Ch 5 Higher-Order Linear Differential Equations

5.1 Second-Order Linear Eqs ($n=2$)

5.2 General Sols. of Linear Eqs. (nth order)

5.3 Homogeneous Eqs. with Constant Coeffs.

5.5 Nonhom. eqs. and Undetermined Coeffs. including Variation of parameter

Ayse Peker $\Rightarrow$ starts with 5.2

5.1 Second-Order Linear Eqs.

General second-order eq.: $G(x, y, y', y'') = 0$ $*$

Eq $*$ is said to be a linear eq. provided it is linear in y, y', y'': $A(x)y'' + B(x)y' + C(x)y = D(x)$

e.g. $e^x y'' + (\cos x) y' + \sqrt{1+x}\, y = \tan^{-1} x$ Linear DE

$y y'' = e^x$: Nonlinear $y'' + (3y')^2 + 4y^3 = 0$

$A(x) y'' + B(x) y' + C(x) y = F(x)$ is

said to be homogeneous if $F(x) \equiv 0$

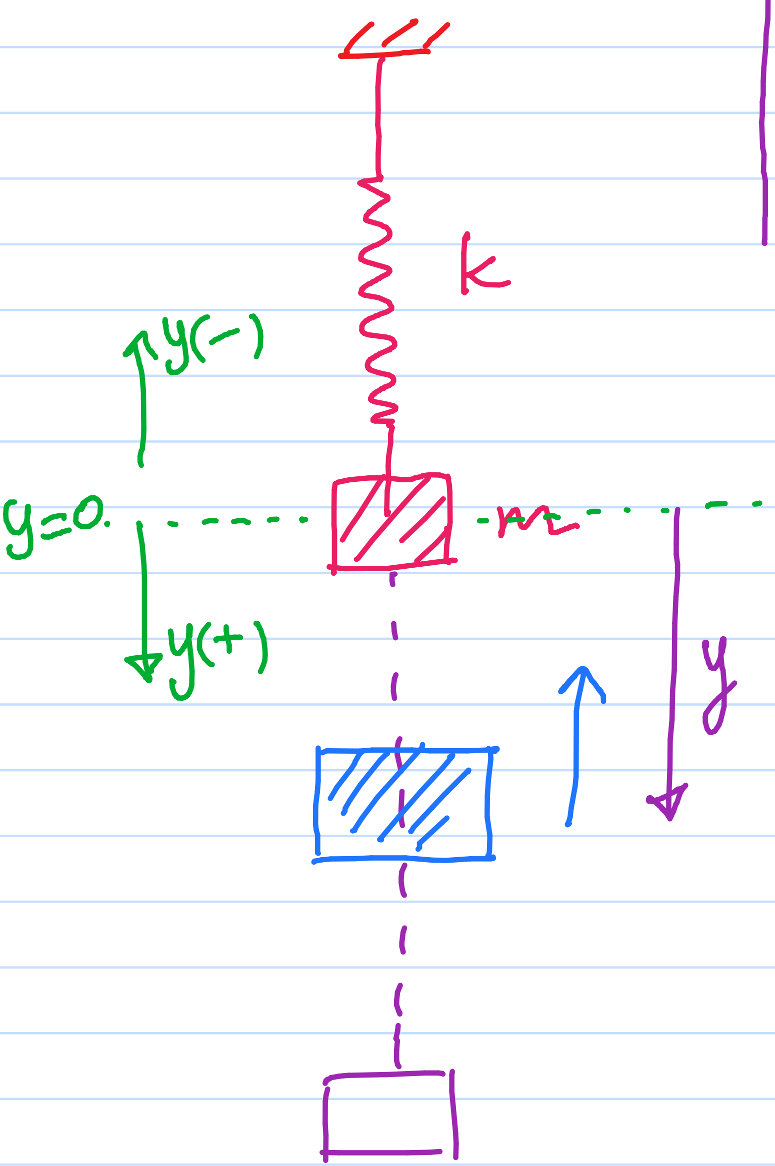
nonhomogeneous if $F(x) \not\equiv 0$.

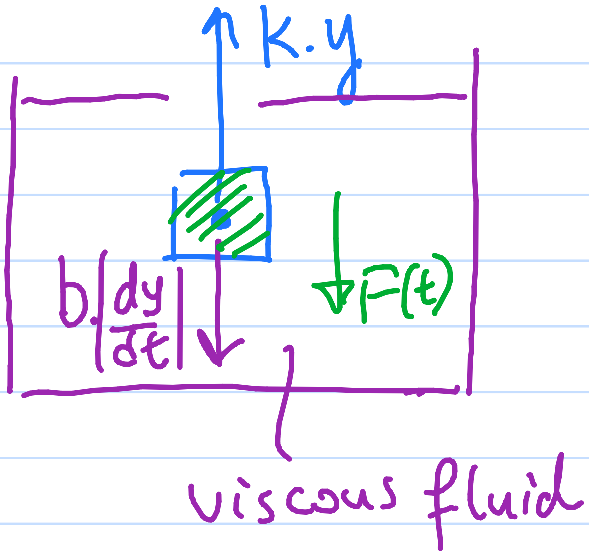
(Eq1) $x^2 y'' + 2xy' + 3y = \cos x$ non-homogeneous eq.

(Eq2) $x^2 y'' + 2xy' + 3y = 0$ homogeneous eq.

