BLG 336E

Analysis of Algorithms II

Lecture 13:

NP-Completeness and Review

Basic reduction strategies

1. Reduction by simple equivalence.
2. Reduction from special case to general case.
3. Reduction by encoding with gadgets.

1. Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.

Pf. We show S is an independent set iff V – S is a vertex cover.

2. Claim. VERTEX-COVER $\leq_P$ SET-COVER.

Pf. Given a VERTEX-COVER instance G = (V, E), k, we construct a set cover instance whose size equals the size of the vertex cover instance.

3.Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Polynomial-Time Reductions

Dick Karp
(1972)
1985 Turing
Award

constraint satisfaction

3-SAT

3-SAT reduces to INDEPENDENT SET

INDEPENDENT SET

DIR-HAM-CYCLE

GRAPH 3-COLOR

SUBSET-SUM

VERTEX COVER

HAM-CYCLE

PLANAR 3-COLOR

SCHEDULING

SET COVER

TSP

packing and covering

sequencing

partitioning

numerical

Decision Problems

Decision problem.

- X is a set of strings.
- Instance: string s.
- Algorithm A solves problem X: A(s) = `yes` iff $s \in X$.

Polynomial time. Algorithm A runs in poly-time if for every string s, A(s) terminates in at most p(|s|) "steps", where p(·) is some polynomial.

length of s

PRIMES: X = { 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, …. }

Algorithm. [Agrawal-Kayal-Saxena, 2002] $p(|s|) = |s|^8$.

Definition of P

P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector x that satisfies Ax = b?	Gauss-Edmonds elimination	$\left[\begin{array}{ccc} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{array}\right], \left[\begin{array}{c} 4 \\ 2 \\ 36 \end{array}\right]$	$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right], \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$

NP

Certification algorithm intuition.

- Certifier views things from "managerial" viewpoint.
- Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.

Def. Algorithm C(s, t) is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that C(s, t) = `yes`.

"certificate" or "witness"

NP. Decision problems for which there exists a poly-time certifier.

C(s, t) is a poly-time algorithm and
$|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.

Remark. NP stands for nondeterministic polynomial-time.

Certifiers and Certificates: Composite

COMPOSITES. Given an integer s, is s composite?

Certificate. A nontrivial factor t of s. Note that such a certificate exists iff s is composite. Moreover $|t| \leq |s|$.

Certifier.

```
boolean C(s, t) {
   if (t ≤ 1 or t ≥ s)
      return false
   else if (s is a multiple of t)
      return true
   else
      return false
}
```

Instance. s = 437,669.

Certificate. t = 541 or 809. ⟵ $437{,}669 = 541 \times 809$

Conclusion. COMPOSITES is in NP.

Certifiers and Certificates: 3-Satisfiability

SAT. Given a CNF formula Φ, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

Ex.

$$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$$

instance s

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

certificate t

Conclusion. SAT is in NP.

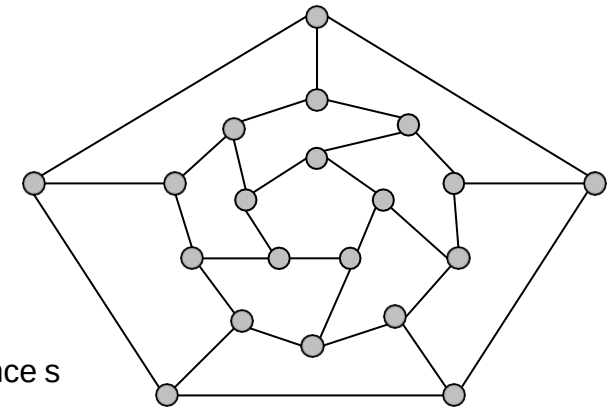
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle C that visits every node?

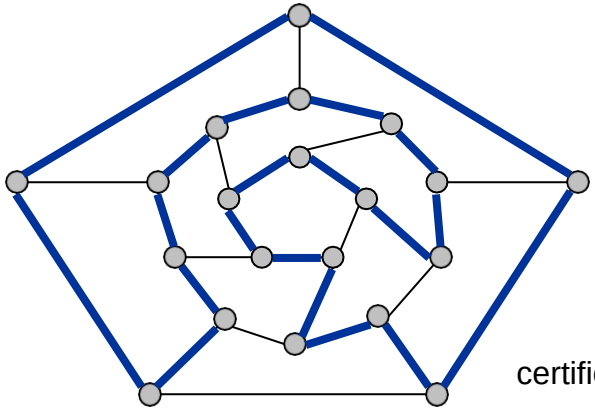
Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

Conclusion. HAM-CYCLE is in NP.



instance s



certificate t

P, NP, EXP

P. Decision problems for which there is a poly-time algorithm.

EXP. Decision problems for which there is an exponential-time algorithm.

NP. Decision problems for which there is a poly-time certifier.

Claim. $P \subseteq NP$.

Pf. Consider any problem X in P.

- By definition, there exists a poly-time algorithm $A(s)$ that solves X.
- Certificate: $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

Claim. $NP \subseteq EXP$.

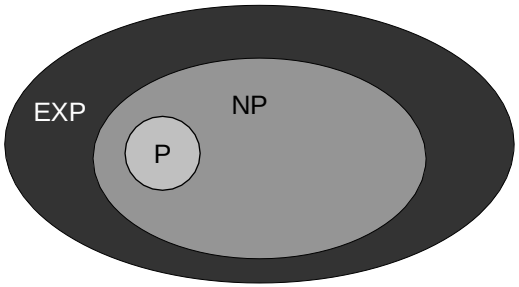
Pf. Consider any problem X in NP.

- By definition, there exists a poly-time certifier $C(s, t)$ for X.
- To solve input s, run $C(s, t)$ on all strings t with $|t| \leq p(|s|)$.
- Return `yes`, if $C(s, t)$ returns `yes` for any of these. ▪

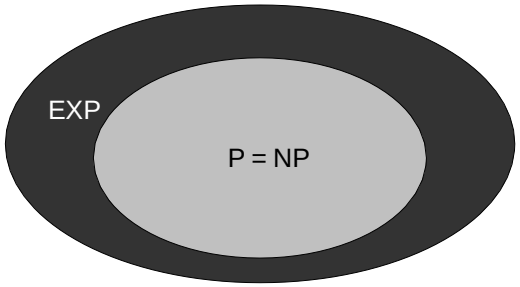
The Main Question: P Versus NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

- Is the decision problem as easy as the certification problem?
- Clay $1 million prize.



If P ≠ NP



If P = NP

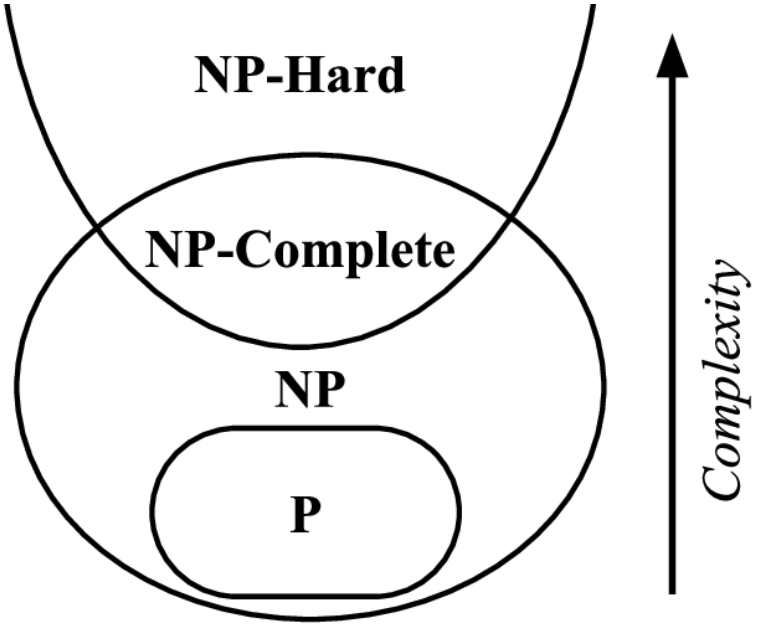
would break RSA cryptography
(and potentially collapse economy)

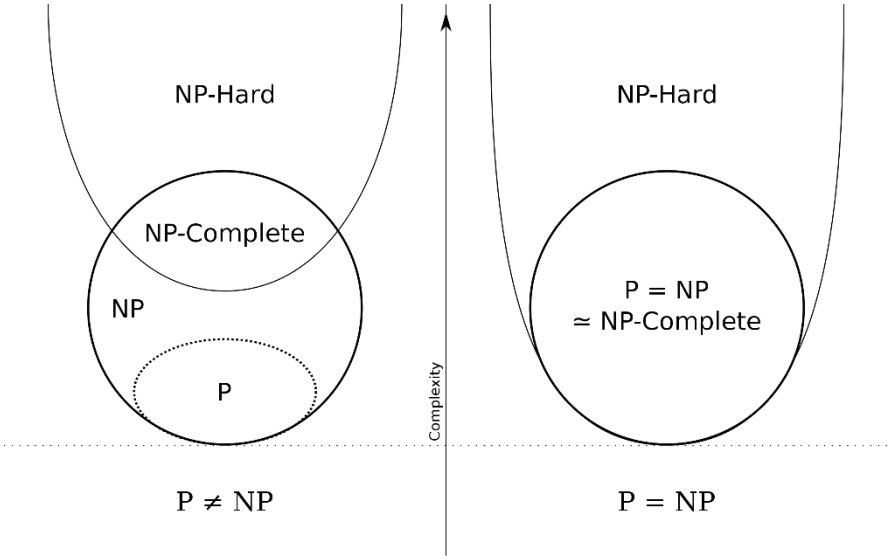
If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

Consensus opinion on P = NP? Probably no.

NP-Completeness





NP-Complete

NP-complete. A problem Y in NP with the property that for every problem X in NP, $X \leq_p Y$.

Theorem. Suppose Y is an NP-complete problem. Then Y is solvable in poly-time iff P = NP.

Pf. $\Leftarrow$ If P = NP then Y can be solved in poly-time since Y is in NP.

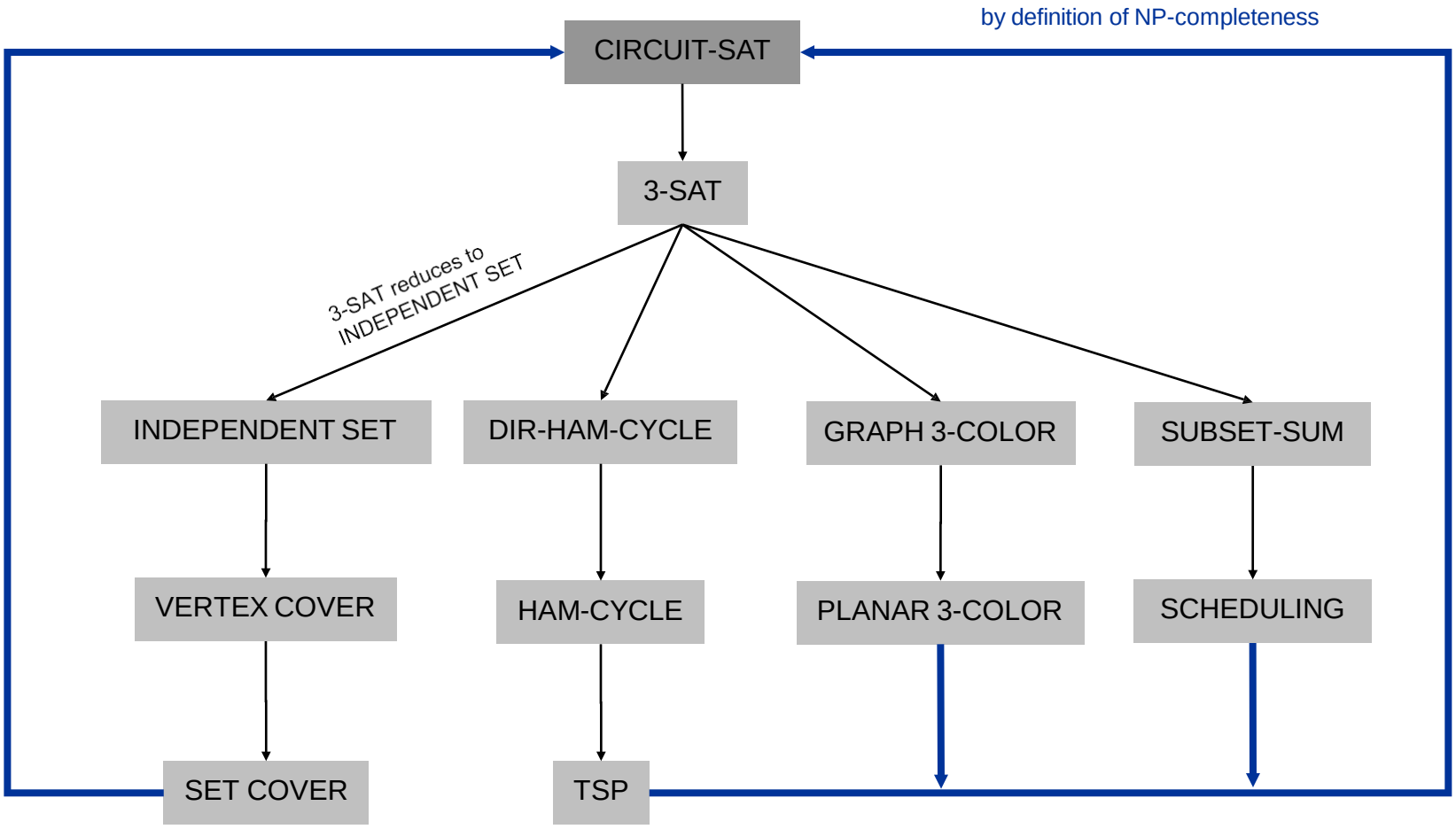
Pf. $\Rightarrow$ Suppose Y can be solved in poly-time.

- Let X be any problem in NP. Since $X \leq_p Y$, we can solve X in poly-time. This implies NP $\subseteq$ P.
- We already know P $\subseteq$ NP. Thus P = NP. ▪

Fundamental question. Do there exist "natural" NP-complete problems?

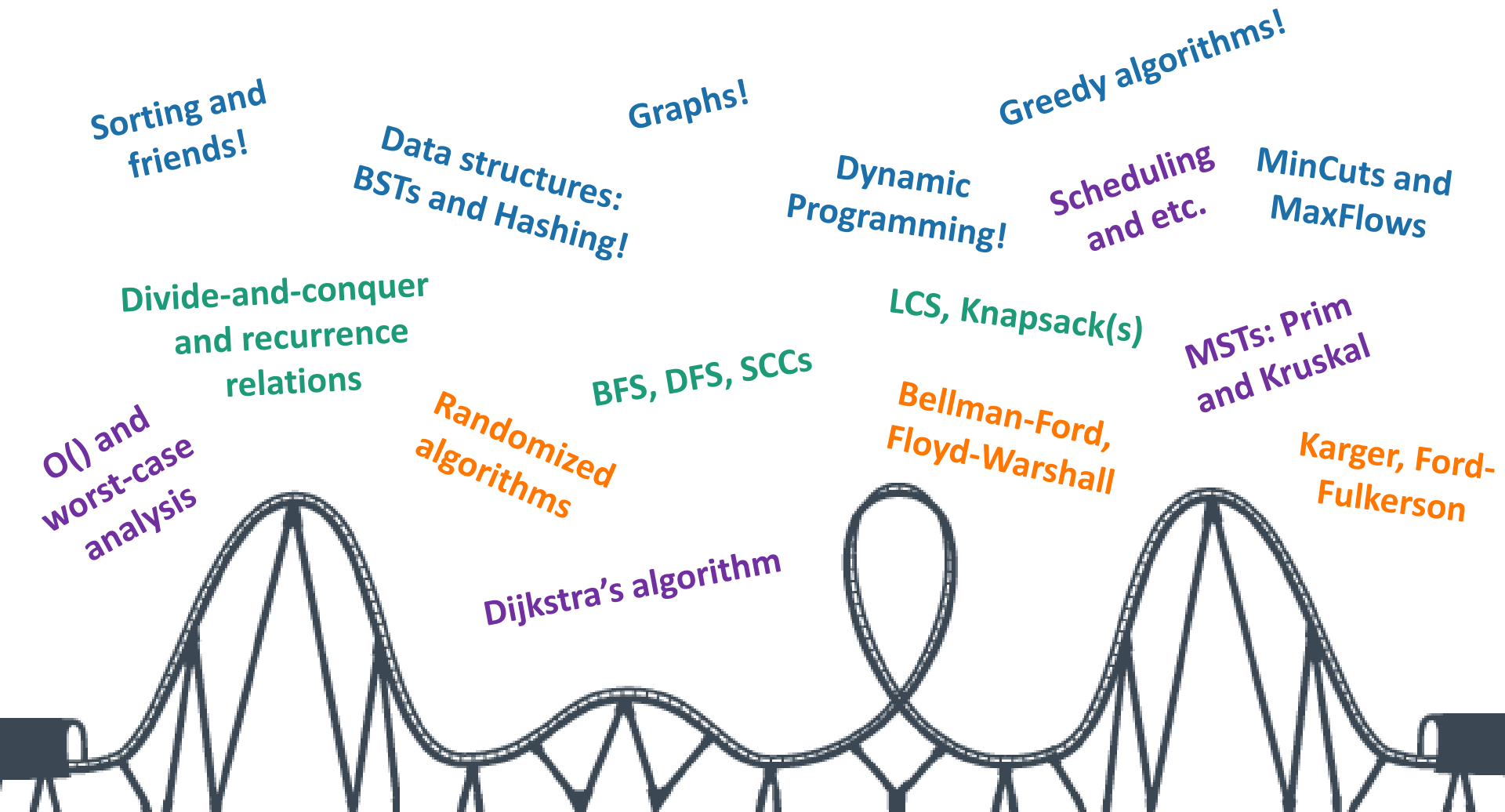
NP-Completeness

Observation. All problems below are NP-complete and polynomial reduce to one another!



