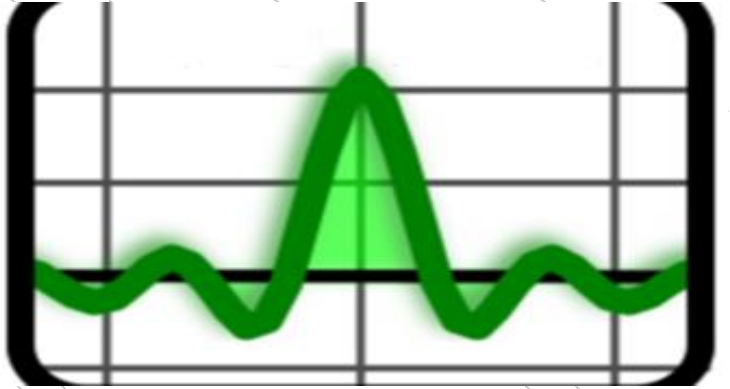

Signals & Systems for Computer Engineering

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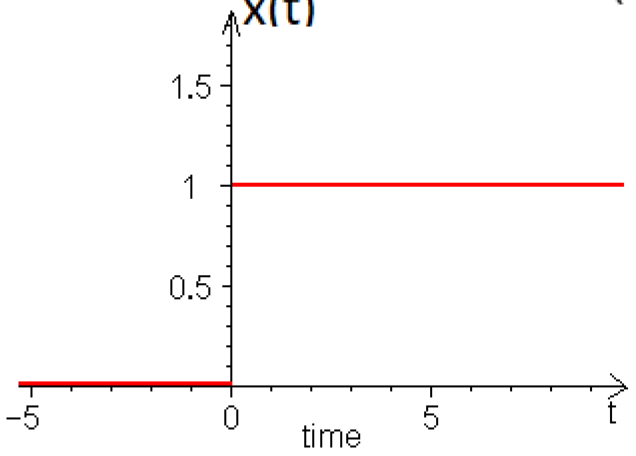
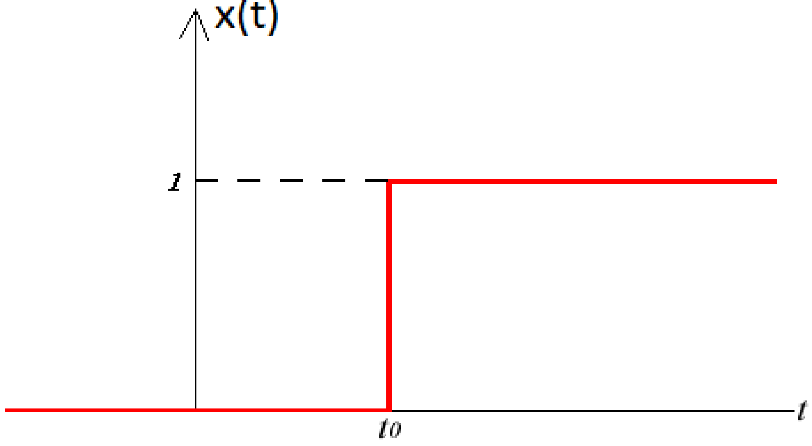
BLG354E / CRN: 21560
2nd Week Lecture

Unit Step Function:

The unit step function u(t) is defined as $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$



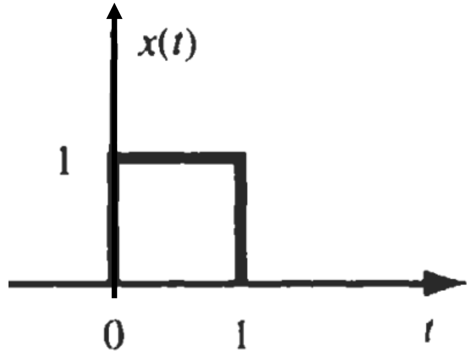
Shifted unit step function $u(t - t_0)$ is defined as $u(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases}$



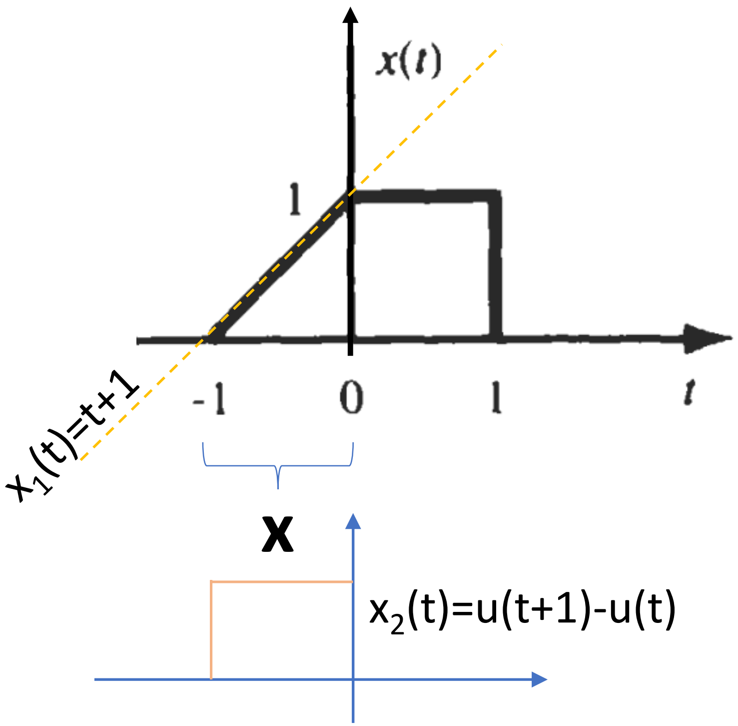
u(t) is discontinuous at t = 0, its value at t = 0 is undefined as different from u[n]

Example:

Express the signals shown in the figures in terms of unit step functions:



$x(t)=u(t)-u(t-1)$



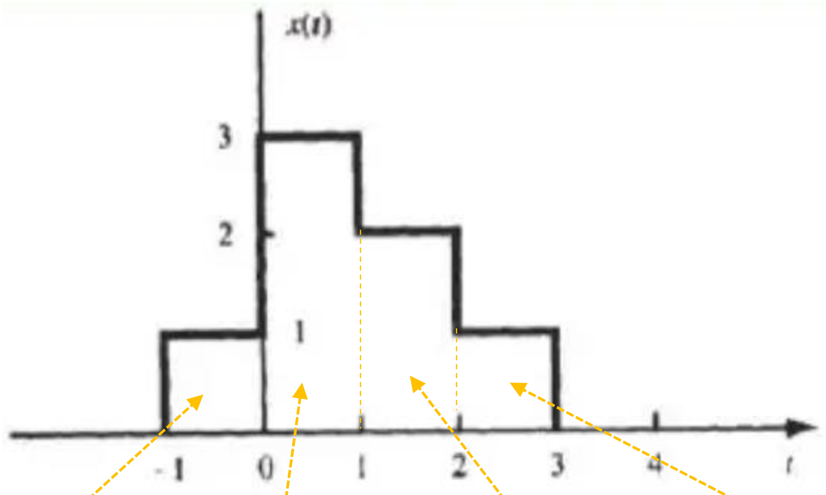
$x(t)=(t+1)[u(t+1)-u(t)]+[u(t)-u(t-1)]$

$x(t)=(t+1)u(t+1)-tu(t)-u(t)+u(t)-u(t-1)$

$x(t)=(t+1)u(t+1)-tu(t)-u(t-1)$

Example:

