Ch 6 Eigenvalues & Eigenvectors (Özdeğerler & Özvektörle

Def The number λ is said to be an eigenvalue of the $n \times n$ matrix A provided there's a nonzero vector $\underset{\sim}{v}$ such that

$$A v = \lambda v.$$

λ: is an eigenvalue with eigenvector $\underset{\sim}{v}$.

The vector $\underset{\sim}{v}$ is called the eigenvector corresponding to / associated with the eigenvalue λ

$$A_{n \times n} V_{n \times 1} = \lambda V_{n \times 1}$$

Ex

$$A = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$$

(i) $v = (2,1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A\underset{\sim}{v} = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\underset{\sim}{v}$$

$\underset{\sim}{v}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

(ii) $v = (3,2) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$Av = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = v = 1 \cdot \underset{\sim}{v}$$

$\underset{\sim}{v}$ is an eigenvector of A with eigenvalue $\lambda = 1$.

Remark 1 For any number λ, $\underset{\sim}{v} = \underset{\sim}{0}$ satisfies the equation

$$A \underset{\sim}{v} = A \underset{\sim}{0} = \underset{\sim}{0} = \lambda \underset{\sim}{0} = \lambda \underset{\sim}{v}$$

for any matrix V. Therefore, since $\underset{\sim}{v} = \underset{\sim}{0}$ satisfies the eq. $Av = \lambda v$ for any λ trivially, it's of no importance.

* Eigenvalue can be equal to zero ($\lambda = 0$) ✓

* Eigenvector must be nonzero, by definition!! ($\underset{\sim}{v} \neq \underset{\sim}{0}$)

Remark 2 If $\underset{\sim}{v}$ is an eigenvector of A with eigenvalue λ, $k \cdot \underset{\sim}{v}$ where $0 \neq k \in \mathbb{R}$ is also an eigenvector of A with eigenvalue λ.

$Av = \lambda v$; claim: $\underset{\sim}{u} = k \cdot \underset{\sim}{v}$ is also an eigenvalue

$$Au = A(k\underset{\sim}{v}) = k(Av) = k(\lambda v) = \lambda(kv)$$
$$= \lambda u$$

$Au = \lambda u \Rightarrow \underset{\sim}{u} = k \cdot \underset{\sim}{v}$ is an eigenvector of A with the same eigenvalue λ.

Moral

Given a matrix A, let $\underset{\sim}{v}$ be an eigenvector of A with eigenvalue λ.

- $\underset{\sim}{v}$
- $-2\underset{\sim}{v}$
- $2\underset{\sim}{v}$
- $-4\underset{\sim}{v}$
- $3\underset{\sim}{v}$

$\Rightarrow$ All eigenvectors of A with eigenvalue λ.

The Characteristic Equation

In case of a 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

we find the eigenvalues & eigenvectors as follows.

We need to solve $A\,v = \lambda\, v$ for $\underset{\sim}{v} \neq \underset{\sim}{0}$

$$A_{2\times2}\, V_{2\times1} = \lambda\, V_{2\times1}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} \neq 0 \quad \text{unique sol.} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we don't want this

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \rightarrow \text{inf. many sols.}$$

This <u>will</u> give me nonzero $\begin{bmatrix} x \\ y \end{bmatrix}$.

The char. eq. in this case

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0$$

$$(\lambda - a_{11})(\lambda - a_{22}) - a_{21}a_{12} = 0$$

$$\boxed{\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0}$$

Trace$(A) = a_{11} + a_{22}$ Det$A = a_{11}a_{22} - a_{21}a_{12}$

{For those who're interested: Look at Cayley-Hamilton Th.}

When A is an $n \times n$ matrix,

$$Av = \lambda v \rightarrow Av = \lambda (I v) = (\lambda I) v$$

$$Av - (\lambda I) v = 0 \Rightarrow (A - \lambda I) v = 0$$

To solve this eq. for $\underset{\sim}{v} \neq 0$, we must put the cond:

$$|A - \lambda I| = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{nn} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$$(-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0 = 0 \quad A.$$

