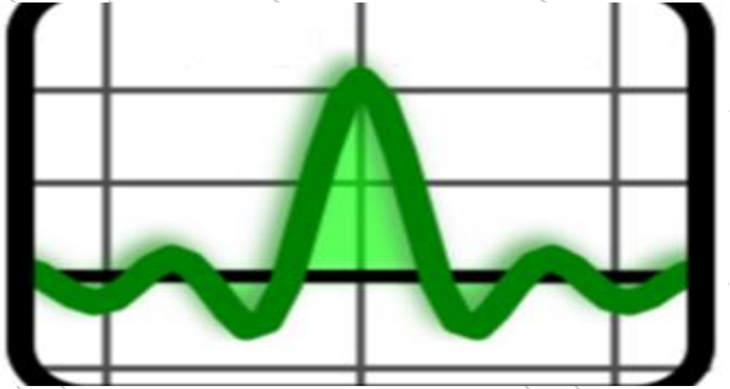

Signals & Systems For Computer Engineering

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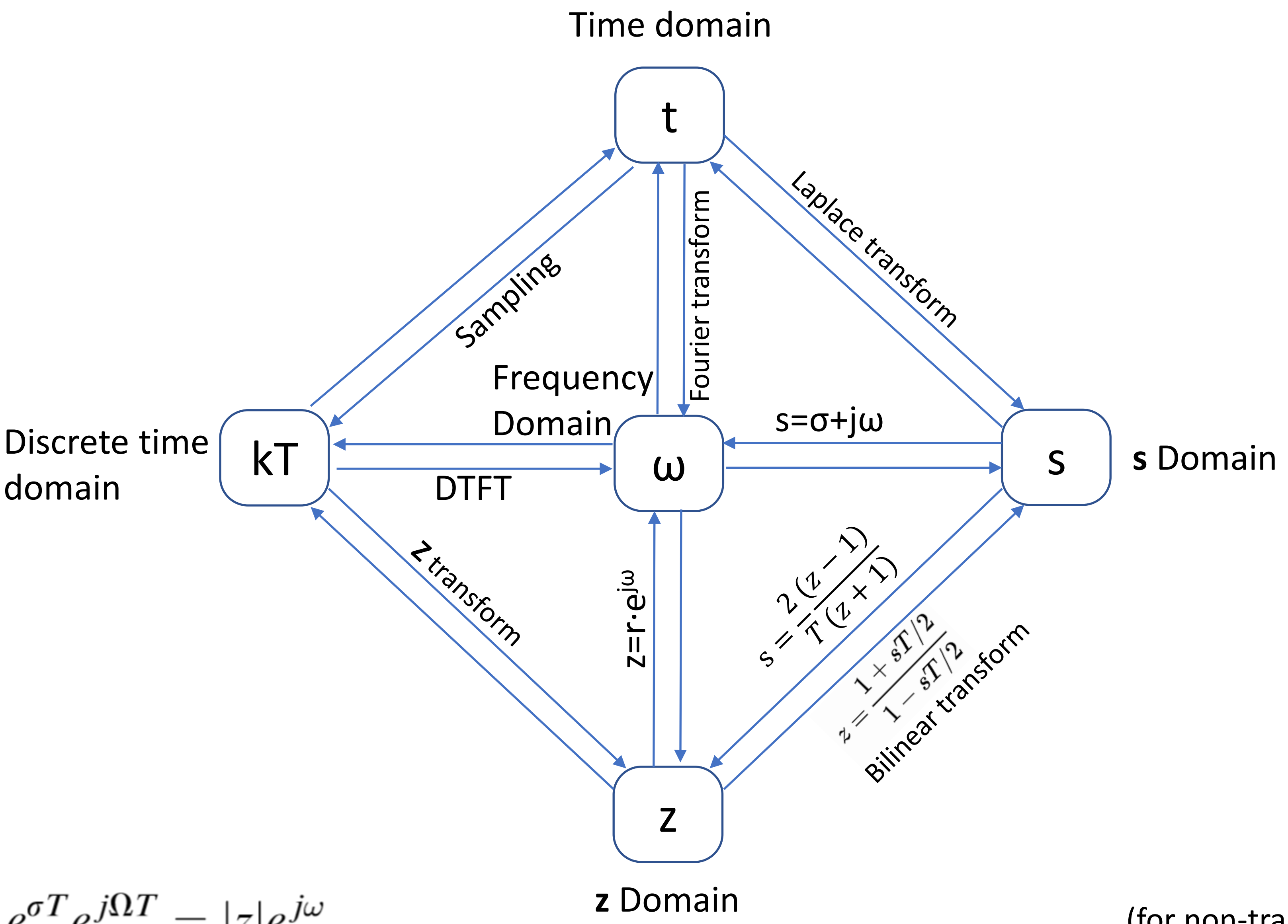
BLG354E / CRN: 21560
7th Week Lecture

