Chapter 1 First-Order Differential Eqs.

1.1 DEs and Mathematical Models
1.2 Integrals as General and Particular Sols.
1.3 Slope Fields and Solution Curves

Read this for an introduction to the subject. I'll do a fast overview of these now!

1.4 Separable Eqs. and Applications
1.5 First Order DEs
1.6 Substitution Methods and Exact Eqs.

- Homogeneous Equations
- Exact Eqs.
- Bernoulli Eq., Riccati eq.
- Reducible eqs.

A differential equation is an equation that includes an unknown function and its derivative(s).

Ex $y' = 0$ $\quad \frac{dy}{dx} = 0$ $\quad y$: unknown function

$$y(x) = C = \text{constant}$$

Ex $y' = x$ $\quad \frac{dy}{dx} = x$ $\quad y$: unknown function
dependent variable

x : independent variable

$$y(x) = \frac{x^2}{2} + C$$

C : arbitrary constant
constant of integration

Ex $\dfrac{d^2y}{dt^2} = -g$, $y(t_0) = y_0$, $y'(t_0) = V_0$

where g, y_0, V_0 are constants.

the independent variable : t
" dependent " : y

Find solution $y(t)$ of this problem.

$y'' = -g$ $\quad y' = -gt + C_1 \xrightarrow{t=0} V_0 = -g \cdot 0 + C_1$

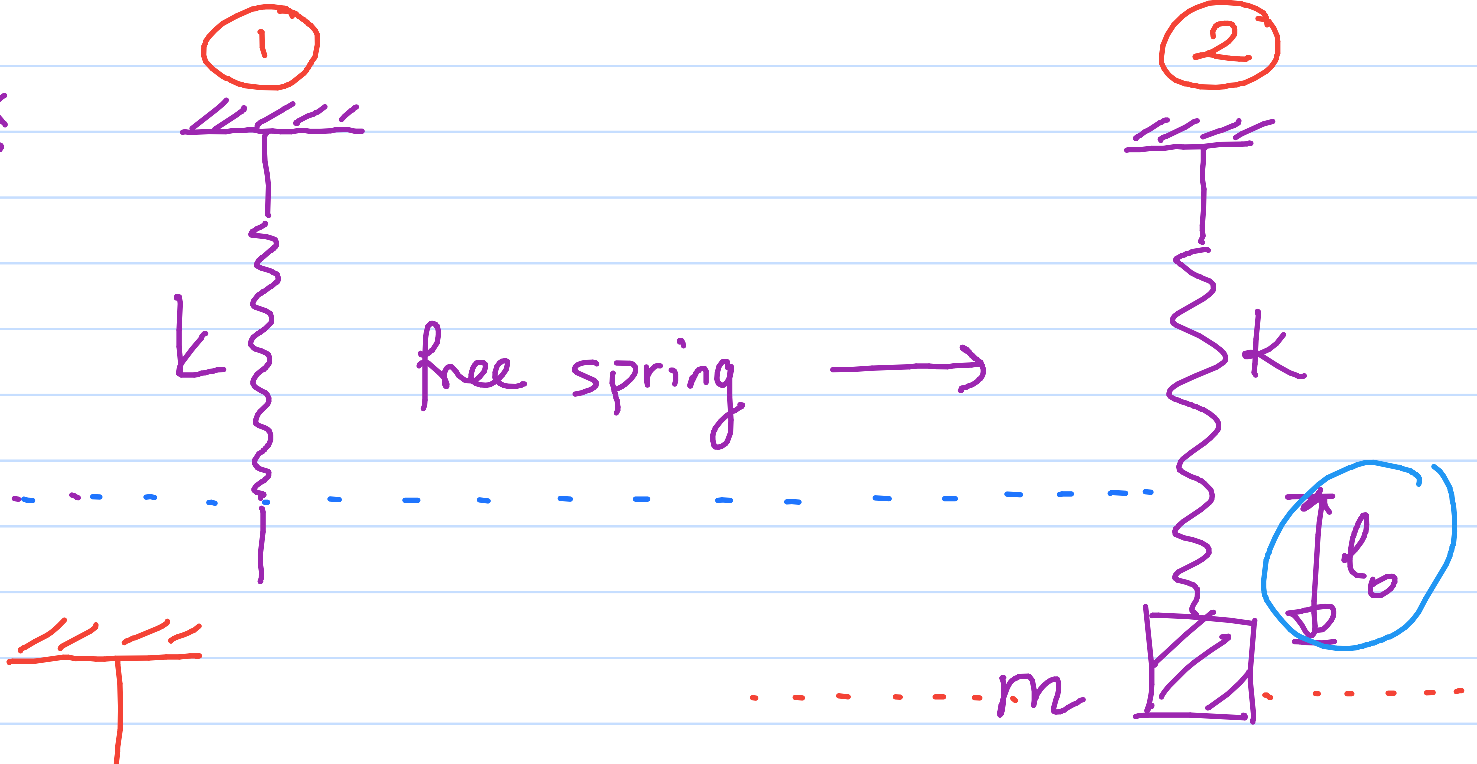
$y' = -gt + V_0$

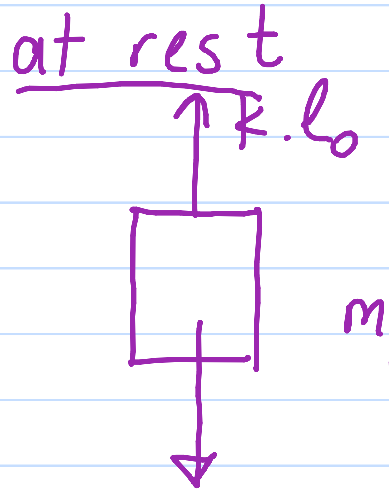
$y = -\frac{1}{2}gt^2 + V_0 t + C_2 \xrightarrow{t=0} y_0 = 0 + 0 + C_2$

$y(t) = -\frac{1}{2}gt^2 + V_0 t + y_0$

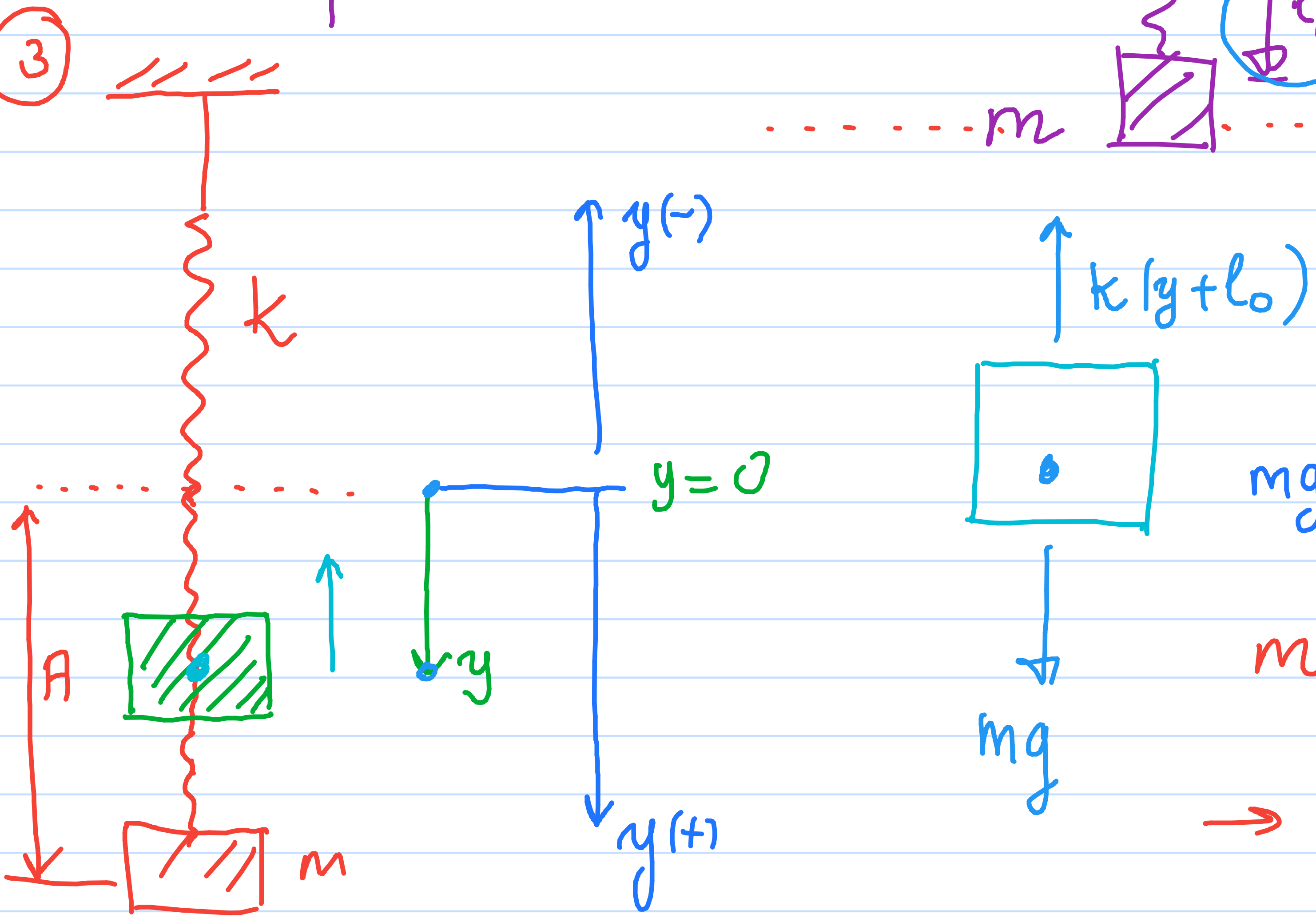
That kinematics problem is actually a DE problem!

Ex





$$mg = k.l_0$$



Newton's sec. law

$$F = m.a$$

$$mg - k(y+l_0) = m.\frac{d^2y}{dt^2}$$

$$m\frac{d^2y}{dt^2} = -k.y$$

$$\rightarrow y = y(t)$$

$$m \frac{d^2y}{dt^2} = -k \cdot y \Rightarrow \frac{d^2y}{dt^2} = -\frac{k}{m} y$$

$$\frac{d^2y}{dt^2} + \frac{k}{m} y = 0 \Rightarrow y'' + a\, y = 0$$

<u>suppose</u> $a = 1$: $\boxed{y'' = -y}$ we're searching for a function $y = y(t)$ for which the second derivative is equal to (-1) times itself!

<u>e.g.</u> $y(t) = \sin t$

$y'(t) = \cos t$

$y'' = -\sin t = -y$ ✓

e.g. $y(t) = \cos t$

$y'(t) = -\sin t$

$y''(t) = -\cos t = -y$

Later, we'll see that $y(t) = C_1 \cos t + C_2 \sin t$

Later, we'll see that $y(t) = c_1 \cos t + c_2 \sin t$

is also a solution to $y'' = -y$. Indeed,

$$y'' = c_1 . (-\cos t) + c_2 . (-\sin t) = -(c_1 \cos t + c_2 \cos t) = -y$$

- $\cos t$
- $\sin t$
- $c_1 \cos t$
- $c_2 \sin t$
- $c_1 \cos t + c_2 \sin t$

solution space of the DE

$y'' = -y$.

Any linear combination of the solutions $\cos t$ and $\sin t$ is also a solution to $y'' = -y$.

$\Rightarrow$ The solution space of the DE $y'' + y = 0$ is a VECTOR SPACE !!

Classification of Differential Equations

1. The number of independent variables
2. The number of dependent variables (unknown functions)
3. The order of the DE
4. Linearity / Nonlinearity

1. The number of independent variables

1 independent variable : Ordinary Differential Equation

>1 independent variables : Partial Differential Equation

Ex

$$\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = x, \quad \frac{d^2y}{dt^2} = -g, \quad y'' + y = 0$$

There is only one independent variable in these examples $\left\{ \begin{array}{lll} x & \text{in the} & \text{first two} \\ t & \text{'' ''} & \text{last two} \end{array} \right\}$

Therefore they're ODEs

Ex

$u = u(x,t)$	$u = u(x,t)$	$u = u(x,y)$
$u_{tt} = c^2 u_{xx}$	$u_t = k u_{xx}$	$u_{xx} + u_{yy} = 0$
Wave Equation	Heat/Diffusion	Laplace's Eq.

② The number of dependent variables / unknown functions

DEs: a single (scalar) DE; a system of DEs

<u>Ex</u> $y'' + y = 0$

<u>Ex</u> $x(t)$: the number of preys in a population

$y(t)$: the # of predators

Lotka-Volterra system

Prey-Predator eqs.

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = -\gamma y + \delta xy \end{cases}$$

A system of eqs. for the unknown functions $x = x(t)$, $y = y(t)$

(3) The Order of the DE

The order of a DE is the order of the highest derivative appearing in the eq.

$y'' + y = 0 \rightarrow$ second-order ODE

$y''' y' + xy^2 = 0 \rightarrow$ third-order ODE

We do not call this thing as "degree".

Ex The most general n-th order ODE is written in the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

(4) Linearity & Nonlinearity of a DE

A DE is called linear if it is linear in the unknown function and all of its derivatives.

By a linear function we mean:

$L(x) = ax + b$ No: x^2, $\sqrt{x}$

$L(x, y) = a_0 + a_1 x + a_2 y$ No: $x^2, y^2, xy, x^{1/2}$ - -

$$L(x_1, x_2, \dots, x_n) = a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

All the variables x_i have the power 1!

They're not multiplied by each other!

Ex $y'' + y' + y = 0$ is linear in the unknown function y and its derivatives y', y''.

$\Rightarrow$ Linear DE

Ex $y'' + y' + y^2 = 0$ is not linear due to the term y^2. NONLINEAR

Ex $y'' + y' + y + x^2 = 0$ is linear as the eq. is linear in y, y', y''.

Ex $y' y'' + y = \sin x$

dep. var. $= x$

ind. var. $= y$

The eq. is linear in y; however is not linear in y' and y''

$\Rightarrow$ NONLINEAR

Ex

Equation	ORDER	N / L
$t^2 y'' + t y' + 2y = \sin t$	2	Linear
$y'' + \sin(t + y) = \sin t$	2	Nonlinear
$(y''')^2 + t y' + 4y = 0$	3	Nonlinear

Now let's see the form of the most general n-th order linear ODE:

$$P_n(t)\, y^{(n)} + P_{n-1}(t)\, y^{(n-1)} + \cdots + P_2(t) y'' + P_1(t) y' + P_0(t) y = g(t)$$

DIFFERENTIAL EQUATION (DE)

An eq. that expresses a relation between an unknown funct and one or more of its derivatives

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

ORDER

Highest order of any diff in the eq.

ORDINARY-PARTIAL

The eq. contains diffs with respect to

- only 1 var → ODE
- several ind. vars → PDE

LINEAR-NONLINEAR

$P_n(x)y^{(n)} + \dots + P_1(x)y' + P_0(x)y = g(x)$

If it can be written in this form → Linear

Otherwise nonlin

HOMOGENEOUS-NONHOM.

If it contains no nondiff terms → homog.

Otherwise nonhom.

diff. term · nondiff t.

DE	ORDER	DEP. V.	IND. V.	ODE-PDE	LIN.-NONLIN	HOM-NONH.
$\frac{dx}{dt} = x^2 + t^3$	1	x	t	ODE	NL	NH
$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$	2	y	x	ODE	L	H
$\left(\frac{dy}{dx}\right)^2 = y + \sin x$	1	y	x	ODE	NL	NH
$\frac{\partial^3 z}{\partial x^3} = 2\frac{\partial^2 z}{\partial x \partial y} + x$	3	z	x, y	PDE	L	NH

Def A "Solution" of a DE

Given the n-th order ODE in the form

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \quad (*)$$

a function $y = \phi(t)$ is a SOLUTION to $(*)$

(i) $\phi'(t), \phi''(t), \ldots, \phi^{(n)}(t)$ exists

(ii) $y = \phi(t)$ satisfies $*$.

$f(t) \in C^n(a,b)$: $f(t), f'(t), f''(t), \ldots, f^{(n)}(t)$ exist and cont. on (a,b).

Ex Show that $y = ce^{x^2}$ satisfies $y' = 2xy$

Indeed, $y' = c \cdot 2x e^{x^2} = 2x \; ce^{x^2} = 2xy$ ✓

1.4 Separable Equations

The most general first-order ODE is written as

$$y' = f(x,y)$$

$y' = x + y$

$y' = x + \sin y + y^2$

Assume that $f(x,y) = \dfrac{H(x)}{Q(y)}$

$$\frac{dy}{dx} = \frac{H(x)}{Q(y)} \Rightarrow Q(y)\,dy = H(x)\,dx$$

$$\Longrightarrow \int Q(y)\,dy = \int H(x)\,dx + C$$

Ex 1 (p 33) Solve the eq. $\frac{dy}{dx} = -6xy$

$$\frac{dy}{y} = -6x\,dx \longrightarrow \int \frac{dy}{y} = -\int 6x\,dx$$

$$\ln|y| = -3x^2 + C \rightarrow y = e^{-3x^2 + C}$$

or $\ln y = -3x^2 + \ln c \rightarrow y = c\, e^{-3x^2}$

They're equivalent, as

$$y = e^{-3x^2+c} = \underbrace{e^{c}}_{\hat{c}}\, e^{-3x^2} = \hat{c}\, e^{-3x^2}$$

Def. A first-order Initial Value Problem (IVP)

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

DE + IC : initial cond.

