Pendulum Project

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Abstract

In this experiment a pendulum was created to investigate the theory of damped harmonic motion of a pendulum. The first thing analyzed about the pendulum was its symmetry. it is essential to have a pendulum, the results of which are invariant under the side the mass is released. It was concluded that the pendulum built in this experiment was highly symmetric. Then, the experimental physical properties of the pendulum, such as the period and the decay constant, were compared to the corresponding theoretical values. According to the theory, the pendulum's period depends only on the length of the string and the initial angle and is independent of the mass of the pendulum. The results of this experiment agrees with the dependency of length but failed to show any correlation for the dependency of initial angle on period. The length dependency was confirmed with a goodness of fit of $\chi^2_{red} = 1.89$. Similarly, the dependency of mass on the decay constant was investigated as suggested by the theory. The relation hypothesized by the theory between mass and decay constant is linear, but during this experiment, the function that best fit the data was a logarithmic function, with the goodness of fit $\chi^2_{red} = 0.24$. In addition to the deviation from the theory of the relation function, the independence of the initial angle and the length of the pendulum was not seen in the experiment. The possibilities of this deviation are discussed in the analysis section.

Introduction

A simple pendulum is a crucial mechanical system that demonstrates periodic motion. Under ideal conditions, it is part of a larger class of model called simple harmonic motion, all systems are mathematically equivalent and are governed by the second-order differential equation 1

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x \tag{1}$$

We call ω the angular frequency of the harmonic motion. The second-order differential equation (1) models the motion of various physical systems, some of which are mass attached to a spring, uniform circular motion, and oscillatory motion, and the system tackled in this report is the simple pendulum. Understanding the theoretical framework of this system is crucial for developing a strong mathematical toolkit as a physics student.

Let a mass be suspended using a string attached to a fixed pivot point. Restrict the movement of the mass to a two-dimensional plane with only one degree of freedom. The system can be simplified by assuming the mass of the string is negligible. Let θ be the angle the string makes with the vertical axis. As the mass moves in the 2-d plane it experiences the gravitational force, air resistance also called the drag force. Furthermore, assuming the resistive force is linearly proportional to the angular velocity of the mass significantly simplifies the differential equation that needs to be solved ([3]). Using Newton's second law this differential equation can be written as

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{b}{m}\dot{\theta} \tag{2}$$

setting $\omega^2 = \frac{g}{l}$ and using the small angle approximation for θ, we can approximate $\sin(\theta) \approx \theta$

$$\ddot{\theta} = -\omega^2\theta - \frac{b}{m}\theta \tag{3}$$

Solution of this ODE with the initial condition $\dot{\theta}(t) = 0$ is given by ([3])

$$\theta(t) = \theta_0 e^{-t/\tau}\cos(\omega_1 t) \tag{4}$$

where $\omega_1 = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$, and $\tau = \frac{m}{b}$ which is the decay coefficient (refer documentation here). For the masses swung in this experiment we can safely assume that $b^2 << m^2$, then angular velocity can be approximated as $\omega_1 = \sqrt{\omega^2 - \frac{b^2}{4m^2}} \approx \omega$. The period is then calculated using

$$T = \frac{2\pi}{\omega} \approx 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{L+D}{g}}$$

l is the length of the string to the point mass, in the experiment we don't have a point mass, in this case, we can treat the mass as a point mass and l then refers to the length of the string L + the distance from the center of mass to the point where the string is attached. This theoretical model implies the decay coefficient is dependent on mass and independent of initial angle θ_0 and length $L + D$. Furthermore, according to this the period T is independent of m and θ_0 and only dependent on length $L + D$. For angles outside the scope of small angle approximation, period does depend on the initial angle and the equation of period is then given by

$$T = 4\sqrt{\frac{L+D}{g}} \int_0^{\pi/2} \frac{d\theta}{1 - \sin^2(\theta_0/2)\sin^2\theta} \tag{5}$$

The integral is called the complete elliptic integral of the first kind and period can be approximated by taking the power expansion of this ellipted integral [2]

$$T \approx 2\pi\sqrt{\frac{L+D}{g}} \left(1 + \frac{1}{16}\theta^2 + \frac{11}{3072}\theta^4 + \frac{173}{737280}\theta^6 + \dots\right) \tag{6}$$

