EHB 211E
Basics of Electrical Circuits

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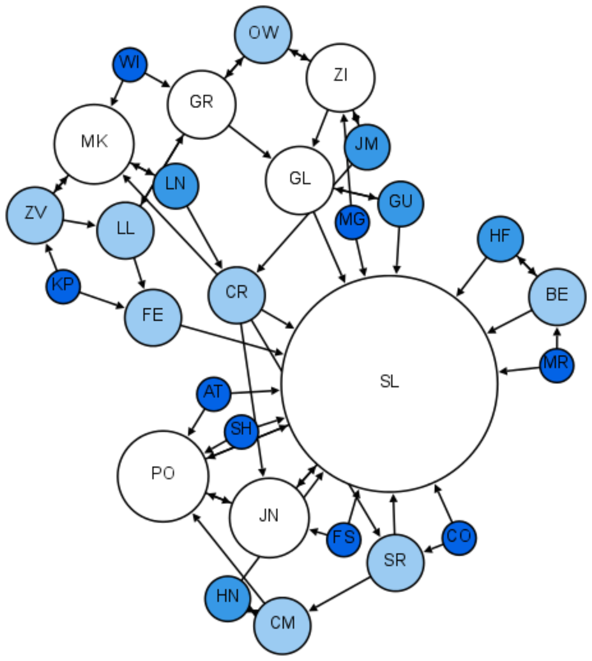
Graph Theory

Linear and Nonlinear Circuit by L.O. Chua, C.A. Desoer, and E.S. Kuh

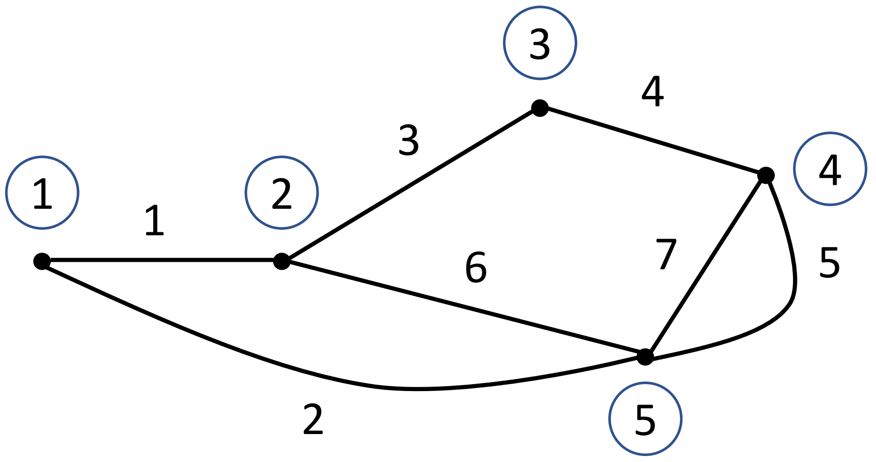
Introduction

- Graph theory (aka network topology): study and model large number of applications in different fields.
  - Computer science: flow of computation, data organization
  - Electrical Engineering: design of circuit connection
  - Physics and chemistry: study molecules (constructing lattice of molecule)
  - Mathematics: minimum cost part, scheduling problem
  - Biology: interaction between species (protein-protein interaction)
  - Linguistics: grammar of language tree
  - Represent routes between cities



Basic Terminology

- What is a graph?
  - Set of nodes connected by branches
- Node (aka vertex): common terminal or point of two or more branches. Node is denoted by n
- Branch (aka edge): line segment that replaces the network element and connects two nodes. Branch is denoted by b.
- Simple example of graph:



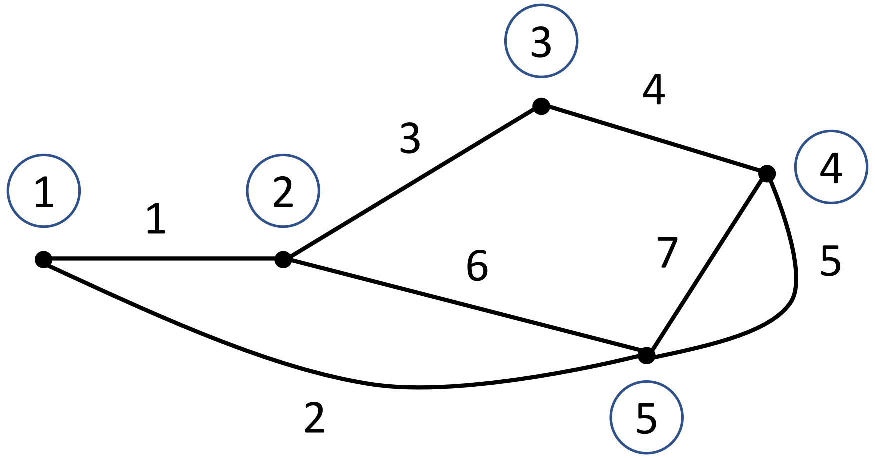
5 nodes and 7 branches

n=5 & b=7

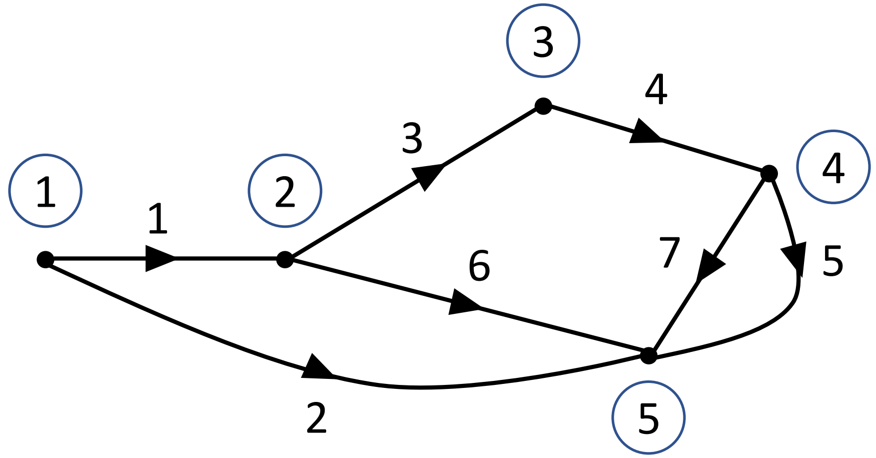
Types of Graphs

- Two types of graphs
  - Undirected graph: direction not specified
  - Directed graph (digraph): all branches are represented with arrows.

Undirected graph



Directed graph

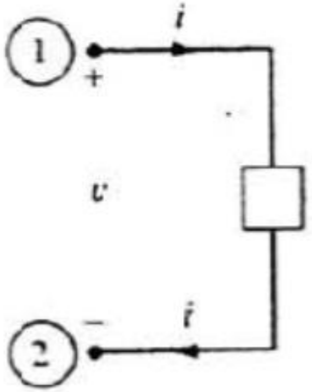


Arrows indicate the direction of current flow in each branch

From Circuits to Graphs

- Graph retains all interconnection properties of the circuit but suppresses the information on the circuit elements.

Two terminal element



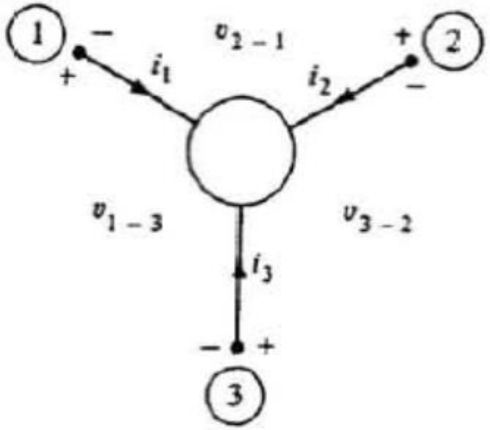
digraph

Digraph representation of two terminal element with two nodes and one branch

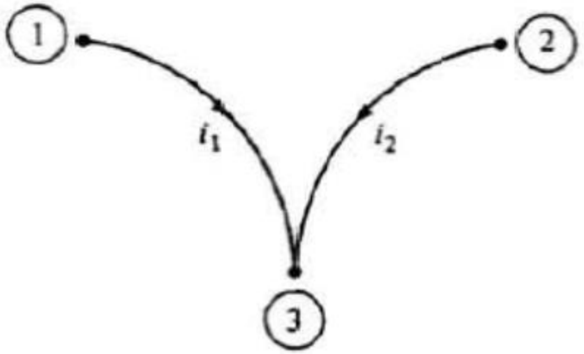
Voltage in branch: branch voltage
Current in branch: branch current

Power delivered to element: $p(t) = v(t)i(t)$

Three terminal element



digraph



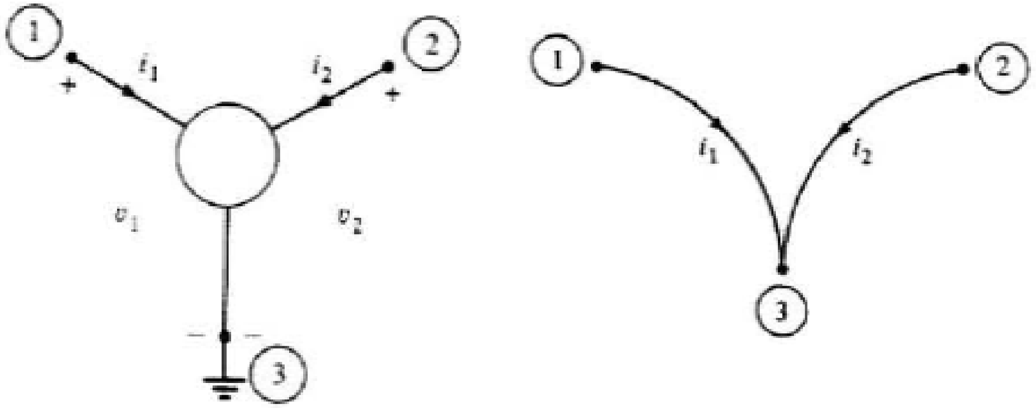
Digraph representation of a three terminal element with node 3 chosen as datum (reference node). Three nodes and two branches.

Apply KVL: Two independent voltages
Apply KCL: Two independent currents

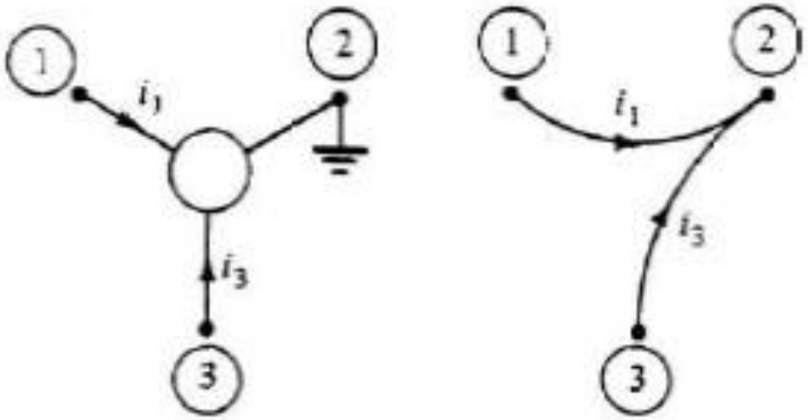
From Circuits to Graphs

- All possibilities for digraph representation of three terminal elements depending on which node chosen as the datum (reference) node.

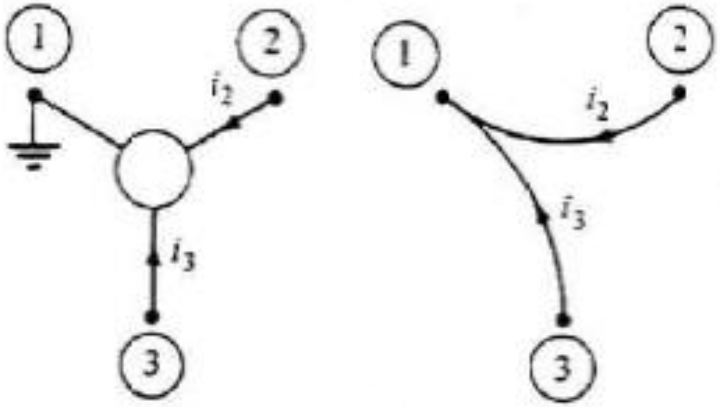
Node 3 (datum node): $v_1 > v_3$ & $v_2 > v_3$



Node 2 (datum node): $v_1 > v_2$ & $v_3 > v_2$



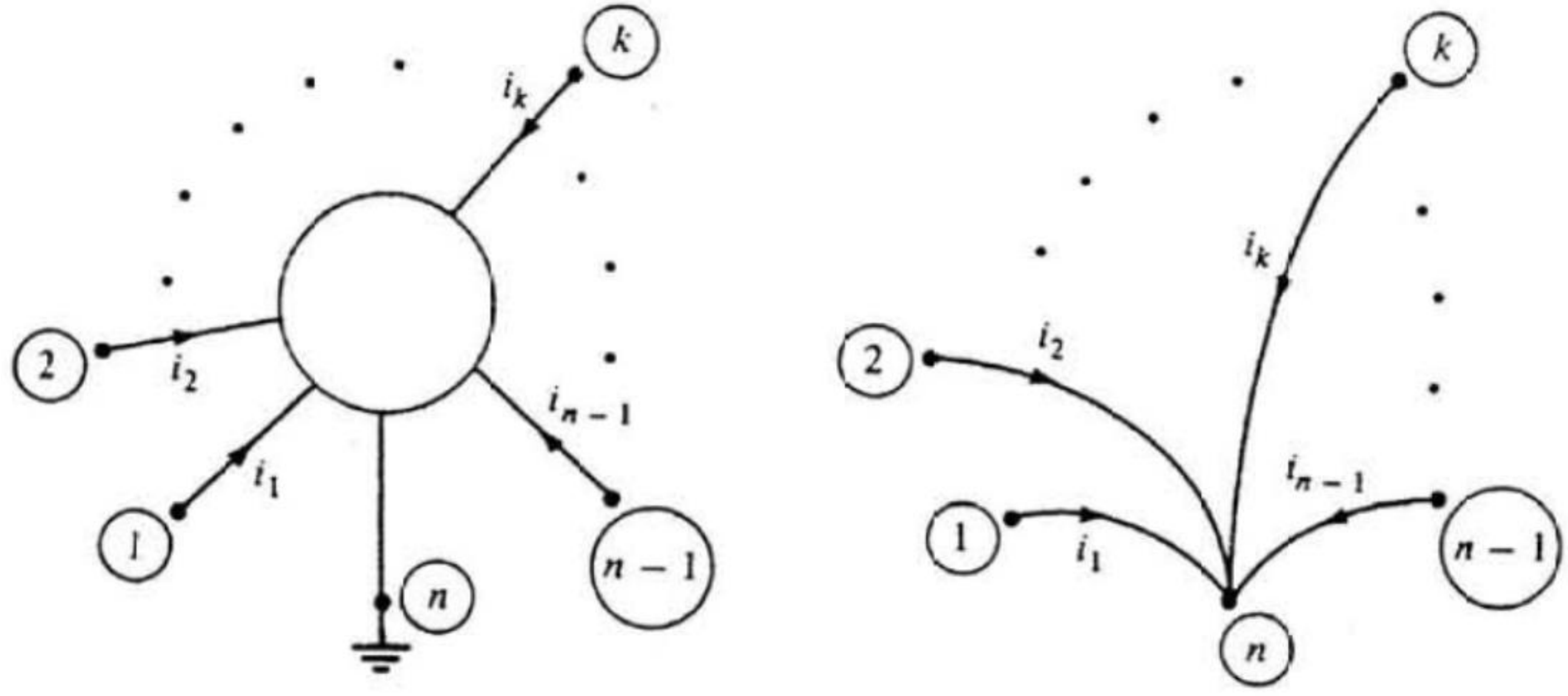
Node 1 (datum node): $v_2 > v_1$ & $v_3 > v_1$



From Circuits to Graphs

- General form of n-terminal elements with digraph representation:

Node n (datum node): n nodes and n-1 branches

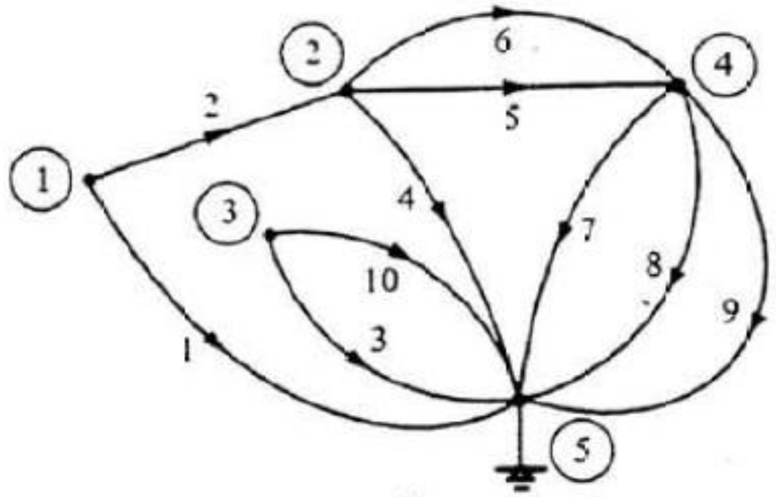


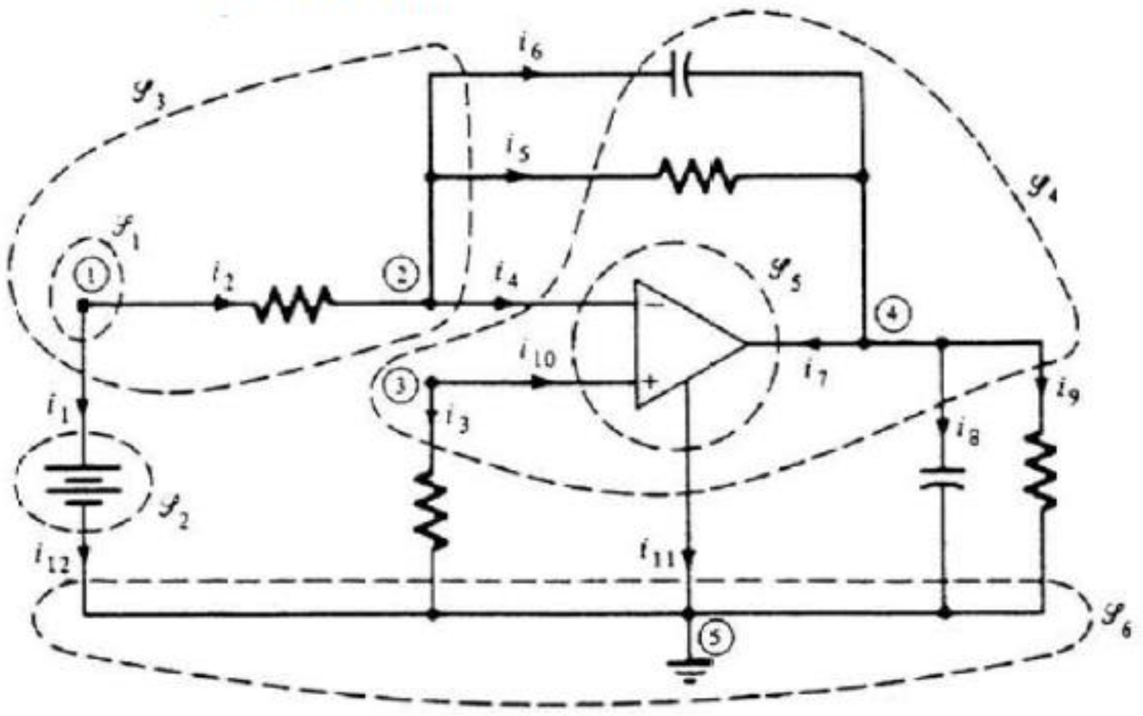
Example

Demonstrate the op amp circuit shown below with the corresponding digraph

Solution:

- Seven two-terminal elements and one four-terminal element
- Recall: n-terminal elements: n-1 branches
- Total number of branches in digraph: 7+4-1=10.
- Circuit has five nodes. Thus, digraph has five nodes with node 5 is datum node.

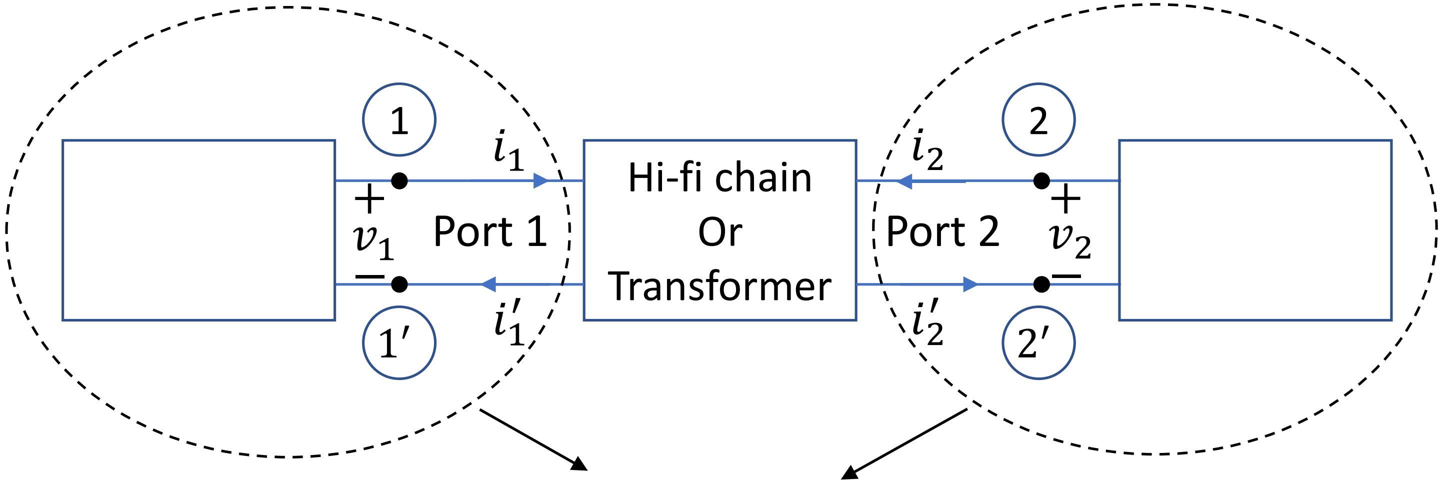




Suppressed circuit elements for drawing the digraph

Two-Ports

- Two-port: circuit element or a circuit with pairs of terminals.
- Example of two-port circuit: Hi-fi chain, two-winding transformer

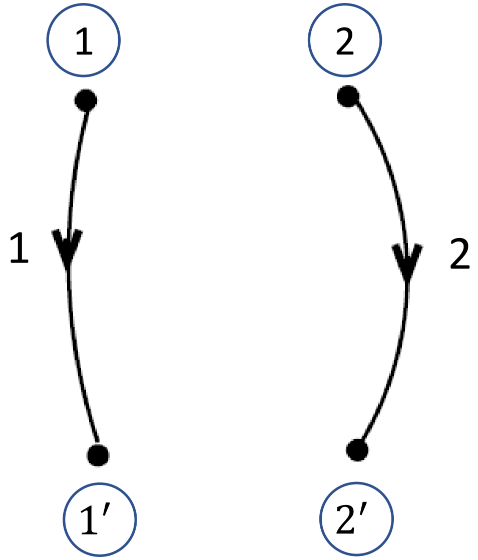


Apply KCL: $i_1 = i_1'$ and $i_2 = i_2'$
(port condition)

Two-port network reduces
number of currents from 4 to 2

Power delivered to two-port:
$p(t) = v_1(t)i_1(t) + v_2(t)i_2(t)$

Digraph representation
of two-port circuit

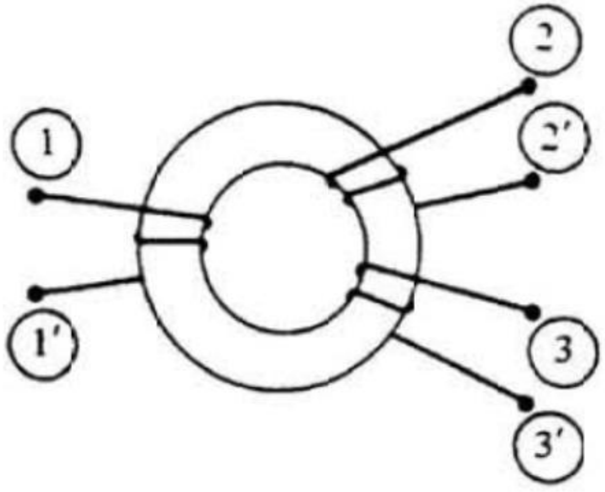


Two-port: 4 nodes and 2 branches

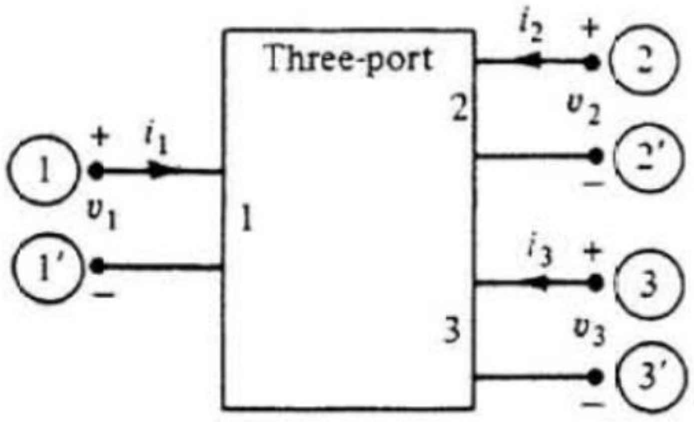
i_1 and i_2: branch currents
v_1 and v_2: branch voltages

Multiport

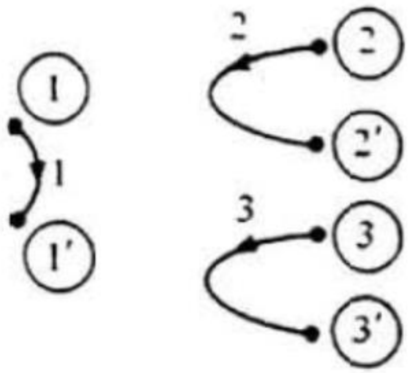
- Generalize the concept of two-port to multiport
- Three-winding transformer: three-port network or circuit.



Three-winding transformer



Three-port



digraph

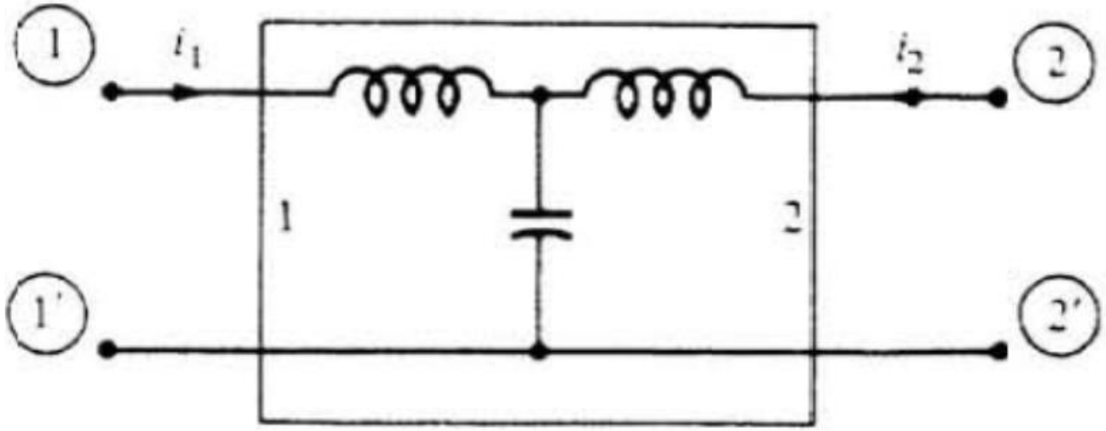
Three-winding transformer: three-port
its digraph: 6 nodes and 3 branches

Recall:
Digraph (directed graph) is a graph that is made up of a set of vertices (nodes) connected by edges (branches)

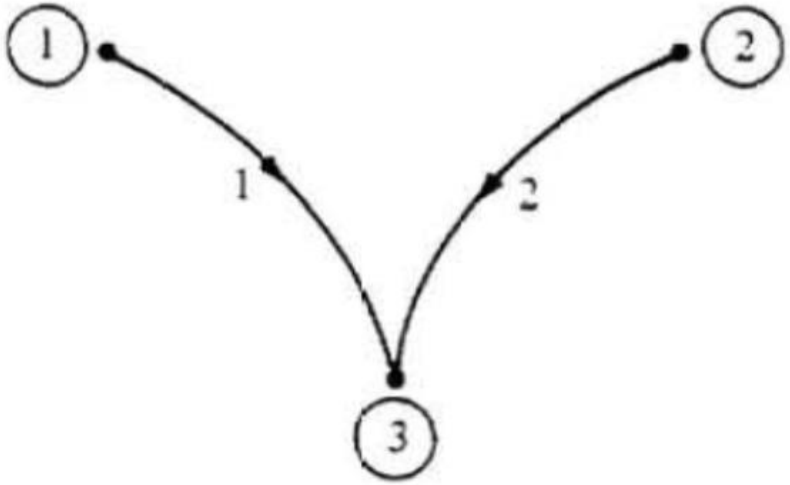
Grounded Two-Port

- If a common connection exists between node $1'$ and $2'$, it is called grounded two-port. This is equivalent to three terminal element

Grounded two-port



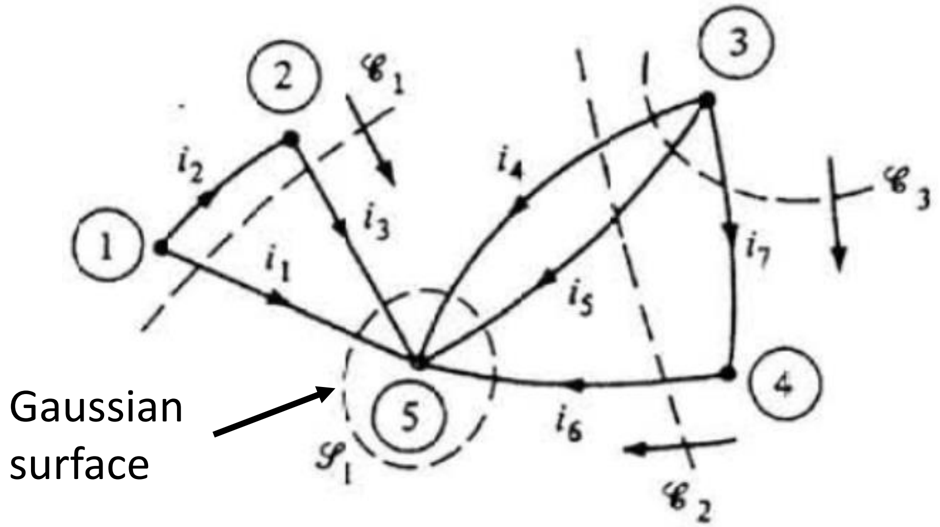
digraph



- Digraph of grounded two-port: 3 nodes and 2 branches which are tied together at the common node

Cut Sets

- Cut set is very useful concept in graph theory
- Given a connected digraph G, a set of branches b of G is called a cut set if and only if:
  - The removal of all the branches of the cut set results in a unconnected digraph (resulted digraph is no longer connected.
  - The removal of all but any one branch of G leaves the digraph connected. In other words, if any branch in the set is left intact, the digraph remains connected.



b_1, b_2, b_3 represent the cut set line.

Arrows for b_1, b_2, b_3 define a reference direction for each cut set. A reference direction is chosen arbitrary.