Signal Processing Domains for Analysis, Design and Implementation of the Systems

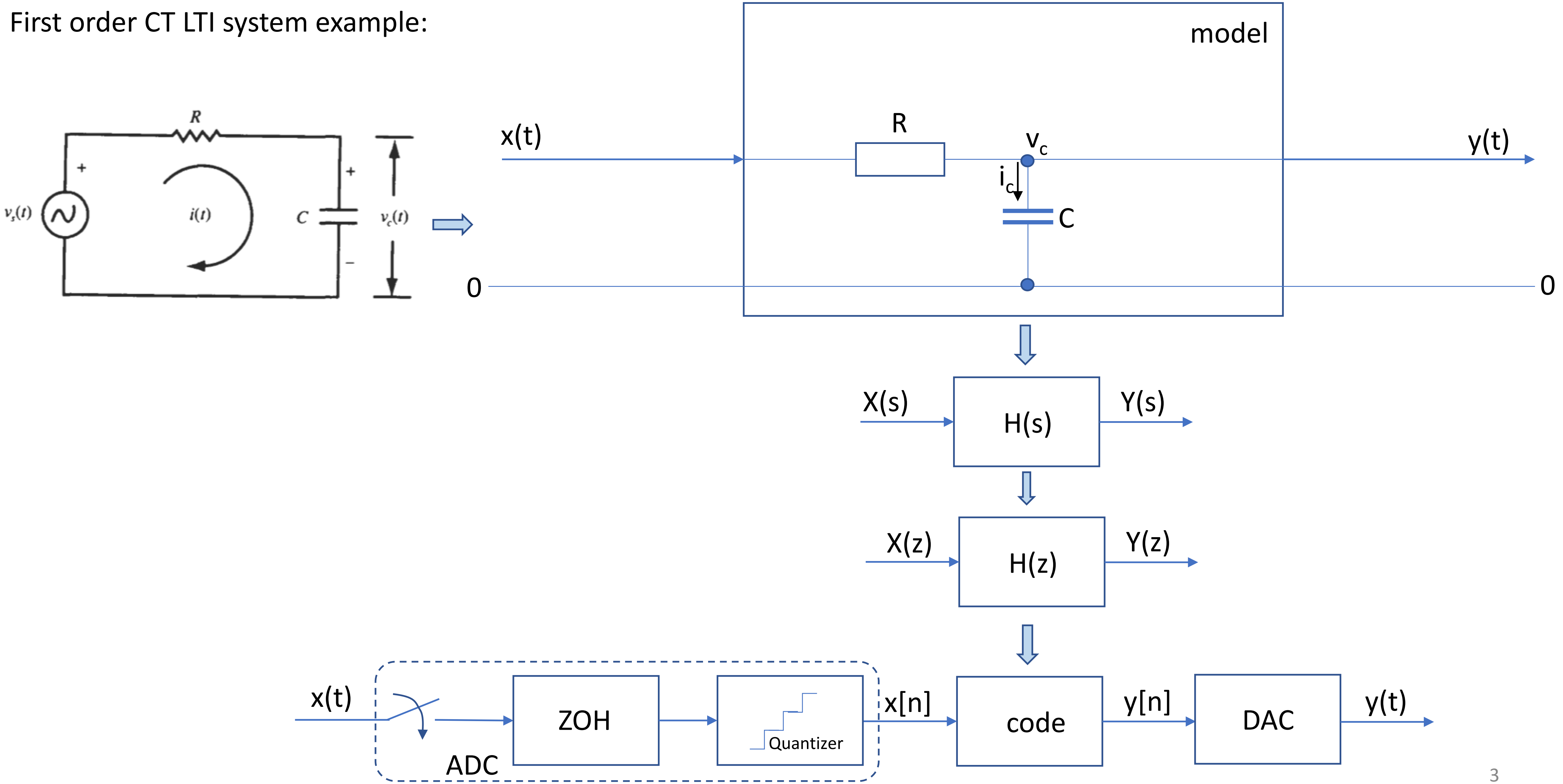


$$z = e^{sT} = e^{\sigma T} e^{j\Omega T} = |z| e^{j\omega}$$

(for non-transients: r=1, σ=0)

Analysis of time domain systems and synthesis of their digital equivalent:

First order CT LTI system example:



Laplace Transform

Laplace transform is the most powerful technique used to describe, represent and analyze analog signals and the systems

h(t) was defined as impulse response of the LTI system.

Output of the system y(t) to the complex exponential input e^{st} is,

Pierre-Simon, marquis de Laplace
(1749- 1827)

$$y(t) = \boldsymbol{T}\{e^{st}\} = H(s)e^{st}$$

where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$

H(s) is referred as Laplace transform of h(t).

For a general continuous-time signal x(t), Bilateral Laplace transform X(s) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$s=\sigma+j\omega$

unilateral (or one-sided) Laplace transform, which is defined as

$$X_I(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

bilateral and unilateral transforms are equivalent only if x(t) = 0 for t < 0

$$X(s) = \mathcal{L}\{x(t)\}$$

$$x(t) \leftrightarrow X(s)$$

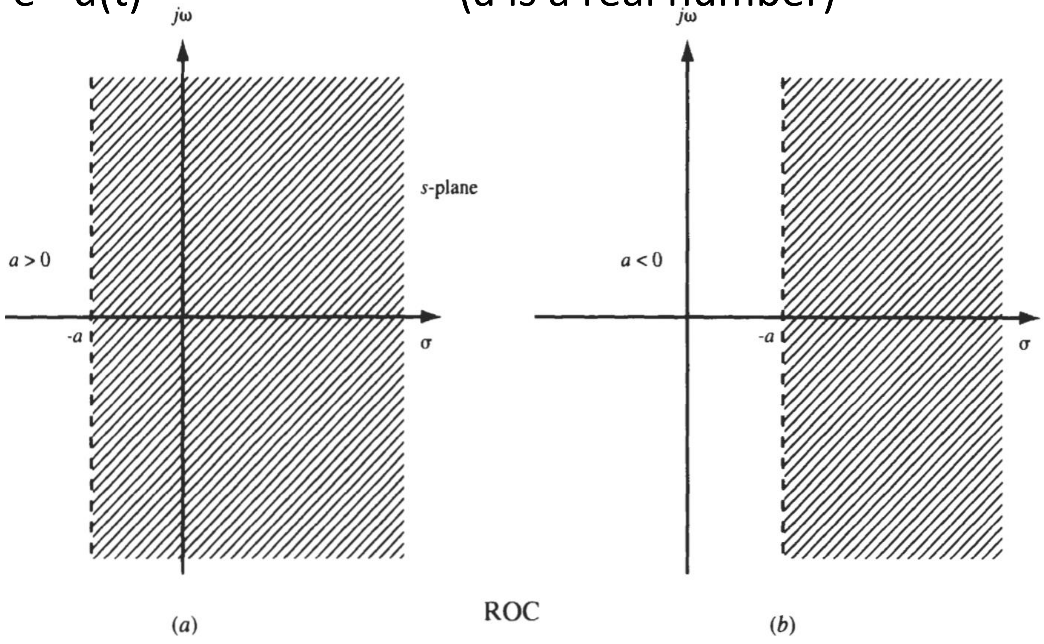
The Region of Convergence (ROC):
The range of values of the complex variables s for which the Laplace transform converges is called the region of convergence

Example: Find the Laplace Transform of the signal $x(t)=e^{-at}u(t)$ (a is a real number)

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(s+a)t}\,dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_{0^+}^{\infty} = \frac{1}{s+a} \qquad \mathrm{Re}(s) > -a$$

because $\lim_{t\to\infty} e^{-(s+a)t} = 0$ only if $\mathrm{Re}(s+a) > 0$ or $\mathrm{Re}(s) > -a$.

