Criteria of Fairness in Envy-Free Rent-Splitting

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1 Introduction

Deciding how to assign rooms and split rent in an apartment can be a perilous first discussion for a new set of roommates. A devious roommate might misrepresent his preferences to try to lower the price of a room he loves, while the shyer among us might say little during the discussion only to later harbor boatloads of resentment in their tiny rooms. Conflict in the room assignment conversation can set the tone for the rest of the roommate relationship, so it's important to choose a process that everyone agrees is fair.

Before we proceed, let's introduce a more formal definition of the fair rent-division problem. A group of n roommates is moving into a home with n rooms and total rent r. Each roommate has a personal valuation for each room in the house. We must assign each roommate to a room and price the rooms in a way the roommates feel is fair, given their valuations. Of course, the prices of the rooms should sum to the total rent of the apartment.

Significant work has been put into designing computational rent-splitting processes that automatically assign rooms and divide rent. Most methods produce envy-free assignments, meaning that in the final assignment no roommate would prefer another roommate's room and price over their own. Envy-freeness is usually the minimum criterion a rent-division algorithm must meet to be considered fair [1]. After all, if a roommate is envious of another room, they probably person won't think the split is very just. Also, under reasonable assumptions, an envy-free split is guaranteed to always exist, so it makes sense that we would expect our rent-splitting algorithm to find one [2]. Interestingly, the challenge is not that envy-free solutions are hard to find, but rather that there are usually far too many!

Consider the following rent-division instance described by Alkan *et al.* [3]: there are two roommates moving into a two-bedroom apartment with a total rent \$100. The first roommate values room A at \$100 and room B at \$0, while the second roommate values room B at \$100 and room A at \$0 – in other words, the two roommates have opposite preferences. It turns out that if we assign each roommate to their preferred room, literally *any* division of the rent constitutes an envy-free solution (including some pretty nonsensical ones, like making one roommate pay the entire rent). Other solutions, like splitting the rent in half, feel quite a bit more just. Clearly not all envy-free solutions are created equal.

This fact has inspired some to devise methods that find "the fairest" division among the many possible envy-free solutions. Of course, designing such a method requires first deciding what makes a solution "the fairest". More formally this means adding an additional criterion of fairness (in addition to envy-freeness) that specifies which of the many possible solutions we should choose. Indeed, Alkan *et al.* showed mathematically that there are many criteria of fairness that specify a unique envy-free solution, and it is straightforward to solve for it in polynomial time [3]. Thus, the real challenge lies in deciding which of those criteria to use.

Recently, Gal *et al.* proposed a rent-division method that uses a criterion of fairness based on quasi-linear utilities [4]. According to their criterion, the fairest envy-free division is the one that maximizes the minimum utility achieved by any roommate. Alkan *et al.* called this the *value-rawlsian* criterion of fairness [3]. In their analysis, Gal *et al.* perform an empirical comparison of the value-rawlsian criterion with arbitrary envy-free solutions. They find that participants in their study consider the value-rawlsian solutions fairer than arbitrary envy-free solutions ($p < 0.04$).

We also consider a different criterion, one in which the fairest division is the envy-free solution that minimizes the maximum price paid by any roommate. Alkan *et al.* called this the *money-rawlsian* criterion of fairness, and Aragones derived a polynomial-time algorithm that solves for it [5].

We introduce a novel criterion of fairness, which we call the *consensus criterion*. Under this criterion, the fairest solution is the one that minimizes the maximum difference between a rooms price and its consensus valuation. For each room, we compute an approximate consensus valuation equal to its average valuation among the roommates. We show how we can find the optimal envy-free solution under this criterion using Gal *et al.*'s algorithmic framework.

To assess the fairness of our new criterion, we perform a user study where we compare our consensus criterion head-on against the value-Rawlsian criterion. Participants found the consensus valuation significantly fairer than the value-Rawlsian ($p < 0.009$). We find no significant preference for the value-Rawlsian over the money-Rawlsian.

Finally, we introduce a new rent-splitting algorithm, which allows users to better specify their priorities. We implement this algorithm along with the existing criteria in a web interface.