(Eq2) will be called as the associated homogeneous eq. of (Eq 1).

A typical application





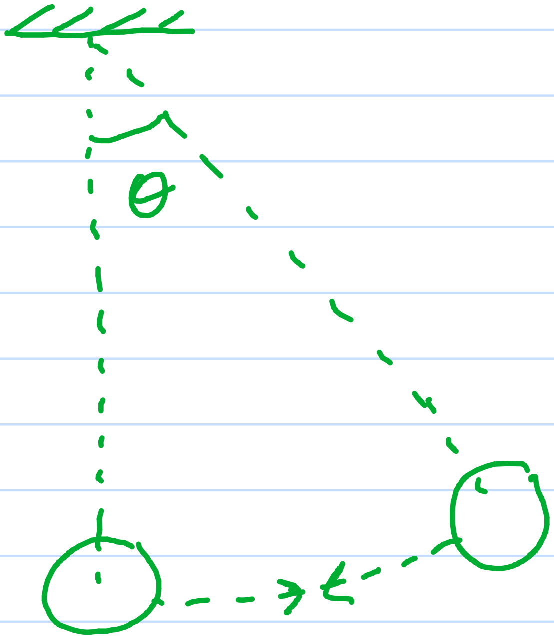
$$\Sigma F = m.a$$

$$m.\frac{d^2y}{dt^2} = -ky + \frac{dy}{dt} + F(t)$$

??(not sure)

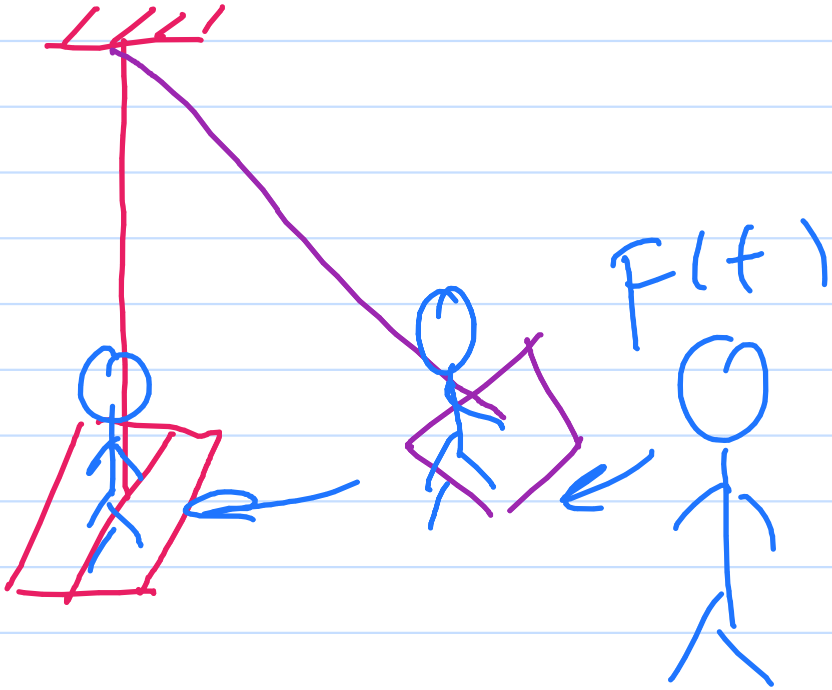
$$m\frac{d^2y}{dt^2} - b\left|\frac{dy}{dt}\right| + ky = F(t)$$

A nonhomogeneous term represents an external force that is applied on the system!!!



$$A \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} + \theta = F(t)$$

$F(t)$: an external force applied on the system as a function of t



$\theta(t)$: angular position

$\frac{d\theta}{dt}$ = angular velocity

$\frac{d^2\theta}{dt^2}$ = angular acceleration

Homogeneous Second-Order Linear Eqs

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = F(x), \qquad A(x) \neq 0$$

$$y'' + \frac{B(x)}{A(x)}\,y' + \frac{C(x)}{A(x)}\,y = F(x)$$

$$y'' + p(x)\,y' + q(x)\,y = f(x) \qquad \text{Nonhom.}$$

$$(*) \quad y'' + p(x)\,y' + q(x)\,y = 0 \qquad \text{Hom.}$$

$\boxed{\text{Th}}$ Principle of Superposition for Linear Homogeneous Eqs.

Let y_1 & y_2 be solutions to $y'' + p(x)y' + q(x)y = 0$.

$y(x) = c_1\,y_1(x) + c_2\,y_2(x)$ is also a solution.

Proof y_1 & y_2 are solutions : $y_1'' + p y_1' + q y_1 = 0$

$y_2'' + p y_2' + q y_2 = 0$

Consider the eq. $y'' + p(x) y' + q(x) y = 0$ (*) in the form

$$L[y] = y'' + p(x) y' + q(x) y = 0 .$$

↑ operator

Any given y solves the eq (*) if the operation specified by L applied to y produces ?

We can explicitly see L:

$$L[y] = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = \left\{\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\right\} y = 0$$

$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$ is a linear differential operator!

