Simplification (Minimization) of Logical Functions

A logic function has many algebraic expressions (see canonical forms and simplified expressions).

The purpose of simplification is to choose the most appropriate expression (with the minimum cost) from the set of all possible expressions according to a cost criterion.

The cost criterion may change and depend on the application.

For example, the design criteria may require the expression to have a minimum number of products (or sums), a minimum number of variables in each product, using only one type of gate (such as NAND), using the gates that are at our disposal.

Objectives of simplification:

- Decreasing the size of the circuit
- Decreasing power consumption (battery, cooling problem)
- Decreasing the delay (increasing the speed) (See 3.21: Propagation Delay)
- Decreasing the cost

Simplification Related Definitions: Prime Implicant

Implicant of a function F is a product P that is covered by this function P ≤ F (2.9).

Reminder: Each minterm (product) of 1st canonical form corresponds to a single 1-generating ("true") point. The minterms are implicants of the function (m ≤ F).

Example:

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$F(A, B, C) = \Sigma m(1,3,5,6,7)$: 1st canonical form

$= A'B'C + A'BC + AB'C + ABC' + ABC$

These products can be simplified into products that have fewer variables, which still cover together all the 1s of the function.

This function was simplified previously (slide 2.33):

$F = \boxed{AB} + \boxed{C}$

While the minterms in the canonical form cover only a single 1, the AB product covers two 1s and C covers four 1s.

Note that, products with fewer variables cover more 1s.

Prime implicant:

Implicants that cannot be simplified any further (i.e., cannot be combined with another term to eliminate a variable) are called **prime implicants**.

For example, products AB and C are prime implicants of this function but minterms are not.

Prime Implicant (cont'd)

A **prime implicant** is a product that cannot be simplified.

A **prime implicant** of F is an implicant that is minimal - that is, the removal of any literal from P results in a non-implicant for F (*Willard Van Orman Quine*).

Example (cont'd):

F(A, B, C)= Σm(1,3,5,6,7) : 1st canonical form
= A'B'C + A'BC + AB'C + ABC' + ABC
= AB + C

- For the given function above, the minterms are not prime implicants.
  For example, ABC' and ABC are NOT prime implicants, because they can be combined together to form AB, which includes fewer literals and covers more 1s.
  If we remove C from ABC the new product AB is still an implicant of F (AB ≤ F).
- AB is a prime implicant because it cannot be simplified as A and B because the function does not have 1s in all the places A and B would require ($A \nleq F$, $B \nleq F$).
- If we remove A or B from AB the new expression (A or B) is not an implicant of F.

Simplification process of a Boolean function:

1. Finding the complete **set of all prime implicants.**
2. Selection of the "most appropriate" subset of the prime implicants that covers all the 1s of the function.

Finding Prime Implicants:

Using Boolean algebra, we can combine minterms to obtain products that have fewer variables and cover more 1s.

It is hard to perform these simplifications manually, especially for complicated function (with many variables). Therefore, a computer program can be used.

A practical procedure (without using the logical expression of the function):
- Investigate the 1s (output = 1) in the truth table,
- Combine 1- generating input combinations with one or more constant variables (Hamming distance = 1).
- Retain the constant variables and remove the rest (variables with changing values).

Example:

Algebraic combining: F = A'B'+AB' = (A'+A)B' = B'

These input combinations are adjacent. Hamming distance =1

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

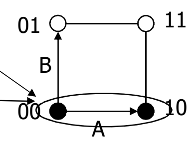
B is constant. For both lines B=0.
Hence, B will be retained in the new product.

The value of A is changing.
Hence, it will be removed from the new product.

Since B=0, the new product will be B' .

Visualization of the process on the Boolean cube

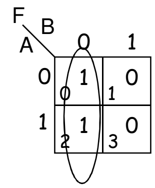
A	B	F
0	0	1
0	1	0
1	0	1
1	1	0



Two points (having 0 dimensions) are combined to obtain a line (having 1 dimension).

This line represents B=0 (B is constant at zero, and A changes) which is the complement of B, namely, B'.

Visualization of the process on the Karnaugh map



Karnaugh maps allow easier grouping of terms.

Neighboring 1s can be grouped together using the adjacency property.

In the grouped column above, B=0 is fixed, and A is changing.

This column represents the complement of B, namely, B'.