Cut set is a way to isolate node or nodes from the circuit

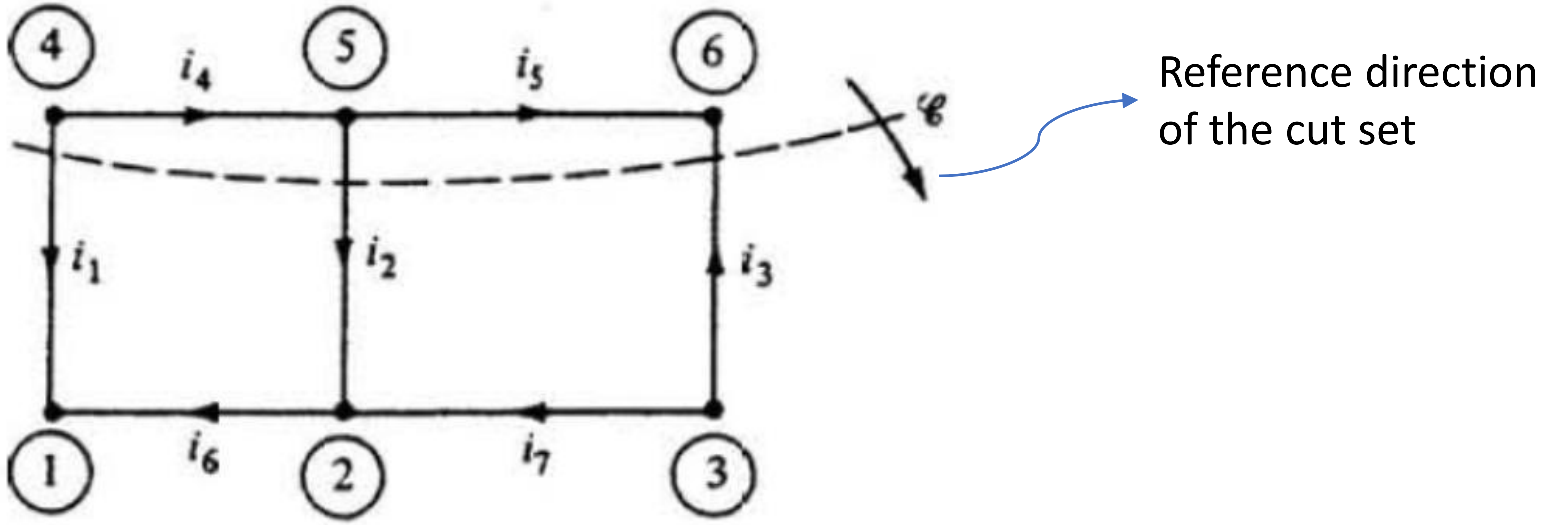
$b_1 = \{1,3\}$ ⟶ b_1 intersects with branch {1,3}

$b_2 = \{4,5,6\}$ ⟶ b_2 intersects with branch {4,5,6}

$b_3 = \{4,5,7\}$ ⟶ b_3 intersects with branch {4,5,7}

Cut Sets and KCL

- KCL: algebraic sum of the currents associated with any cut set is equal to zero.

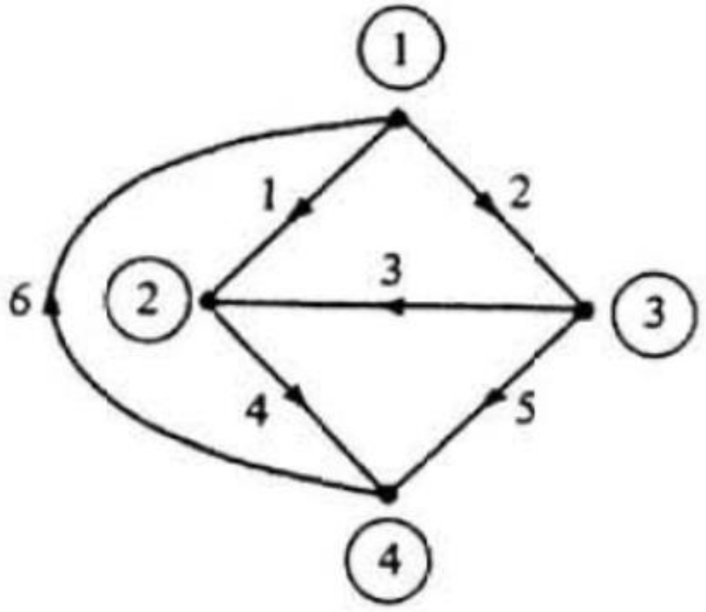


- For the digraph shown in the figure, the cut set $\mathrm{b} = \{1,2,3\}$ is indicated by the dashed line cutting through these branches. (it intersects with three branches {1,2,3}).
- Direction of current is positive if it is in the same direction as the reference direction of the cut set.

$$i_1 + i_2 - i_3 = 0$$

Matrix Formation—KCL

- Formation of incidence matrix of the digraph.
- The incidence matrix is denoted by A_a
- If digraph G has n nodes and b branches, then A_a has n rows (one row to each node) and b columns (one column to each branch).



- Digraph: 4 nodes and 6 branches
- By definition: A_a has 4 rows and 6 columns

- Next step is to apply KCL
- Before writing KCL equations for each node, assign a reference direction for currents
- The direction of current is positive (+) if branch leaves the node. Otherwise it is negative

Matrix Formation—KCL

- Write the KCL equation for each node:

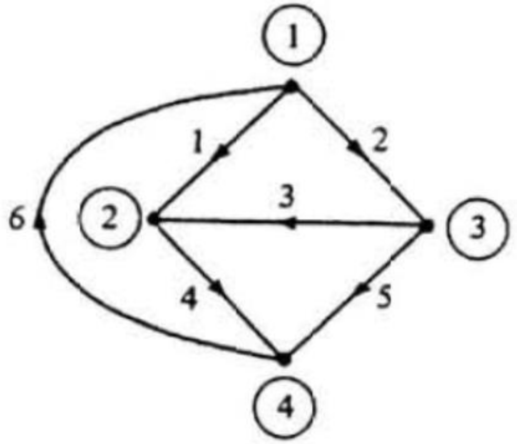
Branches: 1 2 3 4 5 6

$$
\begin{array}{lrrrrrrl}
\text{Node 1:} & i_1 & +i_2 & & & & -i_6 & = 0 \\
\text{Node 2:} & -i_1 & & -i_3 & +i_4 & & & = 0 \\
\text{Node 3:} & & -i_2 & +i_3 & & +i_5 & & = 0 \\
\text{Node 4:} & & & & -i_4 & -i_5 & +i_6 & = 0
\end{array}
$$

- KCL equations in matrix form as:

$$
\begin{array}{c}
① \rightarrow \\ ② \rightarrow \\ ③ \rightarrow \\ ④ \rightarrow
\end{array}
\underbrace{\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 1
\end{bmatrix}}_{\text{branch 1} \quad \cdots \quad \text{branch 6}}
\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

4×6 matrix called incidence matrix A_a



- Node equation of digraph can be written as:

$$A_a i = 0$$

A_a: incidence matrix

i: branch current vector (magnitude and direction)

- Rank: number of linear independent equations in a digraph, denoted by r.

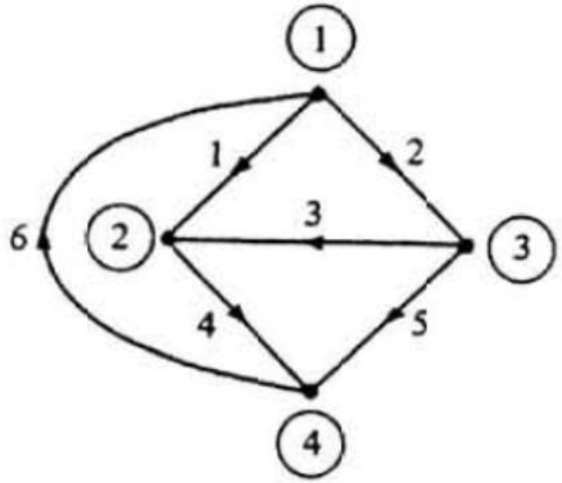
$$r = n - 1$$

n is number of nodes

- For any connected digraph G with n nodes, if we choose a datum node and throw away the corresponding KCL equation, the remaining n-1 KCL equations are linearly independent

Matrix Formation—KCL

- Assume node 4 is a datum (reference) node
- Delete the last equation (row corresponding datum node).



$$\begin{array}{c} 1 \rightarrow \\ 2 \rightarrow \\ 3 \rightarrow \end{array} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{bmatrix}}_{\text{Reduced incidence matrix } A} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow Ai = 0$$

- A (reduced incidence matrix) is obtained from A_a (incidence matrix) by deleting the row which corresponds to the chosen datum node
- Addition of all the KCL equation in above matrix will not cancel out each other. Three KCL equations are linearly independent.
- For connected digraph with n nodes, dimension of reduced incidence matrix is $(n-1) \times b$

Matrix Formation—KCL

- Write KCL equations in a matrix form using cut sets
- First, write all distinct cut sets for the given digraph

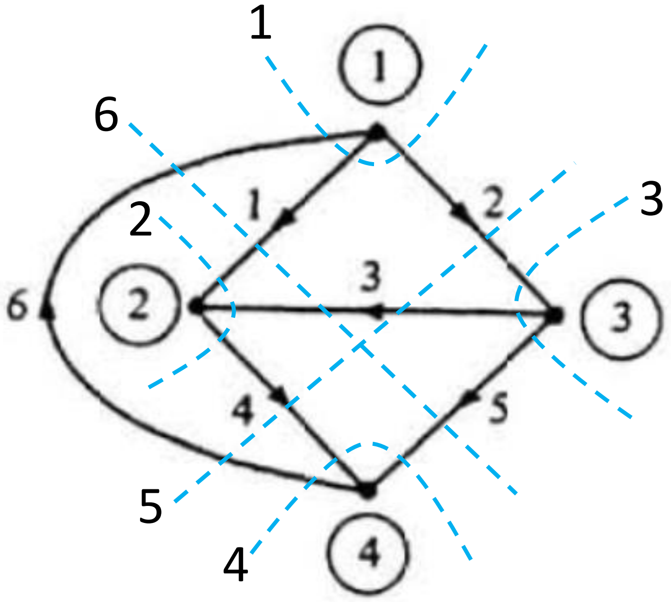
$b_1 = \{1,2,6\}$ $\quad$ $b_4 = \{4,5,6\}$

$b_2 = \{1,3,4\}$ $\quad$ $b_5 = \{2,3,4,6\}$

$b_3 = \{2,3,5\}$ $\quad$ $b_6 = \{1,3,5,6\}$

- Based on the cut sets, the matrix is as follows:

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}}_{\text{Cut-set matrix, } Q_a} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow Q_a i = 0$$



Recall:
Cut set is a partition of the nodes (vertices) of a graph into two disjoint subsets

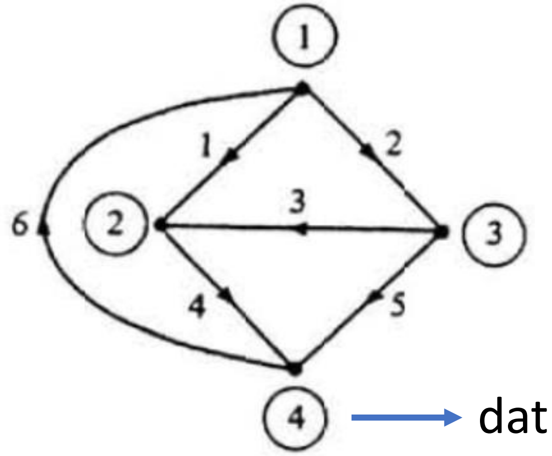
Recall:
Direction of the current is arbitrary. Branch leaving the node is positive (+) and branch entering the node is negative (-)

Rank of cut set matrix: $r = n - 1$

$Q_R i = 0$ (Q_R is called reduced cut-set matrix obtained by deleting redundant equations)

Matrix Formation—KVL

- KVL: another way of obtaining matrix form of the digraph
- Using the associated reference direction, the branch voltages:



datum

$$\left.\begin{aligned}
v_1 &= e_1 - e_2 \\
v_2 &= e_1 \qquad\; - e_3 \\
v_3 &= \quad -e_2 + e_3 \\
v_4 &= \quad\; e_2 \qquad\; - e_4 \\
v_4 &= \quad\; e_2 \\
v_5 &= \qquad\quad\;\; e_3 \\
v_6 &= -e_1
\end{aligned}\right\} \Rightarrow
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{M} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \Rightarrow v = Me$$

$v = Me$

$v = [v_1, v_2, v_3, \dots, v_n]^T$ is the branch voltage vector
$e = [e_1, e_2, e_3, \dots, e_n]^T$ is the node-to-datum voltage vector
$M: b \times (n-1)$ matrix

- Comparing KCL & KVL equations: $M = A^T$ ⟶ M matrix equals to transformation of reduced incidence matrix A

$v = A^T e$

Graph Theory—Subgraph

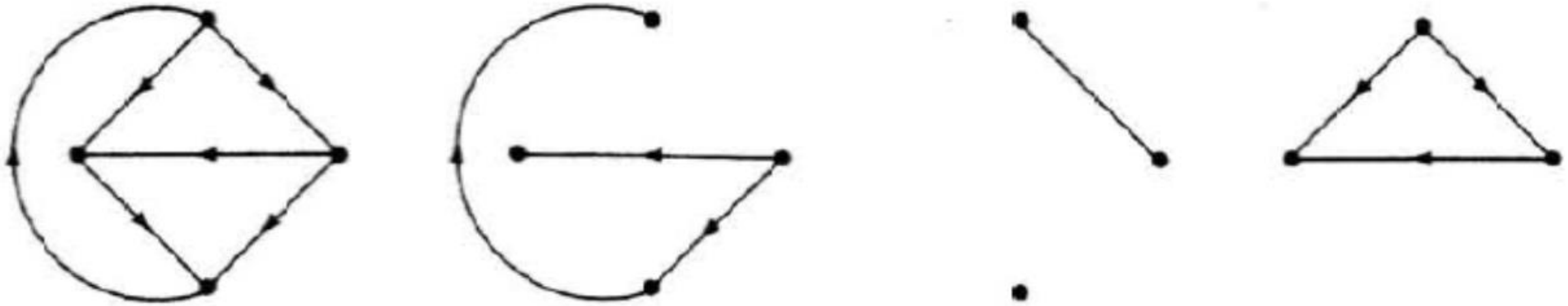
- digraph G is define as a set of vertices (nodes) connected by edges (branches) where the direction of all edges (branches) is specified.

  $G = (v, b)$

- Let G_1 is a subgraph of G if and only if G_1 itself is a graph.

  $G_1 = (v_1, b_1)$ where v_1 is subset of v and b_1 is subset of b

- Example of subgraphs of digraph G:

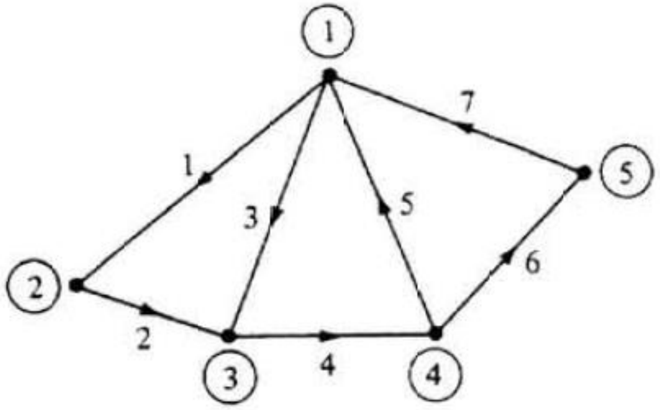


$\mathcal{G}$ $\mathcal{G}_1$ $\mathcal{G}_2$ $\mathcal{G}_3$

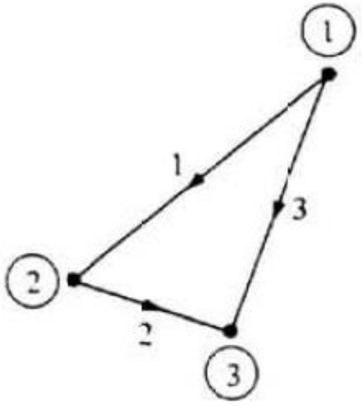
- Graph is connected if there is a path from any point to any other point in the graph.
- Graph G is connected but subgraph does not have to be connected.