The form of the char. eq. for an $n \times n$ matrix

Algorithm: Finding Eigenvalues and Eigenvectors

① Solve the characteristic eq.

$$|A - \lambda I| = 0$$

② For each eigenvalue λ found in the first step, determine the corresponding eigenvector $\underset{\sim}{v}$ by solving the linear system

$$(A - \lambda I)\, \underset{\sim}{v} = \underset{\sim}{0}$$

Ex Find the eigenvalues of $A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{vmatrix} = (5-\lambda)(-4-\lambda) - (-2).7$$

$$= \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$$

$\lambda_1 = -2$ $\qquad$ $\lambda_2 = 3$

$\boxed{\lambda_1 = -2}$ $(A - \lambda_1 I)v = 0$

$$\begin{bmatrix} 5-(-2) & 7 \\ -2 & -4-(-2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{pmatrix} 7 & 7 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 7x + 7y = 0 \\ -2x - 2y = 0 \end{cases} \qquad x + y = 0$$

Let $y = -1 \Rightarrow x = 1$

$\underset{\sim}{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -2$.

$\boxed{\lambda_2 = 3}$ $(A - \lambda_2 I)\underset{\sim}{v} = \underset{\sim}{0}$

$$\begin{bmatrix} 5-3 & 7 \\ -2 & -4-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boxed{\begin{matrix} 2x + 7y = 0 \\ -2x - 7y = 0 \end{matrix}}$$

Let's choose $y = -2$: $2x + 7\cdot(-2) = 0$

$x = 7$

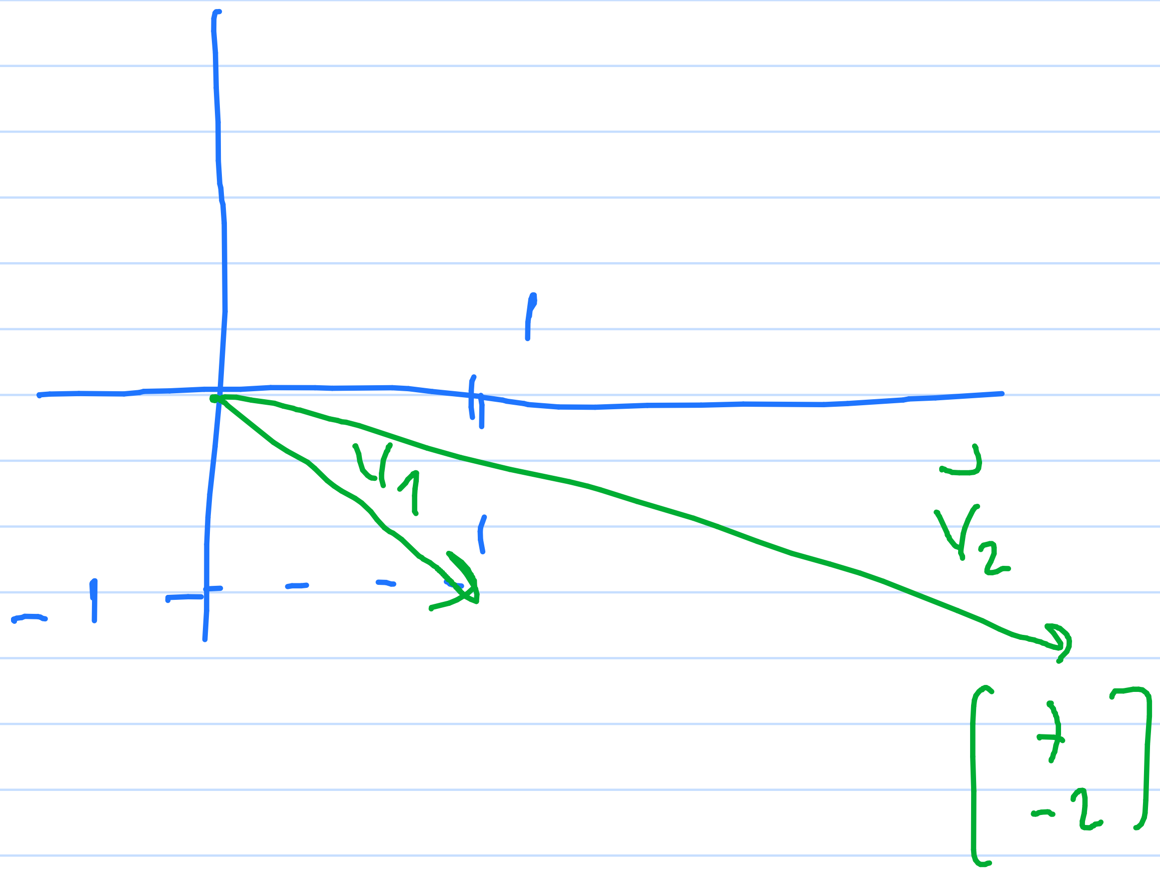
$\underset{\sim}{v_2} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 3$.

