EIGENVALUES AND EIGENVECTORS

Let A be an $n \times n$ matrix. If there exists a nonzero vector v and a number λ such that

$$Av = \lambda v,$$

then v is called an eigenvector whereas λ is called the corresponding eigenvalue.

PROPERTY: If λ is the eigenvalue for the eigenvector v, then $u = kv$ $(k \neq 0)$ is also an eigenvector associated with λ.

$$Av = \lambda v \Rightarrow Au = A(kv) = k(Av) = k(\lambda v) \Rightarrow Au = \lambda(kv) = \lambda u$$

How to find eigenvalues and eigenvectors

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0$$

$\downarrow$

a homog. syst with n eq. and n unknowns

$\downarrow$

This kind of a system has a nontrivial solution $(v \neq 0)$ if and only if

$|A - \lambda I| = 0$ $\rightarrow$ characteristic eq.

$\downarrow$

Find the eigenvalues λ

$\downarrow$

Solve $(A - \lambda I)v = 0$ for each eigenvalue λ to find the corresponding eigenvector v

** Distinct real eigenvalues, each corresponding to a single eigenvector

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \Rightarrow (A-\lambda I)v = 0 \Rightarrow \begin{bmatrix} 1-\lambda & -2 \\ -3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|A-\lambda I| = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0 \Rightarrow 2-\lambda-2\lambda+\lambda^2-6=0 \Rightarrow \lambda^2-3\lambda-4=0$$

$$\Rightarrow (\lambda+1)(\lambda-4) = 0 \Rightarrow \lambda_1 = -1 \text{ and } \lambda_2 = 4$$

$$\lambda_1 = -1 \Rightarrow \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 2x-2y=0 \\ -3x+3y=0 \end{matrix}\right\} x=y,\ x=1 \Rightarrow y=1 \Rightarrow v^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\lambda_2 = 4 \Rightarrow \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -3x-2y=0 \Rightarrow 3x=-2y,\ y=-3 \Rightarrow x=2 \Rightarrow v^{(2)} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

** A single real eigenvalue corresponding to a single eigenvector

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \Rightarrow (A-\lambda I)v = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|A-\lambda I| = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \lambda = 2 \text{ (multiplicity 2)}$$

$$\lambda=2 \Rightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3y=0 \Rightarrow y=0,\ x=1 \Rightarrow v^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

** A single real eigenvalue corresponding to more than one linearly independent eigenvectors

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow |A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \lambda = 2$$

multiplicity 2

$$\lambda=2 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 0x+0y=0 \Rightarrow \begin{matrix} x=1, y=0 \\ y=1, x=0 \end{matrix} \Rightarrow v^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ v^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

★★ Two complex conjugate eigenvalues corresponding to complex conjugate eigenvectors

$$A = \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix} \Rightarrow |A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 5 \\ -1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow -(1-\lambda)(1+\lambda)+5=0$$

$$\Rightarrow 1-\lambda^2 = 5 \Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm 2i$$

$$\lambda_1 = 2i \Rightarrow \begin{bmatrix} 1-2i & 5 \\ -1 & -1-2i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} (1-2i)x + 5y = 0 \\ -x-(1+2i)y = 0 \end{matrix} \Rightarrow \begin{matrix} x = -(1+2i)y \\ y=-1 \Rightarrow x = 1+2i \end{matrix}$$

$$\lambda_2 = -2i \Rightarrow \begin{bmatrix} 1+2i & -5 \\ -1 & -1+2i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} (1+2i)x - 5y = 0 \\ -x+(-1+2i)y = 0 \end{matrix} \Rightarrow \begin{matrix} x = (-1+2i)y \\ y=-1 \Rightarrow x = 1-2i \end{matrix}$$

$$\Rightarrow v^{(1)} = \begin{bmatrix} 1+2i \\ -1 \end{bmatrix}, \quad v^{(2)} = \begin{bmatrix} 1-2i \\ -1 \end{bmatrix}$$ $\Rightarrow$ pay attention that $v^{(2)}$ is also the conjugate of $v^{(1)}$