Graph Theory—Loop

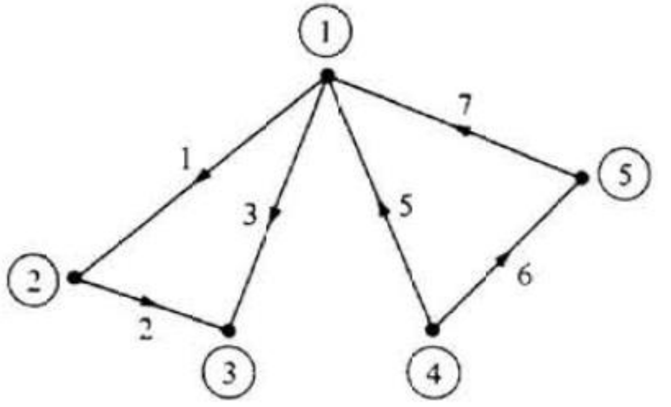
- Loop: connected subgraph of digraph G in which precisely two branches are incident with each other.
- For the loop concept, consider the following digraph G:



Digraph (G)



G_1

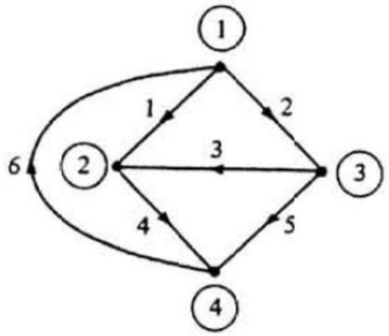


G_2

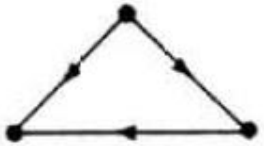
- G: digraph
- G_1: loop
- G_2: not a loop as it violates the definition of loop at node 1.

Graph Theory—KVL Based on Loops

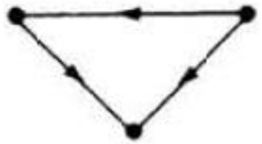
- KVL equations based on loops
- First, identify all possible distinct loops from the digraph
- Following are all distinct loops:



Digraph (G)



L_1

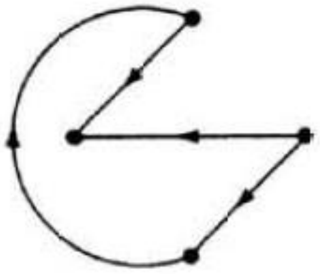


L_2

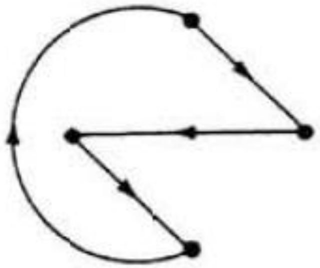
L_3

L_4

L_5



L_6



L_7

- Every node has exactly two branches
- 7 distinct loops of digraph G

Graph Theory—KVL Based on Loops

- Apply KVL to each loop:

$L_1: v_1 - v_2 - v_3 = 0$

$L_2: v_3 + v_4 - v_5 = 0$

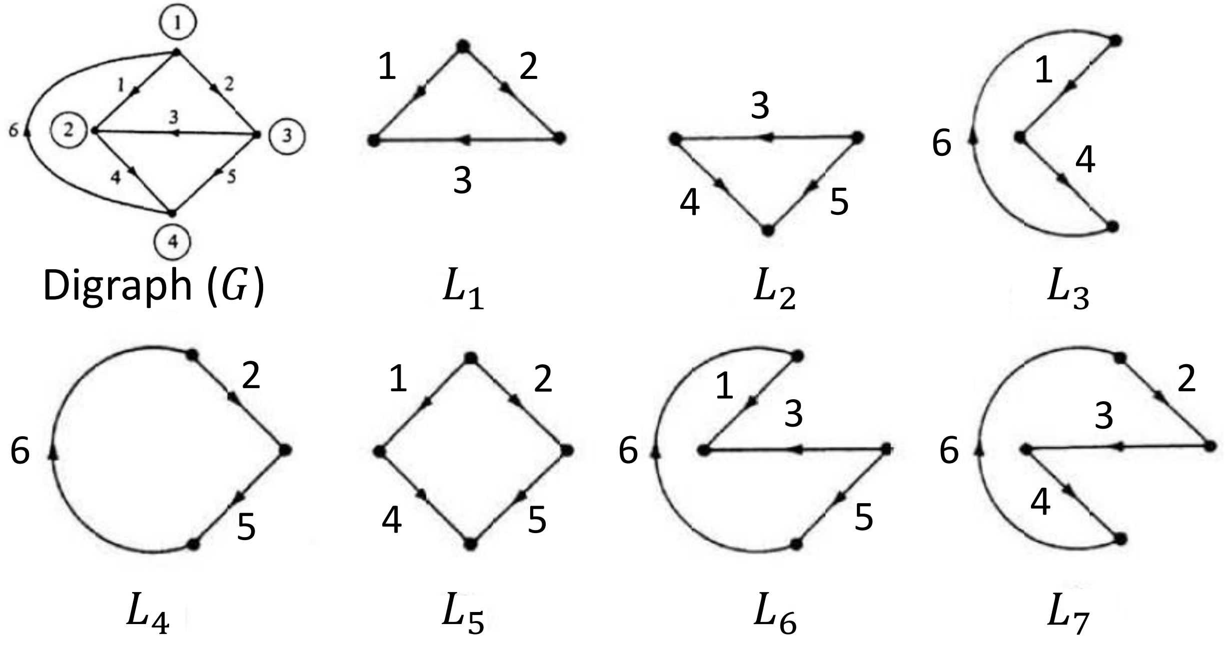
$L_3: v_1 + v_4 + v_6 = 0$

$L_4: v_2 + v_5 + v_6 = 0$

$L_5: v_1 - v_2 + v_4 - v_5 = 0$

$L_6: v_1 - v_3 + v_5 + v_6 = 0$

$L_7: v_2 + v_3 + v_4 + v_6 = 0$



$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}}_{\text{Loop matrix, } B_a} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow B_a v = 0$$

Set of equations in the loop matrix is linearly dependent.

Linear dependency: set of equations are linearly dependent if a linear combination of them is zero.

Graph Theory—KVL Based on Loops

- Minimum set of loops that contains all the information related to KVL equations:

$$l = b - (n - 1)$$

l: minimum number of loops for linearly independent KVL equations
b: number of branches in digraph
n: number of nodes in digraph

- For the digraph, minimum number of linearly independent equations:

$l = 6 - (4 - 1) = 3$ → 3 linearly independent equations contained all the information on the digraph → Choose only 3 loops (or KVL equations)

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}}_{\text{Reduced Loop matrix, } B_R} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

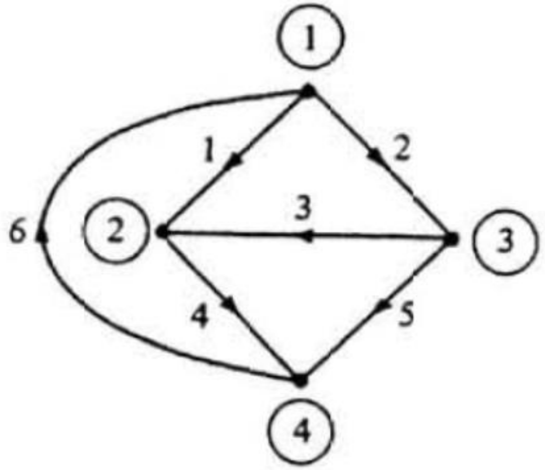
⇒ $L_1, L_2,$ and L_3 are chosen since they contain all the information. Ignore rest of the loops.

$$l = b - (n - 1)$$

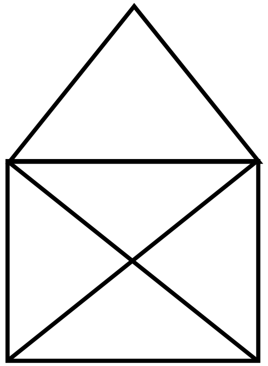
Reduced loop matrix B_R: $l = b - (n - 1)$ row and b columns

Graph Theory—Planar Graph

- What is a planar graph?
  - A graph which can be drawn on a plane in such a way that no two branches intersect at a point which is not a node.
- Example of planar and non-planar graph:



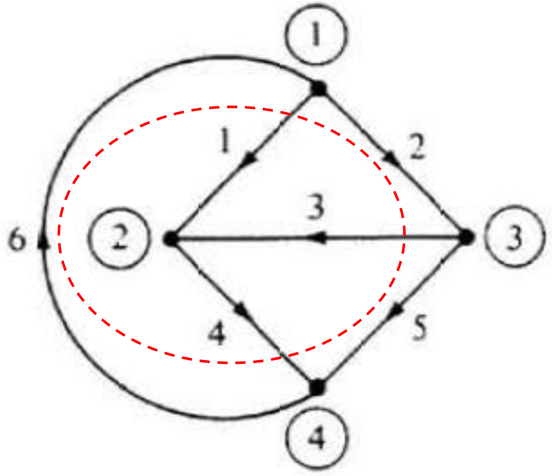
Planar graph (no branches intersect)



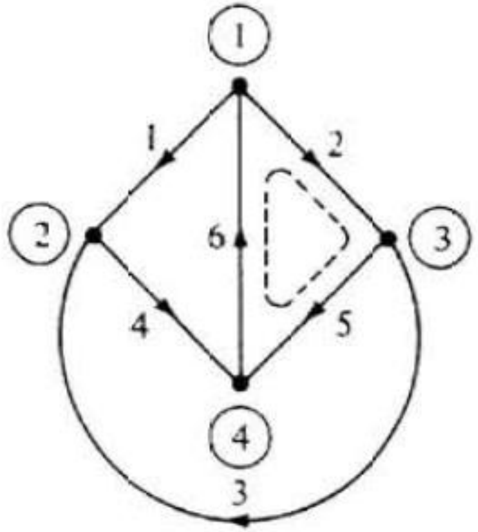
Non-planar graph

Graph Theory—Mesh

- Mesh: a loop consisting of branches encircles nothing in its interior
- Outer mesh: a loop consisting of branches has nothing in its exterior



Graph G

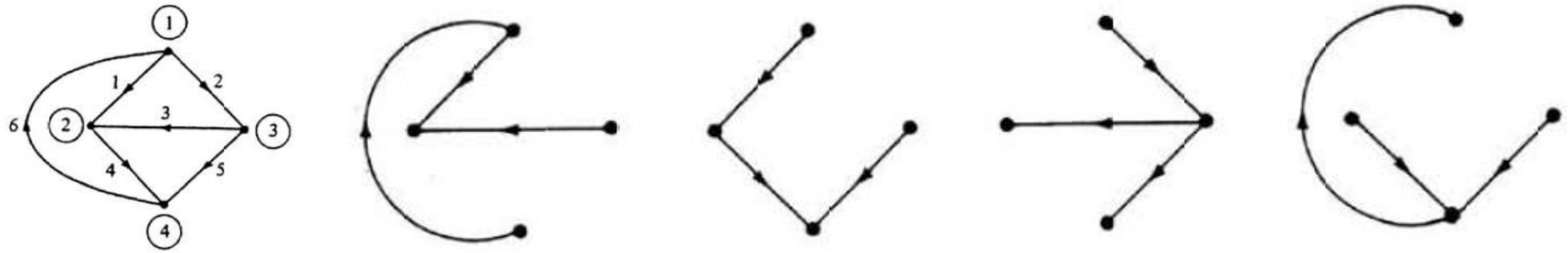


Graph G'

- Graph G': mesh (consisting of branches [2,5,6])
- Graph G: outer mesh (consisting of branches [2,5,6])

Graph Theory—Tree

- Tree: one of the most important concept in graph theory
- Tree T of a connected digraph G is a subgraph which satisfy the following fundamental properties:
  - ❑ It is connected (there is a path from any point to any other point)
  - ❑ It contains all the nodes of the digraph
  - ❑ It has no loops
- Consider the following situation:



Digraph G $\qquad T_1 \qquad T_2 \qquad T_3 \qquad T_4$

- T_1, T_2, T_3, and T_4 are four distinct trees of the digraph G

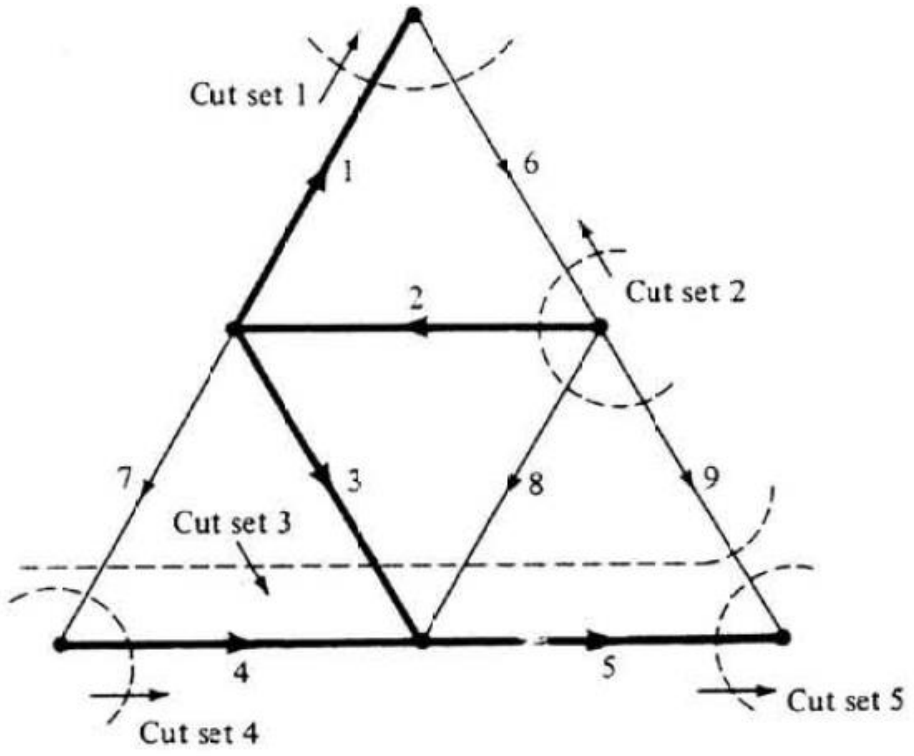
$$\left.\begin{array}{ll} T_1 = \{1,3,6\} & T_3 = \{2,3,5\} \\ T_2 = \{1,4,5\} & T_4 = \{4,5,6\} \end{array}\right\} \text{Each tree consists of three branches}$$

Fundamental Theorem of Graph

- Basic terminology:
  - twig: branches belong to the tree. Also known as tree branches
  - Link: branches that do not belong to the tree. Also known as chord or chord branches
  - Example: $T_1 = \{1,3,6\}$, T_1 has 3 twigs (1,3,6) and 3 links (2,4,5)
- Given a connected digraph G with n nodes, b branches and a tree T;
  - There is a unique path (disregard the branch orientation) along the tree between any pairs of nodes (tree is connected).
  - There are $n-1$ twigs and $l = b-(n-1)$ links.
  - Every twig of T along with some links defines a unique cut set, called the fundamental cut set associated with the twig.
  - Every link of T and the unique path on the tree between its two nodes constitute a unique loop, called the fundamental loop associated with the link.
- Statements mentioned above simply means how to obtain fundamental cut set associated with the twig and fundamental loop associated with link from a given digraph.