In this experiment, we will test the dependencies of T and τ and compare the results to the theory as laid out above. One of the first things that needs to be checked is the symmetry of the pendulum. It must be that the pendulum should behave similarly no matter the side the mass is released from, for example, the period and decay constant should be the same as the initial angle is fixed from both sides. One of the metrics to measure symmetry is to calculate the value in equation (7)

$$\sigma = \sum_{n=1}^{20} d(\theta_{r_n}, \theta_{l_n}) \tag{7}$$

where θ_{r_n} is the n-th angle peak amplitude of the pendulum for mass released from the right, and similar θ_{l_n} is the n-th angle-peak amplitude of the pendulum for mass released from the left. $d(\theta_{r_n}, \theta_{l_n})$ is the absolute value between the n-th peak, and the sum is taken of the first 20 peaks. This sum will give some insight into how symmetric the pendulum is, ideally, this should be very small $<< 1$, then the pendulum would be considered symmetric.

Ideally, the results for T and τ will be within the uncertainty of the theoretical value. Throughout the experiment, various uncertainties will be calculated, the uncertainty of a function of independent values can be computed using propagation of uncertainties given by equation (8). Let $F : \mathbb{R}^n \to \mathbb{R}$ then the uncertainty of the function F is

$$\sigma_F = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i}\right)^2 \sigma_{x_i}^2} \tag{8}$$

Methodology

Building the Pendulum

The pendulum was built using the following materials: a Laboratory stand, a set of brass masses with hooks, a string, and a measuring tape. The clamp was attached to the stand for the string of the pendulum to be attached to it. The pendulum needed to move in a 2d plane with only one degree of freedom, and therefore, the string was attached to the clamp in a triangle formation (can in seen in the figure (1). This provided stability to the pendulum and allowed it to move in a 2-dimensional plane. Configuration of the setup from the side can be seen in the figure (2)

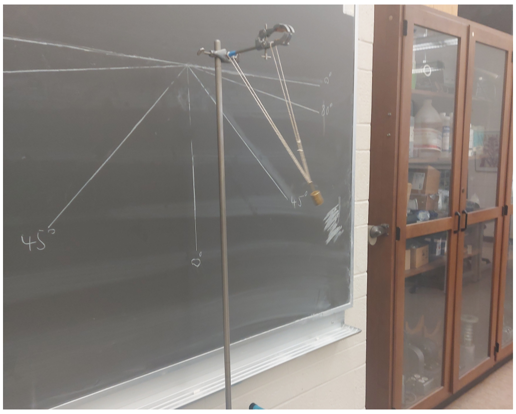

Figure 1: Physical Pendulum set up for the experiment. Some angles are drawn on the blackboard behind the pendulum to assist with the concise initial angle start of the pendulum

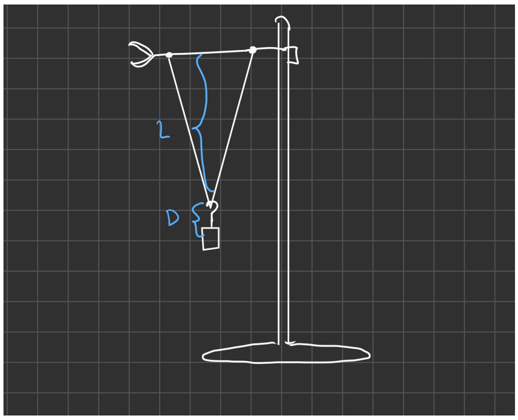

Figure 2: Side point of view of the pendulum set up. The triangle formulation restricts the movement of the pendulum to a 2-dimensional plane

The strings are looped around the clamp instead of being tied to it to make it easier to change the length of the string. Brass masses had hooks on both ends that made it easier to hook the mass on the string and load up multiple masses by hooking the other mass on the second hook. A lightweight string was used in the experiment, so the mass of the string did not affect the results of the pendulum and aligned nicely with the theoretical model, where the mass of the string was assumed to be zero. For this reason, heavier masses were also not used as they might stretch out the string, changing the length more than what a lighter mass would. The heaviest mass used on this pendulum was 300g. It was also important that the initial angle used to drop the mass was measured accurately; for this, some angles were drawn on the blackboard behind as can be seen in the image 1. Then, dropping the mass from the correct initial angle was easier as it only required aligning the string to the line drawn on the blackboard behind.