2 Problem Formulation

Input A group of n roommates $[n] = \{1, ..., n\}$ is moving into an apartment with n rooms $[n] = \{1, ..., n\}$. Let's assume without loss of generality that that the apartment's total cost is 1. Each roommate i submits a valuation $v_{ij} \in \mathbb{R}^+$ for all rooms $j \in [n]$. We will assume *with* loss of generality that each roommate's valuations sum to the total rent

$$\forall i \in [n] \quad \sum_{j \in [n]} v_{ij} = 1. \tag{1}$$

The input to a rent-division method can thus be represented by a row-stochastic matrix $\mathbf{V} \in {\mathbb{R}^+}^{n \times n}$. In fact, the entire rent splitting instance can be summarized by alone.

Output Given a set of valid valuations, the objective is to assign each roommate to a room and divide the total rent among the rooms. Formally, a valid room assignment is a bijection $\sigma : [n] \to [n]$ where roommate $i \in [n]$ is assigned to room $\sigma(i) \in [n]$. Furthermore, a price vector $p \in \mathbb{R}^n$ represents a division of rent where room j costs p_j and $\sum_{j \in [n]} p_j = 1$. Note that this definition allows for negative prices.

A valid solution to the rent-division problem is a pair (σ, p). The *quasi-linear utility* of a roommate i under a solution (σ, p) is the difference between their valuation for their assigned room and its price

$$u_i = v_{i\sigma(i)} - p_{\sigma(i)} \tag{2}$$

Thus, for any solution (σ, p), we have a utility vector $u \in \mathbb{R}^n$. A solution is *envy-free* if each player gets at least as much utility from her room-price pair than from any other. Formally, (σ, p) is envy-free if and only if

$$\forall i, j \in [n] \quad u_i(\sigma, p) \geq v_{ij} - p_j \tag{3}$$

Note that this problem formulation is the same as that of Gal *et al.* and similar to that of Alkan *et al.*

3 Optimizing for fairness criteria

Alkan *et al.* showed that every rent-splitting instance has at at least one envy-free solution [3]. More formally, for all possible valuation matrices $\mathbf{V} \in {\mathbb{R}^+}^{n \times n}$, there exists a solution (σ, p) such that for all $i, j \in [n] \quad u_i(\sigma, p) \geq v_{ij} - p_j$.

In the introduction, we saw that envy-free solutions are sometimes unfair, and that some rent-splitting instances have a huge range of possible envy-free solutions. Given that not all envy-free solutions are equally fair, can we design rent-division algorithms that find the "fairest" one? In this section we'll show that the answer is *yes*, provided we have a mathematically, well-defined criterion of fairness. Specifically, we'll introduce an algorithmic framework that models criteria of fairness by solving a *maximin* optimization problem.

In a recent paper, the creators of a popular rent-splitting service, *Spliddit*, introduced a simple algorithm that can find the envy-free solution that best satisifies the *value-rawlsian* criterion of fairness. Under this criterion, the fairest solution is the one that maximizes the minimum utility among the roommates [4]. Gal *et al.*'s algorithm can be easily modified to model other criteria of fairness. Here, we use the same algorithmic framework as Gal *et al.*, but modify it slightly to accomodate new fairness criteria.

The algorithm finds an optimal solution (σ, p) in two stages:

Assignment Stage

In the first, we find an assignment σ that maximizes welfare among the roommates.

$$\sigma = \text{argmax}_{\pi} \sum_{i=1}^{n} v_{i\pi(i)} \tag{4}$$

This reduces to finding a maximum weight matching on a bipartite graph (roommates on one side, rooms on the other, and each edge (i, j) weighted by valuation v_{ij}). In our implementation, we frame the task as a binary linear program.

Pricing Stage

In the second, we find a price vector p that minimizes the maximum of a set of linear functions $\mathbf{C} = \{f_1(\mathbf{V}, u(\sigma, p), p), ...f_t(\mathbf{V}, u, p)\}$ subject to envy freeness. Different **C** will capture different fairness criteria. For example, if we set $f_i(\mathbf{V}, u(\sigma, p), p) = p_i$, then we minimize the maximum price and model the money-rawlsian criterion of fairness. We can frame the optimization problem as a linear program

$$\begin{aligned}
\text{minimize} \quad & r \\
\text{s.t.} \quad & r \geq f_k(\mathbf{V}, u(\sigma, p), p) && \forall k \in [t] \\
& v_{i\sigma(i)} - p_{\sigma(i)} \geq v_{ij} - p_j && \forall i, j \in [i] \\
& \textstyle\sum_{j=1}^{n} p_j = 1
\end{aligned} \tag{5}$$

The first constraint keeps r larger than all functions in **C**. By minimizing r, we are in effect minimizing the maximum function in **C**. The second constraint ensures envy-freeness and the third enforces that prices must sum to the total rent.

