EHB 211E
Basics of Electrical Circuits

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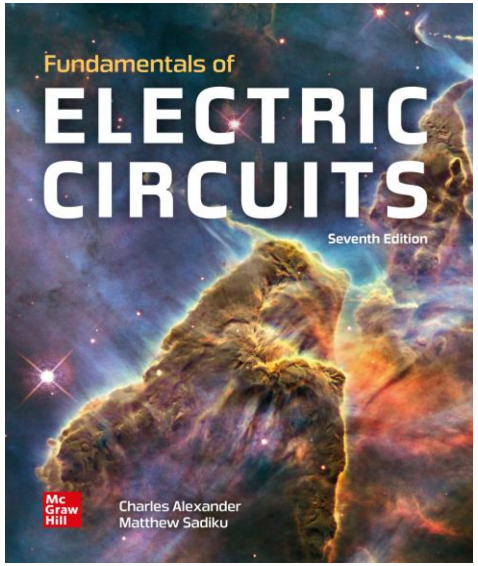
Capacitors and Inductors



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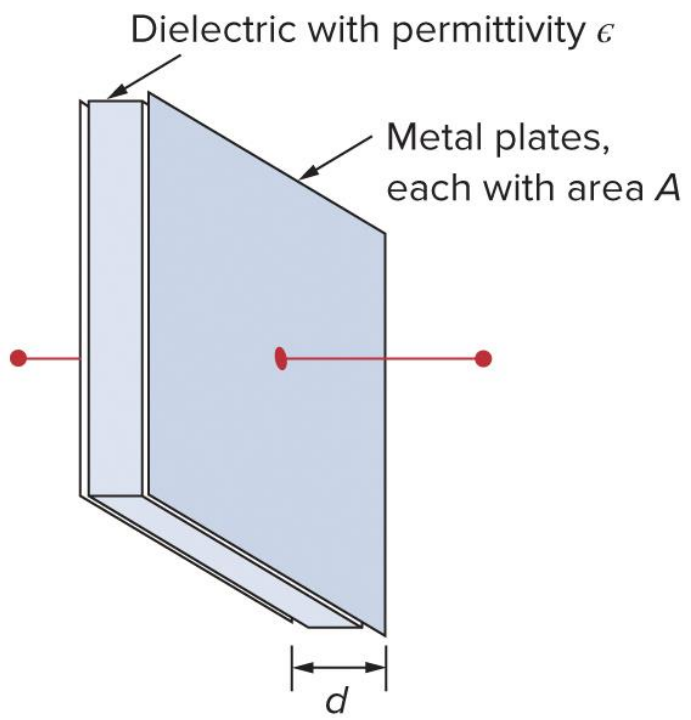
Fundamentals of Electric Circuits by C. Alexander and M. Sadiku, Seventh Edition (McGraw-Hill Education)

Introduction

- Introduce two new and important passive linear elements:
  - Capacitors and inductors
- Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate energy but store energy, which can be retrieved at a later time.
- Capacitors and inductors: storage elements.
- Analyze important and practical circuits with the introduction of capacitors and inductors

Capacitors

- What is a capacitor?
  - Passive element designed to store energy (electrical energy) in an E-field
  - Store and release electrical energy
  - One of the most common electrical components
- Used extensively in electronics, communications, computers, and power systems
- A typical capacitor: two conducting plates separated by an insulator (or dielectric)
- Plates: aluminum
- Dielectric: air, ceramic, paper, or mica.



Capacitors

- Connect a voltage source v to the capacitor: a positive charge q on one plate and a negative charge $-q$ on the other plate.
- During this process, electric charge stored in the capacitor
- Amount of charge stored is directly proportional to the applied voltage v:

$$q = Cv$$

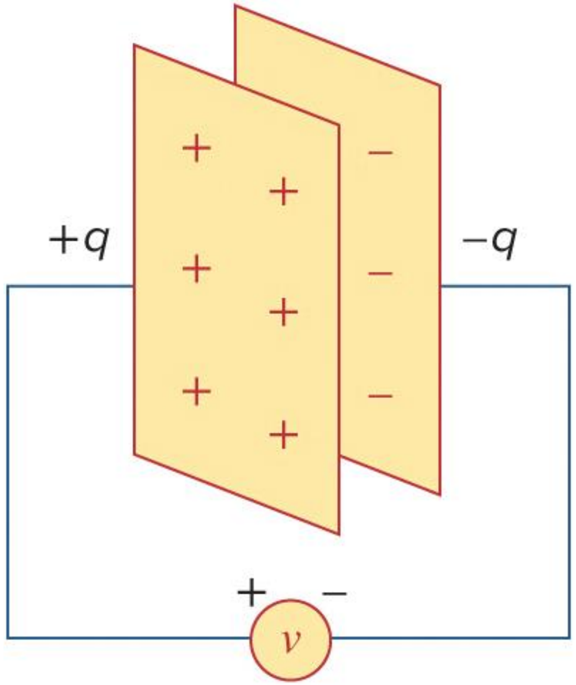
q: the amount of charged stored in capacitor
C: the capacitance of the capacitor, unit is Farad (F)

$$1\,F = 1\,\frac{C}{V}$$

- Capacitance: the ratio of the charge on one plate of a capacitor to voltage difference between the two plates
- Capacitance C depends on physical dimension of the capacitor. For parallel plate, the capacitance:

$$C = \frac{\in A}{d}$$

A: surface area of each plate (m^2)
d: distance (separation) between plates (m)
$\in$: permittivity of dielectric material between plates $\left(\frac{F}{m}\right)$



Capacitors

- Three factors determine the value of capacitance:
  - The surface area of the plates: larger the area, the greater the capacitance
  - The spacing between the plates: the smaller the spacing, the greater the capacitance (d inversely proportional to the C)
  - The permittivity of the material: the higher the permittivity, the greater the capacitance
- Typical value of capacitors: picofarad (pF) to microfarad (μF) range