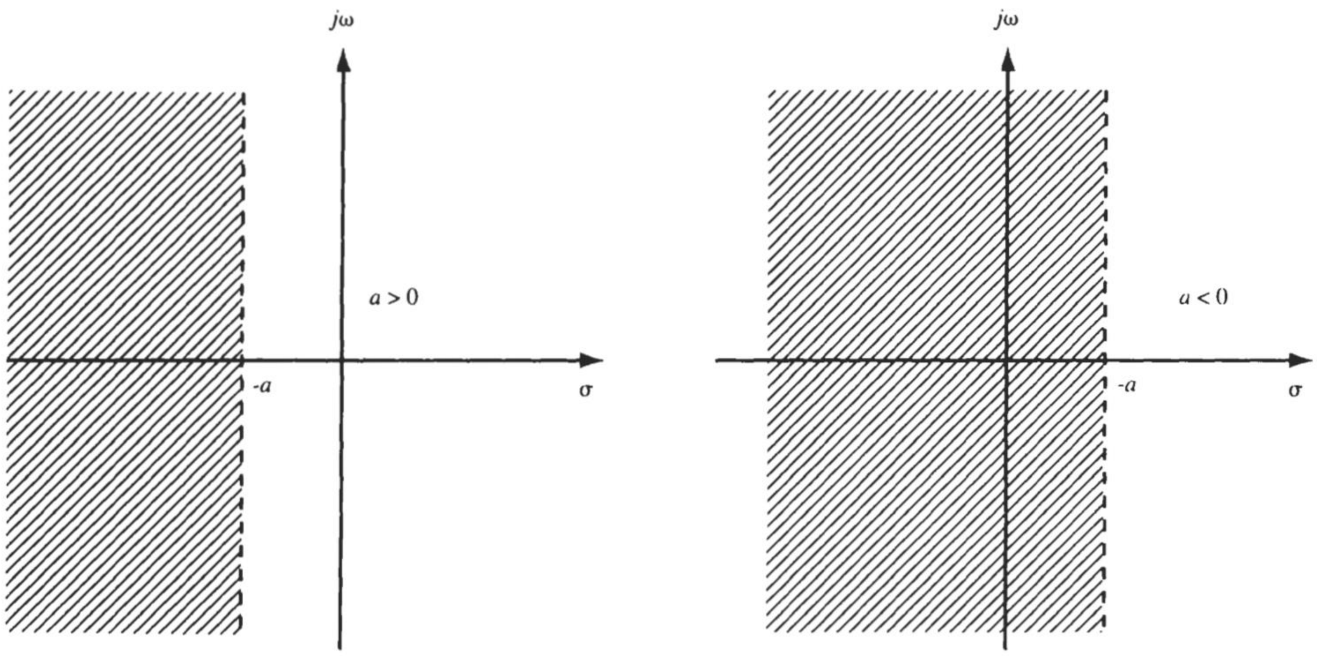


ROC for this example is specified as Re(s) > -a and is displayed in the complex plane as shown in the figure by the shaded area to the right of the line Re(s) = -a. In Laplace transform applications, the complex plane is commonly referred to as the s-plane. The horizontal and vertical axes are sometimes referred to as the σ-axis and the jω-axis, respectively.

Example: Find the Laplace Transform of the signal $x(t)=-e^{-at}u(t)$ (a is a real number)

$$X(s) = \frac{1}{s+a} \qquad \mathrm{Re}(s) < -a$$

ROC for this example is specified as Re(s) < -a and is displayed in the complex plane as shown in the Figure by the shaded area to the left of the line Re(s) = -a. Comparing these two examples, we see that the algebraic expressions for X(s) for these two different signals are identical except for the ROCs. Therefore, in order for the Laplace transform to be unique for each signal x(t), the ROC must be specified as part of the transform.



(a) ROC (b)

Poles and Zeros of X(s):

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0}{b_0} \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

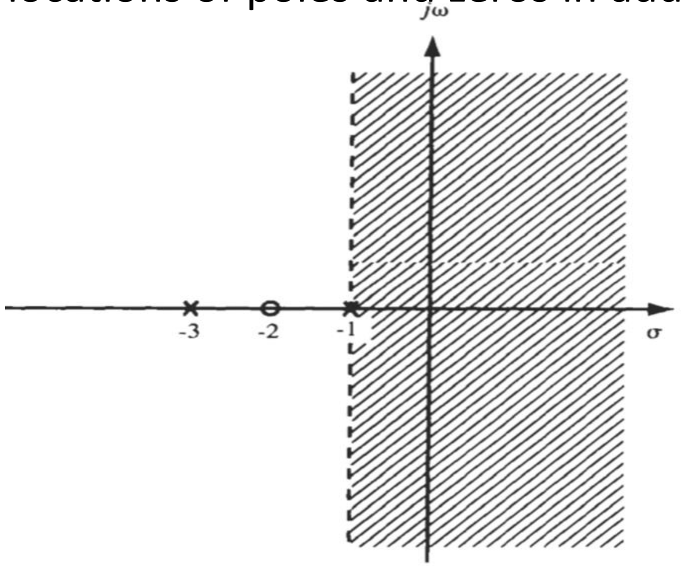
The coefficients a, and b, are real constants, and m and n are positive integers. The X(s) is called a proper rational function if n>m, and an improper rational function if n≤m. The roots of the numerator polynomial, z_k, are called the zeros of X(s) because X(s) = 0 for those values of s.