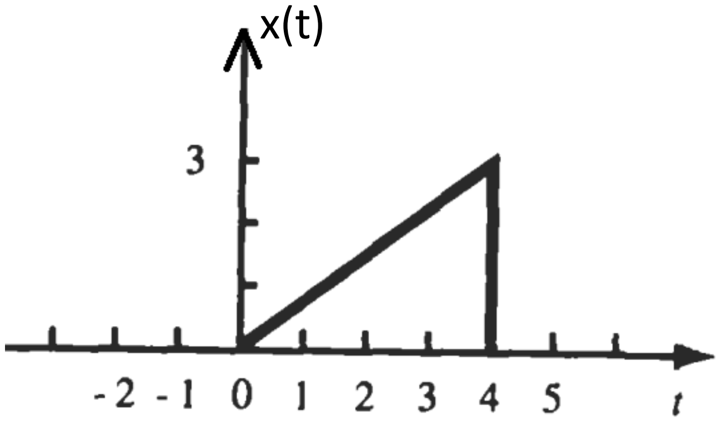
Express the signals shown in the figure in terms of unit step functions:



$$x(t)=(u(t+1)-u(t))+3(u(t)-u(t-1))+2(u(t-1)-u(t-2))+(u(t-2)-u(t-3))$$

$$x(t)=u(t+1)+2u(t)-u(t-1)-u(t-2)-u(t-3)$$

Example: x(t) is a CT signal given in the graph



a) Mathematically express the function

b) Find and sketch $x(t-2)$

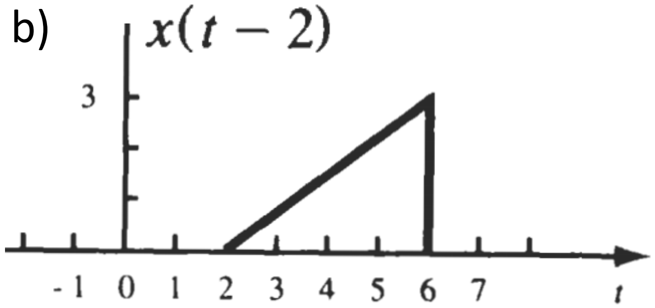
c) Find and sketch $x(2t)$

d) Find and sketch $x(t/2)$

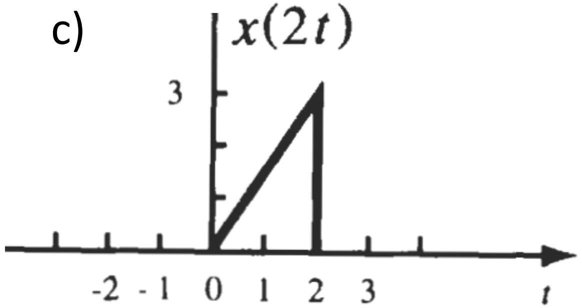
e) Find and sketch $x(-t)$

a) x(t)=0.75t[u(t)-u(t-4)]

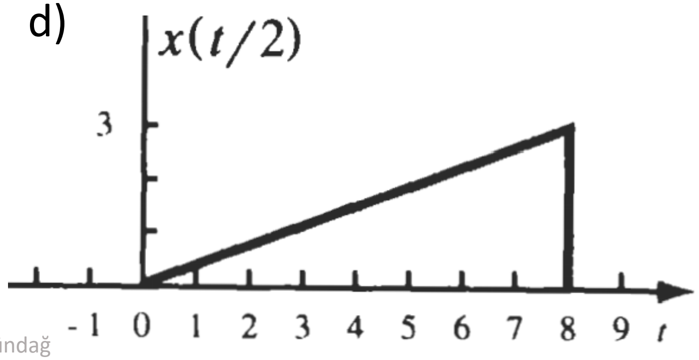
b)



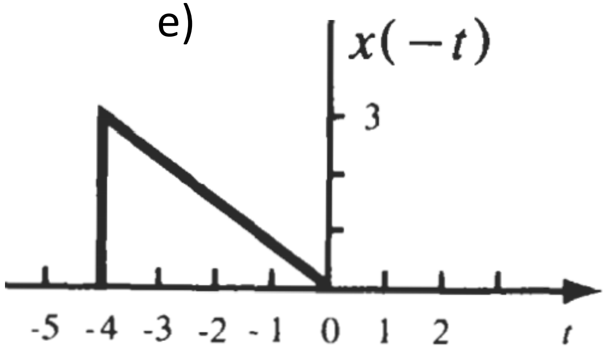
c)



d)

