Week #09: We'll start with 5.2, General Sols. of DEs

Last week: Second-Order Eqs.

$$y'' + p(x)y' + q(x)y = f(x)$$: nonhomogeneous eq.

$$y'' + p(x)y' + q(x)y = 0$$: homogeneous eq.

$$ay'' + by' + cy = 0 \qquad 0 \neq a, b, c \in \mathbb{R} \text{ (constant-coeff.)}$$

$$\Big\downarrow \; y = e^{rx}$$

$$ar^2 + br + c = 0$$ characteristic eq.

(i) $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$ $\qquad y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

(ii) $r_1 = r_2 \in \mathbb{R}$ $\qquad y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

(iii) $r_{1,2} = \alpha \mp \beta i$ $\qquad y = e^{\alpha x} \{ c_1 \cos(\beta x) + c_2 \sin(\beta x) \}$

Principle of Superposition: If $y_1(x)$ and $y_2(x)$ are sols. to $y''+p(x)y'+q(x)y=0$, so is $y=c_1 y_1(x)+c_2 y_2(x)$

Linear independence $c_1 y_1(x) + c_2 y_2(x) = 0 \Rightarrow c_1 = c_2 = 0$

* y_1 and y_2 are linearly ind. on I, if $\exists x_0 \in I$

$$W(y_1, y_2)\Big|_{x_0} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2 \end{vmatrix} \neq 0 \quad \text{at} \quad x = x_0$$

Existence & Uniqueness Th. Given the IVP $\begin{cases} y'' + p(x)y' + q(x)y = f(x) \\ \underline{y(x_0) = y_0}, \ \underline{y'(x_0) = y_1} \end{cases}$

if $p(x)$, $q(x)$ and $f(x)$ are cont. on an interval $I \ni x_0$, then the IVP HAS a UNIQUE solution.

5.2 General Solutions of Linear Equations

$$P_0(x)\, y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_{n-1}(x)\, y' + P_n(x)\, y = F(x)$$

$P_0(x) \not\equiv 0$ divide by $P_0(x)$ and get

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_{n-1}(x)\, y' + p_n(x)\, y = f(x)$$

The homogeneous linear eq. associated with this is

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \cdots + p_{n-1}(x)\, y' + p_n(x)\, y = 0$$

THEOREM Principle of Superposition for Linear Homogeneous Eq.

Assume that $y_1, y_2, \dots, y_n$ are n solutions of the homog. Linear equation

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \dots + P_{n-1}(x)\, y' + P_n(x)\, y = 0$$

on the interval I. Then, the Linear combination

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \qquad (c_1, c_2, \dots, c_n : \text{constants})$$

is also a solution of the d.e. on I.

PROOF: Since $y_1, y_2, \dots, y_n$ are solutions, then

$$y_1^{(n)} + P_1 y_1^{(n-1)} + \dots + P_{n-1} y_1' + P_n y_1 = 0$$
$$y_2^{(n)} + P_1 y_2^{(n-1)} + \dots + P_{n-2} y_2' + P_n y_2 = 0$$
$$\dots$$
$$y_n^{(n)} + P_1 y_n^{(n-1)} + \dots + P_{n-2} y_n' + P_n y_n = 0$$

$$(c_1 y_1 + c_2 y_2 + \dots + c_n y_n)^{(n)} + P_1 (c_1 y_1 + c_2 y_2 + \dots + c_n y_n)^{(n-1)} + \dots$$
$$+ P_{n-1} (c_1 y_1 + c_2 y_2 + \dots + c_n y_n)' + P_n (c_1 y_1 + c_2 y_2 + \dots + c_n y_n)$$
$$= c_1 (y_1^{(n)} + P_1 y_1^{(n-1)} + \dots + P_{n-1} y_1' + P_n y_1)$$
$$+ c_2 (y_2^{(n)} + P_1 y_2^{(n-1)} + \dots + P_{n-1} y_2' + P_n y_2)$$
$$+ \dots$$
$$+ c_n (y_n^{(n)} + P_1 y_n^{(n-1)} + \dots + P_{n-1} y_n' + P_n y_n) = 0 \,//$$

Ex $y_1 = e^{-3x}$, $y_2 = \cos 2x$, $y_3 = \sin 2x$ are solutions of

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

$$y_1 = e^{-3x} \Rightarrow y_1' = -3e^{-3x}, \quad y_1'' = 9e^{-3x}, \quad y_1^{(3)} = -27e^{-3x}$$

$$\Rightarrow y_1^{(3)} + 3y_1'' + 4y_1' + 12y_1 = (-27 + 3.9 - 4.3 + 12)\, e^{-3x} = 0$$

$$y_2 = \cos 2x \Rightarrow y_2' = -2\sin 2x, \quad y_2'' = -4\cos 2x, \quad y_2^{(3)} = 8\sin 2x$$

$$\Rightarrow y_2^{(3)} + 3y_2'' + 4y_2' + 12y_2 = (-3.4 + 12)\cos 2x + (8 - 4.2)\sin 2x = 0$$

$y_3=\sin 2x \Rightarrow y_3'=2\cos 2x,\ y_3''=-4\sin 2x,\ y_3^{(3)}=-8\cos 2x$

$\Rightarrow y_3^{(3)}+3y_3''+4y_3'+12y_3=(-8+4.2)\cos 2x+(-3.4+12)\sin 2x=0$

Thus, any linear combination of y_1, y_2 and y_3 is also a solution of the d.e

$$y=c_1e^{-3x}+c_2\cos 2x+c_3\sin 2x$$

THEOREM Existence and Uniqueness for Linear Eq.

Let $P_1, P_2, \ldots, P_n$ and f be continuous functions on the open interval I containing the point a and let the n numbers $b_0, b_1, \ldots, b_{n-1}$ be given. Then the nth order linear equation

$$y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_{n-1}(x)y'+P_n(x)y=f(x)$$

has a unique (one and only one) solution on I that satisfies the n initial conditions

$$y(a)=b_0,\ y'(a)=b_1,\ \ldots,\ y^{(n-1)}(a)=b_{n-1}.$$

Ex $y^{(3)}+3y''+4y'+12y=0$, $y=c_1e^{-3x}+c_2\cos 2x+c_3\sin 2x$

$y(0)=0,\ y'(0)=1,\ y''(0)=-13$

$y(0)=0 \Rightarrow c_1+c_2=0$

$y'=-3c_1e^{-3x}-2c_2\sin 2x+2c_3\cos 2x,\ y'(0)=1 \Rightarrow -3c_1+2c_3=1$

$y''=9c_1e^{-3x}-4c_2\cos 2x-4c_3\sin 2x,\ y''(0)=-13 \Rightarrow 9c_1-4c_2=-13$

$$\left.\begin{array}{l} c_1+c_2=0 \\ 9c_1-4c_2=-13 \end{array}\right\} c_1=-c_2 \Rightarrow -9c_2-4c_2=-13 \Rightarrow c_2=1 \Rightarrow c_1=-1$$

$-3c_1+2c_3=1 \Rightarrow +3+2c_3=1 \Rightarrow c_3=-1$

$$\boxed{\begin{array}{l} c_1+c_2=0 \\ -3c_1+2c_3=1 \\ 9c_1-4c_2=-13 \end{array}}$$

solve

$$\boxed{y=-e^{-3x}+\cos 2x-\sin 2x}$$

The theorem above implies that there is no other solution which satisfies the same initial values.

Ex Determine the largest interval I so that the IVP

$$y''' + \frac{1}{x-3}\, y'' + \tan x \; y' = \frac{1}{x-5}$$

$y(2) = 1, \quad y'(2) = 2, \quad y''(2) = -1$ has a unique sol.