Proof y_1 & y_2 are sols : $L[y_1] = y_1'' + p y_1' + q y_1 = 0$

$L[y_2] = y_2'' + p y_2' + q y_2 = 0$

Claim : $y = c_1 y_1(x) + c_2 y_2(x)$ is also a sol.

$$L[y] = L[c_1 y_1(x) + c_2 y_2(x)]$$

$$= (c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2)$$

$$= c_1 (y_1'' + p y_1' + q y_1) + c_2 (y_2'' + p y_2' + q y_2)$$

$$= c_1 . L[y_1] + c_2 L[y_2]$$

$$= c_1 . 0 + c_2 . 0 = 0$$

$\Rightarrow c_1 y_1(x) + c_2 y_2(x)$ is also a solution

Question: Is the same true for non-homogeneous eqs?

If y_1 and y_2 solve $y'' + py' + qy = f(x)$, is $c_1 y_1(x) + c_2 y_2(x)$ also a solution?

NO

Observe that given $L[y] = y'' + p(x)y' + q(x)y = 0$,

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)y$$ and L has the property that

$$L[c_1 y_1(x) + c_2 y_2(x)] = c_1 L[y_1] + c_2 L[y_2]$$

(which is the reason we call L a linear diff. op. and also the eq. a linear eq).

Question Is the same true for non-homogeneous eqs?

$(**)$

If y_1 and y_2 solve $y'' + p y' + q y = f(x)$, is $c_1 y_1(x) + c_2 y_2(x)$ also a solution?

NO, in general!!

If y_1 and y_2 are sols. to $(**)$:

$$L[y_1] = y_1'' + p y_1' + q y_1 = f(x) \qquad L[y_1] = f$$

$$L[y_2] = y_2'' + p y_2' + q y_2 = f(x) \qquad L[y_2] = f$$

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] \neq f(x)$$

$$= c_1 \cdot f(x) + c_2 \cdot f(x) = (c_1 + c_2) f(x)$$

For a "Linear" and "Homogeneous" eqs, the principle of superposition IS VALID.

Ex $y_1 = \cos x$ and $y_2 = \sin x$ solve $y'' + y = 0$.

So does $y(x) = c_1 y_1 + c_2 y_2 = c_1 \cos x + c_2 \sin x$:

$$L[y] = y'' + y = 0$$

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= L[c_1 \cos x + c_2 \sin x] \\ &= (c_1 \cos x + c_2 \sin x)'' + c_1 \cos x + c_2 \sin x \\ &= -c_1 \cos x - c_2 \sin x + c_1 \cos x + c_2 \sin x \\ &= 0 \end{aligned}$$

Ex $y_1 = \cos x + x$ and $y_2 = \sin x + x$ are both solutions to $y'' + y = x$, which is a nonhomogeneous eq.!

$$L[y] = y'' + y = x, \qquad L = \frac{d^2}{dx^2} + 1$$

$$L[y_1] = L[\cos x + x] = (\cos x + x)'' + (\cos x + x)$$
$$= -\cos x + 0 + \cos x + x = x$$

y_1 is indeed a solution!

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$
$$= c_1 \cdot x + c_2 \cdot x$$
$$= (c_1 + c_2)x \neq x \Rightarrow \text{the}$$

Superposition is not always a solution!!!!

Remark

Space of all functions

Solution space of $y'' + p(x)\,y' + q(x)\,y = 0$

- $y_1(x)$
- $y_2(x)$
- $c_1 y_1(x) + c_2 y_2(x)$

Solution space of the linear hom. eq. $L[y]=0$ forms a vector space!!

space of all functions

sol. space of $y''+y=x$

- $\cos x + x$
- $\sin x + x$

- $\underline{\cos x + x + \sin x + x}$ $\quad c_1 = 1,\ c_2 = 1$

is not in the sol. space!

$\Rightarrow$ solution space of a nonhom. eq. does not form a vector space!!

Theorem 2 Existence & Uniqueness Theorem for Linear Equations

Suppose $p(x)$, $q(x)$ and $f(x)$ are continuous on the open interval I containing the point $x=a$. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