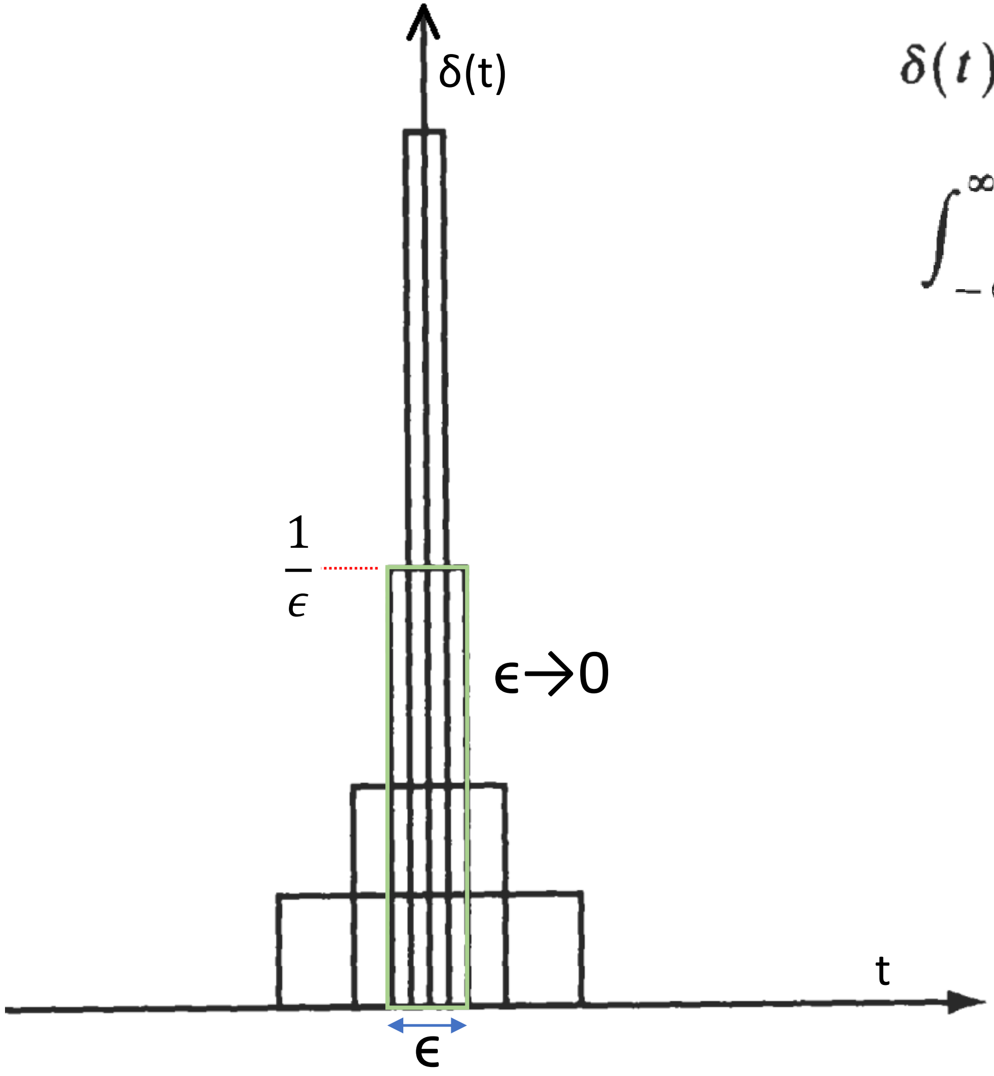


e)



The Unit Impulse Function:

The unit impulse function δ(t), also known as the Dirac delta function, defined as the limit of a suitably chosen conventional function having unity area over an infinitesimal time interval.



$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1$$

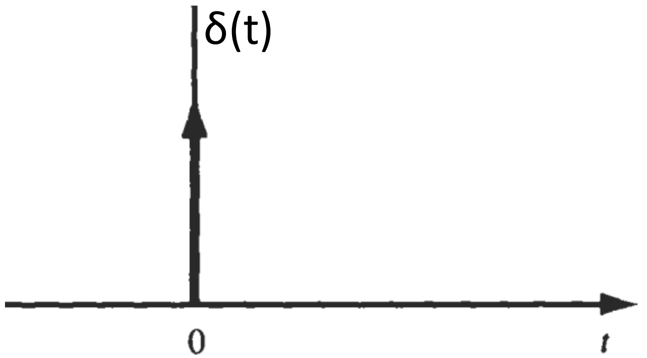
$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\, dt = \phi(0)$$

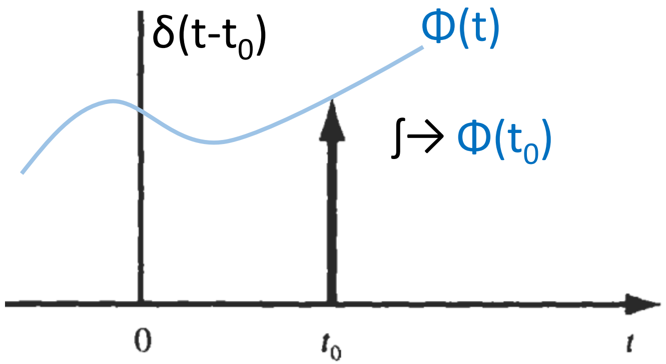
Φ(t) is known as the testing function
δ(t) is called a generalized function

t_0 delayed delta function δ(t- t_0) is defined by

$$\int_{-\infty}^{\infty} \phi(t)\delta(t - t_0)\, dt = \phi(t_0)$$

Φ(t) is any function continuous at $t=t_0$





Properties of delta Dirac Function:

- $$\delta(t) = u'(t) = \frac{du(t)}{dt}$$
- $$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$
- $$\delta(at) = \frac{1}{|a|}\delta(t)$$
- $$x(t)\delta(t) = x(0)\delta(t)$$
- $$\delta(-t) = \delta(t)$$
- $$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

Proof:

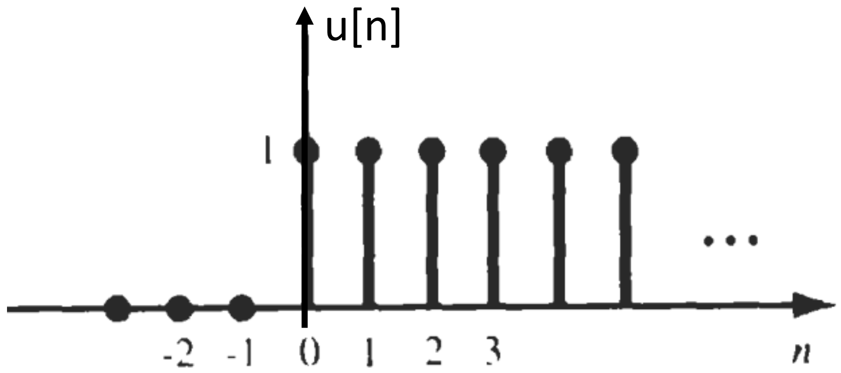
Show that $\delta(t) = u'(t) = \frac{du(t)}{dt} = -[\phi(\infty) - \phi(0)]$

$$\int_{-\infty}^{\infty} \phi(t)u'(t)\, dt = -\int_{-\infty}^{\infty} \phi'(t)u(t)\, dt = -\int_{0}^{\infty} \phi'(t)\, dt = -\phi(t)\Big|_{0}^{\infty}$$

$$= \phi(0) = \int_{-\infty}^{\infty} \phi(t)\,\delta(t)\, dt \rightarrow \delta(t) = u'(t) = \frac{du(t)}{dt}$$