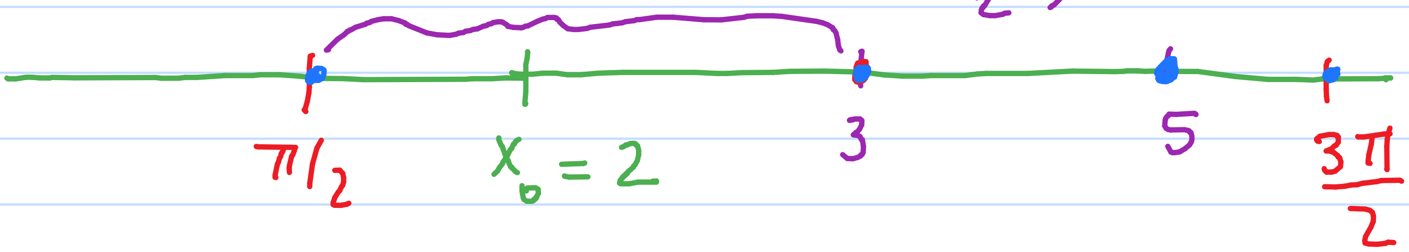
$p_1(x) = \dfrac{1}{x-3}$ is discont. at $x = 3$

$p_2(x) = \tan x = \dfrac{\sin x}{\cos x}$ is disc. at $x = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbb{Z}$.

$f(x) = \dfrac{1}{x-5}$ is discont. at $x = 5$

$I = \left(\dfrac{\pi}{2}, 3\right)$ including $x_0 = 2$

$\dfrac{\pi}{2} \approx \dfrac{3.14}{2} = 1.7$



$3\dfrac{\pi}{2} \simeq 3.1.7$

Ex Determine the largest interval I so that the IVP

$$y''' + \frac{1}{x-3} y'' + \tan x \, y' = \frac{1}{x-5}$$

$$y(2) = 1, \quad y'(2) = 2, \quad y''(2) = -1$$ has a unique sol.

On the interval $I = \left(\frac{\pi}{2}, 3\right)$ which contains $x_0 = 2$; $p_1(x)$, $p_2(x)$ and $f(x)$ are continuous.

The IVP has a unique solution on $I = \left(\frac{\pi}{2}, 3\right)$.

HW Answer the same question if

(a) $x_0 = 0$, (b) $x_0 = 4$, (c) $x_0 = -2$

Ex Existence and uniqueness theorem for linear eq. implies that $y=0$ is the only solution of the hom. eq.

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y = 0 \qquad (*)$$

that satisfies the trivial initial conditions

$$y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0.$$

Read this yourself

Now, let's consider the d.e. $x^2y'' - 4xy' + 6y = 0$ with the initial conditions $y(0) = y'(0) = 0$

As you can easily verify that the trivial solution $y=0$ satisfies the d.e with these initial conditions. But so do $y=x^2$ and $y=x^3$.

Does this contradict with the theorem?

the answer is no. This happens because when we write the d.e in the form of $(*)$ we get

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0$$

for which $P_1(x) = -4/x$ and $P_2(x) = 6/x^2$ are not continuous on an open interval containing the point $x=0$.

LINEAR INDEPENDENCE OF FUNCTIONS

$f_1, f_2, \ldots, f_n$ are linearly independent on I if

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

only when $c_1 = c_2 = \cdots = c_n = 0$. for all $x \in I$.

If you have 2 functions f and g, check if f/g or g/f is a constant valued func. on I. If so, they are linearly dependent

$$\frac{f}{g} = c \Rightarrow f = cg$$

If f & g are linearly dependent;

$\exists\, c_1, c_2$ not both zero such that

$$c_1 f + c_2 g = 0 \rightarrow g = -\frac{c_1}{c_2} f$$

They're constant multiples of each other.

If $f_1, \dots, f_n$ are linearly dependent,

$\exists\, c_1, \dots, c_n$ not all zero:

$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$. Assume $c_i \neq 0$

$$f_i = -\frac{c_1}{c_i} f_1 - \frac{c_2}{c_i} f_2 - \dots - \frac{c_{i-1}}{c_i} f_{i-1} - \frac{c_{i+1}}{c_i} f_{i+1} \dots - \frac{c_n}{c_i} f_n$$

One of the functions can be written as a linear comb. of the others.

Suppose we're given f_1, f_2, f_3:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

$$c_1 f_1' + c_2 f_2' + c_3 f_3' = 0$$

$$c_1 f_1'' + c_2 f_2'' + c_3 f_3'' = 0$$

$$\underbrace{\begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$|A| \neq 0$ $\quad \underset{\sim}{A}\,\underset{\sim}{c} = \underset{\sim}{0} \Longrightarrow A^{-1}AC = A^{-1}\underset{\sim}{0} \Rightarrow \underset{\sim}{c} = 0$

Ex $f(x) = \sin x$, $g(x) = \cos x \Rightarrow \frac{f}{g} = \tan x \Rightarrow f, g$: Lin. indep.

$f(x) = \sin 2x$, $g(x) = \sin x \cdot \cos x \Rightarrow \frac{f}{g} = 2 \Rightarrow f, g$: Lin. dep.

WROSKIAN $f_1, f_2, \dots, f_n$: $(n-1)$ times diff. func.

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \rightarrow \text{Wroskian}$$

Wronski determinant of $f_1, \dots, f_n$.

THEOREM Wroskians of Solutions

Let $y_1, y_2, \dots, y_n$ be the n solutions of the hom. nth order Linear equation

$$y^{(n)} + p_1(x)\, y^{(n-1)} + \dots + p_{n-1}(x)\, y' + p_n(x)\, y = 0$$

on an open int. I where $p_1, p_2, \dots, p_n$ are all continuous

* If $y_1, y_2, \dots, y_n$ are Lin. dependent, then $w(y_1, \dots, y_n) = 0$ on I.
* If $y_1, y_2, \dots, y_n$ are Lin. independent, then $w(y_1, \dots, y_n) \neq 0$ for $\forall x \in I$.

PROOF: $c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$

$c_1 y_1' + c_2 y_2' + \dots + c_n y_n' = 0$

$\vdots$

$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0$

$$\Rightarrow \underbrace{\overbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}}^{A}}_{\text{coeff. matrix}}_{n \times n} \overbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}^{C}_{n \times 1} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{0}_{n \times 1}$$

$\underset{\sim}{A}\, \underset{\sim}{C} = \underset{\sim}{0}$

(i) unique sol. $\longleftrightarrow |A| \neq 0$

$c_1 = c_2 = \dots = c_n = 0 \longleftrightarrow W \neq 0$

(ii) inf. many sols. $\longleftrightarrow |A| = 0$

$c_1, c_2, \dots, c_n$ not all zero $\longleftrightarrow W = 0$

(iii) ~~no sols~~.

We know that a hom. nxn Lin. system of equations has a nontrivial solution (at least one of c_k is nonzero) if and only if its coeff. matrix is not invertible. $\Rightarrow W=0$.

$\Rightarrow y_1, y_2, \dots, y_n$: Lin dependent

$$\boxed{** \quad w' = -p_1(x)\, w \Rightarrow w(x) = k e^{-\int p_1(x)\,dx} \quad \begin{matrix} \nearrow k=0 \Rightarrow w=0 \\ \searrow k\neq 0 \Rightarrow w\neq 0 \end{matrix}}$$

Abel's Th.

Ex $w(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$

$\Rightarrow y_1 = \sin x,\ y_2 = \cos x$ Lin indep.

$w(\sin 2x, \sin x \cdot \cos x) = \begin{vmatrix} \sin 2x & \sin x \cos x \\ 2\cos 2x & \cos^2 x - \sin^2 x \end{vmatrix}$

$= \sin 2x\,(\cos^2 x - \sin^2 x) - \cos 2x\ 2\sin x \cos x$

$= \sin 2x\ \cos 2x - \cos 2x\ \sin 2x = 0$

$\Rightarrow y_1 = \sin 2x,\ y_2 = \sin x \cos x$ Lin dep.

Proof of Abel's Th., $n=3$.

Ex $y^{(3)} + p_1 y'' + p_2 y' + p_3 y = 0,\ y_1, y_2, y_3$

$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1(y_2' y_3'' - y_2'' y_3') - y_2(y_1' y_3'' - y_1'' y_3') + y_3(y_1' y_2'' - y_1'' y_2')$

Read this yourself.