$$y(a) = b_0, \quad y'(a) = b$$

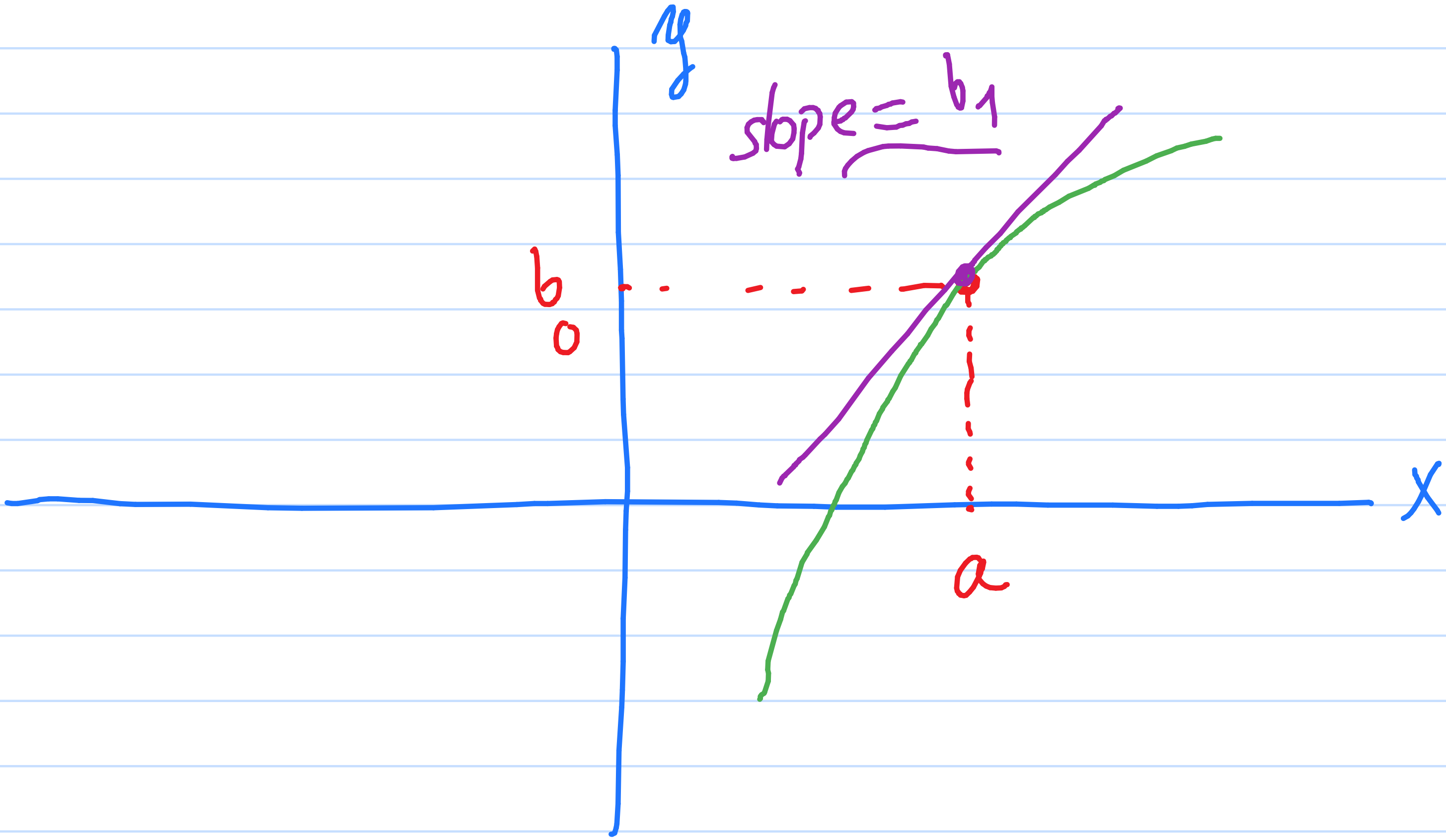
HAS a UNIQUE solution on I.

$$\left.\begin{aligned} & y'' + p(x)\,y' + q(x)\,y = 0 \\ & y(a) = b_0 \\ & y'(a) = b_1 \end{aligned}\right\}$$

→ Find a $y(x)$ that satisfies this

→ passing through (a, b_0)

→ of which tangent line at a has the slope b_1



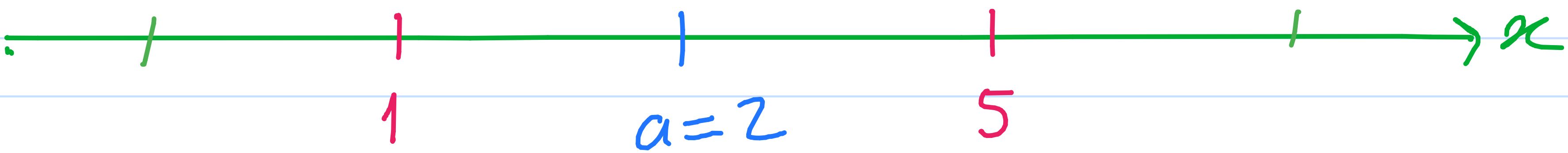
Second-order **IVP**

Ex

$$\begin{cases} y'' + \dfrac{1}{x-1} y' + \dfrac{1}{(x-1)(x-5)} y = x^2 \\[2ex] y(2) = 3, \quad y'(2) = 8 \end{cases}$$

$$p(x) = \frac{1}{x-1}, \qquad q(x) = \frac{1}{(x-1)(x-5)}, \qquad f(x) = x^2$$

$$D = (-\infty, 1) \cup (1, 5) \cup (5, \infty) \qquad \mathbb{R} \setminus \{1, 5\} = D$$



The largest interval on which $p(x)$, $q(x)$, $f(x)$ are all continuous is $I = (1, 5)$. By the existence & uniqueness th., the sol. to the IVP exists uniquely on I.

Linearly Independent Solutions

Def Linear Independence of Two Functions

Two functions defined on an open interval I are said to be linearly independent on I provided neither is a constant multiple of the other.

$f(x) = k \cdot g(x)$ $\quad k \neq 0 \Rightarrow f(x)$ and $g(x)$ are linearly dependent

Def Suppose $f(x)$ and $g(x)$ are defined on I.

$c_1 f(x) + c_2 g(x) = 0 \implies c_1 = c_2 = 0$

Then we say $f(x)$ and $g(x)$ are lin. independent!!

Ex $3\sin x$ and $2\sin x$ are linearly dependent, as there are non zero constants for which the linear combination is zero:

$$(-2).(3\sin x)+(3).(2\sin x)=0$$

$$c_1=-2, \quad c_2=3$$

Ex Is $\sin x$ and $\cos x$ linearly dependent / independent??