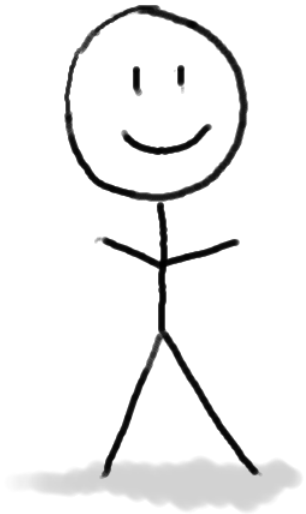
REVIEW

It's been a fun ride…



General approach
to algorithm design and analysis

Can I do better?



Algorithm designer

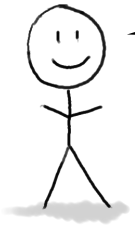
To answer this question we need both **rigor** and **intuition**:

Detail-oriented
Precise
Rigorous

Big-picture
Intuitive
Hand-wavey

We needed more details

python™



Does it work?
Is it fast?

What
does that
mean??

Worst-case analysis

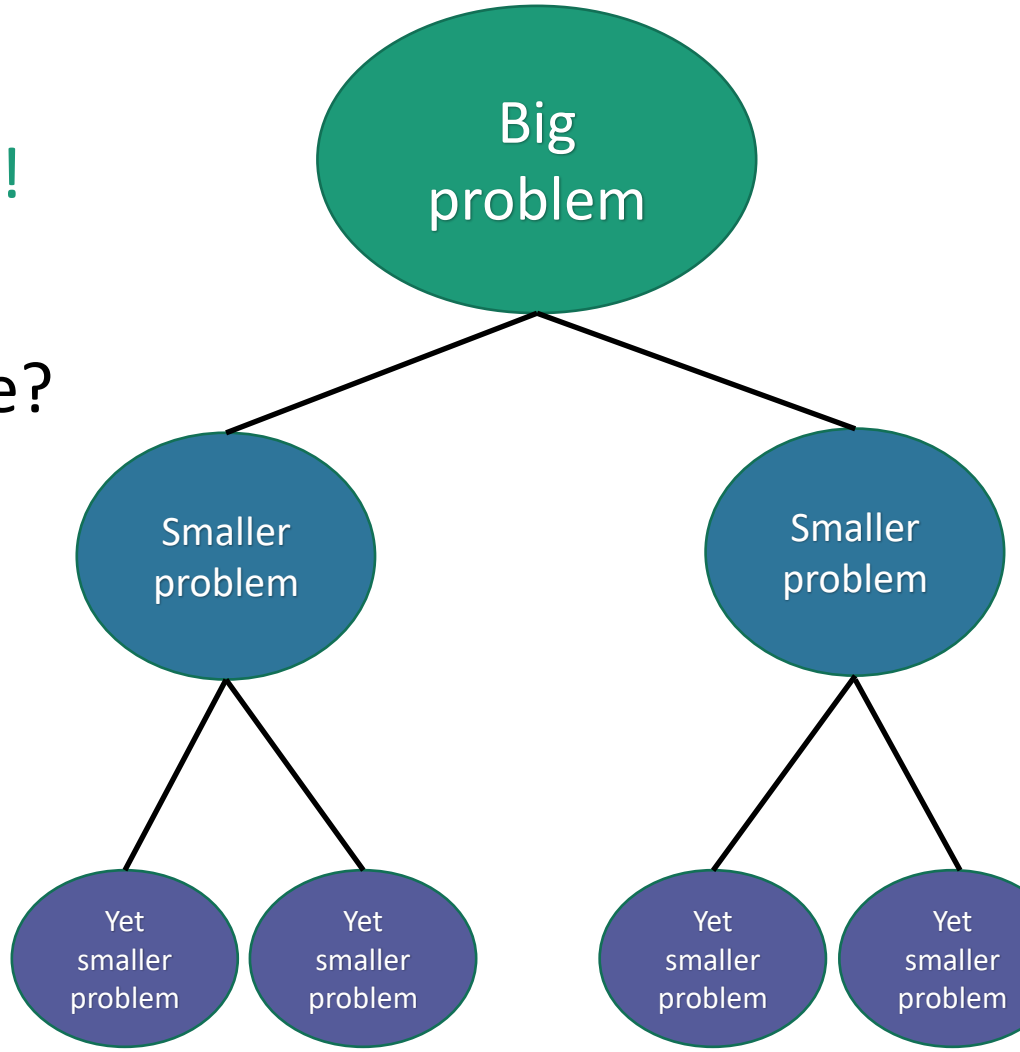
HERE IS AN INPUT!

big-Oh notation

$$T(n) = O\big(f(n)\big)$$
$$\Leftrightarrow$$
$$\exists c, n_0 > 0 \;\; s.t. \;\; \forall n \geq n_0,$$
$$0 \leq T(n) \leq c \cdot f(n)$$

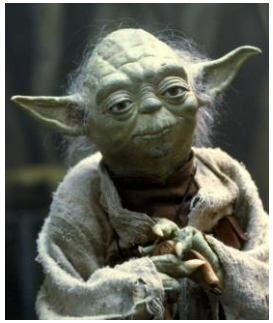
Algorithm design paradigm: divide and conquer

- Like MergeSort!
- Or Karatsuba's algorithm!
- Or SELECT!
- How do we analyze these?



By careful analysis!

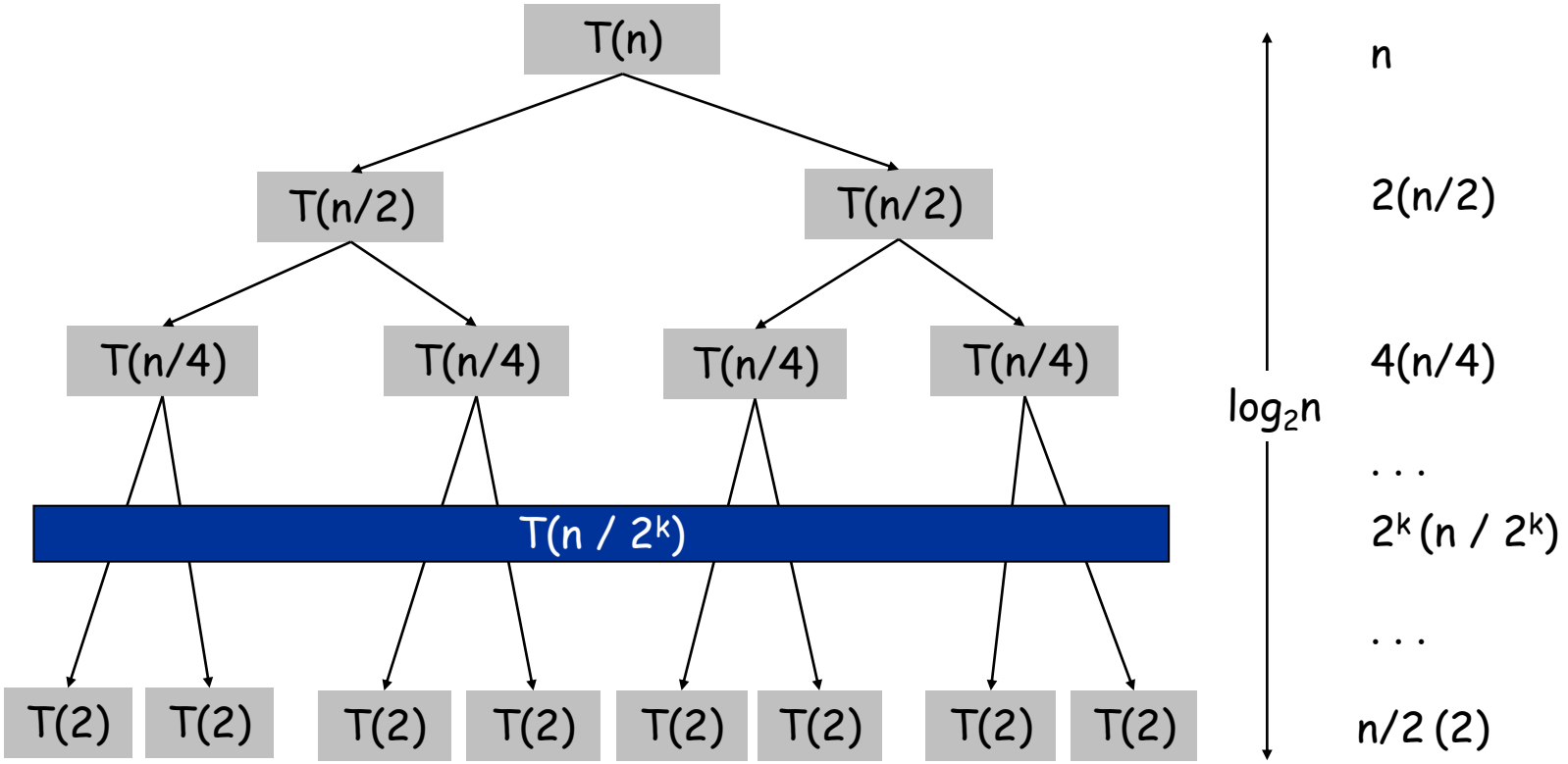
Useful shortcut, the **master method** is.



Jedi master Yoda

Recap- Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Recap-Why Recurrences?

- The complexity of many interesting algorithms is easily expressed as a recurrence – especially divide and conquer algorithms
- The form of the algorithm often yields the form of the recurrence
- The complexity of recursive algorithms is readily expressed as a recurrence.

Recap-Why solve recurrences?

- To make it easier to compare the complexity of two algorithms
- To make it easier to compare the complexity of the algorithm to standard reference functions.

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + O(n^d).$$

Recap-Master Theorem

$$T(n) = \begin{cases} O(n^d \log(n)) & \text{if } a = b^d \\ O(n^d) & \text{if } a < b^d \\ O(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

- Needlessly recursive integer mult.
  - T(n) = 4 T(n/2) + O(n)
  - T(n) = O(n^2)

  a = 4, b = 2, d = 1 — $a > b^d$ ✓

- Karatsuba integer multiplication
  - T(n) = 3 T(n/2) + O(n)
  - T(n) = O($n^{\log\_2(3)} \approx n^{1.6}$)

  a = 3, b = 2, d = 1 — $a > b^d$ ✓

- MergeSort
  - T(n) = 2T(n/2) + O(n)
  - T(n) = O(nlog(n))

  a = 2, b = 2, d = 1 — $a = b^d$ ✓

- That other one
  - T(n) = T(n/2) + O(n)
  - T(n) = O(n)

  a = 1, b = 2, d = 1 — $a < b^d$ ✓

While we're on the topic of sorting

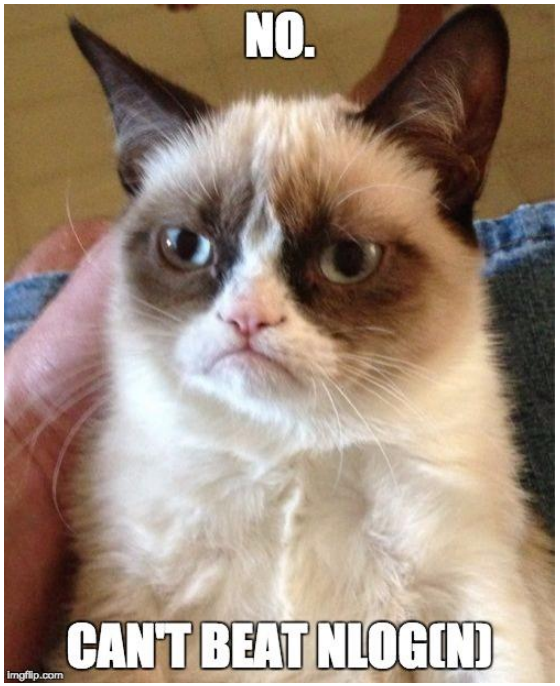
Why not use randomness?

- We analyzed **QuickSort!**
- Still worst-case input, but we use randomness after the input is chosen.
- Always correct, usually fast.
  - This is a Las Vegas algorithm

All this sorting is making me wonder…

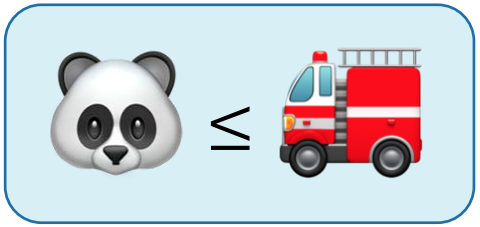
Can we do better?

- Depends on who you ask:



- **RadixSort** takes time O(n) if the objects are, for example, small integers!

- Can't do better in a **comparison-based** model.



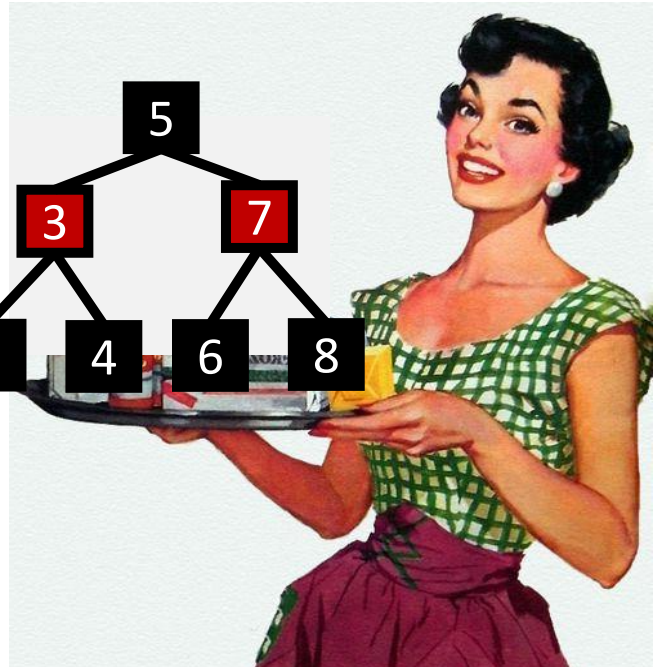
beyond sorted arrays/linked lists:
Binary Search Trees!

- Useful data structure!
- Especially the self-balancing ones!

Red-Black tree!

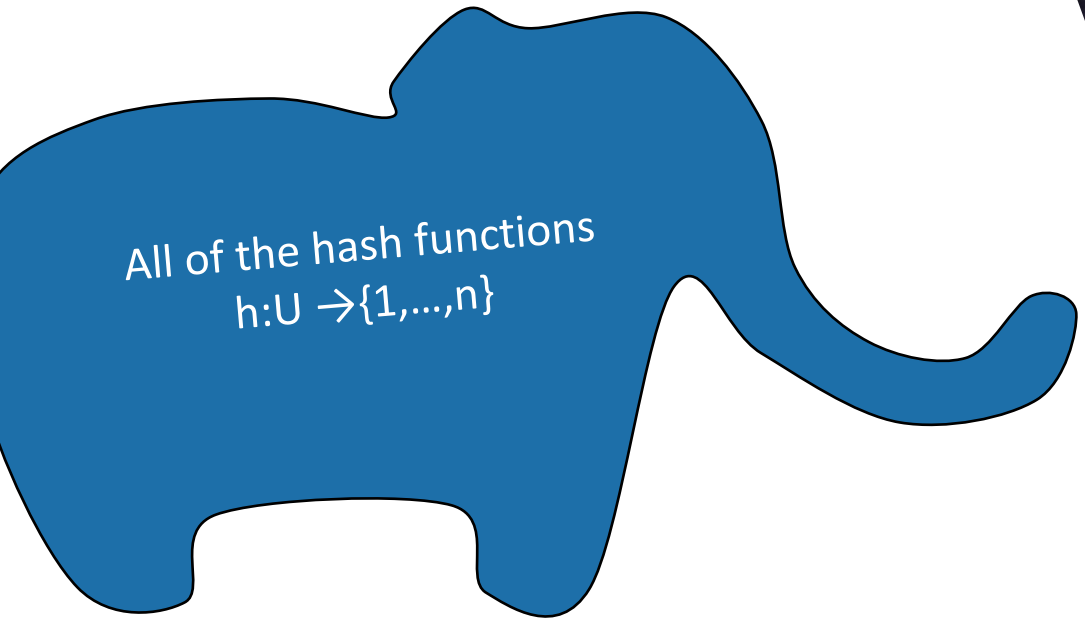
Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.