In Gal *et al.*'s paper, **C** was a set of linear functions of utilities $f_k(u(\sigma, p))$. Here we generalize to linear functions of utilities, valuations and prices $f_k(\mathbf{V}, u(\sigma, p), p)$. This allows us to model more expressive criteria.

All rent-splitting instances are guaranteed to have at least one envy-free solution (σ^*, p^*). However, in the first stage, we compute an arbitrary welfare-maximizing assignment $\hat{\sigma}$, and it is not obvious that there exists an envy free solution $(\hat{\sigma}, p)$ for that assignment. Furthermore, even if an every free solution exists, it is not clear that our two step algorithm will find the optimal one. Gal *et al.* proved that their algorithm will find the envy-free solution that minimizes a set functions of utilities **C**. Here, we present a very similar proof for the more general setting where **C** is a set of functions of utilities, valuations, and prices.

Theorem 1. *Let* $\boldsymbol{C} = \{f_1(\mathbf{V}, u(\sigma, p), p), ...f_t(\mathbf{V}, u(\sigma, p), p)\}$ *and let* σ *be the arbitrary welfare-maximizing assignment computed in stage one. In stage two, there exists a feasible solution* p*, which minimizes the maximum of* **C** *over all envy-free solutions.*

Proof. Consider the welfare-maximizing assignment σ that we computed in stage one of the algorithm and the solution p of the linear program in Equation (4). Furthermore, let (σ^*, p^*) be an envy-free solution that maximizes the minimum of linear functions in **C**.

Gal *et al.* showed that the welfare theorems apply to envy-free allocations. By the 2[nd] Welfare Theorem, (σ, p^*) is envy-free and

$$u_i(\sigma, p^*) = u_i(\sigma^*, p^*) \tag{6}$$

for all $i \in [n]$ [4].

Furthermore, by Equation (6) and the fact that $\mathbf{V}$ and p^* are not functions of the assignment σ, we know that the following maxima are equal:

$$\max_{k\in[t]} f_k(\mathbf{V}, u(\sigma, p^*), p^*) = \max_{k\in[t]} f_k(\mathbf{V}, u(\sigma^*, p^*), p^*) \tag{7}$$

In other words, we can pair the welfare-maximizing assignment σ with the price vector p^* to achieve the same maximum as the minimax solution (σ^*, p^*). By definition, a solution p of our linear program will yield a maximum no greater than that in Equation (7)

$$\min_{k\in[t]} f_k(\mathbf{V}, u(\sigma, p), p) \leq \min_{k\in[t]} f_k(\mathbf{V}, u(\sigma, p^*), p) \tag{8}$$

Thus, (σ, p), the output of our two-stage algorithm, is the envy-free solution that minimizes the maximum of $\mathbf{C}$. $\square$

4 Fairness criteria

In the previous section, we introduced a general algorithmic framework, which allows us to find the envy-free solution that minimizes a maximum of linear functions $\mathbf{C}$. In this section, we'll see three choices of $\mathbf{C}$, each of which models a different criterion of fairness. Most importantly, we introduce a novel criterion of fairness, which we call the *consensus criterion*, and show how it can be applied in our framework.

4.1 The Value-Rawlsian Criterion

The value-Rawlsian criterion is of particular interest because *Spliddit*, a leading fair-division service, uses it in its rent-division algorithm. This criterion is grounded in Rawls' second principle of justice, which argues that we should arrange inequalities so as to give the "greatest benefit" possible to "the least advantaged [6]". Who is the "least advantaged" in the context of rent-splitting? Proponents of the value-Rawlsian criterion would argue that the least advantaged roommate is the one with the smallest utility. By that definition, the Rawlsian thing to do is to maximize the minimum utility among all roommates.

The fairest envy-free solution is the one that maximizes the minimum utility among the roommates.

If we assume quasi-linear utilities, we can easily model this criterion in our algorithmic framework

$$\mathbf{C} = \{-u_i(\sigma, p) \mid i \in [n]\} \tag{9}$$

Several previous studies have argued that the value-Rawlsian is the most attractive criterion of fairness. Alkan *et al.* do so on philosophical grounds, while Gal *et al.* do so via survey [3, 4]. We challenge the notion that the value-Rawlsian ought to be the de facto criterion for rent-splitting.

