30/12/2020, Wednesday

page 373, Example 7, skipped, now to be done:

Ex 7 Find bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & 0-\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix} = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$: if two have an integer sol, it must be a divisor of the constant term: It's possible that the numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ are, possible to be a root of this eq.

Indeed: $\lambda = 2$: $2^3 - 7 \cdot 2^2 + 16 \cdot 2 - 12 = 8 - 28 + 32 - 12 = 0$

$P(\lambda) = \lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda - 2)\, Q(\lambda)$ $Q(\lambda)$: a second-degree polynomial

$$
\begin{array}{r|l}
\lambda^3 - 7\lambda^2 + 16\lambda - 12 & \lambda - 2 \\ \cline{2-2}
\ominus \quad \lambda^3 - 2\lambda^2 \qquad\qquad & \lambda^2 - 5\lambda + 6 \\ \cline{1-1}
-5\lambda^2 + 16\lambda - 12 & \\
-5\lambda^2 + 10\lambda \qquad & \\ \cline{1-1}
6\lambda - 12 & \\
6\lambda - 12 & \\ \cline{1-1}
0 &
\end{array}
$$

$P(\lambda) = (\lambda - 2)(\lambda^2 - 5\lambda + 6)$

$= (\lambda - 2)(\lambda - 2)(\lambda - 3)$

$P(\lambda) = (\lambda - 2)^2 (\lambda - 3)$

Case 1 $\boxed{\lambda = 2}$ $(A - \lambda I)\underset{\sim}{v} = \underset{\sim}{0}$ $\boxed{2x - 2y + z = 0}$

$$
\begin{bmatrix} 4-2 & -2 & 1 \\ 2 & 0-2 & 1 \\ 2 & -2 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$2x-2y+z=0 \rightarrow y=1,\ z=0 \rightarrow x=2$

$y=0 \quad z=2 \rightarrow x=-1$

$V_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ are two eigenvectors corresponding to $\lambda = 2$.

The base of the eigenspace corresponding to $\lambda = 2$ is

$$\{v_1, v_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\} = \{(2,1,0), (-1,0,2)\}$$

$$= \{ [2 \ 1 \ 0]^T, [-1 \ 0 \ 2]^T \}$$

$\boxed{\lambda = 3}$ $(A - \lambda I)\underset{\sim}{V} = \underset{\sim}{0}$

$$\begin{bmatrix} 4-3 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2y + z = 0 \quad (1)$$
$$2x - 3y + z = 0 \quad (2)$$
$$2x - 2y = 0 \quad (3)$$

(3) implies $x = y$: (1) $-x + z = 0$, (2) $-x + z = 0$ $\Big\}$ $x = z$

$x = y = z \Rightarrow$ Let $x = 1 = y = z$ $\quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{x=1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = V_3$$

The eigenspace corresponding to the eigenvalue $\lambda = 3$ is $\{V_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

<u>Comment</u> When finding the eigenvector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corr.

to $\lambda = 2$, we solved $2x - 2y + z = 0$, there's

no other cond. on x, y, z.

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (2y-z)/2 \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

Let $y = 1,\ z = 0$ $\qquad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Let $y = 0,\ z = 2$ $\qquad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$

p 381, 6.2 Diagonalization of Matrices

Th.1 An $n \times n$ A is diagonalizable iff A has n has linearly independent eigenvectors.

Th 2 The eigenvectors of a matrix A corresponding to different eigenvalues are linearly independent.

Th3 If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

page 381, Example 8

Is $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$ diagonalizable ??

A is a 3x3 matrix; $\{A$ is diagonalizable$\} \Leftrightarrow \{A$ has $n=3$ linearly ind. eigenvectors$\}$ Th.1

$\lambda_1 = \lambda_2 = \lambda = 2 \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$\lambda_3 = 3 \rightarrow v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since A has 3 linearly independent eigenvectors, then A is diagonalizable.

$$P = \begin{bmatrix} | & | & | \\ V_1 & V_2 & V_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$D = P^{-1} A P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Theorem 4 Complete independence of eigenvectors

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of the $n \times n$ matrix A $(k \leq n)$. For each k, let $S_1, S_2, \dots, S_k$ be the bases of eigenspaces of A, associated with $\lambda_1, \lambda_2, \dots, \lambda_k$.