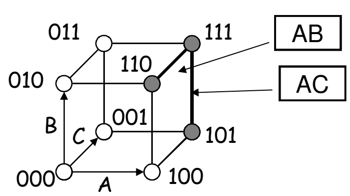
- **If more than one variable is fixed**, each one appears in the product that is the result of their grouping.

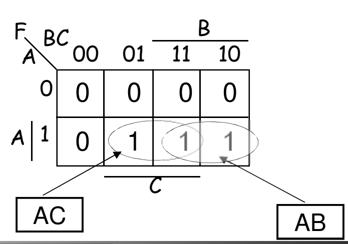
For example:

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A=1 , C=1 are constant. B is changing.
The product AC is formed as the result of this grouping.
Algebraically: AB'C + ABC = AC(B'+B) = AC

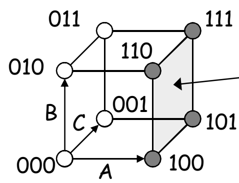
A=1 , B=1 are constant. C is changing.
The product AB is formed as the result of this grouping.
Algebraically: ABC' + ABC = AB(C'+C) = AB





- **More than 2 points can also be combined** to establish new groups.

For example: F(A,B,C) = $\Sigma(4,5,6,7)$



A=1 is constant. B and C are changing.

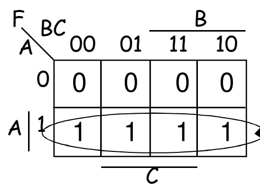
This face of the cube (a plane) is representing A.

Algebraically: AB'C' + AB'C + ABC' + ABC

= AB'+AB

= A

Using the Karnaugh map:



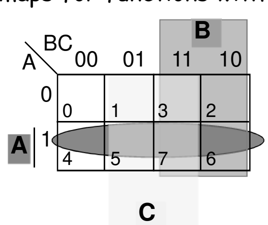
A=1 is constant. B and C are changing.

Finding Prime Implicants Using Karnaugh Maps (Diagrams):

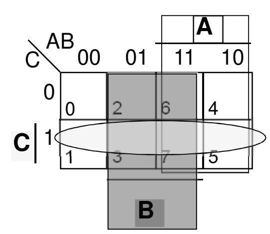
In Karnaugh maps, only a single variable changes between two neighboring cells; the rest remain constant.

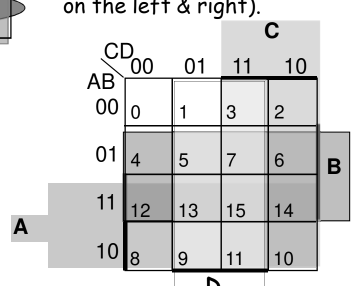
True points (1s) in the neighboring squares (cells) can be clustered into groups that contain 2, 4, 8, … cells.

Below, areas where the variable values stay constant are shown on the Karnaugh maps for functions with 3 and 4 variables.



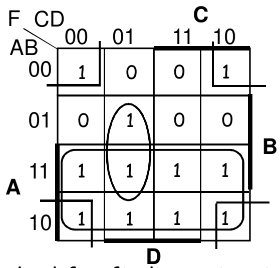
Karnaugh maps for the same function can be drawn in different ways by changing the order of variables (compare the diagrams on the left & right).





Example: Find the prime implicants of the following function.

$F(A,B,C,D) = \Sigma_1(0,2,5,8,9,10,11,12,13,14,15)$



Prime implicants: A , B'D' , BC'D

Method for finding prime implicants (grouping 1s):

- The groups must be rectangular and must have an area that is a power of two (i.e. 1, 2, 4, 8…).
- "True" points (1s) are placed into the largest groups possible.
- Two points within a larger group (or groups) cannot be combined into a smaller subgroup.

  For example, 2 points within different groups of 4 points cannot be combined to form another 2-point group. It is possible to form a new 4-point group.
- However, if one of the points is not contained by any groups (such as 0101 above), that point can be grouped using another point which is already in another group.

The Set of All Prime Implicants and The Sufficient Base:

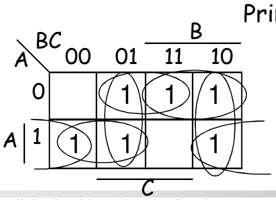
In logic circuit design, the simplification process starts with finding the set of all prime implicants.

In the second step of the simplification, most appropriate prime implicants are selected from the set of all prime implicants.