$W' = y_1'(y_2' y_3'' - y_2'' y_3') + y_1(\cancel{y_2'' y_3''} + y_2' y_3^{(3)} - y_2^{(3)} y_3' - \cancel{y_2'' y_3''})$

$-y_2'(\dots) - y_2(\dots)' + y_3'(\dots) + y_3(\dots)'$

$= y_1'(\cancel{y_2' y_3''} - \cancel{y_2'' y_3'}) + y_1(y_2' y_3^{(3)} - y_2^{(3)} y_3')$

$- y_2'(\cancel{y_1' y_3''} - \cancel{y_1'' y_3'}) - y_2(y_1' y_3^{(3)} - y_1^{(3)} y_3')$

$+ y_3'(\cancel{y_1' y_2''} - \cancel{y_1'' y_2'}) + y_3(y_1' y_2^{(3)} - y_1^{(3)} y_2') = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{vmatrix}$

$$y_1^{(3)} = -p_1 y_1'' - p_2 y_1' - p_3 y_1$$

$$y_2^{(3)} = -p_1 y_2'' - p_2 y_2' - p_3 y_2$$

$$y_3^{(3)} = -p_1 y_3'' - p_2 y_3' - p_3 y_3 .$$

$$w' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1 y_1'' - p_2 y_1' - p_3 y_1 & -p_1 y_2'' - p_2 y_2' - p_3 y_2 & -p_1 y_3'' - p_2 y_3' - p_3 y_3 \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1 y_1'' & -p_1 y_2'' & -p_1 y_3'' \end{vmatrix} = -p_1 w .$$

$$w' = -p_1 w \Rightarrow \frac{dw}{w} = -p_1\,dx \Rightarrow \ln w = -\int p_1\,dx \Rightarrow w = e^{-\int p_1 dx}$$

THEOREM General Solutions of Homogeneous Eq.

Let $y_1, y_2, \ldots, y_n$ be n linearly indep. solutions of the hom. eq.

$$y^{(n)} + p_1(x)\,y^{(n-1)} + \cdots + p_{n-1}(x)\,y' + p_n(x)\,y = 0$$

on an open interval I where $p_1, \ldots, p_n$ are all continuous. If Y is any solution of the d.e, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$ is the general solution!

for all $x \in I$.

($y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$: general sol of the d.e).

Th. Solutions of Nonhomogeneous Eq.

Let $y_p(x)$ be a particular solution to

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = \underline{f(x)}$$

and let $y_1(x), \ldots, y_n(x)$ be linearly independent solutions to the associated homogeneous equation

$$L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0.$$

Then,

$$y = \overbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}^{y_c(x)\,:\,\text{complementary solution}} + y_p(x)$$

is the general solution the nonhomogeneous eq. $L[y]=f(x)$.

$y_c(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$, is called the complementary solution; it's the general solution of the associated hom. eq.

$Ly = f(x)$ is solved by $y = y_p(x)$

$Ly = 0$ is solved by $y = y_c(x)$

Claim : $y = y_c + y_p$ solves $L[y] = f$.

<u>Proof</u>

$$L[y] = L[y_c + y_p] = \underbrace{L[y_c]} + \underbrace{L[y_p]}$$

$$= 0 + f = f(x)$$

Ex Find the general sol. to

$$y'' - 3y' + 2y = 0$$

$y = e^{rx}$ $\quad r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r_1 = 1, \quad r_2 = 2$

$y_1 = e^{1 \cdot x} \quad y_2 = e^{2 \cdot x}$

The general sol. is

$$y = c_1 e^{x} + c_2 e^{2x}$$

Ex Find any particular sol. to

$$y'' - 3y' + 2y = 5$$

Let $y(x) = A$, a constant.

$y' = y'' = 0$: $0 - 3 \cdot 0 + 2A = 5$

$A = \frac{5}{2}$ $\quad y_p(x) = \frac{5}{2}$ solves the eq.

Ex Find the general sol. to

$$y'' - 3y' + 2y = 5$$

Step 1 Solve the associated hom. eq.:

$y'' - 3y' + 2y = 0 \Rightarrow y_c = c_1 e^{x} + c_2 e^{2x}$

Step 2 Find any sol. y_p of the nonhom. eq.

$y'' - 3y' + 2y = 5 \Rightarrow y_p = 5/2$

The general sol. to $y'' - 3y' + 2y = 5$

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} + 5/2$$

<u>HW</u> Verify that $y = c_1 e^{x} + c_2 e^{2x} + \frac{5}{2}$

indeed solves $y'' - 3y' + 2y = 5$.

$$\left(c_1 e^{x} + c_2 e^{2x} + \frac{5}{2}\right)'' - 3\left(c_1 e^{x} + c_2 e^{2x} + \frac{5}{2}\right)$$

$$+ 2\left(c_1 e^{x} + c_2 e^{2x} + \frac{5}{2}\right) = \text{------}$$

Do this, strongly suggested!!

Ex $y''-4y=0$ Read this yourself

$y_1=e^{2x}$ and $y_2=e^{-2x}$ are solutions of the de.

$$W(y_1,y_2)=\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2-2=-4\neq 0 \Rightarrow y_1,y_2 \text{: lin indep.}$$

$\Rightarrow y=c_1e^{2x}+c_2e^{-2x}$: general solution of the de.

$y_3=\cosh 2x$ and $y_4=\sinh 2x$ are also solutions of the de.

$$W(y_3,y_4)=\begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix} = 2(\cosh^2 2x-\sinh^2 2x)=2\neq 0$$

$\Rightarrow y_3,y_4$: lin indep.

$\Rightarrow y_3,y_4$ can be written as a lin. combination of y_1 and y_2.

$$\cosh 2x=\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}, \quad \sinh 2x=\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}$$

This means that there are 2 different basis for the solution space of the d.e. $\{e^{2x},e^{-2x}\}$ and $\{\cosh 2x, \sinh 2x\}$.

$\Rightarrow$ Every particular solution Y can be written as a linear combination of both basis

THEOREM Solutions of Nonhomogeneous Eq. *

Let y_p be a particular solution of the nonhom. eq

$$y^{(n)}+p_1(x)y^{(n-1)}+\dots+p_{n-1}(x)y'+p_n(x)y=f(x) \qquad (*)$$

on an open interval I where $p_1,p_2,\dots,p_n$ and f are all continuous. Let $y_1,y_2,\dots,y_n$ be n linearly independent solutions of the associated homog. eq.

$$y^{(n)}+p_1(x)y^{(n-1)}+\dots+p_{n-1}(x)y'+p_n(x)y=0.$$

If Y is any solution of $(*)$ on I, then

$$Y=\underbrace{c_1y_1(x)+c_2y_2(x)+\dots+c_ny_n(x)}_{y_c(x)\text{: complementary function}}+y_p(x)$$

for all $x\in I$.

HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0 \quad , \quad (a_n \neq 0)$$

$\downarrow$

Offer a solution of the form $\boxed{y = e^{rx}}$

$$\underbrace{(a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0)}_{=0} \underbrace{e^{rx}}_{\neq 0} = 0$$

$\downarrow$

Characteristic Equation

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

$r_1, r_2, \dots, r_n$: roots of the char. eq.