For one, the value-Rawlsian criterion is based on a very strong assumption: that utilities are quasi-linear. Feedback in our user study suggested that this might not be a very safe assumption. For example, several respondents said that their only concern was paying as little as possible, so perhaps $u_i(\sigma, p) = -p_{\sigma(i)}$ would be a better model of their utilities. Moreover, the value-Rawlsian tends to produce results where roommates have very similar utilities. This sounds nice in theory, but in practice, to achieve equality in utility value-Rawlsian algorithms sometimes penalize those with strong preferences, even if no one else shares their preferences. For example, consider the three-roommate rent-splitting instance given by

$$\mathbf{V} = \begin{bmatrix} 0.5 & 0.5 & 0.0 \\ 0.5 & 0.5 & 0.0 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \tag{10}$$

The value-Ralwsian criterion will yield the solution $(\sigma = [1, 2, 3], p = [0.23, 0.23, 0.53])$. Is it fair that roommate three is paying more than twice as much as the other roommates? We find this solution surprising considering she was assigned to a room that others do not want at all.

4.2 The Money-Rawlsian Criterion

One could also argue that the "least advantaged" roommate is the one who pays the the largest share of rent. With this definition, we can produce yet another Rawlsian fairness criterion. This criterion, which Alkan *et al.* calls the money-Rawlsian, can be stated succinctly as

> *The fairest envy-free solution is the one that minimizes the maximum price.*

We can model the money-Rawlsian in our algorithmic framework with the following set of functions

$$\mathbf{C} = \{p_j \mid j \in [n]\} \tag{11}$$

This criterion sometimes produces very attractive solutions, where the $Value - Rawlsian$ cannot. For example, when applied to the rent-splitting instance in Equation (10) V, the $money - Rawsian$ outputs the price vector $p = [0.33, 0.33, 0.33]$. This is, to us, a much more sensible result.

But, like the value-Rawlsian, this criteria is not without drawbacks. If roommates misreport their true valuations, either by accident or intentionally, the money-Rawlsian is much more likely to produce results that are not actually envy-free with respect to the true valuations. Furthermore, in our user study, we found that the money-Rawlsian often under-values highly-demanded rooms.

4.3 The Consensus Criterion

Assume that for every room j, there exists one consensus valuation $\bar{v}_j$ among the roommates. We can think of $\bar{v}_j$ as the valuation the roommates would together come up with if they *had* to agree upon one valuation for room j. Let's introduce the notion of a *bad deal* in a rent splitting solution (σ, p). We say a room is a bad deal when when its price exceeds its consensus valuation $(p_j > \bar{v}_j)$. The *worst deal* in a solution (σ, p) is the room with the largest difference between its price and its consensus valuation $\operatorname{argmax} \quad p_j - \bar{v}_j$.

Here, we present a novel criterion, also based on Rawls' theory of justice. In ours, the worst-off roommate is not the one with with the lowest utility, but rather the one with the worst deal. Stated formally, our criterion, which we call the *consensus criterion*, is

> *The fairest envy-free solution is the one that minimizes the maximum difference between a room's price and its consensus valuation.*

Under the assumption that we know the consensus valuation $\bar{v}_j$ for all rooms j, we can implement the consensus criterion in our algorithmic framework with

$$\mathbf{C} = \{\bar{v}_j - p_j \mid j \in [n]\} \tag{12}$$

Of course, consensus valuations are not guaranteed to exist, and if it were easy to find them, there would not be much use for rent-splitting algorithms. Clearly, it's not realistic to implement our algorithm with Equation (12).

We can, however, approximate consensus valuations as a function of reported valuations $\mathbf{V}$. With some $g_j(\mathbf{V}) \approx \bar{v}_j$, we can easily apply an approximate consensus criterion to our framework

$$\mathbf{C} = \{g_j(\mathbf{V}) - p_j \mid j \in [n]\} \tag{13}$$

A simple and intuitive choice of g_j is the mean valuation of room j

$$g_j(\mathbf{V}) = \frac{1}{n} \sum_{i \in [n]} v_{i,j} \tag{14}$$

Averaging reported valuations is not the only option. Since the valuations are constant with respect to the solution (σ, p), the function $g_j(\mathbf{V})$ need not be linear in $\mathbf{V}$. We experimented with other choices of g_j, but chose the mean valuation for its simplicity and attractive solutions. Future work could explore other more, interesting options.