The minimum set of prime implicants that covers all true points (1s) of a function is called **sufficient base (sufficient covering sum or minimal covering sum)**.
If any of the prime implicants in the sufficient base is removed, some of the true points of the function will not be covered.

Therefore, simplification of a function is the selection of the most appropriate (with minimum cost) sufficient base (**minimal covering sum**).

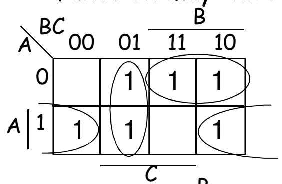
For example: Find the set of all prime implicants of the function.



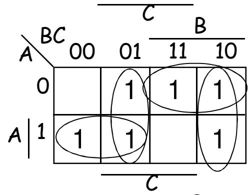
Prime implicants:

BC' , A'B , A'C , AB' , B'C , AC'

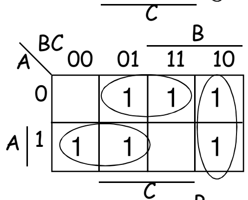
A function may have many sufficient bases (minimal covering sums).



F(A,B,C)= A'B + B'C + AC'

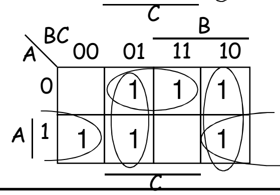


F(A,B,C)= A'B + BC' + B'C + AB'



F(A,B,C)= BC' + A'C + AB'

1. The minimal covering sum covers all 1s.
2. If any of the prime implicants is removed from the minimal covering sum, some of the 1s will not be covered.



F(A,B,C)= BC' + A'C + B'C + AC'

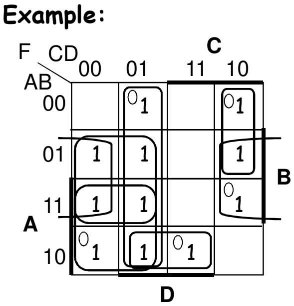
Distinguished Points and Essential Prime Implicants:

Some true points (1s) of a function may be covered only by a single prime implicant. These 1s are called **distinguished points**.

The prime implicant that covers a distinguished point is called an **essential prime implicant**.

Essential prime implicants must be included in the minimal covering sum. Otherwise, it is not possible to cover all true points of a function.

Example:



Set of all prime implicants:

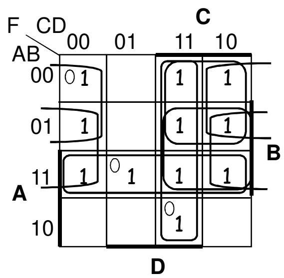
C'D	, BC'	, AC'	, BD'	,A'CD'	, AB'D

Distinguished points	Essential prime implicants
0001	C'D
0010	A'CD'
1000	AC'
1110	BD'
1011	AB'D

Here, essential prime implicants cover all true points of the function. This is a specific case.

F= C'D + A'CD' + AC' + BD' + AB'D

Example: Finding the set of all prime implicants, distinguished points and essential prime implicants.



Set of all prime implicants:

CD , AB , A'C , BC , A'D' , BD'

Distinguished points	Essential prime implicants
0000	A'D'
1101	AB
1011	CD

Simplification: Selection of the Most Appropriate Prime Implicants

Reminder: Simplification process has two steps:

1. Finding the set of all prime implicants
2. Selection of a subset of prime implicants with minimum cost that covers the function (minimal covering sum).

Prime implicant charts are used to select the sufficient base with the minimum cost.

Prime Implicant Chart:

- Simple symbols are assigned to each prime implicant such as A, B, C, …
- Using the given cost criteria, the cost of each prime implicant is calculated.

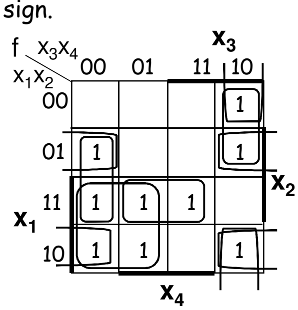
Prime implicant chart is similar to a matrix:

- The symbols of prime implicants are placed to the rows of the chart.
- Number corresponding to the true points of the function are placed to the columns of the chart.
- The costs of the particular prime implicants are placed in the last column.
- If a prime implicant covers a true point their intersection is marked with an 'X'.

Example: Find the set of all prime implicants and form the prime implicant chart for the following function.