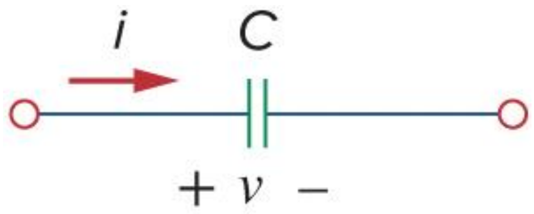
$$C = \frac{\in A}{d}$$

- If $v > 0$ and $i > 0$
- If $v < 0$ and $i < 0$

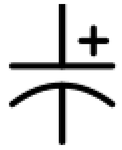
Capacitor is being charged

- If $vi < 0$ (v and i have opposite sign): capacitor is discharging

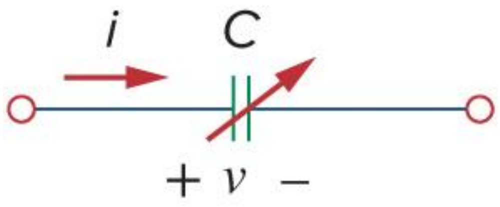
Fixed capacitor



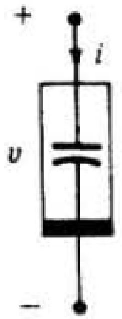
Fixed capacitor



Variable capacitor

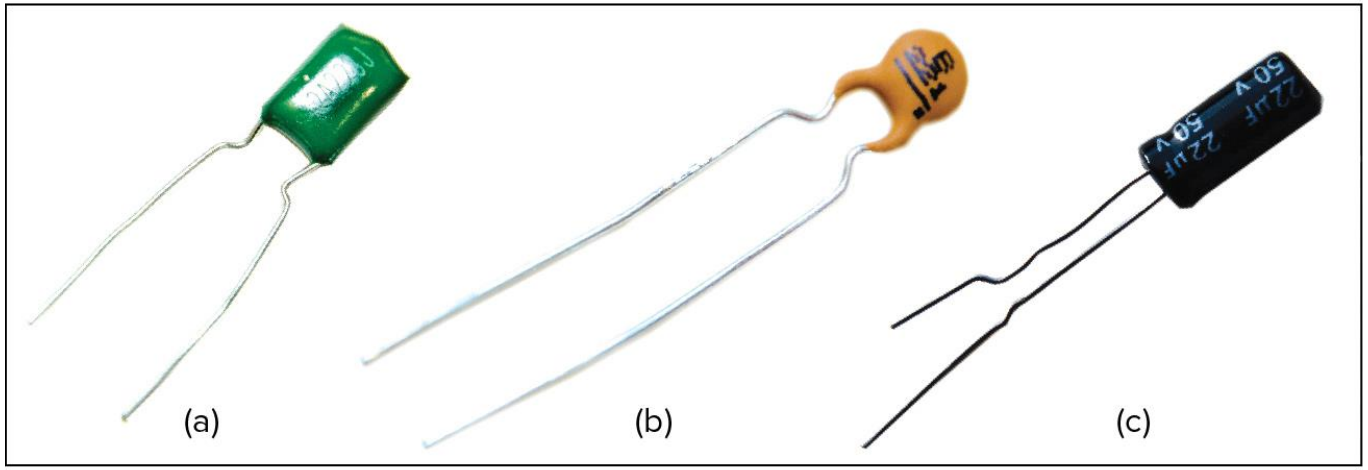


Nonlinear capacitor



Capacitors

Fixed Capacitors



Mark Dierker/McGraw-Hill Education

Variable Capacitor

Charles Alexander

Capacitors

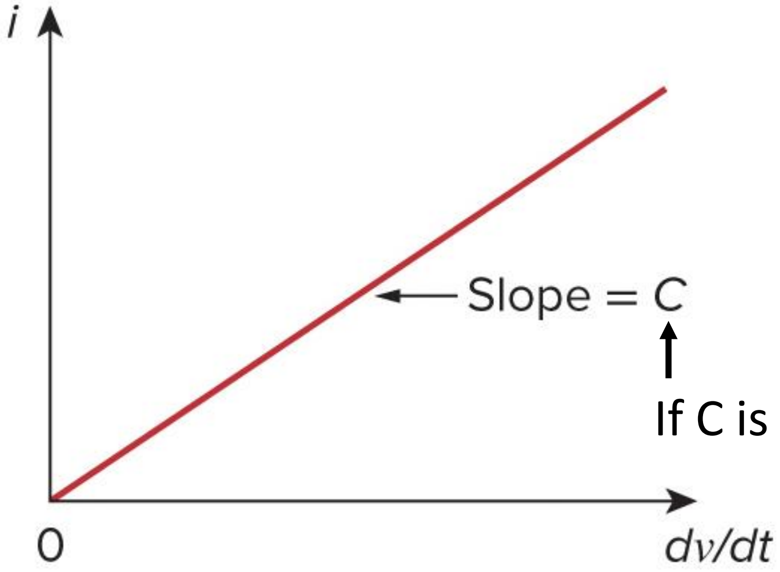
- Charge q stored in a capacitor represented by

$q = Cv$ ⟹ Derivation of both sides

$$\frac{dq}{dt} = C\frac{dv}{dt}$$ ⟹ By definition: $i = \frac{dq}{dt}$ ⟹ $\boxed{i = C\frac{dv}{dt}}$ Current-voltage relationship for a capacitor

Linear capacitor

Plot of current-voltage relationship



If C is large, line will be steeper

- Ideal capacitors are linear.
- For a nonlinear capacitor, the plot of the current-voltage relationship is not a straight line

Capacitors

- How does voltage relate to current for a capacitor?

$$i = C\frac{dv}{dt} \Rightarrow idt = Cdv \Rightarrow dv = \frac{1}{C}idt \Rightarrow$$ Integration of both sides

$$\int dv = \frac{1}{C}\int i\,dt \Rightarrow v = \frac{1}{C}\int i\,(\tau)d(\tau)$$

$$\boxed{v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d(\tau) + v(t_0)}$$

Voltage-current relationship for a capacitor

where $v(t_0) = q(t_0)/C$ is the voltage across capacitor at time t_0

- Capacitor voltage depends on the past history of the capacitor current.
- Capacitor has memory.

Capacitors

- Instantaneous power delivered to the capacitor:

By definition: $p = vi$ and $i = C\dfrac{dv}{dt}$ $\Rightarrow p = Cv\dfrac{dv}{dt}$

- Energy stored in the capacitor:

By definition: $p = \dfrac{\omega}{t} \Rightarrow \omega = pt$ ➡ Derivation of both sides

$d\omega = pdt$ ➡ Integration of both sides

$$\int d\omega = \int_{-\infty}^{t} p\,dt \Rightarrow \omega = \int_{-\infty}^{t} p\,dt \Rightarrow \omega = C\int_{v(-\infty)}^{v} vdv \Rightarrow \omega = C\frac{v^2}{2}\Big|_{v(-\infty)}^{v}$$

$$\boxed{\omega = \frac{1}{2}Cv^2}$$

Charge stored in a capacitor

where $v(-\infty)$ is initial voltage and $v(-\infty) = 0$

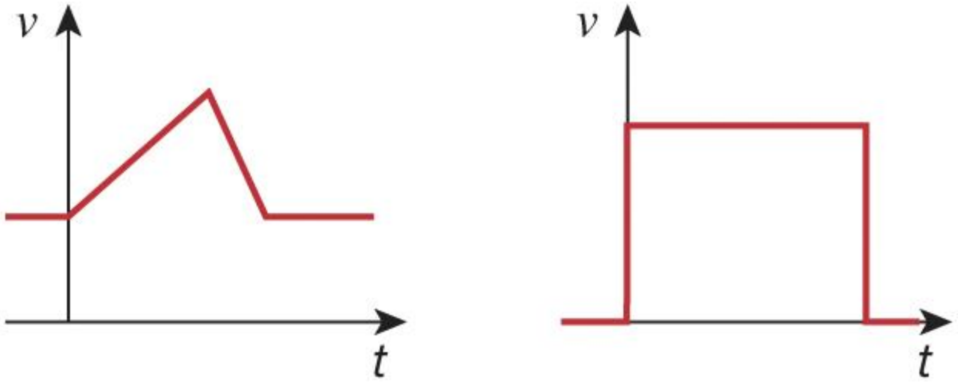
- Since $q = Cv$, $\omega = \dfrac{1}{2}Cv^2 \Rightarrow \omega = \dfrac{1}{2}C\left(\dfrac{q}{C}\right)^2 \Rightarrow \omega = \dfrac{1}{2}\dfrac{q^2}{C}$

Capacitors

- Important properties of a capacitor:
  1. When the voltage across a capacitor is not changing with time (i.e., dc voltage, constant voltage) the current through the capacitor is zero. Therefore, a capacitor is an open circuit to dc.
  2. The voltage on the capacitor must be continuous. The voltage on the capacitor cannot change arbitrary. Discontinuous change in voltage requires an infinite current, which is physically impossible.
  3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy and returns previously stored energy when delivering power to the circuit. (Note that nonlinear capacitor has a leakage resistance which dissipate some energy during discharging).

$$i = C\frac{dv}{dt}$$

Voltage across a capacitor



Allowed

Not allowed: arbitrary change is not possible

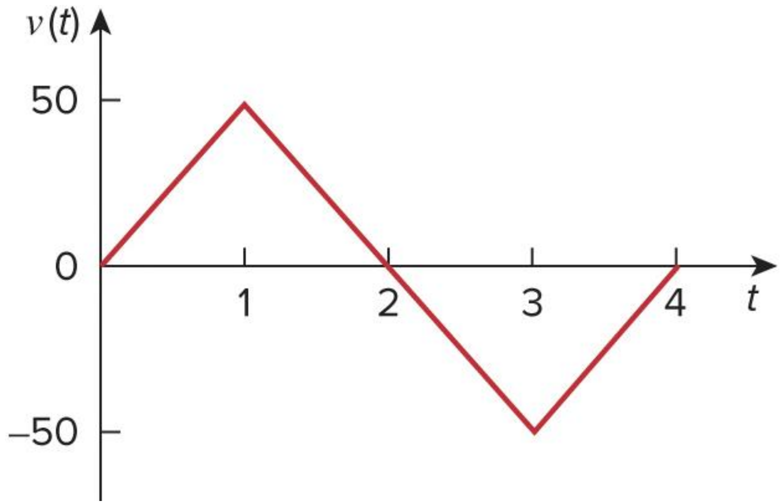
Example 1

Determine the current through a 200-μF capacitor whose voltage is shown below.

Solution:

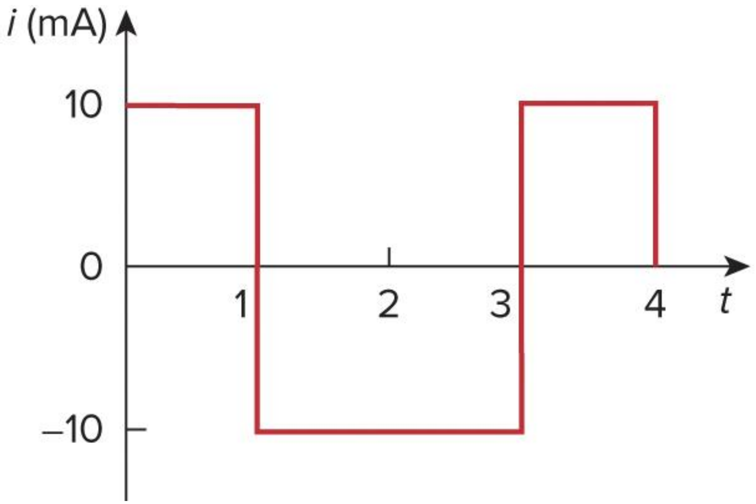
The voltage waveform can be described mathematically as

$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$



Since $i = C\,dv/dt$ and $C = 200\ \mu\text{F}$, we take the derivative of v to obtain

$$i(t) = 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$



Example 2

Obtain the energy stored in each capacitor in the figure below under dc conditions.