It's just good sense!



Another way to store things

Hash tables!



The universe

hash function h

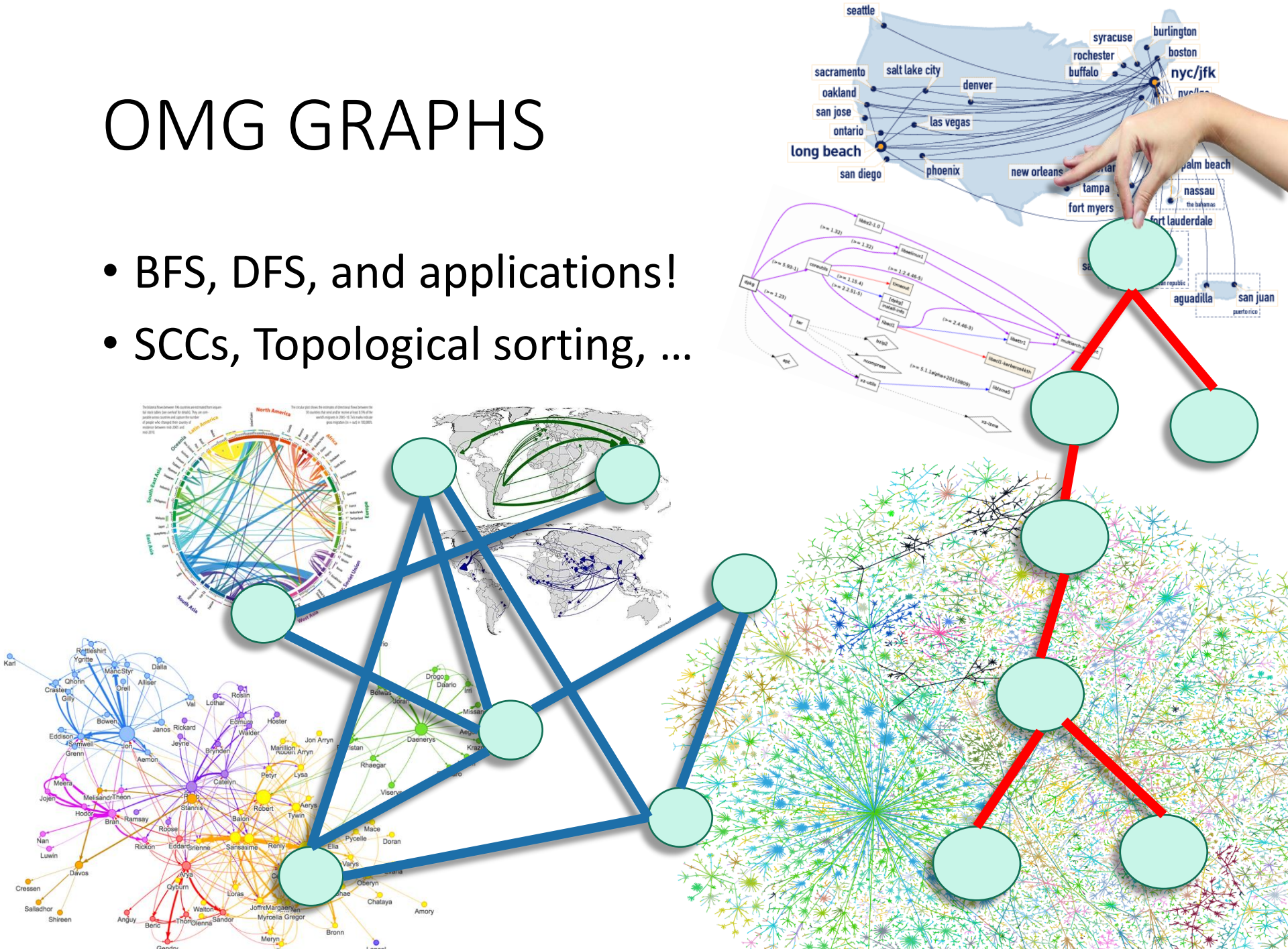
Choose h randomly from a universal hash family.

Some buckets

It's better if the hash family is small! Then it takes less space to store h.

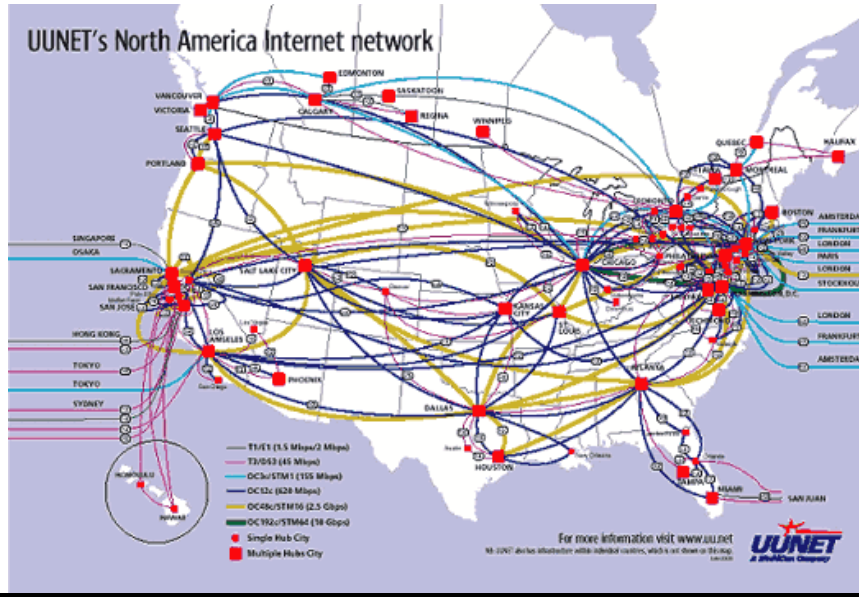
OMG GRAPHS

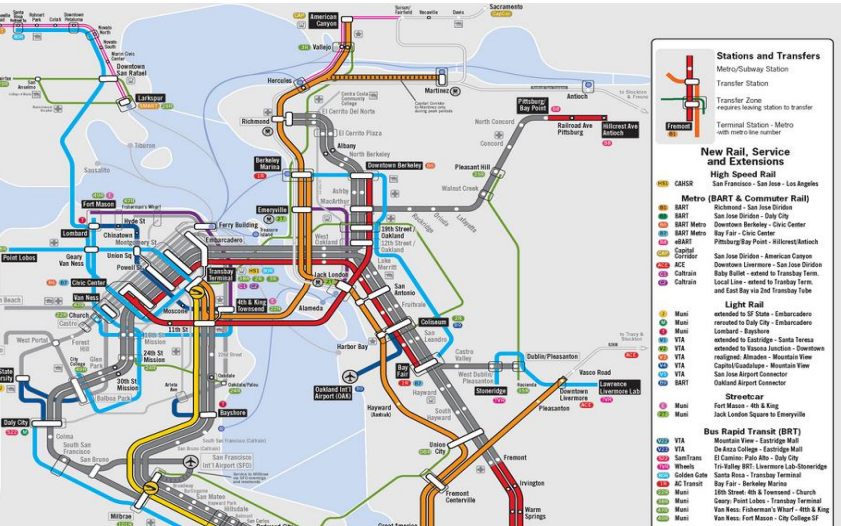
- BFS, DFS, and applications!
- SCCs, Topological sorting, …



A fundamental graph problem:
shortest paths

- Eg, transit planning, packet routing, …
- Dijkstra!






```
DN0a22a0e3:~ mary$ traceroute -a www.ethz.ch
traceroute to www.ethz.ch (129.132.19.216), 64 hops max, 52 byte packets
 1  [AS0] 10.34.160.2 (10.34.160.2)  38.168 ms  31.272 ms  28.841 ms
 2  [AS0] cwa-vrtr.sunet (10.21.196.28)  33.769 ms  28.245 ms  24.373 ms
 3  [AS32] 171.66.2.229 (171.66.2.229)  24.468 ms  20.115 ms  23.223 ms
 4  [AS32] hpr-svl-rtr-vlan8.sunet (171.64.255.235)  24.644 ms  24.962 ms  17.453 ms
 5  [AS2152] hpr-svl-hpr2--stan-ge.cenic.net (137.164.27.161)  22.129 ms  4.902 ms  3.642 ms
 6  [AS2152] hpr-lax-hpr3--svl-hpr3-100ge.cenic.net (137.164.25.73)  12.125 ms  43.361 ms  32.3
 7  [AS2152] hpr-i2--lax-hpr2-r&e.cenic.net (137.164.26.201)  40.174 ms  38.399 ms  34.499 ms
 8  [AS0] et-4-0-0.4079.sdn-sw.lasv.net.internet2.edu (162.252.70.28)  46.573 ms  23.926 ms  17
 9  [AS0] et-5-1-0.4079.rtsw.salt.net.internet2.edu (162.252.70.31)  30.424 ms  25.770 ms  23.1
10  [AS0] et-4-0-0.4079.sdn-sw.denv.net.internet2.edu (162.252.70.8)  47.454 ms  57.273 ms  73.
11  [AS0] et-4-1-0.4079.rtsw.kans.net.internet2.edu (162.252.70.11)  70.825 ms  67.809 ms  62.1
12  [AS0] et-4-1-0.4070.rtsw.chic.net.internet2.edu (198.71.47.206)  77.937 ms  57.421 ms  63.6
13  [AS0] et-0-1-0.4079.sdn-sw.ashb.net.internet2.edu (162.252.70.60)  77.682 ms  71.993 ms  73
14  [AS0] et-4-1-0.4079.rtsw.wash.net.internet2.edu (162.252.70.65)  71.565 ms  74.988 ms  71.0
15  [AS21320] internet2-gw.mx1.lon.uk.geant.net (62.40.124.44)  154.926 ms  145.606 ms  145.872
16  [AS21320] ae0.mx1.lon2.uk.geant.net (62.40.98.79)  146.565 ms  146.604 ms  146.801 ms
17  [AS21320] ae0.mx1.par.fr.geant.net (62.40.98.77)  153.289 ms  184.995 ms  152.682 ms
18  [AS21320] ae2.mx1.gen.ch.geant.net (62.40.98.153)  160.283 ms  160.104 ms  164.147 ms
19  [AS21320] swice1-100ge-0-3-0-1.switch.ch (62.40.124.22)  162.068 ms  160.595 ms  163.095 ms
20  [AS559] swizh1-100ge-0-1-0-1.switch.ch (130.59.36.94)  165.824 ms  164.216 ms  163.983 ms
21  [AS559] swiez3-100ge-0-1-0-4.switch.ch (130.59.38.109)  164.269 ms  164.370 ms  163.929 ms
22  [AS559] rou-gw-lee-tengig-to-switch.ethz.ch (192.33.92.1)  164.082 ms  170.645 ms  165.372
23  [AS559] rou-fw-rz-rz-gw.ethz.ch (192.33.92.169)  164.773 ms  165.193 ms  172.158 ms
```

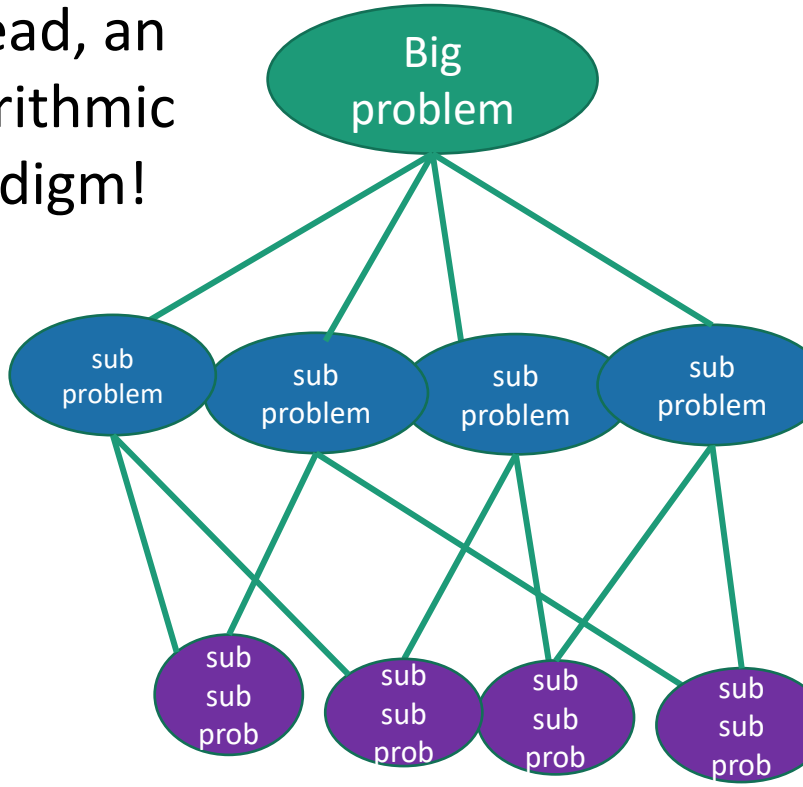
Bellman-Ford and Floyd-Warshall were examples of...

Dynamic Programming!

We saw many other examples, including Longest Common Subsequence and Knapsack Problems.

- Not programming in an action movie.

Instead, an algorithmic paradigm!



- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the value of the optimal solution.
- **Steps 3-5:** Use dynamic programming: fill in a table to find the answer!

Dynamic Programming!

- Dynamic programming is an algorithm design paradigm.
- Basic idea:
  - Identify **optimal sub-structure**
    - Optimum to the big problem is built out of optima of small sub-problems
  - Take advantage of **overlapping sub-problems**
    - Only solve each sub-problem once, then use it again and again
  - Keep track of the solutions to sub-problems in a table as you build to the final solution.

Dynamic Programming-Recap

- We saw examples of how to come up with dynamic programming algorithms.
  - Longest Common Subsequence
  - Knapsack two ways
  - (If time) maximal independent set in trees.
- There is a **recipe** for dynamic programming algorithms.

Recipe for applying Dynamic Programming

- **Step 1:** Identify optimal substructure.
- **Step 2:** Find a recursive formulation for the length of the longest common subsequence.
- **Step 3:** Use dynamic programming to find the length of the longest common subsequence.
- **Step 4:** If needed, keep track of some additional info so that the algorithm from Step 3 can find the actual LCS.
- **Step 5:** If needed, code this up like a reasonable person.

Knapsack problem

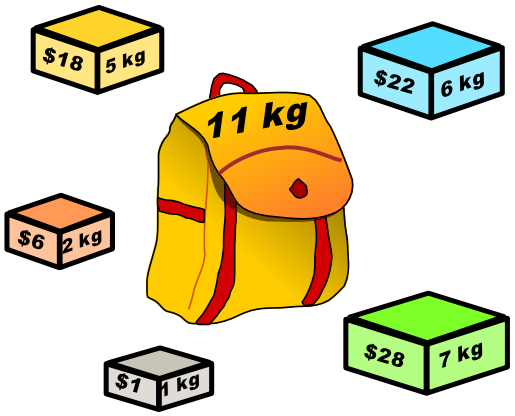
Goal. Pack knapsack so as to maximize total value of items taken.

- There are n items: item i provides value $v_i > 0$ and weighs $w_i > 0$.
- Value of a subset of items = sum of values of individual items.
- Knapsack has weight limit of W.

Ex. The subset $\{ 1, 2, 5 \}$ has value \$35 (and weight 10).

Ex. The subset $\{ 3, 4 \}$ has value \$40 (and weight 11).

Assumption. All values and weights are integral.



i	v_i	w_i
1	\$1	1 kg
2	\$6	2 kg
3	\$18	5 kg
4	\$22	6 kg
5	\$28	7 kg

weights and values can be arbitrary positive integers

knapsack instance (weight limit W = 11)

Dynamic programming: two variables

Def. $OPT(i, w)$ = optimal value of knapsack problem with items $1, \ldots, i$, subject to weight limit w.

Goal. $OPT(n, W)$.

Case 1. $OPT(i, w)$ does not select item i. ← possibly because $w_i > w$

- $OPT(i, w)$ selects best of $\{ 1, 2, \ldots, i-1 \}$ subject to weight limit w.

Case 2. $OPT(i, w)$ selects item i. ← optimal substructure property (proof via exchange argument)

- Collect value v_i.
- New weight limit $= w - w_i$.
- $OPT(i, w)$ selects best of $\{ 1, 2, \ldots, i-1 \}$ subject to new weight limit.