Real and Distinct Roots

$r_1 \neq r_2 \neq \dots \neq r_n$, all real

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$

$\underbrace{\{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}}_{\text{basis}}$

Repeated Real Roots

r: repeated real root of multiplicity k

$y = (c_1 + c_2 x + \dots + c_k x^{k-1}) e^{rx}$

$\underbrace{\{e^{rx}, x e^{rx}, \dots, x^{k-1} e^{rx}\}}_{\text{basis}}$

Complex Roots

$r = a \mp ib \quad (b \neq 0)$

$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$\underbrace{\{e^{ax} \cos bx, e^{ax} \sin bx\}}_{\text{basis}}$

Repeated Complex Roots

$r = a \mp ib$ of multiplicity of k

$y = e^{ax}[(b_1 + b_2 x + \dots + b_k x^{k-1}) \cos bx + (c_1 + c_2 x + \dots + c_k x^{k-1}) \sin bx]$

$\underbrace{\{x^p e^{ax} \cos bx, x^p e^{ax} \sin bx\} \quad p = 0, 1, \dots, k-1}_{\text{basis}}$

Third order, constant-coefficient eq:

$$a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0 \xrightarrow{y=e^{rx}} a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$$

- $r_1, r_2, r_3 \in \mathbb{R}$
- $r_1 \in \mathbb{R}, \quad r_2, r_3 \in \mathbb{C}$

Forth order

$$a_4 y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0$$

$$\Big\downarrow y = e^{rx}$$

$$a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$$

$r_1, r_2, r_3, r_4 \in \mathbb{R}$; $\quad r_1, r_2 \in \mathbb{R}, \; r_{3,4} \in \mathbb{C}$; $\quad r_{1,2}, r_{3,4} \in \mathbb{C}$

- r_1, r_2, r_3, r_4 are all distinct
- $r_1 = r_2 = r_3 = r_4$
- $r_1 = r_2, \quad r_3 = r_4$ - -

EX $y''+5y'+6y=0$, $\boxed{y(0)=2,\ y'(0)=3}$

$y=e^{rx}$ plugged in the eq. gives that

$r^2+5r+6=0 \Rightarrow (r+2)(r+3)=0 \Rightarrow r_1=-2, r_2=-3$ (real dist.)

$\Rightarrow \boxed{y=c_1e^{-2t}+c_2e^{-3t}} \Rightarrow y'=-2c_1e^{-2t}-3c_2e^{-3t}$

$\left.\begin{array}{l} y(0)=2 \Rightarrow c_1+c_2=2 \\ y'(0)=3 \Rightarrow -2c_1-3c_2=3 \end{array}\right\} -c_2=7 \Rightarrow c_2=-7 \Rightarrow c_1=9$

$y_1=e^{-2t}$

$\Rightarrow y=9e^{-2t}-7e^{-3t}$

$y_2=e^{-3t}$

EX $y^{IV}-y=0$

$r^4-1=0 \Rightarrow (r+1)(r-1)(r^2+1)=0 \Rightarrow r_1=-1, r_2=1, r_{3,4}=\pm i$

$\Rightarrow y=c_1e^{-t}+c_2e^{t}+c_3\cos t+c_4\sin t$

EX $9y^{(5)}-6y^{(4)}+y^{(3)}=0$

$9r^5-6r^4+r^3=0 \Rightarrow r^3(9r^2-6r+1)=0 \Rightarrow r^3(3r-1)^2=0$

$\Rightarrow r_{1,2,3}=0$, $r_{4,5}=1/3$

$\Rightarrow y=(c_1+c_2x+c_3x^2)e^{0x}+(c_4+c_5x)e^{x/3}$

$=c_1+c_2x+c_3x^2+(c_4+c_5x)e^{x/3}$

EX $Dy=y'$, $D^2y=y''$, $\dots$, $D^ny=y^{(n)} \Rightarrow (D^2+6D+13)^2y=0$

$\Rightarrow (r^2+6r+13)^2=0 \Rightarrow \Delta=36-4.1.13=-16=16i^2$

$\Rightarrow r_{1,2}=\dfrac{-6\mp4i}{2}=-3\mp2i$, multiplicity $k=2$

$\Rightarrow y=e^{-3x}[(c_1+c_2x)\cos 2x+(d_1+d_2x)\sin 2x]$

Ex $\quad y^{(4)} - y = 0 \quad$ Find the general solution

$y = e^{rx} \longrightarrow r^4 - 1 = 0 \rightarrow (r^2-1)(r^2+1) = 0$

$(r-1)(r+1)(r^2+1) = 0 \qquad r_1 = -1, \qquad r_2 = +1$

$r^2 + 1 = 0 \rightarrow r^2 = -1 = i^2 \rightarrow r_{3,4} = \mp i = 0 \mp i$

$r_1 = -1 \rightarrow y_1 = e^{-x}$

$r_2 = +1 \rightarrow y_2 = e^{x}$

$r_{3,4} = 0 \mp i \longrightarrow y = e^{\alpha x}\left[c_3 \cos(\beta x) + c_4 \sin(\beta x)\right]$

$\alpha = 0, \quad \beta = 1$

$y = e^{0 \cdot x}\left[c_3 \cos(1 \cdot x) + c_4 \sin(1 \cdot x)\right]$

$y = c_3 \cos x + c_4 \sin x$

$r_{3,4} = 0 \mp i \longrightarrow y_3 = \cos x, \quad y_4 = \sin x$

When y_1 is any solution, is $x \cdot y_1$ always a solution? $\longrightarrow$ NO

Valid only for repeated roots, up to the order of multiplicity.

$\underline{\underline{Ex}}$ $\quad y^{(4)} - y = 0$ Find the general solution

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$$
$$= c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x$$

Ex $\quad 9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$ General solution?

$y = e^{rx}$: $\quad 9r^5 - 6r^4 + r^3 = 0 \rightarrow r^3(9r^2 - 6r + 1) = 0$

$$r^3 (3r-1)^2 = 0$$

$r^3 = 0 \longrightarrow r_1 = r_2 = r_3 = 0$; $(3r-1)^2 = 0 \rightarrow r_4 = r_5 = \frac{1}{3}$

$r_1 = r_2 = r_3 = 0 \rightarrow y_1 = e^{0.x} = 1$, $y_2 = x.1 = x$, $y_3 = x.x = x^2$

$r_4 = r_5 = \frac{1}{3} \rightarrow y_4 = e^{\frac{1}{3}x}$, $y_5 = x e^{\frac{1}{3}x}$

$$y = c_1 . 1 + c_2 . x + c_3 . x^2 + c_4 . e^{x/3} + c_5 . x . e^{x/3}$$

Remark $Dy = y'$, $D^2 y = y''$, ..., $D^{(n)} y = y^{(n)}$

In this notation, we can write the eq.

$y'' + 3y' + 2y = 0$ as $D^2 y + 3Dy + 2y = 0$

$(D^2 + 3D + 2)\, y = 0$

$$y'' + 3y' + 2y = 0 \xrightarrow{y = e^{rx}} r^2 + 3r + 2 = 0$$

$$(D^2 + 3D + 2)\, y = 0 \xrightarrow{y = e^{rx}} r^2 + 3r + 2 = 0 \qquad D \to r$$

$$r^2 + 3r + 2 = 0 \longleftrightarrow r^2 + 3r + 2 = 0$$

Ex Find the general sol. to $(D^2 + 6D + 13)^2 y = 0$

$$(D^2 + 6D + 13)^2 = (D^2 + 6D + 13)(D^2 + 6D + 13)$$

$$= D^2 (D^2 + 6D + 13) + 6D(D^2 + 6D + 13) + 13(D^2 + 6D + 13)$$

$$= \text{---} = D^4 + 12D + 169$$

$$(D^2+6D+13)^2 y=0 \xrightarrow[D \to r]{y=e^{rx}} (r^2+6r+13)^2=0$$

$$r^2+6r+13=0 \rightarrow r_{1/2}=\frac{-6\mp\sqrt{6^2-4.13}}{2.1}$$

$$r_{1,2}=\frac{-6\mp\sqrt{-16}}{2}=\frac{-6\mp\sqrt{16i^2}}{2}=-3\mp 4i$$

$$r_{1,2}=r_{3,4}=-3\mp 2i \qquad (r^2+6r+13)(r^2+6r+13)=0$$

$$r_1=r_3=-3+2i$$

$$r_2=r_4=-3-2i$$

$r_{1,2} = r_{3,4} = -3 \mp 2i$ $\qquad y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_5$

$$y = e^{-3x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$
$$+ x e^{-3x}\left[c_3 \cos 2x + c_4 \sin 2x\right]$$