The roots of the denominator polynomial, p_k, are called the poles of X(s) because X(s) is infinite for those values of s. Therefore, the poles of X(s) lie outside the ROC since X(s) does not converge at the poles, by definition. The zeros, on the other hand, may lie inside or outside the ROC. Except for a scale factor a_0/b_0, X(s) can be completely specified by its zeros and poles. Thus, a very compact representation of X(s) in the s-plane is to show the locations of poles and zeros in addition to the ROC.

Example:

$$X(s) = \frac{2s + 4}{s^2 + 4s + 3} = 2\frac{s + 2}{(s + 1)(s + 3)} \qquad \text{Re}(s) > -1$$

$X(s)$ has one zero at $s = -2$ and two poles at $s = -1$ and $s = -3$ with scale factor 2.



s-plane representation of $X(s) = (2s + 4)/(s^2 + 4s + 3)$.

LAPLACE TRANSFORMS OF SOME COMMON SIGNALS:

Unit impulse function δ(t):

$$\mathscr{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st}\, dt = 1$$

Unit step function u(t):

$$\mathscr{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \int_{0^+}^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s} \qquad \mathrm{Re}(s) > 0$$

Example: Find the Laplace Transform of $x(t)=e^{at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} e^{at} u(-t) e^{-st}\, dt = \int_{-\infty}^{0^-} e^{-(s-a)t}\, dt$$

$$= -\frac{1}{s-a} e^{-(s-a)t}\Big|_{-\infty}^{0-} = -\frac{1}{s-a} \qquad \mathrm{Re}(s) < a$$

Some Laplace Transform Pairs:

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(a)$
$\cos \omega_0 t u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$e^{-at}\cos \omega_0 t u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$
$e^{-at}\sin \omega_0 t u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(a)$

Laplace Transform Properties:

1- Linearity: $\left.\begin{array}{ll} x_1(t) \leftrightarrow X_1(s) & \mathrm{ROC} = R_1 \\ x_2(t) \leftrightarrow X_2(s) & \mathrm{ROC} = R_2 \end{array}\right\} \Rightarrow a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(s) + a_2 X_2(s) \qquad R' \supset R_1 \cap R_2$

2- Time shifting: $x(t) \leftrightarrow X(s) \qquad \mathrm{ROC} = R \qquad \Rightarrow x(t - t_0) \leftrightarrow e^{-st_0} X(s) \qquad R' = R$

3- Shifting in s domain: $x(t) \leftrightarrow X(s) \qquad \mathrm{ROC} = R \qquad \Rightarrow e^{s_0 t} x(t) \leftrightarrow X(s - s_0) \qquad R' = R + \mathrm{Re}(s_0)$

4- Time scaling: $x(t) \leftrightarrow X(s) \qquad \mathrm{ROC} = R \qquad \Rightarrow x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad R' = aR$

5- Time reversal: $x(t) \leftrightarrow X(s) \qquad \mathrm{ROC} = R \qquad \Rightarrow x(-t) \leftrightarrow X(-s) \qquad R' = -R$

6- Differentiation in Time domain: $x(t) \leftrightarrow X(s) \qquad \mathrm{ROC} = R \qquad \Rightarrow \frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$

7- Differentiation in s domain: $x(t) \leftrightarrow X(s) \qquad \mathrm{ROC} = R \qquad \Rightarrow -tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$

8- Integration in time domain: $x(t) \leftrightarrow X(s) \qquad \mathrm{ROC} = R \qquad \Rightarrow \int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s} X(s) \qquad R' = R \cap \{\mathrm{Re}(s) > 0\}$

9- Convolution: $\left.\begin{array}{ll} x_1(t) \leftrightarrow X_1(s) & \mathrm{ROC} = R_1 \\ x_2(t) \leftrightarrow X_2(s) & \mathrm{ROC} = R_2 \end{array}\right\} \Rightarrow x_1(t) * x_2(t) \leftrightarrow X_1(s) X_2(s) \qquad R' \supset R_1 \cap R_2$

Question: Prove the convolution property → $x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s) \qquad R' \supset R_1 \cap R_2$

Let $$y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$$

By definition of the Laplace Transform: $$Y(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau\right]e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-st}\,dt\right]d\tau$$

Since bracketed term in the last expression is the Laplace transform of the shifted signal $x_2(t-\tau)$,

$$Y(s) = \int_{-\infty}^{\infty} x(\tau)e^{-s\tau}X_2(s)\,d\tau$$

$$= \left[\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}\,d\tau\right]X_2(s) = X_1(s)X_2(s)$$

$$x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s) \qquad R' \supset R_1 \cap R_2$$

Example: Find Laplace Transform of the signal $x(t)=t\cdot u(t)$ by using properties of the Laplace transform

$$tu(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2} \qquad \mathrm{Re}(s) > 0$$

Example: Find Laplace Transform of the signal $x(t)=e^{-at}\cdot u(t)$ by using properties of the Laplace transform

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a} \qquad \mathrm{Re}(s) > -a$$

Example: Find Laplace Transform of the signal $x(t)=t\cdot e^{-at}\cdot u(t)$ by using properties of the Laplace transform

$$te^{-at}u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2} \qquad \mathrm{Re}(s) > -a$$

Example: Find Laplace Transform of the signal $x(t)=\cos(\omega_0 t)\cdot u(t)$ by using properties of the Laplace transform

$$\cos \omega_0 t u(t) = \tfrac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})u(t) = \tfrac{1}{2}e^{j\omega_0 t}u(t) + \tfrac{1}{2}e^{-j\omega_0 t}u(t)$$

$$\cos \omega_0 t u(t) \longleftrightarrow \frac{1}{2}\frac{1}{s-j\omega_0} + \frac{1}{2}\frac{1}{s+j\omega_0} = \frac{s}{s^2+\omega_0^2} \qquad \mathrm{Re}(s) > 0$$

Example:

The input signal $x(t) = e^{-2t}u(t)$ is applied to an LTI system, and the output of the system is given as

$y(t) = (e^{-t} + e^{-2t} - e^{-3t})u(t)$. Find the system's transfer function $H(s)$ and the impulse response $h(t)$.