Experimental Procedure

To confirm the reliability of the pendulum, the pendulum must be symmetric. This was tested by releasing a mass from the right side at an initial angle of $80^\circ \pm 1^\circ$ and then from the left side with the same initial angle. The idea as stated in the introduction is to measure the difference in peaks of θ amplitude of each side. The difference of the first 20 peaks can be summed, and as long as this sum is small, we can safely assume that the pendulum is symmetric. To ensure the dependencies of one variable to the other, when the data was collected for one variable, the others were held constant.

Data Collection

The oscillations of the pendulum were video recorded using an iPhone 15 camera as data collection. The oscillations in the videos were tracked using tracker software ([1]). This method seemed to be more appropriate compared to human data collection and is less prone to error. The position of the mass in the video was calibrated using a feature called calibration stick from the software. By assigning some length to an object in the video, this tool then calibrates the distances of pixels on the frames relative to some reference point used as the origin. In this experiment, the actual length (L+D) of the string was assigned to the string in the video for the software to calibrate every point on all the frames; this works given the camera is not moved. Then the software can be used to track the position of the mass; it goes through all the frames (30 frames per second in this case) and saves the mass's position in the 2d-plane. The tracker uses the RGB line profiling method, which measures the intensity of red, green, and blue light in the video and tracks the same intensity throughout the frames as it changes. After data were collected using the software, they were saved in an Excel file for analysis later.

Controlling Variables and Uncertainty Analysis

When looking for dependences of some variables on the other, all the rest of the variables in the experiment needs to be fixed. One of the difficulties arose during the experiment was when mass was need to be changed the length D (length from the end of the string to the center of the mass) would change, changing the whole length $l = L + D$. Length l needs to be fixed for parts of the experiment and therefore when changing the mass the length L was also changed fixing the whole length $l = L + D$. For some parts initial angle of the pendulum was fixed, as explained previously this was done with the help of marking angles on the blackboard behind the pendulum to assist in measuring angle while dropping the mass.

Every measurement comes with some uncertainty. The lengths from the fixed point of rotation to the center of mass is given uncertainty of $0.2cm$ which is twice the resolution of the measuring tape used to measure these lengths. Two times the resolution is chosen compared to only the resolution because the center of the masses were eyeballed and thus the measured values carry some extra human error. Initial angles in this experiment are given uncertainties of $1°$ which is the resolution of the protractor used to draw angles on the board. Lastly, the masses carry uncertainties of 10mg per 100g of mass, which is double the uncertainty provided by the manufactures website. Double because the masses are used and most likely not the same mass as unused masses. These uncertainties are propagated to other values calculated, an example of such uncertainty propagation is as follows. In this experiment dependency of length of string on the period was measured and a sqaure root function was fit with the data collected (6). The root function was $T(L) = a\sqrt{L} + b$, every value here has an uncertainty, so to find the uncertainty of for example $T(0.45)$ we use the equation (8)

$$\sigma_{T(0.45m)} = \left(L\sigma_a^2 + \frac{a^2}{L}\sigma_L^2 + \sigma_b^2 \right)^{1/2} = 0.1185s$$

Where the uncertainties of a and b can be found in the caption of the plot (6)

Results and Analysis

Symmetry

To verify that the pendulum gives symmetric results, the pendulum of mass 200g and length of $43.2cm \pm 0.2cm$ was swung from the left side and right side with an initial angle of $\theta_0 = 1.4\text{rad} \pm 0.02\text{rad}$ and the difference of the first 20 corresponding peak amplitudes were computed. The peaks were found using the function scipy.signal.find-peaks() and then the absolute difference between adjacent peaks was calculated. The figure (3) shows the plot of each difference in peak amplitude.