Fundamental Cut Set Associated with a Tree-KCL

- Consider the following digraph with $b = 9$ and $n = 6$
- First, pick (draw) a tree: solid dark line is a tree
- Second, determine twigs and links
  - twig: $n - 1 = 6 - 1 = 5$ twigs (solid dark line)
  - Link: $b - (n - 1) = 9 - 5 = 4$ links (thin solid line)
- Third, determine the fundamental cut set
  - Every twig of tree with some link defines a unique fundamental cut set
- Five fundamental cut sets as:



Cut set 1: $\{1,6\}$

Cut set 2: $\{2,6,8,9\}$

Cut set 3: $\{3,7,8,9\}$

Cut set 4: $\{4,7\}$

Cut set 5: $\{5,9\}$

Recall: Tree

It is connected

It contains all nodes of the digraph

It has no loops

Fundamental Cut Set Associated with a Tree-KCL on Fundamental Cut Set

- KCL equations for five fundamental cut sets:

Cut set 1: $\{1,6\}$ Cut set 2: $\{2,6,8,9\}$ Cut set 3: $\{3,7,8,9\}$ Cut set 4: $\{4,7\}$ Cut set 5: $\{5,9\}$

$$\left.\begin{array}{ll}
\text{Cut set 1:} & i_1 - i_6 = 0 \\
\text{Cut set 2:} & i_2 - i_6 + i_8 + i_9 = 0 \\
\text{Cut set 3:} & i_3 + i_7 + i_8 + i_9 = 0 \\
\text{Cut set 4:} & i_4 - i_7 = 0 \\
\text{Cut set 5:} & i_5 + i_9 = 0
\end{array}\right\}$$

$$\begin{array}{l}
\text{Cut set 1:} \\
\text{Cut set 2:} \\
\text{Cut set 3:} \\
\text{Cut set 4:} \\
\text{Cut set 5:}
\end{array}
\overbrace{\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}}^{Q}
\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ \cdot \\ \cdot \\ \cdot \\ i_9 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$n-1$ twigs: 1_{n-1} (unit matrix)

ℓ links: Q_ℓ

$Qi = 0$ where $Q: (\mathrm{n}-1) \times b$
fundamental cut set matrix

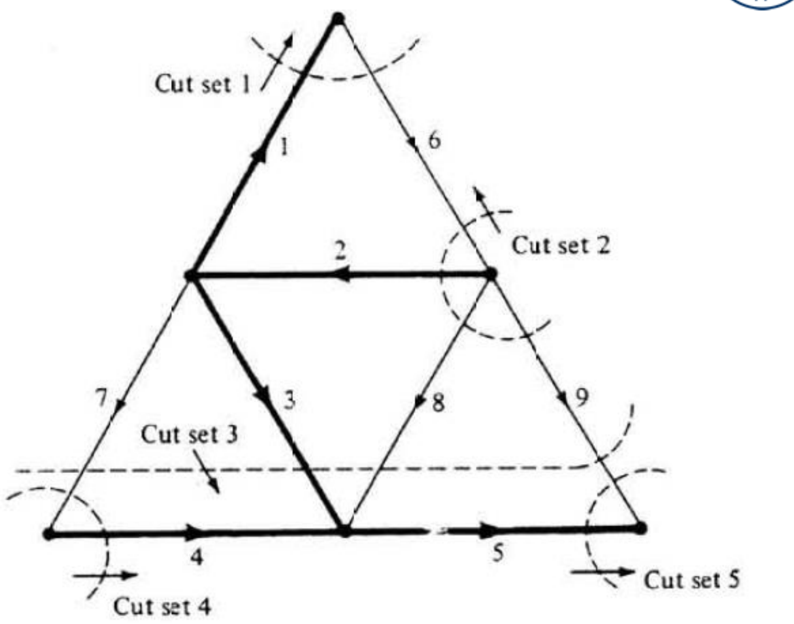
$$Q = [1_{n-1}, Q_\ell]$$

KVL Equations Using Twig Voltages

- Let the twig voltages be:

$v_{t1}, v_{t2}, v_{t3}, v_{t4}, v_{t5}$ (5 twigs).

$v_1 = v_{t1},\ v_2 = v_{t2},\ v_3 = v_{t3},\ v_4 = v_{t4},\ v_5 = v_{t5}$



- Four fundamental loops defined by each link
  - Fundamental loop contains only one link and some twigs
  - Apply KVL for fundamental loops defined by four links

$$v_6 + v_2 + v_1 = 0 \Rightarrow v_6 = -v_1 - v_2 = -v_{t1} - v_{t2}$$

$$v_7 + v_4 - v_3 = 0 \Rightarrow v_7 = v_3 - v_4 = v_{t3} - v_{t4}$$

$$v_8 - v_3 - v_2 = 0 \Rightarrow v_8 = v_2 + v_3 = v_{t2} + v_{t3}$$

$$v_9 - v_5 - v_3 - v_2 = 0 \Rightarrow v_9 = v_2 + v_3 + v_5 = v_{t2} + v_{t3} + v_{t5}$$

- In general, matrix can be written as:

$$v = Q^T v_t$$

v_t: twig voltage vector
Q^T: transpose of fundamental cut-set matrix

$$\begin{bmatrix} v_1 \\ v_2 \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ v_9 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}}_{n-1} \begin{bmatrix} v_{t1} \\ v_{t2} \\ v_{t3} \\ v_{t4} \\ v_{t5} \end{bmatrix}$$

Fundamental Loop Matrix Associated with Tree

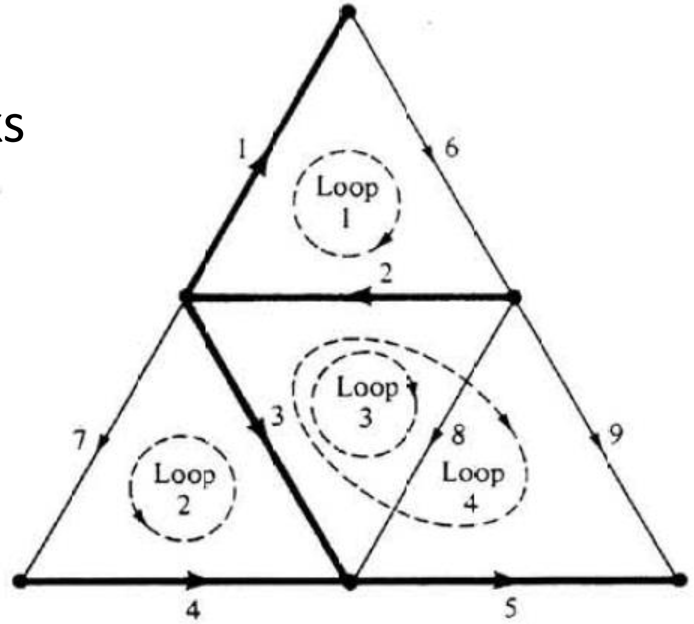
- Determine the KVL equations based on fundamental loops
- Four fundamental loops in the graph since it contains four links
- Apply KVL to each fundamental loop:

Loop 1: $v_6 + v_2 + v_1 = 0$

Loop 2: $v_7 + v_4 - v_3 = 0$

Loop 3: $v_8 - v_3 - v_2 = 0$

Loop 4: $v_9 - v_5 - v_3 - v_2 = 0$



Reference direction of each loop is defined by the direction of its associated link

$$\ell\text{ loops}\ \overbrace{\left[\begin{matrix}\underbrace{\begin{matrix}1&1&0&0&0\\0&0&-1&1&0\\0&-1&-1&0&0\\0&-1&-1&0&-1\end{matrix}}_{n-1\text{ twigs}} & \underbrace{\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}}_{\ell\text{ links}}\end{matrix}\right]}^{B}\begin{bmatrix}v_1\\v_2\\v_3\\ \cdot\\ \cdot\\ \cdot\\v_9\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow Bv = 0$$

$$B = [B_t, 1_\ell]$$

1_ℓ: unit matrix
B_t: submatrix of ℓ(loop) rows and n-1 columns

B: $(\ell \times b)$ fundamental loop matrix associated with the tree T

KCL Equations Using Link Currents

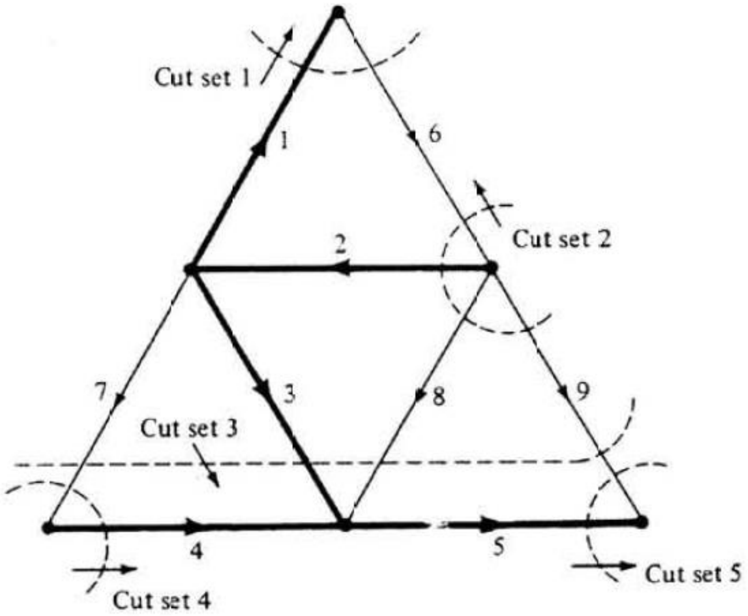
- Let the link currents be: $i_{\ell 1}, i_{\ell 2}, i_{\ell 3}, i_{\ell 4}$ (4 links)
- Four link branches identify as:

$$i_6 = i_{\ell 1},\, i_7 = i_{\ell 2},\, i_8 = i_{\ell 3},\, i_9 = i_{\ell 4}$$

- Determine the fundamental cut sets

Cut set 1: $\{1,6\}$ Cut set 2: $\{2,6,8,9\}$ Cut set 3: $\{3,7,8,9\}$

Cut set 4: $\{4,7\}$ Cut set 5: $\{5,9\}$



Cut set 1: $i_1 - i_6 = 0 \Rightarrow i_1 = i_6 = i_{\ell 1}$

Cut set 2: $i_2 - i_6 + i_8 + i_9 = 0 \Rightarrow i_2 = i_6 - i_8 - i_9$

$\Rightarrow i_2 = i_{\ell 1} - i_{\ell 3} - i_{\ell 4}$

Cut set 3: $i_3 + i_7 + i_8 + i_9 = 0 \Rightarrow i_3 = -i_7 - i_8 - i_9$

$\Rightarrow i_3 = -i_{\ell 2} - i_{\ell 3} - i_{\ell 4}$

Cut set 4: $i_4 - i_7 = 0 \Rightarrow i_4 = i_7 = i_{\ell 2}$

Cut set 4: $i_5 + i_9 = 0 \Rightarrow i_5 = -i_9 = -i_{\ell 4}$

Recall:
Cut set is a way to isolate node or nodes from the circuit. For fundamental cut set, consider one branch and some links one at a time

Direction of current is chosen w.r.t the reference direction of cut set

KCL Equations Using Link Currents

$$
\begin{aligned}
i_1 &= i_{\ell 1} & \qquad i_6 &= i_{\ell 1}\\
i_2 &= i_{\ell 1} - i_{\ell 3} - i_{\ell 4} & \qquad i_7 &= i_{\ell 2}\\
i_3 &= -i_{\ell 2} - i_{\ell 3} - i_{\ell 4} & \qquad i_8 &= i_{\ell 3}\\
i_4 &= i_{\ell 2} & \qquad i_9 &= i_{\ell 4}\\
i_5 &= -i_{\ell 4} & &
\end{aligned}
$$

- Equations can be written in matrix form as:

$$
\begin{bmatrix} i_1 \\ i_2 \\ \\ \\ \vdots \\ \\ \\ \\ i_9 \end{bmatrix}
=
\underbrace{\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 \\
0 & -1 & -1 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}}_{\ell}
\begin{bmatrix} i_{\ell 1} \\ i_{\ell 2} \\ i_{\ell 3} \\ i_{\ell 4} \end{bmatrix}
\quad \Rightarrow \quad i = B^T i_\ell
$$

i_ℓ: link current vector

B^T:transpose of the fundamental loop matrix

Graph Theory—Chord Current Method

- Linear circuits containing two-terminal resistors and independent sources, use the following steps:
  1. Pick a proper tree of the graph of the circuit which includes all voltage sources. Current sources are placed in co-trees
  2. Write the fundamental loop equations which do not correspond to the current sources in co-trees
  3. Write the $v - i$ relations of the resistors in the form of $v_k = R_k i_k$
  4. Substitute voltages in step 3 into step 2
  5. Write the fundamental cut-set equations which do not correspond to the voltage sources
  6. Substitute the fundamental loop equations in step 4 into the equations in step 5
  7. Present the equation in the following form:

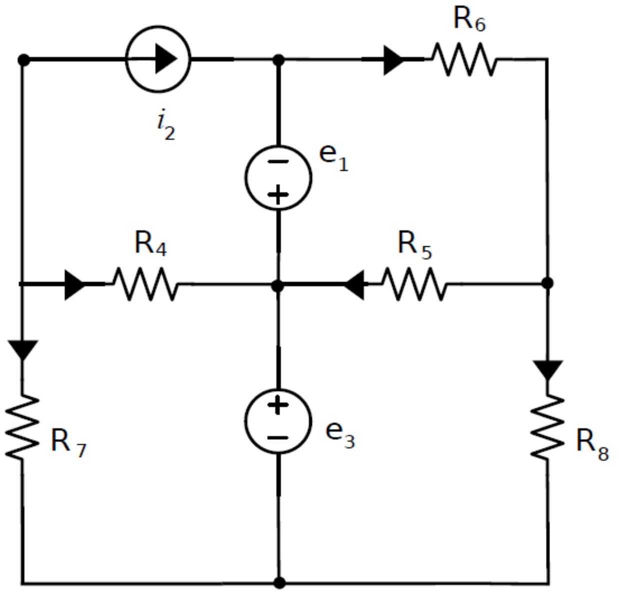
$$B i_{\ell'} + Q i_S + M v_S$$

B: Fundamental loop matrix
$i_{\ell'}$: link current vector
Q: Fundamental cut-set matrix
i_S: current source vector
M: Transpose of reduced incidence matrix
v_S: Voltage source vector

The rest of the slides are based on Müstak E. Yalçın's notes

Example 1

1. Pick (draw) a proper tree of the circuit. All voltage sources will be on the tree and current source will be on the co-tree (link).



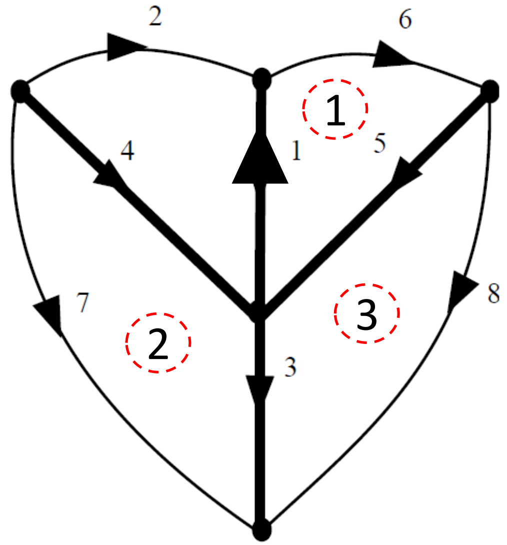
Solid thick line: twig
Solid thin line: link

Direction of voltage source is chosen from + to -

Tree can be written as:

$$T = \{1,3,4,5\}$$

2. Write the fundamental loop equations which do not correspond to the current source in the co-tree.
   - In order to write the fundamental loop equation, the fundamental loops must be determined
   - 3 fundamental loops indicated by dashed red circles. Direction of loop is chosen based on the reference direction of the link



Fundamental loop equations:

Loop 1: $v_6 + v_5 + v_1 = 0$

Loop 2: $v_7 - v_3 - v_4 = 0$

Loop 3: $v_8 - v_3 - v_5 = 0$

Recall:
Fundamental loop contains only one link and some twigs. By definition, do not include the loop that contains current source

Solution:

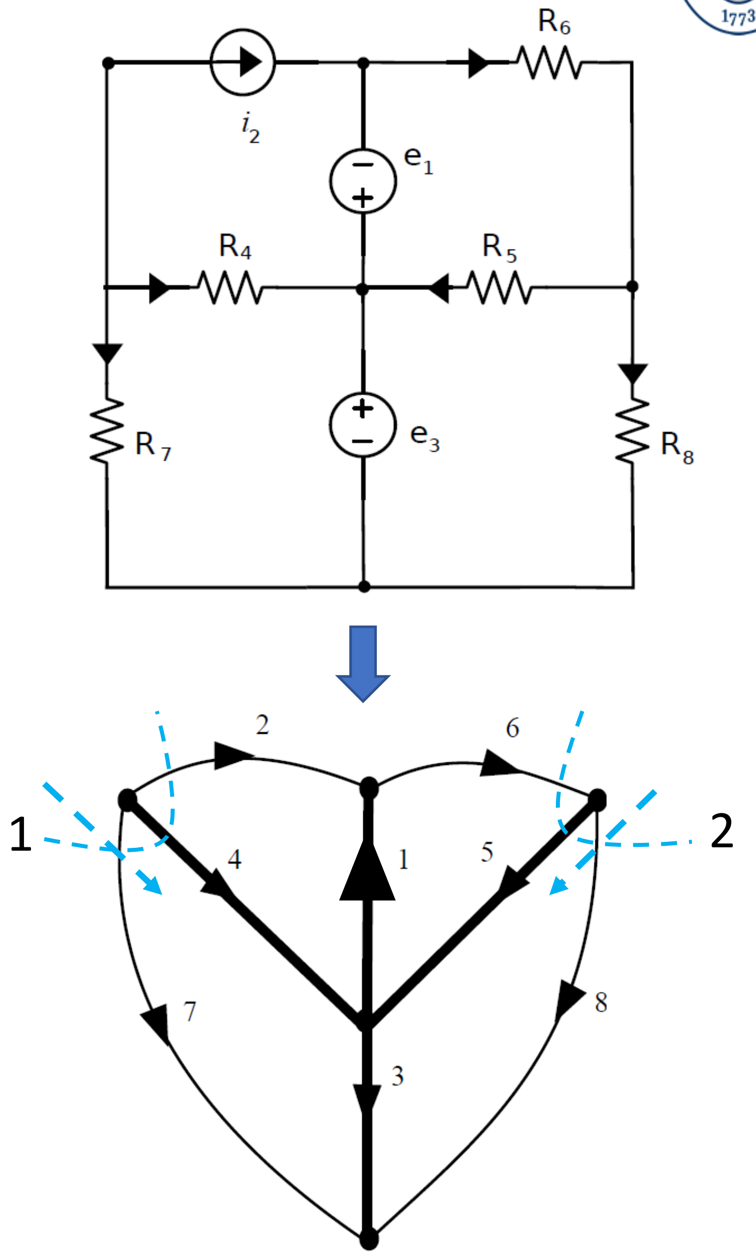
3. Write $v - i$ relations of the resistors: $v = iR$ (Ohm's law)

$$v_8 = R_8 i_8, \quad v_7 = R_7 i_7, \quad v_6 = R_6 i_6, \quad v_5 = R_5 i_5, \quad v_4 = R_4 i_4$$

4. Substitute voltages into the equations in step 2

$$\begin{aligned} v_6 + v_5 + v_1 &= 0 \\ v_7 - v_3 - v_4 &= 0 \\ v_8 - v_3 - v_5 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} R_6 i_6 + R_5 i_5 + v_1 &= 0 \\ R_7 i_7 - v_3 - R_4 i_4 &= 0 \\ R_8 i_8 - v_3 - R_5 i_5 &= 0 \end{aligned}$$

5. Write the fundamental cut-set equations which do not correspond to the voltage sources
   - In order to write the fundamental cut-set equation, the fundamental cut-set of a tree must be determined
   - 2 fundamental cut-sets indicated by dashed blue lines. Direction of cut-sets is defined by the direction of the twig.