Bellman equation.

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max \{ OPT(i-1, w),\ v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Sometimes we can take even better advantage of optimal substructure…with

Greedy algorithms

- Make a series of choices, and commit!

- Intuitively we want to show that our greedy choices never rule out success.
- Rigorously, we usually analyzed these by induction.
- Examples!
  - Activity Selection
  - Job Scheduling
  - Huffman Coding
  - **Minimum Spanning Trees**

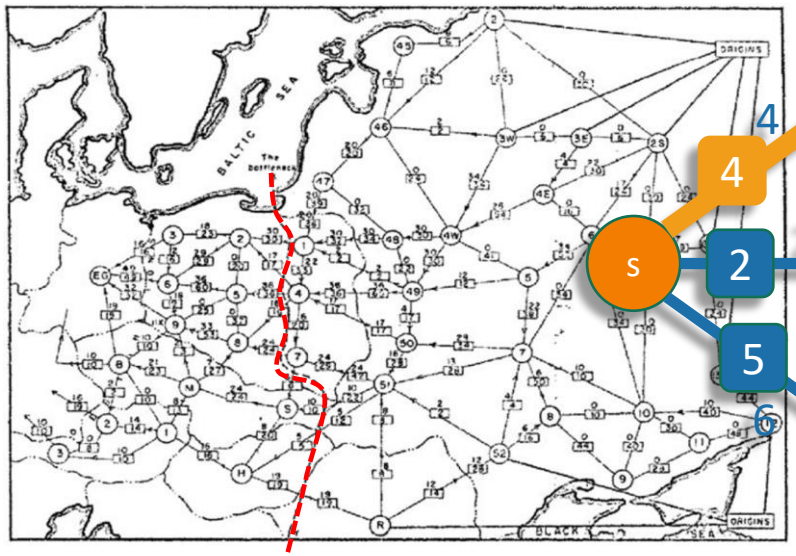
Prim's algorithm:
greedily grow a tree

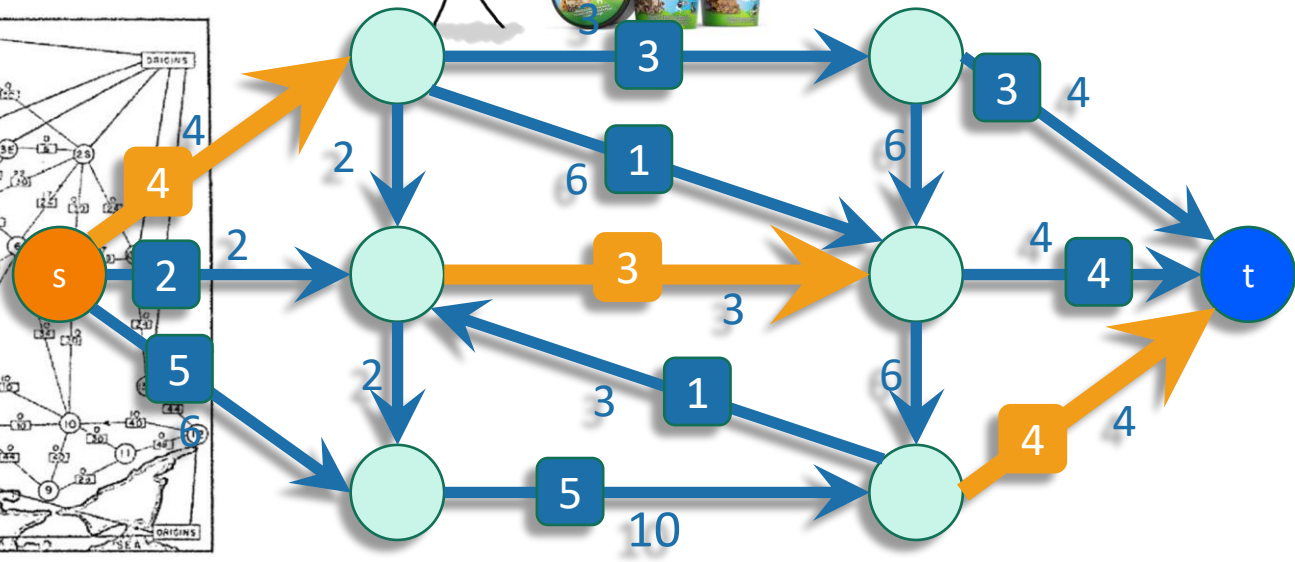
Kruskal's algorithm:
greedily grow a forest

Cuts and flows

- Global minimum cut:
  - Karger's algorithm!
- minimum s-t cut:
  - is the same as maximum s-t flow!
  - Ford-Fulkerson can find them!
    - useful for routing
    - also assignment problems

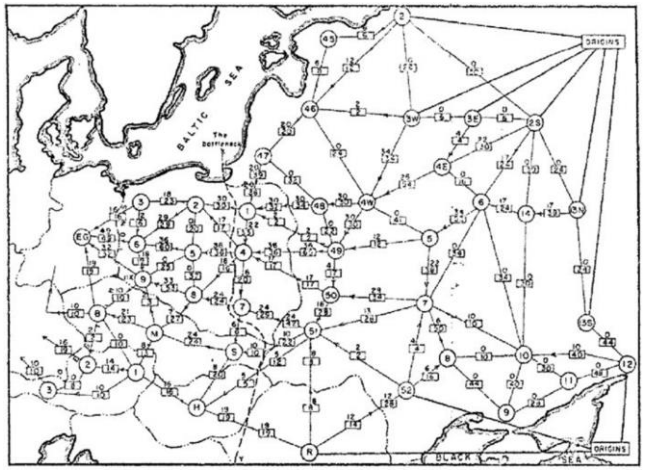
Karger's algorithm is a **Monte-Carlo algorithm**: it is always fast but might be wrong.



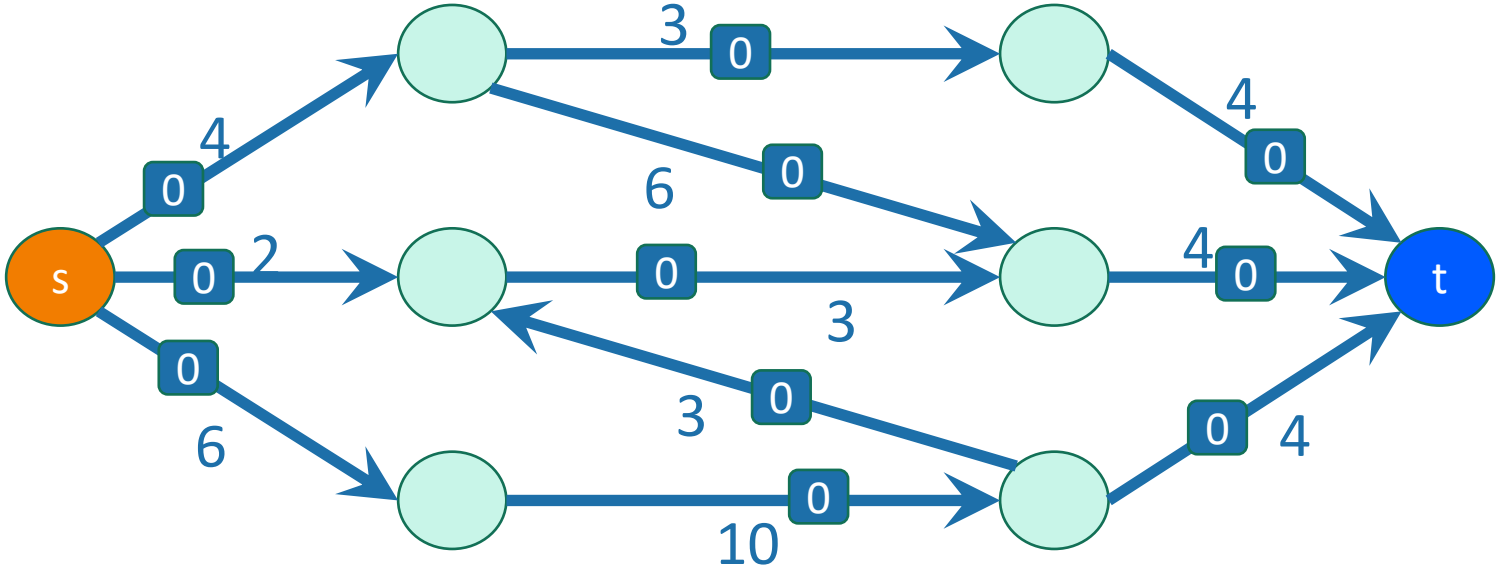


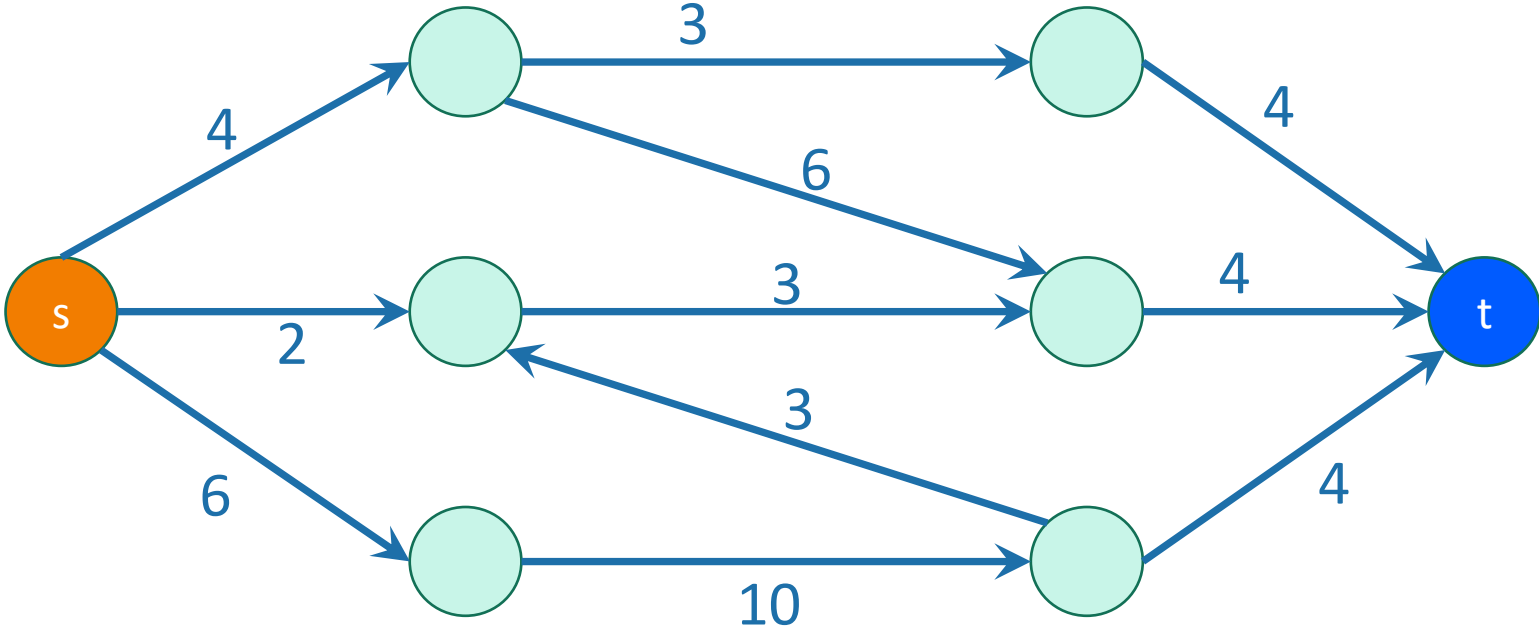
What have we learned?

- Max s-t flow is equal to min s-t cut!
  - The USSR and the USA were trying to solve the same problem…
- The Ford-Fulkerson algorithm can find the min-cut/max-flow.
  - Repeatedly improve your flow along an augmenting path.
- **How long does this take???**



Example of Ford-Fulkerson





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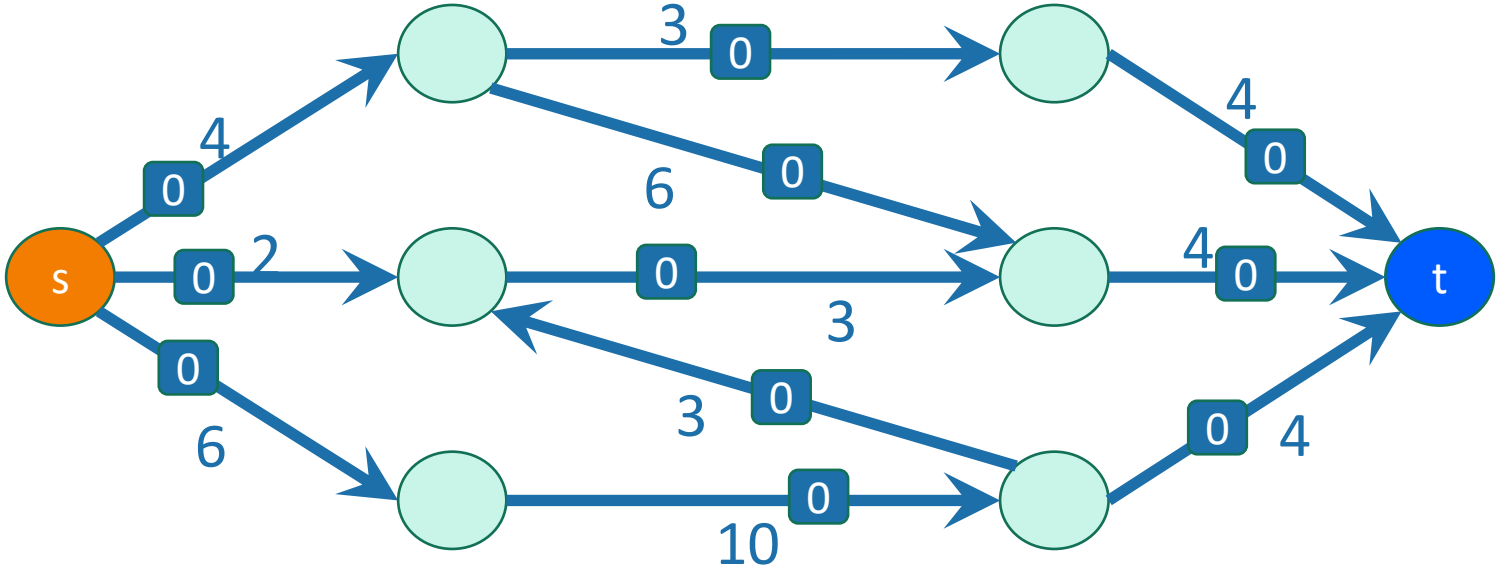

