Review of first order ODEs

19 July 2018

2a) $$\frac{dy}{dx} = \frac{x+2y+2}{2x-4} = \frac{x+2}{2x-4} + \frac{2}{2x-4}y$$

$$y' - \frac{1}{x-2}y = \frac{x+2}{2x-4}$$

$$y' + p(x)y = g(x)$$

$$p(x) = -\frac{1}{x-2}, \quad g(x) = --$$

$$\mu = e^{\int p(x)dx} = e^{\int \frac{-1}{x-2}dx}$$

$$= e^{-\ln|x-2|} = (x-2)^{-1} = \frac{1}{x-2}$$

$$\frac{1}{x-2}y' - \frac{1}{(x-2)^2}y = \frac{x+2}{2(x-2)^2}$$

$$\frac{d}{dx}(\mu y) = \mu g(x)$$

$$\frac{d}{dx}\left[\frac{1}{x-2}\cdot y\right] = \frac{x+2}{2(x-2)^2}$$

$$\frac{y}{x-2} = \int \frac{x+2}{2(x-2)^2}\,dx = \int \frac{x-2+4}{2(x-2)^2}\,dx$$

$$\left\{ \frac{x+2}{2(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} \right\}$$

$$\frac{y}{x-2} = \frac{1}{2}\int \frac{dx}{x-2} + 2\int \frac{dx}{(x-2)^2}$$

$$\frac{y}{x-2} = \frac{1}{2}\ln|x-2| + 2\cdot\frac{(-1)}{x-2} + C$$

$$y(x) = \frac{x-2}{2}\ln|x-2| - 2 + c(x-2)$$

$$\int \frac{dx}{(x-2)^2} = \left\{ \begin{array}{l} u = x-2 \\ du = dx \end{array} \right\} = \int \frac{dx}{u^2} = \int u^{-2}\,du = \frac{-1}{u} + C$$

OR

$$\frac{dy}{dx} = \frac{x+2y+2}{2x-4}$$

We can cancel the constants 2 and 4 by a transformation

$x = X + h$, $y = Y + k$ where h, k are constants

$dx = dX$, $dy = dY$

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)+2}{2(X+h)-4} = \frac{X+2Y+h+2k+2}{2X+2h-}$$

choose $\left.\begin{aligned} h+2k+2 &= 0 \\ 2h-4 &= 0 \end{aligned}\right\} \quad \begin{aligned} h &= 2 \\ k &= -2 \end{aligned}$

$$\frac{dY}{dX} = \frac{X+2Y}{2X} = \frac{1}{2} + \frac{Y}{X} = F\left(\frac{Y}{X}\right)$$

First order hom. eq. $\Rightarrow$ Look at the sol. yourself!! (available in the arxiv!)

or

$$\frac{dy}{dx} = \frac{x+2y+2}{2x-4} \qquad M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\underbrace{(x+2y+2)}_{M}\,dx + \underbrace{(4-2x)}_{N}\,dy = 0 \qquad \left.\begin{aligned} M_y &= 2 \\ N_x &= -2 \end{aligned}\right\} \text{Not exact}$$

14 Sep '20

Q4 $\quad t y'' + 3y' = 3t \qquad (y' = p)$

$F(t, \cancel{y}, y', y'') = 0 \qquad F(t, y', y'') = 0$

$y' = z \Rightarrow y'' = z' \qquad F(t, z, z')$

$t z' + 3z = 3t \qquad t \frac{dz}{dt} + 3z = 3t$

$z' + \frac{3}{t} z = 3 \qquad y' + p(t)\, y = g(t)$

$\mu = e^{\int p(t)\,dt} = e^{\int \frac{3}{t}\,dt} = e^{3\ln t} = t^3$

$$t^3 z' + 3t^2 z = 3t^3$$

$$\frac{d}{dt}\left(t^3 z\right) = 3t^3 \Rightarrow t^3 z = \frac{3}{4} t^4 + C_1$$

$$z = \frac{3}{4} t + c_1 t^{-3}$$

$$\frac{dy}{dt} = \frac{3}{4} t + c_1 t^{-3}$$

$$y = \frac{3}{8} t^2 + c_1 \frac{t^{-3+1}}{-3+1} + C_2$$

$+ c_1$

$$= \frac{3}{8} t^2 - \frac{c_1}{2} \frac{1}{t^2} + C_2$$

11 Nov 2017

1a Show that $(2xy - 9x^2) + (2y + x^2 + 1)y' = 0$ is exact & find the general solution.

$\rightarrow \frac{dy}{dx}$

$y' = \frac{dy}{dx}$

$$\underbrace{(2xy - 9x^2)}_{M} dx + \underbrace{(2y + x^2 + 1)}_{N} dy = 0$$

$M_y = 2x = N_x \Rightarrow$ the eq. is exact!!