Recall:
Cut-set is partition of node (or vertex) of graph into two disjoint subset.
For fundamental cut-set, consider one tree branch with some links one at a time (do not intersect two or more branches at the same time).
Cut-set orientation (direction) is defined by the direction of twig.

Solution:

- Fundamental cut-set equations:

$$i_2 + i_4 + i_7 = 0 \Rightarrow i_4 = -i_2 - i_7$$
$$i_5 + i_8 - i_6 = 0 \Rightarrow i_5 = i_6 - i_8$$

6. Substitute currents into the equations in step 4

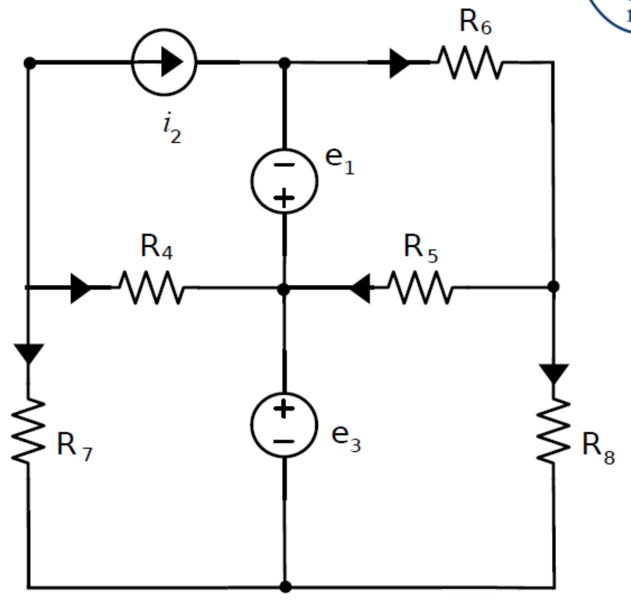
$$\left.\begin{aligned} R_6 i_6 + R_5(i_6 - i_8) + v_1 &= 0 \\ R_7 i_7 - R_4(-i_2 - i_7) - v_3 &= 0 \\ R_8 i_8 - R_5(i_6 - i_8) - v_3 &= 0 \end{aligned}\right\} \begin{aligned} (R_5 + R_6)i_6 - R_5 i_8 + v_1 &= 0 \\ (R_4 + R_7)i_7 + R_4 i_2 - v_3 &= 0 \\ (R_5 + R_8)i_8 - R_5 i_6 - v_3 &= 0 \end{aligned}$$

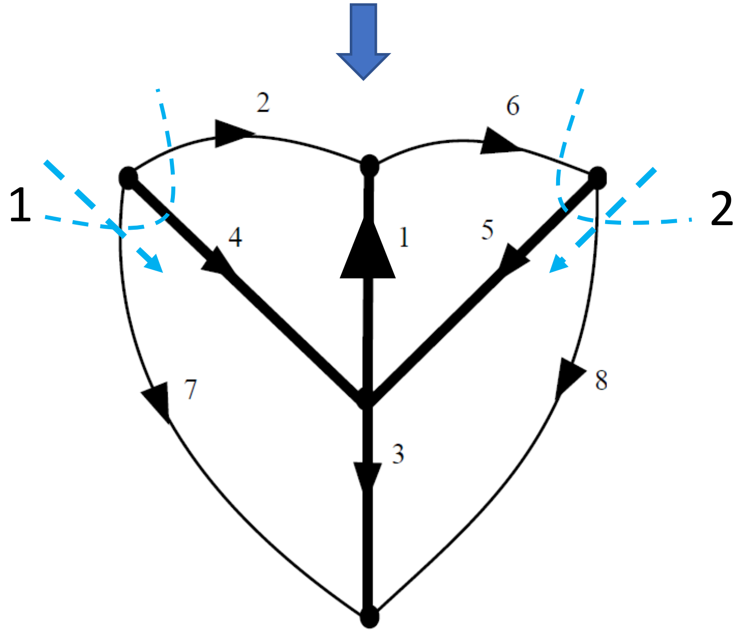
7. Finally, write these equations in matrix form as:

$$B i_{\ell'} + Q i_S + M v_S$$

$$\begin{bmatrix} R_5 + R_6 & 0 & -R_5 \\ 0 & R_4 + R_7 & 0 \\ -R_5 & 0 & R_5 + R_8 \end{bmatrix} \begin{bmatrix} i_6 \\ i_7 \\ i_8 \end{bmatrix} + \begin{bmatrix} 0 \\ R_4 \\ 0 \end{bmatrix} i_2 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = 0$$

Unknown variables: $i_{\ell'} = i_6, i_7, i_8$ (link currents in co trees).





Solution:

- After obtaining link currents (i_6, i_7, i_8), determine branch currents (i_1, i_3, i_4, i_5)
- Write branch currents in terms of link currents using KCL equations
- Apply KCL:

$$i_1 + i_2 = i_6 \Rightarrow i_1 = -i_2 + i_6$$

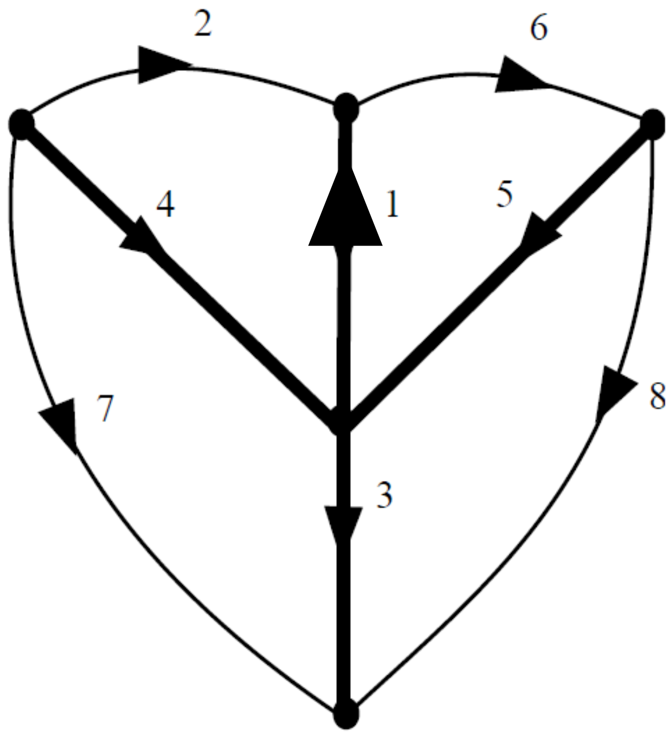
$$i_3 + i_7 + i_8 = 0 \Rightarrow i_3 = -i_7 - i_8$$

$$i_4 + i_2 + i_7 = 0 \Rightarrow i_4 = -i_2 - i_7$$

$$i_5 + i_8 = i_6 \Rightarrow i_5 = i_6 - i_8$$

- In matrix form:

$$\begin{bmatrix} i_1 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_2 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix}$$



Graph Theory—Generalized Chord Current Method

- If linear circuits containing two-terminal resistors along with independent and dependent sources, use the following steps:

  - ❑ Follow the same steps mentioned earlier and treat dependent source as independent sources.

  - ❑ Place dependent voltage sources in a tree and place dependent current sources in a co-tree.

  - ❑ Using $v - i$ relations of the dependent sources, new unknown variables are written in terms of the link current, voltage sources, and current sources.

- If there is a multi-terminal component in the circuit, it can be considered as an independent source.

Example 2

1. Pick (draw) a proper tree of the circuit. All voltage sources will be on the tree and current source will be on the co-tree (link).
   - Treat dependent sources as independent sources

Proper tree can be written as: $T = \{2,3,4,5,7\}$

2. Write the fundamental loop equations which do not correspond to the current source in the co-tree.
   - In this case, ignore link 6 & 8 as current sources located in link 6 & 8
   - Only one loop (direction of loop is same as direction of link)

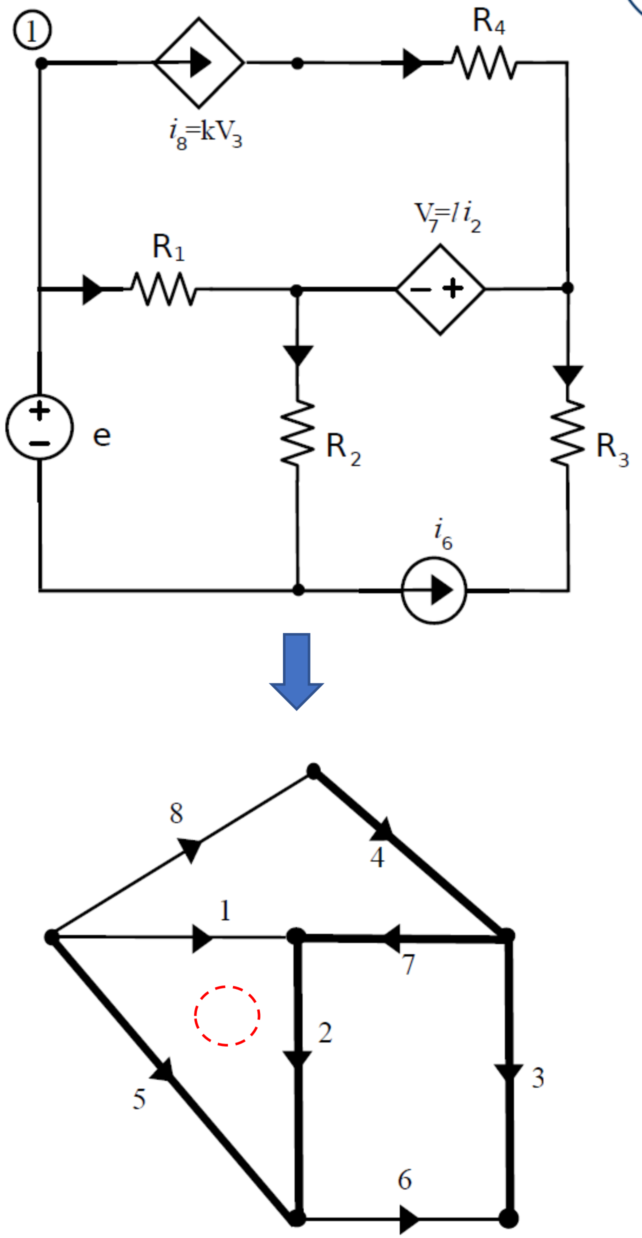
Fundamental loop equations: $v_1 + v_2 - v_5 = 0$

3. Write $v - i$ relations of the resistors: $v = iR$ (Ohm's law)

$v_1 = R_1 i_1$ $\qquad$ $v_2 = R_2 i_2$ $\qquad$ v_5: voltage source

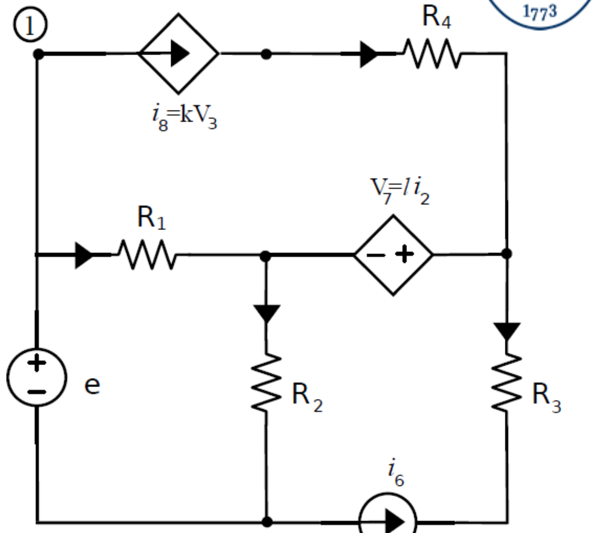
4. Substitute the equations in step 3 into the equations in step 2

$R_1 i_1 + R_2 i_2 - v_5 = 0$ where $v_5 = e$ ➡ $R_1 i_1 + R_2 i_2 - e = 0$



Solution

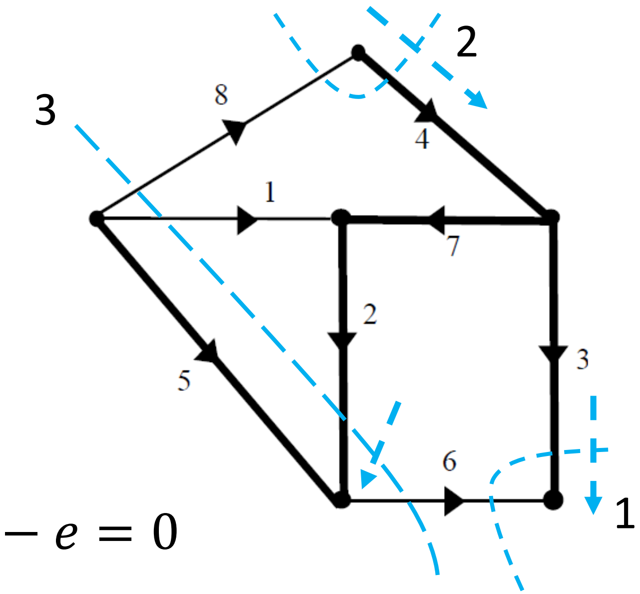
5. Write the fundamental cut-set equations which do not correspond to the voltage sources
   - 3 fundamental loops indicated by dashed blue lines.