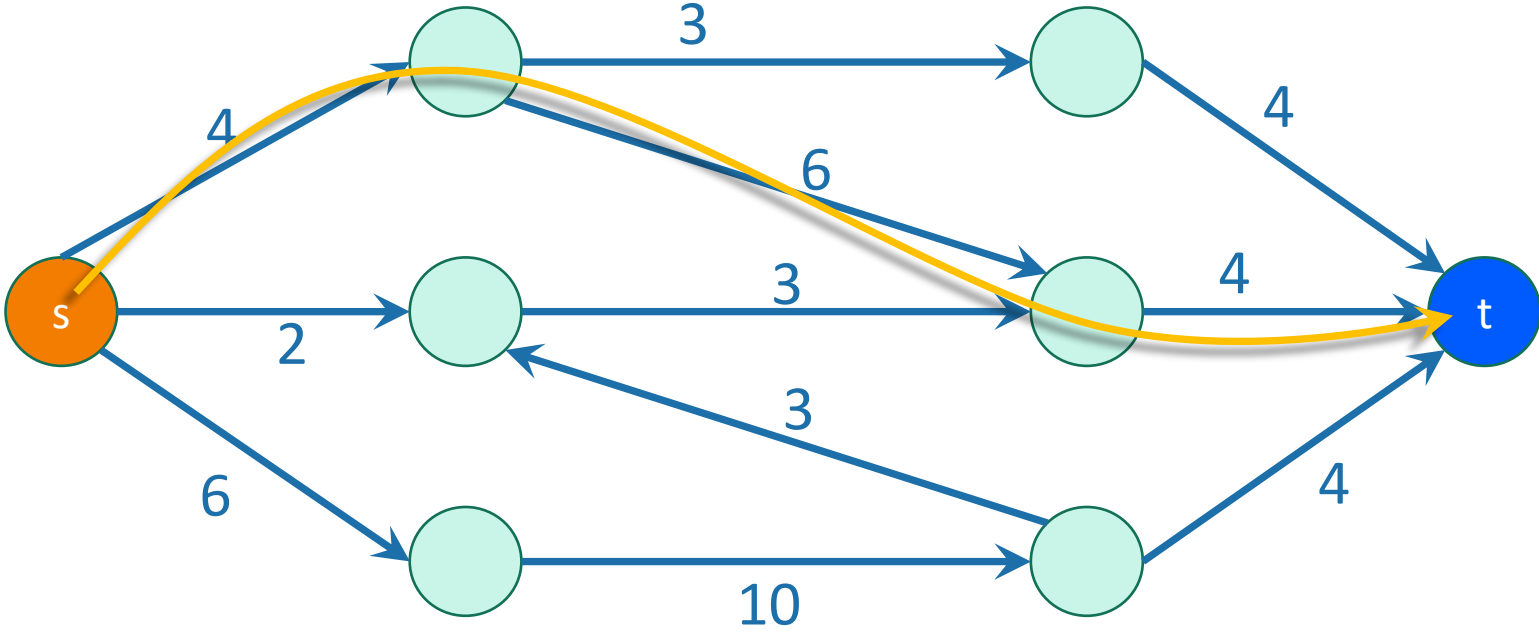

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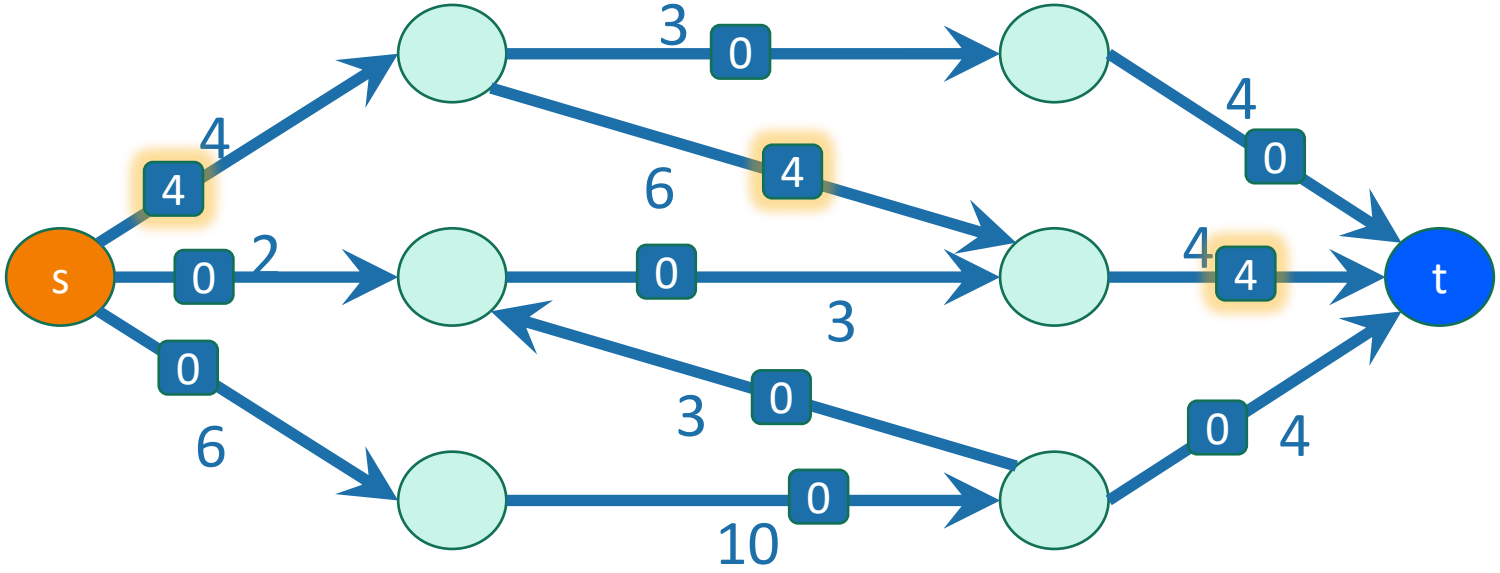

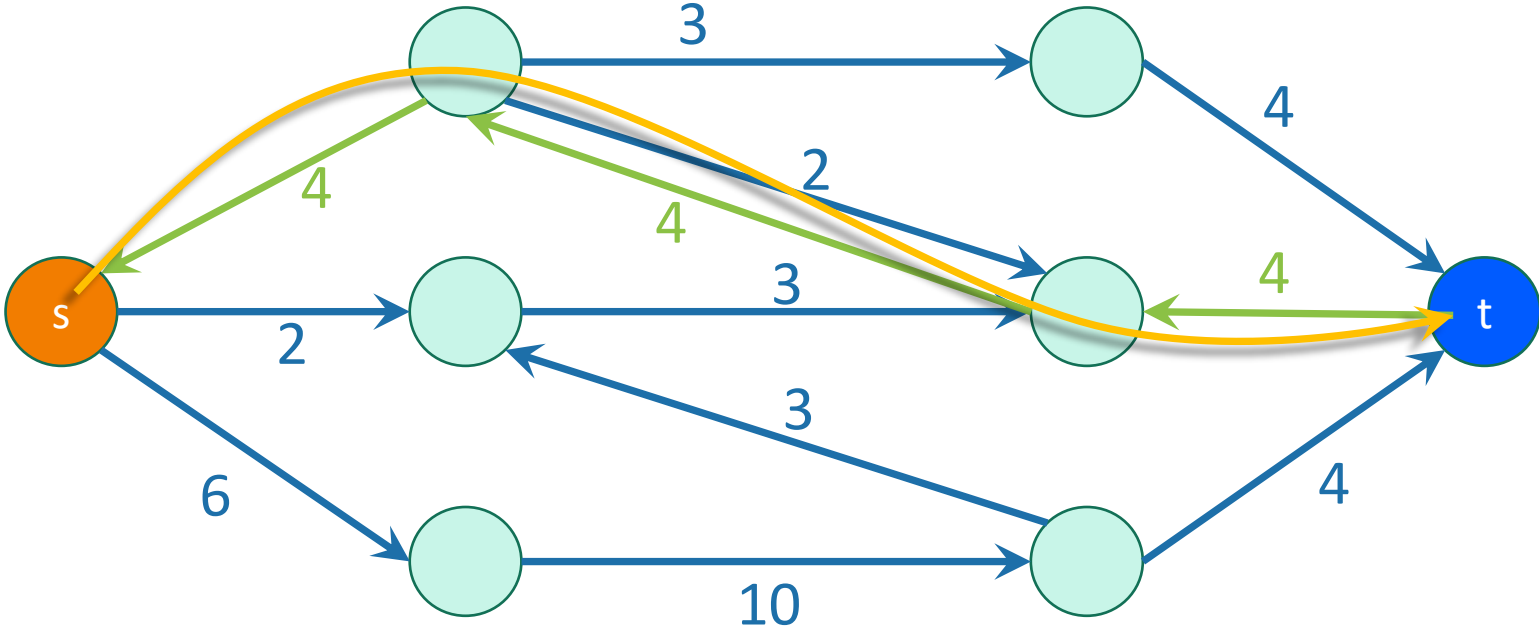

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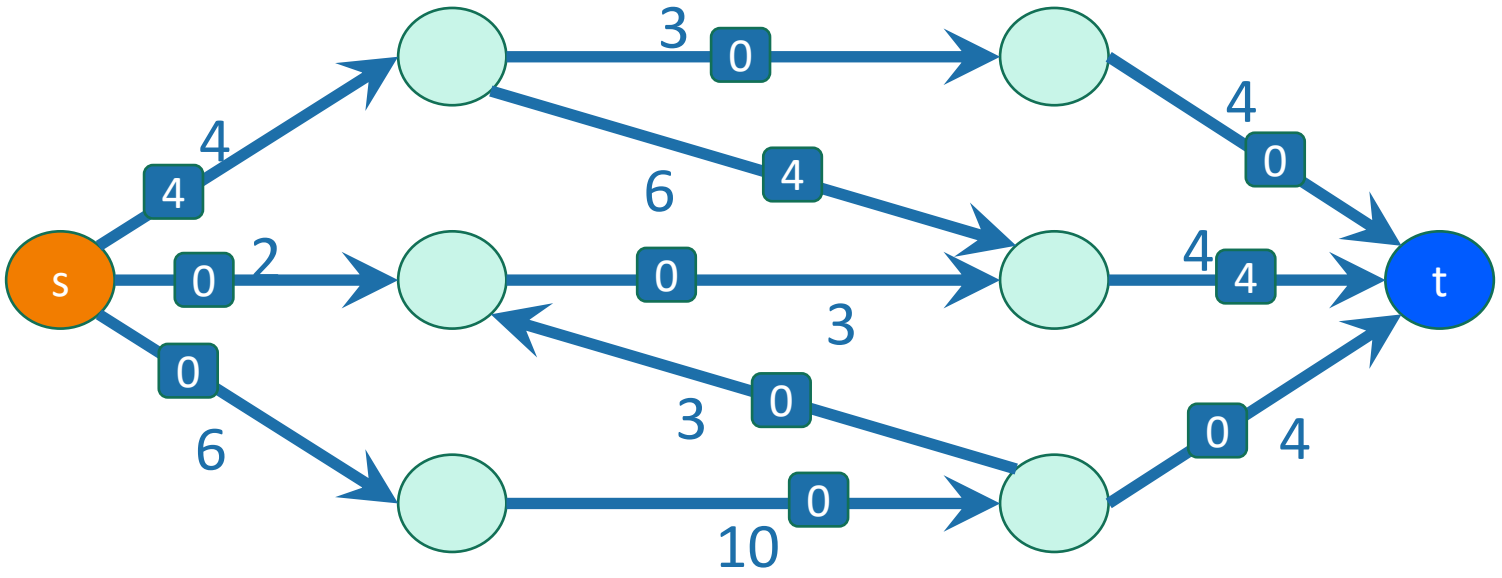

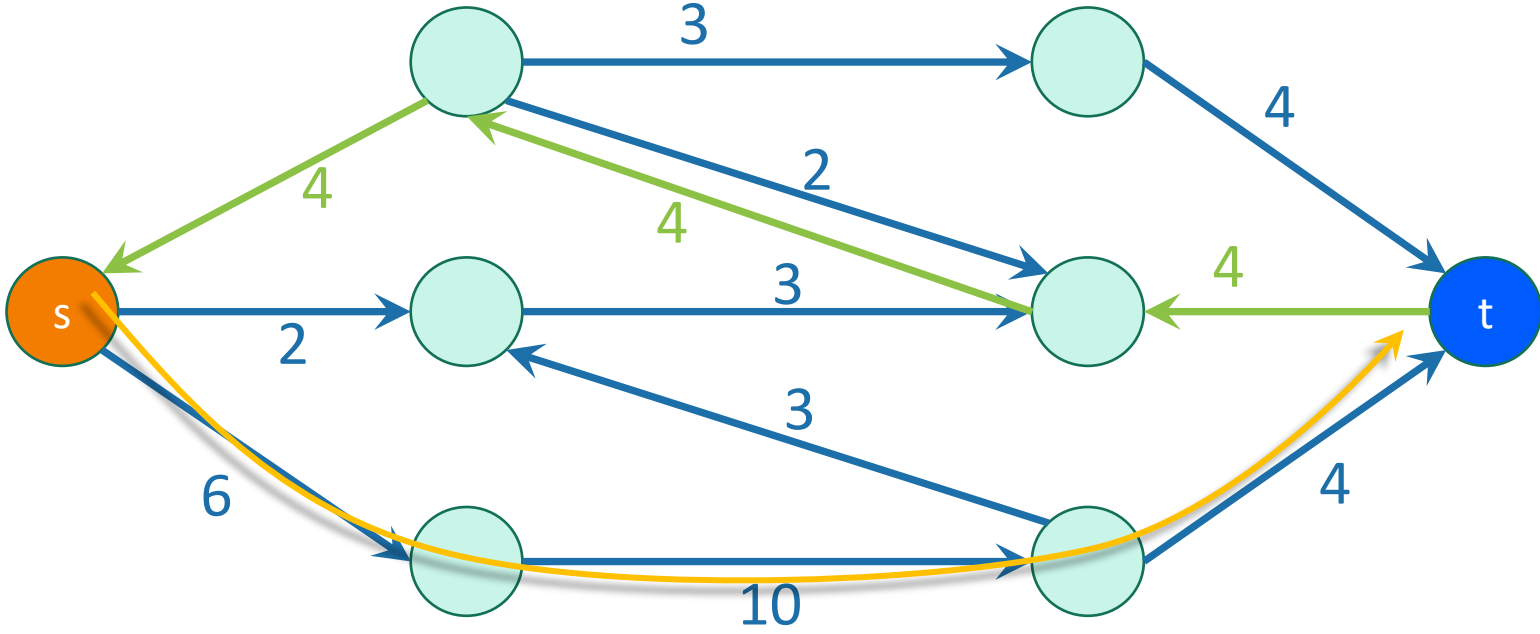

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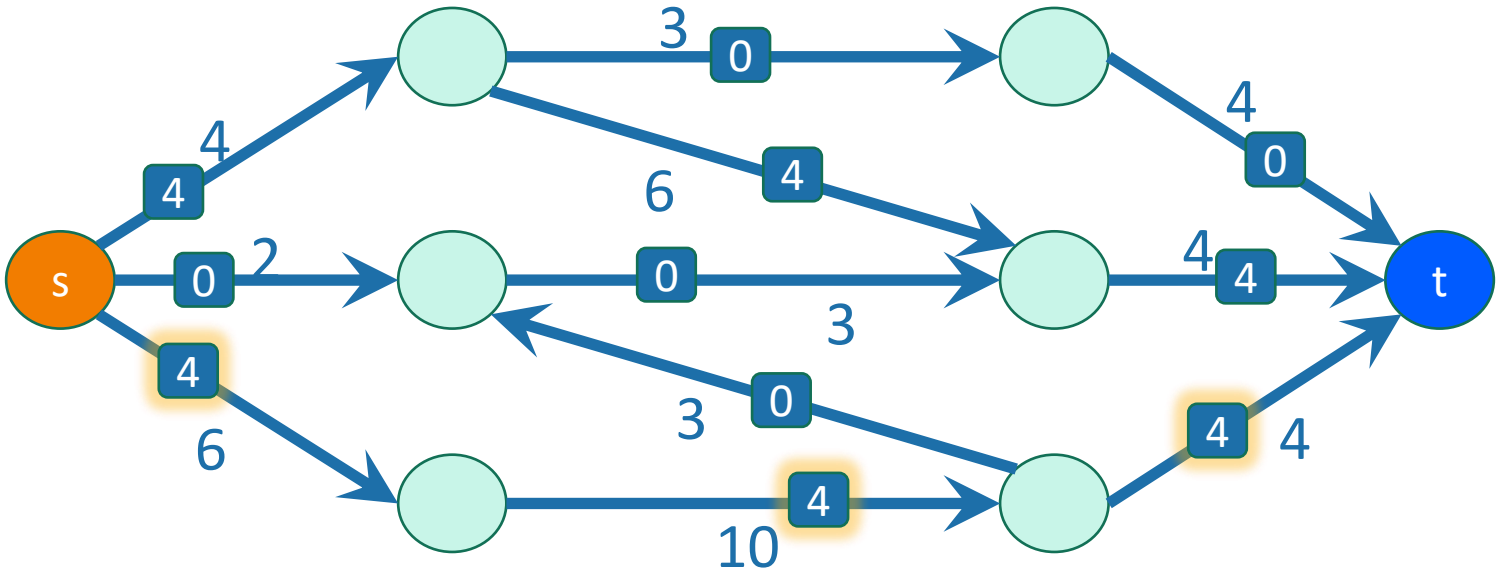