The fundamental sols which span the sol. space are:

$$y_1 = e^{-3x} \cos 2x$$
$$y_2 = e^{-3x} \sin 2x$$
$$y_3 = x e^{-3x} \cos 2x$$
$$y_4 = x e^{-3x} \sin 2x$$

$$r_{1,2} = \alpha \mp i\beta \longrightarrow y = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right]$$

$$r_{1,2} = -3 \mp 2i \longrightarrow \alpha = -3, \quad \beta = 2$$

$$y = e^{-3x}\left[c_1 \cos(2x) + c_2 \sin(2x)\right]$$

what if we say $\alpha = -3, \quad \beta = -2$

$$y = e^{-3x}\left[c_1 \cos(-2x) + c_2 \sin(-2x)\right]$$

$$y = e^{-3x}\left[c_1 \cos 2x - c_2 \sin 2x\right]$$

$$y = e^{-3x}\left[\hat{c}_1 \cos 2x + \hat{c}_2 \sin 2x\right]$$

* The Method of Undetermined Coefficients – 2^{nd} order CASE

<u>Ex 1</u> Find the general solution to $y'' - 3y' - 4y = 3e^{2x}$

<u>Step 1</u> Solve the associated homogeneous eq.

$$y'' - 3y' - 4y = 0 \xrightarrow[y=e^{rx}]{} r^2 - 3r - 4 = 0$$

$(r-4)(r+1) = 0 \rightarrow r_1 = -1,\ r_2 = 4 \qquad y_1 = e^{-x},\ y_2 = e^{4x}$

$y_c = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 e^{4x}$ (homogeneous sol.)

<u>Step 2</u> Find any particular sol. $y_p(x)$ of the full eq.

$y'' - 3y' - 4y = 3e^{2x}$ propose a sol. of the form

$y = Ae^{2x}$ for this eq.

$y = Ae^{2x}$, $y' = 2Ae^{2x}$, $y'' = 4Ae^{2x}$

$\Rightarrow 4Ae^{2x} - 3 \cdot 2Ae^{2x} - 4 . Ae^{2x} = 3e^{2x}$

$-6Ae^{2x} = 3e^{2x} \longrightarrow A = -\frac{1}{2}$

$y_p(x) = -\frac{1}{2}e^{2x}$ is a solution to the hom. eq.

Claim: $y = y_c(x) + y_p(x)$ is the general solution.

$$y = c_1 e^{-x} + c_2 e^{4x} - \frac{1}{2}e^{2x}$$

is the general solution to the nonhom. eq.

Indeed,

$y_c(x) = c_1 e^{-x} + c_2 e^{4x}$ solves $L[y] = y'' - 3y' - 4y = 0$

$y_p(x) = -\frac{1}{2} e^{2x}$ solves $L[y_p] = y'' - 3y' - 4y = 3e^{2x}$

Is $y_c(x) + y_p(x)$ a solution to $L[y] = 3e^{2x}$

$$L[y_c(x) + y_p(x)] = L[y_c(x)] + L[y_p(x)]$$
$$= 0 + 3e^{2x}$$
$$= 3e^{2x}$$

HW : Verify that $y = c_1 e^{-x} + c_2 e^{4x} - \frac{1}{2} e^{2x}$

solves $y'' - 3y' - 4y = 3e^{2x}$.

Ex 2 Find the general sol. to $y'' - 3y' - 4y = 2\sin x$

Step 1 $y'' - 3y' - 4y = 0 \longrightarrow y_c(x) = c_1 e^{-x} + c_2 e^{4x}$

Step 2 Find $y_p(x)$ for $y'' - 3y' - 4y = 2\sin x$

$$y_p(x) = A\cos x + B\sin x$$

$$y' = -A\sin x + B\cos x, \quad y'' = -A\cos x - B\sin x$$

$$-A\cos x - B\sin x - 3(-A\sin x + B\cos x) - 4(A\cos x + B\sin x) = 2\sin x$$

$$\cos x(-A - 3B - 4A) + \sin x(-B + 3A - 4B) = 2\sin x$$

$$-\cos x(5A + 3B) + \sin x(3A - 5B) = 2\sin x$$

$$\left.\begin{aligned} 5A + 3B &= 0 \\ 3A - 5B &= 2 \end{aligned}\right\} \quad A = \frac{3}{17}, \quad B = \frac{-5}{17}$$

$$y_p(x) = \frac{3}{17}\cos x - \frac{5}{17}\sin x$$

The general solution is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{4x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x$$

$e^{t}\cos 2t$

$e^{t}\sin 2t$

Ex 3 $\quad y'' - 3y' - 4y = -8e^{x}\cos 2x$

Step 1 $\quad y'' - 3y' - 4y = 0 \rightarrow y_c = c_1 e^{-x} + c_2 e^{4x}$

Step 2 $\quad y'' - 3y' - 4y = -8e^{x}\cos 2x$

$$y_p(x) = A e^{x}\cos 2x + B e^{x}\sin 2x$$

$y_p' = \dots$ $\quad$ $y_p'' = \dots$ $\quad$ DIY $\quad$ HW

Plug them in the eq., to get

$$\left.\begin{aligned} 10A + 2B &= 8 \\ 2A - 10B &= 0 \end{aligned}\right\} \qquad A = \frac{10}{13}, \quad B = \frac{2}{13}$$

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{4x} + \frac{10}{13} e^{x} \cos 2x + \frac{2}{13} e^{x} \sin 2x$$

Ex 4 $y'' - 3y' - 4y = \underbrace{3e^{2x}}_{Ex1} + \underbrace{2\sin x}_{Ex2} - \underbrace{8e^{x}\cos 2x}_{Ex}$

Find the general solution.

Step 1 $y'' - 3y' - 4y = 0 \rightarrow y_c = c_1 e^{-x} + c_2 e^{4x}$

Step 2 • $y'' - 3y' - 4y = 3e^{2x}$ $\quad y_{p_1}(x) = -\frac{1}{2} e^{2x}$

• $y'' - 3y' - 4y = 2\sin x$ $\quad y_{p_2}(x) = \frac{3}{17}\cos x - \frac{5}{17}\sin x$

• $y'' - 3y' - 4y = -8e^{x}\cos 2x$ $\quad y_{p_3}(x) = \frac{10}{13} e^{x}\cos 2x + \frac{2}{13} e^{x}\sin 2x$

$\Rightarrow y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

Ex 4 $\quad y'' - 3y' - 4y = \underbrace{3e^{2x}}_{f_1} + \underbrace{2\sin x}_{f_2} - \underbrace{8e^{x}\cos 2x}_{f_3}$

Find the general solution.

$$y = c_1 e^{-x} + c_2 e^{4x}$$

$$- \frac{1}{2} e^{2x}$$

$$+ \frac{3}{17}\cos x - \frac{5}{17}\sin x$$

$$+ \frac{10}{13} e^{x}\cos 2x + \frac{2}{13} e^{x}\sin 2x$$

When given $L[y] = ay'' + by' + cy = f_1(x) + f_2(x) + f_3(x)$ follow these steps:

- Solve $L[y] = 0 \rightarrow y = y_c(x)$
- Solve $L[y] = f_1(x) \rightarrow y = y_{p_1}(x)$

  $L[y] = f_2(x) \rightarrow y = y_{p_2}(x)$

  $L[y] = f_3(x) \rightarrow y = y_{p_3}(x)$

$\Rightarrow$ $y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$ is the gen. sol.,

since:

$$L[y_c + y_{p_1} + y_{p_2} + y_{p_3}] = L[y_c] + L[y_{p_1}] + L[y_{p_2}] + L[y_{p_3}]$$

$$= 0 + f_1(x) + f_2(x) + f_3(x)$$

Separate f_1, f_2, f_3 so that they're linearly independent!!!

Remark In all of the previous examples, the non-homogeneous term on the RHS is not included in the complementary/homogeneous solution.

Ex $y'' - y = e^{x}$ Find the general solution.