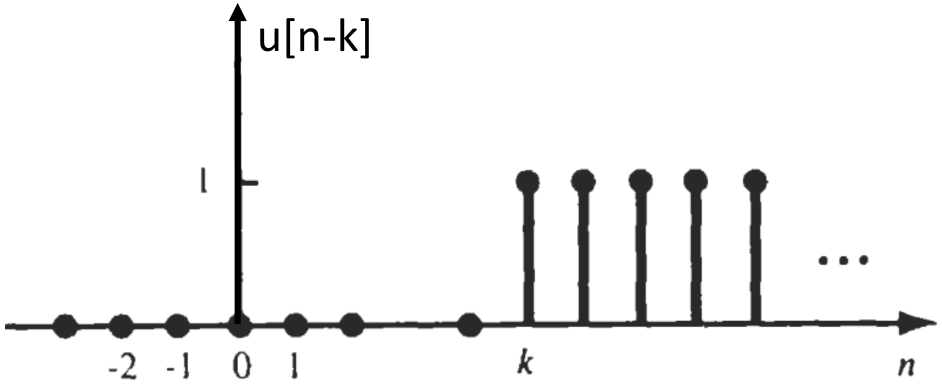
The Unit Step Sequence:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Unlike the continuous-time step function u[n] at n=0, u[0]=1

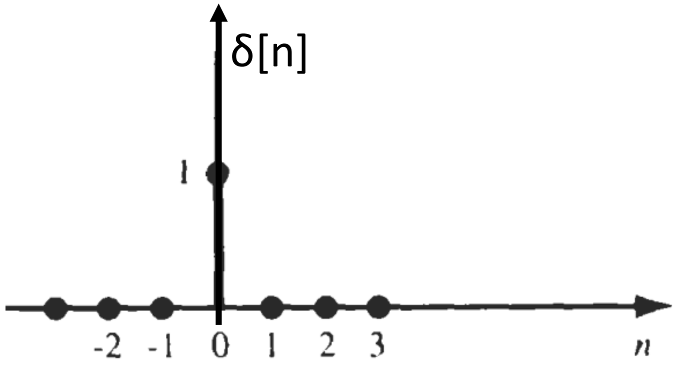
$$u[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$$



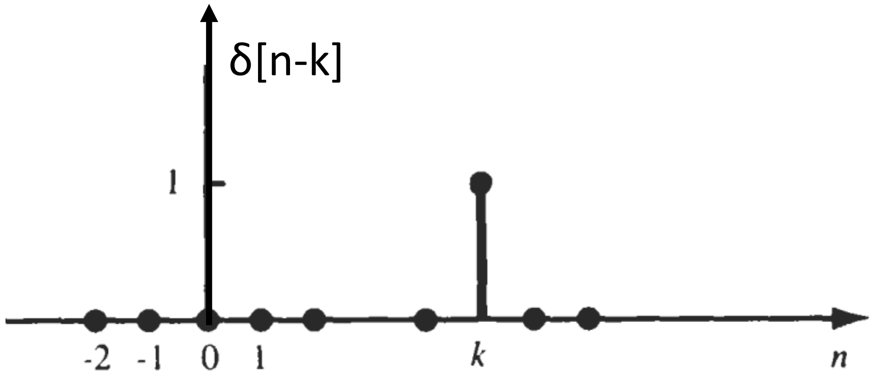
The Unit Impulse Sequence:

Unit impulse sequence δ[n] is defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



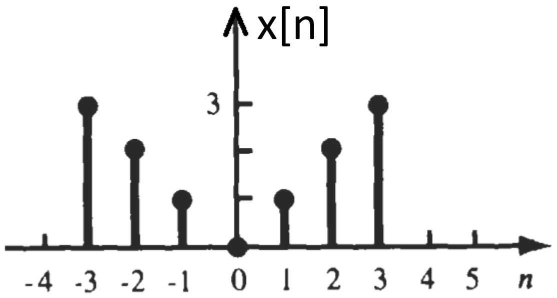
$$\delta[n] = u[n] - u[n-1]$$



$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

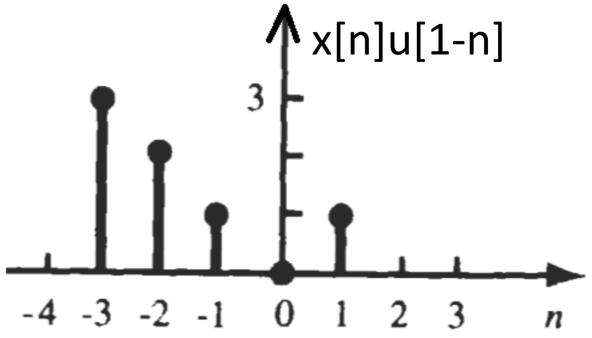
Example:



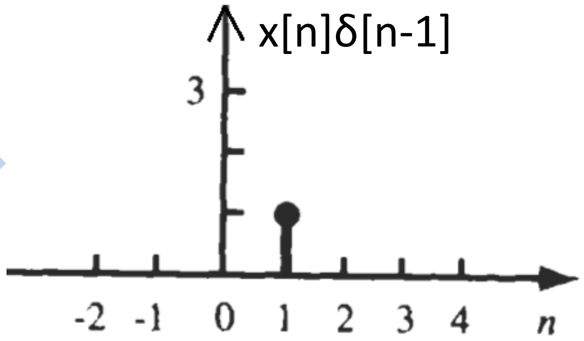
a) Find and sketch $x[n]u[1-n]$

b) Find and sketch $x[n]\delta[n-1]$

a) $$u[1-n] = \begin{cases} 1 & n \le 1 \\ 0 & n > 1 \end{cases}$$



b) $$x[n]\delta[n-1] = x[1]\delta[n-1] = \delta[n-1] = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$



Example:

$x[n]$ is given as $x_n=\{-1,\underset{\uparrow}{0},1,2,0,4\}$. Express $x[n]$ as a function.

$x[n]=-\delta[n+1]+\delta[n-1]+2\delta[n-2]+4\delta[n-4]$ or

$x[n]=n(u[n+1]-u[n-5])-3\delta[n-3]$

Vector Space:

Theorems of linear algebra provides that there is a basis $\{v_k\}$ →A set of signals in terms of which any signal s can be expanded

$$s = \sum_k c_k v_k$$

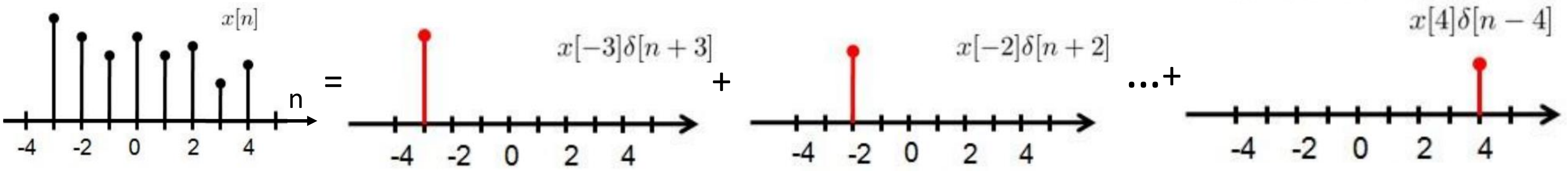
Here summation sigma indicates that there are finite or denumerable number of basis signals

If a nondenumerable infinity of basis signals is required then sum must be replaced by the integration: $s = \int c(k) v(k)\, dk$

Every vector space has a basis. In general, basis is not unique.