In this first example, for a 2x2 matrix, we have found 2 real, distinct eigenvalues and corresponding eigenvectors. What are the other possible cases?

$$\lambda^2 + p\lambda + q = 0$$

- $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$: 2 eigenvectors — linearly ind.
- $\lambda_1 = \lambda_2 \in \mathbb{R}$: How many eigenvectors? (linearly independent)
- $\lambda_1, \lambda_2 \in \mathbb{C}$: How many eigenvectors?

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$ of the first example



Ex $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Eigenvalues & eigenvectors?

(i)

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) = (\lambda-1)^2 = 0$$

$$\lambda_1 = \lambda_2 = 1$$

(ii) $\quad \lambda = 1 \qquad \underline{(A - \lambda I)}\, \underset{\sim}{v} = 0$

$$\begin{bmatrix} 1-1 & 0 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 0 \cdot x + 0 \cdot y &= 0 \\ 0 \cdot x + 0 \cdot y &= 0 \end{aligned}\right\} \quad x \,\&\, y \text{ are arbitrary.}$$

Let $x = 1, \quad y = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let $x = 0, \quad y = 1 \quad \Rightarrow \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Indeed, v_1 & v_2 are linearly independent. We have successfully found two linearly ind. vectors for the unique eigenvalue $\lambda = 1$.

Ex

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

(i)

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$$

$$\lambda_1 = \lambda_2 = 2 = \lambda$$

(ii) $(A - \lambda I) v = 0$

$$\begin{bmatrix} 2-2 & 3 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underset{\sim}{0}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} 0.x + 3.y = 0 \\ 0.x + 0.y = 0 \end{matrix}\right\} \Rightarrow y = 0$$

There's no restr. on x.

choose $x = 1$: $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

In this case, for the multiple eigenvalue $\lambda_1 = \lambda_2 = \lambda = 2$, we could find just one eigenvector!

Ex $A = \begin{bmatrix} 0 & 8 \\ -2 & 0 \end{bmatrix}$

i

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & 8 \\ -2 & 0-\lambda \end{vmatrix} = \lambda^2 + 16 = 0$$

$$\lambda^2 = -16 = 16\, i^2 \Rightarrow \lambda = \mp 4i \quad \begin{cases} \lambda_1 = 4i \\ \lambda_2 = -4i \end{cases}$$

(ii) $\boxed{\lambda_1 = 4i}$ $\quad (A - \lambda_1 I)\underset{\sim}{v} = \underset{\sim}{0}$

$$\begin{bmatrix} 0-4i & 8 \\ -2 & 0-4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -4ix + 8y &= 0 \\ -2x - 4iy &= 0 \end{aligned}\right\} \quad \text{Let } y = -1$$

$$-4ix + 8.(-1) = 0$$

$$x = \frac{-2}{i} = \frac{-2}{i} \cdot \frac{i}{i} = \frac{-2i}{i^2} = \frac{-2i}{-1} = 2i$$

$$V_1 = \begin{bmatrix} 2i \\ -1 \end{bmatrix}$$

For $\lambda_2 = -4i$, you will find $V_2 = \begin{bmatrix} -2i \\ -1 \end{bmatrix}$

