13.01.2021 Linear Systems of First Order Eqs, Cont.

$$\underset{\sim}{X}' = \underset{\sim}{A}\,\underset{\sim}{X} \qquad \underset{\sim}{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\underset{\sim}{X}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 \end{aligned}$$

$x_1(t) = ?$ $\quad x_2(t)$

In general, $X = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $\quad A = \begin{bmatrix} & \\ & \end{bmatrix}_{n \times n}$

$x_1(t) = ?$ $\quad - - - \quad$ $x_n(t) = ?$

$$x' = Ax \Rightarrow \underset{\sim}{x}(t) = \underset{\sim}{v}\, e^{\lambda t} \quad \text{when put in } x' = Ax,$$

$$|A - \lambda I| = 0, \qquad (A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$$

<u>Real, distinct eigenvalues λ</u>: λ_1, $\underset{\sim}{v_1}$ $\qquad$ λ_2, v_2

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{x_1}(t) + c_2 \underset{\sim}{x_2}(t)$$

<u>Complex eigenvalues λ</u>

$$\underset{\sim}{x}(t) = \underset{\sim}{u}(t) + i\,\underset{\sim}{v}(t)$$

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{u}(t) + c_2 \underset{\sim}{v}(t)$$

<u>Multiple eigenvalues λ</u>

$$x = \underset{\sim}{v}\, e^{\lambda t}$$

$$x = (v_1 t + v_2) e^{\lambda t} \quad \begin{cases} (A - \lambda I) v_1 = 0 \\ (A - \lambda I) v_2 = v_1 \end{cases}$$

Now, let's turn back to the beginning (!), and see the theoretical/conceptual arguments regarding first order linear systems.

$$x_1'(t) = p_{11}(t)\,x_1 + p_{12}(t)\,x_2 + \cdots + p_{1n}(t)\,x_n + f_1(t)$$

$$x_2'(t) = p_{21}(t)\,x_1 + p_{22}(t)\,x_2 + \cdots + p_{2n}(t)\,x_n + f_2(t)$$

$$\vdots$$

$$x_n'(t) = p_{n1}(t)\,x_1 + p_{n2}(t)\,x_2 + \cdots + p_{nn}(t)\,x_n + f_n(t)$$

The most general first order LINEAR system of DEs, which can be put into the form

$$\underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}}_{\underset{\sim}{x}'} = \underbrace{\begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & & & \\ p_{n1}(t) & p_{n2}(t) & & p_{nn}(t) \end{bmatrix}}_{\underset{\sim}{P}(t)} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\underset{\sim}{x}(t)} + \underbrace{\begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}}_{\underset{\sim}{f}(t)}$$

$$\Rightarrow \quad \underset{\sim}{x}' = \underset{\sim}{P}(t)\,\underset{\sim}{x} + \underset{\sim}{f}(t)$$

The Existence and Uniqueness Th.

Suppose all functions $p_{ij}(t)$ $1 \le i,j \le n$ and $f_1, \ldots, f_n$ are continuous on an interval I containing $t = a$. Then,

The Existence and Uniqueness Th.

Suppose all functions $p_{ij}(t)$ $1 \le i, j \le n$ and $f_1, \dots, f_n$ are continuous on an interval I containing $t = a$. Then, the IVP,

$$\underset{\sim}{x}' = \underset{\sim}{P}(t)\,\underset{\sim}{x} + \underset{\sim}{f}(t)$$

$$x_1(0) = b_1, \quad x_2(0) = b_2, \; \dots \;, \quad x_n(0) = b_n$$

has a unique solution.

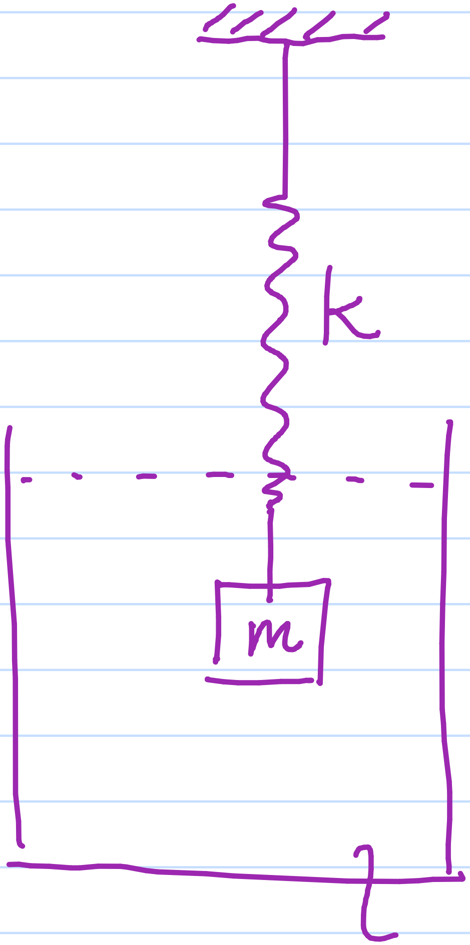
Question Give me an example of an IVP in two unknowns.

Ex

$$m x'' + b x' + c x = 0 \quad *$$

$$x(t=0) = x_0, \qquad x'(t=0) = v_0$$

position of the mass at $t=0$ — velocity of the mass when $t=0$



viscous medium

Let $x(t) = x_1(t) \rightarrow x_1' = x' = x_2$

Let $x'(t) = x_2(t) \rightarrow x''(t) = x_2'$

$$\boxed{m x_2' + b x_2 + c x_1 = 0}$$

$$\begin{cases} x_1' = x_2 & x_1(0) = x(0) = x_0 \\ x_2' = -\dfrac{c}{m} x_1 - \dfrac{b}{m} x_2 & x_2(0) = x'(0) = v_0 \end{cases}$$

b: friction coefficient (Viscosity)

IVP in two-unknowns

Now, we'll say more on the solutions to

$$\underset{\sim}{x}'(t) = P(t)\,\underset{\sim}{x} \qquad (*)$$

a homogeneous ($\underset{\sim}{f} \equiv 0$), first-order, linear system of DEs.

Theorem 1 Let $\underset{\sim}{x}_1(t), \dots, \underset{\sim}{x}_n(t)$ be solutions to $(*)$. Then the superposition

$$\underset{\sim}{x}(t) = c_1\,\underset{\sim}{x}_1(t) + \dots + c_n\,\underset{\sim}{x}_n(t)$$

is also a solution to $(*)$.

Proof: Straightforward. Try it yourself.