Ex Solve the IVP $y' = -6xy$, $y(0) = 7$.

$$y(x) = C e^{-3x^2}$$

$$y(0) = C e^{0} = 7 \rightarrow C = 7$$

The sol. of the IVP is

$$y(x) = 7 e^{-3x^2}$$

Ex 2 $\quad y' = \dfrac{4-2x}{3y^2-5}$ $\quad$ DE

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5} \longrightarrow (4-2x)\,dx = (3y^2-5)\,dy$$

$$4x - x^2 + C = y^3 - 5y \quad //$$

Ex 2* $\quad y' = \dfrac{4-2x}{3y^2-5}, \quad y(1) = 3$

$$4x - x^2 + C = y^3 - 5y$$

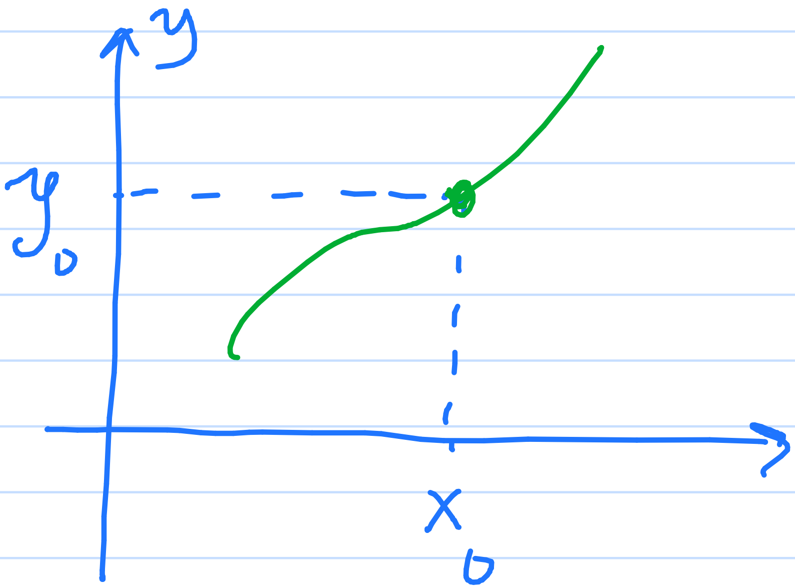
$\left.\begin{array}{l} x = 1 \\ y = 3 \end{array}\right\}$ $\quad 4\cdot 1 - 1^2 + C = 3^3 - 5\cdot 3 \longrightarrow C = 9$

$$\boxed{y^3 - 5y = 4x - x^2 + 9}$$

$$y' = f(x,y)$$

$$y(x_0) = y_0$$

We find $y = y(x) = ?$

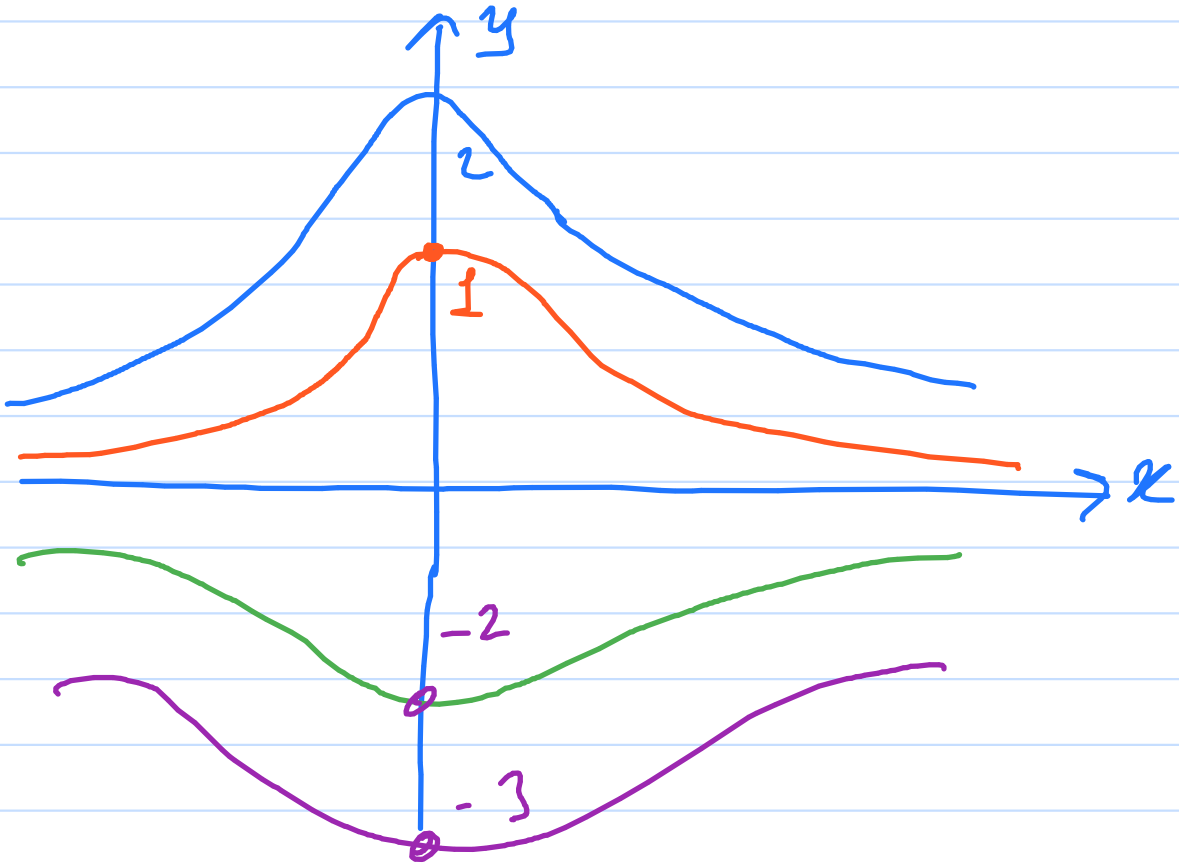


For example;

$$* \quad y' = -6xy \rightarrow y = C e^{-3x^2}$$

$C = 1 \quad y = e^{-3x^2}$; $C = 2 \quad y = 2e^{-3x^2}$

$c = -3 \quad y = -3 e^{-3x^2}$



The solution set of a single DE is a family of curves depending on the integration parameter/constant C. These curves are called integral curves of the DE.

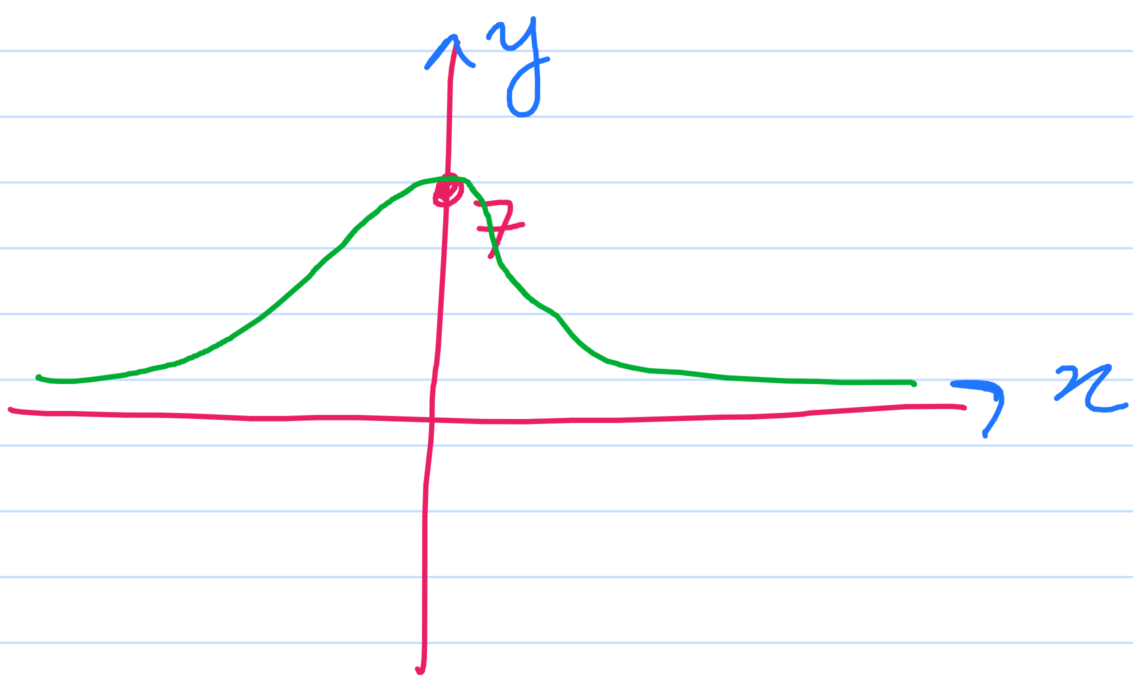
In case of an IVP,

$$y' = -6xy$$

$$y(0) = 7$$

$$\downarrow$$

$$y(x) = 7e^{-3x^2}$$



Ex 4 $\quad y' = 6x(y-1)^{2/3}$

$$\frac{dy}{dx} = 6x(y-1)^{2/3} \Rightarrow \frac{dy}{(y-1)^{2/3}} = 6x\,dx$$

$$u = y-1 \; ; du = dy \rightarrow \frac{du}{u^{2/3}} = 6x\,dx$$

$$\frac{u^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} = 6 \cdot \frac{x^2}{2} + 3C \rightarrow 3u^{1/3} = 3x^2 + 3C$$

$$u^{1/3} = x^2 + C \Rightarrow u = (x^2+C)^3$$

$$y - 1 = (x^2+c)^3$$

$$y = 1 + (x^2+c)^3$$

Ex Problems 1.4, 16 $\quad (x^2+1)\tan y \, y' = x$

$$(x^2+1)\tan y \frac{dy}{dx} = x \longrightarrow \tan y \, dy = \frac{x}{x^2+1} dx$$

$$\frac{\sin y \, dy}{\cos y} = \frac{x}{x^2+1} dx$$

LHS: $u = \cos y \rightarrow du = -\sin y dy$

RHS: $v = x^2+1 \rightarrow dv = 2x \, dx$

$$-\frac{du}{u} = \frac{\frac{dv}{2}}{v} \Rightarrow -\ln u = \frac{1}{2}\ln v + \ln C$$

$$u^{-1} = v^{1/2} C \longrightarrow (\cos y)^{-1} = (x^2+1)^{1/2} C$$

$$\sec y = C(x^2+1)^{1/2} \longrightarrow y = \sec^{-1}\left[C(x^2+1)^{1/2}\right]$$

page 43 of the book $\Rightarrow$ more separable eqs.

1.5 Linear First Order Eqs. ***

The general form of the first-order linear DE is

$$y' + p(x)\,y = g(x)$$

Now we'll find the solution $y = y(x)$ of this DE.

Solution Let's multiply this eq. by some function $\mu(x)$:

$$\mu(x)\,y'(x) + \overbrace{\mu(x)\,p(x)}^{\mu'}\,y(x) = \mu(x)\,g(x)$$

Suppose $\quad \mu(x)\,p(x) = \mu'(x)$

$$\mu(x)\,y'(x) + \mu'(x)\,y(x) = \mu(x)\,g(x) \Rightarrow \frac{d}{dx}\left[\mu y\right] = \mu g$$

$$\frac{d}{dx}\left[\mu(x)\,y(x)\right] = \mu(x)\,g(x)$$

$$\int d\left[\mu(x)\,y(x)\right] = \int \mu(x)\,g(x)\,dx$$

$$\mu(x)\,y(x) = \int \mu(x)\,g(x)\,dx + C$$

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\,g(x)\,dx + C\right]$$

is the solution to $y' + py = g$.

$g(x)$ is given to us with the DE ✓ but $\mu(x) = ??$

$$\mu(x)\,p(x) = \mu'(x) \longrightarrow \frac{d\mu}{dx} = \mu \cdot p$$

$$\frac{d\mu}{\mu} = p(x)\,dx \xrightarrow{\int} \ln\mu = \int p(x)\,dx$$

$$\boxed{\mu(x) = e^{\int p(x)\,dx}}$$

integration factor!!

$$y(x) = \frac{1}{e^{\int p(x)\,dx}}\left[\int e^{\int p(x)\,dx} g(x)\,dx + C\right]$$

is the solution to $y' + p(x)\,y = g(x)$

Ex Find the general sol. to $\frac{dy}{dx} - y = e^{3x}$

1^{st} order eq; Linear $\Rightarrow$ $y' + p(x)\,y = g(x)$

$p(x) = -1, \qquad g(x) = e^{3x}$

$\mu(x) = e^{\int p(x)\,dx} = e^{\int -1 \cdot dx} = e^{-x}$ integrating factor!

$\frac{dy}{dx} - y = e^{3x} \longrightarrow e^{-x}\frac{dy}{dx} - e^{-x}y = e^{2x}$

$\frac{d}{dx}\left[e^{-x}y\right] = e^{2x} \longrightarrow e^{-x}y = \int e^{2x}\,dx$

$\frac{d}{dx}\left[\mu\, y\right] = \mu \cdot g = e^{-x} \cdot e^{3x}$

$$e^{-x} y = \int e^{2x} dx \rightarrow e^{-x} y = \frac{e^{2x}}{2} + C$$

$$y(x) = \frac{1}{2} e^{3x} + c e^{x} //$$

Ex $\quad (x^2+1) \frac{dy}{dx} + 3xy = 6x$

Linear, 1st order : $y' + p(x) y = g(x)$

$$\frac{dy}{dx} + \frac{3x}{x^2+1} y = \frac{6x}{x^2+1}$$

$$p(x) = \frac{3x}{x^2+1}, \quad g(x) = \frac{6x}{x^2+1}$$

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{3x}{x^2+1}\,dx} = e^{\frac{3}{2}\int \frac{2x\,dx}{x^2+1}}$$

$$= e^{\frac{3}{2}\ln(x^2+1)} = e^{\ln(x^2+1)^{3/2}} = (x^2+1)^{3/2}$$

$$\Rightarrow (x^2+1)^{3/2}\frac{dy}{dx} + 3x(x^2+1)^{1/2}\,y = 6x\,(x^2+1)^{1/2}$$

$$\frac{d}{dx}[\mu \cdot y] = \mu \cdot g$$

$$\frac{d}{dx}\left[(x^2+1)^{3/2}\cdot y\right] = 6x\,(x^2+1)^{1/2}$$

$$(x^2+1)^{3/2}\,y = \int 6x\,(\underbrace{x^2+1}_{u})^{1/2}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$(x^2+1)^{3/2}\, y = \int u^{1/2} \cdot 3\, du = 3\,\frac{u^{3/2}}{\frac{3}{2}} + C$$

$$(x^2+1)^{3/2}\, y = 2\,(x^2+1)^{3/2} + C$$

$$y(x) = 2 + C\,(x^2+1)^{-3/2} \quad //$$

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① PS on Saturday, from Vector spaces

② Finish studying this file before next week's classes.