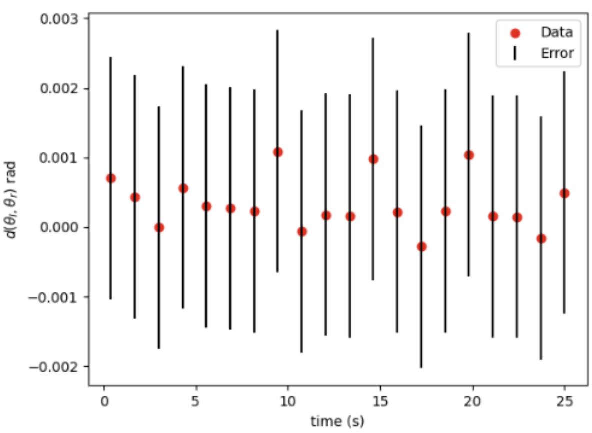

Figure 3: time vs. difference in peak amplitudes of releasing the mass from the left and right

As stated in the introduction, a measure of how symmetric this pendulum is is given by the equation 7, where we sum up the first 20 differences. The sum was $\sum_{n=1}^{20} = d(\theta_{r_n}, \theta_{l_n}) = 0.00474\text{rad}$, which is small compared to the total angle travelled in the 10 periods, by total angle travelled I mean the line integral of $\theta(T)$ for the first 10 periods, which is on the order of magnitude of 10 while the sum of difference in the 20 peaks is on the order of 10^{-3}. Therefore, the difference is negligible, and we can assume the pendulum is symmetric. Is this also seen in the large uncertainties compared to the small difference in amplitudes, alluding that the difference is negligible as it is much smaller than the uncertainties. The difference is also estimated to be much lower for a pendulum with smaller initial angles.

Oscillation Peiod T Dependencies

One of the first things that was concluded in the introduction was that the period should be independent of the initial angle and mass. From the theoretical model it should be $T \approx 2\pi\sqrt{\frac{l}{g}}$. The hypothesis was tested using a pendulum with a mass $100g$ and length $45.0cm \pm 0.2cm$ with 8 different initial angles ranging from $25° \pm 1°$ to $60° \pm 1°$. The result is plotted in the figure 4.

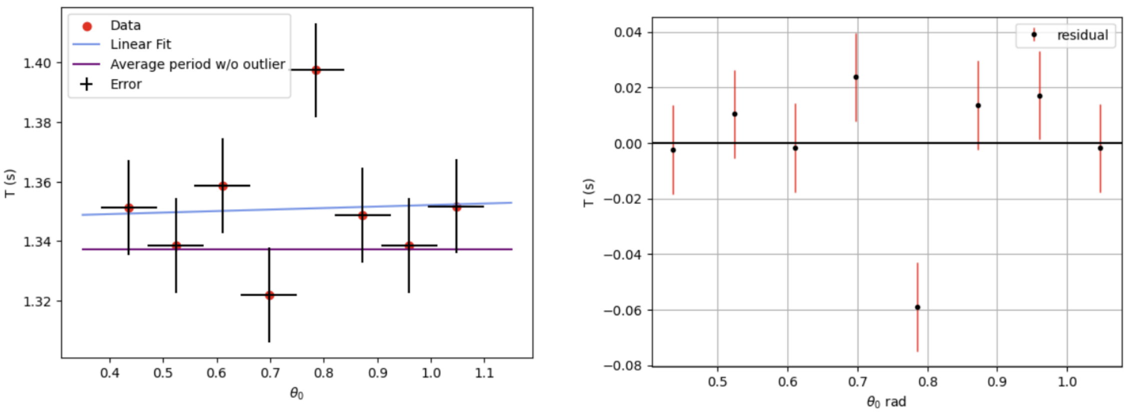

Figure 4: Initial angle θ_0 rad vs. Period T (s) fitted using a linear function $f(\theta_0) = a\theta_0 + b$ with parameters $a = 0.007\frac{s}{rad} \pm 0.04\frac{s}{rad}$ and $b = 1.34s \pm 0.03s$. The reduced chi-squared is $\chi^2/dof = 3.50$. Note that the average period line (at $T = 1.337$) is the average excluding the outlier point of the period around 1.4s. The residuals also show no pattern. The fixed values are $m = 100g \pm 0.1g$ and $l = (45 \pm 0.2)cm$

The plot shows little to no slope in the fit line, supporting the hypothesis that there is no dependency of θ_0 on period. The goodness of fit is higher than what it should be as there is an outlier point skewing the result and contributing the most to $\chi^2/dof = 3.5$. From the theoretical framework, the period should be $T = 2\pi\sqrt{l/g} = 1.346s$, and the average period from the data collected $T_{avg} = 1.337s \pm 0.016s$ lies within the expected period further showing the agreement with the theoretical model.