Def. Wronskian of Vector Functions

$$\underset{\sim}{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \quad \underset{\sim}{x}_2(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix}, \ \cdots, \quad \underset{\sim}{x}_n(t) = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

The Wronskian of $\underset{\sim}{x}_1(t), \cdots, \underset{\sim}{x}_n(t)$ is defined as

$$W(\underset{\sim}{x}_1, \cdots, \underset{\sim}{x}_n) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{12}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{1n}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$

Theorem Linear independence / dependence

n vector functions $\underset{\sim}{x}_1(t), ---, \underset{\sim}{x}_n(t)$ are

* linearly independent $\Longleftrightarrow W(\underset{\sim}{x}_1, ---, x_n) \neq 0$

* linearly dependent $\Longleftrightarrow W(x_1, ---, x_n) = 0$

Theorem If $\underset{\sim}{x}_1(t), ---, x_n(t)$ are linearly independent solutions to the system $\underset{\sim}{x}' = \underset{\sim}{P}(t)\underset{\sim}{x}$, the general solution of is

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{x}_1(t) + --- + c_n x_n(t).$$

Chapter 10 Laplace Transform Methods

10.1 Laplace Transforms and Inverse Transforms

Def. Let $f(t)$ be defined for $t \geq 0$. If the improper integral

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

converges, then it's called the Laplace transform of $f(t)$.

* This will be very useful in solving linear DEs!!

Ex $\quad f(t) = 1 \implies \mathcal{L}[f(t)] = ?$

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, 1\, dt = \lim_{B\to\infty} \int_0^B e^{-st}\, dt$$

$$= \lim_{B\to\infty} \left[-\frac{1}{s} e^{-st} \right] \Bigg|_{t=0}^{t=B}$$

$$= \lim_{B\to\infty} \left[-\frac{1}{s} e^{-sB} + \frac{1}{s} e^{-s\cdot 0} \right]$$

$$= \lim_{B\to\infty} \left[-\frac{1}{s} e^{-sB} + \frac{1}{s} \right] \qquad \text{assume } s>0$$

$$= 0 + \frac{1}{s} = \frac{1}{s} \qquad \boxed{\mathcal{L}[1] = \frac{1}{s}, \quad s>0}$$

Ex $\mathcal{L}[e^{at}] = ?$ $(a \in \mathbb{R})$

$e^{-(-2)B} = e^{2B} \to \infty$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt = \lim_{B\to\infty} \int_0^B e^{-(s-a)t}\, dt$$

$$= \lim_{B\to\infty} \frac{1}{-(s-a)} e^{-(s-a)t} \Big|_{t=0}^{t=B}$$

$\to 0$ provided $s-a>0$

$$= \lim_{B\to\infty} \left\{ \frac{-1}{s-a} e^{-(s-a)B} + \frac{1}{s-a} e^{-(s-a)0} \right\}$$

$$= \frac{-1}{s-a} \cdot 0 + \frac{1}{s-a} \cdot 1 \qquad s>a$$

$$= \frac{1}{s-a}$$

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad s>a}$$

Ex $\mathcal{L}[t] = ?$

$$\mathcal{L}[t] = \int_0^{\infty} e^{-st}\, t \, dt = \lim_{B\to\infty} \int_0^{B} t\, e^{-st} dt$$

$u = t \to du = dt, \quad dv = e^{-st} dt \to v = -\frac{1}{s} e^{-st}$

$$= \lim_{B\to\infty} \left\{ t \cdot \left(-\frac{1}{s}\right) \cdot e^{-st} \Big|_{t=0}^{t=B} - \int_0^{B} -\frac{1}{s} e^{-st} dt \right\}$$

$$= \lim_{B\to\infty} \left\{ -\frac{1}{s} B e^{-sB} + 0 + \frac{1}{s} \cdot \int_0^{B} e^{-st} dt \right\}$$

$$= \lim_{B\to\infty} \left\{ -\frac{1}{s} B e^{-sB} + \frac{1}{s} \cdot \left(-\frac{1}{s}\right) \cdot e^{-st} \Big|_{t=0}^{t=B} \right\}$$

$$= \lim_{B\to\infty} \left\{ -\frac{1}{s} B e^{-sB} - \frac{1}{s^2} \left(e^{-sB} - e^{-s\cdot 0} \right) \right\}$$

$$\lim_{B\to\infty} e^{-sB} = 0 \quad \text{if} \quad s>0$$

$$\lim_{B\to\infty} Be^{-sB} = \lim_{B\to\infty} \frac{B}{e^{sB}} \overset{\frac{\infty}{\infty}}{=} \lim_{B\to\infty} \frac{1}{s\,e^{sB}} = 0$$

$$\mathcal{L}[t] = 0 - \frac{1}{s^2}(0-1) = \frac{1}{s^2}, \quad s>0$$

$$\mathcal{L}[t] = \frac{1}{s^2}, \ldots, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

Indeed, $\mathcal{L}[t^2] = \int_0^\infty e^{-st}t^2\,dt$ evaluate this by int. by parts, twice.

There's another way of calculating $\mathcal{L}[t^n]$:

DIY

Def Gamma Function

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \qquad x>0$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, t^{1-1}\, dt = \int_0^\infty e^{-t}\, dt = e^{-t}\Big|_0^\infty = 1$$

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^{x+1-1}\, dt = \int_0^\infty e^{-t}\, t^{x}\, dt$$

$$= \int_0^\infty t^{x} e^{-t}\, dt = \left\{ \begin{array}{l} u = t^{x} \rightarrow du = x\, t^{x-1} \\ dv = e^{-t} dt \rightarrow v = -e^{-t} \end{array} \right\}$$

$$= t^{x}(-e^{-t})\Big|_{t=0}^{t=\infty} - \int_0^\infty (-e^{-t}) \times t^{x-1}\, dt$$

$$\lim_{t\to\infty} t^{x} e^{-t} = \lim_{t\to\infty} \frac{t^{x}}{e^{t}} = \overset{\text{enough \# of L'Hopital's rule}}{\downarrow} \cdots\cdots = 0$$

$$\Gamma(x+1) = \int_0^\infty e^{-t} \times t^{x-1}\, dt$$

$$= x \int_0^\infty e^{-t}\, t^{x-1}\, dt \qquad \Gamma(x)$$

$$\Gamma(x+1) = x\, \Gamma(x)$$

Suppose x is a positivi integer; $x = n \in \mathbb{Z}^{+}$

$$\Gamma(n+1) = n\,\Gamma(n) = n.(n-1)\,\Gamma(n-1)$$

$$= n.(n-1)(n-2)\,\Gamma(n-2)$$

$$= n(n-1)(n-2)\cdots\cdots 1$$

The function $\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$

coincides with the factorial $n!$ when x is a positive integer!

$\Rightarrow$ $\Gamma(x)$ is a generalization of the factorial operation ! to positive real numbers!!