From the Laplace Transform table, we have: $X(s) = \frac{1}{s+2}$ and $Y(s) = \frac{1}{s+1} + \frac{1}{s+2} - \frac{1}{s+3}$

Transfer function: $H(s) = \frac{Y(s)}{X(s)} = 1 + \frac{s+2}{s+1} - \frac{s+2}{s+3}$

Transfer function can be written as $H(s) = \frac{s^2+6s+7}{(s+1)(s+3)} = 1 + \frac{1}{s+1} + \frac{1}{s+3}$

If we use the table for inverse transform then we get $h(t) = \delta(t) + (e^{-t} + e^{-3t})u(t)$

Laplace transform with Python:

a) Library function method:

```
import sympy

t, s = sympy.symbols('t, s')
a = sympy.symbols('a', real=True, positive=True)
f = sympy.exp(-a*t)
F = sympy.laplace_transform(f, t, s, noconds=True)

print(F)
```

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\mathrm{d}s$$

Output:

```
1/(a + s)
```

b) Direct transform method:

```
F = sympy.integrate(f*sympy.exp(-s*t), (t, 0, sympy.oo))
```

Output:

```
Piecewise((1/(s*(a/s + 1)), Abs(arg(s)) <= pi/2), (Integral(exp(-a*t)*exp(-s*t), (t, 0, oo)), True))
```

$$\begin{cases} \frac{1}{s\left(\frac{a}{s}+1\right)} & \text{for } \left|\arg{(s)}\right| \leq \frac{\pi}{2} \\ \int\limits_0^{\infty} e^{-at}e^{-st}\, dt & \text{otherwise} \end{cases}$$

Example: CT signal x(t) is given as x(t)=0.5t(u(t)-u(t-5)) .

a) Draw x(t) b) Find the Laplace Transform of x(t) by using sympy

```
import sympy
x = 0.5*t*(sympy.Heaviside(t)-sympy.Heaviside(t-5))
sympy.plot(f2);
X = sympy.laplace_transform(x, t, s)
print(X)
```

$\mathcal{L}$\{x(t)\}=X(s)= ((-2.5*s + 0.5*exp(5*s) - 0.5)*exp(-5*s)/s**2, 0, True)

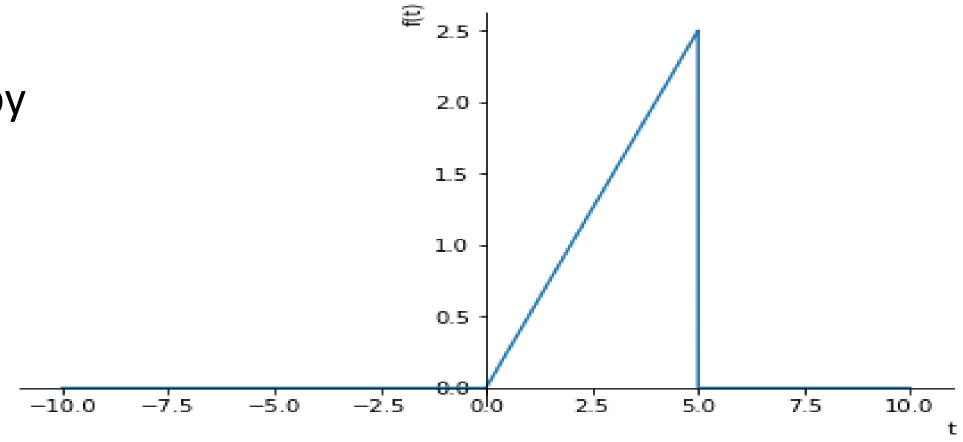


Table of Laplace Transforms by using "sympy":

```
import sympy

def L(f):
   return sympy.laplace_transform(f, t, s, noconds=True)

t, s = sympy.symbols('t, s')
a = sympy.symbols('a', real=True, positive=True)

omega = sympy.Symbol('omega', real=True)
exp = sympy.exp
sin = sympy.sin
cos = sympy.cos

functions = [1,
        t,
        exp(-a*t),
        t*exp(-a*t),
        t**2*exp(-a*t),
        sin(omega*t),
        cos(omega*t),
        1 - exp(-a*t),
        exp(-a*t)*sin(omega*t),
        exp(-a*t)*cos(omega*t),
        ]
print("fonksiyonlar:",functions)
F = [L(f) for f in functions]
print("Laplace dönüşümleri:",F)
```

fonksiyonlar: [1, t, exp(-a*t), t*exp(-a*t), t**2*exp(-a*t), sin(omega*t), cos(omega*t), 1 - exp(-a*t), exp(-a*t)*sin(omega*t), exp(-a*t)*cos(omega*t)]

Laplace dönüşümleri: [1/s, s**(-2), 1/(a + s), (a + s)**(-2), 2/(a + s)**3, omega/(omega**2 + s**2), s/(omega**2 + s**2), a/(s*(a + s)), omega/(omega**2 + (a + s)**2), (a + s)/(omega**2 + (a + s)**2)]

Inverse Laplace Transform

Inversion of the Laplace transform to find the signal x(t) from its Laplace transform X(s) is called the inverse Laplace transform, symbolically denoted as

$$\mathrm{x}(t) = \mathcal{L}^{-1}\{X(s)\}$$

Inverse Laplace transform methods:

a) Inversion formula: $$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

In this integral, the real c is to be selected such that if the ROC of X(s) is $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, then $\sigma_1 < c < \sigma_2$

b) Use of Tables of Laplace Transform Pairs:

for the inversion of X(s), we attempt to express X(s) as a sum $X(s) = X_1(s) + \cdots + X_n(s) \rightarrow x(t) = x_1(t) + \cdots + x_n(t)$