- $y'' - y = 0 \xrightarrow{y=e^{rx}} r^2 - 1 = 0 \to r_1 = -1, \quad r_2 = +1$

$$y_c(x) = c_1 e^{-x} + c_2 e^{x}$$

- $y'' - y = e^{x}$ $\quad y_p(x) = A e^{x}$ will not work;

$y = Ae^{x}, \quad y' = Ae^{x}, \quad y'' = Ae^{x}$

$Ae^{x} - Ae^{x} = e^{x} \Rightarrow 0 = e^{x}$ Not successful!

The reason that the proposed solution, looking at the RHS, $y = Ae^x$ does not work is that it's included in the homogeneous solution; which means, when we perform the ops. on the left for some part of the hom. sol., we'll get 0!!

$$L[y] = y'' - y = e^x \qquad y_c(x) = c_1 e^{-x} + c_2 e^x$$

$$L[Ae^x] = (Ae^x)'' - Ae^x = 0$$

Question: What to do?

$\cdot \quad y'' - y = e^x \qquad y_p(x) = x \cdot Ae^x = Axe^x$

$y' = Ae^x + Axe^x \; ; \; y'' = Ae^x + Ae^x + Axe^x$

$(2Ae^x + Axe^x) - Axe^x = e^x \rightarrow 2Ae^x = e^x$

$A = 1/2 \qquad y_p(x) = \frac{1}{2} x e^x$

$y_c(x) = c_1 e^{-x} + c_2 e^x$

$y_p(x) = \frac{1}{2} x e^x$

$y = y_c + y_p = c_1 e^{-x} + c_2 e^x + \frac{x}{2} e^x$

Ex Find the general sol. to $y'' + 4y = \cos 2x$

- $y'' + 4y = 0 \xrightarrow{y = e^{rx}} r^2 + 4 = 0 \quad r^2 = -4 = 4i^2$

$r_{1,2} = \mp 2i = 0 \mp 2i \;:\; \alpha = 0, \quad \beta = 2$

$y_c(x) = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right] = e^{0.x}\left[C_1 \cos 2x + C_2 \sin 2x\right]$

$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$

- $y'' + 4y = \cos 2x \longrightarrow y_p = x\left(A \cos 2x + B \sin 2x\right)$

$y' = A \cos 2x + B \sin 2x + x\left(-2A \sin 2x + 2B \cos 2x\right)$

$$y'' = -2A\sin 2x + 2B\cos 2x + (-2A\sin 2x + 2B\cos 2x)$$
$$+ x(-4A\cos 2x - 4B\sin 2x)$$

When plugged in the eq. $y'' + 4y = \cos 2x$, this gives:

$$-4A\sin 2x + 4B\cos 2x + x(-4A\cos 2x - 4B\sin 2x)$$
$$+ 4 \cdot x \cdot (A\cos 2x + B\sin 2x) = \cos 2x$$

$$-4A\sin 2x + 4B\cos 2x = \cos 2x \longrightarrow A = 0, \quad 4B = 1$$
$$B = 1/4$$

$$y_p(x) = x \cdot \left[0 \cdot \cos 2x + \frac{1}{4}\sin 2x\right] = \frac{x}{4}\sin 2x$$

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{x}{4}\sin 2x \quad //$$

<u>Ex</u> Find the form of the particular solution to

$$y'' - 6y' + 9y = 3x^4 e^{x} + 2x^3 e^{2x} + x^2 e^{3x}$$

(do not evaluate/find/determine) the constants.

- $y'' - 6y' + 9y = 0 \xrightarrow{y=e^{rx}} r^2 - 6r + 9 = 0 \rightarrow (r-3)^2 = 0$

  $r_1 = r_2 = 3$

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

- $y'' - 6y' + 9y = 3x^4 e^{x}$ $\quad Y_1(x) = (A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0) e^{x}$

$y'' - 6y' + 9y = 2x^3 e^{2x}$ $\quad Y_2(x) = (B_3 x^3 + B_2 x^2 + B_1 x + B_0) e^{2x}$

$y'' - 6y' + 9y = x^2 e^{3x}$ $\quad Y_3(x) = x^2 (D_2 x^2 + D_1 x + D_0) e^{3x}$

$y_p(x) = Y_1(x) + Y_2(x) + Y_3(x)$ $\quad = D_2 x^4 e^{3x} + D_1 x^3 e^{3x} + D_0 x^2 e^{3x}$

Ex Find the form of the general solution to

$$y''' - 4y' = x + 3\cos x + e^{-2x}$$

(do not evaluate the constants)

• $y''' - 4y' = 0 \xrightarrow{y=e^{rx}} r^3 - 4r = 0 \quad r(r-2)(r+2)=0$

$r_1 = 0, \quad r_2 = -2, \quad r_3 = 2$

$y_c(x) = c_1 e^{0.x} + c_2 e^{-2x} + c_3 e^{2x}$

$y_c(x) = c_1 + c_2 e^{-2x} + c_3 e^{2x}$

• $y''' - 4y' = x \longrightarrow Y_1(x) = x(A_1 x + A_0) = A_1 x^2 + A_0 x$

$y''' - 4y' = 3\cos x \longrightarrow Y_2(x) = B_1 \cos x + B_2 \sin x$

$y''' - 4y' = e^{-2x} \longrightarrow Y_3(x) = x.De^{-2x}$

The form of the gen. sol. is $y = y_c + Y_1 + Y_2 + Y_3$

More Examples : Weekend, PS

nth order nonhom. Lin. equations with const. coeff.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots\cdots + a_{n-1} y' + a_n y = f(x) \qquad (1)$$

Find $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

Is $f(x)$ in the form of a polynomial $P_m(x)$, $P_m(x)e^{ax}$, $P_m(x)e^{ax}\cos bt$ or $P_m(x)e^{ax}\sin bt$?

Yes — No

Method of Undetermined Coeff.

$f(x)$	y_p
$P_m(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$	$x^s(A_0 x^m + \cdots + A_m)$
$P_m(x) e^{ax}$	$x^s(A_0 x^m + \cdots + A_m) e^{ax}$
$P_m(x) e^{ax}$ $\cos bx$ or/and $\sin bx$	$x^s[(A_0 x^m + \cdots + A_m)\cos bx + (B_0 x^m + \cdots + B_m)\sin bx] e^{ax}$

Here s is the smallest nonnegative integer for which every term in y_p differs from every term in y_c.

* Then, replace y_p in (1) to determine the values of the coeff in y_p.

Variation of Parameters

- Evaluate $W(y_1, y_2, \ldots, y_n)$
- Find $W_m(y_1, y_2, \ldots, y_n)$ for $m = 1, 2, \ldots, n$ where W_m is the determinant obtained by replacing the mth column of W with $(0, 0, \ldots, 0, 1)$.
- $u_m(x) = \int f(x) \dfrac{W_m}{W} dx$, $m = 1, 2, \ldots, n$
- $y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$

Important Note:
You can use variation of parameters for nth order nonhom. Lin eq even if the coeff. are not constants.
$a_i \to P_i(x)$

$$y = y_c + y_p$$

Ex $y''' - 4y' = e^{3x}$ DIY

$r^3 - 4r = 0 \Rightarrow r(r^2 - 4) = 0 \Rightarrow r_1 = 0,\ r_2 = 2,\ r_3 = -2$

$\Rightarrow y_c = c_1 + c_2 e^{2x} + c_3 e^{-2x}$

$\Rightarrow y_p = Ae^{3x},\ y_p' = 3Ae^{3x},\ y_p'' = 9Ae^{3x},\ y_p''' = 27Ae^{3x}$

$\Rightarrow (27A - 4.3A)e^{3x} = e^{3x} \Rightarrow 15A = 1 \Rightarrow A = 1/15$

$y = y_c + y_p \Rightarrow y = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \frac{1}{15} e^{3x}$

* $y''' - 4y' = 12x + e^{-2x} + 2\cos x$

For $12x$, $y_{p_1} = (Ax + B).x$
↳ because of c_1 in y_c.