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$\left(\frac{5}{2}\right)! \Longrightarrow \Gamma\left(\frac{5}{2}\right) \;!!?$$

Ex $\mathcal{L}[t^a] = ?$ $\quad a > -1, \quad a \in \mathbb{R}.$

$$\mathcal{L}[t^a] = \int_0^\infty e^{-st} t^a \, dt$$

$u = st$

$du = s\,dt$

$t = 0 \to u = 0$

$t = \infty \to u = \infty$

$$= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a \, du$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^{(a-1)+1} \, du = \frac{1}{s^{a+1}} \Gamma(a+1)$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt$$

$a > -1,\quad a \in \mathbb{R}:\quad \mathcal{L}[t^a] = \dfrac{\Gamma(a+1)}{s^{a+1}}$

Remember: $\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1)\ \dots\ = n!$

if $a = n \in \mathbb{Z}^+$

$$\mathcal{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

So far, we have found that:

$$\mathcal{L}[1] = \frac{1}{s},\quad \mathcal{L}[t] = \frac{1}{s^2},\quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^a] = \frac{\Gamma(a+1)}{s^{a+1}},\quad \mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2} \qquad \mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}[\sin(at)] = \int_0^\infty e^{-st} \sin(at)\, dt = \cdots = \frac{a}{s^2+a^2}$$

integration by parts, twice $\downarrow$ DIY

$$I = \int e^{bt} \sin(at)\, dt \longrightarrow$$

$$I = \int e^{bt} \cos(at) \longrightarrow$$

Int. by parts, twice ; turns you back to I again.

Linearity of Laplace Transform

* $\mathcal{L}[a\,f(t) + b\,g(t)] = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$

* If $\mathcal{L}[f(t)] = F(s)$, $\mathcal{L}[g(t)] = G(s)$

$$\mathcal{L}[a\,f(t) + b\,g(t) = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$$

$$= a\,F(s) + b\,G(s)$$

$$\mathcal{L}[a\,f(t) + b\,g(t)] = \int_0^\infty e^{-st}[a\,f(t) + b\,g(t)]\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)] \;//$$

$*\quad \mathcal{L}[1 + 2t^2 + 3\sin(at)] = ?$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$n = 0 \rightarrow \mathcal{L}[1] = \frac{1}{s}$

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}$$

$n = 2 \quad \mathcal{L}[t^2] = \frac{2!}{s^3}$

$$\mathcal{L}[1 + 2t^2 + 3\sin(at)]$$

$$= \mathcal{L}[1] + 2\,\mathcal{L}[t^2] + 3\,\mathcal{L}[\sin(at]$$

$$= \frac{1}{s} + 2 \cdot \frac{2}{s^3} + 3\,\frac{a}{s^2 + a^2}$$

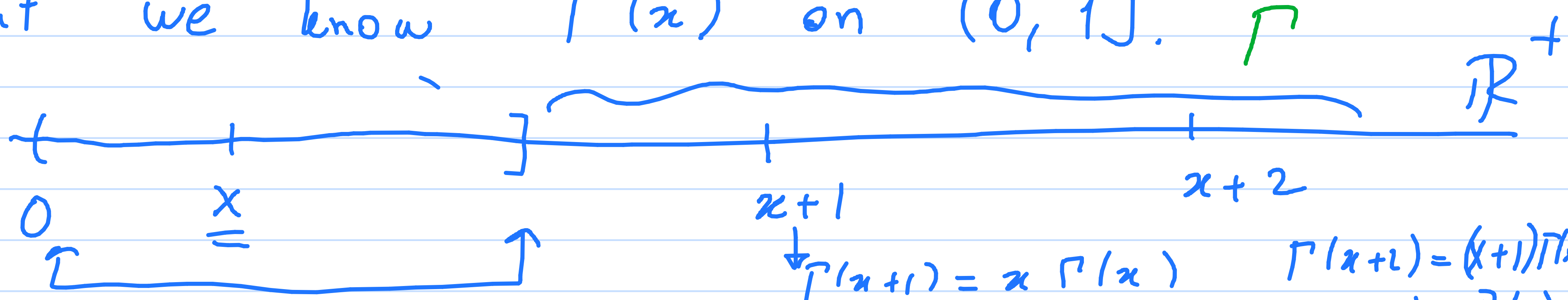
Ex $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \Rightarrow \quad \Gamma\left(\frac{5}{2}\right) = ?$

$$\Gamma(x+1) = x\,\Gamma(x) \qquad \Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt \quad (x>0)$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,\Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{3}{2}\cdot\frac{1}{2}\cdot 1\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\cdot\sqrt{\pi}$$

In order to know $\Gamma(x)$ on $\mathbb{R}^+$, it's enough that we know $\Gamma(x)$ on $(0,1]$.



$\Gamma(x+1) = x\,\Gamma(x)$

$\Gamma(x+2) = (x+1)\Gamma(x+1$

$= (x+1)\,x\,\Gamma(x)$

Ex $\mathcal{L}[\cosh(kt)] = \dfrac{s}{s^2-k^2}$ $\quad (s>k>0)$

$$\mathcal{L}[\cosh(kt)] = \mathcal{L}\left[\frac{1}{2}\left(e^{kt}+e^{-kt}\right)\right]$$

$$= \frac{1}{2}\left\{\mathcal{L}[e^{kt}] + \mathcal{L}[e^{-kt}]\right\} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$= \frac{1}{2}\left\{\frac{1}{s-k} + \frac{1}{s-(-k)}\right\} = \frac{s}{s^2-k^2}$$

Similarly $\mathcal{L}[\sinh(k.t)] = \dfrac{k}{s^2-k^2}$ $\quad (s>k>0)$

$$\mathcal{L}[\sinh(k\cdot t)] = \mathcal{L}\left[\frac{1}{2}\left(e^{k.t}-e^{-kt}\right)\right] = ---$$

Ex

$$\mathcal{L}[\sin^2(kt)] = \mathcal{L}\left[\frac{1-\cos(2kt)}{2}\right]$$

$$= \frac{1}{2}\left\{\mathcal{L}[1] - \mathcal{L}[\cos(2kt)]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+(2k)^2}\right\}$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2+a^2}$$

INVERSE LAPLACE TRANSFORM

$$\mathcal{L}[f(t)] = F(s) \iff \mathcal{L}^{-1}[F(s)] = f(t)$$

$$* \quad \mathcal{L}[1] = \frac{1}{s} \quad \Rightarrow \quad \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$$

$$\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin(at)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^4}\right] = \mathcal{L}^{-1}\left[\frac{1}{6}\,\frac{6}{s^4}\right] = \frac{1}{6}\,\mathcal{L}^{-1}\left[\frac{6}{s^4}\right] = \frac{1}{6}\cdot t^3 \,//$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

$$* \quad \mathcal{L}^{-1}\left[\frac{1}{s^2+3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2+(\sqrt{3})^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right]$$

$$= \frac{1}{\sqrt{3}}\,\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right]$$