They're linearly ind. if $c_1\cos x+c_2\sin x=0$

$$\Rightarrow c_1=c_2=0$$

$$\left.\begin{array}{l} c_1 \cos x + c_2 \sin x = 0 \\ c_1 \cdot (-\sin x) + c_2 \cdot \cos x = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 \sin x \cos x + c_2 \sin^2 x = 0 \\ -c_1 \sin x \cos x + c_2 \cos^2 x = 0 \end{array}$$

$$+ \quad \Rightarrow c_2(\sin^2 x + \cos^2 x) = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$$

$\cos x$ and $\sin x$ are linearly independent!!

or

$$\underbrace{\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underset{\sim}{0}}$$

A hom. linear system of eqs.

$$\underset{\sim}{A}\ \underset{\sim}{c} = \underset{\sim}{0} \qquad |A| = \cos^2 x + \sin^2 x \neq 0 \rightarrow A^{-1} \text{ exists}$$

$$A^{-1} A c = A^{-1} 0 \Rightarrow \underset{\sim}{c} = \underset{\sim}{0} \Rightarrow c_1 = c_2 = 0$$

5.1, Intro. to Second-Order Linear Eqs, contd':

Two functions $f(x)$ & $g(x)$ are linearly independent if

$$c_1 f(x) + c_2 g(x) = 0 \Rightarrow c_1 = c_2 = 0.$$

Suppose $f(x)$ & $g(x) \in C^1(a,b)$, the space of continuously differentiable functions on (a,b).

$$\left.\begin{aligned} c_1 f(x) + c_2 g(x) &= 0 \\ c_1 f'(x) + c_2 g'(x) &= 0 \end{aligned}\right\} x = x_0 \qquad \begin{aligned} c_1 f(x_0) + c_2 g(x_0) &= 0 \\ c_2 f'(x_0) + c_1 g'(x_0) &= 0 \end{aligned}$$

$$\begin{bmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\underset{\sim}{c}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underset{\sim}{0}} \qquad \underset{\sim}{A}\,\underset{\sim}{c} = \underset{\sim}{0}$$

(i) $\det A \neq 0$ $\quad A^{-1}$ exists $\quad A^{-1} A c = A^{-1} 0$

$$\underset{\sim}{c} = \underset{\sim}{0} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad c_1 = c_2 = 0$$

$\det A \neq 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow f$ & g are linearly independent

(ii) $\det A = 0$ $\quad$ inf many sols., other than

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{linearly dependent}$$

$$c_1 f(x_0) + c_2 g(x_0) = 0$$

$$c_1 f'(x_0) + c_2 g'(x_0) = 0$$

$\det A = 0 \Rightarrow f(x_0) g'(x_0) - g(x_0) f'(x_0) = 0$

assume $g'(x_0) \neq 0 \Rightarrow f(x_0) = g(x_0) \dfrac{f'(x_0)}{g'(x_0)}$

$$c_1 \cdot \frac{g(x_0) f'(x_0)}{g'(x_0)} + c_2 g(x_0) = 0$$

$\rightarrow \boxed{c_1 = k c_2}$

$\Rightarrow f$ & g are linearly dependent.

Define
$$W(f(x), g(x)) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}.$$

where $f(x)$ and $g(x)$ are continuously diff. on $I = (a,b)$.

If there's at least one point $x_0 \in I(a,b)$ such that
$$W(f(x), g(x))\Big|_{x=x_0} \neq 0$$

then f & g are linearly independent.

$W(f(x), g(x))$: Wronski determinant / Wronskian of f & g

Ex $\quad f(x) = 2\sin x, \quad g(x) = 3\sin x$

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 2\sin x & 3\sin x \\ 2\cos x & 3\cos x \end{vmatrix}$$

$$= 0 \Rightarrow f \,\&\, g \text{ are linearly dependent}$$

Ex $\quad f(x) = \sin x \quad g(x) = \cos x$

$$W(f,g) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0$$

$\Rightarrow$ $\sin x$ and $\cos x$ are linearly independent

Ex Are $f(x) = e^{r_1 x}$ $g(x) = e^{r_2 x}$

linearly dependent / independent?

$$W(f, g) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= (r_2 - r_1)\, e^{(r_1 + r_2)x} = \begin{cases} \neq 0 & r_1 \neq r_2 \\ = 0 & r_1 = r_2 \end{cases}$$

$\Rightarrow e^{r_1 x}, e^{r_2 x}$ are linearly $\longrightarrow$ dependent if $r_1 = r_2$

$\longrightarrow$ independent if $r_1 \neq r_2$

Linear Second-Order Homogeneous Equations with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad a \neq 0,\ b, c \in \mathbb{R}$$

$$ay'' + by' + cy = 0$$

Let's propose a solution of the form $y = e^{r.x}$, where r is a real constant.