Ex $\quad y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5 \quad$ (IVP)

$y' + p(x)y = g(x) \quad ; \quad \mu(x) = e^{\int p(x)dx}$

$1\, y' - 2x\, y = 3x^2 e^{x^2} \qquad p(x) = -2x, \quad g(x) = 3x^2 e^{x^2}$

$\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$

$\underbrace{e^{-x^2} y' - 2x e^{-x^2} y} = 3x^2 e^{x^2} \cdot e^{-x^2}$

$\frac{d}{dx}\left[e^{-x^2} y\right] = 3x^2 \rightarrow e^{-x^2} y = x^3 + C$

$y = x^3 e^{x^2} + C e^{x^2}$

$\frac{d}{dx}\left[\mu(x)\, y\right] = \mu(x)\, g(x)$

$y(0) = 0 + ce^0 = 5$

$$\boxed{y(x) = x^3 e^{x^2} + 5e^{x^2}}$$

$\boxed{1.6}$ Subtitle: Homogeneous Equations

A first-order equation $y' = f(x,y)$ is called a homogeneous DE if it is in the form

$$y' = F\left(\frac{y}{x}\right).$$

e.g. $y' = \dfrac{y}{x}$, $\quad \dfrac{dy}{dx} = \left(\dfrac{y}{x}\right)^2 + \sin\left(\dfrac{y}{x}\right)$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$F\left(\frac{y}{x}\right) = \frac{y}{x} \qquad\qquad F\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right)^2 + \sin\left(\frac{y}{x}\right)$$

$$F(w) = w \qquad\qquad F(w) = w^2 + \sin w$$

The solution method:

$y' = F\left(\frac{y}{x}\right)$; we'll find a sol. $y = y(x)$

Let $v = \frac{y}{x}$ $\quad v(x) = \frac{y(x)}{x} \rightarrow y = x\,v(x)$

$$\frac{dy}{dx} = 1 . v(x) + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = F(v)$$ 1^{st} order, separable equation

$$x \frac{dv}{dx} = F(v) - v \Rightarrow \frac{dx}{x} = \frac{dv}{F(v) - v}$$

$$\int \frac{dx}{x} = \int \frac{dv}{F(v) - v} \Rightarrow \ldots$$

Example Solve the DE $2xy\,\frac{dy}{dx} = 4x^2 + 3y^2$

$$\frac{dy}{dx} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = 2\,\frac{x}{y} + \frac{3}{2}\,\frac{y}{x}$$

$$= \frac{2}{\frac{y}{x}} + \frac{3}{2}\,\frac{y}{x} = F\left(\frac{y}{x}\right)$$

$$F(w) = \frac{2}{w} + \frac{3}{2}\,w$$

$$\frac{y}{x} = v(x) \rightarrow y = x.v \Rightarrow \frac{dy}{dx} = 1.v + x\,\frac{dv}{dx}$$

$$2x.\,x.v.\left(v + x\,\frac{dv}{dx}\right) = 4x^2 + 3(xv)^2$$

$$2v^2 + 2xv\,\frac{dv}{dx} = 4 + 3v^2 \rightarrow 2xv\,\frac{dv}{dx} = 4 + v^2$$

$$\int \frac{dx}{x} = \int \frac{2v\,dv}{4+v^2} \rightarrow \ln x = \ln(v^2+4) + \ln C$$

$$x = C(4+v^2) \rightarrow x = C\left[4+\left(\frac{y}{x}\right)^2\right]$$

$$4 + \frac{y^2}{x^2} = \frac{x}{C}$$

$$\frac{y^2}{x^2} = \hat{C}x - 4$$

$$y^2 = x^2(\hat{C}x - 4)$$

Ex Solve the IVP $x\dfrac{dy}{dx} = y + \sqrt{x^2-y^2}$, $y(x_0)=0$ $(x_0>0)$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{x}\sqrt{x^2-y^2} = \frac{y}{x} + \sqrt{\frac{x^2}{x^2} - \frac{y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1-\left(\frac{y}{x}\right)^2} = F\left(\frac{y}{x}\right) \quad \text{Hom. eq.}$$

$$\frac{y}{x} = v \rightarrow y = x\cdot v \rightarrow \frac{dy}{dx} = 1\cdot v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v + \sqrt{1-v^2} \Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1-v^2}}$$

$$\ln x = \sin^{-1}(v) - \ln C \Rightarrow \ln(Cx) = \sin^{-1} v$$

Ex Solve the IVP $x\frac{dy}{dx} = y + \sqrt{x^2-y^2}$, $y(x_0)=0$ $(x_0>0)$

$$v = \sin \ln(c \cdot x) \rightarrow \frac{y}{x} = \sin \ln(c \cdot x)$$

$$* \; y(x) = x \sin(\ln c x)$$

$$\Rightarrow y(x_0) = x_0 \sin(\ln c x_0) = 0 \qquad (x_0 > 0)$$

$$\Rightarrow \ln x = \sin^{-1}(v) - \ln c \qquad x = x_0, \; y = 0$$

$$\ln x = \sin^{-1}\left(\frac{y}{x}\right) - \ln c$$

$$\ln x_0 = \underbrace{\sin^{-1}\left(\frac{0}{x_0}\right)}_{0} - \ln c \Rightarrow c = \frac{1}{x_0}$$

$$\boxed{y(x) = x \sin \ln\left(\frac{x}{x_0}\right)}$$

Ex p74, 15 $\quad x(x+y)y' + y(3x+y) = 0$

First-order, Nonlinear

$$y' = -\frac{y}{x}\,\frac{3x+y}{x+y} = -\frac{y}{x}\,\frac{3+\frac{y}{x}}{1+\frac{y}{x}} = F\left(\frac{y}{x}\right)$$

1^{st} order, hom.

$\frac{y}{x} = v, \quad y = x \cdot v$

DIY

$$x(x+y)\frac{dy}{dx} + y(3x+y) = 0$$

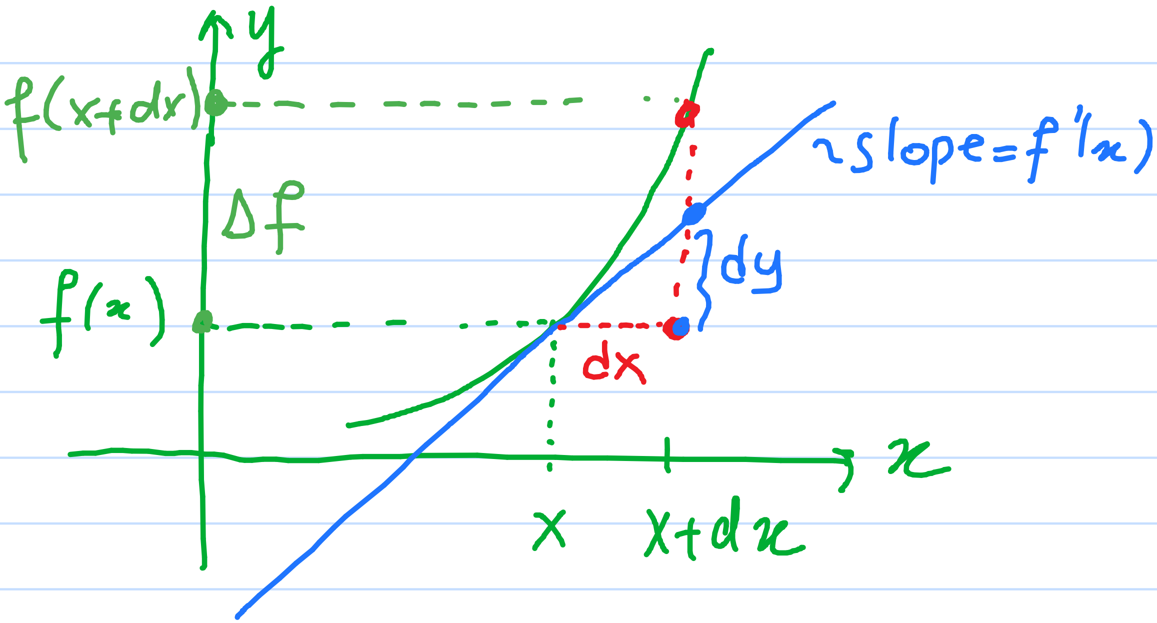
$$y(3x+y)\,dx + x(x+y)\,dy = 0$$

$$(3xy+y^2)\,dx + (x^2+xy)\,dy = 0 \qquad M(x,y)\,dx + N(x,y)\,dy$$

1.6, cont'd : EXACT DIFFERENTIAL EQUATIONS

(subtitle)

$y = f(x) \rightarrow y' = \dfrac{dy}{dx} = f'(x) \rightarrow dy = f'(x)\,dx$



differential of y

when moving from x to x+dx, the real change in the image f is

$\Delta f = f(x+dx) - f(x)$

The approximate change in $y = f(x)$ is defined by the change in the function's tangent line $= dy$

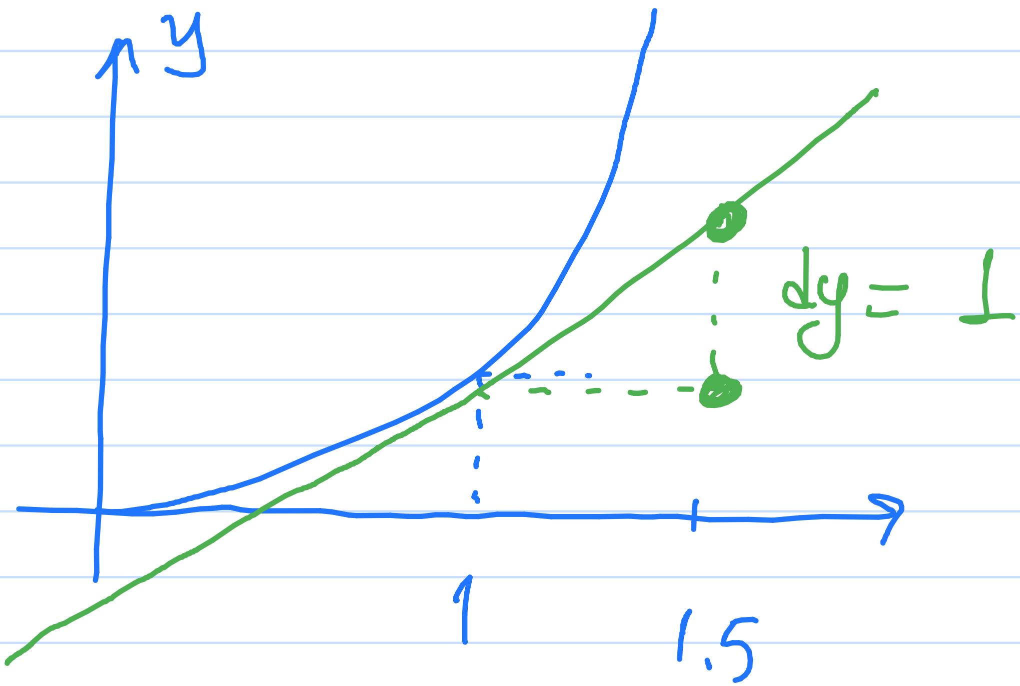
$dy = f'(x)\,dx$ e.g. $f(x) = x^2$, $x = 1$, $ds = 0.5$

$f(x) = x^2, \qquad x = 1, \qquad dx = 0.5$

$f'(x) = 2x, \qquad f'(1) = 2.1 = 2,$

$dy = f'(x)\, dx = f'(1)\, dx = 2.(0{,}5) = 1$

The approximate change in $y = x^2$ when moving 0.5 units from 1 to the right is 1 unit



For a two-variable function $w = f(x,y)$,

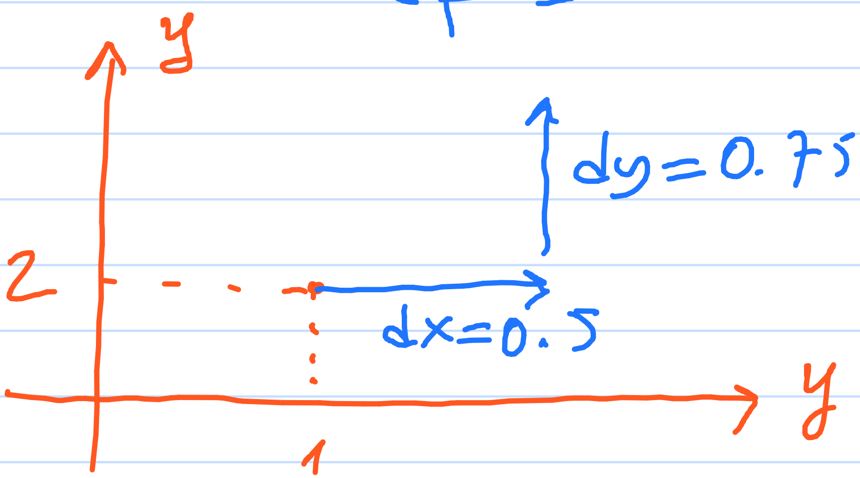
$$dw = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

e.g. $w = f(x,y) = 2xy$

$$dw = df = 2y \,.\, dx + 2x\, dy$$

e.g. , $(x,y) = (1,2)$; $dx = 0.5$, $dy = 0.75$

$$df = 2.2.0{,}5 + 2.1.0{,}75 = 3.5$$



When we move from (1,2)
$dx = 0.5$ units to the right
$dy = 0.75$ units upward
$df = 3.5$ is the approximat
change f experiences.

Example Find the function $f(x,y)$ for which

$$2y\,dx + 2x\,dy = 0.$$

It looks like: Someone calculated the total differential of some function f and made it equal to zero. $\Longleftrightarrow$ For which function f is its total derivative always equal to zero on some curve $y = y(x)$?

Indeed:

$$2y\,dx + 2x\,dy = 0$$

$$\int d(2xy) = \int 0$$

$$2xy = C \longrightarrow \boxed{y = \frac{C}{2x}}$$

Example Solve the DE $y' = -\frac{x}{y}$.

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow 2x\,dx + 2y\,dy = 0$$

$$\int d(x^2+y^2) = \int 0$$

$$x^2 + y^2 = C$$

$$y = \mp\sqrt{c - x^2}$$

In principal, this is the route we'll follow when solving an Exact DE.

Def Given $y' = f(x,y)$, suppose we convert it to the form

$$M(x,y) + N(x,y)\, y' = 0 \qquad y' = \frac{dy}{dx}$$

or

$$M(x,y)\, dx + N(x,y)\, dy = 0.$$

This equation is said to be EXACT DE if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If the eq. is exact, then it's possible to call

$$\underbrace{M(x,y)}_{\frac{\partial \phi}{\partial x}}\, dx + \underbrace{N(x,y)}_{\frac{\partial \phi}{\partial y}}\, dy = 0$$

$$\underbrace{M(x,y)}_{\frac{\partial \phi}{\partial x}}\,dx + \underbrace{N(x,y)}_{\frac{\partial \phi}{\partial y}}\,dy = 0 \Leftarrow$$

$$\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = 0$$

$$d\phi = 0$$

$$\int d\phi = \int 0$$

$\phi(x,y) = C$ is the general sol. to $M(x,y)\,dx + N(x,y)\,dy = 0$; where ϕ is determined by $\frac{\partial \phi}{\partial x} = M$, $\frac{\partial \phi}{\partial x} = N$

Example

$$y' = -\frac{x}{y} \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$x\,dx + y\,dy = 0 \qquad M(x,y) = x \qquad N(x,y) = y$$

$$\frac{\partial}{\partial y} M = \frac{\partial}{\partial y} x = 0, \qquad \frac{\partial}{\partial x} N = \frac{\partial}{\partial x} y = 0 \rightarrow M_y = N_x$$

The eq. is EXACT.

$$\underbrace{x}_{\phi_x}\,dx + \underbrace{y}_{\phi_y}\,dy = 0$$

$$\frac{\partial \phi}{\partial x} = x \rightarrow \phi(x,y) = \frac{x^2}{2} + g(y)$$

$$\frac{\partial \phi}{\partial y} = y \qquad \frac{\partial \phi}{\partial y} = 0 + g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} + C$$

$$\underbrace{x}_{\phi_x}\,dx + \underbrace{y}_{\phi_y}\,dy = 0 \qquad y' = -\frac{x}{y}$$

$$\phi_x\,dx + \phi_y\,dy = 0 \qquad \phi = \frac{x^2+y^2}{2} + C$$

$$\int d\phi = \int 0$$

$$\phi = \hat{c} \qquad \frac{x^2+y^2}{2} = C$$

$$\frac{x^2+y^2}{2} + c = \hat{c}$$

$$\boxed{x^2 + y^2 = \tilde{c}} \qquad \tilde{c} = \hat{c} - 2c$$

$$y = \mp\sqrt{\tilde{c} - x^2}$$

Ex $\underbrace{y^3}_{\phi_x} dx + \underbrace{3xy^2}_{\phi_y} dy = 0$ $M\,dx + N\,dy = 0$

$M = y^3 \longrightarrow M_y = 3y^2$

$N = 3xy^2 \longrightarrow N_x = 3y^2$

$M_y = N_x$ ✓

The eq. is exact. * More comments on next session.

$$\frac{\partial \phi}{\partial x} = M(x,y) = y^3 \longrightarrow \phi(x,y) = y^3 . x + g(y)$$

$$\frac{\partial \phi}{\partial y} = N(x,y) = 3xy^2 \qquad \frac{\partial \phi}{\partial y} = 3y^2 x + g'(y) = 3xy^2$$

$$g'(y) = 0 \rightarrow g(y) = C \Rightarrow \phi(x,y) = \boxed{xy^3} + C$$

The solution of the DE: $\Rightarrow xy^3 = \hat{C}$

$$y = \frac{\hat{C}}{x^{1/3}}$$

We're given $y^3\,dx + 3xy^2\,dy = 0$

* Is this a DE?? Yes, indeed: $\dfrac{dy}{dx} = -\dfrac{y^3}{3xy^2}$

We'll find a function $y(x)$ for which $y' = -\dfrac{y}{3x}$.

