(B) Complex eigenvalues

$$x' = Ax \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{matrix} x_1 = x_1(t) \\ x_2 = x_2(t) \end{matrix}$$

$$\underset{\sim}{x} = \underset{\sim}{v}\, e^{\lambda t} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda} \end{bmatrix}$$

$$|A - \lambda I| = 0 \quad , \quad (A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$$

Ex

$$x' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x$$

Find the general solution

$$\underset{\sim}{x} = \underset{\sim}{v}\, e^{\lambda t} \Rightarrow |A - \lambda I| = \begin{vmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -1/2 - \lambda \end{vmatrix}$$

$$|A-\lambda I| = \lambda^2 + \lambda + \frac{5}{4} = 0 \qquad \Delta = 1^2 - 4 \cdot 1 \cdot \frac{5}{4} = -4$$

$$\lambda_{1,2} = \frac{-1 \mp \sqrt{-4}}{2} = \frac{-1 \mp \sqrt{4i^2}}{2} = -\frac{1}{2} \mp i$$

$$\lambda = -\frac{1}{2} + i \qquad (A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$$

$$\begin{bmatrix} -\frac{1}{2} - \left(-\frac{1}{2} + i\right) & 1 \\ -1 & -\frac{1}{2} - \left(-\frac{1}{2} + i\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-i v_1 + v_2 = 0$$

Let $v_1 = 1 \rightarrow v_2 = i$

$$\underset{\sim}{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_1 = -\frac{1}{2} + i \qquad \underset{\sim}{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_2 = -\frac{1}{2} - i \qquad \underset{\sim}{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

{ If λ is a complex eigenvalue to a real-entry matrix A with eigenvector $\underset{\sim}{v}$, λ^* is also an eigenvalue to A with eigenvector $\underset{\sim}{v}^*$ }

$$\underset{\sim}{x} = \underset{\sim}{v}\, e^{\lambda t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{\left(-\frac{1}{2} + i\right)t}$$ is a "complex vector" solution to the system $x' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -1/2 \end{pmatrix}$

$$\left.\begin{aligned} x_1' &= -\tfrac{1}{2}x_1 + x_2 \\ x_2' &= -x_1 - \tfrac{1}{2}x_2 \end{aligned}\right\}$$ we need to give real solutions $x_1(t)$ & $x_2(t)$.

$$\underset{\sim}{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{\left(-\frac{1}{2}+i\right)t} \\ i\, e^{\left(-\frac{1}{2}+i\right)t} \end{bmatrix} \begin{matrix} \to x_1(t) \\ \to x_2(t) \end{matrix}$$

<u>Claim</u> If $x(t) = \underset{\sim}{u}(t) + i\,\underset{\sim}{v}(t)$ solves $x' = Ax$, so do $\underset{\sim}{u}(t)$ and $\underset{\sim}{v}(t)$; that is, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ also solve $x' = Ax$. Indeed,

$$x = \underset{\sim}{u} + i\underset{\sim}{v} \xrightarrow{x' = Ax} (u + iv)' = A(u + iv)$$

$$\underbrace{u'}\,+\,i\underbrace{v'} = \underbrace{Au}\,+\,i\underbrace{Av} \implies \begin{aligned} u' &= Au \\ v' &= Av \end{aligned}$$

(A is assumed to be with real entries)

$$x(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{\left(-\frac{1}{2}+i\right)t} = e^{-\frac{1}{2}t} \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it}$$

$$= e^{-\frac{t}{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} (\cos t + i \sin t) = e^{-\frac{t}{2}} \begin{bmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{bmatrix}$$

$$= e^{-\frac{t}{2}} \left\{ \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \right\}$$

$$= \underbrace{e^{-\frac{t}{2}} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\underset{\sim}{u}(t)} + i \underbrace{e^{-t/2} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\underset{\sim}{v}(t)}$$

$\underset{\sim}{u}(t)$ and $\underset{\sim}{v}(t)$ are two solutions to

$x' = AX$ and they're linearly ind., as

$$W(u,v) = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} = e^{-t} \neq 0.$$

Hence,

$$\underset{\sim}{x}(t) = c_1\,\underset{\sim}{u}(t) + c_2\,\underset{\sim}{v}(t)$$

$$= c_1 e^{-t/2}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-t/2}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

is the general solution!

