DIFFERENTIAL EQUATION (DE)

An eq. that expresses a relation between an unknown funct and one or more of its derivatives

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

ORDER
Highest order of any diff in the eq.

ORDINARY-PARTIAL
The eq. contains diffs with respect to

only 1 var → ODE

several ind. vars → PDE

LINEAR-NONLINEAR
$P_n(x)y^{(n)} + \dots + P_1(x)y' + P_0(x)y = g(x)$

If it can be written in this form → Linear

Otherwise nonlin

HOMOGENEOUS-NONHOM.
If it contains no nondiff terms → homog.

Otherwise nonhom.

DE	ORDER	DEP. V. (diff. term)	IND. V. (nondiff t.)	ODE-PDE	LIN.-NONLIN	HOM-NONH.
$\frac{dx}{dt} = x^2 + t^3$	1	x	t	ODE	NL	NH
$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$	2	y	x	ODE	L	H
$\left(\frac{dy}{dx}\right)^2 = y + \sin x$	1	y	x	ODE	NL	NH
$\frac{\partial^3 z}{\partial x^3} = 2\frac{\partial^2 z}{\partial x \partial y} + x$	3	z	x, y	PDE	L	NH

Ex1 Show that $\overset{(*)}{y=ce^{x^2}}$ satisfies the DE $\overset{(**)}{y'=2xy}$.

$y' = c\,2x e^{x^2} \overset{?}{=} 2x\,ce^{x^2}$ for all $x \Rightarrow$ $(*)$ is a solution of the DE since it satisfies $(**)$
→ general solution

(pay attention $y=0$ also satisfies $(**)$; thus it is another solution but you can get it for $c=0$

$y=ce^{x^2}$: infinite family of solutions since $c \in \mathbb{R}$ is an arbitrary constant

$\Rightarrow$ introduce a condition $y(0)=2 \rightarrow$ initial condition, $\overset{(***)}{y'=2xy,\ y(0)=2}$ → initial value pr.

$y=ce^{x^2}$, $y(0)=2 \Rightarrow 2=ce^0 \Rightarrow c=2 \Rightarrow y=2e^{x^2}$ → unique solution

$y=ce^{x^2} \Rightarrow$ general sol. of $(**)$, $y=2e^{x^2}$: particular solution

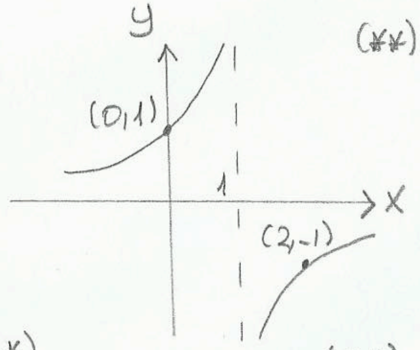
Ex2 $\frac{dy}{dx}=y^2$, $y=\frac{1}{c-x} \Rightarrow -(c-x)^{-2}(-1) \overset{?}{=} (c-x)^{-2} \Rightarrow y=(c-x)^{-1}$ is a solution of the DE
→ also $y=0$

in any interval which doesn't contain $x=c \Rightarrow (-\infty,c)\cup(c,\infty)$

$\overset{(*)}{y'=y^2,\ y(0)=1} \Rightarrow 1=(c-0)^{-1} \Rightarrow c=1 \Rightarrow y=(1-x)^{-1}$

$(**)$ $y(2)=-1 \Rightarrow (c-2)^{-1}=-1 \Rightarrow c=1 \Rightarrow y=(1-x)^{-1}$

Ex3 $(y')^2+y^2=-1 \Rightarrow$ no solution



$(*)$ $x \in (-\infty,1)$ $\quad$ $(**)$ $x \in (1,\infty)$

Assume that $u=u(x)$ is cont on an int. I and $u', u'', \ldots, u^{(n)}$ exist on I.
Then, $u=u(x)$ is a sol. of the DE $F(x,y,y',\ldots,y^{(n)})=0$ if it satisfies the DE.

