3.6 DETERMINANTS

TRIANGULAR MATRIX

If a square matrix has only zeros below (above) its main diagonal, then it is called an upper (lower) triangular matrix.

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix} \to \text{upper tr. matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 1 & 2 & 5 \end{bmatrix} \to \text{lower tr. matrix}$$

TRANSPOSE OF A MATRIX

It is the matrix obtained by changing the rows of A into columns.

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 5 & 3 \end{bmatrix} \qquad A = [a_{ij}]_{m\times n} \Rightarrow A^T = [a_{ji}]_{n\times m}$$

A, B: matrices with appropriate sizes
c: number

$$\text{rows} \xrightarrow{T} \text{columns} \xrightarrow{T} \text{rows}$$

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $(A \cdot B)^T = B^T \cdot A^T$

DETERMINANTS

$A_{ij} = [a_{ij}]_{n \times n}$

M_{ij} : ij-th minor of A is the determinant obtained by deleting the ith row and jth column of A

A_{ij} : the ijth cofactor of A , $A_{ij} = (-1)^{i+j} M_{ij}$

or signed minor

$\Rightarrow \det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$: cofactor expansion along the ith row

or

$= a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$: cofactor expansion along the jth column

and if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = |A| = ad - bc$

Ex:

$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{bmatrix}$

When finding det A , always choose the row or column expansion where the number of zeros is maximum.

→ $\boxed{\text{2nd row}}$

$\begin{bmatrix} - & - & - & - \\ a_{21} & a_{22} & a_{23} & a_{24} \\ - & - & - & - \\ - & - & - & - \end{bmatrix}$

$\det A = \cancel{a_{21}}^{0} A_{21} + a_{22}A_{22} + \cancel{a_{23}}^{0} A_{23} + \cancel{a_{24}}^{0} A_{24}$

$= (-1)A_{22} = -(-1)^{2+2} M_{22}$

$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$= - \begin{vmatrix} 2 & 0 & -3 \\ 7 & 3 & 5 \\ -6 & 2 & 4 \end{vmatrix} = -\det B = -[\cancel{b_{12}}^{0} B_{12} + b_{22}B_{22} + b_{32}B_{32}]$

$= -[3(-1)^{2+2} M^*_{22} + 2(-1)^{3+2} M^*_{32}]$

$= -3 \begin{vmatrix} 2 & -3 \\ -6 & 4 \end{vmatrix} + 2 \begin{vmatrix} 2 & -3 \\ 7 & 5 \end{vmatrix} = -3(8-18) + 2(10+21)$

$= 30 + 62 = 92$ //

$$A = [a_{ij}]_{n \times n}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$$A_{ij} = (-1)^{i+j} M_{ij}$$

↳ cofactor of element ij

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{32}\, a_{23}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}\, a_{33} - a_{31}\, a_{13}$$

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{22} = (-1)^{2+2} M_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$
A = \begin{array}{cl}
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & & a_{2j} & & a_{2n} \\
\vdots & & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & & & \vdots & & \\
a_{n1} & a_{n2} & & a_{nj} & & a_{nn}
\end{bmatrix}_{n \times n} &
\begin{array}{l} \\ \\ \\ \rightarrow i\text{-th row} \\ \\ \\ \end{array} \\
\hookrightarrow j\text{-th column} &
\end{array}
$$

$$\det A = a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{ij} A_{ij} + \cdots + a_{in} A_{in}$$

$$\det = a_{1j} A_{1j} + a_{2j} A_{2j} + \cdots + a_{ij} A_{ij} + \cdots + a_{nj} A_{nj}$$

PROPERTIES OF DETERM.

Thursday

① B : a matrix obtained from A by multiplying a single row (or a column) of A by the constant k

$$\Rightarrow |B| = k\,|A|.$$

Ex: $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 15 & 1 \end{bmatrix}$ → obtained by multiplying the 1st col. of A by 3

$|A| = 2-15 = -13$, $|B| = 6-45 = -39 = 3.|A|$

② B : a matrix obtained from A by interchanging two rows (or columns) of A

$$\Rightarrow |B| = -|A|$$

Ex: $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ → obtained by interchanging the 1st and the 3rd rows.

$|A| = a_{11}A_{11} + \cancel{a_{21}}^{0}A_{21} + \cancel{a_{31}}^{0}A_{31} = 1.(-1)^{1+1}M_{11} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3-2 = 1$

$|B| = \cancel{b_{11}}^{0}B_{11} + \cancel{b_{21}}^{0}B_{21} + b_{31}B_{31} = 1.(-1)^{3+1}M_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2-3 = -1$

③ A : two rows (or two colums) are identical $\Rightarrow |A| = 0$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow |A| = a_{21}A_{21} + a_{22}A_{22} + \cancel{a_{23}}^{0}A_{23}$

$|A| = 1.(-1)^{2+1}M_{21} + (-1)(-1)^{2+2}M_{22} = -\underbrace{\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}}_{(6-6)} - \underbrace{\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}}_{(3-3)} = 0$

④ A_1, A_2, B: identical except their ith row (or column)
ith row (or col) of B = sum of the ith rows (cols) of A_1, A_2

$$\Rightarrow \det B = \det A_1 + \det A_2$$

Ex: $A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$

$|A_1| = 3-4 = -1$, $|A_2| = 4-6 = -2$, $|B| = 5-8 = -3 = |A_1| + |A_2|$

⑤ B: matrix obtained by adding a constant multiple of one row (or col.) of A to another row (or col.) of A

$$\Rightarrow \det B = \det A$$

Ex:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \xrightarrow[\to R_2]{2R_1 + R_2} \begin{bmatrix} 1 & 3 \\ 4 & 10 \end{bmatrix} = B$$