Similarly, to check the hypothesis that the period T is independent of the mass, multiple masses were swung on the pendulum with length $l = 45.0cm \pm 0.2cm$ from an initial angle of $\theta_0 = 60° \pm 1° \approx 1.05 \pm 0.02$rad. The relation is plotted in figure 5

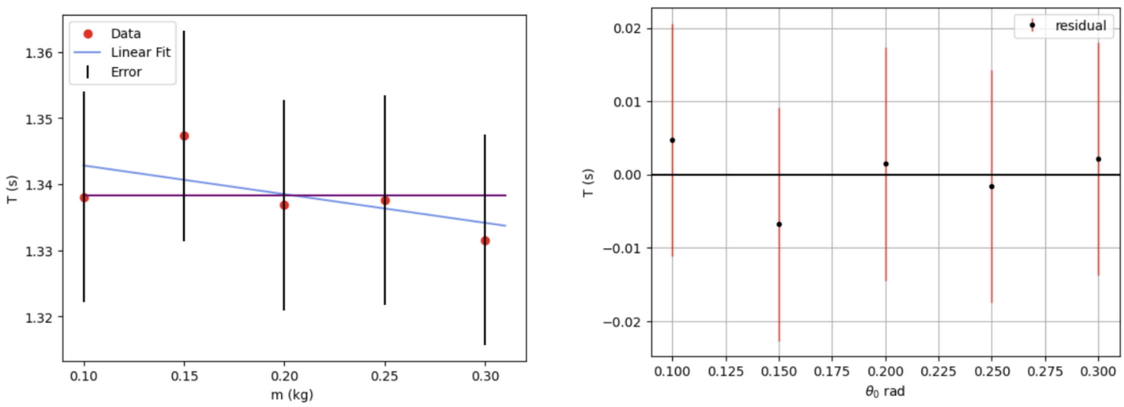

Figure 5: mass (kg) vs. Period T (s), fitted using a linear function $f(m) = am + b$ where $a = -0.04\frac{s}{kg} \pm 0.03\frac{s}{kg}$ and $b = 1.347s \pm 0.006s$. The reduced chi-squared is given by $\chi^2/dof = 0.081$. The purple line indicates the average period which is $T_{avg} = 1.338s \pm 0.015s$. Fixed values are $l = (45 \pm 0.02)cm$ and $\theta_0 = 60° \pm 1°$

Note that, again the average period (purple line on the plot) is within the uncertainty of the theoretical value of $T = 1.346s \pm 0.015s$. Therefore it can be concluded that this pendulum agrees with the theory of the period being independent of the mass. The experimental data failed to show that it depends on the initial angle.
The only variables on which period T should depend is the length of the pendulum, and the initial angle according to the theory, the period should increase as fast as $\approx l^{1/2}$. The exact equation is given by the equation (6). This will be compared to the results of the experiment. For this section of the experiment, 6 different lengths of string were used to swing a mass of $200g \pm 0.1g$ with initial angle $\theta_0 = 45° \pm 1°$. The results are plotted in figure 6