$K(x,y)=0$ implicit solution $\left(x^2+y^2=4,\ x+yy'=0 \Rightarrow y=\pm\sqrt{4-x^2}\ !!!\right)$

Ex 4 $y' = \frac{1}{x} \Rightarrow dy = \frac{dx}{x} \Rightarrow y = \ln|x| + c$

$y(0) = 0 \Rightarrow$ no solution since it is not defined at $x=0$

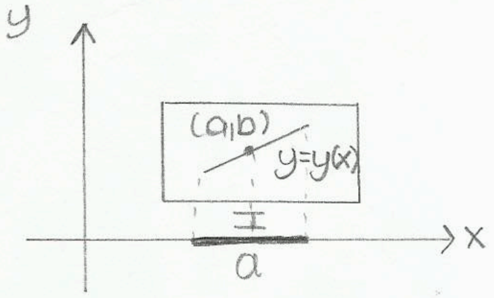
$\Rightarrow$ nonexistence of a solution

Ex 5 $y' = 2\sqrt{y} \Rightarrow \frac{dy}{2\sqrt{y}} = dx \Rightarrow \sqrt{y} = x + c \Rightarrow y = (x+c)^2$
$y = 0$

$y(0) = 0 \Rightarrow c = 0 \Rightarrow y = x^2,\ y = 0 \Rightarrow$ not a unique solution

EXISTENCE & UNIQUENESS OF SOLUTIONS

Assume that $f(x,y)$ and its partial derivative $D_y f(x,y)$ are continuous on some rectangle R in the xy-plane that contains the point (a,b) in its interior. Then, for some open interval I containing the point a, the initial value problem $\frac{dy}{dx} = f(x,y),\ y(a) = b$ has one and only one solution on I.



Ex 4: $f(x,y) = \frac{1}{x}$ and $f_y = 0 \Rightarrow f$: not cont at $(0,0)$

Ex 5: $f = 2\sqrt{y}$, $f_y = \frac{1}{\sqrt{y}} \Rightarrow f_y$: not cont at $(0,0)$

Ex 1: $f = 2xy$, $f_y = 2x \Rightarrow f, f_y$: cont at any (a,b).
Solution exists and it is unique.
$\downarrow$
$y = ce^{x^2}$

Ex: $\overset{(*)}{x\dfrac{dy}{dx} = 2y} \Rightarrow \dfrac{dy}{dx} = 2\dfrac{y}{x}$

$f = 2\dfrac{y}{x}$, $f_y = \dfrac{2}{x}$: cont at any $x \neq 0$

$\rightarrow$ (*) has a unique solution for all $x \neq 0$.

$x\dfrac{dy}{dx} = 2y$, $y(0) = 0$ initial value problem

$y = cx^2 \Rightarrow x \cdot 2cx = 2cx^2$ ✓

$\rightarrow$ IVP has infinitely many solutions.

$x\dfrac{dy}{dx} = 2y$, $y(0) = b$

$\rightarrow$ IVP has no solution if $b \neq 0$

$$x\frac{dy}{dx} = 2y, \quad y(a) = b$$

* $a \neq 0 \Rightarrow$ un. sol
* $a = b = 0 \Rightarrow$ inf. many sol.
* $a = 0, b \neq 0 \Rightarrow$ no sol

1ST ORDER DE. $\frac{dy}{dx} = F(x,y)$

LINEAR 1ST ORD

$\frac{dy}{dx} + M(x)y = N(x)$

μ: int. fact.

$\mu y' + \mu M y = \mu N$

$(\mu y)' = \mu y' + \mu M y$

$= \mu y' + \mu' y$

$\mu' = \mu M$

$\int \frac{d\mu}{\mu} = \int M\,dx$

$\mu = u(x) = e^{\int M dx}$

$uy' + uMy = uN$

$\frac{d}{dx}(uy) = uN$

$\int d(uy) = \int uN\,dx$

$uy = v(x) + c$

$y = u^{-1}(v(x) + c)$

SEPARABLE

$M(y)dy = N(x)dx$

$\int M dy = \int N dx$

$y = u(x) + c$

SUBSTITUTION

HOMOGENEOUS

$y' = F(y/x)$

$v = y/x \Rightarrow y = xv$

$v + xv' = F(v)$

$\frac{dv}{F(v) - v} = \frac{dx}{x}$

$\downarrow$

separable

$\downarrow$

$Ax^m y^n y' = Bx^p y^q + Cx^r y^s$

$m+n = p+q = r+s$

$\Rightarrow$ hom.