Any digital signal can be expanded by unit impulse functions as long as overlapping is avoided (orthogonality and linear independence):

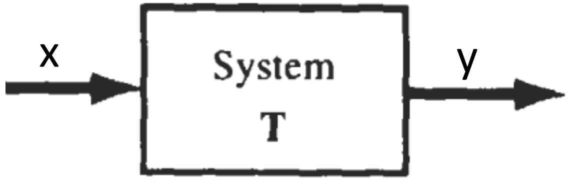
$$s[n] = \sum_{k=-\infty}^{\infty} s[k]\delta[n-k]$$



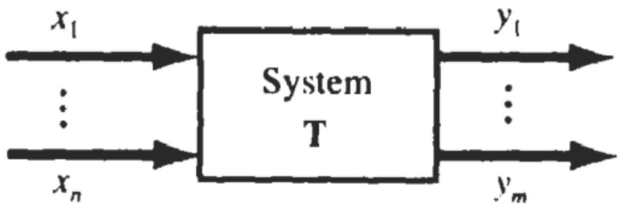
SYSTEMS and CLASSIFICATION of the SYSTEMS

The system is a transformation (or mapping) of x into y where x and y are the input and output signals, respectively. This transformation is stated as:

$$y=\mathbf{T}x$$

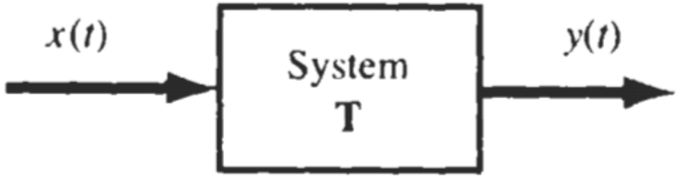


Single input-single output system

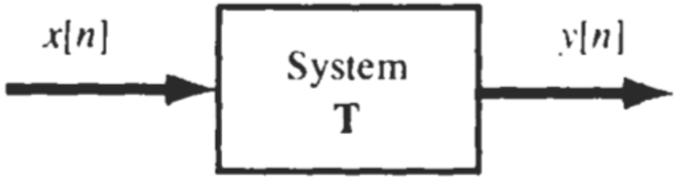


Multiple input-multiple output system

If the input and output signals x and p are continuous-time signals, then the system is called a continuous-time system



If the input and output signals are discrete-time signals or sequences, then the system is called a discrete-time system

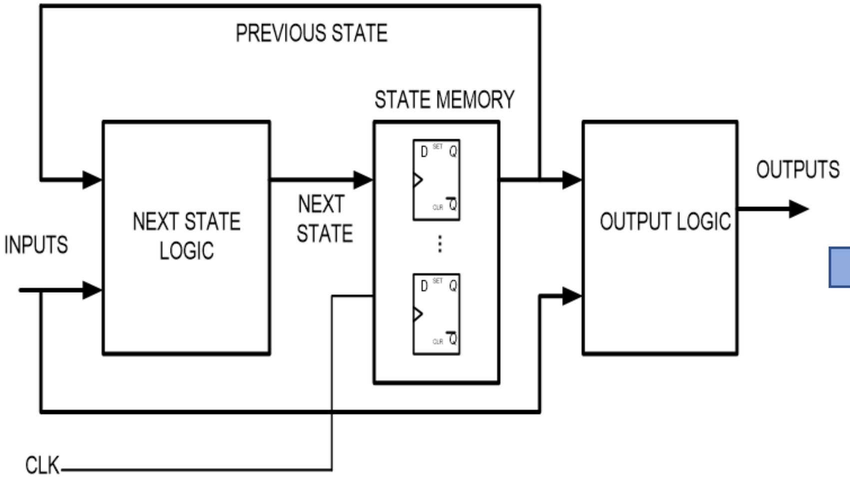


Systems with Memory and without Memory

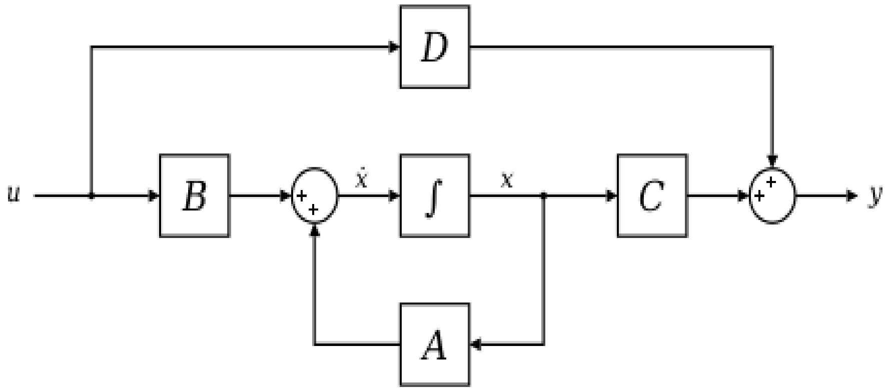
A system is said to be memoryless if the output at any time depends on only the input at that same time. Otherwise, the system is said to have memory.

Is there a contradiction with the term “memoryless” due to definition of a system ?