Solution:

Under dc conditions, we replace each capacitor with an open circuit, current division

$$i = \frac{3}{3 + 2 + 4}(6 \text{ mA}) = 2 \text{ mA}$$

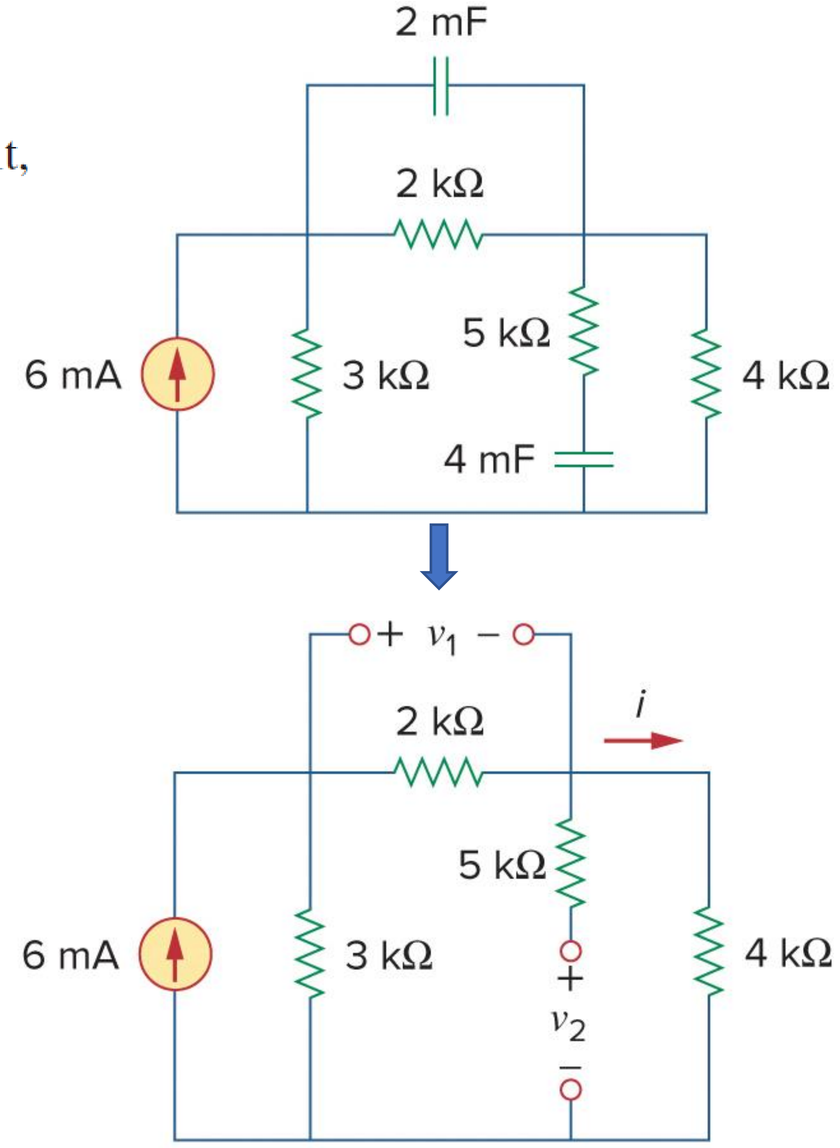
the voltages v_1 and v_2 across the capacitors are

$$v_1 = 2000i = 4 \text{ V} \qquad v_2 = 4000i = 8 \text{ V}$$

the energies stored in them are

$$w_1 = \frac{1}{2}C_1v_1^2 = \frac{1}{2}(2 \times 10^{-3})(4)^2 = 16 \text{ mJ}$$

$$w_2 = \frac{1}{2}C_2v_2^2 = \frac{1}{2}(4 \times 10^{-3})(8)^2 = 128 \text{ mJ}$$



Series and Parallel Capacitors

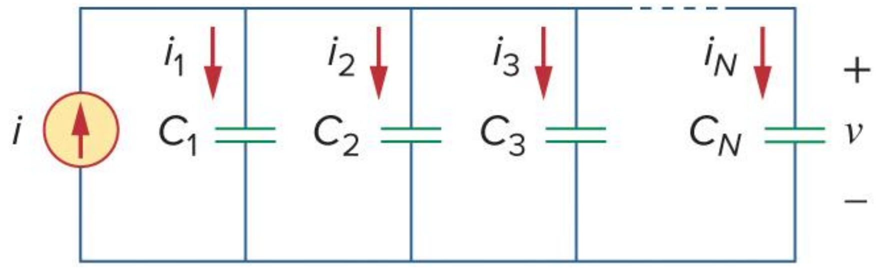
- Capacitors in parallel:
  - N capacitors connected in parallel as shown in the figure below
- Apply KCL:

$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$i = C\frac{dv}{dt}$$

$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt}$$

$$i = \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{eq}\frac{dv}{dt} \quad \text{where } \boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N}$$



Equivalent circuit

- The equivalent capacitance of N parallel-connected capacitors is the sum of the individual capacitances.
- Capacitors in parallel combine in the same manner as resistors in series.
- Note that voltage across each capacitor is same.

Series and Parallel Capacitors

- Capacitors in series:
  - N capacitors connected in series as shown in the figure below
- Apply KVL:

$$v = v_1 + v_2 + v_3 + \cdots + v_N \qquad v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\, d(\tau) + v(t_0)$$

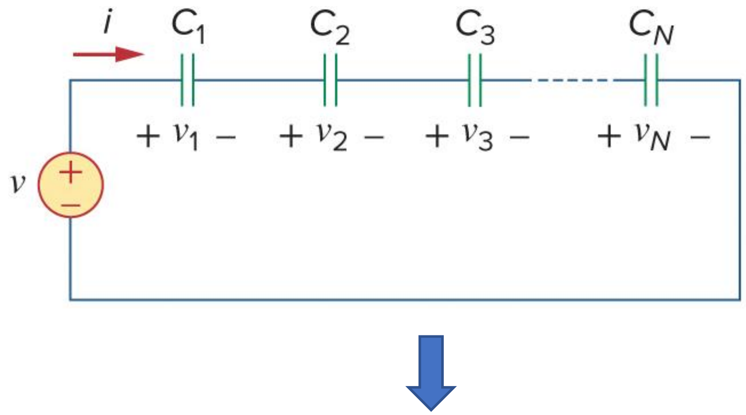
$$v = \frac{1}{C_1}\int_{t_0}^{t} i(\tau)\, d(\tau) + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(\tau)\, d(\tau) + v_2(t_0) + \cdots + \frac{1}{C_N}\int_{t_0}^{t} i(\tau)\, d(\tau) + v_N(t_0)$$

$$v = \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(\tau)\, d(\tau) + v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

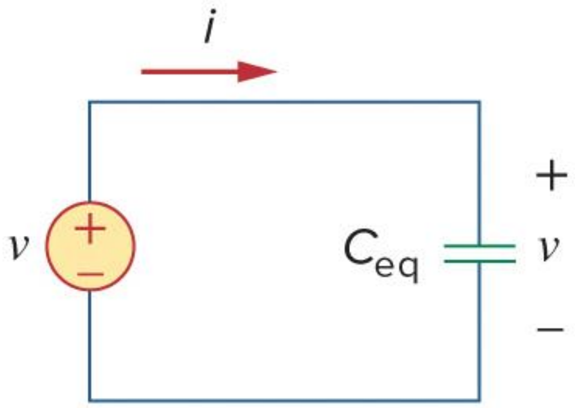
$$v = \frac{1}{C_{eq}}\int_{t_0}^{t} i(\tau)\, d(\tau) + v(t_0)$$

where $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$

$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$



Equivalent circuit

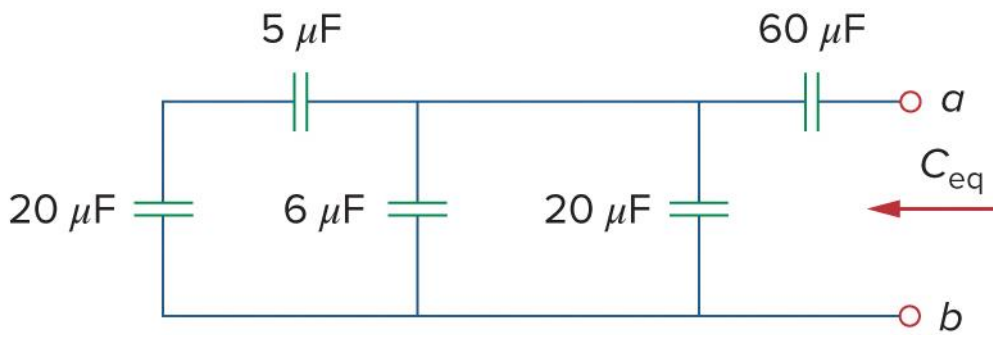


- The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocal individual capacitances.
- Capacitors in series combine in the same manner as resistors in parallel.
- Note that current flows through all capacitors is same.