* What have we done?? We have shown that someone has calculated the total dif. of $f = xy^3$, and made it equal to zero;

$$d(xy^3) = 0$$

$$\frac{\partial}{\partial x}(xy^3)\,dx + \frac{\partial}{\partial y}(xy^3)\,dy = 0$$

$$y^3\,dx + 3xy^2\,dy = 0 \quad \checkmark$$

$$y^3\,dx + 3xy^2\,dy = 0$$

$$\Downarrow$$

$$d(x \cdot y^3) = 0$$

integration is here.

$$x \cdot y^3 = C$$

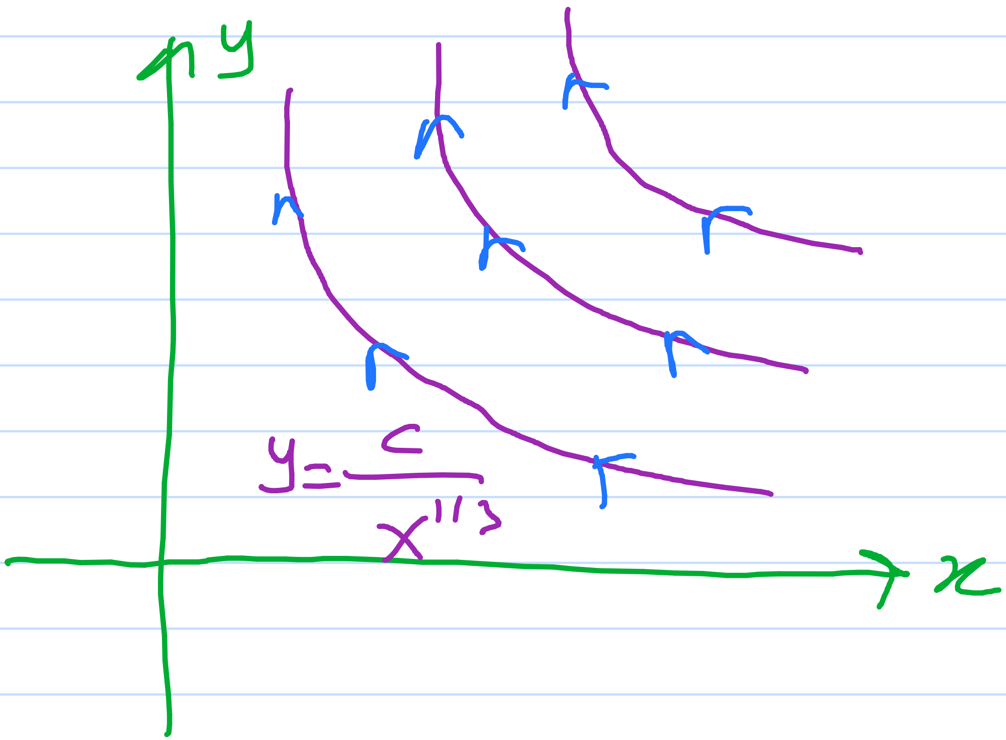
$$y(x) = \frac{\tilde{C}}{x^{1/3}}$$

(*) What is the relation between the potential

$\phi(x,y) = xy^3 + C$ and $y(x) = \dfrac{\hat{C}}{x^{1/3}}$

???

$\phi(x,y) = xy^3 + C$ is a two-variable function.



If we calculate total differential of $\phi(x,y) = xy^3$

$$d\phi = y^3 dx + 3xy^2 dy$$

and evaluate this expression on any point belonging to the curves $y = \frac{c}{x^{1/3}}$ we get

$$d\phi \Big|_{y = \frac{c}{x^{1/3}}} = \left(\underline{y^3 dx + 3xy^2 dy} \right) \Big|_{y = \frac{c}{x^{1/3}}} = 0$$

$\underline{\underline{Ex}}$ $\underbrace{(6xy - y^3)}_{\phi_x} dx + \underbrace{(4y + 3x^2 - 3xy^2)}_{\phi_y} dy = 0$

$M = 6xy - y^3 \rightarrow M_y = 6x - 3y^2$

$N = 4y + 3x^2 - 3xy^2 \rightarrow N_x = 6x - 3y^2$

$M_y = N_x$

The eq. is exact

$$\frac{\partial \phi}{\partial x} = M = 6xy - y^3 \qquad (A)$$

$$\frac{\partial \phi}{\partial y} = N = 4y + 3x^2 - 3xy^2 \qquad (B)$$

From (A) : $\phi = 6y \cdot \frac{x^2}{2} - y^3 \cdot x + g(y)$

$$\downarrow$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - 3y^2 x + g'(y) = 4y + 3x^2 - 3xy^2$$

$g'(y) = 4y \Rightarrow g(y) = 2y^2 + C$

Ex $\underbrace{(6xy - y^3)}_{\phi_x}\,dx + \underbrace{(4y + 3x^2 - 3xy^2)}_{\phi_y}\,dy = 0$

$-$ $\qquad$ 2

$$\phi(x,y) = 3x^2y - xy^3 + 2y^2 + C$$

$$\phi = \tilde{C} \Rightarrow \boxed{3x^2y - xy^3 + 2y^2 = \hat{C}}$$

This is a cubic eq. in y; of which solution for $y(x)$ determines $y = y(x)$.

See exercises on: page: 74

26. 11. 2020, Thursday

Ex page 74, Ex 35 : Solve $\left(x^3+\frac{y}{x}\right)+\left(y^2+\ln x\right)y'=0$

$$y'=\frac{dy}{dx} \rightarrow \underbrace{\left(x^3+\frac{y}{x}\right)}_{M=\phi_x}dx+\underbrace{\left(y^2+\ln x\right)}_{N=\phi_y}dy=0$$

$$M=x^3+\frac{y}{x} \rightarrow M_y=\frac{1}{x}$$

$$N=y^2+\ln x \rightarrow N_x=\frac{1}{x}$$

$M_y=N_x$: The eq. is exact.

$$\frac{\partial\phi}{\partial x}=M=x^3+\frac{y}{x} \rightarrow \phi(x,y)=\frac{x^4}{4}+y\ln x+g(y)$$

$$\frac{\partial\phi}{\partial y}=y^2+\ln x$$

$$\phi_y=\ln x+g'(y)=y^2+\ln x$$

Ex page 74, Ex 35 : Solve $\left(x^3+\frac{y}{x}\right)+(y^2+\ln x)y'=0$

$g'(y)=y^2 \Rightarrow g(y)=\frac{y^3}{3}+C$

$\phi(x,y)=\frac{x^4}{4}+y\ln x+\frac{y^3}{3}+C$

$\phi(x,y)=\hat{C} \Rightarrow \boxed{\frac{x^4}{4}+y\ln x+\frac{y^3}{3}=\tilde{C}}$.

$y'=-\frac{x^3+\frac{y}{x}}{y^2+\ln x}$

$\frac{1}{3}y^3+(\ln x)y+\frac{x^4}{4}-\tilde{C}=0$

$\downarrow$

$y=y(x)=\cdots$

Ex $M(x,y)\,dx + x\tan^{-1}y\,dy = 0$ Determine $M(x,y)$ such that this eq. is exact.

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x}(x\tan^{-1}y)$$

$$M_y = \tan^{-1}y \Rightarrow M(x,y) = \int \tan^{-1}y\,dy + h(x)$$

Integration by parts, **DIY**

$$M(x,y) = y\tan^{-1}y - \frac{1}{2}\ln(1+y^2) + h(x)$$

where h is an arbitrary function.

Remark Why the def. of exactness ??

The eq. $\underbrace{M(x,y)}_{\frac{\partial\phi}{\partial x}}dx + \underbrace{N(x,y)}_{\frac{\partial\phi}{\partial y}}dy = 0$ is exact if

$M_y = N_x$.

Why this cond??

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0$$

$$d\phi = 0 \Rightarrow \phi = c$$

$$\phi(x,y) = c$$

$M = \phi_x$, $N = \phi_y$ is possible if

$$\phi_{xy} = \phi_{yx} \to (\phi_x)_y = (\phi_y)_x \Rightarrow \boxed{M_y = N_x}$$

For more exercises on Exact D.Es: page 74, pr. 31-42

Ch. 1.6, cont'd: **BERNOULLI EQ** (Nonlinear eq.)

$$y' + p(x)\, y = g(x)\, y^{n} \qquad n \neq 0,\ n \neq 1$$

The solution methodology: $y = y(x)$, $V = V(x)$

$$V = y^{1-n} \Rightarrow \frac{dV}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$$

$$y^{-n} y' + p(x)\, y^{1-n} = g(x)\, y^{-n+n}$$

$$\frac{1}{1-n}\frac{dV}{dx} + p(x) \cdot V = g(x) \qquad \text{First order, linear}$$

$$\Rightarrow \frac{dV}{dx} + (n-1)\, p(x)\, V = (n-1)\, g(x)$$

Ex $2xy\,y' = 4x^2 + 3y^2$

$$y' = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = 2\,\frac{x}{y} + \frac{3}{2}\,\frac{y}{x}$$

$$y' - \frac{3}{2x}\,y = 2x\;y^{-1}$$

$y' + p(x)\,y = g(x)\,y^n$

$n = -1$ Bernoulli eq.

$$v = y^{1-n} = y^{1-(-1)} = y^2 \rightarrow \frac{dv}{dx} = 2y\,\frac{dy}{dx}$$

$$2y\,y' - 2y\,\frac{3}{2x}\,y = 2x\,y^{-1}\,2y$$

$$2y\,y' - \frac{3}{x}\,y^2 = 4x$$

$$\frac{dv}{dx} - \frac{3}{x}\,v = 4x$$

First order, linear.

Ex $\quad 2xyy' = 4x^2 + 3y^2$

$$v' - \frac{3}{x}v = 4x \qquad p(x) = -\frac{3}{x}, \qquad g(x) = 4x$$

$$\mu(x) = e^{\int p(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$$

$$x^{-3}v' - x^{-3}\frac{3}{x}v = 4x \cdot x^{-3}$$

$$x^{-3}v' - 3x^{-4}v = 4x^{-2}$$

$\boxed{\text{HW solve it as a homogeneous eq! } y' = F\left(\frac{y}{x}\right)}$

$$\frac{d}{dx}\left[x^{-3}v\right] = 4x^{-2} \Rightarrow x^{-3}v = 4\cdot\left(\frac{-1}{x}\right) + C$$

$$v = x^3\cdot\left(\frac{-4}{x}\right) + x^3\cdot C \Rightarrow v = Cx^3 - 4x^2$$

$$\boxed{y^2 = Cx^3 - 4x^2}$$

$\underline{\underline{Ex}}$ $\quad xy' + 6y = 3x\, y^{4/3}$ $\qquad n \neq 0, 1$

$\Rightarrow \; y' + \frac{6}{x} y = 3\, y^{4/3}$ $\qquad y' + p(x)\, y = g(x)\, y^{n}$

$n = \frac{4}{3}$: $\quad v = y^{1-n} = y^{1-\frac{4}{3}} = y^{-\frac{1}{3}}$: $\quad v' = -\frac{1}{3}\, y^{-4/3}\, y'$

$$-\frac{1}{3}\, y^{-4/3}\, y' - \frac{1}{3}\, y^{-4/3}\, \frac{6y}{x} = -\frac{1}{3}\, y^{-4/3} \cdot 3\, y^{4/3}$$

$$\underbrace{-\frac{1}{3}\, y^{-4/3}\, y'} - \frac{2}{x}\, y^{-1/3} = -1$$

$$v' - \frac{2}{x}\, v = -1$$

First order, linear!

Ex $\quad xy' + 6y = 3xy^{4/3} \qquad n \neq 0, 1$

$$v' - \frac{2}{x} v = -1 \qquad p(x) = -\frac{2}{x}, \quad g(x) = -1$$

$$\mu(x) = e^{\int p(x)dx} = e^{\int -\frac{2}{x} dx} = e^{-2\ln x} = x^{-2}$$

$$\frac{1}{x^2} v' - \frac{2}{x^3} v = -\frac{1}{x^2} \Rightarrow \frac{d}{dx}\left[x^{-2} v\right] = -x^{-2}$$

$$x^{-2} v = \int -x^{-2} dx = -\frac{x^{-2+1}}{-2+1} + C$$

$$x^{-2} v = \frac{1}{x} + C$$

$$v = Cx^2 + x \qquad y^{-1/3} = x + Cx^2$$

$$y = (x + Cx^2)^{-3}.$$

Reducible Second-Order ODEs

(A) If the dependent variable is missing

$F(x, y, y', y'') = 0$: The most general second-order ODE

If the term y does not appear in the eq. explicitly, the eq. is reduced in order by 1:

$$F(x, y', y'') = 0 \xrightarrow[z = y']{} F(x, z, z') = 0$$

$z(x) = \frac{dy}{dx}$

↑ A 1^{st} order ODE

Ex $\quad x y'' + 2y' = 6x$: y is missing.

Let $y' = z \rightarrow y'' = z'$

$$x z' + 2z = 6x \rightarrow \frac{dz}{dx} + \frac{2}{x} z = 6$$

Linear, 1st order eq. $\quad p(x) = \frac{2}{x}$

$$\mu = e^{\int p(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$$

$$x^2 \frac{dz}{dx} + x^2 \cdot \frac{2}{x} \cdot z = x^2 \cdot 6 \rightarrow x^2 z' + 2xz = 6x^2$$

$$\frac{d}{dx}\left[x^2 \cdot z\right] = 6x^2 \Rightarrow \quad x^2 z = \int 6x^2\,dx + C_1$$

Ex $\quad x y'' + 2y' = 6x$: y is missing.

$$x^2 z = 6\frac{x^3}{3} + c_1 \Rightarrow x^2 z = 2x^3 + c_1$$

$$z = 2x + c_1 x^{-2} \qquad z = y'$$

$$\frac{dy}{dx} = 2x + c_1 x^{-2} \Rightarrow \boxed{y = x^2 - \frac{c_1}{x} + c_2}$$

(B) If the independent variable is missing

$$F(x, y, y', y'') = 0 \Longrightarrow F(y, y', y'') = 0$$

Let $y' = p$ $\quad$ p : new dependent variable $\quad$ we'll

y : " independent variable $\quad$ assume

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\frac{dy}{dx} = \frac{dy}{dx}\frac{dy'}{dy} = p\frac{dp}{dy}$$

$$F(y, y', y'') = 0 \xrightarrow{p = y'} F\left(y, p, p\frac{dp}{dy}\right) = 0$$

The order is reduced to 1 !!!

Ex $\quad y\, y'' = (y')^2 \quad : \quad y$ is missing

$$y' = p \qquad y'' = \frac{dy'}{dx} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}\, p = p\frac{dp}{dy}$$

$$y \cdot p \cdot \frac{dp}{dy} = p^2 \Rightarrow \frac{dp}{p} = \frac{dy}{y}$$ separable equation

$$\ln p = \ln y + \ln c_1 \Rightarrow p = c_1 \cdot y$$

$$y' = c_1 \cdot y \longrightarrow \frac{dy}{dx} = c_1 \cdot y \Rightarrow \frac{dy}{y} = c_1\, dx$$

$$\int \frac{dy}{y} = \int c_1\, dx \rightarrow \ln y = c_1 x + \ln c_2 \Rightarrow \boxed{y = c_2 e^{c_1 x}}$$

Ex1 Show that $\overset{(*)}{y=ce^{x^2}}$ satisfies the DE $\overset{(**)}{y'=2xy}$.

$y' = c\,2x e^{x^2} \overset{?}{=} 2x\,ce^{x^2}$ for all $x \Rightarrow (*)$ is a solution → general solution of the DE since it satisfies $(**)$

(pay attention $y=0$ also satisfies $(**)$; thus it is another solution but you can get it for $c=0$

$y=ce^{x^2}$: infinite family of solutions since $c\in\mathbb{R}$ is an arbitrary constant.

$\Rightarrow$ introduce a condition $y(0)=2$ → initial condition, $\overset{(***)}{y'=2xy,\ y(0)=2}$ → initial value pr.

$y=ce^{x^2}$, $y(0)=2 \Rightarrow 2=ce^0 \Rightarrow c=2 \Rightarrow y=2e^{x^2}$ → unique solution

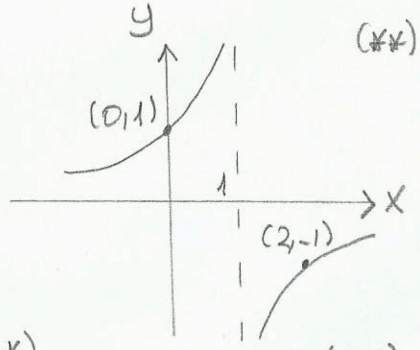
$y=ce^{x^2} \Rightarrow$ general sol. of $(**)$, $y=2e^{x^2}$: particular solution

Ex2 $\frac{dy}{dx}=y^2$, $y=\frac{1}{c-x} \Rightarrow -(c-x)^{-2}(-1) \overset{?}{=} (c-x)^{-2} \Rightarrow y=(c-x)^{-1}$ is a solution of the DE → also $y=0$

in any interval which doesn't contain $x=c \Rightarrow (-\infty,c)\cup(c,\infty)$

$y''=y^2$, $\overset{(*)}{y(0)=1} \Rightarrow 1=(c-0)^{-1} \Rightarrow c=1 \Rightarrow y=(1-x)^{-1}$

$(**)$ $y(2)=-1 \Rightarrow (c-2)^{-1}=-1 \Rightarrow c=1 \Rightarrow y=(1-x)^{-1}$



$(*)$ $x\in(-\infty,1)$ $(**)$ $x\in(1,\infty)$

Ex3 $(y')^2+y^2=-1 \Rightarrow$ no solution

Assume that $u=u(x)$ is cont on an int. I and $u', u'', \dots, u^{(n)}$ exist on I.
Then, $u=u(x)$ is a sol. of the DE $F(x,y,y',\dots,y^{(n)})=0$ if it satisfies the DE.