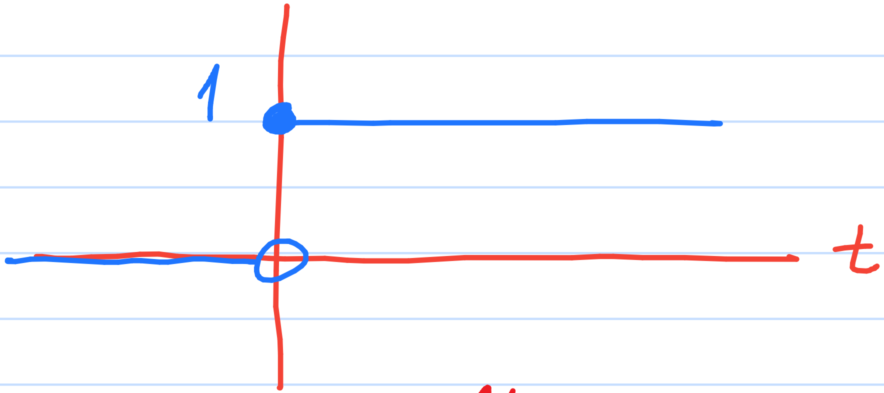
$$\mathcal{L}[\sin(at)] = \frac{a}{s^2+a^2}$$

$$= \frac{1}{\sqrt{3}}\,\sin(\sqrt{3}t)$$

Piecewise Continuous Functions

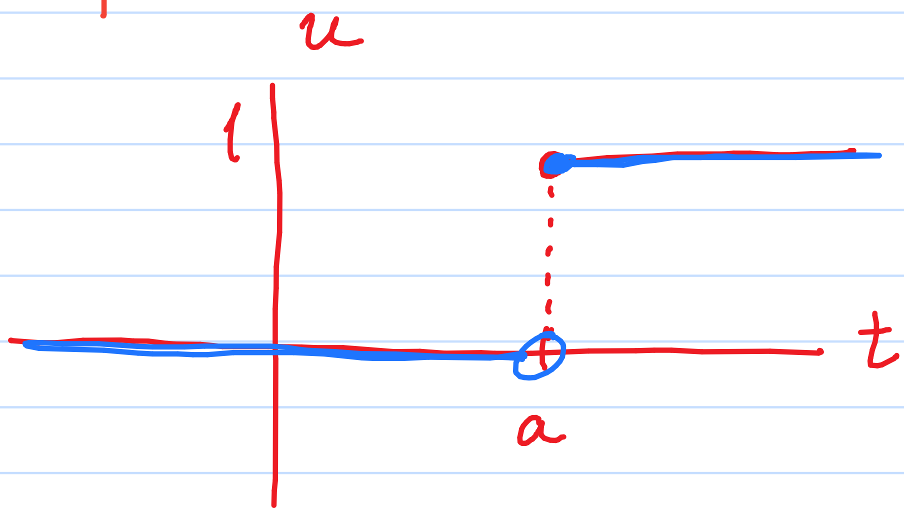
STEP FUNCTION

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

$(a > 0)$



$$\mathcal{L}[u(t)] = \frac{1}{s}$$

$(s > 0)$

$$\mathcal{L}[u_a(t)] = e^{-as}\frac{1}{s}$$

$(s > 0,\ a > 0)$

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st} \underbrace{u(t)}_{1} dt = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

$$0 < t < \infty$$

$$\mathcal{L}[u_a(t)] = \int_0^\infty e^{-st} u_a(t)\, dt =$$

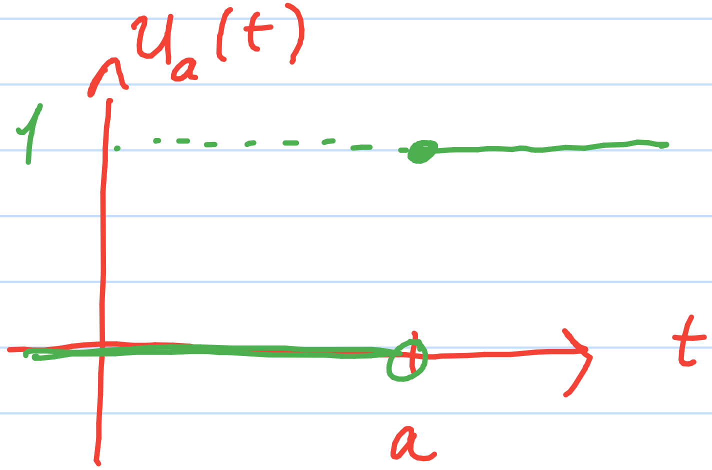
$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} \cdot 1 \cdot dt$$