Example 3

Find the equivalent capacitance seen between terminals a and b of the circuit shown below

Solution:



The 20-μF and 5-μF capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is

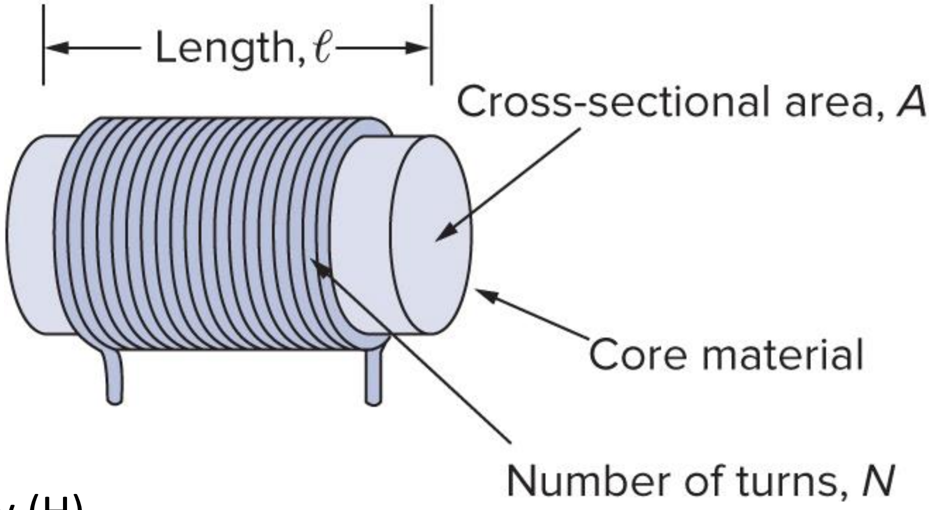
$$4 + 6 + 20 = 30\ \mu\text{F}$$

This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{\text{eq}} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

Inductors

- What is an inductor?
  - Passive element designed to store energy in its magnetic field.
- Numerous applications in electronics and power systems: power supplies, transformers, TVs, electric motors, etc.
- Inductor: formed into a cylindrical coil with many turns of conducting wire
- Inductor: simply a coil of conducting wire
- If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current:



$$\boxed{v = L\frac{di}{dt}}$$

where L: inductance of the inductor, unit is henry (H)

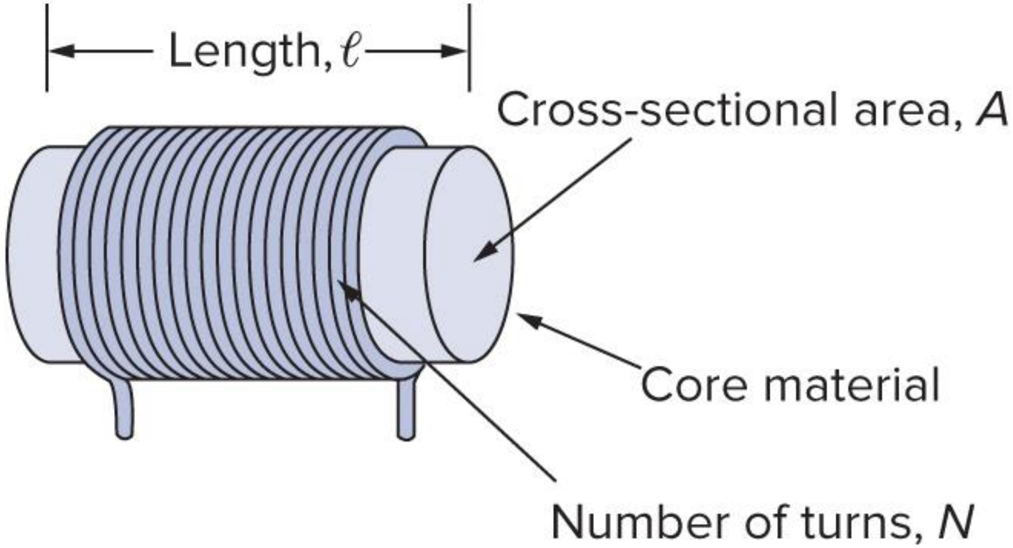
1 henry (H)=1 volt-second per ampere $\left(H = \frac{Vs}{A}\right)$

Inductors

- The inductance of an inductor depends on its physical dimension and construction
- Inductor (solenoid) is given by

$$L = \frac{N^2 \mu A}{\ell}$$

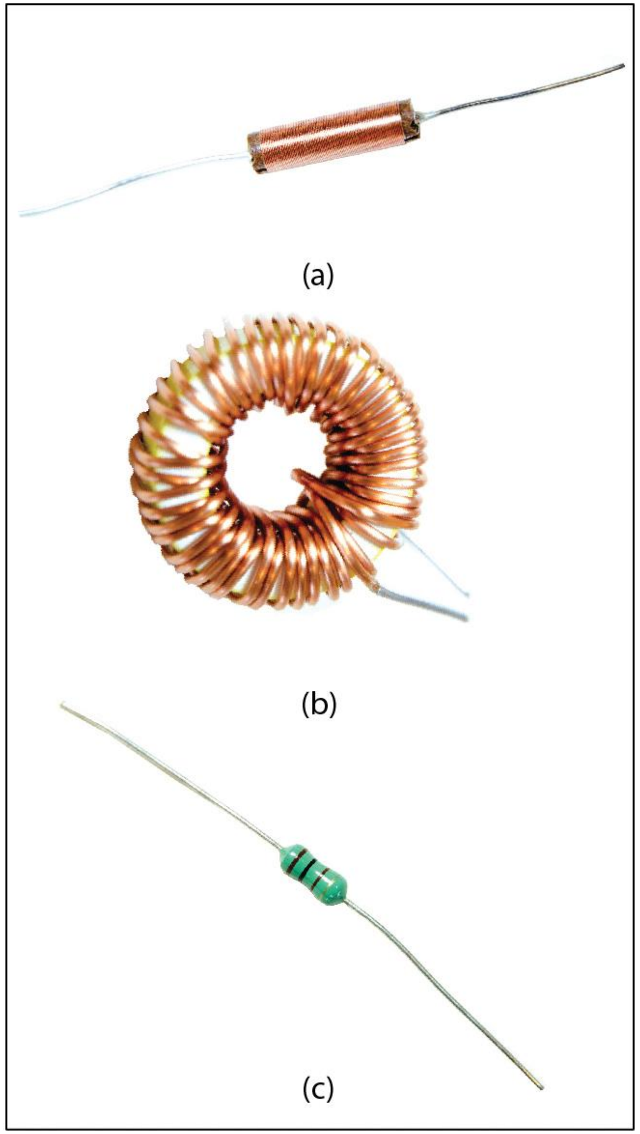
N: number of turns
ℓ: length
A: cross-sectional area
μ: permeability



- Inductance increases with N, increases with A, and using material with higher μ.
- Inductance reduces with increasing ℓ

Inductors

- Commercially available inductors: different values and types
- Typical inductance value of inductors: from a few μH to tens of henry (H)
- Inductors: fixed or variable
- Various types of inductors:
  a) Solenoidal wound inductor
  b) Toroidal inductor
  c) Chip inductor



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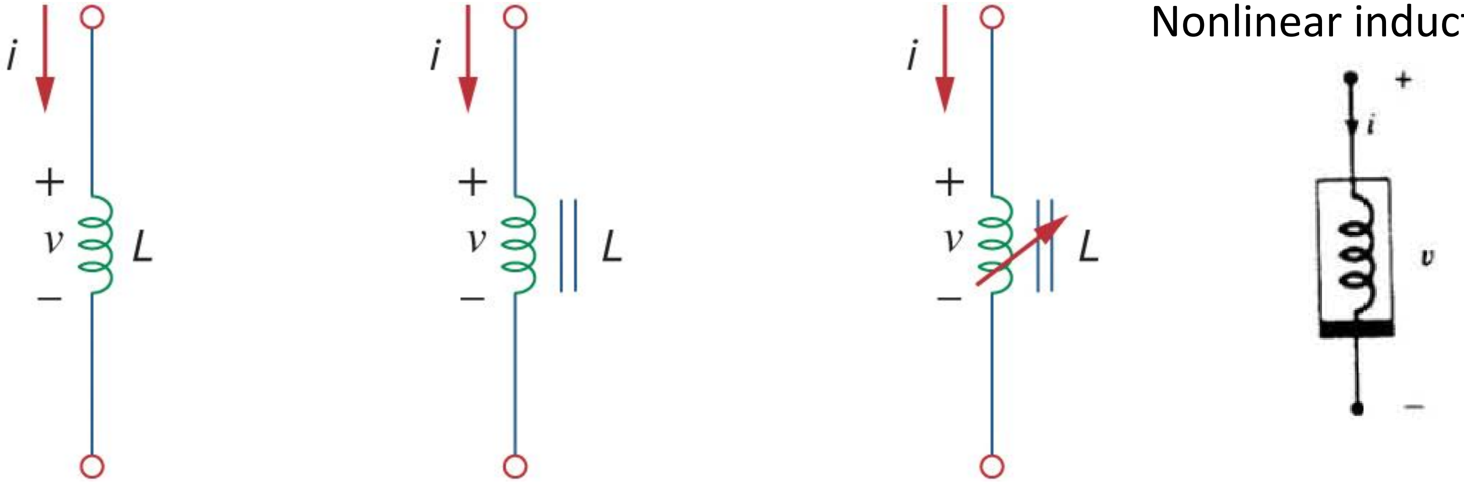
Circuit symbol of inductor