$K(x,y)=0$ implicit solution $(x^2+y^2=4,\ x+yy'=0 \Rightarrow y=\pm\sqrt{4-x^2}$!!!$)$

Ex 4 $y' = \frac{1}{x} \Rightarrow dy = \frac{dx}{x} \Rightarrow y = \ln|x| + c$

$y(0) = 0 \Rightarrow$ no solution since it is not defined at $x=0$

$\Rightarrow$ nonexistence of a solution

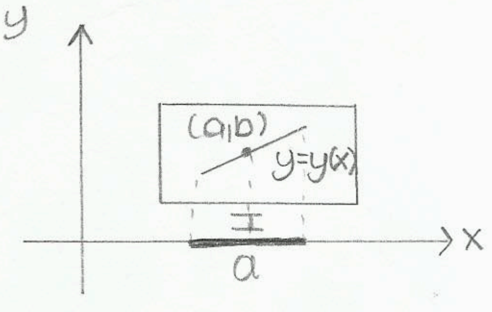
Ex 5 $y' = 2\sqrt{y} \Rightarrow \frac{dy}{2\sqrt{y}} = dx \Rightarrow \sqrt{y} = x + c \Rightarrow y = (x+c)^2$

$y = 0$

$y(0) = 0 \Rightarrow c = 0 \Rightarrow y = x^2,\ y = 0 \Rightarrow$ not a unique solution

EXISTENCE & UNIQUENESS OF SOLUTIONS

Assume that $f(x,y)$ and its partial derivative $D_y f(x,y)$ are continuous on some rectangle R in the xy-plane that contains the point (a,b) in its interior. Then, for some open interval I containing the point a, the initial value problem $\frac{dy}{dx} = f(x,y),\ y(a) = b$ has one and only one solution on I.



Ex 4: $f(x,y) = \frac{1}{x}$ and $f_y = 0 \Rightarrow f$: not cont at $(0,0)$

Ex 5: $f = 2\sqrt{y}$, $f_y = \frac{1}{\sqrt{y}} \Rightarrow f_y$: not cont at $(0,0)$

Ex 1: $f = 2xy$, $f_y = 2x \Rightarrow f, f_y$: cont at any (a,b).

Solution exists and it is unique.

$\downarrow$

$y = ce^{x^2}$

$(*)$

Ex: $x\dfrac{dy}{dx} = 2y \Rightarrow \dfrac{dy}{dx} = 2\dfrac{y}{x}$

$f = 2\dfrac{y}{x}$, $f_y = \dfrac{2}{x}$: cont at any $x \neq 0$

$\rightarrow$ $(*)$ has a unique solution for all $x \neq 0$.

$x\dfrac{dy}{dx} = 2y$, $y(0) = 0$ initial value problem

$y = cx^2 \Rightarrow x \cdot 2cx = 2cx^2$ ✓

$\rightarrow$ IVP has infinitely many solutions.

$x\dfrac{dy}{dx} = 2y$, $y(0) = b$

$\rightarrow$ IVP has no solution if $b \neq 0$

$$x\frac{dy}{dx} = 2y, \quad y(a) = b$$

- $a \neq 0 \Rightarrow$ un. sol
- $a = b = 0 \Rightarrow$ inf. many sol.
- $a = 0,\ b \neq 0 \Rightarrow$ no sol

1ST ORDER DE . $\frac{dy}{dx} = F(x,y)$

LINEAR 1ST ORD

$$\frac{dy}{dx} + M(x)y = N(x)$$

μ : int. fact.

$$\mu y' + \mu M y = \mu N$$

$$(\mu y)' = \mu y' + \mu M y$$

$$= \mu y' + \mu' y$$

$$\mu' = \mu M$$

$$\int \frac{d\mu}{\mu} = \int M\,dx$$

$$\mu = u(x) = e^{\int M dx}$$

$$uy' + uMy = uN$$

$$\frac{d}{dx}(uy) = uN$$

$$\int d(uy) = \int uN\,dx$$

$$uy = v(x) + C$$

$$y = u^{-1}(v(x) + C)$$

SEPARABLE

$$M(y)dy = N(x)dx$$

$$\int M dy = \int N dx$$

$$y = u(x) + C$$

SUBSTITUTION

HOMOGENEOUS

$$y' = F(y/x)$$

$$v = y/x \Rightarrow y = xv$$

$$v + xv' = F(v)$$

$$\frac{dv}{F(v) - v} = \frac{dx}{x}$$

$\downarrow$

separable

$\downarrow$

$$Ax^m y^n y' = Bx^p y^q + Cx^r y^s$$

$$m + n = p + q = r + s$$

$\Rightarrow$ hom.

BERNOULLI

$$y' + M(x)y = N(x)y^n$$

$$(1-n)y^{-n}y' + (1-n)My^{1-n} = (1-n)N$$

$$v = y^{1-n} \Rightarrow v' = (1-n)y^{-n}y'$$

$$v' + (1-n)Mv = (1-n)N$$

$\downarrow$

Linear 1st order

EXACT

$$M(x,y)dy + N(x,y)\,dx = 0$$

if $M_x = N_y$

$$\mu_y = M, \quad \mu_x = N$$

$$\mu_y = M \Rightarrow \mu = \int M dy + h(x)$$

$$\left(\int M dy + h(x)\right)_x = N$$

$$\Rightarrow \mu = f(x,y)$$

$$\Rightarrow f_y dy + f_x\,dx = 0$$

$$df = 0$$

$$f = C$$

Follow these steps:

* Sep. ?
* Lin. ?
* Ex. ?
* Hom. or Bern ?

Ex: Solve $y' - 2y = e^{-x}$, $y(0) = 1$

Linear 1st order $\Rightarrow \mu . y' - 2\mu . y = \mu e^{-x} = (\mu y)'$

$(\mu y)' = \mu y' + \mu' y = \mu y' - 2\mu y \Rightarrow \mu' = -2\mu \Rightarrow \frac{d\mu}{\mu} = -2dx$

$\Rightarrow \ln \mu = -2x \Rightarrow \mu = e^{-2x}$

$(e^{-2x} y)' = e^{-2x} e^{-x} \Rightarrow e^{-2x} y = \int e^{-3x} dx \Rightarrow e^{-2x} y = -\frac{1}{3} e^{-3x} + c$

$\Rightarrow y = -\frac{1}{3} e^{-x} + c e^{2x}$

$y(0) = 1 \Rightarrow 1 = -\frac{1}{3} + c \Rightarrow c = 4/3 \Rightarrow y = -\frac{1}{3} e^{-x} + \frac{4}{3} e^{2x}$

Ex: $(x^2+1) \frac{dy}{dx} + 3xy = 6x$ Lin. 1st order

$y' + \frac{3x}{x^2+1} y = \frac{6x}{x^2+1} \Rightarrow \mu = (x^2+1)^{3/2} \Rightarrow y = 2 + c(x^2+1)^{-3/2}$

Ex: $(x + y e^y) \frac{dy}{dx} = 1$

$\Rightarrow \frac{dx}{dy} = x + y e^y \Rightarrow \frac{dx}{dy} - x = y e^y$ (Lin 1st ord)

$\Rightarrow (\mu x)' = \mu x' - \mu x = \mu y e^y \Rightarrow (\mu x)' = \mu x' + \mu' x = \mu x' - \mu x$

$\mu' = -\mu \Rightarrow \frac{d\mu}{\mu} = -dy \Rightarrow \ln \mu = -y \Rightarrow \mu = e^{-y}$

$(e^{-y} x)' = y \Rightarrow e^{-y} x = \int y \, dy = \frac{1}{2} y^2 + c \Rightarrow x = \frac{1}{2} y^2 e^y + c e^y$

THEOREM: If $M(x)$ and $N(x)$ are cont on the open int I containing the point x_0, then the IVP

$$\frac{dy}{dx} + M(x) y = N(x), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on I given by

$$y(x) = e^{-\int M dx} \left[\int N e^{\int M dx} dx + c \right]$$

for an appropriate value of c.

Ex: $\dfrac{dy}{dx} = -2xy \;,\; y(0)=2$ (sep. d.e)

$\dfrac{dy}{y} = -2x\,dx \Rightarrow \ln|y| = -x^2+c$

$y(0)=2>0$ near $x=0 \Rightarrow \ln y = -x^2+c \Rightarrow y = e^{-x^2+c} = e^c e^{-x^2}$

$\Rightarrow y = Ae^{-x^2}$

$y(0)=2 \Rightarrow 2 = Ae^0 \Rightarrow A=2 \Rightarrow y = 2e^{-x^2}$

Suppose that $y(0)=-2$. Then $y<0$ near $x=0$

$\Rightarrow \ln(-y) = -x^2+c \Rightarrow -y = e^{-x^2+c} \Rightarrow y = -Ae^{-x^2}$

$y(0)=-2 \Rightarrow -2 = -Ae^0 \Rightarrow A=2 \Rightarrow y = -2e^{-x^2}$

Ex: $\dfrac{dy}{dx} = \dfrac{4-2x}{3y^2-5}$ (sep. d.e)

$(3y^2-5)\,dy = (4-2x)\,dx \Rightarrow y^3-5y = 4x-x^2+c$ (impl. sol)

$y(1)=3 \Rightarrow 27-15 = 4-1+c \Rightarrow c=9 \Rightarrow y^3-5y = 4x-x^2+9$

Ex: $2\sqrt{x}\,\dfrac{dy}{dx} = \cos^2 y \;,\; y(4)=\pi/4$ (sep. d.e)

$\dfrac{dy}{\cos^2 y} = \dfrac{dx}{2\sqrt{x}} \Rightarrow \tan y = \sqrt{x}+c$ gen. sol. (impl. sol.)

$y(4)=\pi/4 \Rightarrow \overset{1}{\cancel{\tan(\pi/4)}} = \overset{2}{\cancel{\sqrt{4}}}+c \Rightarrow c=-1 \Rightarrow \tan y = \sqrt{x}+c$

part. sol.

Ex: $\dfrac{dy}{dx} = y^2 \Rightarrow \dfrac{dy}{y^2} = dx \Rightarrow -\dfrac{1}{y} = x+c \Rightarrow y = -\dfrac{1}{x+c}$ gen. sol $(x \neq -c)$

$y=0 \Rightarrow 0=0 \Rightarrow y=0$: sing. s $\left(-\dfrac{1}{x+c} \neq 0\right.$ for any choice of $\left.c\right)$

Ex: $xy\dfrac{dy}{dx} = \dfrac{3}{2}y^2 + x^2$ (hom. eq)

$$\frac{dy}{dx} = \frac{3}{2}\frac{y}{x} + \frac{x}{y} \quad , \quad \frac{y}{x} = v \Rightarrow y = xv$$

$$\Rightarrow v + xv' = \frac{3}{2}v + \frac{1}{v} \Rightarrow xv' = \frac{1}{2}v + \frac{1}{v} = \frac{v^2+2}{2v}$$

$$\frac{2v\,dv}{v^2+2} = \frac{dx}{x} \Rightarrow \quad \ln(v^2+2) = \ln|x| + \ln c \Rightarrow v^2+2 = c|x|$$

$$v = \frac{y}{x} \Rightarrow \frac{y^2}{x^2} + 2 = c|x| \Rightarrow y^2 + 2x^2 = cx^3 \quad (x<0 \Rightarrow c<0,\ x>0 \Rightarrow c>0)$$

Ex: $x\dfrac{dy}{dx} = y + \sqrt{x^2-y^2}$, $y(x_0) = 0$, $x_0 > 0$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1-(y/x)^2} \text{ hom. eq} \Rightarrow \frac{y}{x} = v ,\ y = xv$$

$$v + xv' = v + \sqrt{1-v^2} \Rightarrow xv' = \sqrt{1-v^2} \Rightarrow \frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$$

$$\sin^{-1} v = \ln x + c \Rightarrow \sin^{-1}\frac{y}{x} = \ln x + c$$

$$y(x_0) = 0 \Rightarrow 0 = \ln x_0 + c \Rightarrow c = -\ln x_0 \Rightarrow \sin^{-1}\frac{y}{x} = \ln x - \ln x_0$$

$$\Rightarrow y = x\sin\left(\ln\frac{x}{x_0}\right)$$

Ex: $\dfrac{dy}{dx} = \dfrac{x-y-1}{x+y+3}$ Solve the d.e by finding h and k so that the substitutions $x = u+h$, $y = v+k$ transform it into the hom. eq $\dfrac{dv}{du} = \dfrac{u-v}{u+v}$.

$$\frac{dv}{du} = \frac{u+h-v-k-1}{u+h+v+k+3} = \frac{u-v+h-k-1}{u+v+h+k+3} \quad , \quad \left.\begin{array}{l} h-k-1=0 \\ h+k+3=0 \end{array}\right\} \begin{array}{l} h=-1 \to x = u-1 \\ k=-2 \to y = v-2 \end{array}$$

$$\frac{dv}{du} = \frac{1-(v/u)}{1+(v/u)} \quad , \quad \frac{v}{u} = z \Rightarrow v = uz$$

$$z + uz' = \frac{1-z}{1+z} \Rightarrow uz' = \frac{1-z}{1+z} - z = \frac{1-2z-z^2}{1+z} \Rightarrow \frac{1+z}{1-2z-z^2}\,dz = \frac{du}{u}$$

$$-\frac{1}{2}\ln|1-2z-z^2| = \ln|u| + \ln c \Rightarrow |1-2z-z^2| = \frac{1}{u^2c^2} \Rightarrow \left|1 - 2\frac{y+2}{x+1} - \frac{(y+2)^2}{(x+1)^2}\right| = \frac{1}{c^2(x+1)^2}$$

Ex: $2xyy' = 4x^2 + 3y^2$

$\Rightarrow \frac{dy}{dx} = 2\frac{x}{y} + 3\frac{y}{x} \Rightarrow \frac{dy}{dx} - 3\frac{1}{x}\cdot y = 2xy^{-1}$ Bernoulli eq.
$n=-1 \Rightarrow 1-n=2$

$(1-n)y^{-n} = 2y \Rightarrow 2y\frac{dy}{dx} - \frac{6}{x}y^2 = 4x$

$v = y^{1-n} = y^2 \Rightarrow v' - \frac{6}{x}v = 4x$ (Linear 1st order)

$\mu v' - \frac{6}{x}\mu v = 4\mu x \Rightarrow (\mu v)' = \mu v' + \mu' v = \mu v' - \frac{6}{x}\mu v$

$\mu' = -\frac{6}{x}\mu \Rightarrow \frac{\mu'}{\mu} = -\frac{6}{x} \Rightarrow \ln\mu = -6\ln x \Rightarrow \mu = x^{-6}$

$x^{-6}v' - 6x^{-7}v = 4x^{-5} \Rightarrow (x^{-6}\cdot v)' = 4x^{-5} \Rightarrow x^{-6}v = -x^{-4} + C$

$v = -x^2 + Cx^6 \Rightarrow y^2 = -x^2 + Cx^6$

Ex: $x\frac{dy}{dx} + 6y = 3xy^{4/3} \Rightarrow \frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}$, Ber. eq
$n = 4/3 \Rightarrow 1-n = -1/3$

$(1-n)y^{-n} = -\frac{1}{3}y^{-4/3} \Rightarrow -\frac{1}{3}y^{-4/3}\frac{dy}{dx} - \frac{2}{x}y^{-1/3} = -1$

$v = y^{1-n} = y^{-1/3} \Rightarrow v' - \frac{2}{x}v = -1$ (1st ord. Lin)

$\mu v' - \frac{2}{x}\mu v = -\mu \Rightarrow (\mu v)' = \mu v' + \mu' v = D_x(\mu\cdot v)$

$\mu' = -\frac{2}{x}\mu \Rightarrow \frac{d\mu}{\mu} = -\frac{2}{x}dx \Rightarrow \ln\mu = \ln x^{-2} \Rightarrow \mu = x^{-2}$

$D_x(x^{-2}v) = -x^{-2} \Rightarrow x^{-2}v = x^{-1} + C \Rightarrow v = x + Cx^2$

$\Rightarrow y^{-1/3} = x + Cx^2 \Rightarrow y = (x + Cx^2)^{-3}$

Ex: $y^3 dx + 3xy^2 dy = 0 \quad (*)$

$M = y^3 \,,\, N = 3xy^2 \Rightarrow M_y = 3y^2 = N_x$ (exact d.e)

$M = f_x = y^3 \,,\, N = f_y = 3xy^2$

$\Rightarrow f(x,y) = xy^3 + h(y) \Rightarrow 3xy^2 + h'(y) = 3xy^2 \Rightarrow h'(y) = 0 \Rightarrow h(y) = c$

$\Rightarrow f = xy^3 + c$

$\Rightarrow xy^3 = c \Rightarrow y = k\,x^{-1/3}$

Pay attention : $y\,dx + 3x\,dy = 0$ not exact (divide $(*)$ by y^2)

Ex: $\underbrace{(6xy - y^3)}_{=M} dx + \underbrace{(4y + 3x^2 - 3xy^2)}_{=N} dy = 0$

$M_y = 6x - 3y^2 \,,\, N_x = 6x - 3y^2 \Rightarrow M_y = N_x$ (exact d.e)

$M = f_x = 6xy - y^3 \,,\, N = f_y = 4y + 3x^2 - 3xy^2$

$\Rightarrow f(x,y) = 3x^2y - y^3x + h(y)$

$\Rightarrow 4y + \cancel{3x^2} - \cancel{3xy^2} = \cancel{3x^2} - \cancel{3y^2x} + h'(y) \Rightarrow h(y) = 2y^2 + c$

$\Rightarrow f(x,y) = 3x^2y - y^3x + 2y^2 = c$

REDUCIBLE SECOND ORDER EQ.

$$F(x, y, y', y'') = 0$$

Dependent variable y missing

$$F(x, y', y'') = 0$$

$$p = y' = \frac{dy}{dx} \Rightarrow y'' = \frac{dp}{dx}$$

$$\Rightarrow F(x, p, p') = 0$$

1st order diff. eq.