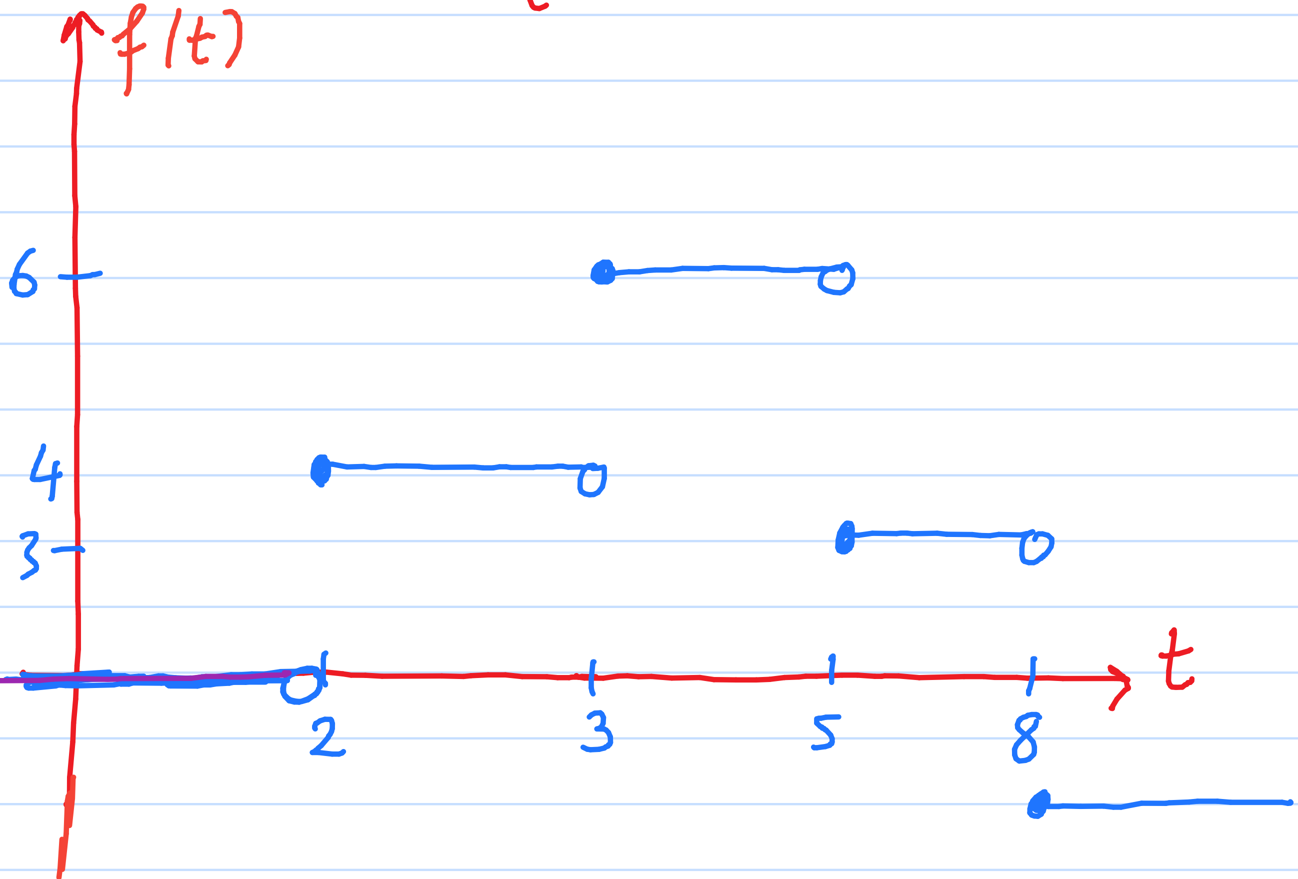
$$= \int_a^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{t=0}^{t=\infty} = \frac{e^{-as}}{s}$$

Ex

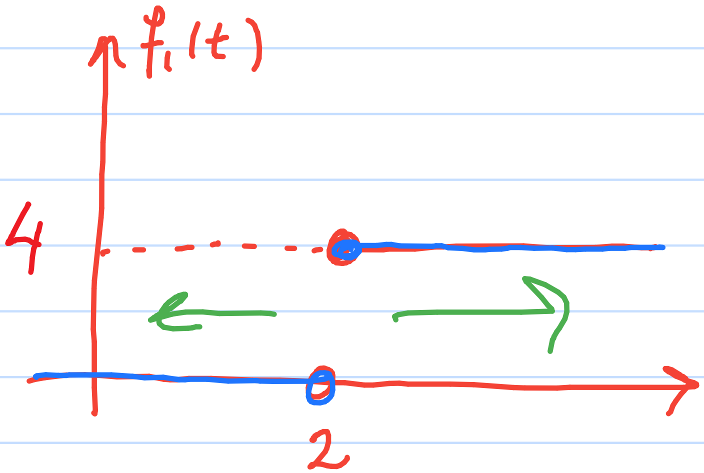
Given $f(t) = \begin{cases} 0 & t < 2 \\ 4 & 2 \leq t < 3 \\ 6 & 3 \leq t < 5 \\ 3 & 5 \leq t < 8 \\ -1 & 8 \leq t \end{cases}$

Calculate $\mathcal{L}[f(t)]$.