Nonlinear inductor



Air-core Iron-core Variable iron-core

Inductors

$$\boxed{v = L\frac{di}{dt}}$$

Voltage-current relationship for an inductor

$$di = \frac{1}{L}vdt$$ ➡ Integration of both sides

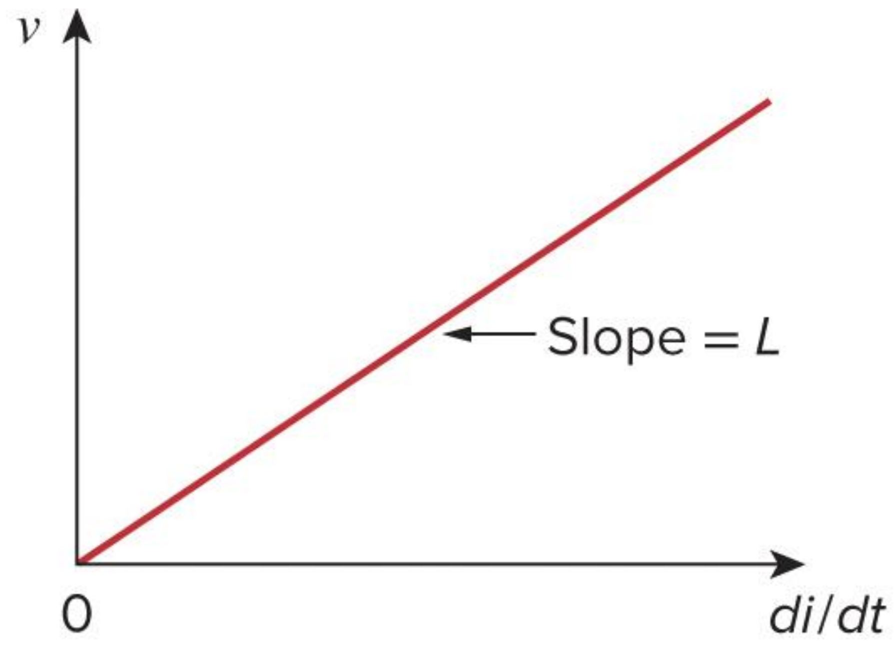
$$\int di = \int \frac{1}{L}vdt \Rightarrow i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

$$\boxed{i = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + i(t_0)}$$

Current-voltage relationship for an inductor

where $i(t_0)$: total current $-\infty < i < t_0$ and $i(-\infty) = 0$ (there must be time in the past where there is no current).

Graphical Representation of linear inductor



Inductors

- Energy stored in the inductor
- By definition:

$$p = vi \qquad v = L\frac{di}{dt} \Rightarrow \text{Substitute into power eq} \Rightarrow p = \left(L\frac{di}{dt}\right)i$$

$$p = \frac{w}{t} \Rightarrow w = pt \Rightarrow \text{derivation of both sides} \Rightarrow dw = pdt \Rightarrow \text{Integration of both sides}$$

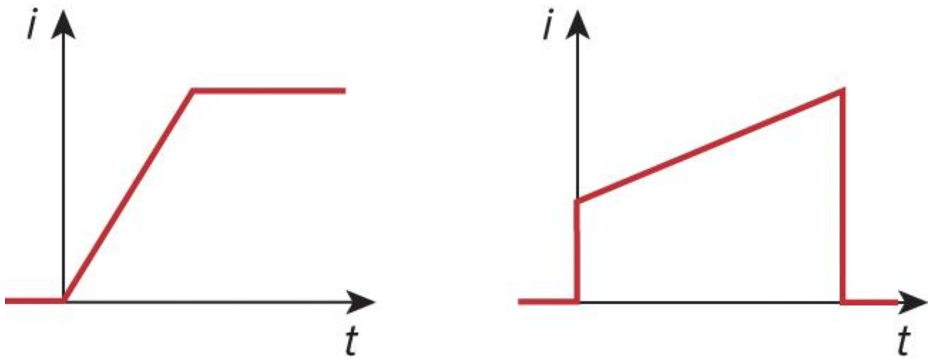
$$\int dw = \int_{-\infty}^{t} pdt \Rightarrow w = \int_{-\infty}^{t} pdt \Rightarrow w = L\int_{-\infty}^{t} \frac{di}{dt} idt$$

$$w = L\int_{-\infty}^{t} idi \Rightarrow w = L\frac{i^2(t)}{2}\bigg|_{-\infty}^{t} \Rightarrow L\left(\frac{i^2(t)}{2} - \frac{i^2(-\infty)}{2}\right)$$

Since $i(-\infty) = 0$ $\Rightarrow$ $\boxed{w = \frac{1}{2}Li^2}$ Energy stored in inductor

Inductors

- Important properties of an inductor:
  1. Voltage across an inductor is zero when current is constant $$v = L\frac{di}{dt}$$
     - Inductor acts like a short circuit to dc
  2. Current through an inductor cannot change instantaneously.



Possible Not possible

  3. Just like capacitors, ideal inductor does not dissipate energy.

- Note that nonideal inductor has a significant resistive component which dissipate very small energy.

Example 4

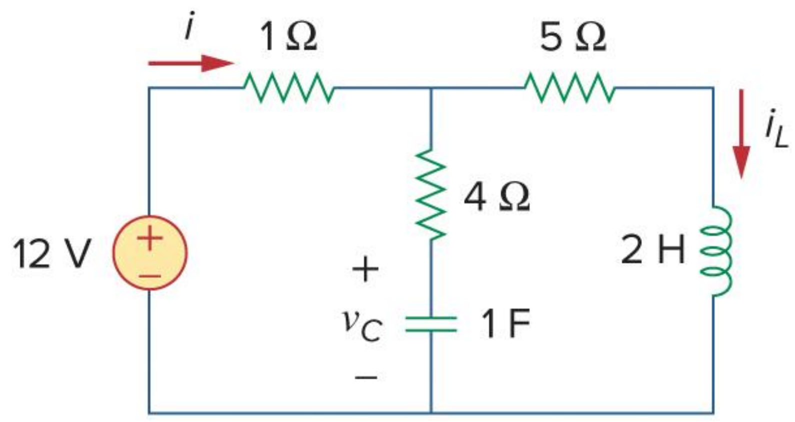
Consider the circuit below. under dc condition, find:

a-) i, v_c , $and\ i_L$

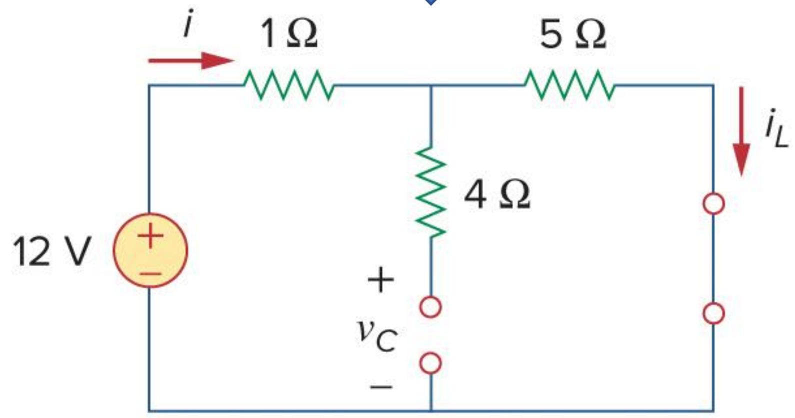
b-) The energy stored in the capacitor and inductor.

Solution:

a-) under dc condition, replace capacitor with an open circuit and inductor with short circuit



$$i = i_L = \frac{12}{1 + 5} = 2\text{ A}$$



The voltage v_c is same as the voltage across 5-Ω resistor since they are in parallel.

$$v_C = 5i = 10\text{ V}$$

b-) The energy stored in the capacitor:

$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50\text{ J}$$

The energy stored in the inductor: $w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4\text{ J}$

Series and Parallel Inductors

- Inductors in series:
  - N inductors connected in series as shown in the figure below

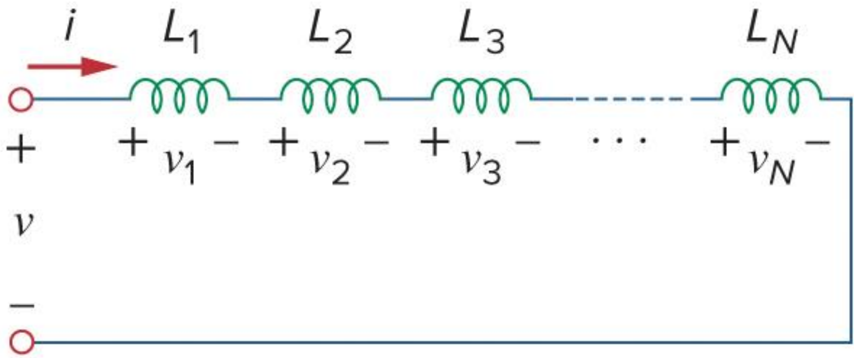
- Apply KVL:

$$v = v_1 + v_2 + v_3 + \cdots + v_N \qquad\qquad v = L\frac{di}{dt}$$

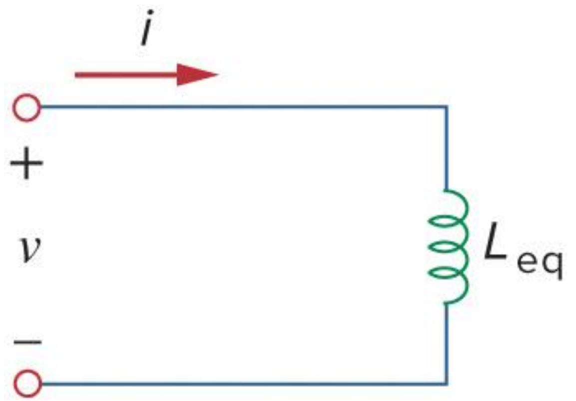
$$i = L_1\frac{di}{dt} + L_2\frac{di}{dt} + \cdots + L_N\frac{di}{dt}$$

$$v = (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt}$$

$$v = \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{eq}\frac{di}{dt} \quad \text{where } \boxed{L_{eq} = L_1 + L_2 + \cdots + L_N}$$



Equivalent circuit



- The equivalent inductance of N series-connected inductors is the sum of the individual inductances.
- Inductors in series are combined in exactly the same as resistors in series.