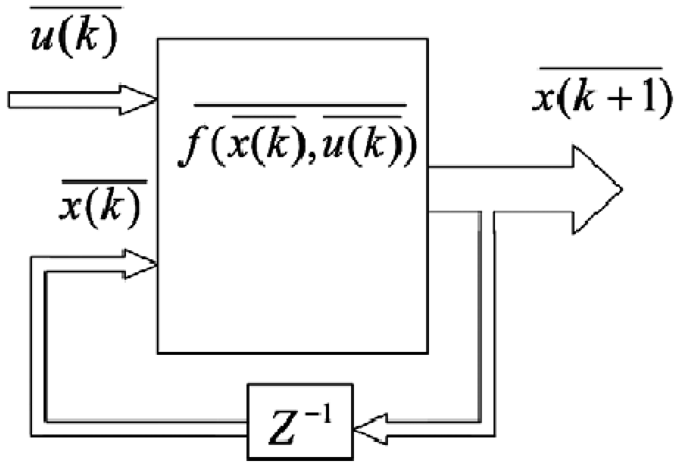
Mealy Machine Model for Logical Systems



State Space Model for Continuous Time Systems



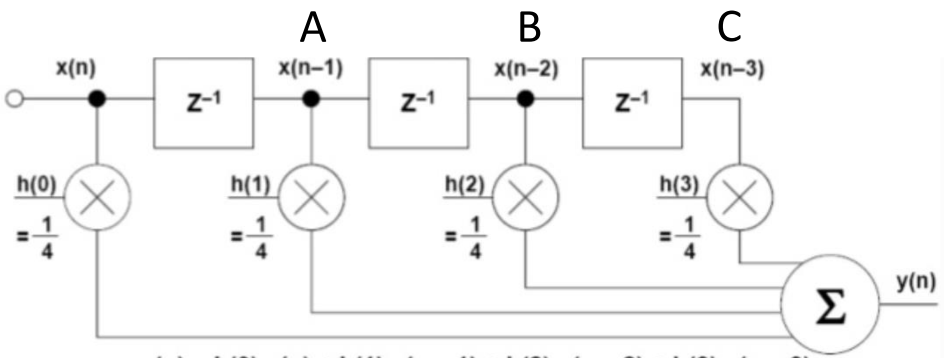
State Space Model for Discrete Time Systems



Output is not only a function of current input but also it depends on internal states, so the past input value(s)

System examples having memory:

Moving Average Filter as a system with memory:



$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + h(3)\,x(n-3)$$
$$= \frac{1}{4}x(n) + \frac{1}{4}x(n-1) + \frac{1}{4}x(n-2) + \frac{1}{4}x(n-3)$$
$$= \frac{1}{4}\left[x(n) + x(n-1) + x(n-2) + x(n-3)\right]$$

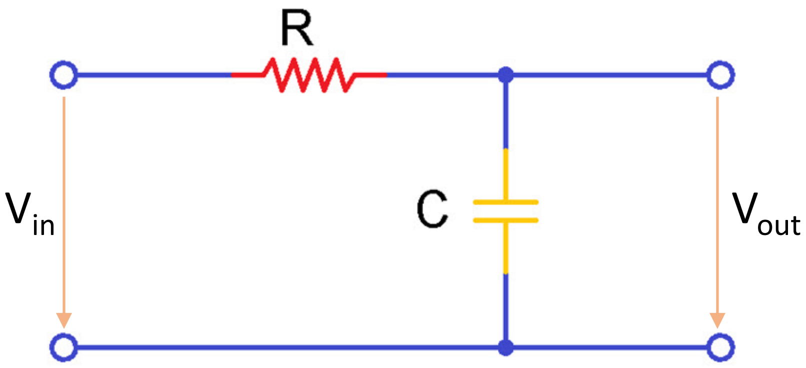
$\mathbf{z^{-1}} = e^{-sT}$

x[n] , A=x[n-1] , B=x[n-2] , C=x[n-3]

$z^{-1} \rightarrow$ C=B , B=A , A=x[n]

Order of a system: Memory Units keeping the states

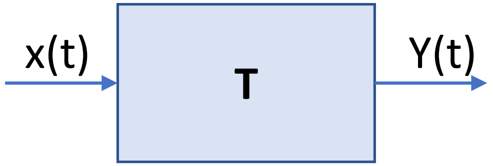
RC low-pass filter circuit as a system with memory:



$$\left.\begin{aligned} v_s(t) &= Ri(t) + v_c(t) \\ i(t) &= C\frac{dv_c(t)}{dt} \end{aligned}\right\} \quad \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

$x(t)=V_{in}$
$y(t)=V_c=V_{out}$

Transfer function



Can you write a difference equation that simulates the RC filter Transfer function?

Linear and Non-Linear Systems:

A system is called linear if it satisfies additivity and scaling conditions

Additivity → for any signals x_1 and x_2 , $\mathbf{T}x_1 = y_1$ and $\mathbf{T}x_2 = y_2$ then $\mathbf{T}\{x_1 + x_2) = y_1 + y_2$
where **T** is the transfer function of the system

Scaling → for any signals x and scalar α , $T\{\alpha x_1\} = \alpha y_1$

Superposition property for Linearity test:

A system is said to be linear if $\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 y_1 + \alpha_2 y_2$ is valid for any signals x_1, x_2 and scalar α_1 and α_1

Example:

If transfer function of a system is given as $y = \mathbf{T}\{x\} = x^2$, show that this system is nonlinear.

$$\mathbf{T}\{x_1 + x_2\} = (x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1 x_2$$

$$\mathbf{T}\{x_1\} + \mathbf{T}\{x_2\} = x_1^2 + x_2^2$$

$\mathbf{T}\{x_1 + x_2\} \neq \mathbf{T}\{x_1\} + \mathbf{T}\{x_2\}$ → system is nonlinear

Causal and Noncausal Systems:

* If output of a system y(t) at an arbitrary time $t = t_0$ depends on only the input x(t) for $t \le t_0$ then it is called causal system. Output of a causal system does not depend on its future values, it is only dependent on the present and/or past values of the input.

* A system is called noncausal if it is not causal.

For example, y(t)=x(t+1) is not a casual system
because it is not possible to obtain an output before an input is applied to the system.

Example:

Compare the casuality of DT systems having the transfer function $T_1(z) = \dfrac{z^2 - 1}{z}$ and $T_2(z) = \dfrac{z}{z^2 - 1}$

$$T_1(z) = \frac{y_1(z)}{x_1(z)} = \frac{z^2 - 1}{z} = z - z^{-1} \rightarrow y_1(k)=x_1(k+1)+x_1(k-1) \quad \text{this system is not casual}$$

$$T_2(z) = \frac{y_2(z)}{x_2(z)} = \frac{z}{z^2 - 1} = \frac{z^{-1}}{1 - z^{-1}} \rightarrow y_2(k)-y_2(k-1)=x_2(k-1) \quad \text{this system is casual}$$

$$T(.) = \frac{y(.)}{x(.)}$$

Numerator polynomial

Denominator polynomial

System is casual if the order of numerator polynomial ≤ order of denominator polynomial
(the number of zeros must not exceed the number of poles)

Time-varying and Time-invariant Systems:

* A CT system is called time-invariant if any time shift τ in the input signal as x(t- τ) causes the same time shift in the output signal as y(t- τ):

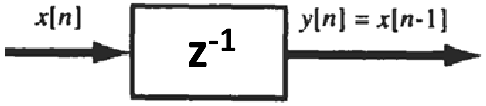
$$\mathbf{T}\{x(t-\tau)\} = y(t-\tau)$$

* A DT system is called time-invariant if any integer amount of delay, k, in the input signal as x(n-k) causes the same amount of delay at the output signal as y(n-k):

$$\mathbf{T}\{x[n-k]\} = y[n-k]$$

* A DT or CT system which is not time-invariant is called a time-varying system

Proof: Show that unit delay operator is time-invariant



Let $y_1[n]$ be the response to $x_1[n] = x[n-n_0]$ then $y_1[n] = \mathbf{T}\{x_1[n]\} = x_1[n-1] = x[n-1-n_0]$