★★ Be careful, we're looking for a nonzero vector v but λ can be 0.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix} \Rightarrow |A-\lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -4 & 6-\lambda & 2 \\ 16 & -15 & -5-\lambda \end{vmatrix} = \lambda(\lambda-1)(3-\lambda) = 0 \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 3 \end{matrix}$$

$$\lambda_1 = 0 \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 3x = 0 \Rightarrow x = 0 \\ \left.\begin{matrix} 6y+2z=0 \\ -15y-5z=0 \end{matrix}\right\} \begin{matrix} z=-3y \\ y=1 \\ z=-3 \end{matrix} \end{matrix} \Rightarrow v^{(1)} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

$$\lambda_2 = 1 \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2x = 0 \Rightarrow x = 0 \\ \left.\begin{matrix} 5y+2z=0 \\ -15y-6z=0 \end{matrix}\right\} \begin{matrix} 2z=-5y \\ z=-5 \\ y=2 \end{matrix} \end{matrix} \Rightarrow v^{(2)} = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}$$

$$\lambda_3 = 3 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ 16 & -15 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x: \text{free var} \\ x = 1 \\ \left.\begin{matrix} 3y+2z=4 \\ 15y+8z=16 \end{matrix}\right\} \begin{matrix} 2z=4 \\ z=2 \\ y=0 \end{matrix} \end{matrix} \Rightarrow v^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

★ $\lambda=0$ if and only if $|A|=0$. $(|A-\lambda I|=|A|)$

EIGENSPACES

Let λ be a fixed eigenvalue of the $n\times n$ matrix A. Then, the set of all eigenvectors is the set of all nonzero solution vectors of the system $(A-\lambda I)v=0$. The solution space of this system is called the eigenspace of A associated with the eigenvalue λ.

Ex $A=\begin{bmatrix}4&-2&1\\2&0&1\\2&-2&3\end{bmatrix} \Rightarrow |A-\lambda I|=\begin{vmatrix}4-\lambda&-2&1\\2&-\lambda&1\\2&-2&3-\lambda\end{vmatrix}=-\lambda^3+7\lambda^2-16\lambda+12$

$-(\lambda^3-7\lambda^2+16\lambda-12)=-(\lambda-2)(\lambda^2-5\lambda+6)=-(\lambda-2)^2(\lambda-3)=0$

$\lambda_1=2$ (multiplicity 2) $\Rightarrow \begin{bmatrix}2&-2&1\\2&-2&1\\2&-2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow 2x-2y+z=0$

$\left.\begin{array}{l} z=0 \Rightarrow 2x-2y=0 \Rightarrow x=y=1 \\ y=0 \Rightarrow 2x+z=0 \Rightarrow z=-2x,\ x=1 \Rightarrow z=-2 \end{array}\right\} v_1^{(1)}=\begin{bmatrix}1\\1\\0\end{bmatrix} \quad v_2^{(1)}=\begin{bmatrix}1\\0\\-2\end{bmatrix}$

$\Rightarrow$ the eigenspace of A associated with $\lambda=2$ has basis $\{v_1^{(1)}, v_2^{(1)}\}$

$\lambda_2=3 \Rightarrow \begin{bmatrix}1&-2&1\\2&-3&1\\2&-2&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{array}{l} 2x-2y=0 \Rightarrow x=y \\ \left.\begin{array}{l} x-2y+z=0 \\ 2x-3y+z=0 \end{array}\right] \begin{array}{l}-x+z=0\\ z=x\end{array} \Rightarrow x=y=z=1 \end{array}$

$v^{(2)}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\Rightarrow$ the eigenspace of A associated with $\lambda=3$ has basis $\{v^{(2)}\}$.

DIAGONALIZATION

$$P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad Av = \lambda v$$

P ↓ eigenvector matrix

D ↓ diagonal eigenvalue matrix

$\lambda_1, \lambda_2, \ldots, \lambda_n$: n eigenvalues , $v_1, v_2, \ldots, v_n$: corresponding n linearly independent eigenvectors

$$AP = \begin{bmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix} = PD$$

$$\Rightarrow A = PDP^{-1} \quad \text{or} \quad D = P^{-1}AP$$

Ex $A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}, \quad v^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v^{(2)} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \lambda_1 = -1, \ \lambda_2 = 4$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}, \quad P^{-1} = -\frac{1}{5}\begin{bmatrix} -3 & -2 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$P.D.P^{-1} = \begin{bmatrix} -1 & 8 \\ -1 & -12 \end{bmatrix}\begin{bmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{bmatrix} = A$$

SIMILAR MATRICES

Let A and B be $n \times n$ matrices. A and B are called similar if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

A is called diagonalizable if it's similar to a diagonal matrix D.

THEOREM The $n\times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof: We showed before that if A has n lin. indep. eigenvectors, then A is diagonalizable.

Let A be the $n\times n$ diagonalizable matrix and let $D=P^{-1}AP$ where $P=[v_1\ v_2 \cdots v_n]$ and $D=[d_1 d_2 \cdots d_n]$.

$$AP=PD \Rightarrow [Av_1\ Av_2 \ldots Av_n] = [v_1 d_1\ v_2 d_2 \ldots v_n d_n]$$

$$\Rightarrow Av_i = v_i d_i\ ,\quad i=1,2,\ldots,n$$

This means that $v_1, v_2, \ldots, v_n$ are eigenvectors of A.