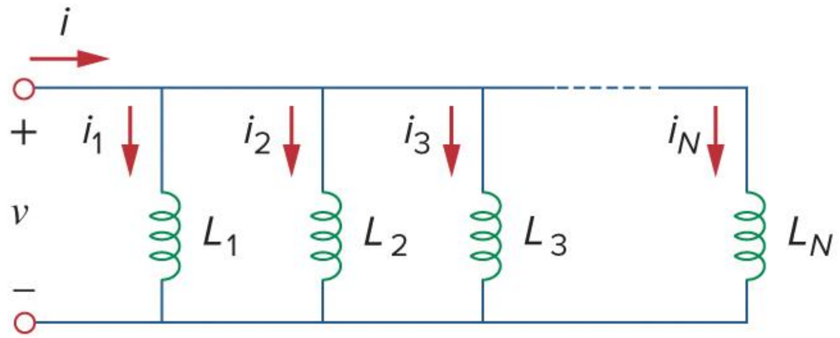
- Note that current flows through all inductors is same.

Series and Parallel Inductors

- Inductors in parallel:
  - N inductors connected in parallel as shown in the figure below
- Apply KCL:

$$i = i_1 + i_2 + i_3 + \cdots + i_N \quad , \quad i = \frac{1}{L}\int_{-\infty}^{t} vdt + i(t_0)$$

$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} vdt + i_2(t_0) + \cdots + \frac{1}{L_N}\int_{t_0}^{t} vdt + i_N(t_0)$$



$$i = \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} vdt + i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

Series and Parallel Inductors

$$i = \left(\sum_{k=1}^{N} \frac{1}{L_k}\right) \int_{t_0}^{t} v dt + \left(\sum_{k=1}^{N} i_k\,(t_0)\right) = \left(\sum_{k=1}^{N} \frac{1}{L_k}\right) \int_{t_0}^{t} v dt + i(t_0)$$

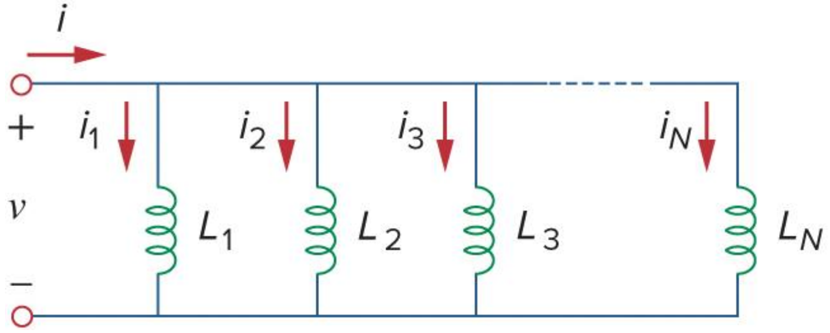
$$i = \left(\sum_{k=1}^{N} \frac{1}{L_k}\right) \int_{t_0}^{t} v dt + \left(\sum_{k=1}^{N} i_k\,(t_0)\right) = \left(\sum_{k=1}^{N} \frac{1}{L_k}\right) \int_{t_0}^{t} v dt + i(t_0)$$

where $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}$

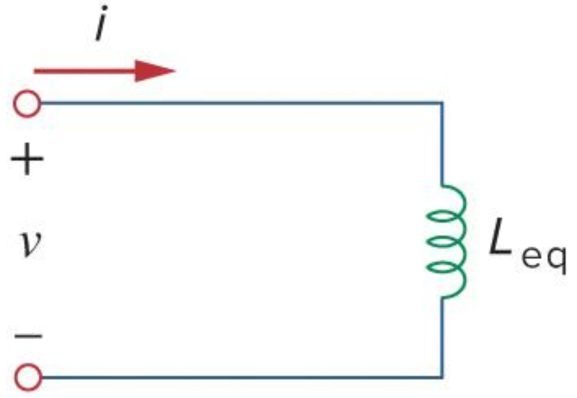
$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$

- The equivalent inductance of N parallel-connected inductors is the reciprocal of the individual inductances.
- Inductors in parallel are combined in the same as resistors in parallel.

- Note that voltage across all inductors is same.



Equivalent circuit



Summary

TABLE 6.1

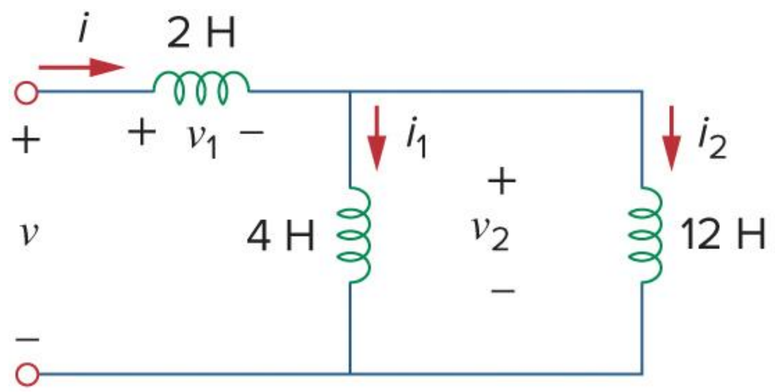
Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)\, d\tau + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)\, d\tau + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

Example 5

For the circuit shown below, $i(t) = 4(2 - e^{-10t})mA$. If $i_2(0) = -1mA$, find:

a-) $i_1(0)$; b-) v(t), $v_1(t)$, and $v_2(t)$; c-) $i_1(t)$ and $i_2(t)$.



Solution:

(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + (4 \,\|\, 12) = 2 + 3 = 5 \text{ H}$$

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5 \text{ mA}$$

$$= -3e^{-10t}\Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA}$$

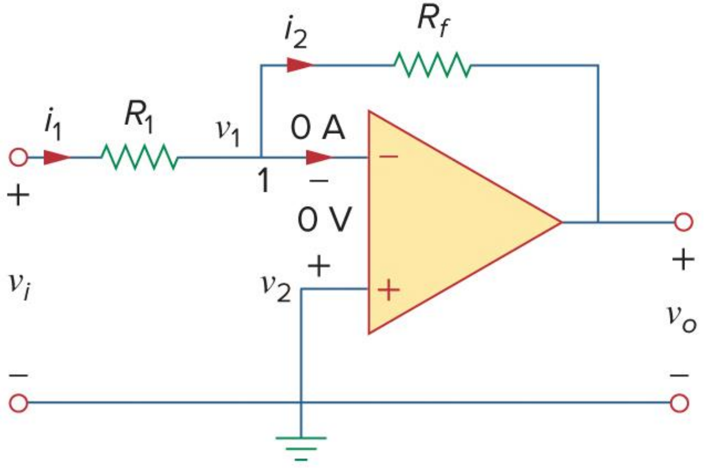
$$i_2(t) = \frac{1}{12}\int_0^t v_2\,dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\,dt - 1 \text{ mA}$$

$$= -e^{-10t}\Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA}$$

Application: Integrator

- What is an integrator?
  - Op amp circuit whose output is proportional to the integral of the input signal.
  - Replace R_f (feedback resistor) with a capacitor.

Ideal inverting amplifier

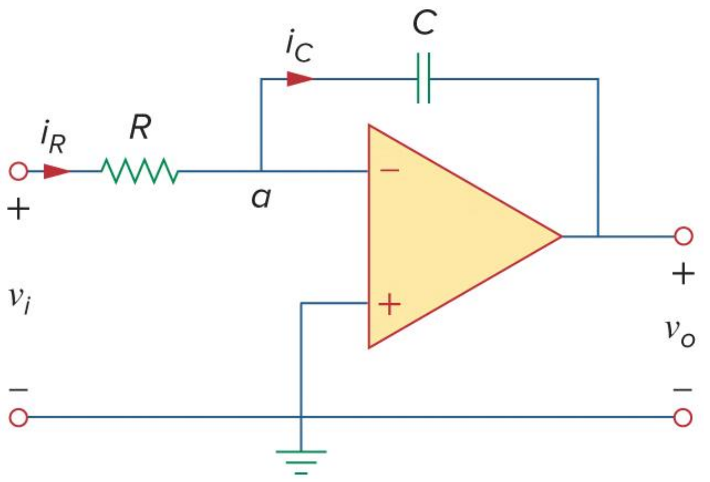


$i_R = i_C$ (No current flows into op amp)

$$i_R = \frac{v_i - v_a}{R} \quad , \quad i_C = -C\frac{dv_0}{dt} \quad , \quad v_a = v_b = 0$$

$$\frac{v_i}{R} = -C\frac{dv_0}{dt} \Rightarrow dv_0 = -\frac{1}{RC}v_i dt \Rightarrow$$ Integration of both sides