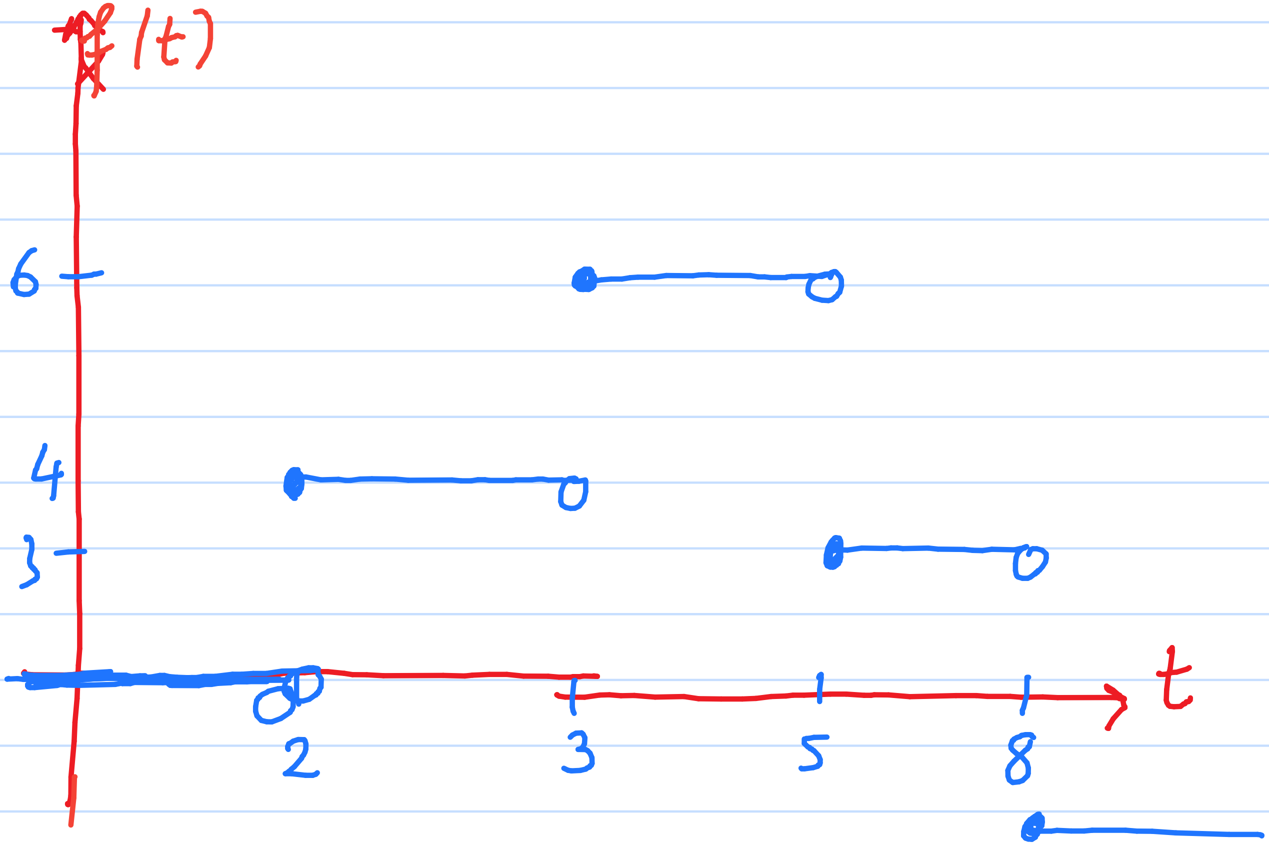


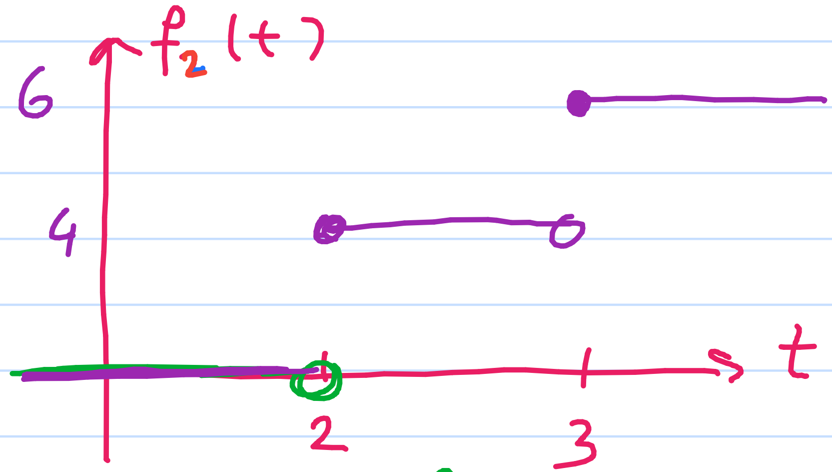
$f_1(t) = 4\,u_2(t)$



$f_1 = 4\, u_2(t)$

$f_2(t) = 4\, u_2(t) + 2\, u_3(t)$





$t < 2$ $\quad f_2(t) = 4 \cdot 0 + 2 \cdot 0 = 2$

$2 \le t < 3$ $\quad f_2(t) = 4 \cdot 1 + 2 \cdot 0 = 4$

$3 \le t$ $\quad f_2(t) = 4 \cdot 1 + 2 \cdot 1 = 6$

$$f_1 = 4\,u_2(t)$$

$$f_2(t) = 4\,u_2(t) + 2\,u_3(t)$$

$$f_3(t) = 4\,u_2(t) + 2\,u_3(t) - 3\,u_5(t)$$

$$f_4(t) = 4\,u_2(t) + 2u_3(t) - 3\,u_5(t) - 4\,u_8(t) = f(t)$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[4u_2(t) + 2u_3(t) - 3u_5(t) - 4u_8(t)\right]$$

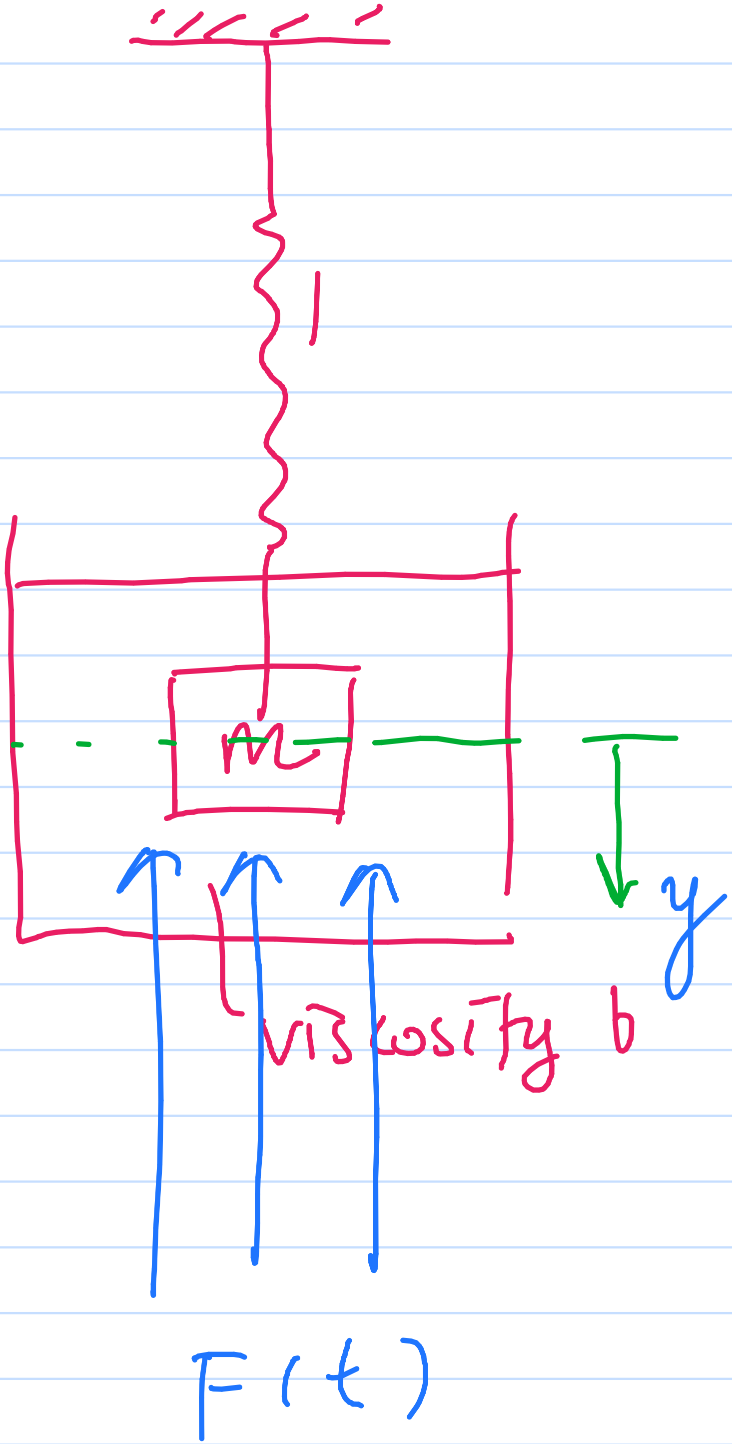
$$= 4\mathcal{L}[u_2(t)] + 2\mathcal{L}[u_3(t)] - 3\mathcal{L}[u_5(t)] - 4\mathcal{L}[u_8(t)]$$

$$= 4 \cdot \frac{e^{-2s}}{s} + 2\frac{e^{-3s}}{s} - 3\frac{e^{-5s}}{s} - 4\frac{e^{-8s}}{s}$$

* We can calculate $\mathcal{L}$-transform of piecewise-defined functions.

Q: What's the aim?