BERNOULLI

$y' + M(x)y = N(x)y^n$

$(1-n)y^{-n}y' + (1-n)My^{1-n} = (1-n)N$

$v = y^{1-n} \Rightarrow v' = (1-n)y^{-n}y'$

$v' + (1-n)Mv = (1-n)N$

$\downarrow$

Linear 1st order

EXACT

$M(x,y)dy + N(x,y)\,dx = 0$

if $M_x = N_y$

$\mu_y = M$, $\mu_x = N$

$\mu_y = M \Rightarrow \mu = \int M dy + h(x)$

$(\int M dy + h(x))_x = N$

$\Rightarrow \mu = f(x,y)$

$\Rightarrow f_y dy + f_x\,dx = 0$

$df = 0$

$f = c$

Follow these steps:

* Sep.?
* Lin.?
* Ex.?
* Hom. or Bern?

Ex: Solve $y' - 2y = e^{-x}$, $y(0) = 1$

Linear 1st order $\Rightarrow \mu \cdot y' - 2\mu \cdot y = \mu e^{-x} = (\mu y)'$

$(\mu y)' = \mu y' + \mu' y = \mu y' - 2\mu y \Rightarrow \mu' = -2\mu \Rightarrow \frac{d\mu}{\mu} = -2dx$

$\Rightarrow \ln\mu = -2x \Rightarrow \mu = e^{-2x}$

$(e^{-2x}y)' = e^{-2x}e^{-x} \Rightarrow e^{-2x}y = \int e^{-3x}dx \Rightarrow e^{-2x}y = -\frac{1}{3}e^{-3x} + c$

$\Rightarrow y = -\frac{1}{3}e^{-x} + ce^{2x}$

$y(0) = 1 \Rightarrow 1 = -\frac{1}{3} + c \Rightarrow c = 4/3 \Rightarrow y = -\frac{1}{3}e^{-x} + \frac{4}{3}e^{2x}$

Ex: $(x^2+1)\frac{dy}{dx} + 3xy = 6x$ Lin. 1st order

$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1} \Rightarrow \mu = (x^2+1)^{3/2} \Rightarrow y = 2 + c(x^2+1)^{-3/2}$

Ex: $(x + ye^y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dx}{dy} = x + ye^y \Rightarrow \frac{dx}{dy} - x = ye^y$ (Lin 1st ord)

$\Rightarrow (\mu x)' = \mu x' - \mu x = \mu ye^y \Rightarrow (\mu x)' = \mu x' + \mu' x = \mu x' - \mu x$

$\mu' = -\mu \Rightarrow \frac{d\mu}{\mu} = -dy \Rightarrow \ln\mu = -y \Rightarrow \mu = e^{-y}$

$(e^{-y}x)' = y \Rightarrow e^{-y}x = \int y\,dy = \frac{1}{2}y^2 + c \Rightarrow x = \frac{1}{2}y^2e^y + ce^y$

THEOREM: If $M(x)$ and $N(x)$ are cont on the open int I containing the point x_0, then the IVP

$$\frac{dy}{dx} + M(x)y = N(x), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on I given by

$$y(x) = e^{-\int M dx}\left[\int N e^{\int M dx} dx + c\right]$$

for an appropriate value of c.

Ex: $\frac{dy}{dx} = -2xy$, $y(0) = 2$ (sep. d.e)

$\frac{dy}{y} = -2x\,dx \Rightarrow \ln|y| = -x^2 + c$

$y(0) = 2 > 0$ near $x = 0 \Rightarrow \ln y = -x^2 + c \Rightarrow y = e^{-x^2+c} = e^c e^{-x^2}$

$\Rightarrow y = Ae^{-x^2}$

$y(0) = 2 \Rightarrow 2 = Ae^0 \Rightarrow A = 2 \Rightarrow y = 2e^{-x^2}$

Suppose that $y(0) = -2$. Then $y < 0$ near $x = 0$

$\Rightarrow \ln(-y) = -x^2 + c \Rightarrow -y = e^{-x^2+c} \Rightarrow y = -Ae^{-x^2}$