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2e^{rx}$$

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$e^{rx}(ar^2 + br + c) = 0 \xrightarrow{e^{rx} \neq 0} ar^2 + br + c = 0$$

$$ay'' + by' + cy = 0 \xrightarrow{y = e^{rx}} ar^2 + br + c = 0$$

$$r_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a} \qquad \Delta = b^2 - ac$$

$r_1, r_2 \in \mathbb{R},\ r_1 \neq r_2$

$y_1(x) = e^{r_1 x}$

$y_2(x) = e^{r_2 x}$

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

(general solution)

$r_1 = r_2 \in \mathbb{R}$

$y_1 = e^{r_1 x}$

$y_2 = x e^{r_1 x}$

$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

$r_{1,2} = \alpha \mp \beta i \in \mathbb{C}$

$y_1 = e^{\alpha x} \cos(\beta x)$

$y_2 = e^{\alpha x} \sin \beta(x)$

$$y = e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$$

$ar^2 + br + c = 0$ is called the "characteristic eq"

⊛ Real, Distinct Roots of the Characteristic Equation

Ex Find the general solution of $2y'' - 7y' + 3y = 0$

$y = e^{rx} \Rightarrow 2r^2 e^{rx} - 7re^{rx} + 3e^{rx} = 0$

$2r^2 - 7 + 3 = 0$

$(2r-1)(r-3) \Rightarrow r_1 = \frac{1}{2}, \quad r_2 = 3$

$y_1(x) = e^{r_1 x} = e^{\frac{1}{2}x}$

$y_2(x) = e^{r_2 x} \quad e^{3x}$

The eq. $2y'' - 7y' + 3y = 0$ belongs to the family $y'' + p(x)\, y' + q(x)\, y = 0$, which is a linear, homogeneous eq.

If $y_1(x)$ and $y_2(x)$ solve $y'' + p(x)\, y' + q(x) y = 0$, so does $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$

$$y_1(x) = e^{\frac{1}{2}x}$$
$$y_2(x) = e^{3x}$$

$\Rightarrow$ $y(x) = c_1 e^{\frac{x}{2}} + c_2 e^{3x}$

is also a solution to

$2y'' - 7y' + 3y = 0$, and is the general solution (that is: all the solutions are expressible in this form; proof: later!!)

Ex $y'' + 2y' = 0$ $\quad y = e^{rx}$

$a = 1, \quad b = 2, \quad c = 0 \rightarrow ar^2 + br + c = 0$

$1 \cdot r^2 + 2 \cdot r + 0 = 0 \qquad r(r+2) = 0$

$r_1 = 0, \quad r_2 = -2 \qquad y_1 = e^{r_1 x} = e^{0 \cdot x} = 1$

$y_2 = e^{r_2 x} = e^{-2x}$

$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 \cdot 1 + c_2 e^{-2x}$

* Repeated Roots of the Characteristic Equation

$$ay'' + by' + cy = 0 \xrightarrow{y=e^{rx}} ar^2 + br + c = 0 \qquad r_{1,2} = \frac{-b \mp \sqrt{\Delta}}{2a}$$

In this case, we (must) have $\Delta = b^2 - 4ac = 0$

$$r_1 = r_2 = \frac{-b}{2a}$$

So, one solution we can find is $y_1(x) = e^{-\frac{b}{2a}x}$

Question Can we find a second solution $y_2(x)$ to the eq. $ay'' + by' + cy = 0$, which is linearly independent from $y_1(x)$?

Reduction of order Assume $y_2(x) = u(x)\, y_1(x)$

$$y(x) = u(x)\, e^{-\frac{b}{2a}x} \qquad ay'' + by' + cy = 0$$

$$y' = u' e^{-\frac{b}{2a}x} - \frac{b}{2a} u e^{-\frac{b}{2a}x}$$

$$y'' = u'' e^{-\frac{b}{2a}x} - \frac{b}{2a} u' e^{-\frac{b}{2a}x} - \frac{b}{2a} u' e^{-\frac{b}{2a}x} + \left(\frac{b}{2a}\right)^2 u e^{-\frac{b}{2a}x}$$

$$y'' = u'' e^{-\frac{b}{2a}x} - \frac{b}{a} u' e^{-\frac{b}{2a}x} + \frac{b^2}{4a^2} u e^{-\frac{b}{2a}x}$$

$$a\left\{u'' e^{-\frac{b}{2a}x} - \frac{b}{a} u' e^{-\frac{b}{2a}x} + \frac{b^2}{4a^2} u e^{-\frac{b}{2a}x}\right\}$$

$$+ b\left\{u' e^{-\frac{b}{2a}x} - \frac{b}{2a} u e^{-\frac{b}{2a}x}\right\} + c u e^{-\frac{b}{2a}x} = 0$$

$$a u'' - b u' + \frac{b^2}{4a} u + b u' - \frac{b^2}{2a} u + c u = 0$$

$$a u'' + u\left(c - \frac{b^2}{4a}\right) = 0$$

$$a u'' - u\overbrace{\left(\frac{b^2 - 4ac}{4ac}\right)}^{=0} = 0 \qquad u'' = 0$$

$$u(x) = c_2 x + c_1$$

$$y(x) = u(x)\, y_1(x) = (c_2 x + c_1)\, e^{-\frac{b}{2a}x}$$

$$y(x) = c_1 \underbrace{e^{-\frac{b}{2a}x}}_{y_1(x)} + c_2 \underbrace{x\, e^{-\frac{b}{2a}x}}_{y_2(x)}$$

HW show that $y_1 = e^{-\frac{b}{2a}x}$ $y_2 = x e^{-\frac{b}{2a}x}$

are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \cdots \neq 0$$

Ex $y'' - 2y' + y' = 0$

$y = e^{rx} \longrightarrow r^2 - 2r + 1 = 0 \rightarrow (r-1)^2 = 0$

$r_1 = r_2 = 1$ $\qquad y_1(x) = e^{1 \cdot x}$

$y_2 = x\, y_1 = x e^{x}$

$y = c_1 y_1 + c_2 y_2 = c_1 e^{x} + c_2 x e^{x}$

* Complex Roots of the Characteristic Equation

$$ay'' + by' + cy = 0 \xrightarrow{y=e^{rx}} ar^2 + br + c = 0$$

$$r_1 = \alpha + \beta i \quad , \quad r_2 = \alpha - \beta i \qquad r_{1,2} = \alpha \mp \beta i$$