Ideal integrator



$$\int_{v_0(0)}^{v_0(t)} dv_0 = -\frac{1}{RC}\int_0^t v_i dt$$

Assume initially capacitor is discharged $v_0(0) = 0$

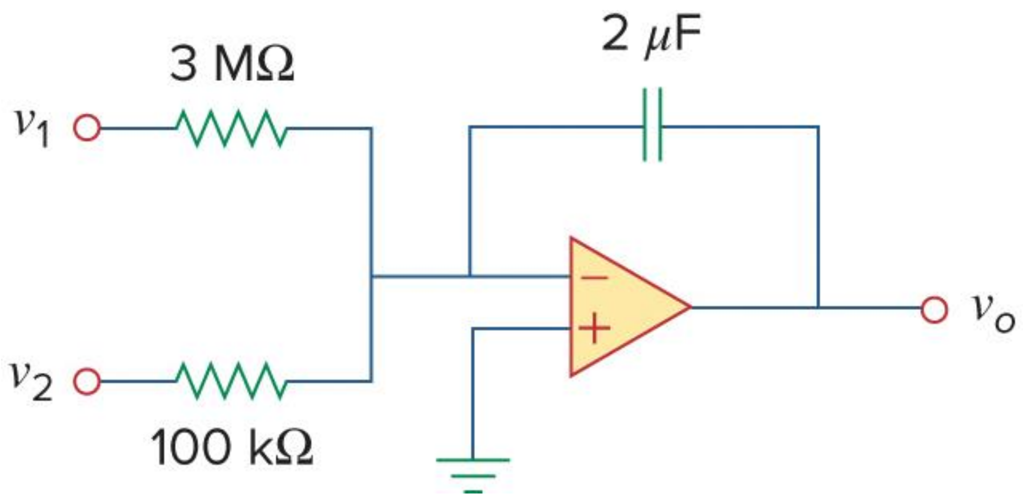
$$\boxed{v_0(t) = -\frac{1}{RC}\int_0^t v_i dt}$$

Example 6

If $v_1 = 10cos2t\ mV$ and $v_2 = 0.5t\ mV$, find v_0 in the op amp circuit shown below. Assume that the voltage across the capacitor is initially zero.

Solution:

This is a summing integrator.



$$v_o = -\frac{1}{R_1C}\int v_1\,dt - \frac{1}{R_2C}\int v_2\,dt$$

$$= -\frac{1}{3\times 10^6 \times 2\times 10^{-6}}\int_0^t 10\cos(2\tau)d\tau$$

$$-\frac{1}{100\times 10^3\times 2\times 10^{-6}}\int_0^t 0.5\tau d\tau$$

$$= -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0.2}\frac{0.5t^2}{2} = -0.833\sin 2t - 1.25t^2\ \text{mV}$$

Recall

$$\int cosx\,dx = sinx$$

$$\int cos2x\,dx = \frac{sin2x}{2}$$

Application: Differentiator

- What is a differentiator?
  - Op amp circuit whose input is proportional to the rate of change of the input signal.
  - Replace R_1 (input resistance) with a capacitor.

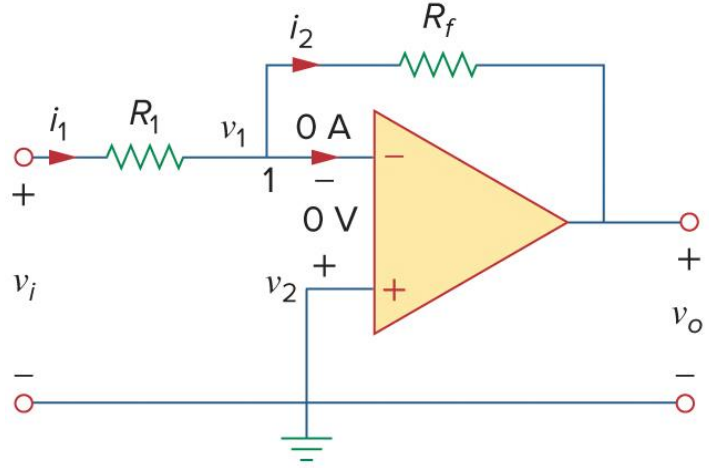
$i_C = i_R$ (No current flows into op amp)

$$i_C = C\frac{dv_i}{dt} \quad , \quad i_R = \frac{v_a - v_0}{R} \quad , \quad v_a = v_b = 0$$

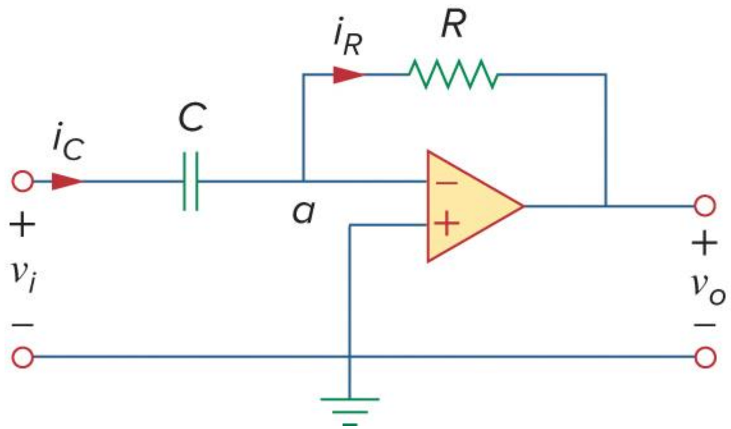
$$i_R = -\frac{v_0}{R} \quad , \quad i_C = i_R \quad \Rightarrow \quad C\frac{dv_i}{dt} = -\frac{v_0}{R}$$

$$\boxed{v_0 = -RC\frac{dv_i}{dt}}$$

Ideal inverting amplifier



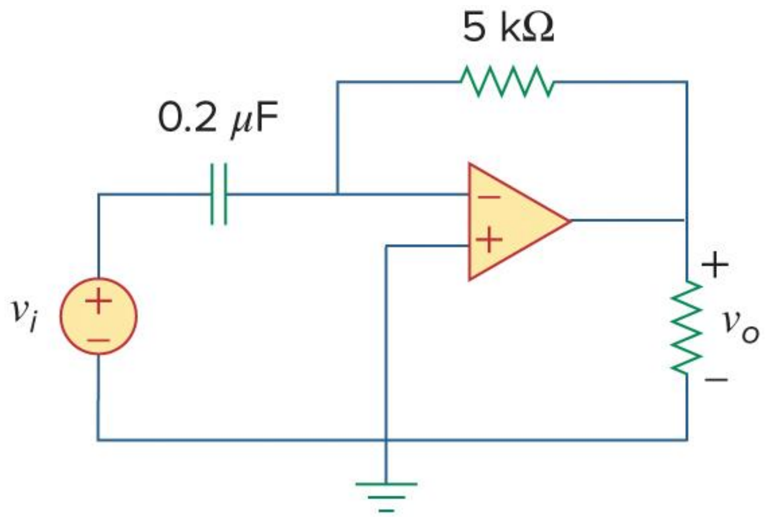
Differentiator



Example 7

Sketch the output voltage for the circuit shown below, given the input voltage below as well. Take $v_0 = 0$ at t = 0.

Solution:

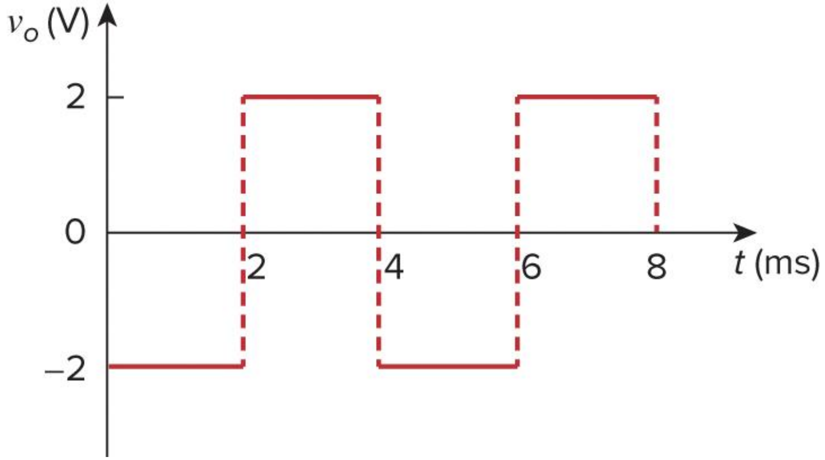


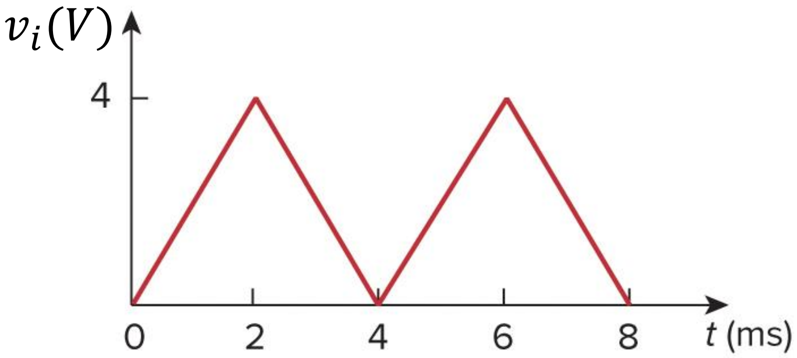
$$v_0 = -RC\frac{dv_i}{dt}\ ,\ RC = 5\times10^3\times0.2\times10^{-6} = 10^{-3}s$$

$$v_i = \begin{cases} 2000t & 0 < t < 2 \text{ ms} \\ 8 - 2000t & 2 < t < 4 \text{ ms} \end{cases}$$

This is repeated for $4 < t < 8$ ms.

$$v_o = -RC\frac{dv_i}{dt} = \begin{cases} -2\text{ V} & 0 < t < 2 \text{ ms} \\ 2\text{ V} & 2 < t < 4 \text{ ms} \end{cases}$$