We designed the consensus criterion to address some of the issues, with the existing criteria that we outlined above. For example, in the rent splitting instance from Equation (10), the consensus criterion outputs $p = [0.36, 0.36, 0.26]$. This is, in our view, a more reasonable response than the value-Rawlsian's lopsided output. Furthermore, since highly-demanded rooms usually have high average valuation, under the consensus criterion high-demand rooms get high prices.

All of this being said, our discussion of fairness criteria has so far been highly anecdotal, theoretical and subjective. To get a better sense of how these fairness criteria behave in the real world, we performed a user study, which we describe in the next section.

5 User Studies

So far, we have introduced additional criteria of fairness and find the envy-free solution that best satisfies each of them. The next step is to find which envy-free method comes off as the fairest to public. This is a critical point in qualitative analysis as our goal is to develop an algorithm that people will use in real life setting, something that will make sense to them. Since each criterion involves some value system in fairness that does not have the absolute measure of accuracy, we decided to conduct a series of user studies to gauge public's preference.

As a reference point, the creators of Spliddit also conducted user studies asking participants to judge the fairness of *value-rawlsian* solutions against arbitrary envy-free solutions [3]. Forty-six participants in their survey preferred significantly the *value-rawlsian* solution over the arbitrary envy-free one. The survey result was used to justify *value-rawlsian* criterion is fairer than the others. However, the new criteria we have discussed above were not competed against *value-rawlsian* solution in the survey. Therefore, we conducted a user-study that compares the *value-rawlsian* criterion head on with the *money-rawlsian* and *consensus* criteria. Our results indicate that users find the consensus criterion significantly fairer than the *value-rawlsian* ($p < 0.009$).

5.1 Method

We collected room valuation data from residents who currently share house rent together. Data come from seven houses - two houses with two rooms, two houses with three rooms, two houses with four rooms, and one house with five rooms - with total of 23 participants. No compensation was given to the participants. After receiving basic information about the house, including current total rent and identifiable names of rooms (usually based on name of roommate who currently lives there), we asked each participant to give valuations for the rooms in their house using sliders. We formatted the survey so that the valuations should sum to the total rent to be submitted.

The fairness evaluation data come from the second round of survey. After collecting data from all residents in each house, we ran the rent-splitting algorithm using three different criteria – Value-Rawlsian, Money-Rawlsian, and Demand-Rawlsian – and calculated three different rent-splitting results. In the followup survey we presented visulisation of the results, and asked the participants to show preference over Money-Rawlsian and Consenus-Rawlsian against Value-Rawlsian. In the survey, we did not reveal the name of the criteria, but instead use A, B, and C randomly to represent each result. The order of question ('Value-Rawlsian vs Consenus-Rawlsian') and ('Value-Rawlsian vs Money-Rawlsian') was randomized as well.

For the second round of survey, we also included participants who are not residents of the houses, to evaluate the result without the bias of living in the house. We evaluated that while residents can provide more realistic data, non-residents can evaluate fairness in a more objective way. We phrased the question not to indicate any arbitrary definition of fairness:

> For this one, you are going to evaluate possible rent split situations A, B, and C. For each question, we want you to take a look at two and choose the one you think your house should use or the one that you think is "fairer"

After the participant selects a choice, we asked them to input rationale for their choice to understand the participant's definition of fairness.

Figure 1 shows an example visualization we provided for the survey. Guide-words on how to read it was included as followed

The following bar charts are visualisations of the results. One chart for one room, and the height of a bar is valuation the person has given for that room.If the person's bar is green, that means the person is assigned to the room. Dotted lines shows prices each results gave to each room.