$y[n-n_0] = x[n-n_0-1] = x[n-1-n_0] = y_1[n]$ → system is time-invariant

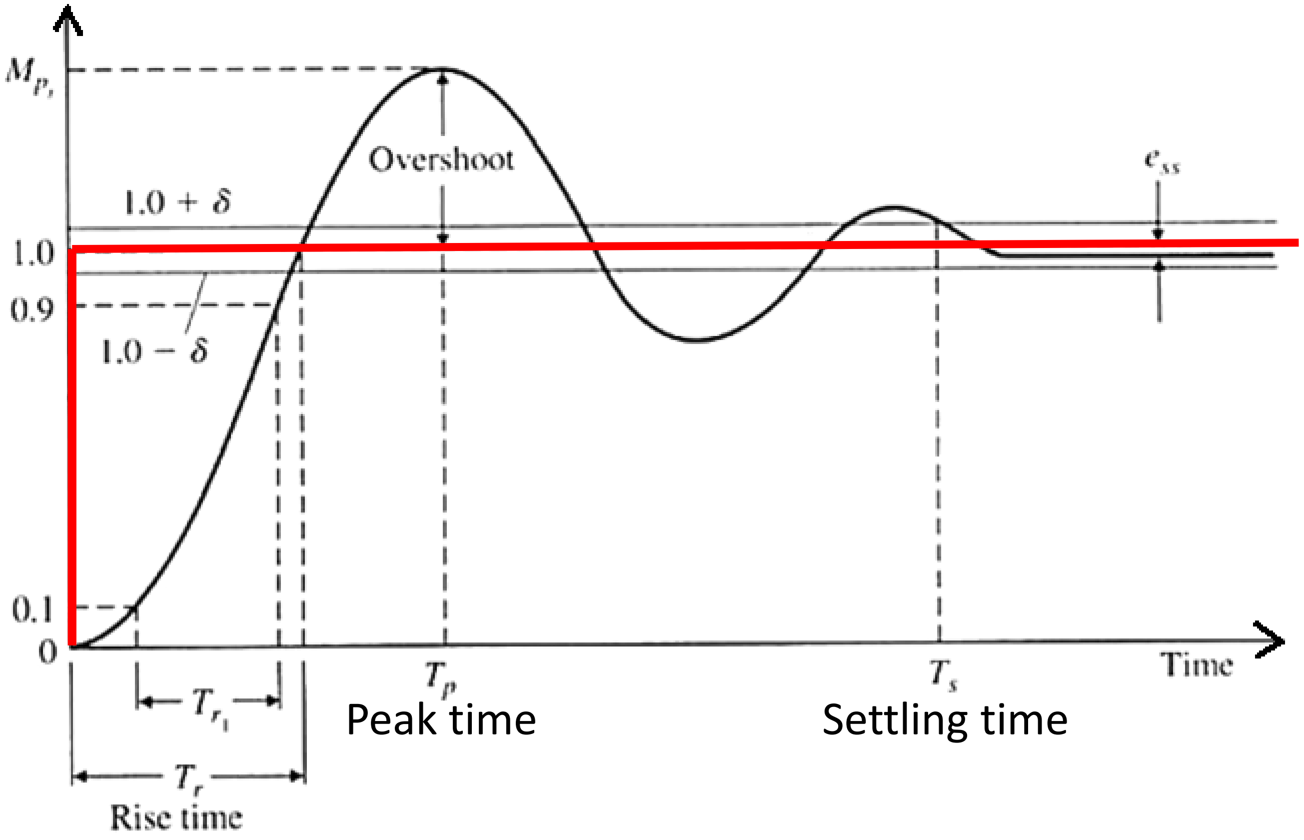
Stable Systems:

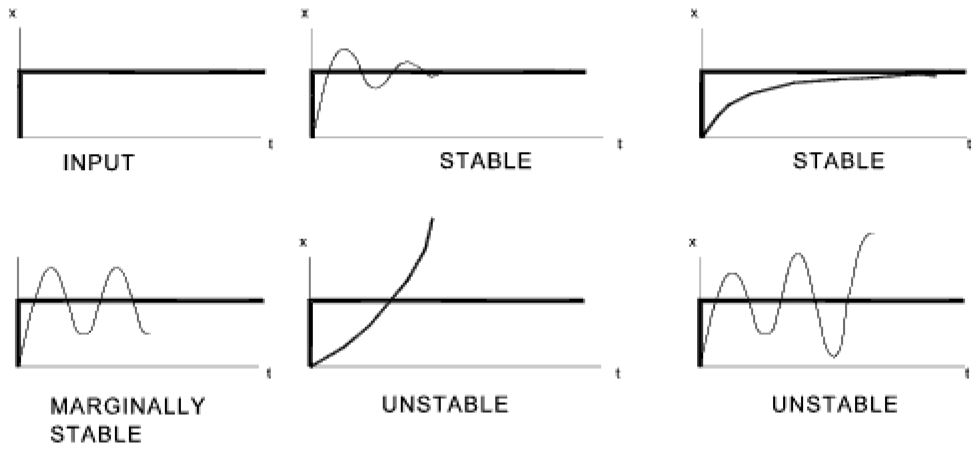
There are many definitions of stability. **BIBO stability**:

A system is bounded-input/bounded-output (BIBO) stable if for any bounded input x defined by $|x| \leq k_1$ $|y| \leq k_2$

where k_1 and k_2 are finite real constants

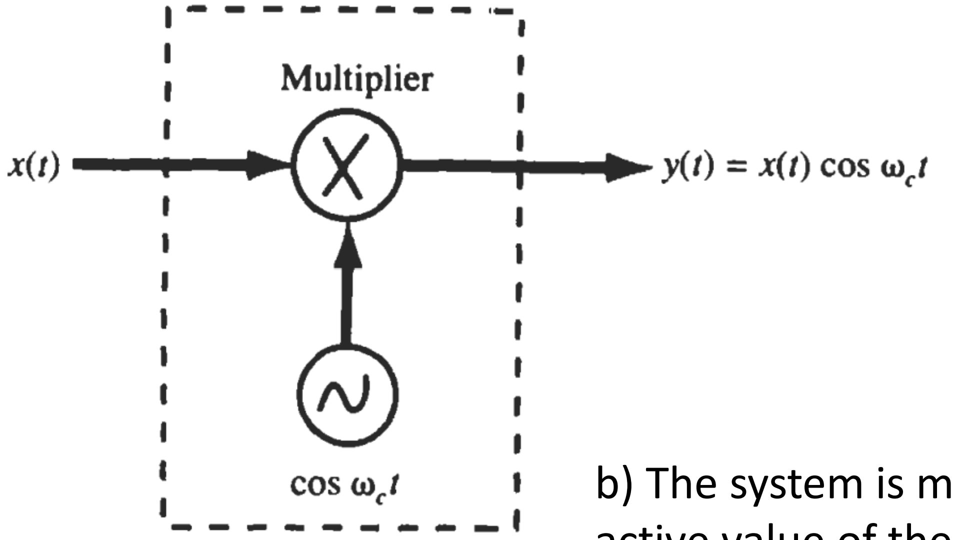
Stability definitions in control systems:





If a system is **asymptotic**ally stable, then it is also BIBO stable.