$(p = p(x))$

Independent variable x missing

$$F(y, y', y'') = 0$$

$$p = y' = \frac{dy}{dx} \Rightarrow y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

$$\Rightarrow F\left(y, p, p\frac{dp}{dy}\right) = 0$$

1st order diff eq.

$(p = p(y))$

Ex: $xy'' + 2y' = 6x \Rightarrow y$ is missing

$$p = y' = \frac{dy}{dx} \Rightarrow y'' = \frac{dp}{dx} \quad , \; p(x)$$

$$x\frac{dp}{dx} + 2p = 6x \quad \text{(Linear 1st order)}$$

$$\Rightarrow \frac{dp}{dx} + \frac{2}{x}p = 6 \Rightarrow \underbrace{\mu p' + \mu\frac{2}{x}p}_{(\mu.p)' = \mu p' + \mu' p} = 6\mu$$

$$\mu' = \mu\frac{2}{x} \Rightarrow \frac{d\mu}{\mu} = \frac{2}{x}dx \Rightarrow \ln\mu = 2\ln x = \ln x^2 \Rightarrow \mu = x^2$$

$$x^2p' + 2xp = 6x^2 \Rightarrow (x^2p)' = 6x^2 \Rightarrow \int d(x^2p) = \int 6x^2dx$$

$$\Rightarrow x^2p = 2x^3 + c_1 \Rightarrow x^2\frac{dy}{dx} = 2x^3 + c_1 \Rightarrow \int dy = \int\left(2x + \frac{c_1}{x^2}\right)dx$$

$$\Rightarrow y = x^2 - \frac{c_1}{x} + c_2$$

Ex: $yy'' = (y')^2 \Rightarrow x$ is missing

$$p = y' = \frac{dy}{dx} \Rightarrow y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} \quad , \; p(y)$$

$$yp\frac{dp}{dy} = p^2 \Rightarrow y\frac{dp}{dy} = p \Rightarrow \frac{dp}{p} = \frac{dy}{y} \quad \text{(sep. d.e)}$$

$$\ln p = \ln y + \ln c_1 \Rightarrow p = c_1 . y \qquad (y>0,\, p>0)$$

$$\frac{dy}{dx} = c_1 y \Rightarrow \frac{dy}{y} = c_1 dx \Rightarrow \ln y = c_1 x + c_2$$

$$\Rightarrow y = e^{c_1x + c_2} = e^{c_1 x}e^{c_2} \Rightarrow y = ke^{cx}$$

* Note: Even if $k<0$ $(y<0)$ the d.e. is still satisfied.

Arguments on Existence & Uniqueness of Sols.

1. Show that $y = C e^{x^2}$ satisfies $y' = 2xy$.

$$y' = C \cdot 2x\, e^{x^2} = 2x \underbrace{C e^{x^2}} = 2xy$$

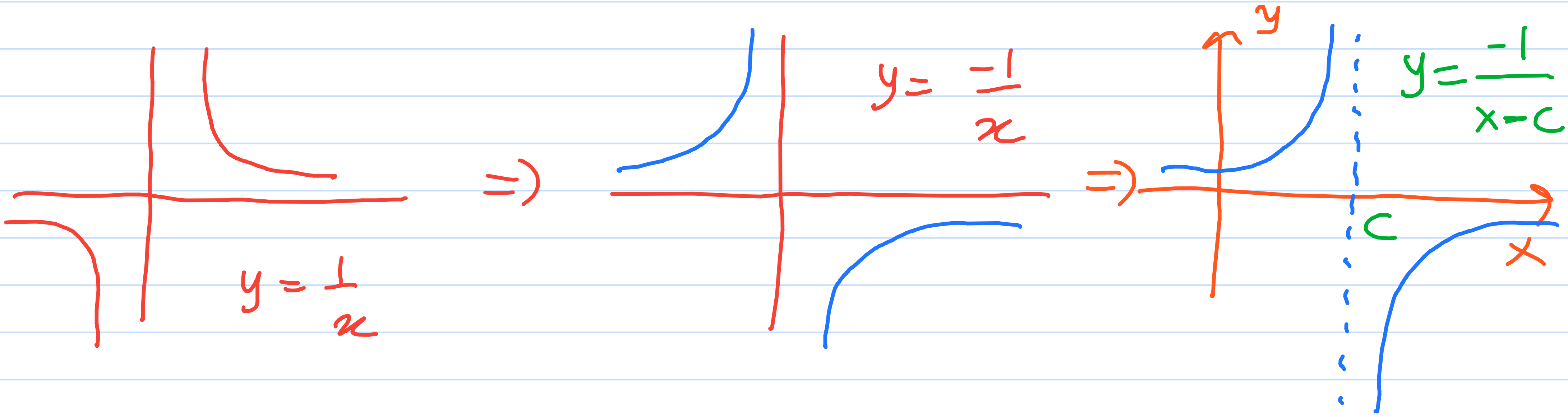
the function indeed satisfies the equality.

2. $\frac{dy}{dx} = y^2$ $\quad \frac{dy}{y^2} = dx \rightarrow \int \frac{dy}{y^2} = \int dx$

$$-\frac{1}{y} = x - C \Rightarrow y = \frac{-1}{x-C} = \frac{1}{C-x}$$

is the general solution of this DE.

$$y = \frac{1}{c-x} = \frac{-1}{x-c}$$



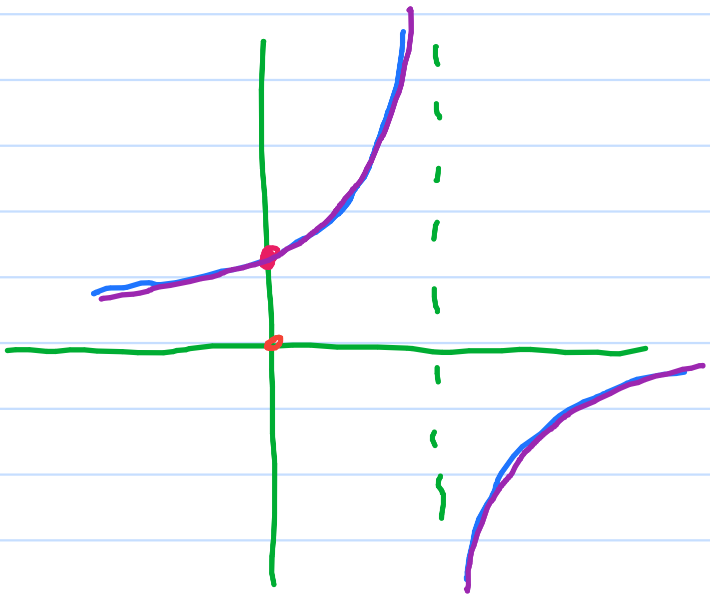
For this DE, the sol. exists on

$(-\infty, c) \cup (c, \infty)$.

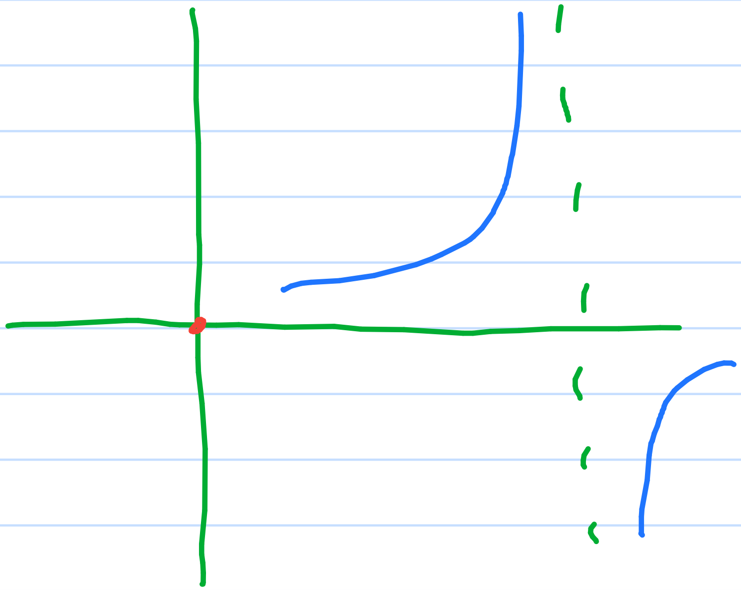
③ $\dfrac{dy}{dx} = y^2, \quad y(0) = 1 \quad \Big\}$ IVP

$$y(x) = \frac{1}{c-x} \qquad y(0) = \frac{1}{c-0} = 1 \qquad c = 1$$

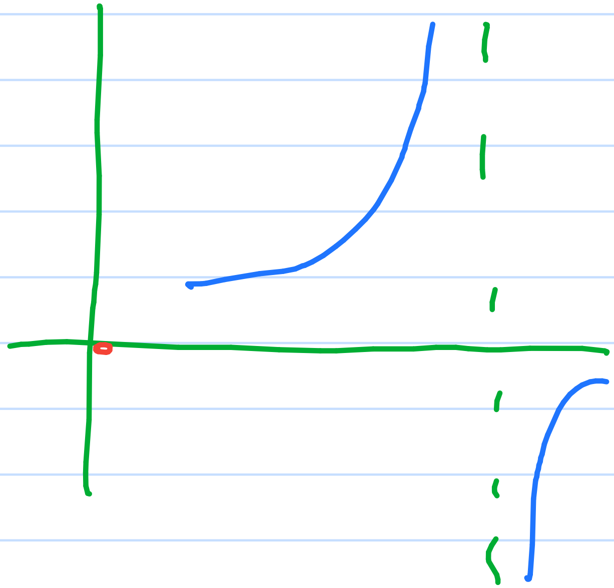
$$y(x) = \frac{1}{1-x}$$



c = 1

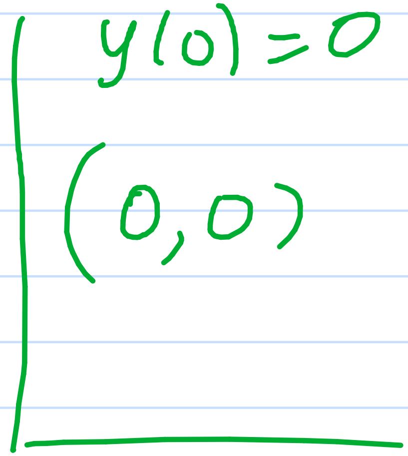


c = 2



c = 3

(4) $\frac{dy}{dx} = y^2$, $y(0) = A$ for which values of A this IVP does not have a solution?



$y(x) = \frac{1}{c-x}$ $\qquad y(0) = \frac{1}{c} = A$

$c = \frac{1}{A}$ $\quad (A \neq 0)$ $\qquad$ when $A=0$, the cond. $y(0) = \frac{1}{c} = 0$ cannot be satisfied by any number c. Therefore in case $A=0$, the IVP has no solution!!

Theorem Existence & Uniqueness Th. for First-Order Linear DEs

If the functions $p(x)$ and $g(x)$ are continuous on an interval I containing $x = x_0$, then the IVP

$$y' + p(x)\,y = g(x)$$

$$y(x_0) = y_0$$

has a unique solution on I.

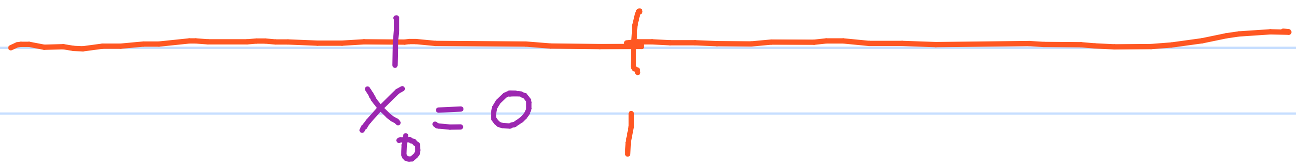
<u>Ex</u> Determine the interval such that the solution to the IVP

$$(x-1)\, y' + x y = \sin x$$
$$y(0) = 3$$

is to exist uniquely.

$$p(x) = \frac{x}{x-1}, \qquad g(x) = \frac{\sin x}{x-1}$$

$\Big\}$ $p(x)$ and $g(x)$ are cont. on $(-\infty, 1) \cup (1, \infty)$.



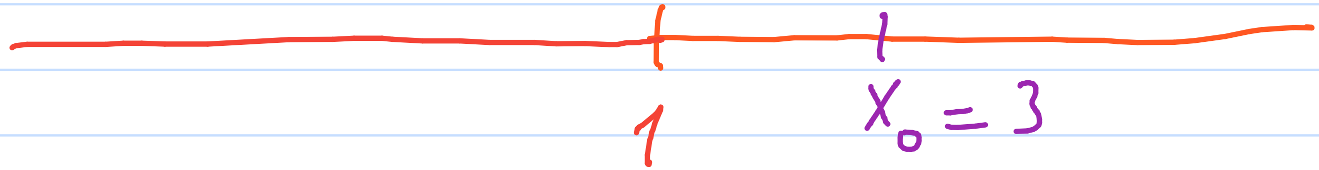
$p(x)$ and $g(x)$ are cont. on $I = (-\infty, 1) \ni x_0 = 0$

therefore, the sol. of the IVP exists uniquely on $I = (-\infty, 1)$.

* change the IVP to $\begin{cases} (x-1)\, y' + x\, y = \sin x \\ y(3) = 5 \end{cases}$

$$p(x) = \frac{x}{x-1}, \qquad g(x) = \frac{\sin x}{x-1}$$

$p(x)$ and $g(x)$ are cont. on $(-\infty, 1) \cup (1, \infty)$.



$p(x)$ and $g(x)$ are cont. on $I = (1, \infty) \ni x_0 = 3$

The IVP has a unique solution on $(1, \infty)$.

