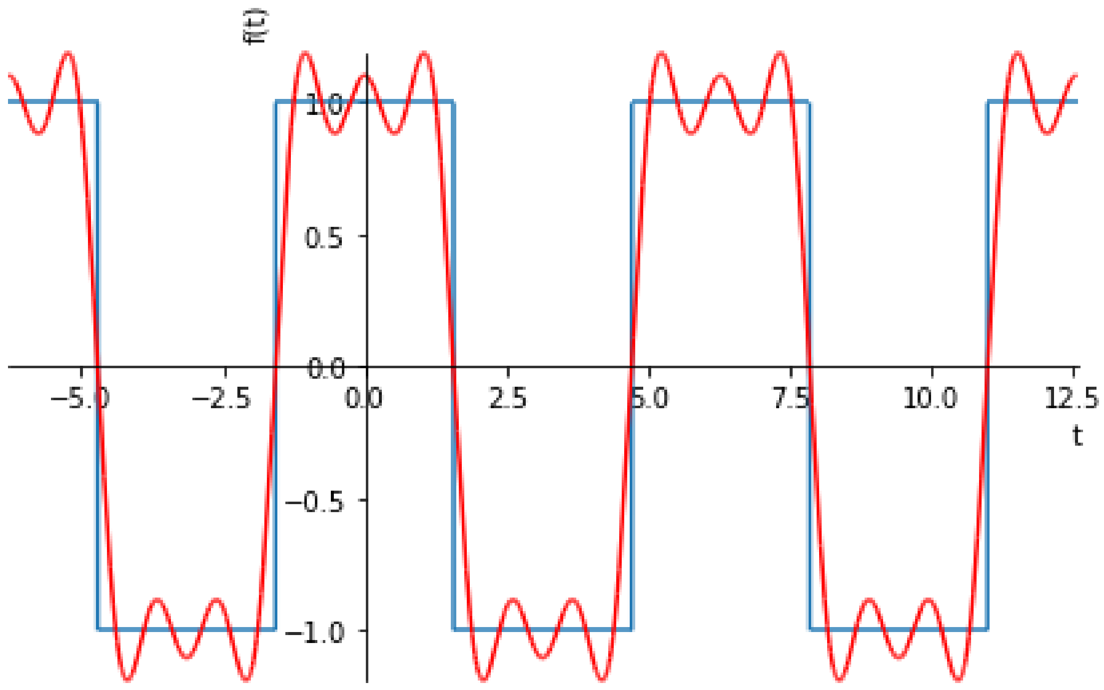
3.18309886183791*sin(pi*t/5) - 1.59154943091895*sin(2*pi*t/5) + 1.06103295394597*sin(3*pi*t/5) - 0.795774715459477*sin(4*pi*t/5) + 0.636619772367581*sin(pi*t)



3.18309886183791*sin(pi*t/5) - 1.59154943091895*sin(2*pi*t/5) + 1.06103295394597*sin(3*pi*t/5) - 0.795774715459477*sin(4*pi*t/5) + 0.636619772367581*sin(pi*t) - 0.530516476972985*sin(6*pi*t/5) + 0.454728408833987*sin(7*pi*t/5) - 0.397887357729738*sin(8*pi*t/5) + 0.353677651315323*sin(9*pi*t/5) - 0.318309886183791*sin(2*pi*t) + 0.289372623803446*sin(11*pi*t/5) - 0.265258238486492*sin(12*pi*t/5) + 0.244853758602916*sin(13*pi*t/5) - 0.227364204416993*sin(14*pi*t/5) + 0.212206590789194*sin(3*pi*t) - 0.198943678864869*sin(16*pi*t/5) + 0.187241109519877*sin(17*pi*t/5) - 0.176838825657662*sin(18*pi*t/5) + 0.16753151904411*sin(19*pi*t/5) - 0.159154943091895*sin(4*pi*t) + 0.151576136277996*sin(21*pi*t/5) - 0.144686311901723*sin(22*pi*t/5) + 0.138395602688605*sin(23*pi*t/5) - 0.132629119243246*sin(24*pi*t/5) + 0.127323954473516*sin(5*pi*t)

Fourier Series Expansion of a Square Waveform by Python

```
import sympy
exp = sympy.exp
j2pi = sympy.I*2*sympy.pi

def S(N):
   return sum(c(n)*exp(j2pi*n*t/T) for n in range(-N, N+1)).expand(complex=True).simplify()
def c(n):
   return (sympy.integrate(f(t)*exp((-j2pi * n * t)/T),(t, t0, t0 + T))/T)

a = sympy.Symbol('a', positive=True)
def f(t):
   return sympy.sign(sympy.cos(t))
T = 2*sympy.pi
t0 = 0
t = sympy.Symbol('t', real=True)
N = 5
analytic_app = S(N).expand()

interval = (t, t0-T, t0+2*T)
p1 = sympy.plot(f(t), interval, show=False)
p1.linestyle='dashed'
p2 = sympy.plot(analytic_app, interval, show=False)
p2[0].line_color = 'red'
p1.extend(p2)
p1.show()
print(sympy.N(analytic_app))
```