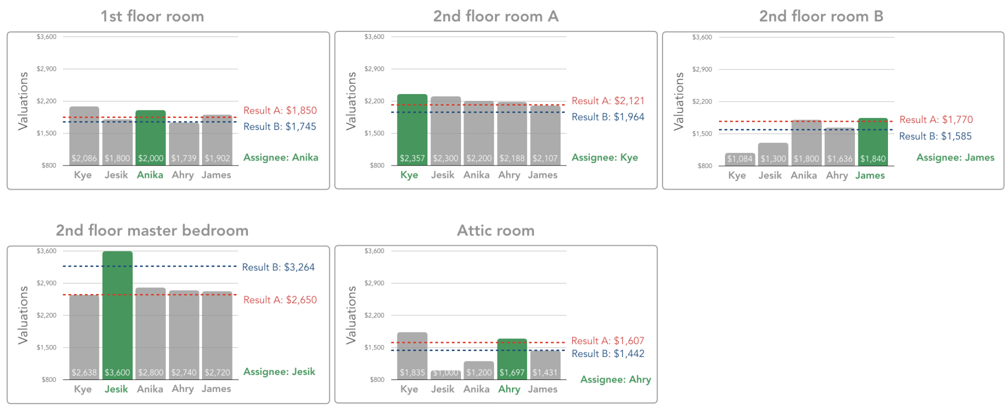

Figure 1: Visualization provided in the follow up survey

5.2 Results

We have gathered total of 29 responses for Value-Rawlsian and Demand-Rawlsian comparison question, and total of 33 responses for Value-Rawlsian and Money-Rawlsian comparison question. Figure 2 shows the result of the follow-up survey. We found that participants prefer the consensus criterion to the value-Rawlsian passing a binomial test with $p < 0.009$. Many participants chose

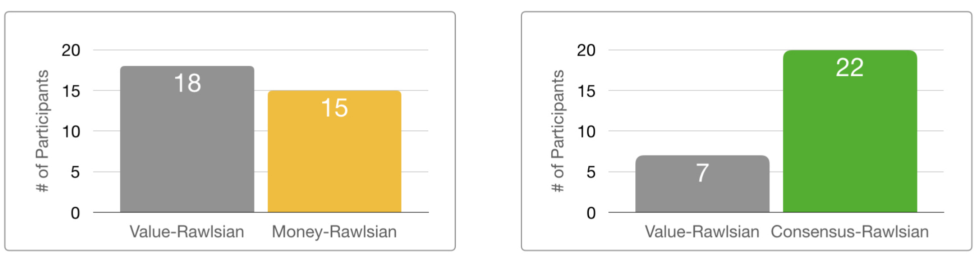

Figure 2: Result of the follow-up survey

Value-Rawlsian criterion over Money-Rawlsian criterion but chose Consensus-Rawlsian criterion when compared with the same Value-Rawlsian criterion. Based on the participants' input on rationale, this pattern of choices largely came from the notion of fairness in minimizing discrepancy if there is no harm to others. One answer, for example was "in each option [in Demand-Rawlsian], the assignee is paying closer to what the other people evaluated it as.".

6 Beyond Universal Utility Functions

When asking participants to input valuations during the first stage of our study, we often received feedback to the effect of "I truly don't care which room I get. I'd just like to pay as little as possible."

In Gal *et al.*'s algorithmic framework, roommates have no way of prioritizing price over room, since valuations must sum to the total rent.

Here, we present a novel rent-splitting algorithm that takes priorities into account. The algorithm is a straightforward extension of the Gal *et al.*'s algorithm, which we described above.

Suppose each roommate i has a priority $\alpha_i \in \mathbb{R}^+$. A roommate with priority $\alpha_i = 0$ cares only about price and has no regard for which room they're assigned to, while a roommate with priority $\alpha_i = \infty$ cares only for the room they're placed in with no regard for money. Just as we elicit valuations from the roommates, we can also ask users to input their personal priority at the beginning of the rent-splitting process. This gives us a priority vector $\alpha \in \mathbb{R}^+ n$.

With α, we can define personal utility function for each roommate, based off their priorities

$$u_i(\sigma, p) = \alpha_i * v_{i\sigma(i)} - p_{\sigma(i)} \tag{15}$$

In this new setting, an α-envy-free solution is a pair (σ, p) that satisfies

$$\forall i, j \in [n] \quad \alpha_i * v_{i\sigma(i)} - p_{\sigma(i)} \geq \alpha_{ij} - p_j \tag{16}$$

With our new definition of α-envy-freeness and utility, we're ready to present the two stages of our new algorithm:

Assignment Stage

In the first stage, we find the assignment σ that maximizes the α-*weighted* welfare among the roommates.

$$\sigma = \text{argmax}_\pi \sum_{i=1}^{n} v_{i\pi(i)} \tag{17}$$

Weighting by alpha in assignment stage discourages players who actually care about their room assignment from reporting very low α.