$y(0) = -2 \Rightarrow -2 = -Ae^0 \Rightarrow A = 2 \Rightarrow y = -2e^{-x^2}$

Ex: $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$ (sep. d.e)

$(3y^2-5)\,dy = (4-2x)\,dx \Rightarrow y^3 - 5y = 4x - x^2 + c$ (impl. sol)

$y(1) = 3 \Rightarrow 27 - 15 = 4 - 1 + c \Rightarrow c = 9 \Rightarrow y^3 - 5y = 4x - x^2 + 9$

Ex: $2\sqrt{x}\,\frac{dy}{dx} = \cos^2 y$, $y(4) = \pi/4$ (sep. d.e)

$\frac{dy}{\cos^2 y} = \frac{dx}{2\sqrt{x}} \Rightarrow \tan y = \sqrt{x} + c$ gen. sol. (impl. sol.)

$y(4) = \pi/4 \Rightarrow \overbrace{\tan(\pi/4)}^{1} = \overbrace{\sqrt{4}}^{2} + c \Rightarrow c = -1 \Rightarrow \tan y = \sqrt{x} + c$

part. sol.

Ex: $\frac{dy}{dx} = y^2 \Rightarrow \frac{dy}{y^2} = dx \Rightarrow -\frac{1}{y} = x + c \Rightarrow y = -\frac{1}{x+c}$ gen. sol $(x \neq -c)$

$y = 0 \Rightarrow 0 = 0 \Rightarrow y = 0$: sing. s $\left(-\frac{1}{x+c} \neq 0\right.$ for any choice of $\left.c\right)$

Ex: $xy\dfrac{dy}{dx} = \dfrac{3}{2}y^2 + x^2$ (hom. eq)

$$\frac{dy}{dx} = \frac{3}{2}\frac{y}{x} + \frac{x}{y} \quad , \quad \frac{y}{x} = v \Rightarrow y = xv$$

$$\Rightarrow v + xv' = \frac{3}{2}v + \frac{1}{v} \Rightarrow xv' = \frac{1}{2}v + \frac{1}{v} = \frac{v^2+2}{2v}$$

$$\frac{2v\,dv}{v^2+2} = \frac{dx}{x} \Rightarrow \ln(v^2+2) = \ln|x| + \ln c \Rightarrow v^2+2 = c|x|$$

$$v = \frac{y}{x} \Rightarrow \frac{y^2}{x^2} + 2 = c|x| \Rightarrow y^2 + 2x^2 = cx^3 \quad (x<0 \Rightarrow c<0,\ x>0 \Rightarrow c>0)$$

Ex: $x\dfrac{dy}{dx} = y + \sqrt{x^2-y^2}$, $y(x_0) = 0$, $x_0 > 0$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1-(y/x)^2} \text{ hom. eq} \Rightarrow \frac{y}{x} = v,\ y = xv$$

$$v + xv' = v + \sqrt{1-v^2} \Rightarrow xv' = \sqrt{1-v^2} \Rightarrow \frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$$

$$\sin^{-1} v = \ln x + c \Rightarrow \sin^{-1}\frac{y}{x} = \ln x + c$$

$$y(x_0) = 0 \Rightarrow 0 = \ln x_0 + c \Rightarrow c = -\ln x_0 \Rightarrow \sin^{-1}\frac{y}{x} = \ln x - \ln x_0$$

$$\Rightarrow y = x\sin\left(\ln\frac{x}{x_0}\right)$$

Ex: $\dfrac{dy}{dx} = \dfrac{x-y-1}{x+y+3}$ Solve the d.e by finding h and k so that the substitutions $x = u+h$, $y = v+k$ transform it into the hom. eq $\dfrac{dv}{du} = \dfrac{u-v}{u+v}$.