Cut-set 1: $i_3 + i_6 = 0$
Cut-set 2: $i_4 - i_8 = 0$
Cut-set 3: $i_2 - i_6 - i_1 - i_8 = 0$

Although three equations are found, only one of them will be used as there is only one equation in step 4. Since i_1 is new unknown in this example, only cut-set 3 equation will be used.

$$i_2 - i_6 - i_1 - i_8 = 0 \Rightarrow i_2 = i_1 + i_6 + i_8$$



6. Substitute equations in step 5 into the equation in step 4

$$R_1 i_1 + R_2(i_1 + i_6 + i_8) - e = 0 \Rightarrow (R_1 + R_2)i_1 + R_2 i_6 + R_2 i_8 - e = 0$$

$$v_3 = R_3 i_3 \text{ where } i_3 + i_6 = 0 \Rightarrow i_3 = -i_6 \Rightarrow v_3 = -R_3 i_6$$

$$i_8 = k v_3 \Rightarrow i_8 = -k R_3 i_6$$

$$(R_1 + R_2)i_1 + R_2 i_6 - k R_2 R_3 i_6 - e = 0 \Rightarrow (R_1 + R_2)i_1 + (R_2 - k R_2 R_3)i_6 - e = 0$$

Graph Theory—Branch Voltage Method

- For a given circuit, use the following steps:
  1. Pick a proper tree which includes all voltage sources. Current sources are placed in a co-trees. Complete the tree with resistors.
  2. Write the fundamental cut-set equations for branches. Do not include branches that contain voltage sources.
  3. Write $v - i$ relations of resistors in the form of $i_k = G_k v_k$
  4. Substitute currents found in step 3 into the fundamental cut set equations in step 2
  5. Write fundamental loop equations. Do not include links that contain current sources.
  6. Substitute the fundamental loop equations in step 5 into the equations in step 4
  7. Present the equation in the following form:

$$MV_{b'} + Qi_S + Mv_S$$

$V_{b'}$: branch voltage vector
Q: Fundamental cut-set matrix
i_S: current source vector
v_S: Voltage source vector

Example 3

1. Pick (draw) a proper tree of the circuit. All voltage sources will be on the tree and current source will be on the co-tree (link).
   - Start from the voltage sources. Direction of voltage sources from + to –
   - Tree should be connected, i.e., contains all nodes and has no loop

   Proper tree can be written as: $T = \{1,3,4,5\}$

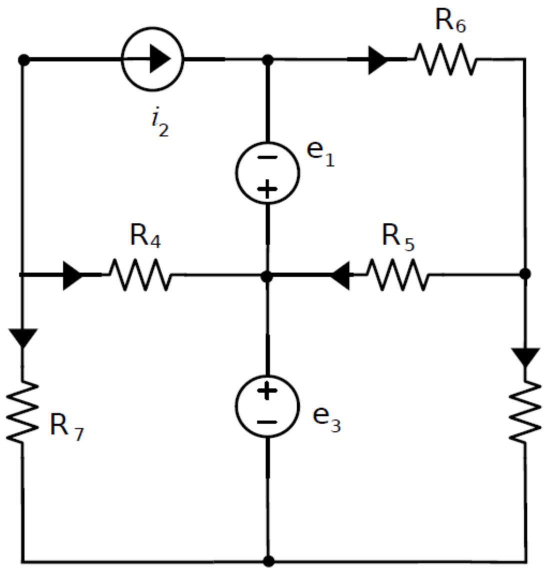
2. Write the fundamental cut set equations for branches.
   - Do not include branches that contain voltage sources
   - Write fundamental cut set equations for branches 4 & 5

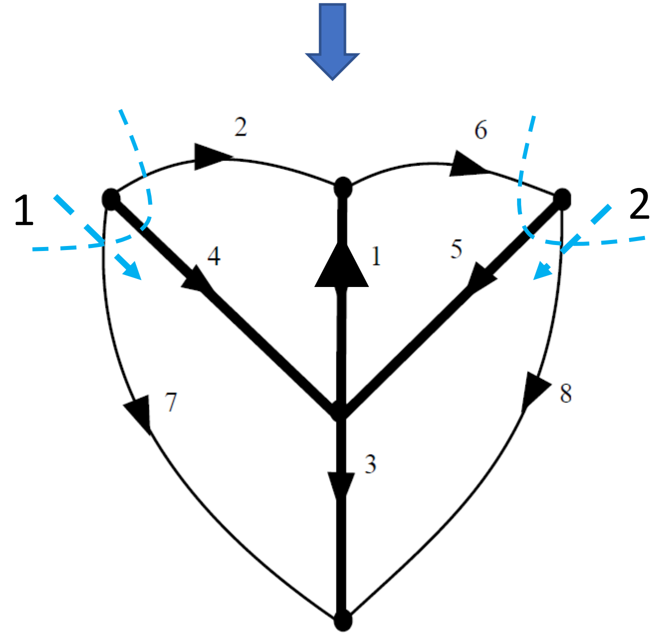
   Cut-set 1: $i_2 + i_4 + i_7 = 0$
   Cut-set 2: $i_5 - i_6 + i_8 = 0$

3. Write $v - i$ relations of resistors: $i_k = G_k v_k$

$$i_4 = G_4 v_4 \qquad i_5 = G_5 v_5 \qquad i_6 = G_6 v_6$$

$$i_7 = G_7 v_7 \qquad i_8 = G_8 v_8$$



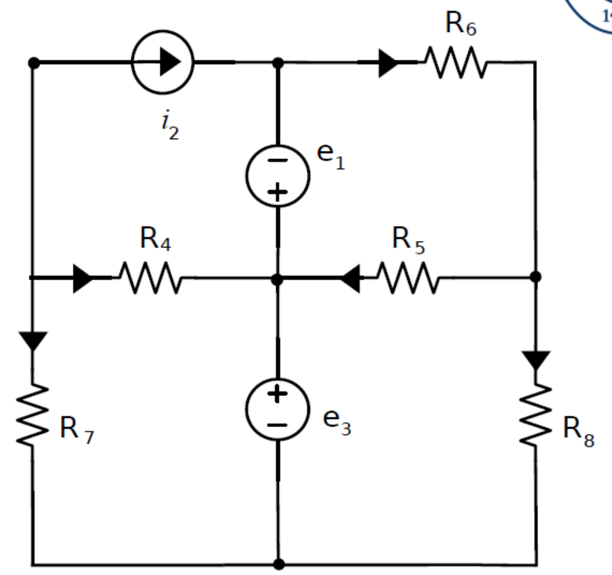


Solution

4. Substitute current found in step 3 into the fundamental cut set equations in step 2

$$i_2 + i_4 + i_7 = 0 \Rightarrow i_2 + G_4 v_4 + G_7 v_7 = 0$$

$$i_5 - i_6 + i_8 = 0 \Rightarrow G_5 v_5 - G_6 v_6 + G_8 v_8 = 0$$

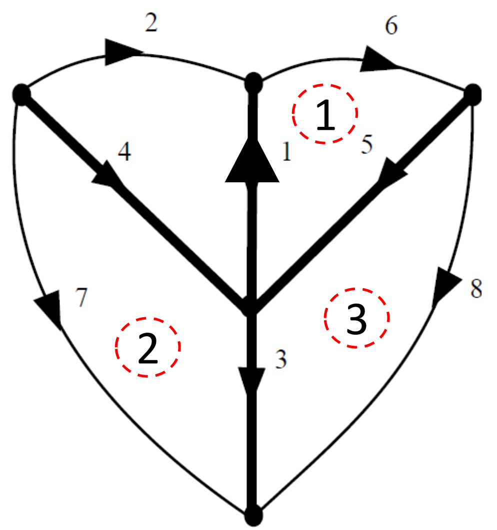


5. Write the fundamental loop equations which do not include link that contains current source.
   - 3 fundamental loops indicated by dashed red circles.

Fundamental loop equations:

Loop 1: $v_6 + v_5 + v_1 = 0$
Loop 2: $v_7 - v_3 - v_4 = 0$
Loop 3: $v_8 - v_3 - v_5 = 0$

$$v_6 = -v_1 - v_5$$
$$v_7 = v_3 + v_4$$
$$v_8 = v_3 + v_5$$



Solution

6. Substitute fundamental loop equations in step 5 into the equations in step 4

$$\left.\begin{aligned} v_6 &= -v_1 - v_5 \\ v_7 &= v_3 + v_4 \\ v_8 &= v_3 + v_5 \end{aligned}\right\} \Rightarrow \begin{aligned} & i_2 + G_4 v_4 + G_7 v_7 = 0 \\ & G_5 v_5 - G_6 v_6 + G_8 v_8 = 0 \end{aligned}$$

$$i_2 + G_4 v_4 + G_7 v_7 = 0 \Rightarrow i_2 + G_4 v_4 + G_7 (v_3 + v_4) = 0$$

$$G_5 v_5 - G_6 v_6 + G_8 v_8 = 0 \Rightarrow G_5 v_5 - G_6(-v_1 - v_5) + G_8(v_3 + v_5) = 0$$

$$i_2 + G_4 v_4 + G_7 v_3 + G_7 v_4 = 0 \Rightarrow i_2 + G_7 v_3 + (G_4 + G_7) v_4 = 0$$

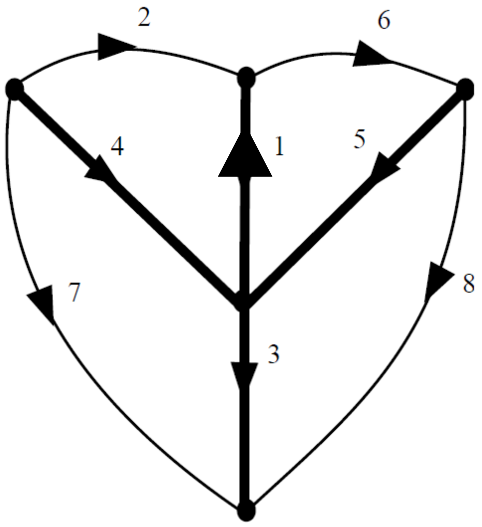
$$G_5 v_5 + G_6 v_1 + G_6 v_5) + G_8 v_3 + G_8 v_5 = 0 \Rightarrow G_6 v_1 + G_8 v_3 + (G_5 + G_6 + G_8) v_5 = 0$$

Solution

7. Finally, present equations in the matrix form as: $MV_{b'} + Qi_S + Mv_S$

$$i_2 + G_7 v_3 + (G_4 + G_7)v_4 = 0$$

$$G_6 v_1 + G_8 v_3 + (G_5 + G_6 + G_8)v_5 = 0$$



$$\underbrace{\begin{bmatrix} G_4 + G_7 & 0 \\ 0 & G_5 + G_6 + G_8 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} v_4 \\ v_5 \end{bmatrix}}_{V_b'} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{Q} \underbrace{i_2}_{i_S} + \underbrace{\begin{bmatrix} 0 & G_7 \\ G_6 & G_8 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} v_1 \\ v_3 \end{bmatrix}}_{v_S} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- If values of resistors, current source and independent voltage sources (v_1 & v_3) are given, branch voltages v_4 & v_5 can be found from the above matrix.
- After finding v_4 & v_5, link voltages (v_2, v_6, v_7, v_8) can be as follows:
- Apply KVL for each link on the digraph:

$$\left.\begin{aligned} v_2 - v_1 - v_4 &= 0 \\ v_6 + v_5 + v_1 &= 0 \\ v_7 - v_3 - v_4 &= 0 \\ v_8 - v_3 - v_5 &= 0 \end{aligned}\right\} \quad \begin{aligned} v_2 &= v_1 + v_4 \\ v_6 &= -v_1 - v_5 \\ v_7 &= v_3 + v_4 \\ v_8 &= v_3 + v_5 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} v_2 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

Obtaining State Equations

- For solving the state equations, use the following steps:

1. Pick (draw) a proper tree
   - The voltage sources must be placed in the tree
   - If the tree is not complete, the edges corresponding to as many capacitors as possible must be placed in the tree. If a capacitor in a loop which consisting entirely of capacitors and voltage sources, the capacitor must not placed in the tree.
   - If the tree is not complete, the edges corresponding to the resistors must be chosen and as many resistors as possible must be included.
   - If the tree is still not complete, then the edges corresponding to the inductors will be chosen until the tree is completed. If an inductor on a cut set which consisting entirely of inductors and current sources, the inductor must be placed in the tree.
   - All the edges corresponding to the current sources must be placed in the co-tree.
2. After selection of proper tree, the state variables are branch capacitor voltages and chord inductor currents.

Obtaining State Equations

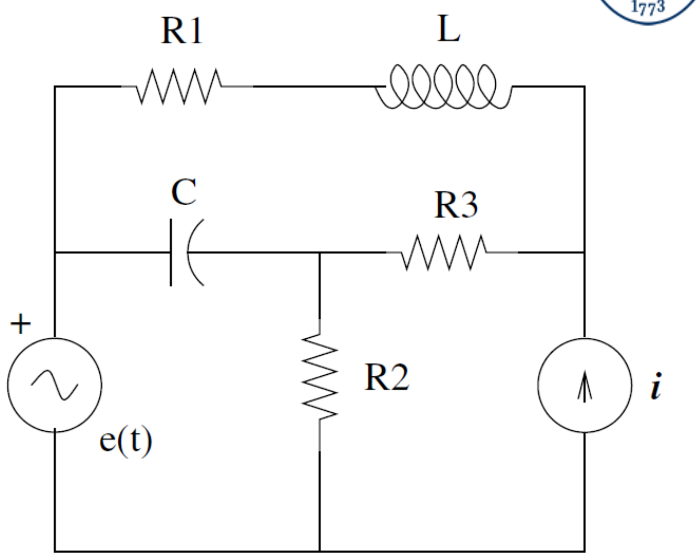
3. Obtaining state equations from the circuit: Express the voltage across each element corresponding to a branch and the current through each element corresponding to non-branch edge in terms of voltage sources, current sources, and state variables. If not possible, assign a new voltage variable to a resistor corresponding to a branch and a new current variable to a resistor corresponding to a non-branch edge.
   a) Apply KVL to the fundamental loop determined by each non-branch inductor
   b) Apply KCL to the fundamental cut-set determined by each branch capacitor
   c) Apply KVL to the fundamental loop determined by each resistor with a new current variable assigned in
   d) Apply KCL to the node or super-node corresponding to the fundamental cut-set determined by each resistor with a new voltage variable assigned in
   e) Solve the simultaneous equations obtained from step c and d for the new variables in terms of the voltage sources, current sources, and the state variables.
   f) Substitute the expression obtained in step e into equations determined in step a and b

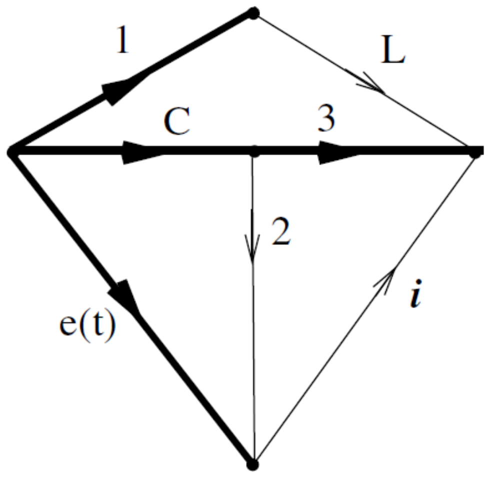
Example 4

1. Pick (draw) a proper tree of the circuit. All voltage sources will be on the tree and current source will be on the co-tree (link).
   - Identify nodes. There are 5 nodes in the circuit.
   - Since direction of branches is not given, choose a direction.
   - Direction of resistor and inductor is arbitrary but direction for capacitor and voltage source is chosen from + to –
   - Place voltage source in a tree
   - If the tree is not complete, place capacitor in the tree.
   - If the tree is still not complete, use as many resistors as possible to complete the tree
   - If the tree is still not complete, select inductor to complete the tree
   - Place current source in the co-tree.

