# K-Theory for C* Algebra

Shumail Farooqi

14th August, 2024

## Review From Oral Report

On 17th of July I did an oral report where I introduced the $K_0$ group for a unital C*-algebra using the Grothedieck construction. Here I will present a very short review of that day which might be useful for this written report. No proofs or explanations will be given in the review.

**Definition 0.1.** Let $A$ be a $C^*$ algebra then define $P_n(A) = P(M_n(A))$, as set of all projection from $n \times n$ matrices with entries from $A$ and let $P_\infty(A) = \bigcup_{n=1}^{\infty} P_n(A)$ (disjoint union)

**Definition 0.2** (Murray-Von Neumann Equivalent $\sim_0$ on $P_\infty(A)$)**.**
For $p, q \in P_\infty(A)$ if $p \in P_n(A)$ and $q \in P_m(A)$ then $p \sim_0 q$ when there exist $v \in M_{m,n}(A)$ such that

$$p = v^*v \text{ and } q = vv^*$$

If $p \sim_0 q$ then we know that the subspace $p$ and $q$ project on have the same dimension. We can see this by the following short computation. First note that the dimension of the projected subspace is the rank of the projection meaning $\dim(p(A^n)) = \text{rank}(p)$. Then from linear algebra we know that trace of a projection gives the rank of the projection, thus

$$\begin{aligned} rank(p) &= Tr(p) \\ &= Tr(v^*v) \text{ ( since } p \sim_0 q \text{ )} \\ &= Tr(vv^*) \text{ ( property of trace } Tr(AB) = Tr(BA)) \\ &= Tr(q) \\ &= \text{rank}(q) \end{aligned}$$

$P_\infty(A)$ is an abelian semi group with the following binary operation

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

This was proven in the oral report so it will not be proven here but it is a good short exercise.

**Definition 0.3.** $D(A) = P_\infty(A)/{\sim_0}$ is an abelian semi group with the binary operation

$$[p]_D + [q]_D = [p \oplus q]_D$$

again it is a good exercise to show this is a semi group and it is abelian, it was proved in the oral report.

### 0.0.1 Grothedieck contruction

Let $B$ be an abelian semi group then define an equivalence relation on $B \times B$ by

$$(x_1, y_1) \sim (x_2, y_2) \text{ if } \exists c \in B \text{ st.}$$

$$x_1 + y_2 + c = x_2 + y_1 + c$$

denote the equivalence class by $\langle x, y \rangle$, $(G(B), +)$ is an Abelian group with the operation defined by.

$$\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$$

**Definition 0.4** (Grothedieck Group).
Let $B$ be an abelian semi group then the Grothedieck group is denoted by
$$G(B) = (B \times B)/\sim$$

Now we can finally define the $K_0$ for a $C^*$ algebra.

**Definition 0.5** ($K_0$ group for a $C^*$ algebra).
Let $A$ be a $C^*$ algebra then the $K_0$ group for $A$ is
$$K_0(A) = G(D(A)) = G(P_\infty(A)/\sim_0)$$

It is a good exercise to compute the $K_0(\mathbb{C})$ (answer: $K_0(\mathbb{C}) = \mathbb{Z}$).

Define a function $\gamma_B : B \to G(B)$ by $G(x) = \langle x + y, y\rangle$, it is not hard to see that this function is well-defined and independent of the choice of $y$. We can define the Grothedieck group of an abelian semi group using this map, more specifically
$$G(B) = \{\gamma_B(x) - \gamma_B(y) : x, y \in B\}$$
This concludes the short review of the oral report, now we can continue with new material.

# $K_0$ Group and Homotopy Equivalence

**Definition 0.6.** For the sake of clean notation we define a map $[\cdot]_0 : P_\infty \to K_0(A)$ by
$$[p]_0 = \gamma([p]_D) \in K_0(A),\ p \in P_\infty(A)$$

Now as shown in the review we can write the $K_0$ group as

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in P_\infty(A)\} \tag{1}$$

**Lemma 0.7.** A $^*-$ homomorphism (homomorphism with the property $\psi(x^*) = (\psi(x))^*$) maps projections to projections

**Proof.** let $p \in A$ be a projection then
$$\psi(p) = \psi(p^2) = \psi(p)\psi(p) = (\psi(p))^2$$
😎

For unital $C^*$ algebras $A$ and $B$ let $\psi : A \to B$ be a $^*-$homomorphism. Associate to $\psi$ a group homomorphism $K_0(\psi) : K_0(A) \to K_0(B)$ as follows. In fact this is a unique group homomorphism given by
$$K_0(\psi)([p]_0) = [\psi(p)]_0,\ p \in P_\infty(A) \tag{2}$$

**Theorem 0.8** (Properties of $K_0$ for unital $C^*-$ algebras).

1. For each unital $C^*-$ algebra $A$, $K_0(id_A) = id_{k_0(A)}$
2. If $A, B$, and $C$ are unital $C^*-$algebras, if $\psi : A \to B$ and $\phi : B \to C$ are $^*-$homomorphisms then $K_0(\phi \circ \psi) = K_0(\phi) \circ K_0(\psi)$
3. $K_0(\{0\}) = \{0\}$

**Proof.** By equation 2, first bullet point is obvious,
$$K_0(id_A)([p]_0) = [id_A(p)]_0 = [p]_0$$
$K_0$ group for a $C^*-$algebra is also a covariant functor thus it has the property that, $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$.