$|A| = 4-6 = -2$, $|B| = 10-12 = -2 = |A|$

⑥ B: triangular matrix $\Rightarrow |B|$ = product of its main diagonal elements

Ex:

$$\begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 11 & 7 & 4 \end{vmatrix} = 2 \cdot (-1)^{1+1} \begin{vmatrix} -1 & 0 \\ 7 & 4 \end{vmatrix} = 2(-4) = -8$$
$$= 2 \cdot (-1) \cdot 4$$

⑦ $\det(A^T) = \det A$

Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 4 & -1 \end{vmatrix} = 1 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + (-1)(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -1-3+4-2 = -2$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & -1 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & 4 \\ -1 & -1 \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -1+4-2-3 = -2$$

THEOREM

* A is invertible $\Leftrightarrow |A| \neq 0$
* $|A.B| = |A|\,|B|$

} A, B: nxn matrices

$$B = A^{-1} \Rightarrow |A A^{-1}| = |I| = 1 = |A|\,|A^{-1}| \Rightarrow |A^{-1}| = \frac{1}{|A|}$$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & 5 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 3 & 2 & 1 \end{bmatrix} \Rightarrow |A^{-1}| = ?$

$$|A| = 1(-1)^{3+2} \begin{vmatrix} 1 & 3 & -1 \\ 2 & 5 & 0 \\ -1 & 2 & 1 \end{vmatrix} + 2(-1)^{3+4} \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= -\left\{-1.(-1)^{1+3} \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} + 1(-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}\right\}$$

$$-2\left\{1.(-1)^{1+1} \begin{vmatrix} -1 & 5 \\ 3 & 2 \end{vmatrix} + 2.(-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} - 1.(-1)^{3+1} \begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}\right\}$$

$$= -\{-(4+5) + (5-6)\} - 2\{-2-15-2(4-9)-(10+3)\} = 10-28 = -18$$

$|A^{-1}| = -1/18$

CRAMER'S RULE

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1}$$

$$Ax = b \Rightarrow x_i = \frac{1}{|A|} \begin{array}{c} \quad\quad\quad \text{ith column} \\ \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \end{array}$$

replace the vector b on the ith column of |A|

Ex. $\left.\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ -4x_1 + x_2 + 3x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= 1 \end{aligned}\right\} \quad x_1, x_2, x_3 = ?$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ -4 & 1 & 3 \\ -2 & -1 & 1 \end{bmatrix} , \quad b = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$|A| = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} + 3 \cdot (-1)^{1+2} \begin{vmatrix} -4 & 3 \\ -2 & 1 \end{vmatrix} + (-1)(-1)^{1+3} \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}$$

$$= 1 + 3 - 3(-4+6) - (4+2) = -8$$

$$x_1 = -\frac{1}{8} \begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & 3 \\ 1 & -1 & 1 \end{vmatrix} = -\frac{1}{8} \left\{ 1(-1)^{3+1} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} - (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} + (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \right\}$$

$$= -\frac{1}{8} \{ 9 + 1 + 6 + 4 + 2 - 12 \} = -\frac{10}{8} = -\frac{5}{4}$$

$$x_2 = -\frac{1}{8} \begin{vmatrix} 1 & 2 & -1 \\ -4 & 4 & 3 \\ -2 & 1 & 1 \end{vmatrix} = -\frac{1}{8} \left\{ (-2)(-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} + (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ -4 & 3 \end{vmatrix} + (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ -4 & 4 \end{vmatrix} \right\}$$

$$= -\frac{1}{8} \{ -2(6+4) - (3-4) + 4 + 8 \} = \frac{7}{8}$$

$$x_3 = -\frac{1}{8} \begin{vmatrix} 1 & 3 & 2 \\ -4 & 1 & 4 \\ -2 & -1 & 1 \end{vmatrix} = -\frac{1}{8} \left\{ (-2)(-1)^{3+1} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} - (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ -4 & 4 \end{vmatrix} + (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ -4 & 1 \end{vmatrix} \right\}$$

$$= -\frac{1}{8} \{ -2(12-2) + (4+8) + 1 + 12 \} = -\frac{5}{8}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5/4 \\ 7/8 \\ -5/8 \end{bmatrix}$$

ADJOINT MATRIX AND INVERSE

$$A^{-1} = \frac{[A_{ij}]^T}{|A|} \quad , \ [A_{ij}]^T : \text{adjoint matrix, adj } A$$
(transposed cofactor matrix)

Ex

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 4 & 1 & 0 \\ -3 & -1 & 3 \end{bmatrix} \Rightarrow A^{-1} = ?$$

$$|A| = 4.(-1)^{1+2} \begin{vmatrix} 4 & 0 \\ -3 & 3 \end{vmatrix} + 1.(-1)^{1+3} \begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix}$$

$$= -4\ (12) + (-4+3) = -48 - 1 = -49$$

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} = 3 \quad , \ A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 4 & 0 \\ -3 & 3 \end{vmatrix} = -12$$

$$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix} = -1, \quad A_{21} = (-1)^{2+1} \begin{vmatrix} 4 & 1 \\ -1 & 3 \end{vmatrix} = -(12+1) = -13$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} = 3, \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 0 & 4 \\ -3 & -1 \end{vmatrix} = -(0+12) = -12$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = 4$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 0 & 4 \\ 4 & 1 \end{vmatrix} = -16 \quad \Rightarrow [A_{ij}] = \begin{bmatrix} 3 & -12 & -1 \\ -13 & 3 & -12 \\ -1 & 4 & -16 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{49} \begin{bmatrix} 3 & -13 & -1 \\ -12 & 3 & 4 \\ -1 & -12 & -16 \end{bmatrix}$$