$y_{p_1}' = 2Ax + B,\ y_{p_1}'' = 2A,\ y_{p_1}''' = 0$

$\Rightarrow 0 - 4(2Ax + B) = 12x \Rightarrow -8Ax - 4B = 12x + 0 \Rightarrow A = -12/8 = -3/2,\ B = 0$

$\Rightarrow y_{p_1} = -\frac{3}{2}x^2$

For e^{-2x}, $y_{p_2} = Ae^{-2x}.x$
↳ because of $c_3 e^{-2x}$ in y_c.

$y_{p_2}' = A(1 - 2x)e^{-2x},\ y_{p_2}'' = A(-2 - 2 + 4x)e^{-2x} = A(-4 + 4x)e^{-2x}$

$y_p''' = A(4 + 8 - 8x)e^{-2x} = A(12 - 8x)e^{-2x}$

$\Rightarrow A[12 - \cancel{8x} - 4(1 - \cancel{2x})]e^{-2x} = e^{-2x} \Rightarrow 8A = 1 \Rightarrow A = 1/8$

$\Rightarrow y_{p_2} = \frac{1}{8} e^{-2x} x$

For $2\cos x$, $y_{p_3} = A\cos x + B\sin x$

$y_{p_3}' = -A\sin x + B\cos x,\ y_{p_3}'' = -A\cos x - B\sin x,\ y_{p_3}''' = A\sin x - B\cos x$

$A\sin x - B\cos x - 4[-A\sin x + B\cos x] = 2\cos x$

$\Rightarrow 5A\sin x - 5B\cos x = 2\cos x \Rightarrow A = 0,\ B = -2/5$

$\Rightarrow y_{p_3} = -\frac{2}{5}\sin x$

$\Rightarrow y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

$y = c_1 + c_2 e^{2x} + c_3 e^{-2x} - \frac{3}{2}x^2 + \frac{1}{8} e^{-2x} x - \frac{2}{5}\sin x$

Variation of Parameters : The Case for 2^{nd} order eqs.

(Eq1) $y'' + p(x)\, y' + q(x)\, y = f(x)$ → General solution??

Consider the associated hom. eq.

(Eq2) $y'' + p(x)\, y' + q(x)\, y = 0$

Suppose we can find two linearly independent sols. $y_1(x)$ and $y_2(x)$ of this eq. and write the complementary solution as

$$y_c(x) = c_1\, y_1(x) + c_2\, y_2(x).$$

For (Eq1), propose a solution of the form

$$y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

$$y' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}$$

choose : $u_1' y_1 + u_2' y_2 = 0$

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Plug in Eq1 :

$$\begin{aligned} y'' + p(x) y' + q(x) y = \; & u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' \\ & + p(x) (u_1 y_1' + u_2 y_2') \\ & + q(x) (u_1 y_1 + u_2 y_2) = f(x) \end{aligned}$$

$$\begin{aligned} u_1' y_1' + u_2' y_2' + u_1 & \underbrace{(y_1'' + p(x) y_1' + q(x) y_1)}_{=0} \\ & + u_2 \underbrace{(y_2'' + p(x) y_2' + q(x) y_2)}_{=0} = f(x) \end{aligned}$$

$$u_1' y_1' + u_2' y_2' = f(x)$$

$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}\right\} \quad u_1(x) = ? \quad u_2(x) = ?$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(y_1(x), y_2(x)) \neq 0 \quad \left\{ \text{y_1 and y_2 are linearly independent} \right\}$$

$\Rightarrow$ this system has a unique sol. u_1' and u_2'!!

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2(x)\, f(x)}{W(y_1, y_2)}$$

$$u_1(x) = -\int \frac{y_2(x)\, f(x)}{W(y_1, y_2)}\, dx + C_1$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1(x)\, f(x)}{W(y_1, y_2)}$$

$$u_2(x) = \int \frac{y_1(x)\, f(x)}{W(y_1, y_2)}\, dx + C_2$$

$$y = u_1(x)\, y_1(x) + u_2(x)\, y_2$$

$$= \left\{ -\int \frac{y_2 f}{W(y_1, y_2)}\, dx + c_1 \right\} y_1 + \left\{ \int \frac{y_1 f}{W(y_1, y_2)}\, dx + c_2 \right\} y_2$$

$$= c_1\, y_1(x) + c_2\, y_2(x)$$

$$+ \; y_2 \int \frac{y_1 f}{W(y_1, y_2)}\, dx - y_1 \int \frac{y_2 f}{W(y_1, y_2)}\, dx$$

$y_p(x)$

for any eq!!!

Find the general sol. to $y'' + 4y = 3\operatorname{cosec} x$

(i) $y'' + 4y = 0 \xrightarrow[y = e^{rx}]{} r^2 + 4 = 0 \rightarrow r_{1,2} = 0 \mp 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

$y_1(x) = \cos 2x \quad , \quad y_2(x) = \sin 2x$

(ii) $y'' + 4y = 3\operatorname{cosec} x \qquad y = u_1 y_1 + u_2 y_2$

$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}\right\} \rightarrow \begin{aligned} u_1' \cos 2x + u_2' \sin 2x &= 0 \\ u_1'(-2\sin 2x) + u_2' \, 2\cos 2x &= 3\operatorname{cosec} x \end{aligned}$$

$$\left.\begin{aligned} u_1' \cos 2x + u_2' \sin 2x &= 0 \\ u_1'(-2\sin 2x) + u_2' \, 2\cos 2x &= 3\,\mathrm{cosec}\,x \end{aligned}\right\} \begin{aligned} &/\, 2\sin 2x \\ &/\, \cos 2x \end{aligned}$$

$$u_1' \, 2\sin 2x \cos 2x + u_2' \, 2\sin^2 2x = 0$$

$$-u_1' \, 2\sin 2x \cos 2x + u_2' \, 2\cos^2 2x = 3\,\mathrm{cosec}\,x \cos 2x$$

$$u_2' = \frac{3}{2\sin x}\cos 2x = \frac{3}{2\sin x}\left(1 - 2\sin^2 x\right)$$

$$u_2' = \frac{3}{2}\,\mathrm{cosec}\,x - 3\sin x$$

$$u_2(x) = -\frac{3}{2}\ln\left|\mathrm{cosec}\,x + \cot x\right| + 3\cos x + C_2$$

$$\left.\begin{aligned} u_1' \cos 2x + u_2' \sin 2x &= 0 \\ u_1'(-2\sin 2x) + u_2' \, 2\cos 2x &= 3\,\mathrm{cosec}\,x \end{aligned}\right\}$$

$$u_1' \cdot \cos 2x + \frac{3}{2\sin x} \cos 2x \sin 2x = 0$$

$$u_1' = -\frac{3}{2} \frac{1}{\sin x} 2\sin x \cos x = -3\cos x$$

$$\boxed{u_1 = -3\sin x + c_1}$$

$$u_2(x) = -\frac{3}{2} \ln\left|\mathrm{cosec}\,x + \cot x\right| + 3\cos x + c_2$$

$$y = \left(c_1 - 3\sin x\right) \cos 2x$$
$$+ \left(c_2 - \frac{3}{2} \ln\left|\mathrm{cosec}\,x + \cot x\right| + 3\cos x\right) \sin 2x$$

$$y = (c_1 - 3\sin x)\cos 2x$$
$$+ \left(c_2 - \frac{3}{2}\ln|\operatorname{cosec} x + \cot x| + 3\cos x\right)\sin 2x$$

$$y = c_1 \cos 2x + c_2 \sin 2x - 3\sin x \cos 2x$$
$$+ 3\cos x \sin 2x - \frac{3}{2}\ln|\operatorname{cosec} x + \cot x| \quad //$$

$$\int \operatorname{cosec} x \, dx = \int \operatorname{cosec} x . \frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x}$$

$$= \ldots = -\ln|\operatorname{cosec} x + \cot x| + C$$

VARIATION OF PARAMETERS

$$y^{(n)}+P_1(x)y^{(n-1)}+\dots+P_n y=f(x)$$

$$y_c=c_1y_1+c_2y_2+\dots+c_ny_n$$

$$y_p=u_1y_1+u_2y_2+\dots+u_ny_n$$

$$y_p'=(u_1y_1'+u_2y_2'+\dots+u_ny_n')+\underbrace{(u_1'y_1+u_2'y_2+\dots+u_n'y_n)}_{=0}$$