The aim is to solve linear ODEs with constant coefficients, such as:

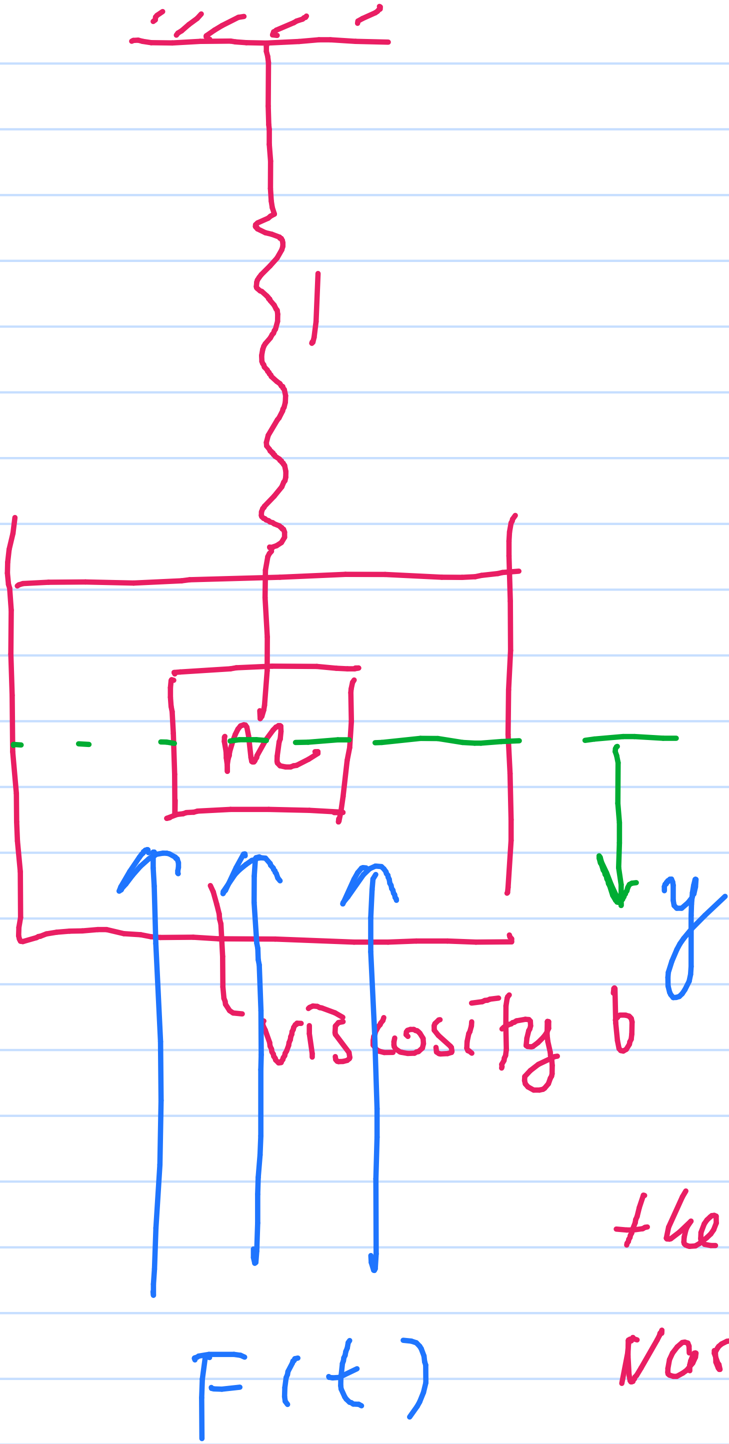


$$my'' + by' + ky = 0$$

y : the displacement from equilibrium. In this model, there's no outer force $F(t)$ that's applied on the system. If there's such a force, the eq. becomes

$$my'' + by' + ky = F(t)$$

$$y' = \frac{dy}{dt}$$



$$my'' + by' + ky = 0$$

Suppose

$F(t) = 0$: $\quad y'' + y' + y = 0$

$F(t) = t^2 + t + 1$: $\quad y'' + y' + y = t^2 + t + 1$

$F(t) = e^{2t}$ $\quad y'' + y' + y = e^{2t}$

$F(t) = \sqrt{1+t^2}$ $\quad y'' + y' + y = \sqrt{1+t^2}$

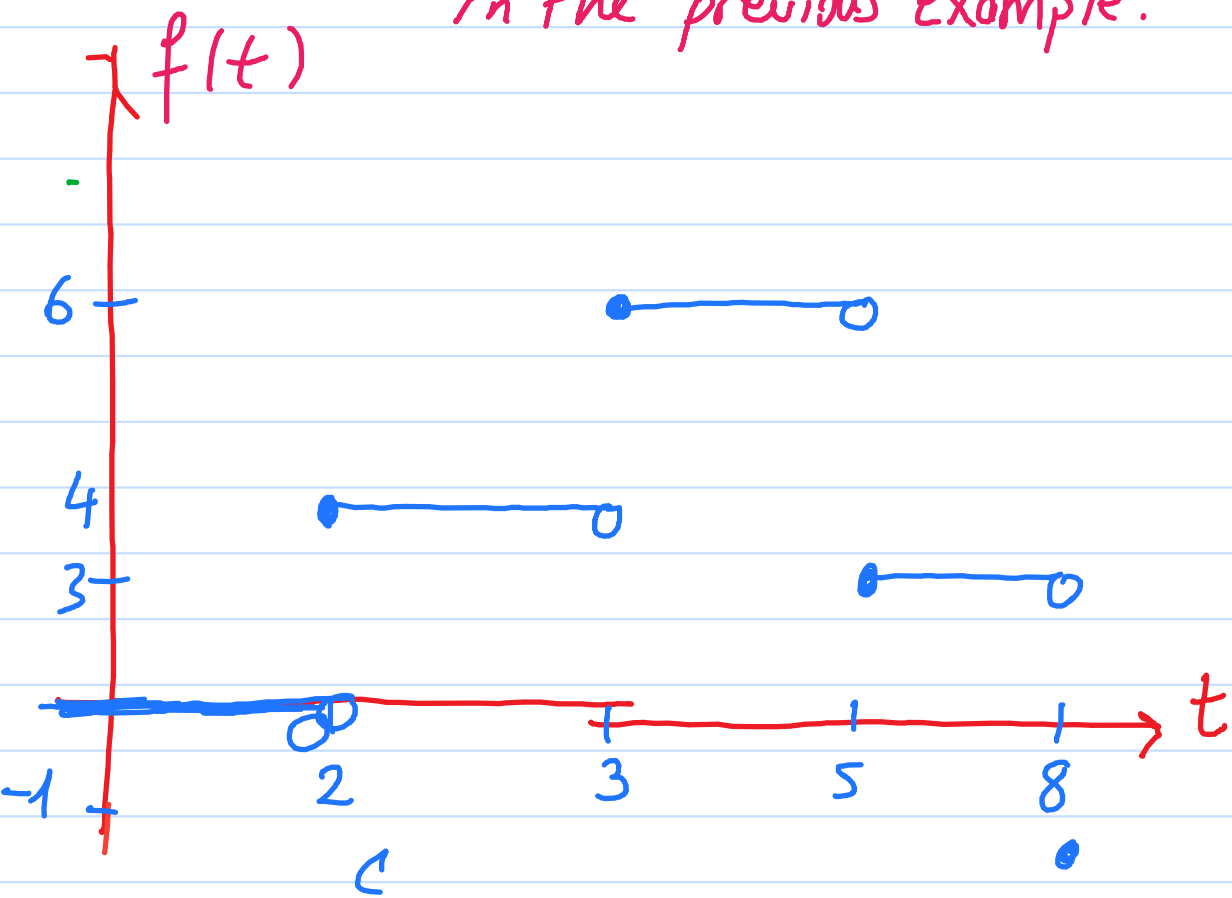
All these cases are solvable by the method of undetermined coeffs. or variation of parameters.