$$\frac{dv}{du} = \frac{u+h-v-k-1}{u+h+v+k+3} = \frac{u-v+h-k-1}{u+v+h+k+3} \quad , \quad \left.\begin{matrix} h-k-1=0 \\ h+k+3=0 \end{matrix}\right\} \begin{matrix} h=-1 \rightarrow x = u-1 \\ k=-2 \rightarrow y = v-2 \end{matrix}$$

$$\frac{dv}{du} = \frac{1-(v/u)}{1+(v/u)} \quad , \quad \frac{v}{u} = z \Rightarrow v = uz$$

$$z + uz' = \frac{1-z}{1+z} \Rightarrow uz' = \frac{1-z}{1+z} - z = \frac{1-2z-z^2}{1+z} \Rightarrow \frac{1+z}{1-2z-z^2}dz = \frac{du}{u}$$

$$-\frac{1}{2}\ln|1-2z-z^2| = \ln|u| + \ln c \Rightarrow |1-2z-z^2| = \frac{1}{u^2c^2} \Rightarrow \left|1 - 2\frac{y+2}{x+1} - \frac{(y+2)^2}{(x+1)^2}\right| = \frac{1}{c^2(x+1)^2}$$

Ex: $2xyy' = 4x^2 + 3y^2$

$\Rightarrow \frac{dy}{dx} = 2\frac{x}{y} + 3\frac{y}{x} \Rightarrow \frac{dy}{dx} - 3\frac{1}{x}\cdot y = 2xy^{-1}$ Bernoulli eq.

$n=-1 \Rightarrow 1-n=2$

$(1-n)y^{-n} = 2y \Rightarrow 2y\frac{dy}{dx} - \frac{6}{x}y^2 = 4x$

$v = y^{1-n} = y^2 \Rightarrow v' - \frac{6}{x}v = 4x$ (Linear 1st order)

$\mu v' - \frac{6}{x}\mu v = 4\mu x \Rightarrow (\mu v)' = \mu v' + \mu' v = \mu v' - \frac{6}{x}\mu v$

$\mu' = -\frac{6}{x}\mu \Rightarrow \frac{\mu'}{\mu} = -\frac{6}{x} \Rightarrow \ln\mu = -6\ln x \Rightarrow \mu = x^{-6}$

$x^{-6}v' - 6x^{-7}v = 4x^{-5} \Rightarrow (x^{-6}\cdot v)' = 4x^{-5} \Rightarrow x^{-6}v = -x^{-4} + c$

$v = -x^2 + cx^6 \Rightarrow y^2 = -x^2 + cx^6$

Ex: $x\frac{dy}{dx} + 6y = 3xy^{4/3} \Rightarrow \frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}$, Bern. eq

$n = 4/3 \Rightarrow 1-n = -1/3$

$(1-n)y^{-n} = -\frac{1}{3}y^{-4/3} \Rightarrow -\frac{1}{3}y^{-4/3}\frac{dy}{dx} - \frac{2}{x}y^{-1/3} = -1$

$v = y^{1-n} = y^{-1/3} \Rightarrow v' - \frac{2}{x}v = -1$ (1st ord. Lin)

$\mu v' - \frac{2}{x}\mu v = -\mu \Rightarrow (\mu v)' = \mu v' + \mu' v = D_x(\mu\cdot v)$

$\mu' = -\frac{2}{x}\mu \Rightarrow \frac{d\mu}{\mu} = -\frac{2}{x}dx \Rightarrow \ln\mu = \ln x^{-2} \Rightarrow \mu = x^{-2}$

$D_x(x^{-2}v) = -x^{-2} \Rightarrow x^{-2}v = x^{-1} + c \Rightarrow v = x + cx^2$

$\Rightarrow y^{-1/3} = x + cx^2 \Rightarrow y = (x+cx^2)^{-3}$

Ex: $y^3 dx + 3xy^2 dy = 0$ $(*)$

$M = y^3$, $N = 3xy^2 \Rightarrow M_y = 3y^2 = N_x$ (exact d.e)

$M = f_x = y^3$, $N = f_y = 3xy^2$.