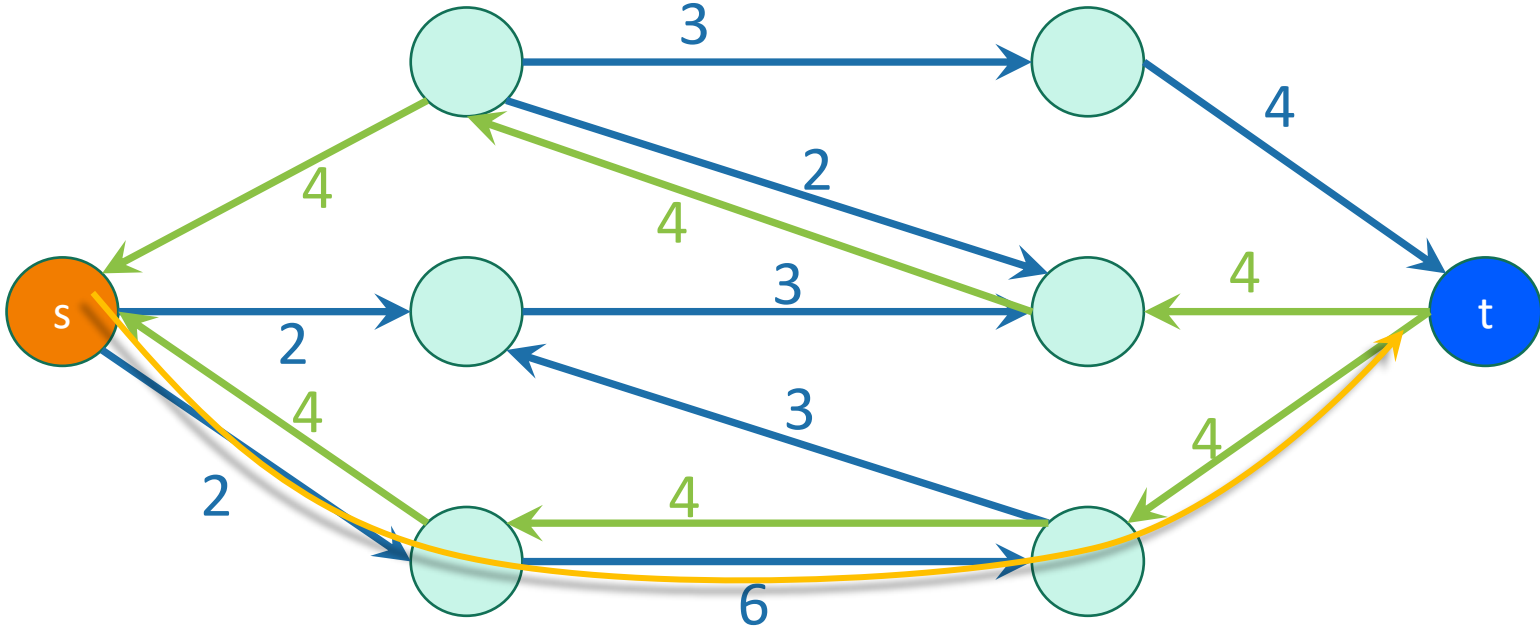

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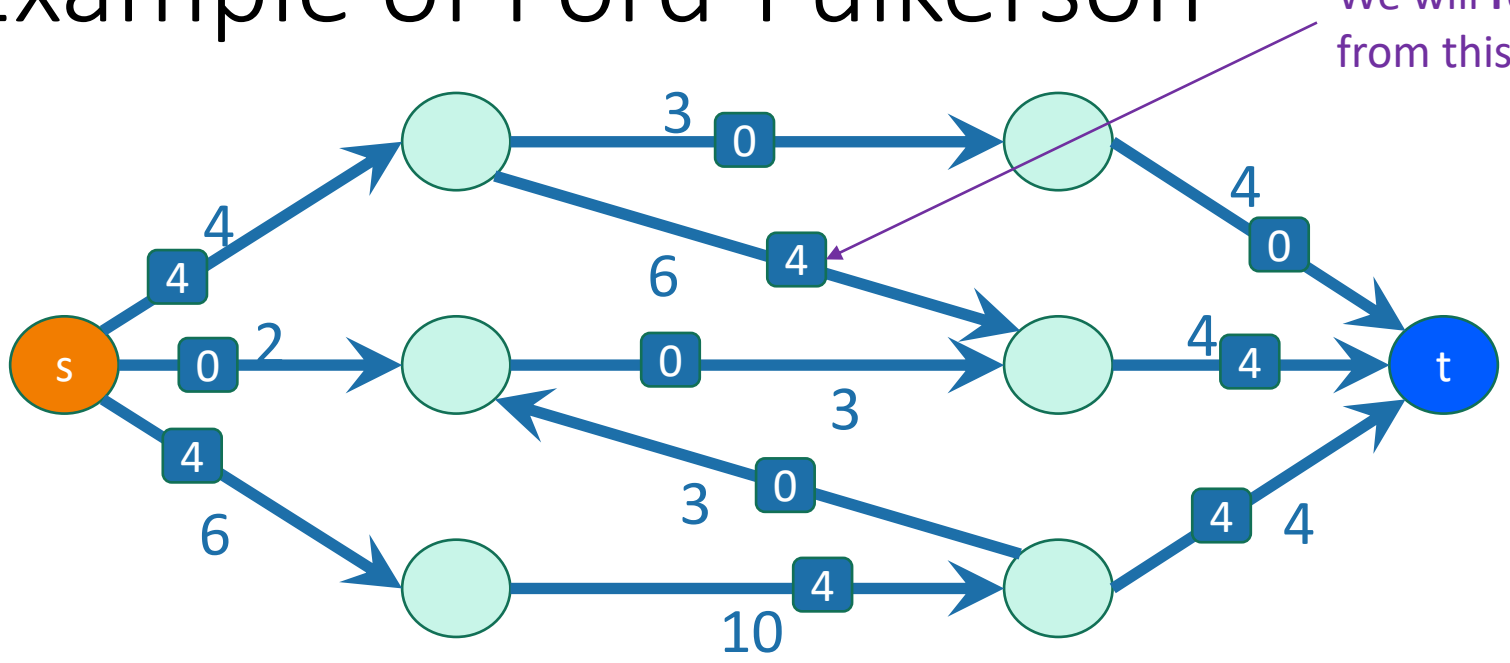
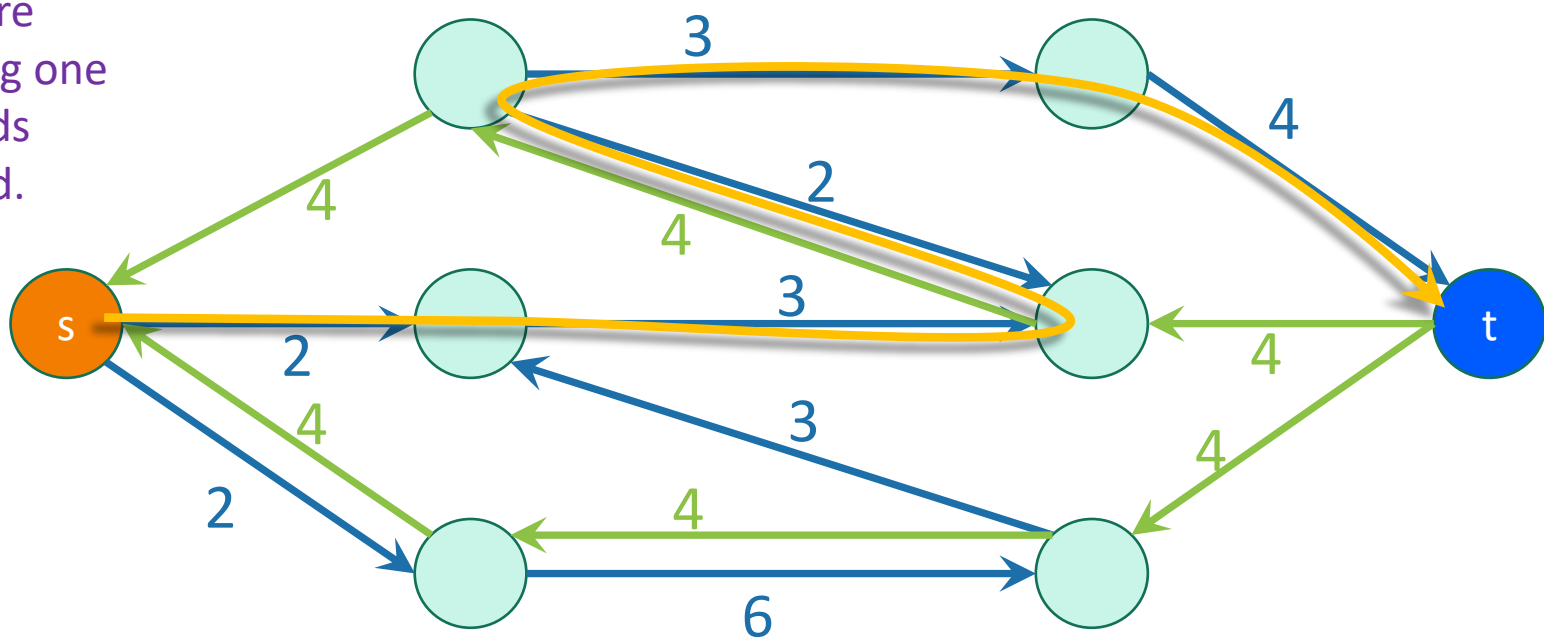


Notice that we're going back along one of the backwards edges we added.



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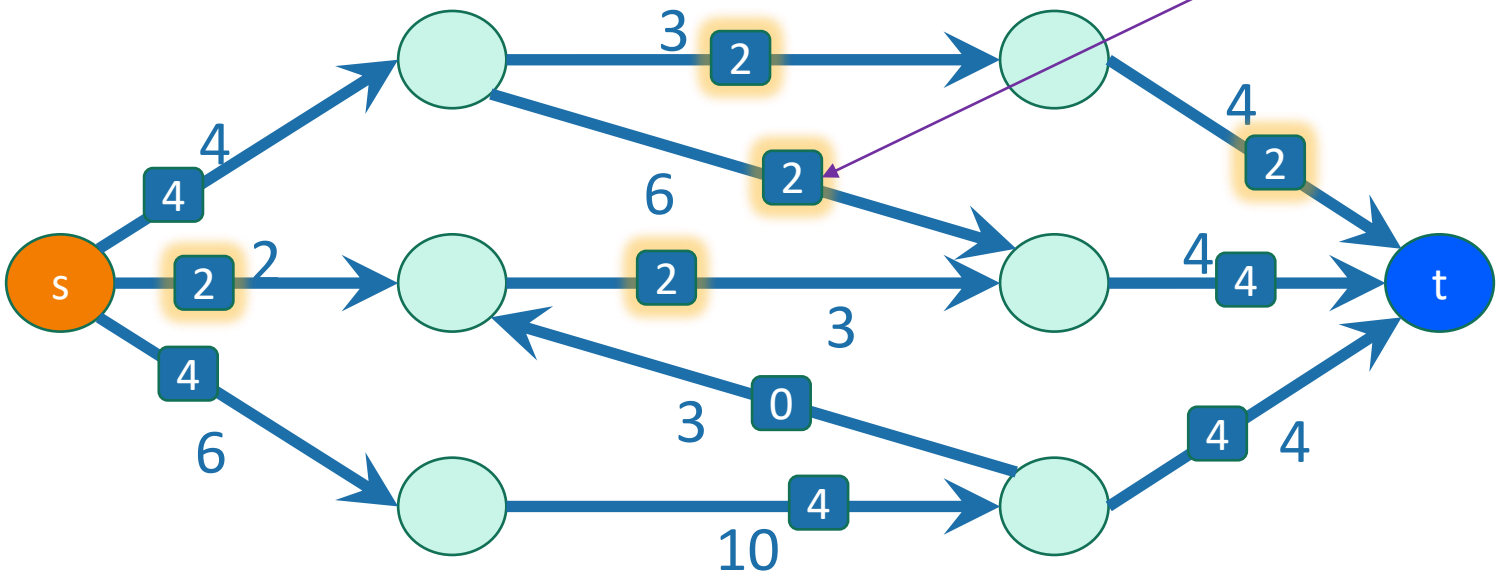

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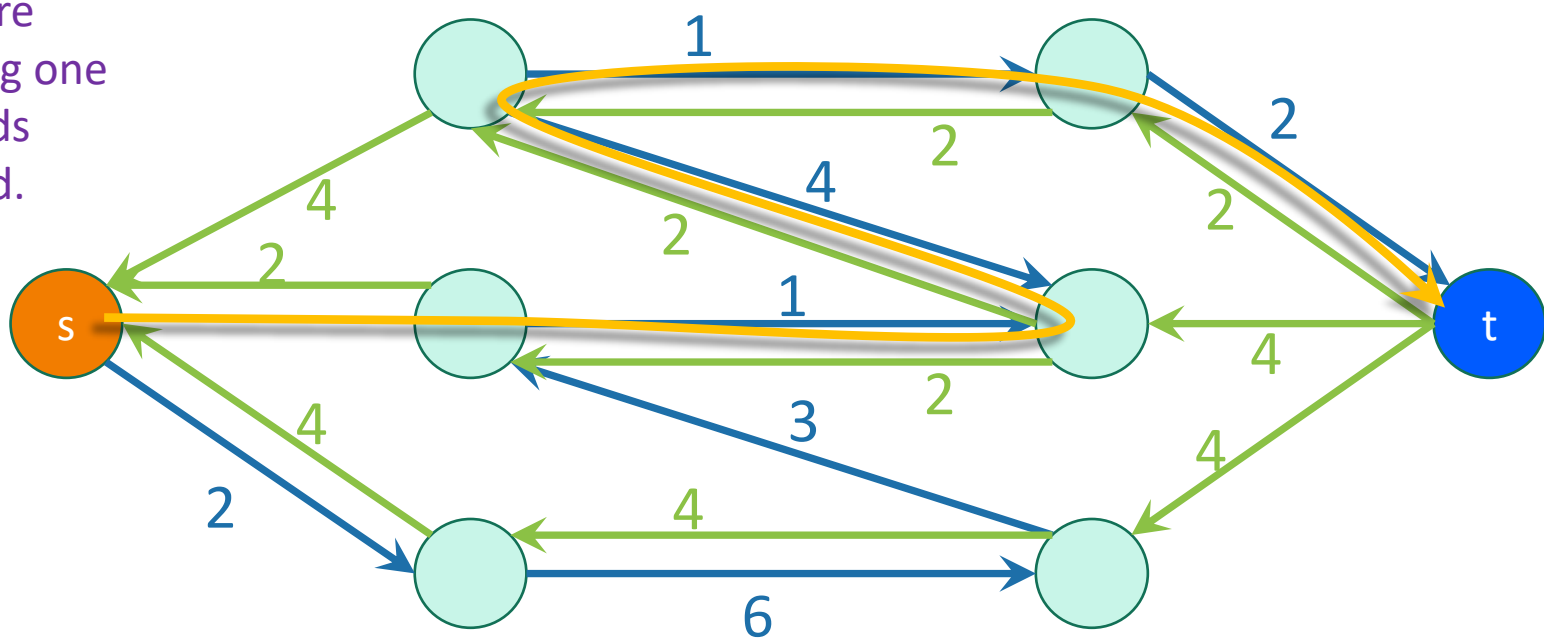

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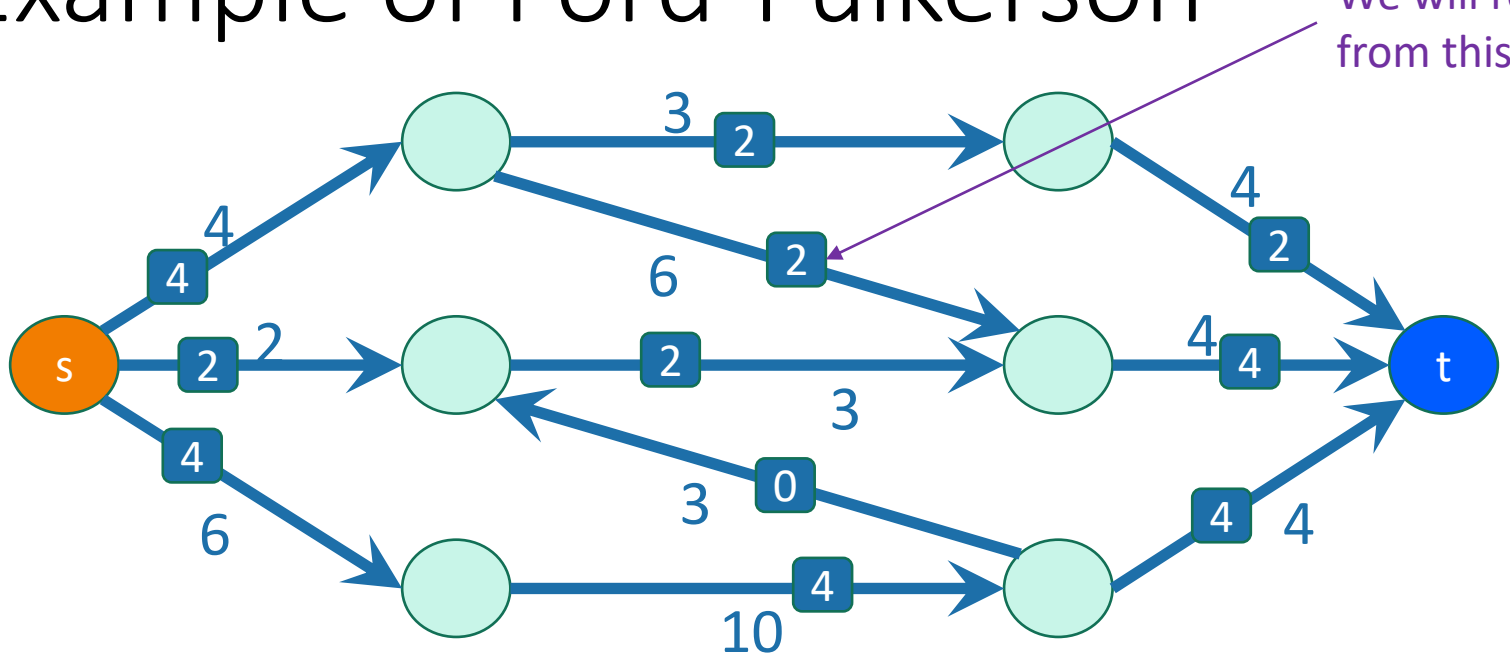