$f(x_1, x_2, x_3, x_4)=\Sigma m(2, 4, 6, 8, 9, 10, 12, 13, 15)$

The cost criteria are: 2 units for each variable and 1 unit for each complement sign.



Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1' x_2 x_4'$	$x_1 x_2 x_4$	$x_1' x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Cost:	5	8	8	6	8	8	8
Covered points:	8,9,12,13	4,12	4, 6	13, 15	2, 6	2, 10	8, 10

Example (cont'd):

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1' x_2 x_4'$	$x_1 x_2 x_4$	$x_1' x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Covered points:	8,9,12,13	4,12	4, 6	13, 15	2, 6	2, 10	8, 10

True points of the function

Prime implicants

	2	4	6	8	9	10	12	13	15	Cost
A				X	X		X	X		5
B		X					X			8
C		X	X							8
D								X	X	6
E	X		X							8
F	X					X				8
G				X		X				8

Simplification of Prime Implicant Chart:

1. Distinguished points are determined. If there is a single X in a column, that is a distinguished point.

   The prime implicant that covers the distinguished point (essential prime implicant) is necessarily selected.

   The row of this essential prime implicant and columns that are covered by this implicant are removed from the chart.

2. If there is an X in the i^{th} row for each X in the j^{th} row, then row i covers row j. In other words, all points covered by row j are also covered by row i.

i	X		X	4
~~j~~			~~X~~	~~5~~

   If row i covers row j AND the cost at row i is smaller or equal to the cost at row j, then row j (covered row) is removed from the chart.

3. If a column covers another column, the covering column (with more X) is removed from the chart.

i	X	~~X~~	
j	X	~~X~~	
k		~~X~~	

These rules are applied successively until all true points are covered with the least cost.

Example: Simplification of prime implicant chart of the following function

$f(x_1, x_2, x_3, x_4)=\Sigma m(2, 4, 6, 8, 9, 10, 12, 13, 15)$

True points of the function

		2	4	6	8	9	10	12	13	15	Cost
√ $x_1 x_3'$	A				X	X		X	X		5
$x_2 x_3' x_4'$	B		X					X			8
$x_1' x_2 x_4'$	C		X	X							8
√ $x_1 x_2 x_4$	D								X	X	6
$x_1' x_3 x_4'$	E	X		X							8
$x_2' x_3 x_4'$	F	X					X				8
$x_1 x_2' x_4'$	G				X		X				8

1. step: In this chart 9 and 15 are the distinguished points.

As A and D are essential prime implicants, their rows and the columns that they cover are removed from the chart.

These products are marked to show their inclusion into the final set.

	2	4	6	10	Cost
~~B~~		~~x~~			~~8~~
C		x	x		8
E	x		x		8
F	x			x	8
~~G~~				~~x~~	~~8~~

2. step: In this chart, C covers B. As the cost of C is equal to B, B (as the covered row) is removed from the chart.

Similarly, F covers G and they have the same cost. So the row of G is removed from the chart. These products (B and G) will not be in the final set.

		2	4	6	10	Cost
√	C		x	x		8
	E	x		x		8
√	F	x			x	8

3. step: In this chart 4 and 10 are distinguished points. Therefore, C and F are selected (and marked). With this selection all true points of the function are covered.

Result:

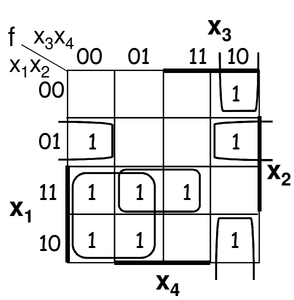
Marked prime implicants for the expression of the function with the least cost.

Selected prime implicants: A + D + C + F

Total cost= 5 + 6 + 8 + 8 = 27

$f(x_1, x_2, x_3, x_4) = x_1 x_3' + x_1 x_2 x_4 + x_1'x_2 x_4' + x_2' x_3 x_4'$

It is possible to see selected prime implicants using the Karnaugh map.



Selected prime implicants should cover all true points and there should be no redundancy.

Selected prime implicants should form a sufficient base. Therefore, removal of any implicant should result in uncovered true point(s).

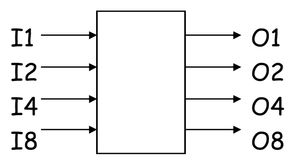
$\boxed{x_1 x_3'}$ $\boxed{x_1'x_2 x_4'}$ $\boxed{x_1 x_2 x_4}$ $\boxed{x_2' x_3 x_4'}$

Simplification of Incomplete Functions

Reminder: In incomplete functions, for some input combinations the function result is undetermined (we do not care about it).