$\phi_x = M = 2xy - 9x^2$

$\phi_y = N = 2y + x^2 + 1$

$\rightarrow \boxed{\phi = yx^2 - 3x^3 + g(y)}$

$\phi_y = x^2 - 0 + g'(y)$

$= 2y + x^2 + 1$

$g'(y) = 2y + 1 \Rightarrow g(y) = y^2 + y + C$

$\boxed{\phi(x,y) = C}$

$$\phi(x,y) = yx^2 - 3x^3 + y^2 + y + C$$

$$yx^2 - 3x^3 + y^2 + y = \hat{C}$$

$$y^2 + (x^2+1)y + \hat{C} - 3x^3 = 0$$

$$y_{1,2}(x) =$$

$$\left.\begin{aligned} \phi_x &= M = 2xy - 9x^2 \\ \phi_y &= N = 2y + x^2 + 1 \end{aligned}\right\} \rightarrow \phi = yx^2 - 3x^3 + g(y)$$

$$\phi_y = x^2 - 0 + g'(y)$$

$$= 2y + x^2 + 1$$

$$\Downarrow$$

$$\boxed{\phi(x,y) = C}$$

$$g'(y) = 2y + 1 \Rightarrow g(y) = y^2 + y + C$$

16. Solve the eq. $xy' + 4y = x^4 y^2$, $x>0$

$$y' + \frac{4}{x} y = x^3 y^2 \qquad y' + p(x) y = g(x) y^n$$

$n \neq 0, 1$

$$n = 2 \qquad v = y^{1-n} = y^{1-2} = y^{-1}$$

$$v' = -1 \cdot y^{-2} \cdot y'$$

$$-y^{-2} y' - y^{-2} \frac{4}{x} y = -y^{-2} \cdot x^3 \cdot y^2$$

$$\underbrace{-y^{-2} y'}_{v'} - \frac{4}{x} y^{-1} = -x^3 \longrightarrow \boxed{v' - \frac{4}{x} v = -x^3}$$

$$\mu = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = x^{-4}$$

$$x^{-4} v' - 4x^{-5} v = -x^{-1}$$

$$\left(x^{-4} v\right)' = -\frac{1}{x} \Rightarrow x^{-4} v = -\ln x + C$$

$$v = -x^4 \ln x + C x^4$$

$$y^{-1} = ---$$

$$y = \frac{1}{x^4 (c - \ln x)}$$

$$\boxed{v' - \frac{4}{x} v = -x^3}$$

Ex (a) Show that $y_1(x) = \frac{2}{x}$ solves the DE

$$y' + y^2 = \frac{2}{x^2} \quad (*)$$

(b) Find the general solution of $(*)$ by applying a transformation of the form

$$y(x) = y_1(x) + \frac{1}{u(x)}$$

A Riccati eq. is $y' = A(x) + B(x)\,y + C(x)\,y^2$

In $(*)$, $C(x) = 1$, $B(x) = 0$, $A(x) = \frac{2}{x^2}$

$y = y_1 = \frac{2}{x}$ $\quad y' + y^2 - \frac{2}{x^2} = -\frac{2}{x^2} + \left(\frac{2}{x}\right)^2 - \frac{2}{x^2} = 0$

$y_1(x)$ indeed solves the eq.

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$y_1(x)$ indeed solves the eq.

$$y' + y^2 = \frac{2}{x^2} \qquad y = y_1 + \frac{1}{u} = \frac{2}{x} + \frac{1}{u(x)}$$

$$y' = -\frac{2}{x^2} - \frac{u'}{u^2}$$

$$-\frac{2}{x^2} - \frac{u'}{u^2} + \left(\frac{2}{x} + \frac{1}{u}\right)^2 = \frac{2}{x^2}$$

$$-\frac{2}{x^2} - \frac{u'}{u^2} + \frac{4}{x^2} + \frac{4}{xu} + \frac{1}{u^2} = \frac{2}{x^2}$$

$$-\frac{u'}{u^2} + \frac{4}{xu} + \frac{1}{u^2} = 0 \implies u' - \frac{4u}{x} - 1 = 0$$

$$\boxed{u' - \frac{4}{x}u = 1}$$

Solve this yourself!

Look at 11 Nov 2017, Question: 2a

<u>Ex</u> Determine $f(y)$ so that

$$y' = \frac{2xy - f(y)}{xy^2}$$ is an exact

DE.

$$\frac{dy}{dx} = \frac{2xy - f(y)}{xy^2} \Rightarrow \left[2xy - f(y)\right]dx - xy^2\,dy = 0$$

$$\frac{\partial}{\partial y}\left[2xy - f(y)\right] = \frac{\partial}{\partial x}\left[-xy^2\right]$$

$$2y - f'(y) = -y^2 \Rightarrow \quad f'(y) = 2y + y^2$$

$$f(y) = y^2 + \frac{y^3}{3} + C$$

24 Mar '18

(1a) Solve the IVP $\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} = 0$

$y'(0) = 1, \quad y(0) = 0$

$y'' - 2yy' = 0$ x is missing

$F(y, y', y'') = 0$ $y' = p$ $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$

$p\frac{dp}{dy} - 2yp = 0 \Rightarrow p\left(\frac{dp}{dy} - 2y\right) = 0$

(A) $p = 0 \to y' = 0$ $y(x) = C$ does not satisfy the ICs.

$$\frac{dp}{dy} - 2y = 0 \qquad \frac{dp}{dy} = 2y \rightarrow p = y^2 + C$$

$$y' = y^2 + C \xrightarrow[y'=1]{x=0:\ y=0} 1 = 0^2 + C$$

$$y' = y^2 + 1 \longrightarrow \frac{dy}{dx} = y^2 + 1$$

$$\frac{dy}{y^2+1} = dx \rightarrow \int \frac{dy}{y^2+1} = dx$$

$$\tan^{-1} y = x + C \Rightarrow y = \tan^{-1}(x + C) \,//$$