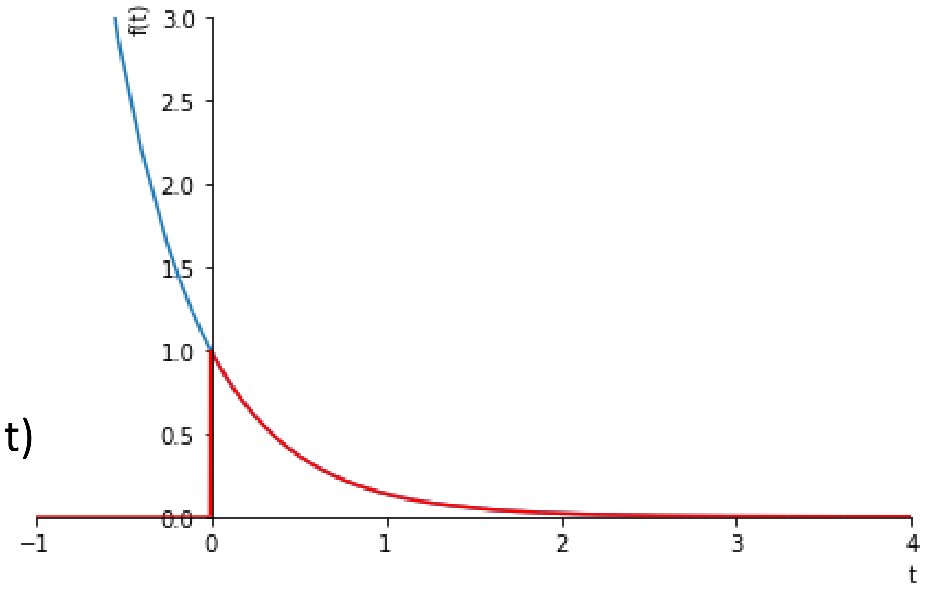
c) Partial-Fraction Expansion for $X(s) = \dfrac{N(s)}{D(s)} = k\dfrac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$ for m<n:

* Simple Pole Case: $X(s) = \dfrac{c_1}{s-p_1} + \cdots + \dfrac{c_n}{s-p_n}$ $\qquad c_k = (s-p_k)X(s)\big|_{s=p_k}$

* Multiple Pole Case: $\dfrac{\lambda_1}{s-p_i} + \dfrac{\lambda_2}{(s-p_i)^2} + \cdots + \dfrac{\lambda_r}{(s-p_i)^r}$

where $\lambda_{r-k} = \dfrac{1}{k!}\dfrac{d^k}{ds^k}\left[(s-p_i)^r X(s)\right]\Big|_{s=p_i}$

Inverse Laplace transform with Python:




```
import sympy
import matplotlib.pyplot as plt

def invL(F):
    return sympy.inverse_laplace_transform(F, s, t)

t, s = sympy.symbols('t, s')
a = sympy.symbols('a', real=True, positive=True)
```

Function in time domain ←

```
f = sympy.exp(-a*t)
```

Laplace transform of f(t) ←

```
F = sympy.laplace_transform(f, t, s, noconds=True)
```

1/(a + s) ←

```
print(F)

p = sympy.plot(f.subs({a: 2}), invL(F).subs({a: 2}),
               xlim=(-1, 4), ylim=(0, 3), show=False)
p[1].line_color = 'red'
p.show()
```

Example:

Find the inverse Laplace transform of $X(s) = \dfrac{s}{s^2+4}$, $\mathrm{Re}(s) > 0$

From the transform pairs table we obtain: $x(t) = \cos 2t u(t)$

```
import sympy
t, s = sympy.symbols('t, s’)
x=sympy.inverse_laplace_transform(s/(s**2+4), s, t)
print(‘x(t)=,’x)
```

→ cos(2*t)*Heaviside(t)

Example:

Find the inverse Laplace transform of $X(s) = \dfrac{e^{-as}s}{s^2+a^2}$

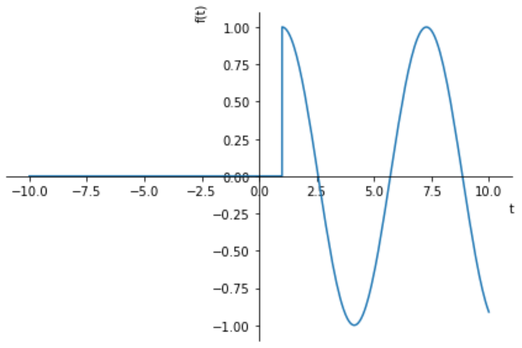
```
import sympy
t, s = sympy.symbols('t, s')
a = sympy.symbols('a', real=True, positive=True)
exp = sympy.exp
x=sympy.inverse_laplace_transform((exp(-a * s)*s)/(s**2+a**2), s, t)
print(‘x(t)=,’x)
```

→ x(t)=cos(a*(a - t))*Heaviside(-a + t)

As an example, If a=1 then x(t)→

```
import sympy
from sympy.plotting import plot
t, s = sympy.symbols('t, s')
a = sympy.symbols('a', real=True, positive=True)
exp = sympy.exp
x=sympy.inverse_laplace_transform((exp(-1 * s)*s)/(s**2+1), s, t)
print(‘x(t)=,’x)
plot(x, show=True)
```

→ cos(t - 1)*Heaviside(t - 1) →



Example: Find the inverse Laplace transform of $X(s) = \dfrac{2s+4}{s^2+4s+3}$, $\mathrm{Re}(s) > -1$