Pricing Stage

In the pricing stage, we use the consensus criterion with $g_j(\mathbf{V}, \alpha_i) = \frac{1}{n}\sum_{i \in [n]} \alpha_i * v_{i,j}$. We then find the minimax of the consensus objective subject to α-envy-freeness.

$$\begin{array}{lll} \text{minimize} & r & \\ \text{s.t.} & r \geq \left(\frac{1}{n}\sum_{i\in[n]} \alpha_i * v_{i,j}\right) - p_j & \forall j \in [n] \\ & \alpha_i * v_{i\sigma(i)} - p_{\sigma(i)} \geq \alpha_i * v_{ij} - p_j & \forall i, j \in [i] \\ & \sum_{j=1}^{n} p_j = 1 & \end{array} \tag{18}$$

Note that our $\bar{v}$-approximating function g_j gives more weight to the valuations of those who report high α. This might incentivize players to report higher α so that their valuations are given more weight when approximating the consensus valuation.

Since we've changed settings and our definition of utility is different, we must show that our algorithm can still find the optimal envy-free solution.

Theorem 2. *Let* $\boldsymbol{C} = \{f_1(\mathbf{V}, \alpha), ...f_n(\mathbf{V}, \alpha)\}$ *and let* σ *be the arbitrary* α*-weighted welfare-maximizing assignment computed in stage one. In stage two, there exists a feasible solution* p*, which minimizes the maximum of* $\boldsymbol{C}$ *over all* α*-envy-free solutions.*

Proof. Consider the α-*weighted* welfare-maximizing assignment σ that we compute during the assignment stage and the solution p of the linear program from the pricing stage. Let $(\sigma*, p^*)$ be the envy-free solution that minimizes the maximum of $\mathbf{C}$.

To complete our proof, we'll need to show that the welfare theorems apply to envy-free solutions in our new setting with α. Let us consider a more general setting where the welfare theorems are known to apply: a set of $|n|$ buyers, a bundle of goods G, and valuation functions $v'_i : 2^G \to \mathbb{R}$. A Walrasian equilibrium in this setting is any allocation $\{A_1, ...A_n\}$ and price vector p where

$\forall i \in [n],\ S \subset G,\ v'_i(A_i) - p(A_i) \geq v'_i(S) - p(S)$. An α-envy-free solution in our setting is a Walrasian equilibrium in this more general setting where the goods are the rooms and valuations for a subset of rooms S is given by $v'_i(S) = \max_j \alpha_i * v_ij$. Furthermore, note that an α-weighted welfare-maximizing solution σ is simply a welfare-maximizing solution in the more general setting.

By the 2nd Welfare Theorem, if $(A*, p*)$ is a Walrasian equilibrium , and A is a welfare-maximizing solution, then $(A, p*)$ is a Walrasian equilibrium as well. Translating this to our setting, given that $(\sigma*, p*)$ is an α-envy-free solution, (σ, p^*) is also an α-envy-free solution, since σ is an α-weighted welfare-maximizing.

This proves that (σ, p^*) is an α-envy-free solution. Furthermore, since our objective $\max\big(\frac{1}{n}\sum_{i\in[n]} \alpha_i * v_{i,j}\big) - p_j$ is not a function of σ, we know that

$$\max\Big(\frac{1}{n}\sum_{i\in[n]} \alpha_i * v_{i,j}\Big) - p_j \leq \max\Big(\frac{1}{n}\sum_{i\in[n]} \alpha_i * v_{i,j}\Big) - p^*_j \tag{19}$$

Thus, (σ, p), is the output of our two-stage algorithm, is an α-envy-free solution that minimizes the maximum of $\mathbf{C}$. □

We implemented our algorithm in our web interface described below, but due to timing, were not able to include it in the user study.

7 Web Interface

We implemented a simple website which puts the rent-splitting criteria we've discussed into practice. Unlike existing websites, ours allows users to choose which criterion of fairness to use. We hope that our interface will enable users to reflect more on their own definition of fairness and proactively choose a method they find the fairest. The application asks users for their total rent, room names, valuations and priority over price and room if there is any, and algorithm method they want to use. The interface returns the corresponding result with graphical visualizations. The repository is available at: https://github.com/seyuboglu/free-the-envy.

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