<u>Question</u> Can we find linearly ind. sol. $y_1(x)$ $y_2(x)$ which are real valued functions?

$$y_1(x) = e^{(\alpha+\beta i)x} \quad , \quad y_2(x) = e^{(\alpha-\beta i)x}$$

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$

$$y = e^{\alpha x}\left[c_1 e^{i(\beta x)} + c_2 e^{i(-\beta x)}\right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$y = e^{\alpha x}\left\{ c_1\left[\cos\beta x + i\sin\beta x\right] + c_2\left[\cos(-\beta x) + i\sin(-\beta x)\right]\right\}$$

$$= e^{\alpha x}\left\{ c_1\left[\cos\beta x + i\sin\beta x\right] + c_2\left[\cos\beta x - i\sin\beta x\right]\right\}$$

$$= e^{\alpha x}\left\{(c_1 + c_2)\cos(\beta x) + i(c_1 - c_2)\sin(\beta x)\right\}$$

$$= \underbrace{(c_1 + c_2)}\, e^{\alpha x}\cos\beta x + \underbrace{i(c_1 - c_2)}\, e^{\alpha x}\sin\beta x$$

$$= A\, e^{\alpha x}\cos\beta x + B\, e^{\alpha x}\sin\beta x$$

Let $\quad y_1(x) = e^{\alpha x}\cos\beta x \qquad y_2(x) = e^{\alpha x}\sin\beta x$

$y_1(x)$ and $y_2(x)$ are indeed solutions, with $W(y_1, y_2) \neq 0$

DIY

To sum up; if $r_{1,2} = \alpha \mp i\beta$

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$$

$$y = e^{\alpha x} \left[c_1 \cos(\beta x) + c_2 \sin\beta(x) \right]$$

Ex $y'' + 3y' + 4y = 0$ Write the general sol.

$y = e^{rx} \rightarrow r^2 + 3r + 4 = 0$

$$r_{1,2} = \frac{-3 \mp \sqrt{3^2 - 4.1.4}}{2.1} = \frac{-3 \mp \sqrt{-7}}{2}$$

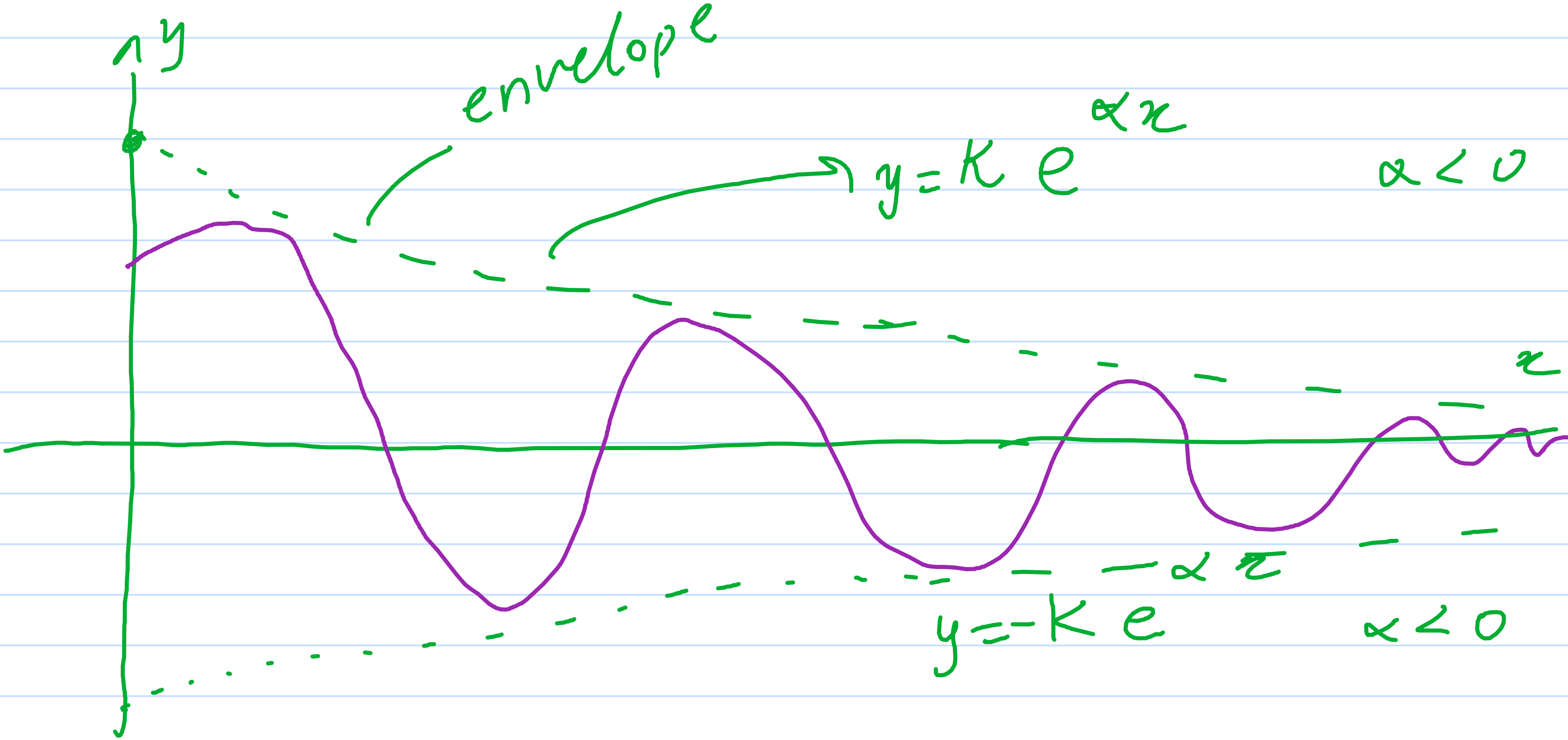
$$= \frac{-3 \mp \sqrt{7 i^2}}{2} = \overset{+}{\frac{-3}{2}} \mp \frac{\sqrt{7}}{2} i$$