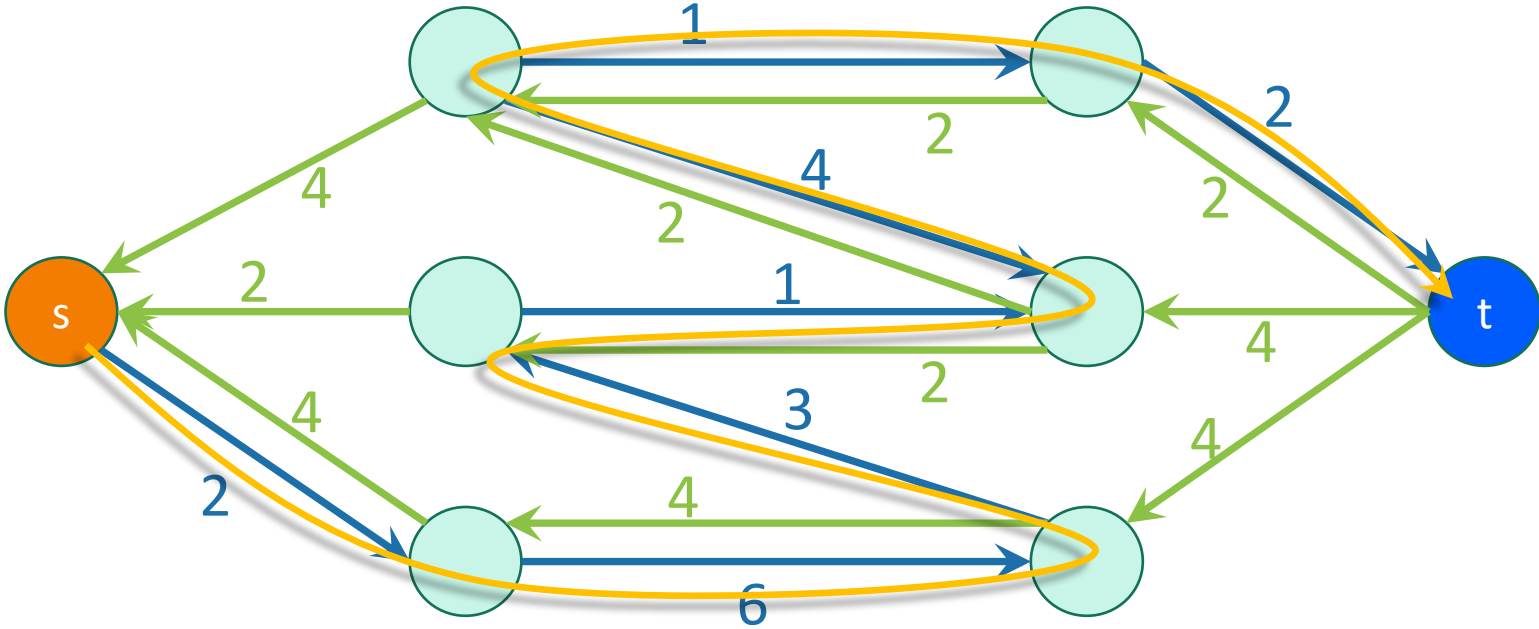

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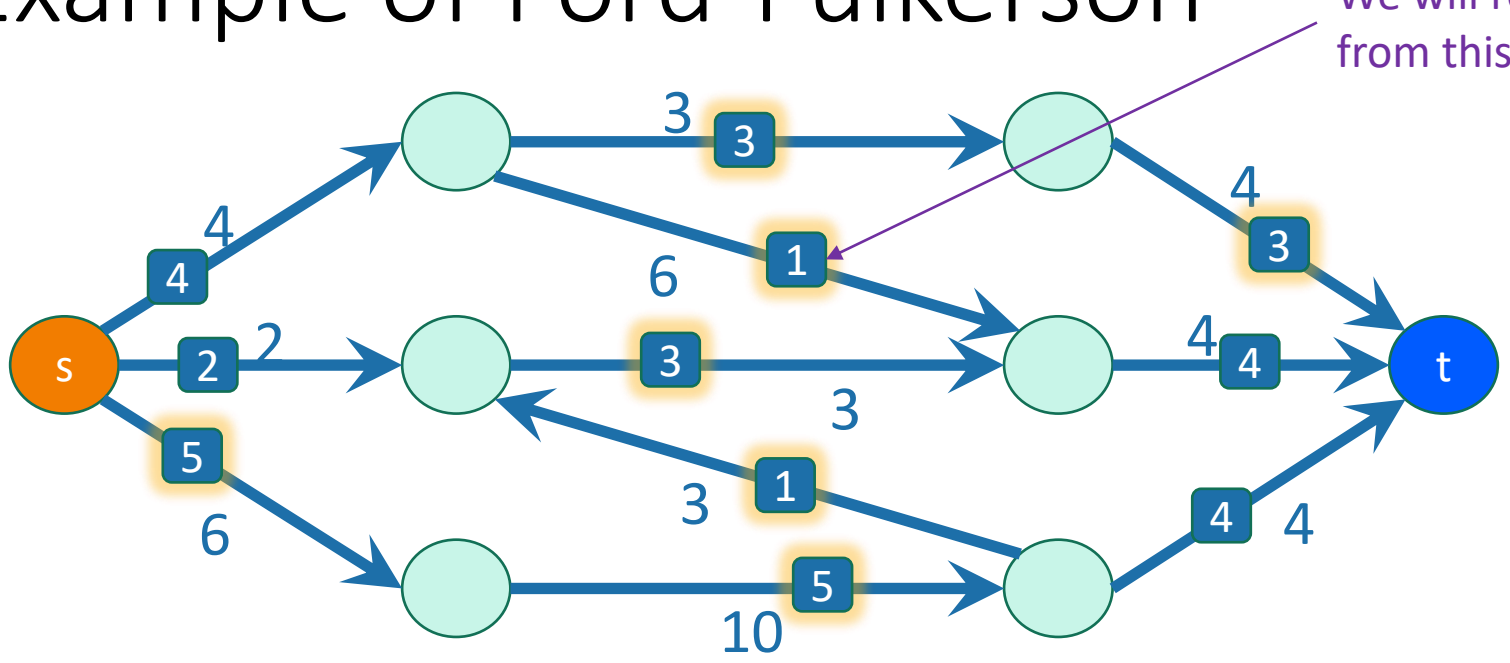

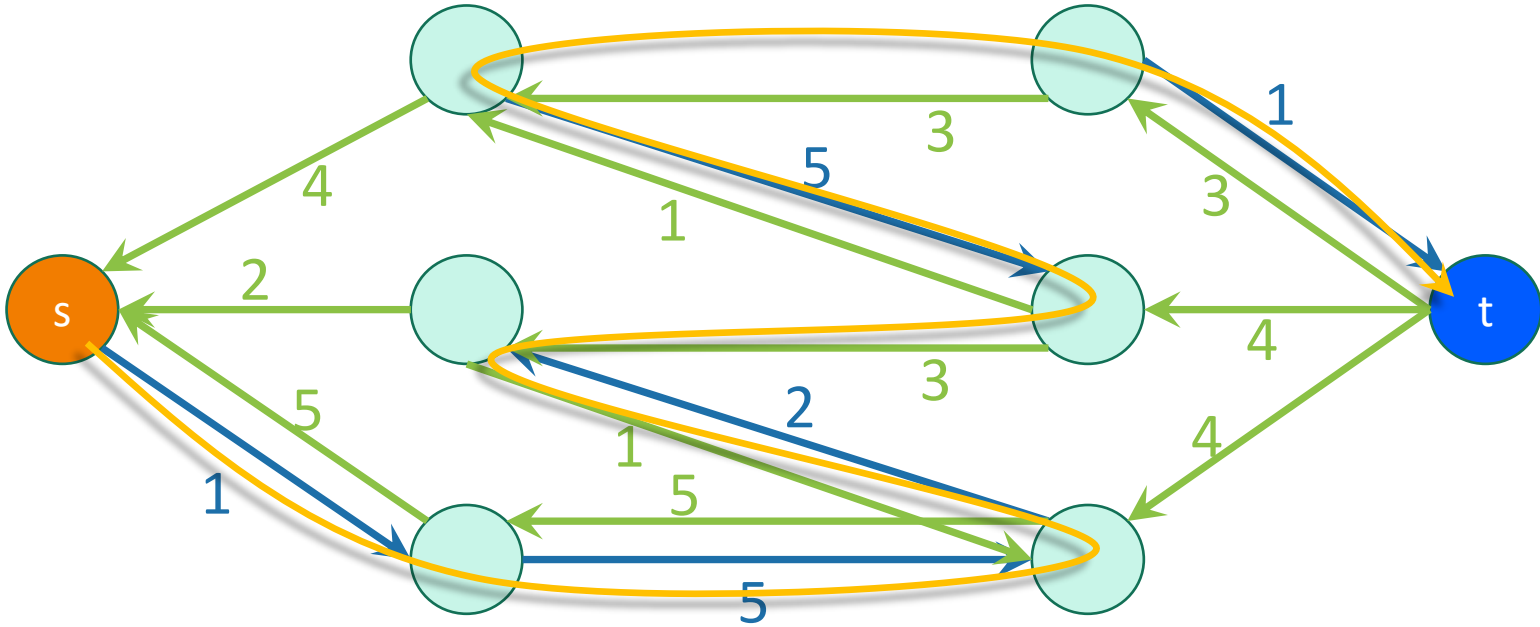

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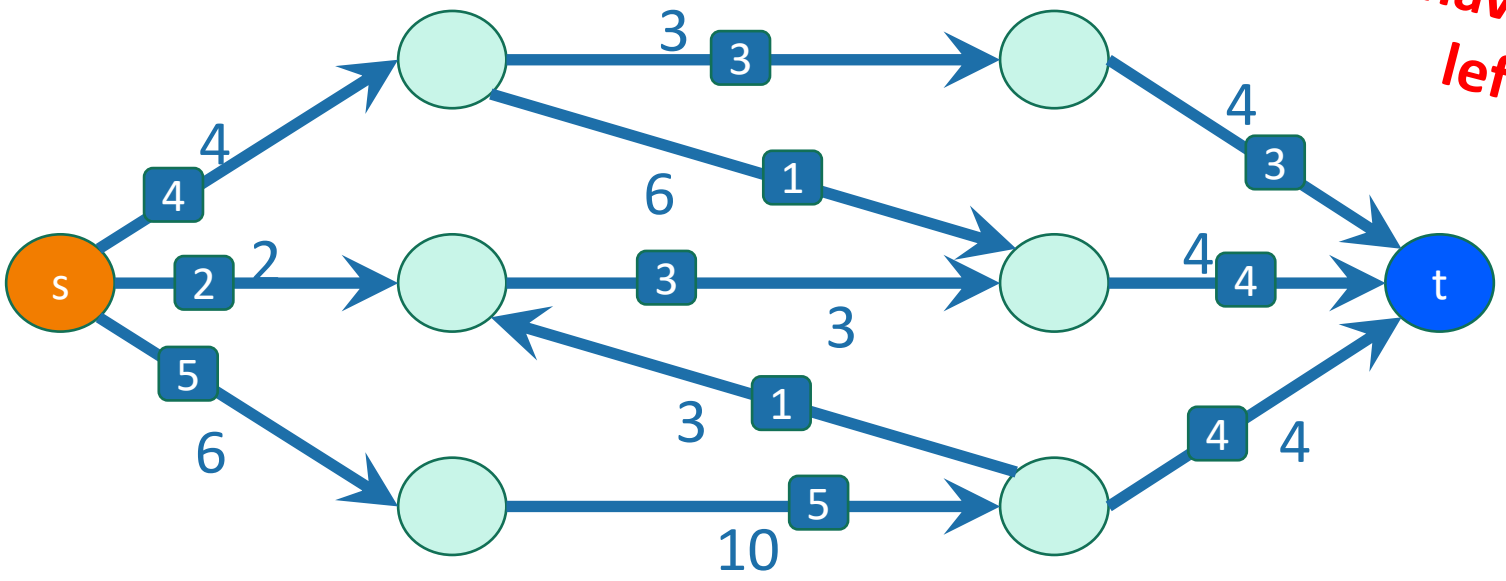
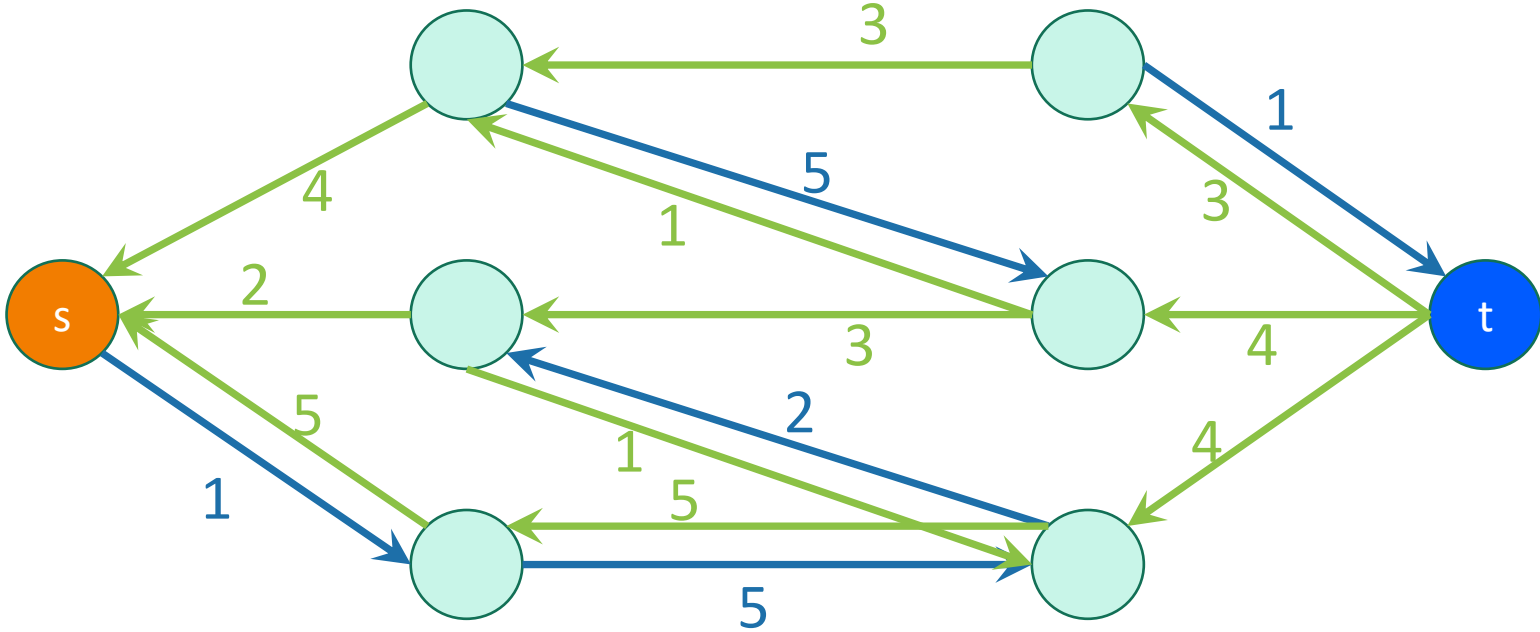
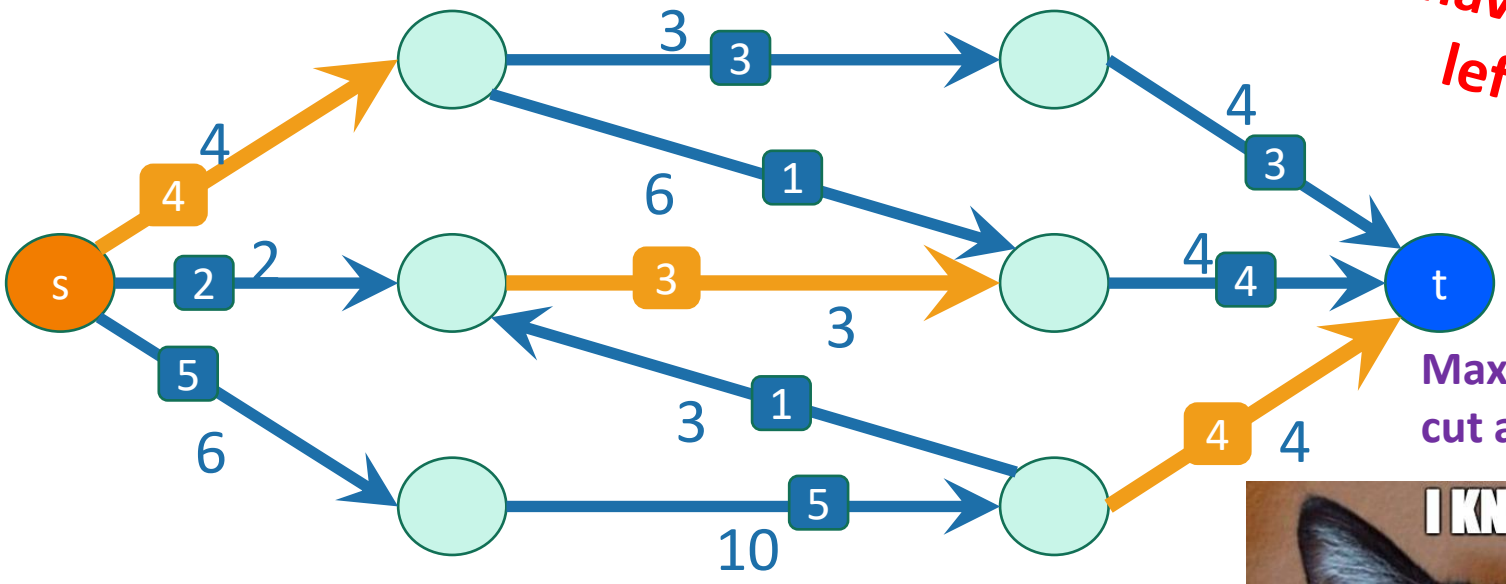


Now we have nothing left to do!