$$(x-1)y' + xy = \sin x$$

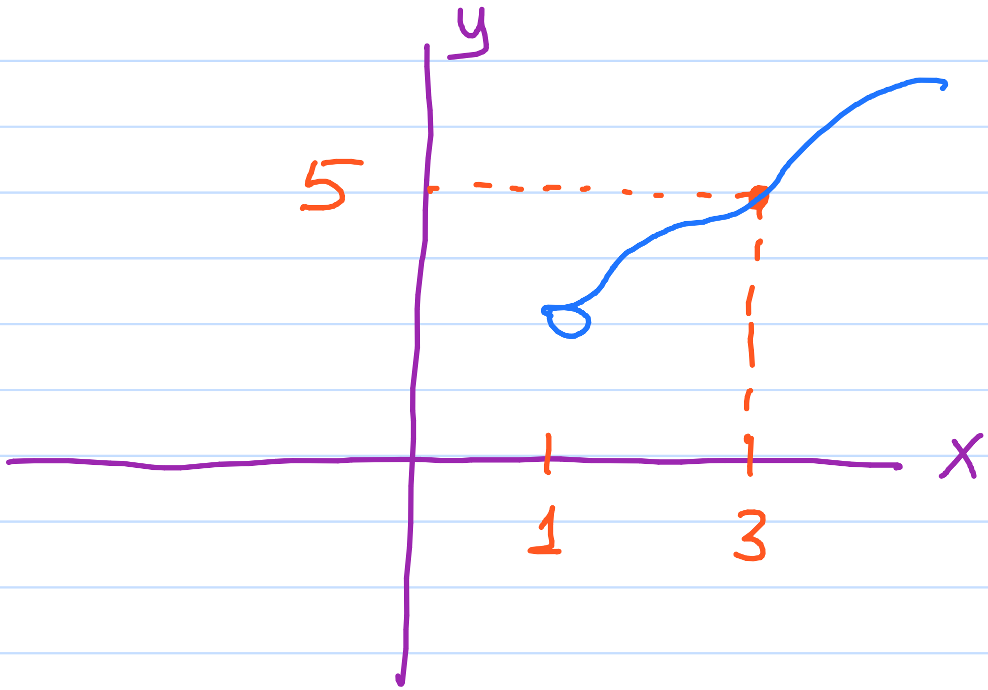
$$y(0) = 1$$

$x = 0, \quad y = 1 \rightarrow (0, 1)$

$I_1 = (-\infty, 1)$

$$(x-1)y' + xy = \sin x$$

$$y(3) = 5$$



$I_2 = (1, \infty)$

Next week

Existence & Uniqueness Th. for First-Order IVPs

$$\boxed{\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}}$$

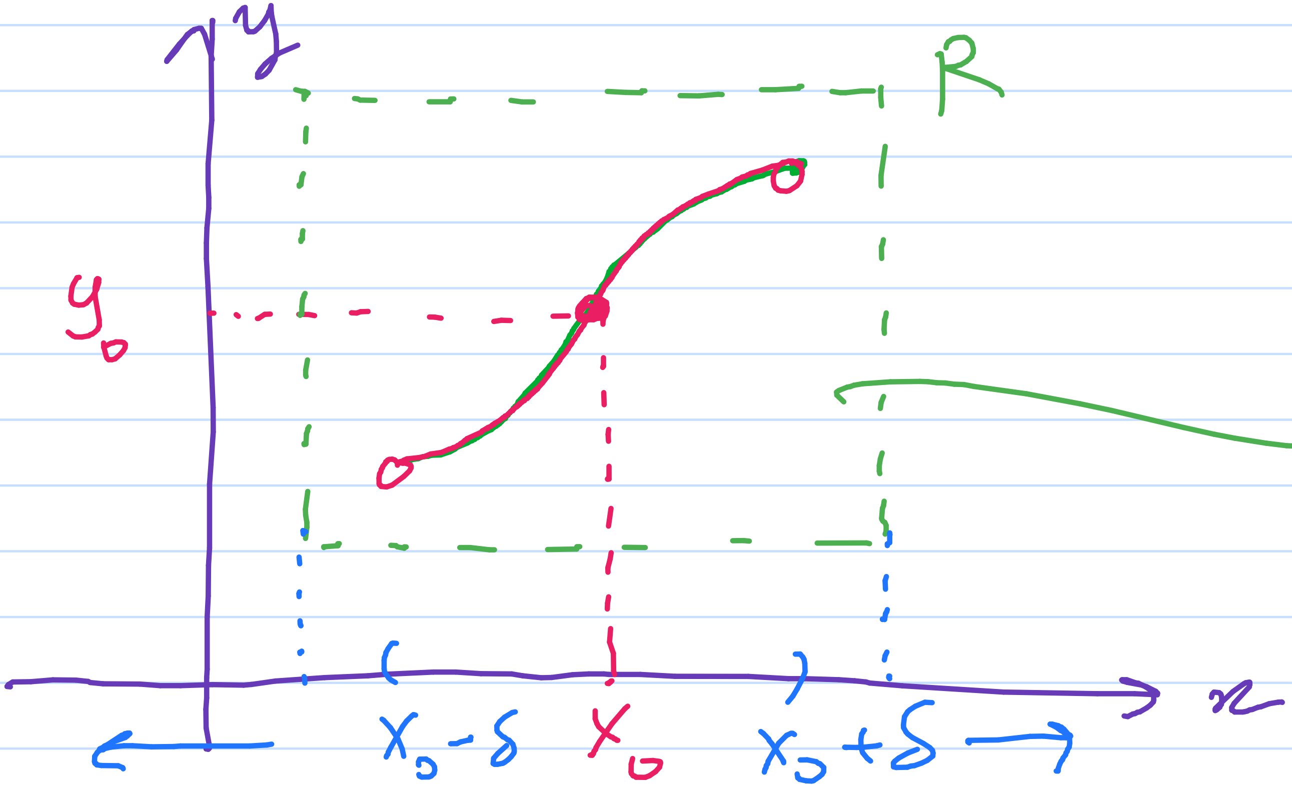
$\rightarrow$ Available in teaching files.

Existence & Uniqueness Th. for First Order IVP

$$y' = f(x,y) \quad , \quad y(x_0) = y_0 \qquad (*)$$

$*$ If $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on a rectangle R containing the point (x_0, y_0), then there's a neighbourhood $(x_0 - \delta, x_0 + \delta) = I$ such that the solution $y = y(x)$ of the IVP $(*)$ exists uniquely on I.

$$\left.\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}\right\} \longrightarrow \begin{aligned} &y = y(x) \checkmark \\ &\Downarrow \\ &y(x_0) = y_0 \end{aligned}$$



$y = y(x)$

$f(x,y)$ is cont.

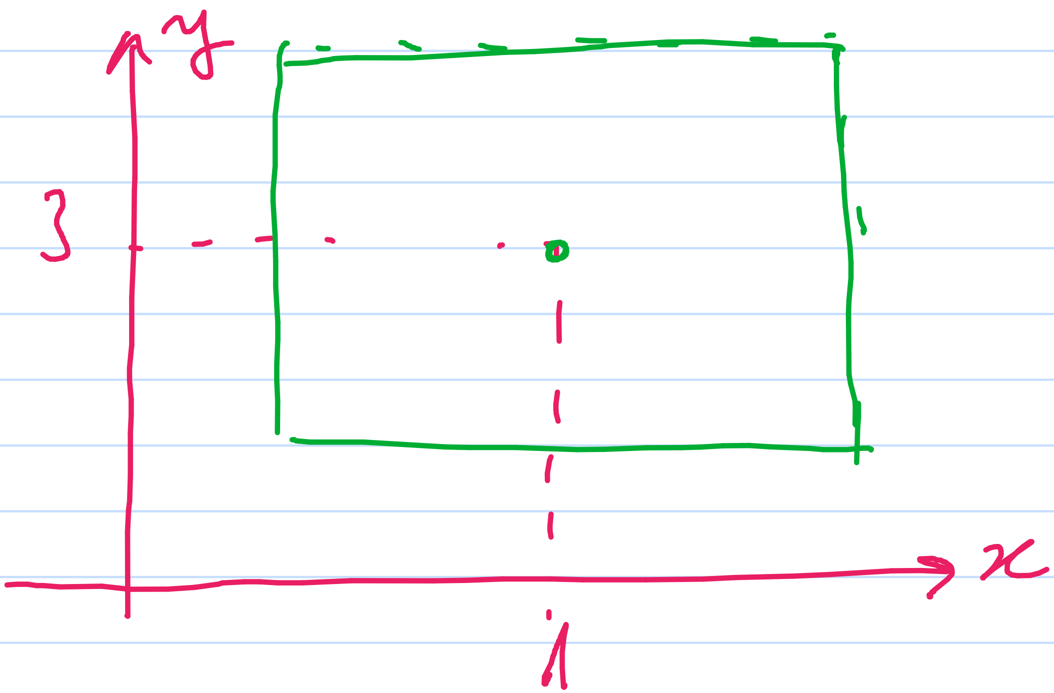
$\dfrac{\partial f}{\partial y}$ is cont.

$\Rightarrow$ the sol. of the IVP exists uniquely on some interval $\underline{I} = (x_0 - \delta,\ x_0 + \delta)$

Ex

$$y' = y^2 + x^2 \qquad y(1) = 3$$

$$f(x,y) = x^2 + y^2, \qquad \frac{\partial f}{\partial y} = 2y$$

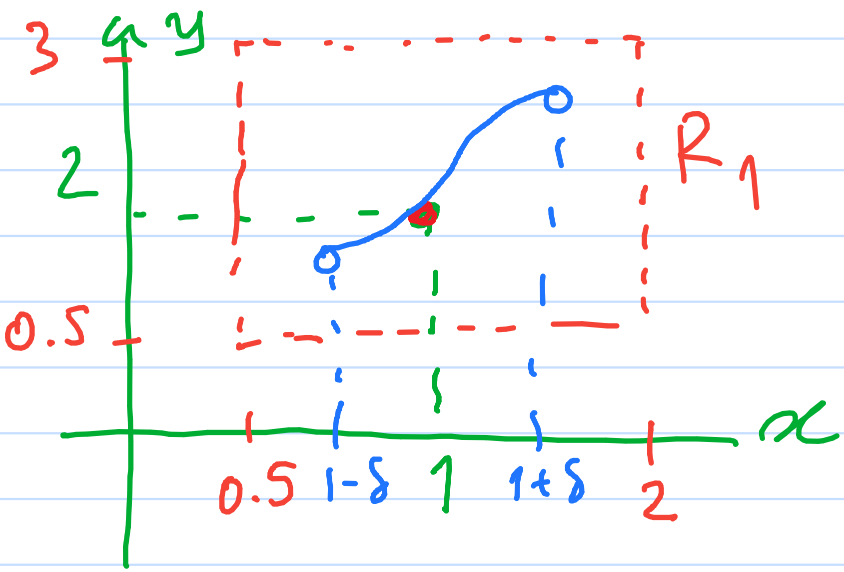


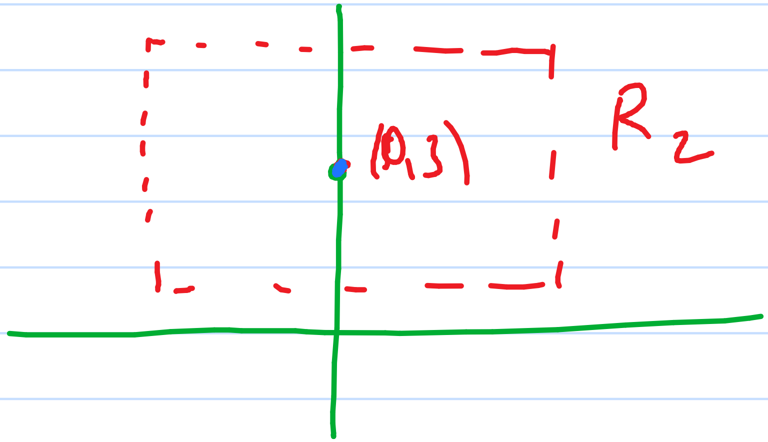
f and f_y are cont. on any ope rectangle containing $(x_0, y_0) = (1, 3)$

$\Rightarrow$ the sol. of the IVP exists uniquely on some open interval $(1-\delta, 1+\delta)$.

Ex

$$\left.\begin{aligned} y' &= \frac{1}{x} \\ y(1) &= 2 \end{aligned}\right\} \text{IVP\_1} \qquad \left.\begin{aligned} y' &= \frac{1}{\underline{\underline{x}}} \\ y(0) &= 3 \end{aligned}\right\} \text{IVP\_2}$$





$f = \frac{1}{x}$ is not cont. on any rectangle containing $(0,3)$, since any such rectangle contains points where $x=0$.

Therefore $\Rightarrow$ we cannot say anything about existence & uniqueness

$$\left.\begin{aligned} f(x,y) &= \frac{1}{x} \\ f_y &= 0 \end{aligned}\right\} \text{are cont. on } \left.\begin{aligned} 0.5 < x < 2 \\ 0.5 < y < 3 \end{aligned}\right\} R$$

The solution to the IVP_1 exists uniquely on $(1-\delta, 1+\delta) \subset (0.5, 2)$

For the solution to the IVP_2, we cannot say anything from the existence & uniqueness th, as the conds. are not satisfied.

However;

$$\left.\begin{aligned} y' &= \frac{1}{x} \\ y(0) &= 3 \end{aligned}\right\} \quad \begin{aligned} &y(x) = \ln|x| + c \\ &\text{is not defined at } x = 0 !! \end{aligned}$$

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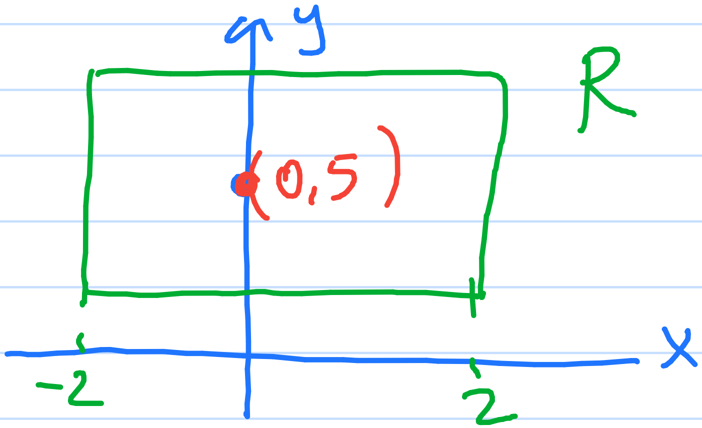
the IVP_2 does not have a solution.

Ex

$$y' = 2\sqrt[3]{y}, \qquad y(0) = 5.$$

$$f(x,y) = 2\sqrt[3]{y}, \qquad \frac{\partial f}{\partial y} = \frac{2}{3}\frac{1}{\sqrt[3]{y^2}}$$



f and f_y are cont. on R

$\exists\ (0-\delta, 0+\delta) \subset (-2,2)$

such that the sol. to the

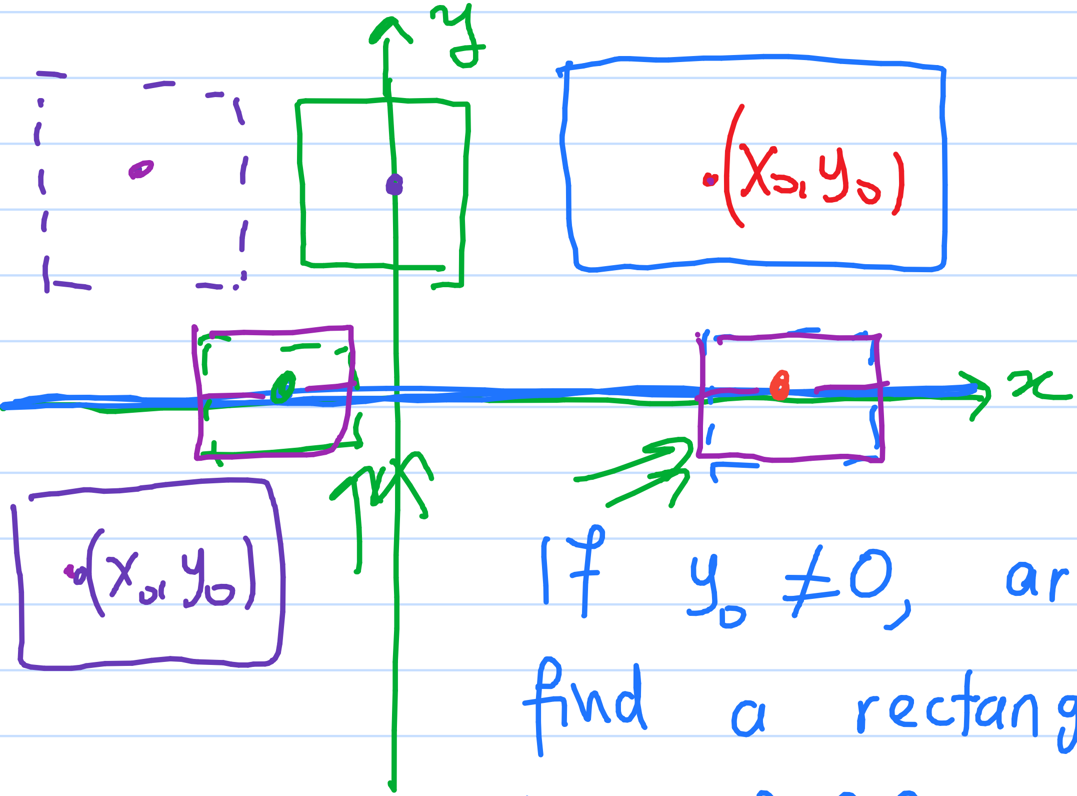
IVP exists uniquely!

Ex $y' = \sqrt[3]{y}$, $y(x_0) = y_0$

If $y_0 \neq 0$ the IVP has a unique solution.

For which values of y_0 the solution to the IVP exists uniquely?

If $y_0 = 0$, we cannot say anything.



$f = \sqrt[3]{y}$ $\qquad f_y = \frac{1}{3} \frac{1}{y^{2/3}}$

f_y is disc. when $y = 0$

If $y_0 \neq 0$, around any point (x_0, y_0) we can find a rectangle R such that on R, both f & f_y are cont. If $y_0 = 0$,

$(x_0, y_0) = (x_0, 0) \Rightarrow$ there's no such rectangle R!!

Ex $\quad x\dfrac{dy}{dx} = 2y, \quad y(a) = b$

$\dfrac{dy}{dx} = \dfrac{2y}{x} \qquad f(x,y) = \dfrac{2y}{x}, \quad \dfrac{\partial f}{\partial y} = \dfrac{2}{x}$: f & $\dfrac{\partial f}{\partial y}$ are both cont. when $x \neq 0$.

$x\dfrac{dy}{dx} = 2y, \quad y(0)=0$

$\dfrac{dy}{y} = \dfrac{2}{x}dx \to \ln y = 2\ln x + \ln c$

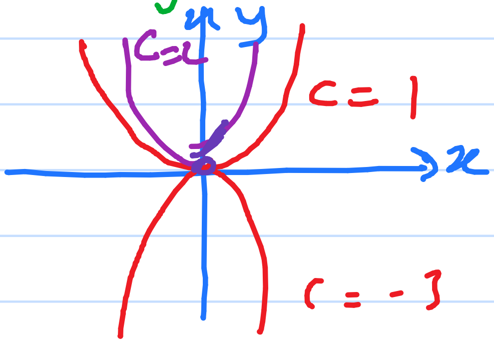
$y(x) = Cx^2$

For any $C \in \mathbb{R}$, $y(0) = 0$
the IVP has inf. many sols.