These combinations may never appear in the circuit or they are prohibited by the designer.

Example: BCD incrementer circuit



I8	I4	I2	I1	O8	O4	O2	O1
0	0	0	0	0	0	0	1
0	0	0	1	0	0	1	0
0	0	1	0	0	0	1	1
0	0	1	1	0	1	0	0
0	1	0	0	0	1	0	1
0	1	0	1	0	1	1	0
0	1	1	0	0	1	1	1
0	1	1	1	1	0	0	0
1	0	0	0	1	0	0	1
1	0	0	1	0	0	0	0
1	0	1	0	X	X	X	X
1	0	1	1	X	X	X	X
1	1	0	0	X	X	X	X
1	1	0	1	X	X	X	X
1	1	1	0	X	X	X	X
1	1	1	1	X	X	X	X

For these input combinations, the output of the circuit (function) is undetermined.
The symbols X or Φ are used to show the undetermined (don't care) outputs.

Selection of Undetermined (Don't Care) Values (Φ):

Undetermined (don't care) values (Φ) can be chosen to be 0 or 1 in order to utilize the least costly expression in the simplification process.

- While searching **for the set of all prime implicants**, undetermined values are taken **as 1** in order to have (larger groups in the Karnaugh map) simpler products (**Φ = 1**).
- While **forming the prime implicant chart**, undetermined values are taken **as 0** because there is no need to cover these points (**Φ = 0**).

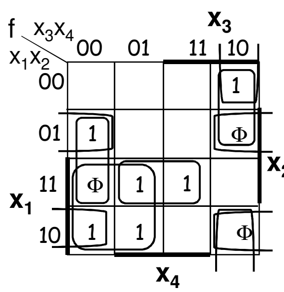
Example: Implement the following incomplete function with the least possible cost.

$f(x_1, x_2, x_3, x_4)=\Sigma_m(2, 4, 8, 9, 13, 15) + \Sigma_\Phi(6,10,12)$

Remark: it can also be written as
$f(x_1, x_2, x_3, x_4)=\cup_1(2, 4, 8, 9, 13, 15) + \cup_\Phi(6,10,12)$

Cost criteria: 2 units for each variable and 1 unit for each complement.

Finding the prime implicants:



Φ's are assumed to be 1, when we are finding the set of all prime implicants.

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1'x_2 x_4'$	$x_1 x_2 x_4$	$x_1'x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Points covered:	8,9,13	4	4	13,15	2	2	8

Forming the prime implicant chart:

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1'x_2 x_4'$	$x_1 x_2 x_4$	$x_1'x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Points covered:	8,9,13	4	4	13,15	2	2	8

True points of the function

Prime implicants

	2	4	8	9	13	15	Cost
A			X	X	X		5
B		X					8
C		X					8
D					X	X	6
E	X						8
F	X						8
G			X				8

Φ's are assumed to be 0, when we are forming the prime implicant chart.

As there is no need to cover the points with undetermined values, these points are not placed into the prime implicant chart.

True points of the function

	Prime implicants	2	4	8	9	13	15	Cost
√	A			X	X	X		5
	B		X					8
	C		X					8
√	D					X	X	6
	E	X						8
	F	X						8
	G			X				8

Step 1: In this chart, points 9 and 15 are distinguished points.

As A and D are the essential prime implicants, they are selected. The rows and columns covered by A and D are removed.

A and D are marked to show that they will be in the final set of prime implicants.

	2	4	Cost
B		x	8
C		x	8
E	x		8
F	x		8

Step 2: B and C are covering the same points and they have the same cost. Therefore, it is not possible to make a choice between B and C. One of B and C can be selected.

Same situation exists for prime implicants E and F.

At the end, the same function can be implemented using any of the following expressions which have the same (lowest) cost.

f= A + D + B + E = $x_1 x_3' + x_1 x_2 x_4 + x_2 x_3' x_4' + x_1' x_3 x_4'$

f= A + D + B + F = $x_1 x_3' + x_1 x_2 x_4 + x_2 x_3' x_4' + x_2' x_3 x_4'$

f= A + D + C + E = $x_1 x_3' + x_1 x_2 x_4 + x_1' x_2 x_4' + x_1' x_3 x_4'$

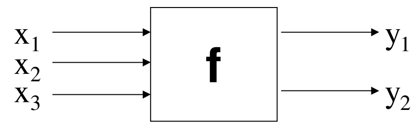
f= A + D + C + F = $x_1 x_3' + x_1 x_2 x_4 + x_1' x_2 x_4' + x_2' x_3 x_4'$

All designs have the same cost (27).