Ex Find the general sol. of

$$\frac{dx_1}{dt} = 4x_1 - 3x_2$$

$$\frac{dx_2}{dt} = 3x_1 + 4x_2$$

$$\underset{\sim}{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \underset{\sim}{x}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \underset{\sim}{x}$$

$$\underset{\sim}{x} = \underset{\sim}{v} e^{\lambda t} \qquad (A - \lambda I)\underset{\sim}{v} = 0$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 0$$

$$\Rightarrow (\lambda - 4)^2 = -9 = (3i)^2 \Rightarrow \lambda - 4 = \mp 3i \rightarrow \boxed{\lambda = 4 \mp 3i}$$

$$\underline{\lambda = 4-3i} \qquad (A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$$

$$\begin{bmatrix} 4-(4-3i) & -3 \\ 3 & 4-(4-3i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 3i\,v_1 - 3\,v_2 = 0$$

Let $v_1 = 1 \rightarrow v_2 = i$

$$\lambda = 4-3i, \qquad \underset{\sim}{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\underset{\sim}{x} = \underset{\sim}{v}\, e^{\lambda t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(4-3i)t}$$

$$\underset{\sim}{x}(t) = e^{4t}\begin{bmatrix} 1 \\ i \end{bmatrix} e^{i(-3t)} = e^{4t}\begin{bmatrix} 1 \\ i \end{bmatrix}\left(\cos(-3t) + i\sin(-3t)\right)$$

$$= e^{4t}\begin{bmatrix} \cos(3t) - i\sin(3t) \\ i\cos(3t) + \sin 3t \end{bmatrix}$$

$$= e^{4t}\left\{\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + i\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}\right\}$$

$$= \underbrace{e^{4t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}}_{\underset{\sim}{x_1}(t)} + i\underbrace{e^{4t}\begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}}_{\underset{\sim}{x_2}(t)}$$

$x' = Ax$.

$x_1(t)$ & $x_1(t)$ are lin. ind. real vector sols. to ↑

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{x_1}(t) + c_2 \underset{\sim}{x_2}(t)$$

$$= c_1 \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} e^{4t}$$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= 4x_1 - 3x_2 \\ \frac{dx_2}{dt} &= 3x_1 + 4x_2 \end{aligned}\right\} \qquad \begin{aligned} x_1(t) &= e^{4t}(c_1 \cos 3t + c_2 \sin 3t) \\ x_2(t) &= e^{4t}(c_1 \sin 3t + c_2 \cos 3t) \end{aligned}$$

Remark

$$ay'' + by' + cy = 0$$

$$y = e^{rt}$$

$$ar^2 + br + c = 0$$

$$r_{1,2} = \alpha \mp \beta i$$

$$y(t) = e^{\alpha t}\left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right]$$

$ay'' + by' + cy = 0$ second-order, linear, single equation.

Let: $y = x_1$, $y' = x_2 \longleftrightarrow x_1' = x_2$

$\downarrow$

$y'' = x_2'$

$$a x_2' + b x_2 + c x_1 = 0$$

$$ay'' + by' + cy = 0 \xrightarrow{\text{converted to}} \boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{b}{a} x_2 - \frac{c}{a} x_1 \end{aligned}}$$

(c) Repeated / Multiple Eigenvalues

Given the system $x' = Ax$, upon substituting $x = \underset{\sim}{v} e^{\lambda t}$, suppose we have found an eigen-value of multiplicity $k = 2$.

$\lambda_1 = \lambda_2 = \lambda$. Solving $(A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$,

we can find an eigenvector $\underset{\sim}{v}$ and a

solution $\underset{\sim}{x}_1(t) = \underset{\sim}{v} e^{\lambda t}$

<u>Question</u> Can we find a second solution $\underset{\sim}{x}_2(t)$ for the same eigenvalue λ ??

Actually, this case is the same as:

$y'' - 2y' + y = 0 \qquad y = e^{rt} \qquad r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \qquad r_1 = r_2 = 1 \qquad y = c_1 e^t + c_2 t e^t$

Let's suggest that $\underset{\sim}{x}(t) = \underset{\sim}{v}_2 t e^{\lambda t}$ solves $x' = Ax$

$x' = \underset{\sim}{v}_2 e^{\lambda t} + v_2 \lambda t e^{\lambda t}$

$\underset{\sim}{v}_2 e^{\lambda t} + \underset{\sim}{v}_2 \lambda t e^{\lambda t} = A v_2 e^{\lambda t}$

$\lambda v_2 (t e^{\lambda t}) - (A - I) v_2 e^{\lambda t} = 0$

which is possible iff $\underline{v_2} = \underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ No result!!

