대학원 신입생 세미나 과제 1

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Theorem 1 ($\pi - \lambda$ Theorem) *If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.*

Proof 1 *We will show that*

(a) if $l(\mathcal{P})$ is the smallest λ-system containing $\mathcal{P}$, then $l(\mathcal{P})$ is a σ-field.

The desired result follows from (a). To see this, note that since $\sigma(\mathcal{P})$ is the smallest σ-field and $l(\mathcal{P})$ is the smallest λ-system containing $\mathcal{P}$, we have

$$\sigma(\mathcal{P}) \subset l(\mathcal{P}) \subset \mathcal{L}$$

To prove (a), we begin by noting that a λ-system that is closed under intersection is a σ-field, since

$$\begin{aligned}
&\text{if } A \in \mathcal{L} \text{ then } A^c = \Omega - A \in \mathcal{L} \\
&A \cup B = (A^c \cap B^c)^c \\
&\cup_{i=1}^{n} A_i \uparrow \cup_{i=1}^{\infty} A_i \text{ as } n \uparrow \infty
\end{aligned}$$

Thus, it is enough to show

(b) $l(\mathcal{P})$ is closed under intersection.

To prove (b), we let $\mathcal{G}_A = \{B : A \cap B \in l(\mathcal{P})\}$ and prove

(c) if $A \in l(\mathcal{P})$, then $\mathcal{G}_A$ is a λ-system.

To check this, we note: (i) $\Omega \in \mathcal{G}$, since $A \in l(\mathcal{L})$.

(ii) if $B, C \in \mathcal{G}_A$ and $B \supset C$, then $A \cap (B - C) = (A \cap B) - (A \cap C) \in l(\mathcal{P})$, since $A \cap B, A \cap C \in l(\mathcal{P})$ and $l(\mathcal{P})$ is a λ-system.

(iii) if $B_n \in \mathcal{G}_A$ and $B_n \uparrow B$, then $A \cap B_n \uparrow A \cap B \in l(\mathcal{P})$, since $A \cap B_n \in l(\mathcal{P})$ and $l(\mathcal{P})$ is a λ-system.

To get from (c) to (b), we note that since $\mathcal{P}$ *is a* π*-system,*

if $A \in \mathcal{P}$*, then* $\mathcal{G}_A \supset \mathcal{P}$ *and so (c) implies* $\mathcal{G}_A \supset l(\mathcal{P})$

i.e., if $Ain\mathcal{P}$ *and* $B \in l(\mathcal{P})$*, then* $A \cap B \in l(\mathcal{P})$*. Interchanging* A *and* B *in the last sentence: if* $A \in l(\mathcal{P})$ *and* $B \in \mathcal{P}$*, then* $A \cap B \in l(\mathcal{P})$ *but this implies*

if $A \in l(\mathcal{P})$*, then* $\mathcal{G}_A \supset \mathcal{P}$*, and so (c) implies* $\mathcal{G}_A \supset l(\mathcal{P})$*.*

This conclusion implies that if $A, B \in l(\mathcal{P})$*, then* $A \cap B \in l(\mathcal{P})$*, which proves (b) and completes the proof.*