Expanding by partial fractions, we have $X(s) = \dfrac{2s+4}{s^2+4s+3} = 2\dfrac{s+2}{(s+1)(s+3)} = \dfrac{c_1}{s+1} + \dfrac{c_2}{s+3}$

$$c_1 = (s+1)X(s)\big|_{s=-1} = 2\frac{s+2}{s+3}\bigg|_{s=-1} = 1$$

$$c_2 = (s+3)X(s)\big|_{s=-3} = 2\frac{s+2}{s+1}\bigg|_{s=-3} = 1$$

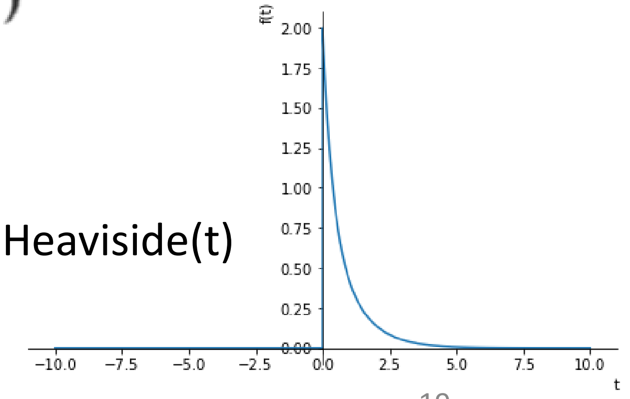
Hence $X(s) = \dfrac{1}{s+1} + \dfrac{1}{s+3}$

ROC of X(s) is Re(s) > - 1. Therefore, x(t) is a right-sided signal and from the Transform pairs table we get,

$$x(t) = e^{-t}u(t) + e^{-3t}u(t) = (e^{-t} + e^{-3t})u(t)$$

```
import sympy
from sympy.plotting import plot
t, s = sympy.symbols('t, s')
x=sympy.inverse_laplace_transform((2*s+4)/(s**2+4*s+3), s, t)
print('x(t)=',x)
plot(x, show=True)
```

x(t)=(exp(2*t) + 1)*exp(-3*t)*Heaviside(t)



Partial Fractions Expansion by Python:

Code:

```
import sympy

s = sympy.symbols('s’)

G = ((s + 1)*(s + 2)* (s + 3))/((s + 4)*(s + 5)*(s + 6))
H = G.apart(s)
print(H)
```

$$\frac{(s+1)(s+2)(s+3)}{(s+4)(s+5)(s+6)}$$

Output:

```
1 - 30/(s + 6) + 24/(s + 5) - 3/(s + 4)
```

$$1-\frac{30}{s+6}+\frac{24}{s+5}-\frac{3}{s+4}$$

Example: Find the inverse Laplace transform of $X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2} \qquad \mathrm{Re}(s) > -3$

Since X(s) has one simple pole at s=-3 and one multiple pole at s=-5 → $X(s) = \frac{c_1}{s+3} + \frac{\lambda_1}{s+5} + \frac{\lambda_2}{(s+5)^2}$

$$c_1 = (s+3)X(s)\big|_{s=-3} = \frac{s^2+2s+5}{(s+5)^2}\bigg|_{s=-3} = 2$$

$$\lambda_2 = (s+5)^2 X(s)\big|_{s=-5} = \frac{s^2+2s+5}{s+3}\bigg|_{s=-5} = -10$$

$$\lambda_1 = \frac{d}{ds}\left[(s+5)^2 X(s)\right]\Big|_{s=-5} = \frac{d}{ds}\left[\frac{s^2+2s+5}{s+3}\right]\bigg|_{s=-5} = \frac{s^2+6s+1}{(s+3)^2}\bigg|_{s=-5} = -1$$

$$X(s) = \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$$

The ROC of X(s) is Re(s) > -3. Thus, x(t) is a right-sided signal and from the Transformation pairs table,

$$x(t) = 2e^{-3t}u(t) - e^{-5t}u(t) - 10te^{-5t}u(t) = [2e^{-3t} - e^{-5t} - 10te^{-5t}]u(t)$$

Example: The output y(t) of a continuous-time LTI system is found to be $2e^{-3t}\cdot u(t)$ when the input $x(t)=u(t)$

Find the output y(t) when the input x(t) is $e^{-t}\cdot u(t)$.

Finding the output for a different input signal requires the transfer function to be known. when $x(t)=u(t)$, output $y(t)=2e^{-3t}\cdot u(t)$

Taking the Laplace transforms of x(t) and y(t) we obtain $X(s) = \dfrac{1}{s} \qquad \mathrm{Re}(s) > 0$

$$Y(s) = \frac{2}{s+3} \qquad \mathrm{Re}(s) > -3$$

The system transfer function is $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{2s}{s+3} \qquad \mathrm{Re}(s) > -3$

$$H(s) = \frac{2s}{s+3} = \frac{2(s+3)-6}{s+3} = 2 - \frac{6}{s+3} \qquad \mathrm{Re}(s) > -3$$

By taking the Inverse Laplace Transform: $h(t) = 2\delta(t) - 6e^{-3t}u(t)$

$$x(t) = e^{-t}u(t) \longleftrightarrow \frac{1}{s+1} \qquad \mathrm{Re}(s) > -1$$

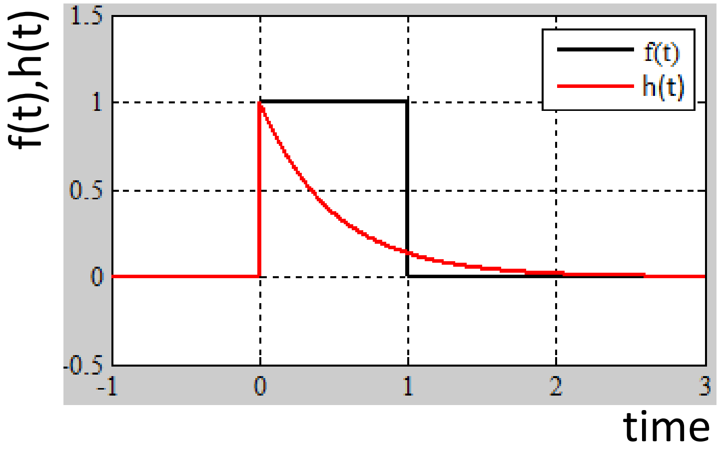
$$Y(s) = X(s)H(s) = \frac{2s}{(s+1)(s+3)} \qquad \mathrm{Re}(s) > -1$$