Let's suggest a solution of the form

$$\underset{\sim}{x}(t) = (\underset{\sim}{v}_1 t + \underset{\sim}{v}_2)\, e^{\lambda t} \quad \text{for} \quad x' = Ax$$

$$\underset{\sim}{x}(t) = \underset{\sim}{v}_1 t e^{\lambda t} + \underset{\sim}{v}_2 e^{\lambda t}$$

$$\underset{\sim}{v}_1 e^{\lambda t} + \underset{\sim}{v}_1 \lambda t e^{\lambda t} + \underset{\sim}{v}_2 \lambda e^{\lambda t} = A(\underset{\sim}{v}_1 t + \underset{\sim}{v}_2) e^{\lambda t}$$

Let's cancel $e^{\lambda t}$'s.

$$\underset{\sim}{v}_1 + \underset{\sim}{v}_1 \lambda t + \underset{\sim}{v}_2 \lambda = A v_1 t + \underset{\sim}{A} v_2$$

$$t \cdot \underbrace{(A - \lambda I) v_1}_{= 0} + \underbrace{(A - \lambda I)\underset{\sim}{v}_2 - \underset{\sim}{v}_1}_{= 0} = \underset{\sim}{0}$$

$$(A - \lambda I)\underset{\sim}{v_1} = \underset{\sim}{0} \Longleftarrow \quad v_1 \text{ is the eigenvector found previously}$$

$$(A - \lambda I)\underset{\sim}{v_2} = v_1$$

$\boxed{\text{ALGORITHM}}$

1. First find a nonzero solution v_1 of

$$(A - \lambda I)v_1 = 0$$

2. and then solve

$$(A - \lambda I)v_2 = v_1 \quad \text{to find } v_2.$$

Example Find a general sol. of the system

$$\underset{\sim}{x}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \underset{\sim}{x} .$$

$$x = v e^{\lambda t} \rightarrow (A - \lambda I) \underset{\sim}{v} = 0$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = (\lambda - 4)^2 = 0$$

$$\lambda_1 = \lambda_2 = 4$$

$$\begin{bmatrix} 1-4 & -3 \\ 3 & 7-4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3a + 3b = 0 \rightarrow a + b = 0$ let $b = 1 \rightarrow a = -1$

$$\underset{\sim}{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \boxed{\underset{\sim}{x}_1(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}}$$

To find the second solution, we propose

$$\underset{\sim}{x}(t) = (\underset{\sim}{v}_1 t + \underset{\sim}{v}_2) e^{\lambda t}$$

$$(A - \lambda I) v_1 = 0 \longrightarrow v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad (\lambda = 4)$$

$$(A - \lambda I) V_2 = v_1$$

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{aligned} -3c - 3d &= -1 \\ 3c + 3d &= 1 \end{aligned}$$

Let's choose $3c + 3d = 1 \xrightarrow{c=0} d = 1/3$

$$v_2 = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} \qquad \underset{\sim}{x}_1(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$$

$$\underset{\sim}{x}_2(t) = (\underset{\sim}{v}_1 t + v_2)\, e^{\lambda t}$$

$$= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} \right) e^{4t}$$

$$x(t) = c_1 \underset{\sim}{x}_1(t) + c_2 \underset{\sim}{x}_2(t)$$

$$= c_1 \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}}_{\underset{\sim}{x}_1(t)} + c_2 \underbrace{\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} \right\}}_{x_2(t)} e^{4t}$$

The solution, explicitly, is

$$x_1(t) = -c_1 e^{4t} + c_2(-t)e^{4t}$$

$$x_2(t) = c_1 e^{4t} + c_2\left(t + \frac{1}{3}\right)e^{4t}$$

<u>Question</u> In calculating v_2, when solving $3c + 3d = 1$, we have chosen $c = 0$. Does this cause any loss of information in the solution?

Let's say $d = \frac{1-3c}{3}$. So we have

$$V_2 = \begin{bmatrix} c \\ (1-3c)/3 \end{bmatrix}$$

$$\Rightarrow \quad \underset{\sim}{x}_2(t) = (\underset{\sim}{v}_1\, t + \underset{\sim}{v}_2)\, e^{\lambda t}$$

$$x_2(t) = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} c \\ (1-3c)/3 \end{bmatrix} \right\} e^{4t}$$

$$\underset{\sim}{x}(t) = c_1\, \underset{\sim}{x}_1(t) + c_2\, \underset{\sim}{x}_2(t)$$

$$= c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} c \\ (1-3c)/3 \end{bmatrix} \right\} e^{4t}$$

$$= c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} t e^{t} + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} \right\} e^{4t} - c_2 c \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$$

$$= \underbrace{(c_1 - c_2 c)}_{\tilde{c}_1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} \right\} e^{4t}$$

$$\underset{\sim}{x}(t) = \tilde{c}_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} \right\} e^{4t}$$

There's no loss of inf. on choosing $c = 0$!!