Application: Analog Computers:

- Program to solve mathematical models of mechanical and electrical systems.
- Models expressed in terms of differential equations.
- To solve simple differential equations using the analog computers requires cascading three types of op amp circuits:
  - Integrator
  - Summing amplifiers
  - Inverting (negative scaling)/noninverting (positive scaling) amplifiers

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

Second order differential equation

- Here a, b, and c are constant and $f(t)$ is an arbitrary forcing function.

$$\frac{d^2x}{dt^2} = \frac{f(t)}{a} - \frac{b}{a}\frac{dx}{dt} - \frac{c}{a}x \quad \Rightarrow$$

Solve for x

Application: Analog Computers:

$$\frac{d^2x}{dt^2} = \frac{f(t)}{a} - \frac{b}{a}\frac{dx}{dt} - \frac{c}{a}x \Rightarrow$$ Solve for x

- To solve for x, take integral of each derivative to reduce to the x.
- To obtain $\frac{dx}{dt}$ in the equation, integrate and inverted $\frac{d^2x}{dt^2}$ (inverted means we have $-\frac{dx}{dt}$ in the equation and we'll change the sign of it).
- Finally, to obtain x, the $\frac{dx}{dt}$ term is integrated and inverted.
- Forcing function is injected at the proper point.
- Thus, the analog computer for solving differential equation is implemented by connecting the necessary summers, inverters, and integrators.
- If a plotter or an oscilloscope is connected to the system, we can view the output x, $\frac{dx}{dt}$, or $\frac{d^2x}{dt^2}$ depending on where we connect the oscilloscope in the system.

Example 8

Design an analog computer circuit to solve the differential equation

$$\frac{d^2v_0}{dt^2} + 2\frac{dv_0}{dt} + v_0 = 10sin4t, \qquad t > 0$$

Subject to $v_0(0) = -4, v_0'(0) = 1$, where the prime refers to the time derivative.

Solution:

- Design an analog computer circuit: integrator circuit combined with a summing capability and inverter circuits.
- Pick proper value of resistances and capacitors, many which result in correct solution (infinite number of possibilities for picking resistance and capacitors)

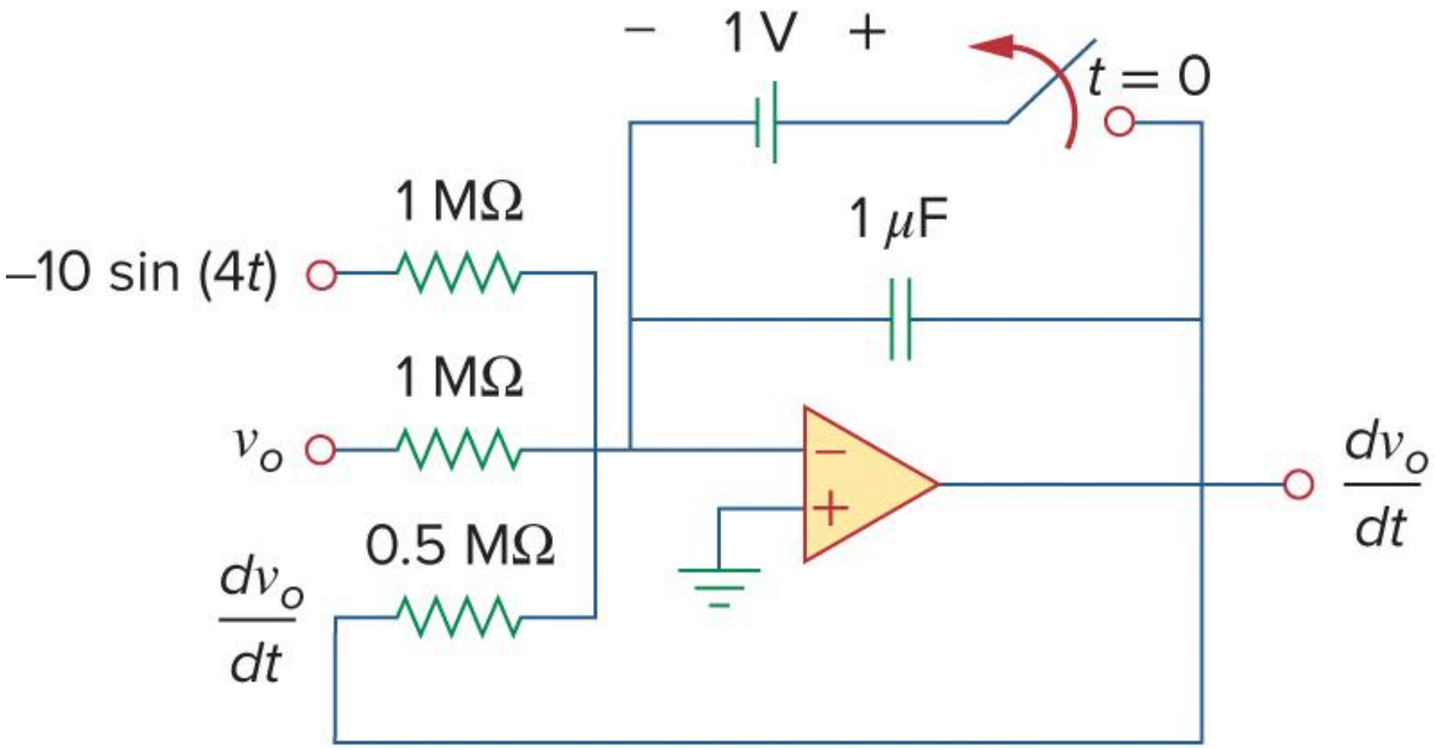
$v_0(0) = -4$ ➡ Output voltage

$v_0'(0) = 1$ ➡ 1st derivative of output voltage

$$\frac{d^2v_0}{dt^2} = 10sin4t - 2\frac{dv_0}{dt} - v_0$$ ➡ Take the integral of both sides

Solution

$$\frac{dv_o}{dt} = -\int_0^t \underbrace{\left(-10\ \sin(4\tau) + 2\frac{dv_o(\tau)}{d\tau} + v_o(\tau)\right)}_{\text{Three inputs: summing integrator}} d\tau + v_o'(0) \qquad \text{where } v_o'(0) = 1$$

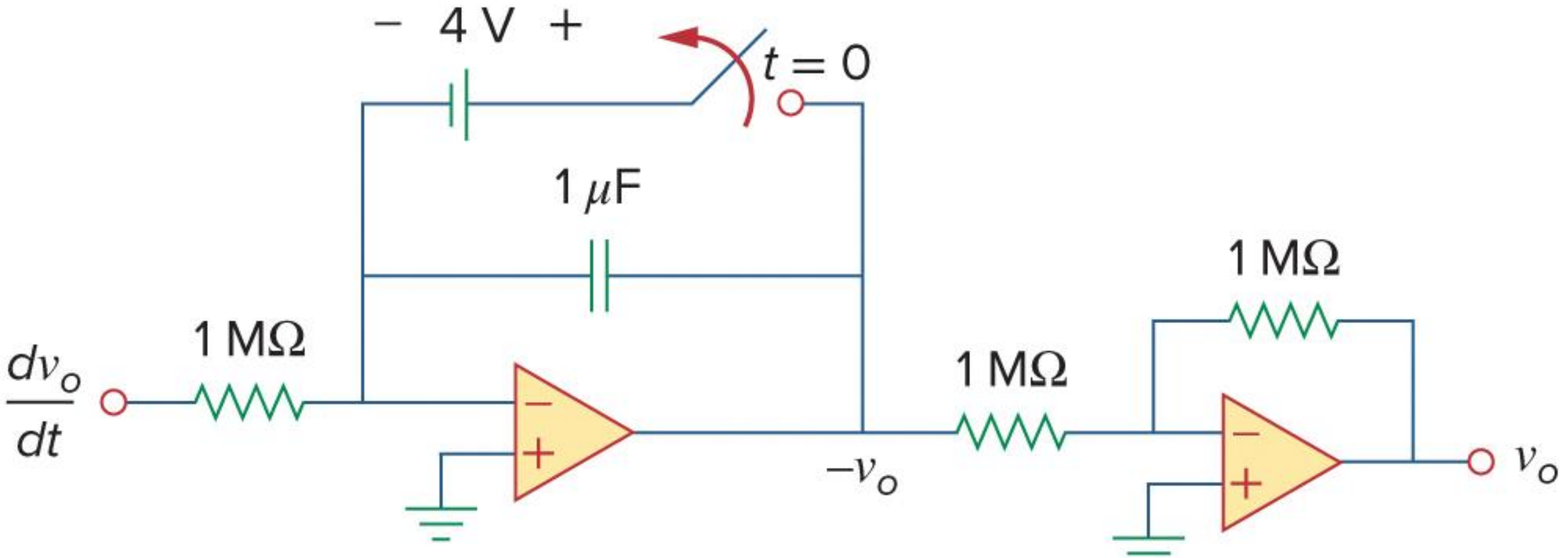


Solution

- Next step is to obtain v_0 by integrating $\frac{dv_0}{dt}$ and inverting.

$$v_o = -\int_0^t \left(-\frac{dv_o(\tau)}{d\tau}\right) d\tau + v(0)$$

$$v_0(0) = -4$$



Solution

- Combine two circuits to obtain complete circuit as shown below.