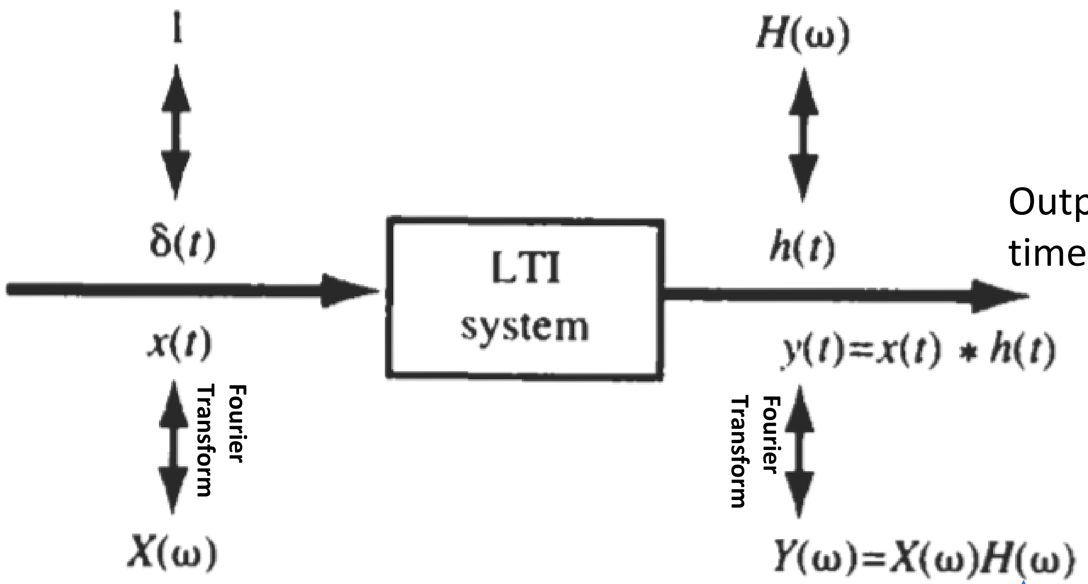
1.27323954473516*cos(t) - 0.424413181578388*cos(3*t) + 0.254647908947033*cos(5*t)

Fourier Spectra: The Fourier transform $X(\omega)$ of $x(t)$ is, in general, complex, and it can be expressed as

$$X(\omega)=|X(\omega)|\cdot e^{j\theta(\omega)}$$

Magnitude Spectra

Phase Spectra



Output of the system h(t) for impulse signal in time domain → Transfer function

Output of the system for given input signal in time domain

Output of the system for given input signal in frequency domain

Transfer function of the system in frequency domain

$$Y(\omega)=|H(\omega)|\cdot|X(\omega)|\cdot e^{j(\Theta+\Theta_H(\omega))}$$

Find the Fourier transform of the signal $x(t) = e^{-at}u(t) \qquad a > 0$

Laplace of the signal x(t) is,

$$\mathscr{L}\{x(t)\} = X(s) = \frac{1}{s+a} \qquad \mathrm{Re}(s) > -a$$

Fourier transform of the signal can be expressed as,

$$\mathscr{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0^+}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{1}{a+j\omega}$$

$$\boxed{X(\omega) = X(s)|_{s=j\omega}}$$

Fourier Transform of the signal can also be obtained from its Laplace Transform by substituting s with jω

Properties of the Fourier Transform

Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
Real signal	$x(t) = x_e(t) + x_o(t)$	$X(\omega) = A(\omega) + jB(\omega)$
		$X(-\omega) = X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\} = A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)\,d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)\,d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\,d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

Prove the time shifting property $x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$

By the definition $\mathscr{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$

By the variable change $\tau = t - t_0$ $\mathscr{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau + t_0)}\, d\tau$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = e^{-j\omega t_0} X(\omega)$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$$

Exercise:

Asymptotic frequency response (magnitude) graph of a 1st order low pass filter is given in the below diagram.
Corner frequency of this LPF system is f_c=1000Hz.

a) Find the transfer function H(s)

b) Find Magnitude spectra and Phase spectra of the system

c) If the square wave with amplitude A=10V and the frequency f=1000Hz is applied to this system then find the first three non-zero harmonics both at the input and output the system