$\Rightarrow f(x,y) = xy^3 + h(y) \Rightarrow 3xy^2 + h'(y) = 3xy^2 \Rightarrow h'(y) = 0 \Rightarrow h(y) = c$

$\Rightarrow f = xy^3 + c$

$\Rightarrow xy^3 = c \Rightarrow y = k\,x^{-1/3}$

Pay attention : $y\,dx + 3x\,dy = 0$ not exact (divide $(*)$ by y^2)

Ex: $(\underbrace{6xy - y^3}_{=M})\,dx + (\underbrace{4y + 3x^2 - 3xy^2}_{=N})\,dy = 0$

$M_y = 6x - 3y^2$, $N_x = 6x - 3y^2 \Rightarrow M_y = N_x$ (exact d.e)

$M = f_x = 6xy - y^3$, $N = f_y = 4y + 3x^2 - 3xy^2$

$\Rightarrow f(x,y) = 3x^2y - y^3x + h(y)$

$\Rightarrow 4y + \cancel{3x^2} - \cancel{3xy^2} = \cancel{3x^2} - \cancel{3y^2x} + h'(y) \Rightarrow h(y) = 2y^2 + c$

$\Rightarrow f(x,y) = 3x^2y - y^3x + 2y^2 = c$

REDUCIBLE SECOND ORDER EQ.

$$F(x, y, y', y'') = 0$$

Dependent variable y missing

$$F(x, y', y'') = 0$$

$$p = y' = \frac{dy}{dx} \Rightarrow y'' = \frac{dp}{dx}$$

$$\Rightarrow F(x, p, p') = 0$$

1st order diff. eq.

$(p = p(x))$

Independent variable x missing

$$F(y, y', y'') = 0$$

$$p = y' = \frac{dy}{dx} \Rightarrow y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

$$\Rightarrow F\left(y, p, p\frac{dp}{dy}\right) = 0$$

1st order diff eq.

$(p = p(y))$

Ex: $xy'' + 2y' = 6x \Rightarrow y$ is missing

$p = y' = \dfrac{dy}{dx} \Rightarrow y'' = \dfrac{dp}{dx}$, $p(x)$

$x\dfrac{dp}{dx} + 2p = 6x$ (Linear 1st order)

$\Rightarrow \dfrac{dp}{dx} + \dfrac{2}{x}p = 6 \Rightarrow \underbrace{\mu p' + \mu \dfrac{2}{x} p}_{(\mu . p)' = \mu p' + \mu' p} = 6\mu$

$\mu' = \mu \dfrac{2}{x} \Rightarrow \dfrac{d\mu}{\mu} = \dfrac{2}{x}dx \Rightarrow \ln\mu = 2\ln x = \ln x^2 \Rightarrow \mu = x^2$

$x^2p' + 2xp = 6x^2 \Rightarrow (x^2p)' = 6x^2 \Rightarrow \int d(x^2p) = \int 6x^2 dx$

$\Rightarrow x^2p = 2x^3 + c_1 \Rightarrow x^2\dfrac{dy}{dx} = 2x^3 + c_1 \Rightarrow \int dy = \int \left(2x + \dfrac{c_1}{x^2}\right) dx$

$\Rightarrow y = x^2 - \dfrac{c_1}{x} + c_2$

Ex: $yy'' = (y')^2 \Rightarrow x$ is missing

$p = y' = \dfrac{dy}{dx} \Rightarrow y'' = \dfrac{dp}{dx} = \dfrac{dp}{dy}\dfrac{dy}{dx} = p\dfrac{dp}{dy}$, $p(y)$

$yp\dfrac{dp}{dy} = p^2 \Rightarrow y\dfrac{dp}{dy} = p \Rightarrow \dfrac{dp}{p} = \dfrac{dy}{y}$ (sep. d.e)

$\ln p = \ln y + \ln c_1 \Rightarrow p = c_1 . y$ $(y>0, p>0)$

$\dfrac{dy}{dx} = c_1 y \Rightarrow \dfrac{dy}{y} = c_1 . dx \Rightarrow \ln y = c_1 x + c_2$

$\Rightarrow y = e^{c_1 x + c_2} = e^{c_1 x} e^{c_2} \Rightarrow y = ke^{cx}$

* Note: Even if $k<0$ $(y<0)$ the d.e. is still satisfied.