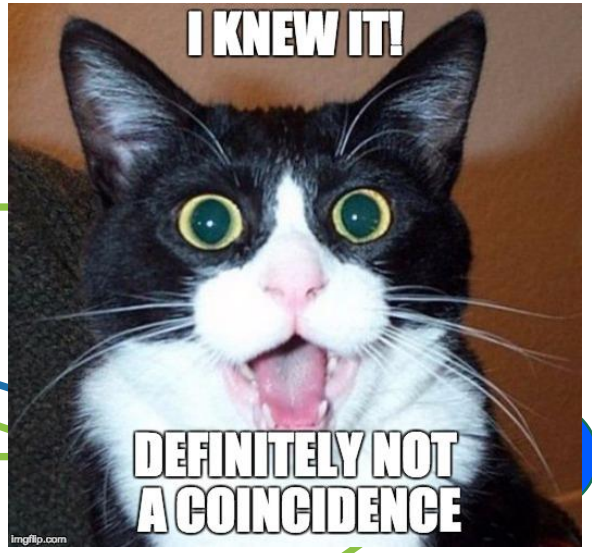


Example of Ford-Fulkerson



Now we have nothing left to do!

Max flow and min cut are both 11.



There's no path from s to t, and here's the cut to prove it.

Doing Ford-Fulkerson with BFS is called the **Edmonds-Karp algorithm**.

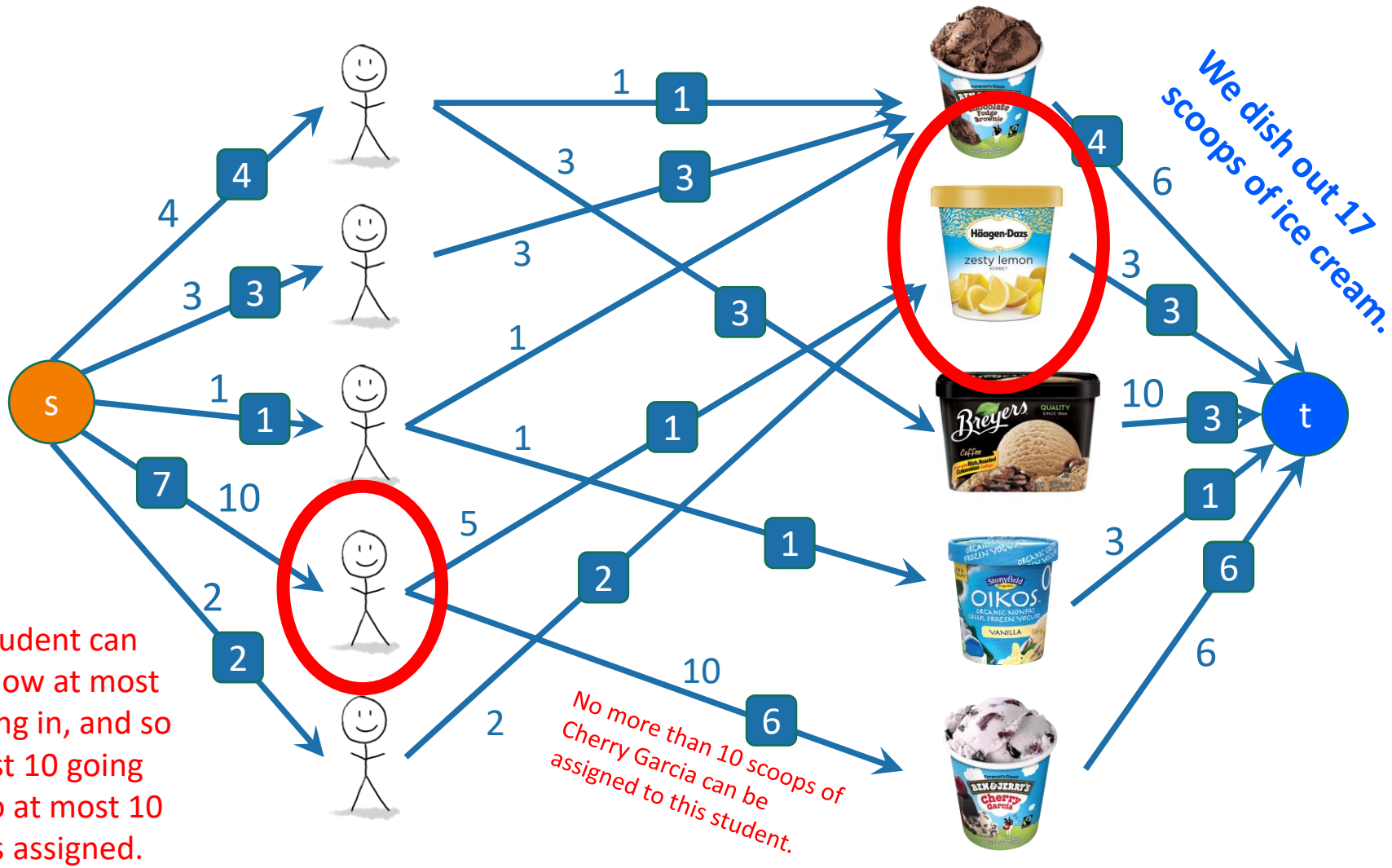
Theorem

- If you use BFS, the Ford-Fulkerson algorithm runs in time **$O(nm^2)$.**

Doesn't have anything to do with the edge weights!

- We will skip the proof in class.
- Basic idea:
  - The number of times you remove an edge from the residual graph is O(n).
    - This is the hard part
  - There are at most m edges.
  - Each time we remove an edge we run BFS, which takes time O(n+m).
    - Actually, O(m), since we don't need to explore the whole graph, just the stuff reachable from s.

Solution via max flow



As before, flows correspond to assignments, and max flows correspond to max assignments.

Recap

- Today we talked about s-t cuts and s-t flows.
- The **Min-Cut Max-Flow Theorem** says that minimizing the cost of cuts is the same as maximizing the value of flows.
- The Ford-Fulkerson algorithm does this!
  - Find an augmenting path
  - Increase the flow along that path
  - Repeat until you can't find any more paths and then you're done!
- An important algorithmic primitive!
  - eg, assignment problems.

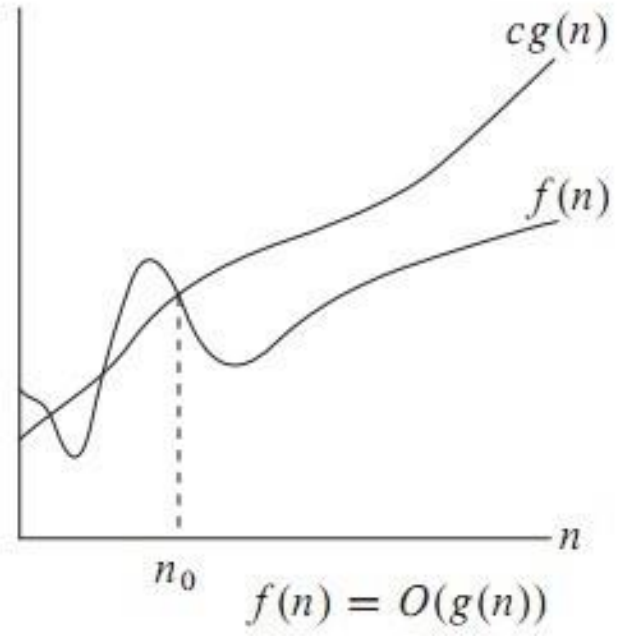
Complexity definitions

- Big-Oh
- Big-Theta
- Big - Omega

Big Oh (*O*)

f(n)= O(g(n)) *iff* there exist positive constants c and n0 such that f(n) ≤ cg(n) for all n ≥ n0

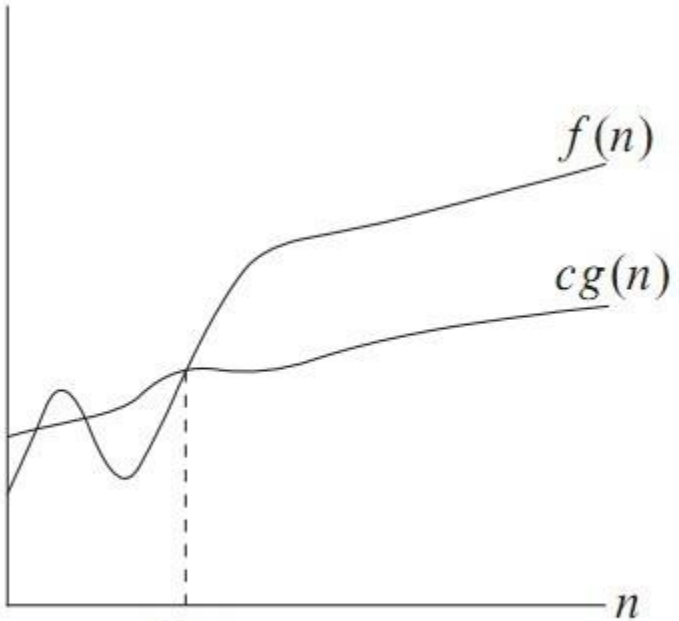
O-notation to give an upper bound on a function



Omega Notation

Big oh provides an asymptotic upper bound on a function.
Omega provides an asymptotic lower bound on a function.

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}.$$

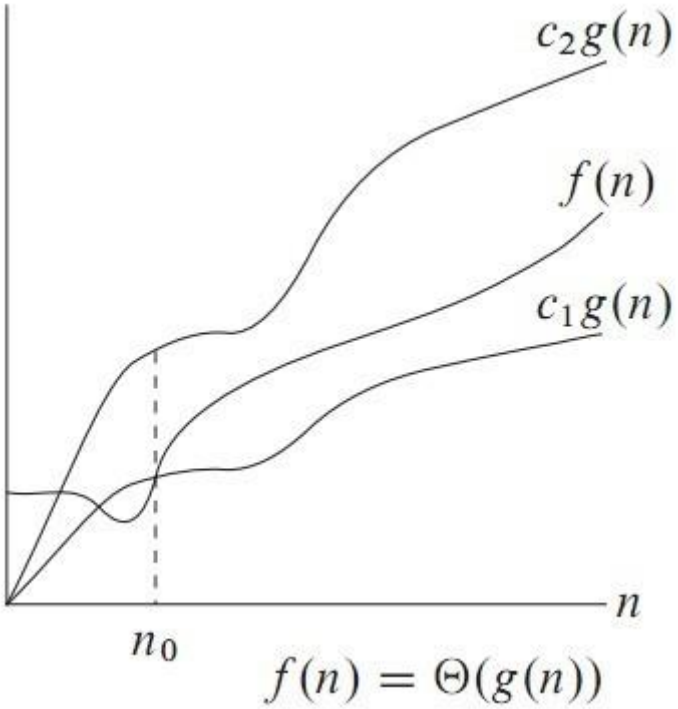


$f(n) = \Omega(g(n))$

Theta Notation

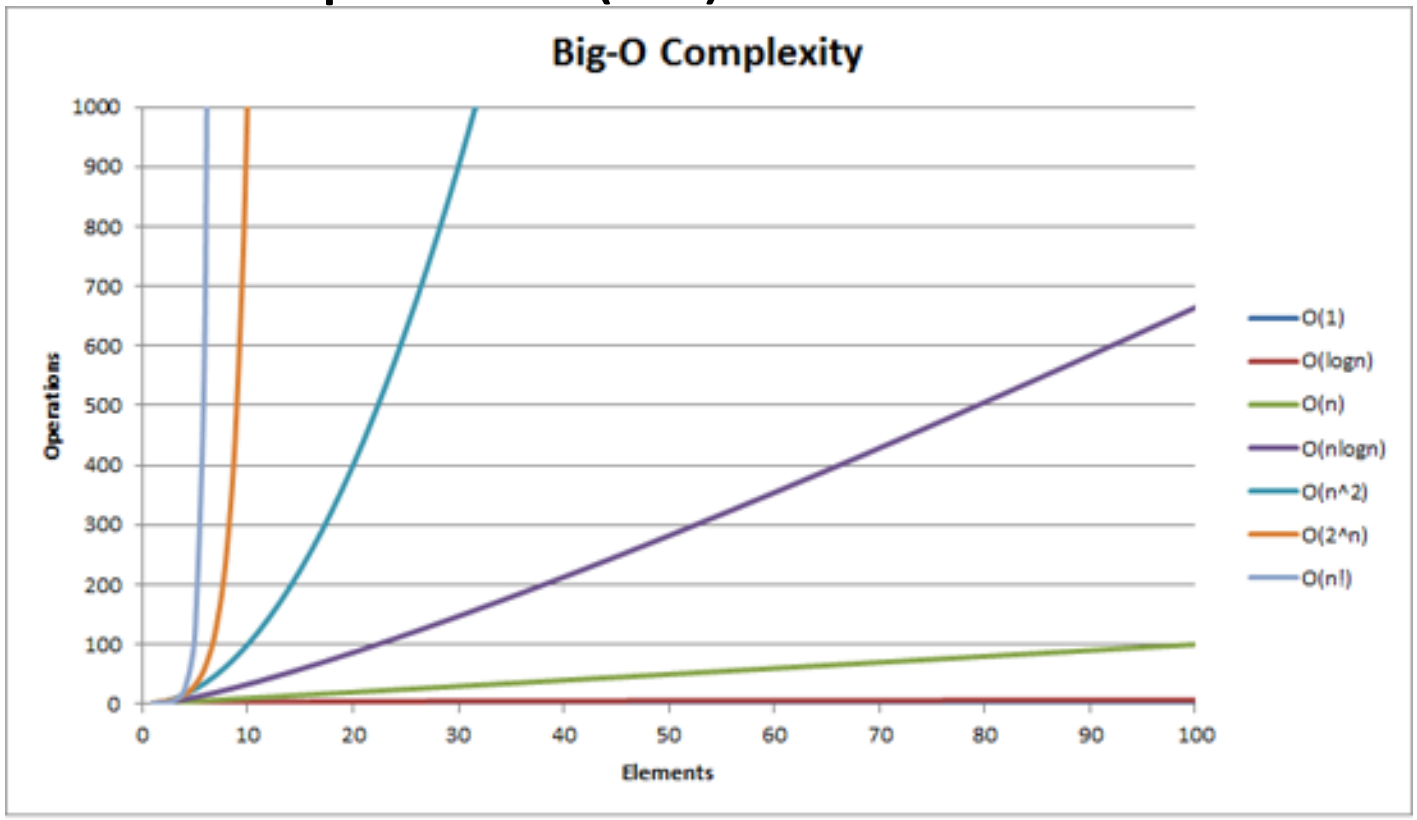
Theta notation is used when function *f* can be bounded both from above and below by the same function *g*

$$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}.^1$$



How bad is exponential complexity

- Fibonacci example – the recursive fib cannot even compute fib(50)



What have we learned?

We've filled out a toolbox

- Tons of examples give us intuition about what algorithmic techniques might work when.
- The technical skills make sure our intuition works out.

But there’s lots more out there

THANK YOU ALL!!!

Week	Date	Topics
1	22 Feb	Introduction. Some representative problems
2	1 March	Stable Matching
3	8 March	Basics of algorithm analysis.
4	15 March	Graphs (Project 1 announced)
5	22 March	Greedy algorithms I
6	29 March	Greedy algorithms II (Project 2 announced)
7	5 April	Divide and conquer
8	12 April	Midterm
9	19 April	Dynamic Programming I
10	26 April	Dynamic Programming II (Project 3 announced)
11	3 May	BREAK
12	10 May	Network Flow-I
13	17 May	Network Flow II
14	24 May	NP and computational intractability I
15	31 May	NP and computational intractability II