2. After selection of proper tree, the state variables are branch capacitor voltages and chord (link) inductor currents.
   - 1 capacitor on the tree branch and 1 inductor on the co-tree (chord or link). Thus, v_C and i_L are state variables.
   - RC circuit: 1st order differential equation

$$\dot{V}_C = f(V_C, i_L, e(t), i(t)) \quad \text{and} \quad \dot{i}_L = f(V_C, i_L, e(t), i(t))$$





Solution

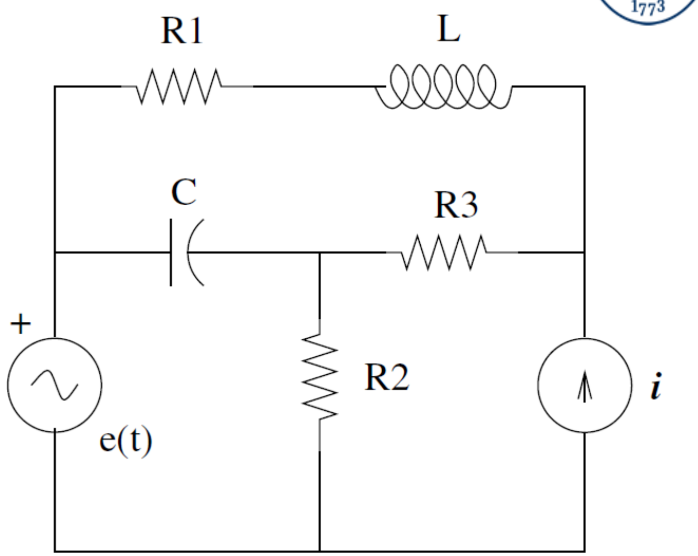
3. Obtaining state equations:
   a) Apply KVL to the fundamental loop determined by each link inductor
      - Since only one link inductor, circuit has one fundamental loop
      - Fundamental loop indicated by dashed red circle

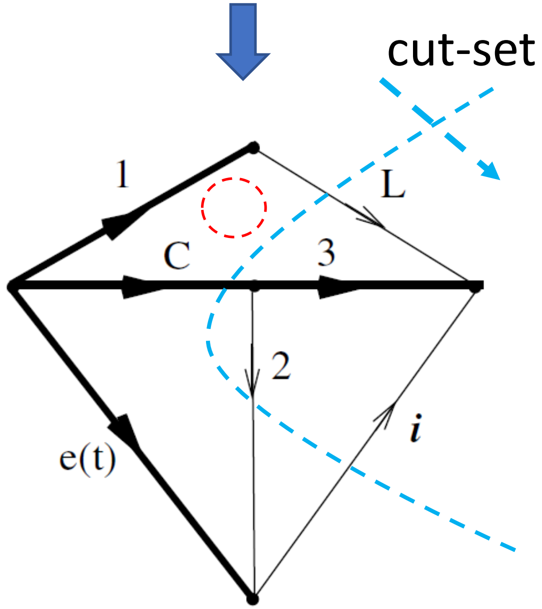
$$V_L - V_3 - V_c + V_1 = 0$$

   b) Apply KCL to the fundamental cut-set determined by each tree branch capacitor
      - Since one branch capacitor, circuit has one fundamental cut-set
      - Fundamental cut-set indicated by dashed blue line

$$i_L + i_C - i_2 + i = 0$$

$$\left.\begin{aligned} i_L + i_C - i_2 + i &= 0 \\ V_L - V_3 - V_c + V_1 &= 0 \end{aligned}\right\} \text{State equations}$$

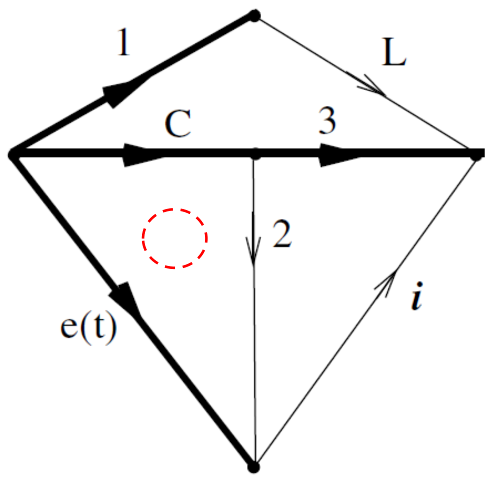




Solution

$$i_L + i_C - i_2 + i = 0 \Rightarrow i_C = -i_L + i_2 - i$$

$$V_L - V_3 - V_c + V_1 = 0 \Rightarrow V_L = V_3 + V_c - V_1$$



- By definition: $i_C = C\dfrac{dV_C}{dt}$ and $V_L = L\dfrac{di_L}{dt}$

$$C\frac{dV_C}{dt} = -i_L + i_2 - i \qquad L\frac{di_L}{dt} = V_3 + V_c - V_1$$

c) Apply KVL to the fundamental loop determined by each resistor with a new current variable assigned in

- Fundamental loop determined by each resistor that contains a new loop
- Fundamental loop indicated by dashed red circle

$$V_2 - e + V_c = 0 \text{ where } V_2 = R_2 i_2 \Rightarrow \underbrace{R_2 i_2}_{\text{New variable}} = e - V_c$$

Solution

d) Apply KCL to the node or super-node corresponding to the fundamental cut-set determined by each resistor with a new voltage variable assigned in

- Two fundamental cut-set as there are two resistors on tree branch
- Fundamental cut-sets indicated by dashed blue lines

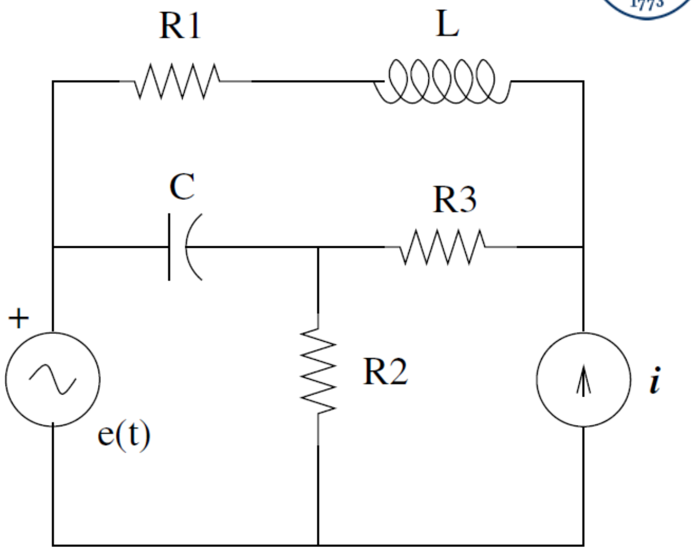
Cut set 1: $i_1 - i_L = 0$

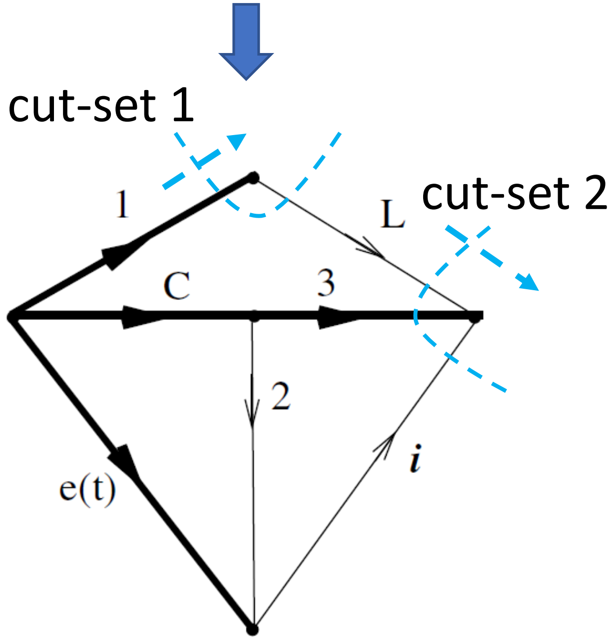
Cut set 2: $i_3 + i_L + i = 0$

$$\boxed{i_1} = i_L$$

$$\boxed{i_3} = -i_L - i$$

New variables





- i_1 & i_3 can be written in terms of voltage variables:

$$i = GV \quad \Rightarrow \quad i_1 = G_1V_1 \qquad i_3 = G_3V_3$$

$$G_1\boxed{V_1} = i_L$$

$$G_3\boxed{V_3} = -i_L - i$$

New variables

Solution

$$R_2 i_2 = e - V_c$$
$$G_1 V_1 = i_L$$
$$G_3 V_3 = -i_L - i$$

Solve these equations in terms of new variables (i_2, V_1, V_3)

$$\boxed{G = \frac{1}{R}}$$

$$i_2 = \frac{e}{R_2} - \frac{V_c}{R_2}$$
$$V_1 = R_1 i_L$$
$$V_3 = -R_3 i_L - R_3 i$$

Substitute new variables into state equations

$$i_L + i_C - i_2 + i = 0 \Rightarrow i_L + C\frac{dV_C}{dt} - \left(\frac{e}{R_2} - \frac{V_c}{R_2}\right) + i = 0$$

$$V_L - V_3 - V_c + V_1 = 0 \Rightarrow L\frac{di_L}{dt} - (-R_3 i_L - R_3 i) - V_c + R_1 i_L = 0$$

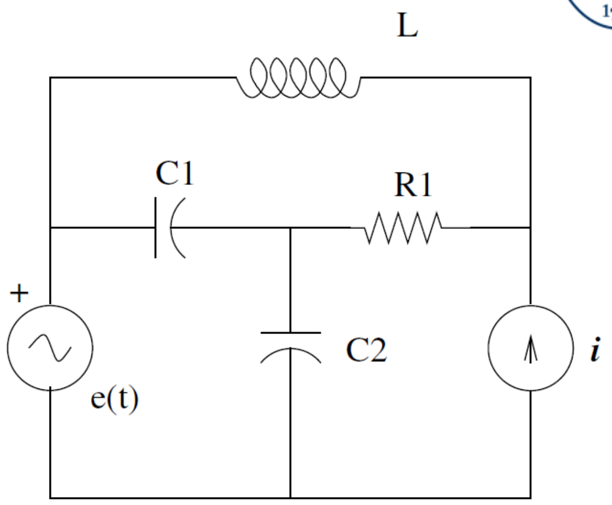
$$\frac{dV_C}{dt} = -\frac{1}{R_2 C}V_c - \frac{1}{C}i_L + \frac{1}{R_2 C}e - \frac{1}{C}i$$

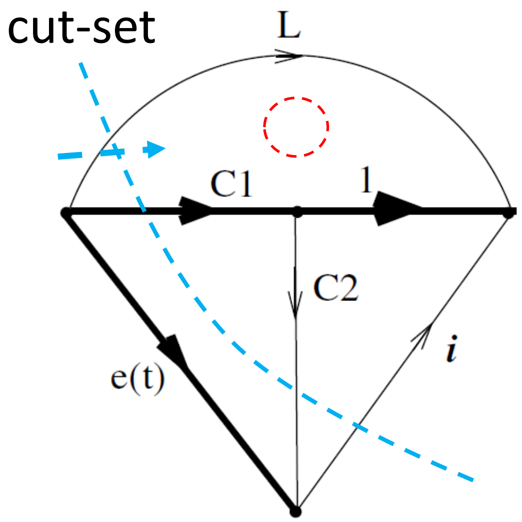
$$\frac{di_L}{dt} = \frac{1}{L}V_c - \frac{(R_1 + R_3)}{L}i_L - \frac{R_3}{L}i$$

$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_2 C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{(R_1 + R_3)}{L} \end{bmatrix}\begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2 C} \\ 0 \end{bmatrix} e(t) + \begin{bmatrix} -\frac{1}{C} \\ -\frac{R_3}{L} \end{bmatrix} i$$

Example 5

1. Pick (draw) a proper tree of the circuit. All voltage sources will be on the tree and current source will be on the co-tree (link).
   - Identify nodes. There are 4 nodes in the circuit.
   - Since direction of branches is not given, choose a direction.
   - Place C_1 in the tree but C_2 in the co-tree (link) as two capacitors and voltage source make a loop in the circuit.
   - If the tree is still not complete, use as many resistors as possible to complete the tree
   - If cut-set consists of entirely inductors and current sources, place inductor in the tree. Otherwise, place inductor in the co-tree.





2. After selection of proper tree, the state variables are branch capacitor voltages and chord (link) inductor currents.
   - 1 capacitor on the tree (C_1) and 1 inductor (L) on the co-tree (chord or link). Thus, v_{c1} and i_L are state variables.

3. Obtaining state equations: Apply KVL and KCL

Loop: $V_L - V_1 - V_{C1} = 0$

Cut set: $i_L + i_{C1} - i_{C2} + i = 0$

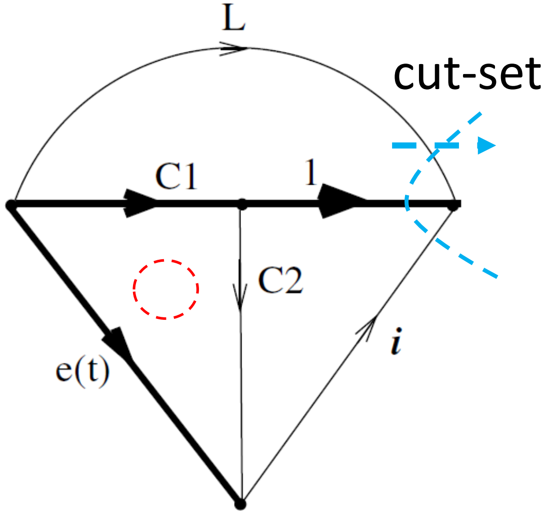
Solution

$$i_{C1} = -i_L - i + i_{C2}$$

$$V_L = V_1 + V_{C1}$$

- By definition: $i_C = C\dfrac{dV_C}{dt}$ and $V_L = L\dfrac{di_L}{dt}$

$$\left.\begin{aligned} C_1\frac{dV_{C1}}{dt} &= -i_L - i + i_{C2} \\ L\frac{di_L}{dt} &= V_1 + V_{C1} \end{aligned}\right\} \text{State equations}$$



- Apply KVL to the fundamental loop determined by capacitor C_2

$$V_{C1} + V_{C2} - e = 0 \Rightarrow V_{C2} = e - V_{C1}$$

- Apply KCL to the fundamental cut-set determined by resistor R_1

$$i_1 + i_L + i = 0 \Rightarrow i_1 = -i_L - i$$

$$i_1 = G_1V_1 \Rightarrow G_1V_1 = -i_L - i \Rightarrow G = \frac{1}{R} \Rightarrow V_1 = -R_1i_L - R_1i$$

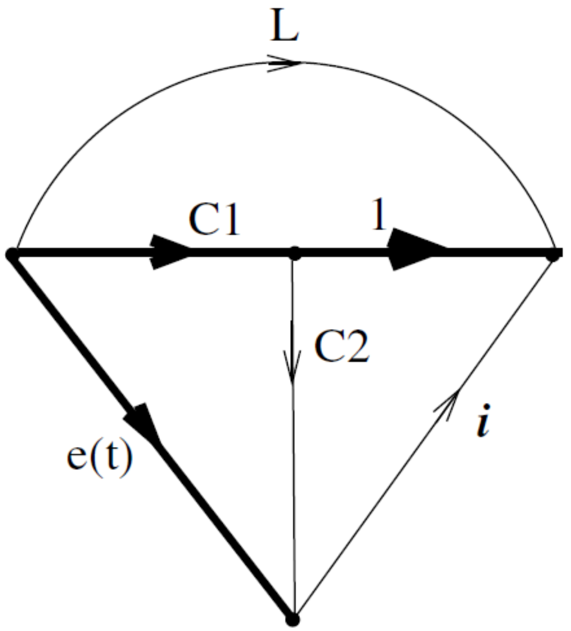
Solution

- Determine i_{C2} in state equations as:

$V_{C2} = e - V_{C1}$ ⟶ Take the derivative of both sides

$\frac{dV_{C2}}{dt} = \frac{de}{dt} - \frac{dV_{C1}}{dt}$ ⟶ Multiply both sides by C_2

$$\underbrace{\boxed{C_2 \frac{dV_{C2}}{dt}}}_{i_{C2}} = C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt}$$



- State equations in standard form:

$$C_1 \frac{dV_{C1}}{dt} = -i_L - i + C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt}$$

$$L \frac{di_L}{dt} = \underbrace{-R_1 i_L - R_1 i}_{V_1} + V_{C1}$$

Solution

$$C_1 \frac{dV_{C1}}{dt} = -i_L - i + C_2 \frac{de}{dt} - C_2 \frac{dV_{C1}}{dt}$$

$$L \frac{di_L}{dt} = -R_1 i_L - R_1 i + V_{C1}$$

Rearrange the state equations

$$\frac{dV_{C1}}{dt} = -\frac{1}{C_1 + C_2} i_L + \frac{C_2}{C_1 + C_2} \frac{de}{dt} - \frac{1}{C_1 + C_2} i$$

$$\frac{di_L}{dt} = \frac{1}{L} V_{C1} - \frac{R_1}{L} i_L - \frac{R_1}{L} i$$

State equations in standard form

- In matrix form:

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C_1 + C_2} \\ \frac{1}{L} & -\frac{R_1}{L} \end{bmatrix} \begin{bmatrix} V_{C1} \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{C_2}{C_1 + C_2} \\ 0 \end{bmatrix} \frac{de}{dt} + \begin{bmatrix} -\frac{1}{C_1 + C_2} \\ -\frac{R_1}{L} \end{bmatrix} i$$