Since $v_1, v_2, \ldots, v_n$ are column vectors of an invertible matrix P ($\det P \neq 0$), then they are lin. independent.

Ex $A=\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda=2$ (mult. 2), $v^{(1)}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

A does not have $n=2$ lin. indep. eigenvectors $\Rightarrow A$ is not diagonalizable.

Ex $A=\begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$, $\begin{array}{l}\lambda_1=0\\ \lambda_2=1\\ \lambda_3=3\end{array}$, $v^{(1)}=\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, $v^{(2)}=\begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}$, $v^{(3)}=\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

A has 3 lin. indep. vect $\Rightarrow A$ is diagonalizable

$P=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ -3 & -5 & 2 \end{bmatrix}$, $D=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $P^{-1}=\begin{bmatrix} 4 & -5 & -2 \\ -2 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$\Rightarrow D=P^{-1}A\,P$$

$$=P^{-1}\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

THEOREM Let the eigenvectors $v_1, v_2, \dots, v_k$ be associated with the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. Then, these k eigenvalues are linearly independent.

Proof: To prove the theorem, let's use mathematical induction

* The case is true for $k=1$ since any single eigenvector constitutes a lin. indep. set ($c_1 v_1 = 0$, $v_1 \neq 0 \Rightarrow c_1 = 0$)
* Let's assume that k-1 eigenvectors $v_2, v_3, \dots, v_k$ are lin indep and let's prove that k eig.vec. are also lin ind.

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$$

$$\Rightarrow c_1 \underbrace{(A - \lambda_1 I) v_1}_{=0 \text{ since } A_1 v_1 = \lambda_1 v_1} + c_2 (A - \lambda_1 I) v_2 + \dots + c_k (A - \lambda_1 I) v_k = 0$$

$Av_2 = \lambda_2 v_2 \Rightarrow Av_2 - \lambda_1 v_2 = \lambda_2 v_2 - \lambda_1 v_2 = (\lambda_2 - \lambda_1) v_2$ and so on.

$$\Rightarrow c_2 \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} v_2 + c_3 \underbrace{(\lambda_3 - \lambda_1)}_{\neq 0} v_3 + \dots + c_k \underbrace{(\lambda_k - \lambda_1)}_{\neq 0} v_k = 0$$

since $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct

$$\Rightarrow c_2 = c_3 = \dots = c_k = 0$$

$$\Rightarrow c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0 \Rightarrow c_1 v_1 = 0 \Rightarrow c_1 = 0$$

THEOREM If the nxn matrix A has n distinct eigenvalues, then it is diagonalizable.

Important Note: This theorem does not say that if the nxn matrix has k, (k<n) distinct eigenvalues, the it is not diagonalizable. Be careful!!!

Ex $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$, $\lambda_1 = 2$ (mult. 2), $v_1^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$\lambda_2 = 3 \Rightarrow \quad v^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

There are 2 eigenvalues but 3 lin ind. eigenvectors

$\Rightarrow$ A is diagonalizable

THEOREM Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of the $n \times n$ matrix A. Let S_i be the basis for the eigenspace associated with λ_i. Then the union S of the basis $S_1, S_2, \dots, S_k$ is a linearly independent set of eigenvectors of A.

POWERS OF MATRICES

Let A be diagonalizable

$$A = PDP^{-1}$$

$$A^2 = (PDP^{-1})(P.DP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PD^2P^{-1}$$

....

$$A^k = PD^kP^{-1}$$

and

if $D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$, then $D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$

Ex

$$A=\begin{bmatrix}4 & -2 & 1\\ 2 & 0 & 1\\ 2 & 2 & 3\end{bmatrix} \qquad \lambda_1=3 \Rightarrow v^{(1)}=[1\ 1\ 1]^T$$

$$\underset{\text{mult. } 2}{\lambda_2=2} \Rightarrow v_1^{(2)}=[1\ 1\ 0]^T,\ v_2^{(2)}=[-1\ 0\ 2]^T$$

$$\Rightarrow P=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & 0\\ 1 & 0 & 2\end{bmatrix},\quad D=\begin{bmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 2 & 0 & 0 & 1\end{array}\right] \xrightarrow[R_3-R_1\to R_3]{R_2-R_1\to R_2} \left[\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 1 & 0\\ 0 & -1 & 3 & -1 & 0 & 1\end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|ccc}1 & 1 & -1 & 1 & 0 & 0\\ 0 & -1 & 3 & -1 & 0 & 1\\ 0 & 0 & 1 & -1 & 1 & 0\end{array}\right]$$