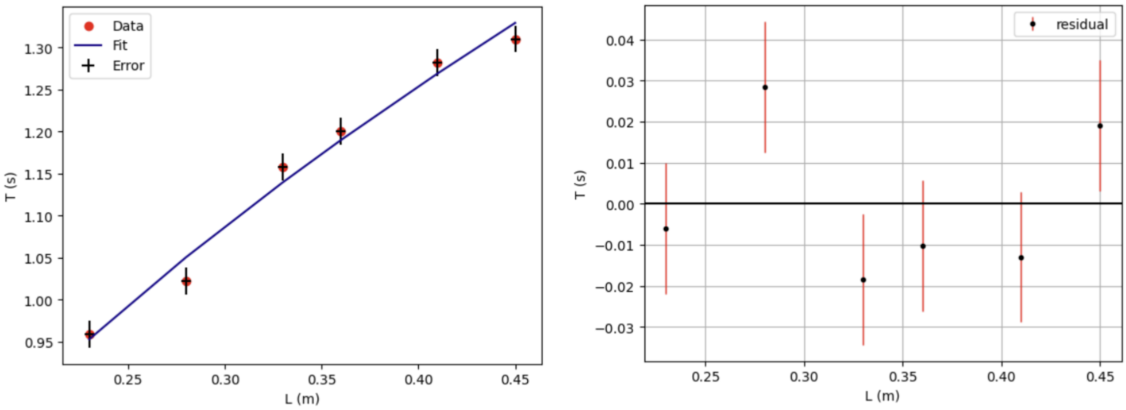

Figure 6: Length (m) vs. Period T(s) fitted with function $f(L) = a\sqrt{L} + b$ where $a = 1.96\frac{s}{\sqrt{m}} \pm 0.13\frac{s}{\sqrt{m}}$ and $b = 0.01s \pm 0.08s$. The reduced chi-squared is given by $\chi^2/dof = 1.89$, and the residual shows no pattern. Fixed values are $m = 200g \pm 0.1g$ and $\theta_0 = 45° \pm 1°$

From curvefit, we get that the constant in front of $\sqrt{l}$ is $a = (1.96 \pm 0.13)\frac{s}{\sqrt{m}}$, which is within the uncertainty of the theoretical value of 2. The reduced chi-squared and no pattern in the residual also indicate that the fit function is good. This concludes that the period of the oscillations is independent of mass and dependent on the length of the string, the results agree with the theory within the uncertainties. Although it was a negative result to show that it also depends on the initial angle as given by the equation (6).

Decay Constant τ Dependencies

It is expected that the decay constant would be independent of the initial angle and length of the string since from the theoretical model it only depends on mass, the relation given by $\tau = m/b$ where b is the friction term from the ode in the equation 3.

To test the theory, the relation between the decay constant and initial angle was plotted, the length of the string was held constant at $l = 37cm \pm 0.02cm$, and mass was fixed at $m = 200g \pm 0.2g$. Multiple initial angles from $20° \pm 1°$ to $60° \pm 1°$ were chosen. The figure 7 below plots the relation.