Suppose we apply a discontinuous force

$$y'' + y' + y = f(t)$$

$F(t)$, the force is taken as this one in the previous example.

$$f(t) = \begin{cases} 0. & t < 2 \\ 4. & 2 \le t < 3 \\ 6. & 3 \le t < 5 \\ 3. & 5 \le t < 8 \\ -1. & 8 \le t \end{cases}$$



The solution of this DE requires solution of 5 separate DEs. However, since we can calculate the $\mathcal{L}$-transform of the discontinuous function on the RHS, we'll be able to solve this DE at just 1 attempt!!

Before passing to solve DEs by $\mathcal{L}$-transform, there is just 1 remaining step: How to evaluate the $\mathcal{L}$-transforms of the derivatives y', y'', ... etc. appearing in an equation?

TRANSFORMS of DERIVATIVES

$$\mathcal{L}[y'] = ? \qquad \mathcal{L}[y''] = ? \quad \ldots \quad \mathcal{L}[y^{(n)}] = ?$$

(in terms of $\mathcal{L}[y]$)

Suppose $f(t)$ is a continuous, piecewise smooth for $t \geq 0$ and is of exponential order as $t \to \infty$.

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$u = e^{-st} \to du = -s e^{-st} dt$; $dv = f'(t)dt \to v = f(t)$

$$= e^{-st} \cdot f(t) \Big|_{t=0}^{t=\infty} - \int_0^\infty f(t) \cdot (-s) \cdot e^{-st}\, dt$$

$$= \underbrace{e^{-st} f(t) \Big|_{t=\infty}}_{=0} - e^{-s \cdot 0} f(0) + s \int_0^\infty e^{-st} f(t)\,dt$$

$$= s \int_0^\infty e^{-st} f(t)\,dt - f(0) = s \cdot \mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}[f'(t)] = s\,\mathcal{L}[f(t)] - f(0)$$

In this expression, replace $f \to f'$

$$\mathcal{L}[f''(t)] = s\,\mathcal{L}[f'(t)] - f'(0)$$

$$= s\,\{s\,\mathcal{L}[f(t)] - f(0)\} - f'(0)$$

$$\mathcal{L}[f''(t)] = s^2\,\mathcal{L}[f(t)] - s\,f(0) - f'(0)$$

$$\mathcal{L}[f'''(t)] = s^3\,\mathcal{L}[f(t)] - s^2 f(0) - s f'(0) - f''(0)$$

Let $\mathcal{L}[y(t)] = Y(s)$

$$\mathcal{L}[y'] = sY(s) - y(0)$$

$$\mathcal{L}[y''] = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}[y'''] = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)$$

$$\vdots$$

$$\mathcal{L}[y^{(n)}] = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0)$$

$$\cdots - s y^{n-2}(0) - y^{(n-1)}(0)$$

Example Solve $y'' - y' - 6y = 0$, $y(0) = 2$, $y'(0) = -1$.

$$y'' - y' - 6y = 0$$

$$\mathcal{L}[y'' - y' - 6y] = \mathcal{L}[0]$$

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 6\,\mathcal{L}[y] = 0 \qquad \text{Let } \mathcal{L}[y] = Y(s)$$

$$[s^2 Y(s) - s\,y(0) - y'(0)] - [s\,Y(s) - y(0)] - 6\,Y(s) = 0$$

$$s^2 Y(s) - s.2 - (-1) - [s\,Y(s) - 2] - 6.Y(s)$$

$$(s^2 - s - 6)\,Y(s) = 2s - 3$$

$$\mathcal{L}[y] = Y(s) = \frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)}$$

$$\frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$A = 3/5$

$B = 7/5$

$$\mathcal{L}[y] = Y(s) = \frac{\frac{3}{5}}{s-3} + \frac{\frac{7}{5}}{s+2} = \frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}\right]$$

$$= \frac{3}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + \frac{7}{5}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = \frac{3}{5}\cdot e^{3t} + \frac{7}{5}\cdot e^{-2t} \;//$$

Ex Solve the IVP $x'' + 4x = \sin(3t)$, $x(0) = x'(0) = 0$.

Suppose $\mathcal{L}[\,x(t)\,] = X(s)$

$$\mathcal{L}[\,x''(t) + 4\,x(t)\,] = \mathcal{L}[\sin(3t)]$$

$$\mathcal{L}[\,x''(t)\,] + 4\,\mathcal{L}[x(t)] = \mathcal{L}[\,\sin(3t)\,]$$

$$s^2 X(s) - s\,x(0) - x'(0) + 4\,X(s) = \frac{3}{s^2 + 3^2}$$

$$s^2 X(s) - s \cdot 0 - 0 + 4X = \frac{3}{s^2 + 3^2}$$

$$X(s) = \frac{3}{(s^2 + 9)(s^2 + 4)}$$

$$\frac{3}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$A = C = 0, \quad B = 3/5, \quad D = -3/5$$

$$\mathcal{L}[x(t)] = X(s) = \frac{3}{5}\,\frac{1}{s^2+2^2} - \frac{3}{5}\,\frac{1}{s^2+3^2}$$

$$\mathcal{L}[x(t)] = X(s) = \frac{3}{5}\cdot\frac{1}{2}\cdot\frac{2}{s^2+2^2} - \frac{1}{5}\cdot\frac{3}{s^2+3^2}$$

$$x(t) = \frac{3}{10}\sin(2t) - \frac{1}{5}\sin(3t)$$