$$\xrightarrow[-R_2\to R_2]{R_1+R_2\to R_1} \left[\begin{array}{ccc|ccc}1 & 0 & 2 & 0 & 0 & 1\\ 0 & 1 & -3 & 1 & 0 & -1\\ 0 & 0 & 1 & -1 & 1 & 0\end{array}\right] \xrightarrow[3R_3+R_2\to R_2]{-2R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 2 & -2 & 1\\ 0 & 1 & 0 & -2 & 3 & -1\\ 0 & 0 & 1 & -1 & 1 & 0\end{array}\right]$$

$$\underbrace{\begin{bmatrix}2 & -2 & 1\\ -2 & 3 & -1\\ -1 & 1 & 0\end{bmatrix}}_{=P^{-1}}$$

$$D^5=\begin{bmatrix}3^5 & 0 & 0\\ 0 & 2^5 & 0\\ 0 & 0 & 2^5\end{bmatrix}$$

$$\Rightarrow A^5=\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & 0\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}243 & 0 & 0\\ 0 & 32 & 0\\ 0 & 0 & 32\end{bmatrix}\begin{bmatrix}2 & -2 & 1\\ -2 & 3 & -1\\ -1 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}243 & 32 & -32\\ 243 & 32 & 0\\ 243 & 0 & 64\end{bmatrix}\begin{bmatrix}2 & -2 & 1\\ -2 & 3 & -1\\ -1 & 1 & 0\end{bmatrix}=\begin{bmatrix}454 & -422 & 211\\ 422 & -390 & 211\\ 422 & -422 & 243\end{bmatrix}$$

CAYLEY-HAMILTON

If the $n \times n$ matrix A has the characteristic polynomial

$$p(\lambda) = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_2 \lambda^2 + c_1 \lambda + c_0 ,$$

then

$$p(A) = (-1)^n A^n + c_{n-1} A^{n-1} + \cdots + c_2 A^2 + c_1 A + c_0 I = 0.$$

Proof: We will prove this when A is diagonalizable.

$A^k = P.D^k P^{-1}$

$$p(D) = (-1)^n D^n + c_{n-1} D^{n-1} + \cdots + c_2 D^2 + c_1 D + c_0 I$$

$$= (-1)^n \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n^n \end{bmatrix} + c_{n-1} \begin{bmatrix} \lambda_1^{n-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{n-1} & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n^{n-1} \end{bmatrix} + \cdots + c_1 \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} + c_0 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & p(\lambda_2) & \cdots & 0 \\ 0 & 0 & \cdots & p(\lambda_n) \end{bmatrix} = 0 \text{ since } p(\lambda_k) = 0 \text{ for } k = 1, 2, \ldots, n.$$

$$p(A) = (-1)^n P D^n P^{-1} + c_{n-1} P D^{n-1} P^{-1} + \cdots + c_1 PDP + c_0 PIP^{-1}$$

$$= P \{ \underbrace{(-1)^n D^n + c_{n-1} D^{n-1} + \cdots + c_1 D + c_0 I}_{= p(D)} \} P^{-1}$$

$$= P.0.P^{-1} = 0$$

Ex $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$p(\lambda) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \Rightarrow -A^3 + 7A^2 - 16A + 12I = 0$

$A^2 = \begin{bmatrix} 14 & -10 & 5 \\ 10 & -6 & 5 \\ 10 & -10 & 9 \end{bmatrix}$

$\Rightarrow A^3 = 7A^2 - 16A + 12I = \begin{bmatrix} 46 & -38 & 19 \\ 38 & -30 & 19 \\ 38 & -38 & 27 \end{bmatrix}$

$\Rightarrow A^4 = 7A^3 - 16A^2 + 12A = 7(7A^2 - 16A + 12I) - 16A^2 + 12A$

$= 33A^2 - 100A + 84I$

$= \begin{bmatrix} 146 & -130 & 65 \\ 130 & -114 & 65 \\ 130 & -130 & 81 \end{bmatrix}$

$-A^3 + 7A^2 - 16A + 12I = 0 \Rightarrow 12I = A^3 - 7A^2 + 16A$

$\Rightarrow 12A^{-1} = A^2 - 7A + 16I$

$\Rightarrow A^{-1} = \frac{1}{12}(A^2 - 7A + 16I) = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ -2 & 5 & -1 \\ -2 & 2 & 2 \end{bmatrix}$