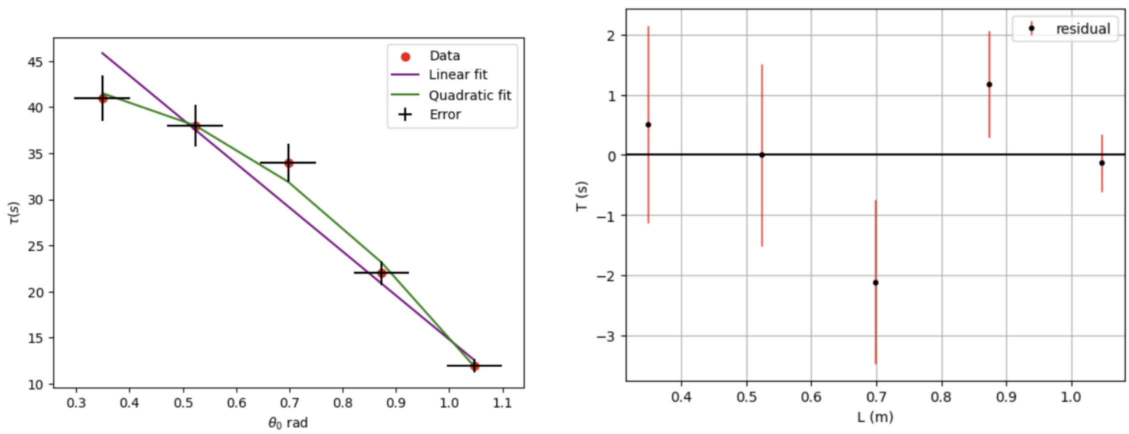

Figure 7: Figure on the left is plotting initial angle θ_0 rad vs. decay constant τ (s), the plot is fitted using a linear function and a quadratic function, parameters for the linear function $f(\theta_0) = a\theta_0 + b$ are $a = (-47 \pm 5)\frac{s}{rad}$ and b $= 62s \pm 5s$ and the parameters for the quadratic fit $f(\theta_0) = a\theta_0^2 + b\theta_0 + c$ are $a = (-42 \pm 14)\frac{s}{rad^2}$, b $= (17s \pm 20)\frac{s}{rad}$, and $c = 40s \pm 8s$, the goodness of fit for linear fit is $\chi^2/dof = 640$ and for quadratic fit is 431. Plot on the right are residuals for the quadratic fit. Values held fixed are $l = (37 \pm 0.02)cm$ and $m = 200g \pm 0.2g$

As seen from the figures above, the data does not align with the experimental predictions. A decreasing relation can be seen in the graph when it was expected to be no slope. The data was fit using a linear and a quadratic function but both gave a bad goodness of fit $\chi^2/dof = 640$ and $\chi^2/dof = 431$ respectively. The relation between the initial value and the decay constant can be studied further with a higher sample size but with this data, not much can be concluded further. While fitting the raw data of time t vs. θ using the equation

4 I noticed that the curve fit function results were sensitive to the initial parameters guess p_0 and sometimes the function would return parameters that were quite off, this tells me that there are additional uncertainties associated to the fitting method but am not sure how to quantify it. Regardless of the uncertainties, the sample size is too small to conclude an exact relation between the initial angle and decay constant.

The relation between the length of the string and its impact on the decay constant was investigated next. The fixed variables were now initial angle $\theta_0 = (0.5 \pm 0.02)$rad and mass $m = 200g \pm 0.2g$. The length was varied from $L = (20 \pm 0.2)cm$ to $(45 \pm 0.2)cm$

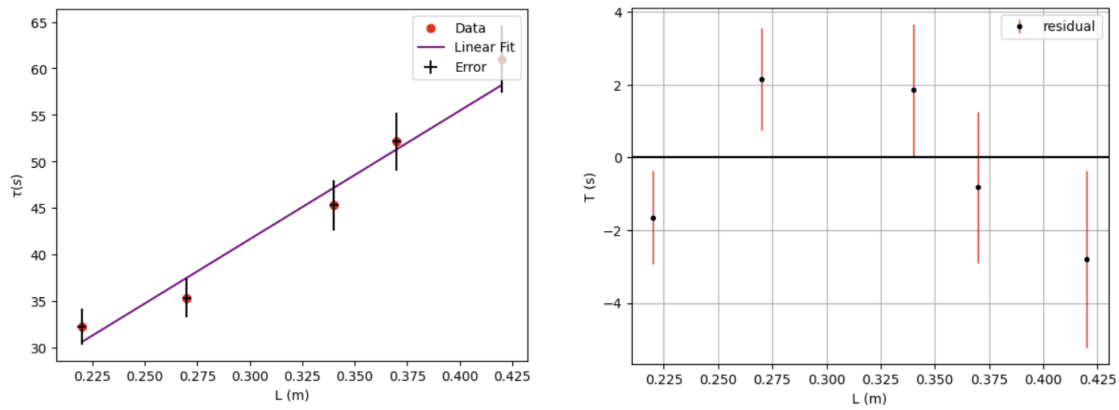

Figure 8: Length (m) vs. Decay constant (s) fitted using a linear function $f(L) = aL + b$ with parameters $a = (138 \pm 16)\frac{s}{m}$ and $b = 0.2s \pm 5s$. The goodness of fit is $\chi^2/dof = 10.6$. The plot on the right is for linear fit, they show no patterns, at least there are enough points to see any pattern. Values held fixed are $\theta_0 = (0.5 \pm 0.02)$rad and $m = 200g \pm 0.2g$

Data again differed from the theory, no correlation between length and the decay constant was expected but the decay constant seems to be proportional to the length. The relation can be looked into further with a larger sample size, but it is still clear that the decay constant is positively correlated to the length of the string. Lastly, the correlation between decay constant and mass was studied. The length was held constant at $L = (0.35 \pm 0.002)m$ and the initial angle was fixed at $45° \pm 1°$.