$y = x^2$

$y = 2x^2$

$y = -3x^2$



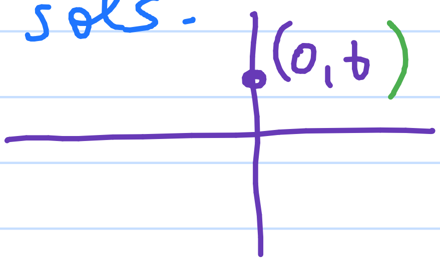
$x\dfrac{dy}{dx} = 2y$

$y(0) = b \neq 0$

$y(x) = Cx^2$

$y(0) = 0 = b \neq 0$

The IVP has no sols.



$x\dfrac{dy}{dx} = 2y, \quad y(a) = b$

$a \neq 0$

$y(a) = Ca^2 = b$

$C = \dfrac{b}{a^2} \quad (a \neq 0)$

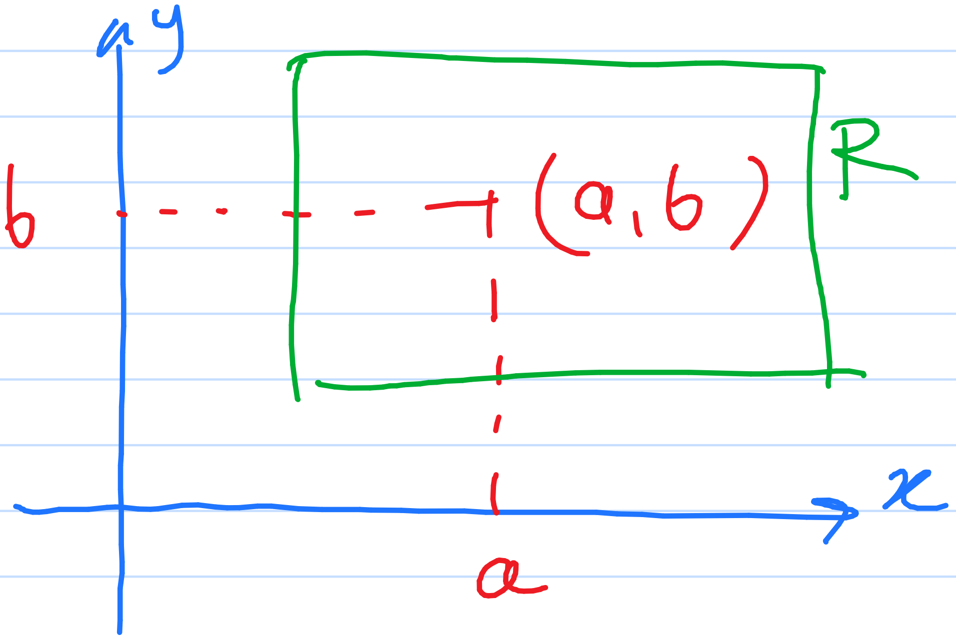
$y(x) = \dfrac{b}{a^2}x^2$

is the unique solution!!

Indeed, by using the existence & uniqueness th.

$$x\frac{dy}{dx} = 2y, \quad y(a) = b, \quad a \neq 0:$$

$f = \dfrac{2y}{x}$, $\quad f_y = \dfrac{2}{x}$ are both cont. for $x \neq 0$.



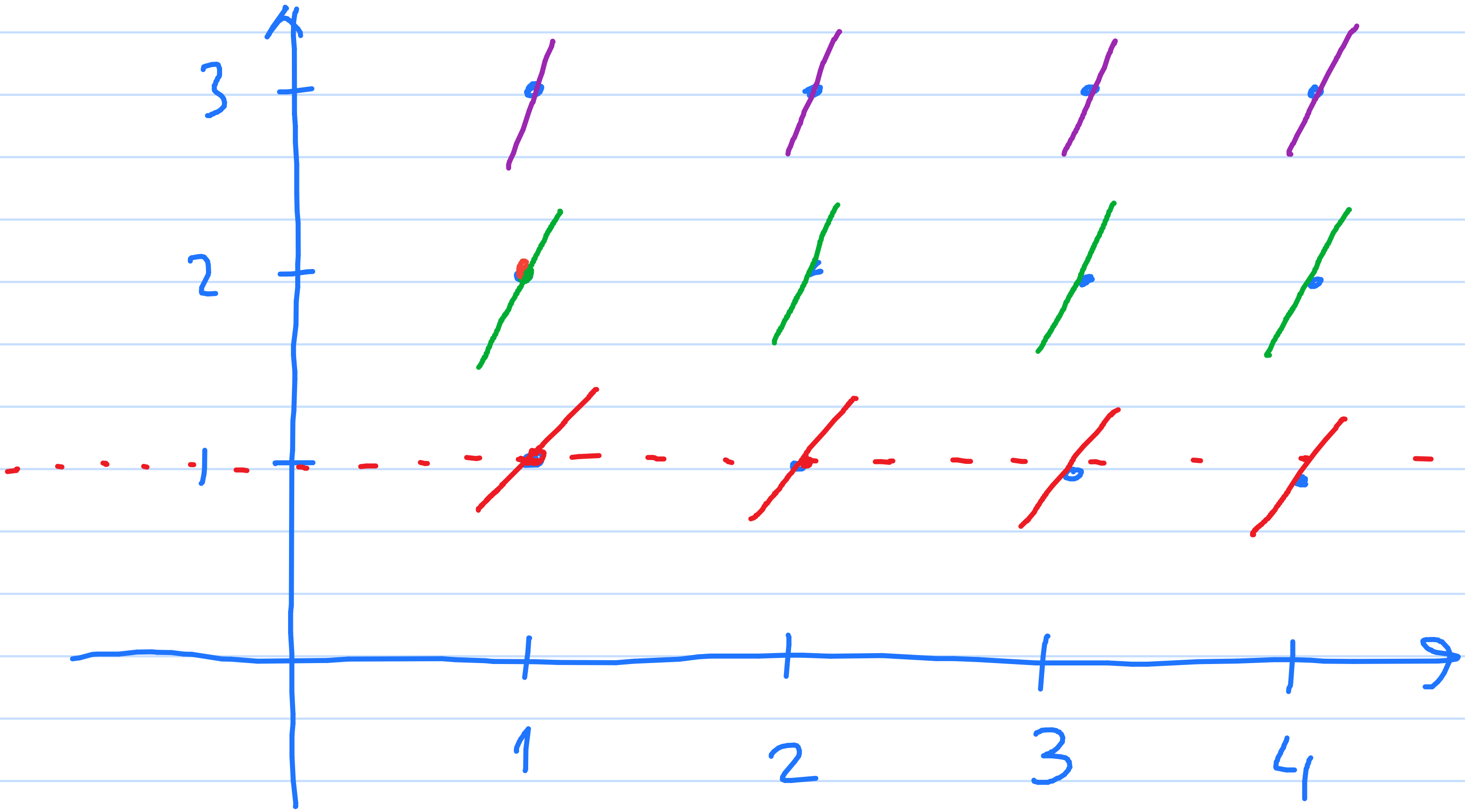
R ✓

f and f_y are cont. on R

$\Rightarrow$ The solution of the IVP exists uniquely!

Ex $\quad \frac{dy}{dx} = y \quad \to \quad y = y(x)$

We must find a solution curve for which, at every point (x_0, y_0) of the xy-plane, $\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = y_0$



$\left.\frac{dy}{dx}\right|_{(1,1)} = 1$

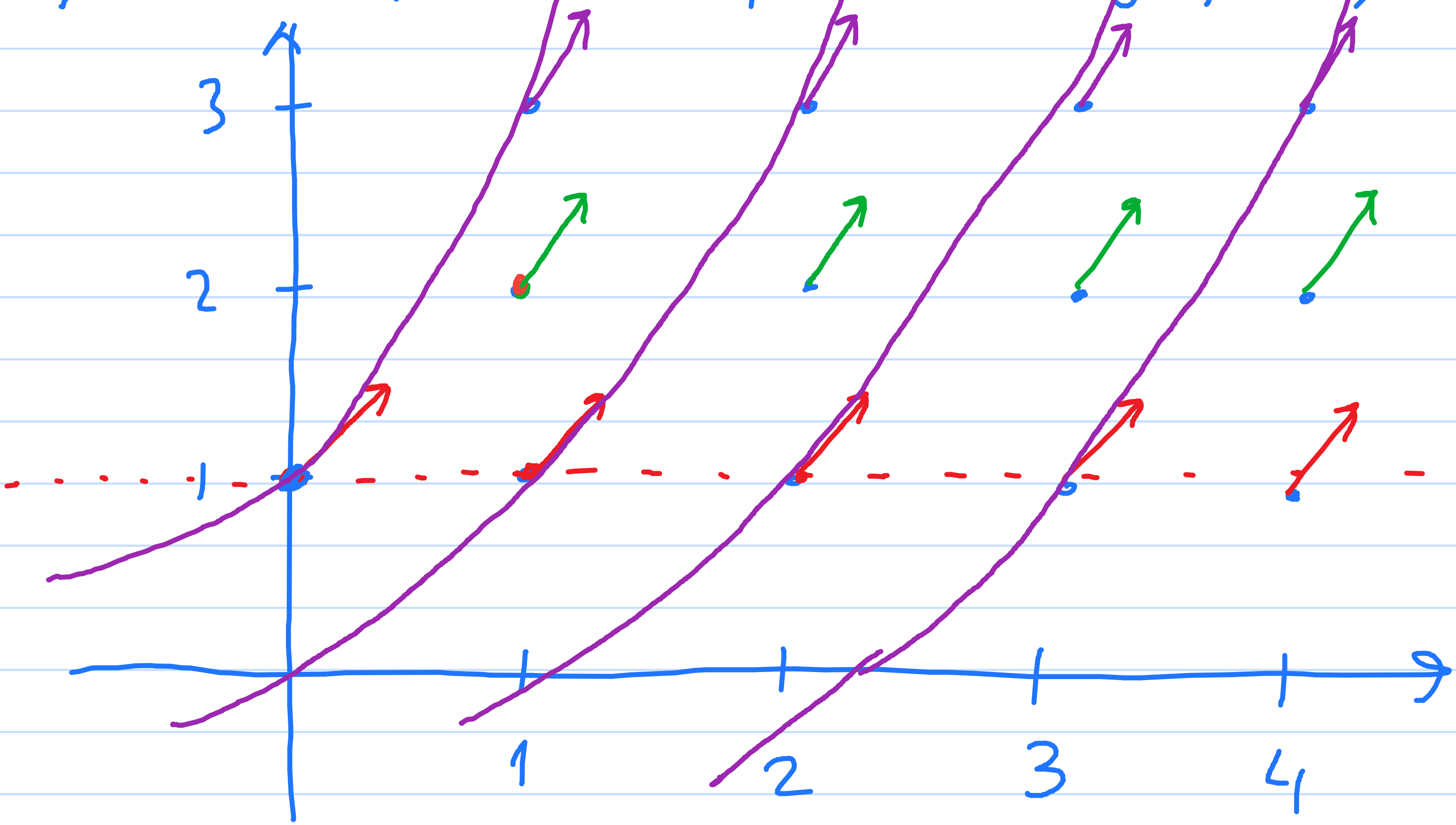
$\left.\frac{dy}{dx}\right|_{(1,2)} = 2$

$\left.\frac{dy}{dx}\right|_{(1,3)} = 3$

Each differential eq. $y' = f(x,y)$ defines a tangent (vector) field on $\mathbb{R}^2$.

Ex $\quad \frac{dy}{dx} = y \rightarrow y = y(x) \quad \{ y = y(x) = Ce^x \}$

We must find a solution curve for which, at every point (x_0, y_0) of the xy-plane, $\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = y_0$



$$\left.\frac{dy}{dx}\right|_{(1,1)} = 1$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = 2$$

$$\left.\frac{dy}{dx}\right|_{(1,3)} = 3$$

Each differential eq. $y' = f(x,y)$ defines a tangent (vector) field on $\mathbb{R}^2$.

(F4) $\ln\left(\dfrac{y'}{e^{3x}+x^2}\right) = -2y \quad , \quad y(0) = 0$

$$\frac{y'}{e^{3x}+x^2} = e^{-2y} \Rightarrow \frac{\frac{dy}{dx}}{e^{3x}+x^2} = e^{-2y}$$

$$\frac{dy}{e^{-2y}} = (x^2+e^{3x})\,dx \Rightarrow \int e^{2y}\,dy = \int (x^2+e^{3x})\,dx$$

F5

$$\frac{dy}{dx} = \frac{y\sin x - 3x^2y^2}{2x^3y + \cos x + \cos y}$$

$M(x,y)\,dx + N(x,y)\,dy = 0$

$M_y = N_x$

$$\underbrace{(y\sin x - 3x^2y^2)}_{\underbrace{M}_{\phi_x}}dx \; \underbrace{-(2x^3y + \cos x + \cos y)}_{\underbrace{N}_{\phi_y}}dy = 0$$

$\phi_x = y\sin x - 3x^2y^2 \rightarrow \phi(x,y) = -y\cos x - x^3y^2 + g(y)$

$\phi_y = -(2x^3y + \cos x + \cos y)$

$\phi_y = -\cos x - 2x^3y + g'(y)$

$= -2x^3y - \cos x - \cos y$

$g'(y) = -\cos y$

$g(y) = -\sin y + C$

$\phi(x,y) = \boxed{-y\cos x - x^3y^2 - \sin y} + C$

$-y\cos x - x^3y^2 - \sin y = \tilde{C}$

$y\cos x + x^3y^2 + \sin y = \hat{C}$

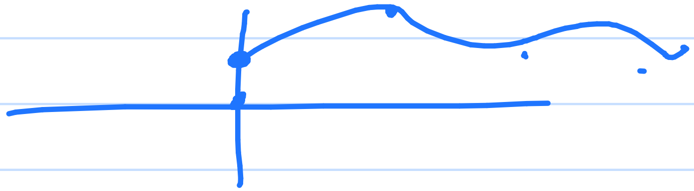
F5 $\quad \dfrac{dy}{dx} = \dfrac{y\sin x - 3x^2y^2}{2x^3y + \cos x + \cos y} \Rightarrow \underline{\underline{y = y(x)}}$

$\Rightarrow \quad y\cos x + x^3y^2 + \sin y = \hat{C} \quad \leftarrow$

This defines the sol. implicitly.

$c = 1$

$\underline{x=0} \quad y\cos 0 + 0 + \sin y = 1 \to y + \sin y = 1$



$\underline{y = \dots}$

$\boxed{x=1} \quad y\cos 1 + y^2 + \sin y = 1 \to \underline{y = \dots}$

23 March '19

$$y^2 y'' - (y')^3 = 0 \qquad y(1)=1, \quad y'(1)=3$$

Reducible 2^{nd} order ODES $F(x, y, y', y'') = 0$ $\Bigg\}$ — x is missing $F(y, y', y'') = 0$; y is missing $F(x, y', y'') = 0$

In the question $\Rightarrow$ x is missing!

Let $y' = p \Rightarrow y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\frac{dy}{dx} = \frac{dp}{dy} p = p\frac{dp}{dy}$

In this setting, p is the new dependent variable
y " " " independent "

$$y^2 \cdot p \frac{dp}{dy} = p^3 \longrightarrow \frac{dp}{p^2} = \frac{dy}{y^2} \Rightarrow -\frac{1}{p} = -\frac{1}{y} + C_1$$

$x=1$: $y=1$, $y'=3 \longleftrightarrow p=3$ $\qquad$ $x=1$: $-\frac{1}{3} = \frac{-1}{1} + C_1$ $\quad C_1 = \frac{2}{3}$

23 March '19

$$y^2 y'' - (y')^3 = 0 \qquad y(1)=1, \quad y'(1)=3$$

$$-\frac{1}{p} = -\frac{1}{y} + \frac{2}{3} \qquad \frac{1}{p} = \frac{1}{y} - \frac{2}{3} = \frac{3-2y}{3y}$$

$$\frac{1}{y'} = \frac{3-2y}{3y} \rightarrow y' = \frac{3y}{3-2y}$$

$$\boxed{y = y(x)}$$

$$\frac{dy}{dx} = \frac{3y}{3-2y} \longrightarrow \frac{3-2y}{3y}\, dy = dx$$

$$\left(\frac{1}{y} - \frac{2}{3}\right) dy = dx \longrightarrow \ln y - \frac{2}{3} y = x + C_2$$

$$x=1,\ y=1: \quad \ln 1 - \frac{2}{3} 1 = 1 + C_2 \qquad C_2 = -\frac{5}{3}$$

$$\Rightarrow \boxed{\ln y - \frac{2}{3} y = x - \frac{5}{3}}$$

$$\ln y - \frac{2}{3}y = x - \frac{5}{3} \qquad y = y(x)$$

$x=0$ $\quad \ln y - \frac{2}{3}y = 0 - \frac{5}{3} \rightarrow (0, y_1)$

$x=1$ $\quad \ln y - \frac{2}{3}y = 1 - \frac{5}{3} \rightarrow (1, y_2)$

$x=2$ $\quad \ln y - \frac{2}{3}y = 2 - \frac{5}{3} \rightarrow (2, y_3)$