$\alpha = \frac{-3}{2}$

$\beta = \frac{\sqrt{7}}{2}$

$$y = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right]$$

$$y = e^{-\frac{3}{2}x}\left[C_1 \cos \frac{\sqrt{7}}{2} x + C_2 \sin \frac{\sqrt{7}}{2} x\right]$$



$$y(x) = e^{\alpha x}\left[C_1 \cos(\beta x) + C_2 \sin(\beta x)\right]$$

Question We said that, if $y_1(x)$ and $y_2(x)$ are linearly independent solutions to

$$ay'' + by' + cy = 0$$

then, $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution to this equation, without proof.

Now comes the proof. We will show that, for any $\phi(x)$ that solves

$$\boxed{y'' + p(x)y' + q(x)y = 0},$$

there's a choice of numbers c_1 & c_2 such that

$$\phi(x) = c_1 y_1(x) + c_1 y_2(x)$$

where y_1 & y_2 are linearly ind. sols. to $y'' + py' + qy = 0$

Theorem Suppose y_1 and y_2 solve

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad (1)$$

we assign initial conds

$$y(x_0) = a, \qquad y'(x_0) = b \qquad (2)$$

It's always possible to choose c_1 and c_2 to satisfy ICs (2) if and only if y_1 and y_2 are linearly indepedent.

Proof (1) is linear and homogeneous. y_1 & y_2 solve (1) $\Rightarrow$ $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution. Now, can this superposition satisfy ICs (2) for some choice of the numbers c_1 & c_2 ??

$$\left.\begin{aligned} y(x_0) &= c_1\, y_1(x_0) + c_2\, y_2(x_0) = a \\ y'(x_0) &= c_1\, y_1'(x_0) + c_2\, y_2'(x_0) = b \end{aligned}\right\} \quad \begin{aligned} c_1 &= ? \\ c_2 &= ? \end{aligned}$$

$$\underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

If $W(y_1(x), y_2(x))\Big|_{x=x_0} \neq 0 \Rightarrow A^{-1}$ exists

and we can find c_1 and c_2 uniquely.

In that case, the IVP has a unique solution.

This is possible, if y_1 & y_2 are linearly independent!!!

Theorem Suppose $y_1(x)$ and $y_2(x)$ are two solutions to

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad (*)$$

Then, the family of solutions

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

is the general solution; it includes every solution, if and only if $W(y_1, y_2) \neq 0$. It's enough that we find two linearly ind. sols. of $(*)$; the other sols. are linear combinations of y_1 & y_2.

Proof Let $\phi(x)$ be any solution to

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad (*)$$

for which we know two linearly independent sols. y_1 & y_2.
We must show $\phi(x)$ is included in the linear combos. $c_1 y_1(x) + c_2 y_2(x)$; in other words, we must prove that, for some selection of the constants c_1 and c_2, we have $c_1 y_1(x) + c_2 y_2(x) = \phi(x)$.

Since y_1 and y_2 are linearly independent, there's at least one point x_0 such that

$$W\big(y_1(x), y_2(x)\big)\Big|_{x=x_0} \neq 0.$$

Define $a = \phi(x_0)$; $b = \phi'(x_0)$ and set up the problem

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad *$$

$$y(x_0) = a, \quad y'(x_0) = b$$

$\phi(x)$ is a solution to this problem.

$$\phi'' + p(x)\, \phi' + q(x)\, \phi = 0$$

$$\phi(x_0) = a, \quad \phi'(x_0) = b$$

$y = c_1 y_1(x) + c_2 y_2(x)$ solves $*$

Can $y = c_1 y_1(x) + c_2 y_2(x)$ satisfy the ICs

$$c_1 \, y_1(x_0) + c_2 \, y_2(x_0) = a$$
$$c_1 \, y_1'(x_0) + c_2 \, y_2'(x_0) = b$$

Can we determine c_1 & c_2 successfully,

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$c_1 = \ldots$

$c_2 = \ldots$

$\hookrightarrow \det \neq 0$

By the existence & uniqueness th., if $p(x)$ & $q(x)$ are continuous, the IVP must have 1 and only 1 sol. Therefore, for the values of c_1 and c_2 found above

$$c_1 \, y_1(x) + c_2 \, y_2(x) = \phi(x)$$

Since we can do this for any sol. $\phi(x)$, any sol. is a linear comb. of y_1 & y_2. ∎