Using partial-fraction expansions, we get $Y(s) = -\dfrac{1}{s+1} + \dfrac{3}{s+3} \xrightarrow{\text{ILT}} y(t) = (-e^{-t} + 3e^{-3t})u(t)$

Example: Find the convolution of h(t) and f(t) by using Laplace transform where

$$h(t) = e^{-2t}, \quad t > 0$$

$$f(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t \le 1 \\ 0, & 1 < t \end{cases}$$



$$y(t) = f(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = F(s)H(s)$$

$$F(s) = \frac{1}{s} - e^{-s}\frac{1}{s} \qquad H(s) = \frac{1}{s+a}$$

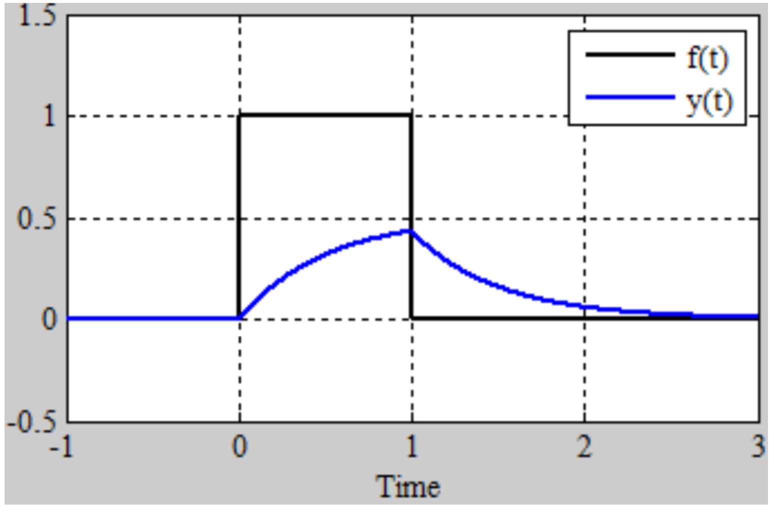
$$Y(s) = F(s)H(s) = \left(\frac{1}{s} - e^{-s}\frac{1}{s}\right)\left(\frac{1}{s+a}\right)$$

$$Y(s) = \frac{1}{s(s+2)} - e^{-s}\frac{1}{s(s+2)}$$

$$\frac{1}{s(s+2)} \xleftrightarrow{\mathcal{L}} \frac{1}{2}\left(1-e^{-2t}\right) \qquad \frac{1}{s(s+2)}e^{-s} \xleftrightarrow{\mathcal{L}} \frac{1}{2}\left(1-e^{-2(t-1)}\right)u(t-1)$$

$$Y(s) = \frac{1}{s(s+2)} - e^{-s}\frac{1}{s(s+2)}$$

$$y(t) = \frac{1}{2}\left(1-e^{-2t}\right) - \frac{1}{2}\left(1-e^{-2(t-1)}\right)u(t-1)$$



* Verify the result by using the convolution integral

Example: Use the convolution theorem to find the inverse Laplace transform of $H(s) = \frac{1}{(s^2+a^2)^2}$

$$H(s) = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right)$$

$$y(t) = f(t) * h(t) \xleftrightarrow{\mathcal{L}} Y(s) = F(s)H(s)$$

$F(s)=H(s)=\frac{1}{s^2+a^2}$ $\xrightarrow[\text{table}]{}$ $f(t)=h(t)= \frac{1}{a}\sin(at)$

$$f(t)*h(t) = \frac{1}{a^2}\int_0^t \sin(at-as)\sin(as)ds = \frac{1}{2a^3}(\sin(at) - at\cos(at))$$

Let

$$x(t) = v_s(t) \qquad y(t) = v_c(t)$$

In this case, the *RC* circuit is described by

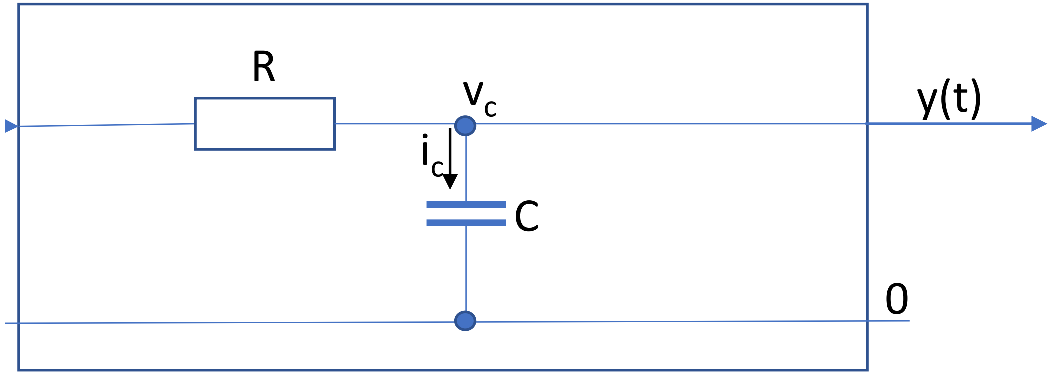
$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

Taking the Laplace transform of the above equation, we obtain

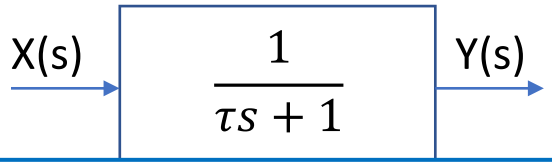
$$sY(s) + \frac{1}{RC}Y(s) = \frac{1}{RC}X(s)$$

or

$$\left(s + \frac{1}{RC}\right)Y(s) = \frac{1}{RC}X(s)$$



$$\tau = RC$$



What is the Bandwidth if R and C are known ? What is the filtering rate (dB/dec)?

How can we implement real-time software equivalent of this analog system? (Next week)

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \frac{1}{RC}e^{-t/RC}u(t)$$