Simplification of General Functions

Remark: General functions have more than one output.

x_1 x_2 x_3	y_1 y_2
0 0 0	1 1
0 0 1	1 Φ
0 1 0	0 0
0 1 1	Φ 0
1 0 0	1 Φ
1 0 1	0 1
1 1 0	0 1
1 1 1	Φ 0



$y_1 = f_1(x_1, x_2, x_3)$

$y_2 = f_2(x_1, x_2, x_3)$

During the simplification of general functions, set of prime implicants for each output is found independently, and prime implicants are selected from these sets.

An important point is to select the common prime implicants of both outputs.

Simplification of general functions is not in the scope of this course.

Finding the Set of All Prime Implicants Using Tabular (Quine-McCluskey) Method

It is hard to use Karnaugh maps for the functions with too many variables as it becomes harder to visualize.

Especially for the functions with 5 or more variables, it is hard to draw and visualize the adjacency of the points.

In tabular method (Quine-McCluskey), systematic (algorithmic) operations are performed successively.

Performing these operations manually may be time consuming. However, it is possible to implement this method as a computer program.

Tabular (Quine-McCluskey) **Method:**

Remember, to find the set of all prime implicants, true points (minterms) of the function are merged (grouped). Adjacent minterms where a single variable changes are taken into the same group (See the figure at 4.4).

In the tabular method, minterms (corresponding to 1-generating input combinations) are compared to all other minterms.

If a single variable (input) changes between two minterms, they are merged.

The variable with the changing value is removed, and a new term is obtained.

This process is repeated until no further groups can be formed.

Terms that cannot be grouped are the prime implicants.

Method (Algorithm): *Willard Van Orman Quine (1908-2000), Philosophy, logic*
Edward J. McCluskey (1929-2016) Electric engineer.

1st Step: Finding the set of all prime implicants:

- Consider 1-generating input combinations (true points) in the truth table.
- Cluster the 1-generating input combinations depending on the number of 1s included in the combination. For example, 1011 has three 1s.

  This will shorten the running time of the algorithm.
- Compare combinations that are in the neighboring clusters. Merge the combinations where a single variable changes.
- Variable with changing value will be removed.
- Mark the combinations that are grouped.
- Repeat the grouping on the newly formed combinations until no further groups can be formed.
- Combinations that are not grouped (items that are not signed) form the set of all prime implicants.

2nd Step: Finding the minimal covering sum

The prime implicant chart is used to select the subset of prime implicants with minimum cost that covers the function (minimal covering sum) (See 4.14).

Example: Find the set of all prime implicants of the following function using Quine-McCluskey method.

$f(x_1, x_2, x_3, x_4)= \Sigma_m(0, 1, 2, 8, 10, 11, 14, 15)$

1-generating (true) input combinations

Num.	x_1 x_2 x_3 x_4	
0	0 0 0 0	√
1	0 0 0 1	√
2	0 0 1 0	√
8	1 0 0 0	√
10	1 0 1 0	√
11	1 0 1 1	√
14	1 1 1 0	√
15	1 1 1 1	√

Groups with 2 points

Num.	x_1 x_2 x_3 x_4	
0,1	0 0 0 -	
0,2	0 0 - 0	√
0,8	- 0 0 0	√
2,10	- 0 1 0	√
8,10	1 0 - 0	√
10,11	1 0 1 -	√
10,14	1 - 1 0	√
11,15	1 - 1 1	√
14,15	1 1 1 -	√

Groups with 4 points

Num.	x_1 x_2 x_3 x_4
0,2,8,10	- 0 - 0
0,8,2,10	- 0 - 0
10,11,14,15	1 - 1 -
10,14,11,15	1 - 1 -

No need to rewrite the same items.

Set of all prime implicants (Not marked): $\boxed{x_1' x_2' x_3'}$, $\boxed{x_2' x_4'}$, $\boxed{x_1 x_3}$

To find the minimal covering sum (lowest cost), prime implicant chart is used.