The reason is, if $\underset{\sim}{v}$ is a complex eigenvector of A with complex eigenvalue λ, then $\underset{\sim}{v}^*$ is an eigenvector of A with eigenvalue λ^*; where A is a matrix with real entries. Indeed,

$$A v = \lambda v \longrightarrow (A v)^* = (\lambda v)^*$$

$$A^* v^* = \lambda^* v^* \longrightarrow \underbrace{A v^* = \lambda^* v^*}_{v^* \text{ is an eigenvector with eigenvalue } \lambda^*}$$

$\boxed{A^* = A}$ (A is a real matrix)

Example Find the eigenvalues & eigenvectors of

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$$

(i) $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ -4 & 6-\lambda & 2 \\ 16 & -15 & -5-\lambda \end{vmatrix} = \lambda(\lambda-1)(3-\lambda) = 0$$

verify this yourself

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3$$

$\boxed{\lambda_1 = 0}$ $\qquad \underline{(A - \lambda_1 I)} \, \underset{\sim}{v} = \underset{\sim}{0}$

$$\begin{bmatrix} 3-0 & 0 & 0 \\ -4 & 6-0 & 2 \\ 16 & -15 & -5-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 3x = 0 \\ -4x + 6y + 2z = 0 \\ 16x \;\; -15y \;\; -5z = 0 \end{array}\right\} \to \begin{array}{l} x = 0 \\ \left.\begin{array}{l} 3y + z = 0 \\ -15y - 5z = 0 \end{array}\right\} \end{array}$$

Let $y = 1 \implies z = -3$ $\qquad v_1 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

$\boxed{\lambda_2 = 1}$ $\qquad (A - \lambda_2 I)\underset{\sim}{v} = \underset{\sim}{0}$

$$\begin{bmatrix} 3-1 & 0 & 0 \\ -4 & 6-1 & 2 \\ 16 & -15 & -5-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 2x &= 0 \\ -4x + 5y + 2z &= 0 \\ 16x - 15y - 6z &= 0 \end{aligned}\right\} \rightarrow x = 0$$

$$\left.\begin{aligned} 5y + 2z &= 0 \\ -15y - 6z &= 0 \end{aligned}\right\}$$

Let $y = 2 \rightarrow 5.2 + 2z = 0$

$z = -5$

$$V_2 = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$\boxed{\lambda_3 = 3} \qquad (A - \lambda_3 I)\,\underset{\sim}{v} = \underset{\sim}{0}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ 16 & -15 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x = 1, \quad y = 0, \quad z = 2$

$$V_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

: verify this yourself

Def The solution space of

$$(A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$$

for a fixed λ is called eigenspace of A associated with the eigenvalue λ.

For the matrix A of the prev. example,

Eigenspace of A associated with $\lambda_1 = 0$ is

$$\{ k v_1 \mid k \neq 0 \} = \left\{ k \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

For $\lambda_2 = 1$: $\left\{ k \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix} \mid k \neq 0, k \in \mathbb{R} \right\}$

For $\lambda_3 = 3$ $\left\{ k \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \mid k \neq 0, k \in \mathbb{R} \right\}$

This Saturday, at 13.00

1^{st} hour : Review on Higher Order Diff. Eqs

2^{nd} hour : continue with Eigenvalues &

Eigenvectors !

$$y'' + y = 0 \qquad y = c_1 \cos x + c_2 \sin x$$

$$\frac{d^2 y}{dx^2} = -y \qquad y_1(x) = \cos x \rightarrow \boxed{c_1 \cos x}$$

$$y_2(x) = \sin x$$

$$\underbrace{\frac{d^2}{dx^2}}_{L} y = \underbrace{(-1)}\, y$$

y_1, y_2 are sols. to this DE.

$$L y = \lambda y$$

y_1 and y_2 are two eigen "functions" of the operator $L = \frac{d^2}{dx^2}$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\,\psi = E\,\psi$$ Schrödinger eq. of quantum mechanics

ψ : wave function of a particle

$V(x)$: is the potential the particle is subjected to (electrical, gravitational

E : energy of the particle.

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]}\psi = E\,\psi$$

$$L\,\psi = E\,\psi$$

The energy E of the particle is nothing but an eigenvalue of the diff. op L!!!!