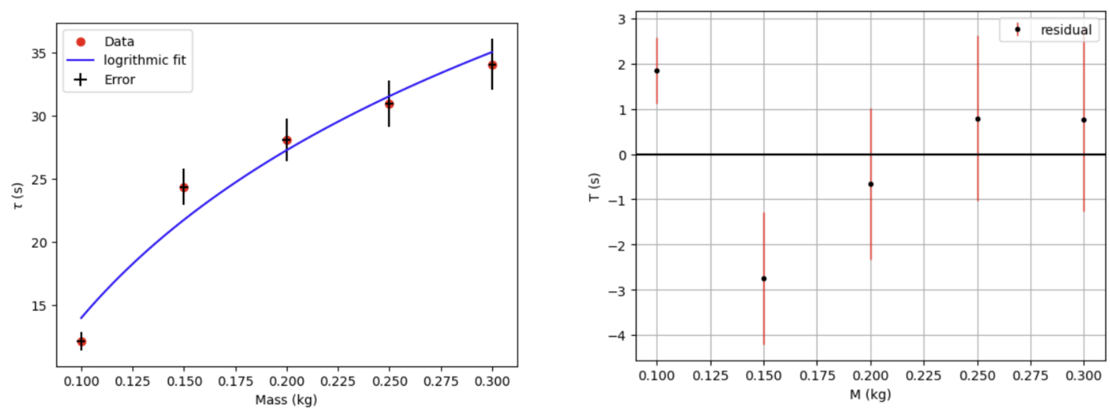

Figure 9: Mass (kg) vs Decay Constant (s), fitted using a logarithmic function $f(m) = a\log(m) + b$, where $a = (18.9 \pm 0.4)\frac{s}{log(kg)}$ and $b = 57s \pm 8s$, the goodness of fit is $\chi^2/dof = 0.24$. The residuals plot also shows no pattern. Values that were fixed are $l = (35 \pm 0.2)cm$ and $\theta_0 = 45° \pm 1°$

Figure 9 shows a good fit of the correlation between the mass of the pendulum and the decay constant. The only result for the decay constant that somewhat aligns with the theory. As discussed in the introduction section the decay constant should increase as mass increases, but the relation was hypothesized by a linear function $\tau = m/b$. The relation was not found to be linear but this can be explained easily by the friction in the pendulum, as the mass was increasing the friction force on the string attached to the lab stand also increased, resulting in a logarithmic relation instead of a linear one.

Conclusion

Some Results from the experiment agreed that the oscillations of the pendulum is modeled by the equation 4. While other results gave negative results. The analysis of the pendulum started by checking the reliability of the pendulum by making sure the pendulum was symmetric. This was done by checking the difference in amplitudes with different starting sides. The sum of the first 20 differences was 0.0047, which is a low number considering the line integral of the curve $\theta(T)$ for the first 10 periods is on the order of 10. Results for the period of the pendulum somewhat aligned with the theory, confirming that the period depends on the length of the string and is independent of the mass, and the initial angle in the case of small angle approximation. The value of the period agreed with the theoretical value of $2\pi\sqrt{\frac{l}{g}}$ within uncertainty. In the large angle approximation the results of the experiment failed to show that that the period is described by the equation (6). Lastly results on the decay constant were not as clear due to low sample size, and possibly high uncertainties that were not taken into consideration and not quantified. Decay constant related quadratically with the initial angle, linear with the length, and logarithmically with the mass. The fits on decay constant data yielded considerably high χ^2 values.

References

[1] Douglas Brown, Wolfgang Christian, and Robert M Hanson. *Tracker*. URL: `https://opensourcephysics.github.io/tracker-[]website/`.

[2] L. P. Fulcher; B. F. Davis. "Theoretical and experimental study of the motion of the simple pendulum". In: *American Journal of Physics* 44 (1976), pp. 51–55.

[3] John R. Taylor. *Classical Mechanics*. University Science Books, 2003.

1 AI statement

No AI was used in writing this report.