Example: A basic amplitude modulator is shown in the figure below. Determine whether if this system is (a) linear, (b) memoryless, (c) causal, (d) time-invariant, or (e) stable



a) Lets apply the linearity test: $\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} \overset{?}{=} \alpha_1 y_1 + \alpha_2 y_2$

$$y(t) = \mathbf{T}\{x(t)\} = [\alpha_1 x_1(t) + \alpha_2 x_2(t)] \cos \omega_c t$$

$$= \underbrace{\alpha_1 x_1(t) \cos \omega_c t}_{y_1(t)} + \underbrace{\alpha_2 x_2(t) \cos \omega_c t}_{y_2(t)}$$

$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$ → system is linear

b) The system is memoryless because the output value $y(t)=x(t)\cos \omega_c t$ only depends on the active value of the input x(t)

c) It is a causal system since the output y(t) does not depend on the future values of the input x(t),

d) If we shift the input x(t) for t_0 as $x_1(t)=x(t-t_0)$ then the output will be $y_1(t) = \mathbf{T}\{x(t-t_0)\} = x(t-t_0) \cos \omega_c t$

The output y(t) at the time $+t_0$ is $y(t-t_0) = x(t-t_0) \cos \omega_c (t-t_0)$

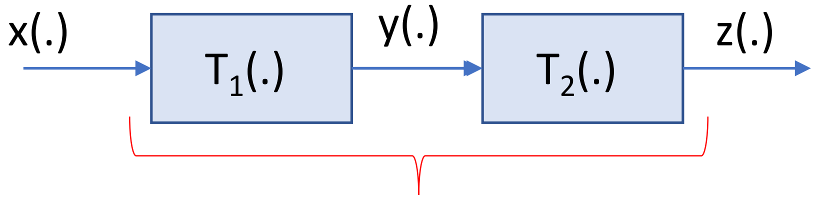
Since $y(t-t_0) \neq y_1(t)$ the system is not time invariant

e) System is BIBO stable because $|y(t)| = |x(t)\cos(\omega_c t)| \leq |x(t)|$ since $\cos(\omega_c t) \leq 1$

so if the input x(t) is bounded then the output y(t) is also bounded

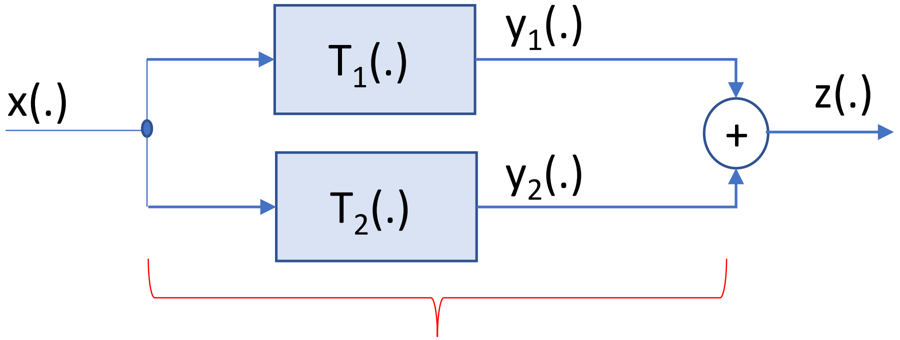
Interconnectivity of the systems:

Cascaded Systems



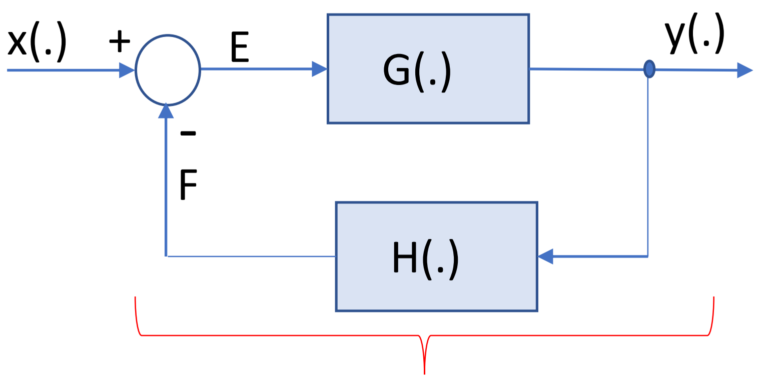
$$T = \frac{Z}{Y}\frac{Y}{X} = \frac{Z}{X} = (T_1 \cdot T_2)$$

Parallel Systems



$$T = \frac{Y_1}{X} + \frac{Y_2}{X} = \frac{Z}{Y} = (T_1 + T_2)$$

Feedback Systems:



F=H·Y

E=X-F

Y=E·G

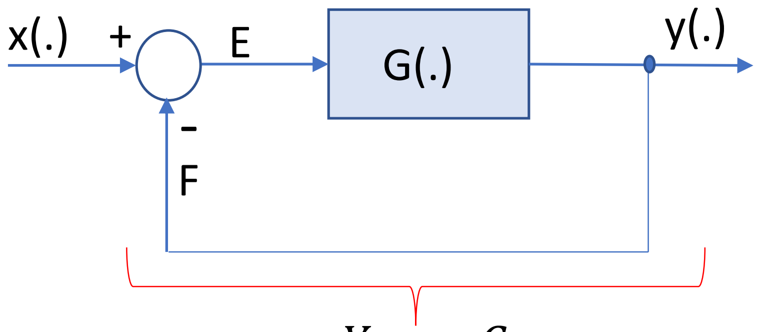
Y= (X-F)·G

Y= (X-Y·H)·G

Y= X·G-Y·H·G

$$T = \frac{Y}{X} = \frac{G}{1 + GH}$$

Negative Unit Feedback System:



$$T = \frac{Y}{X} = \frac{G}{1 + G}$$