We assume that the 2nd term on the right hand side is 0 to make the calculation as simple as possible.

$$y_p''=(u_1y_1''+u_2y_2''+\dots+u_ny_n'')+\underbrace{(u_1'y_1'+u_2'y_2'+\dots+u_n'y_n')}_{=0}$$

If we go on like this, we'll eventually get

$$y_p^{(n)}=(u_1y_1^{(n)}+u_2y_2^{(n)}+\dots+u_ny_n^{(n)})+(u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\dots+u_n'y_n^{(n-1)}).$$

Since $y^{(n)}+P_1(x)\,y^{(n-1)}+\dots+P_n(x)\,y=f(x)$, then

$$y_p^{(n)}+P_1\,y_p^{(n-1)}+\dots+P_ny_p=f(x).$$

$$(u_1y_1^{(n)}+u_2y_2^{(n)}+\dots+u_ny_n^{(n)})+(u_1'y_1^{(n-1)}+\dots+u_n'y_n^{(n-1)})$$

$$+P_1\,(u_1y_1^{(n-1)}+\dots+u_ny_n^{(n-1)})+\dots+P_n(u_1y_1+\dots+u_ny_n)=f(x)$$

$$\Rightarrow u_1(y_1^{(n)}+P_1y_1^{(n-1)}+\dots+P_ny_1)+u_2(y_2^{(n)}+P_1y_2^{(n-1)}+\dots+P_ny_2)$$

$$+\dots+u_n(y_n^{(n)}+P_1y_n^{(n-1)}+\dots+P_ny_n)+(u_1'y_1^{(n-1)}+\dots+u_n'y_n^{(n-1)})=f(x)$$

Each paranthesis except the last one is zero since $y_1, y_2, \dots, y_n$ are all solutions of the homog. equation $y^{(n)}+P_1y^{(n-1)}+\dots+P_ny=0$. Then $u_1'y_1^{(n-1)}+\dots+u_n'y_n^{(n-1)}=f(x)$

Now, we have n unknowns and n equations.

$$y_1u_1'+y_2u_2'+\dots+y_nu_n'=0$$

$$y_1'u_1'+y_2'u_2'+\dots+y_n'u_n'=0$$

$$\dots$$

$$\dots$$

$$y_1^{(n-1)}u_1'+y_2^{(n-1)}u_2'+\dots+y_n^{(n-1)}u_n'=f(x).$$

Since $W(y_1, y_2, \dots, y_n) \neq 0$ ($y_1, y_2, \dots, y_n$ are all Lin. indep), we can find $u_1', u_2', \dots, u_n'$ by using Cramer's rule

$$\Rightarrow u_m' = \frac{f(x)\, W_m}{W} \Rightarrow u_m = \int f(x) \cdot \frac{W_m}{W}\, dx$$

$$y_p = \sum_{m=1}^{n} y_m u_m = \sum_{m=1}^{n} y_m \int f(x) \frac{W_m}{W}\, dx$$

Ex $y'' + y = \tan x$

$$\Rightarrow r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i \Rightarrow y_c = c_1 \underbrace{\cos x}_{=y_1} + c_2 \underbrace{\sin x}_{=y_2}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x, \quad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$$

$$u_1 = \int \tan x \cdot \frac{-\sin x}{1}\, dx = -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1 - \cos^2 x}{\cos x}\, dx$$

$$= \int \cos x\, dx - \int \sec x\, dx = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \tan x \frac{\cos x}{1}\, dx = \int \sin x\, dx = -\cos x$$

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2 = (\cancel{\sin x} - \ln|\sec x + \tan x|)\cos x + \cancel{(-\cos x)}\sin x$$

$$= -\ln|\sec x + \tan x| \cdot \cos x$$

$$\Rightarrow y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \cos x \cdot \ln|\sec x + \tan x|$$

EULER'S EQUATION

$$ax^2y'' + bxy' + cy = 0 \rightarrow \text{2nd order Euler eq.}$$

a, b, c: constants

Let's assume that $x>0$ and let's use the substitution $v = \ln x$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dv}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dv}\right) = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d}{dv}\left(\frac{dy}{dv}\right)\underbrace{\frac{dv}{dx}}_{\frac{1}{x}}$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}$$

$$\Rightarrow ax^2\left[-\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}\right] + bx\left[\frac{1}{x}\frac{dy}{dv}\right] + cy = 0$$

$$\Rightarrow a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0 \rightarrow \text{constant coefficient linear eq.}$$

Assume that r_1 and r_2 are two distinct real roots of the characteristic equation. Then

$$y = c_1e^{r_1v} + c_2e^{r_2v}$$

$$= c_1e^{r_1\ln x} + c_2e^{r_2\ln x}$$

$$= c_1e^{\ln x^{r_1}} + c_2e^{\ln x^{r_2}}$$

$$\Rightarrow y = c_1x^{r_1} + c_2x^{r_2} : \text{gen. sol. of the Euler eq.}$$

Ex $x^2y'' + xy' - y = 0$ Eul eq. $a=1, b=1, c=-1 \Rightarrow \frac{d^2y}{dv^2} - y = 0$

$\Rightarrow r^2 - 1 = 0 \Rightarrow r_{1,2} = \pm 1$

$\Rightarrow y = c_1x + c_2x^{-1}$

REDUCTION OF ORDER

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Suppose that one solution $y_1(x)$ of the homogeneous 2nd order Linear diff eq is known

$\Rightarrow$ $y(x) = v(x)\,y_1(x)$ where y is the gen. sol. of the diff eq

$$y = v'y_1 + vy_1' \Rightarrow y'' = v''y_1 + 2v'y_1' + vy_1''$$

$$\Rightarrow v''y_1 + 2v'y_1' + vy_1'' + p\,(v'y_1 + vy_1') + q\,v\,y_1 = 0$$

$$v\underbrace{(y_1'' + py_1' + q\,y_1)}_{=0 \text{ since } y_1 \text{ is the solution of diff eq}} + v''y_1 + 2v'y_1' + pv'y_1 = 0$$

$$\Rightarrow y_1 v_1'' + (2y_1' + py_1)\,v' = 0$$

$$\Rightarrow \frac{dv'}{v'} = -\frac{2y_1' + py_1}{y_1}\,dx \rightarrow \text{separable diff eq.}$$

EX $x^2y'' + xy' - 9y = 0$, $x > 0$, $y_1(x) = x^3$

$$y_1 = x^3 \Rightarrow y_1' = 3x^2,\ y_1'' = 6x \Rightarrow x^2\,6x + x\cdot 3x^2 - 9x^3 = 0$$

$$\Rightarrow x^2y_1'' + xy_1' - 9y_1 = 0$$

$$y'' + \frac{1}{x}y' - \frac{9}{x^2}y = 0 \Rightarrow p = \frac{1}{x}$$

$$\frac{dv'}{v'} = -\frac{2.3x^2 + \frac{1}{x}x^3}{x^3}\,dx = -\frac{7}{x}\,dx \Rightarrow \ln v' = -7\ln x + \ln c_1$$

$$\ln v' = \ln(c_1x^{-7}) \Rightarrow v' = c_1x^{-7} \Rightarrow dv = c_1x^{-7}\,dx$$

$$\Rightarrow v = -c_1\frac{x^{-6}}{6} + c_2 \Rightarrow y = v\,y_1 = -c_1\underbrace{\frac{x^{-3}}{6}}_{y_2} + c_2\underbrace{x^3}_{y_1}$$

$$\Rightarrow y_2 = x^{-3}$$