Then, the union S of the bases $S_1, S_2, \dots, S_k$ is a linearly independent set of eigenvectors of A.

6.3 Applications Involving Powers of Matrices, p 383

Remark Suppose A is an $n \times n$ diagonalizable matrix.

$$\boxed{D = P^{-1} A P} \Rightarrow PD = AP \Rightarrow PDP^{-1} = A$$

$$A = P D P^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$A^3 = A^2 A = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$$

$$\vdots$$

$$A^k = P D^k P^{-1}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

In order to calculate A^k, we need to perform k matrix product ops.

For this 3×3 matrix A, $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$D^2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$$

$$A^k = \underbrace{P \, D^k \, P^{-1}}_{\text{requires 2 products as we directly know } D^k \text{ !!!}}$$

requires k ops. of matrix product

This reduces computational cost drastically !!!

Example

Find A^5 if $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

Remember that $\lambda_1 = 3 \quad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$

$\lambda_2 = 2 \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$$\begin{array}{cc} & \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ P = & \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \end{array}, \quad P^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$D = P^{-1} A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow D^5 = \begin{bmatrix} 3^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 2^{-} \end{bmatrix}$$

__Example__

Find A^5 if $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$$A^k = P D^k P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 243 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 454 & -422 & 211 \\ 422 & -390 & 211 \\ 422 & -422 & 243 \end{bmatrix}$$

$$D = P^{-1} A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow D^5 = \begin{bmatrix} 3^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 2^5 \end{bmatrix}$$

Remember that the eigenvalue equation of $n \times n$ matrix A

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

gives us an n-th degree algebraic equation

$$(-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0 = 0.$$

Let's call the LHS as $p(\lambda)$, the characteristic polynomial:

$$p(\lambda) = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0.$$

* $p(\lambda) = 0 \iff \lambda$ is an eigenvalue of A.

The Cayley-Hamilton Theorem

If the $n \times n$ matrix A has the characteristic polynomial

$$p(\lambda) = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0 ,$$

then

$$p(\underset{\sim}{A}) = (-1)^n \underset{\sim}{A}^n + c_{n-1} \underset{\sim}{A}^{n-1} + \cdots + c_1 \underset{\sim}{A} + c_0 \underset{\sim}{I} = \underset{\sim}{0}$$

This means: Any matrix A satisfies its own eigenvalue equation, when considered as a matrix equation!!

Example

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = 0 \quad \begin{array}{l} \lambda_1 = -2 \\ \lambda_2 = 3 \end{array}$$

$$p(\lambda) = \lambda^2 - \lambda - 6.1$$

$$p(A) = A^2 - A - 6I =$$

$$= \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} - \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$$

$$-6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{-----} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example 6, p 392

Consider $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$$p(\lambda) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12$$

$$p(A) = -A^3 + 7A^2 - 16A + 12I = \underset{\sim}{0}$$

We can calculate $A^2 = \begin{bmatrix} 14 & -10 & 5 \\ 10 & -6 & 5 \\ 10 & -10 & 9 \end{bmatrix}$

$$A^3 = 7A^2 - 16A + 12I = \begin{bmatrix} 46 & -38 & 19 \\ 38 & -30 & 19 \\ 38 & -38 & 27 \end{bmatrix}$$

$A^4 = A^3 A$, $\quad A^6 = A^3 \cdot A^3$, ...

Furthermore, $\quad -A^3 + 7A^2 - 16A + 12I = \underset{\sim}{0}$

Multiply by A^{-1} $\quad -A^{-1}A^3 + 7A^{-1}A^2 - 16A^{-1}A + 12A^{-1}I = \underset{\sim}{0}$

to get

$$A^{-1} = \frac{1}{12}\left(A^2 - 7A + 16I\right)$$

$$= \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ -2 & 5 & -1 \\ -2 & 2 & 2 \end{bmatrix}$$

Remark

$$(-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

$$(-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + (-1)^n \det A = 0$$

$\det A = 0 \Rightarrow c_0 = 0 \Rightarrow$ Cayley-Ham. does not give A^{-1},
AS IT SHOULD NOT!!!

Chapter 7

Next week Linear System of Diff. Eqs

Read 7.1, 7.2