Now for (3) note that $P_n(\{0\}) = \{0_{n\times n}\}$, where $0_{n\times n}$ is the $0_{th} \in M_n(\{0\})$, then it is clear to see that all $0_1, 0_2, \dots, 0_n, \cdots$ are neumann equivalent thus $D(\{0\}) = P_\infty(\{0\})/\sim_0 = \{[0]_D\}$ and therefore $K_0(\{0\}) = G(D(\{0\})) = \{0\}$ 😎

**Definition 0.9.** Let $\phi, \psi : A \to B$ be two $^*-$homomorphisms between $C^*-$algebras then we say that $\phi$ and $\psi$ are homotopy equivalent if there exist a continuous function $F : [0, 1] \times A \to B$ such that
$$F(0, a) = \psi(a)\ \forall a \in A \text{ and } F(1, a) = \phi(a)$$
and we denote this by $\psi \sim_h \phi$

Additionally we say that the $C^*-$algebras $A$ and $B$ are homotopy equivalence if there exist two $^*-$homomorphisms $\phi : A \to B$ and $\psi : B \to A$ such that

$$\phi \circ \psi \sim_h id_B \text{ and } \psi \circ \phi \sim_h id_A$$

Then $A \sim_h B$

Now we will see why homotopy equivalence is preferred to compute the first K group $K_0$

**Theorem 0.10** (Homotopy invariance of $K_0$).

1. If $\phi, \psi : A \to B$ are homotopy equivalent $^*-$homomorphisms, then $K_0(\phi) = K_0(\psi)$
2. If $A$ and $B$ are homotopy equivalent then $K_0(A)$ and $K_0(B)$ are isomorphic. More specifically if

$$A \xrightarrow{\phi} B \xrightarrow{\psi} A$$

is a homotopy then $K_0(\phi) : K_0(A) \to K_0(B)$ and $K_0(\psi) : K_0(B) \to K_0(A)$ are isomorphisms, and $K_0(\phi)^{-1} = K_0(\psi)$

**Proof.** To prove the first claim let $F : [0,1] \times A \to B$ be the continuous function such that $F(0,a) = \phi(a)$ and $F(1,a) = \psi(a)$ then we can extend this homotopy to $M_n(A),$ and $M_n(B)$ so there exist a function $F : [0,1] \times M_n(A) \to M_n(B)$ and note that this is a continuous function connecting $\phi$ and $\psi$, then for every projection $p \in P_\infty(A)$

$$\phi(p) = F(0,p) \sim_h F(1,p) = \psi(p)$$

Therefore

$$K_0(\phi)([p]_0) = [\phi(p)]_0 = [\psi(p)]_0 = K_0(\psi)([p]_0)$$

Now to show the second part since $\phi \circ \psi \sim_h id_B$, we apply (i) to this to get

$$K_0(\phi \circ \psi) = K_0(id_B)$$
$$K_0(\phi) \circ K_0(\psi) = id_B \text{ by theorem 0.8}$$
$$K_0(\psi) = K_0(\phi)^{-1}$$

😎

Another good example to compute is $K_0(M_n(\mathbb{C}))$

$K_0(M_n(\mathbb{C}))$ is isomorphic to the integers. Specifically given the standard trace $\tau$ on $M_n(\mathcal{C})$, the function

$$K_0(\tau) : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$$

is an isomorphism, note that $\tau$ on $M_n(\mathbb{C})$ is a $^*-$homomorphism since it has the property that for all $A, B \in M_n(\mathbb{C})$

$$\tau(A+B) = \tau(A) + \tau(B) \text{ and } \tau(A^*) = \tau(A)^*$$

Where the adjoint map is the complex conjugate of the matrix entries and transpose of the matrix.

**Proof.** Let $x \in K_0(M_n(\mathbb{C}))$ then by the standard picture of $K_0$ from equation 1 we can write $x = [p]_0 - [q]_0$ for $p, q \in P_k(M_n(\mathbb{C}))$ so

$$K_0(Tr)(g) = K_0(Tr)([p]_0 - [q]_0) = [Tr(p-q)]_0 \text{ but since Tr(p - q) is an integer}$$
$$= Tr(p-q) = Tr(p) - Tr(q) = dim(p(\mathbb{C}^{kn})) - dim(q(\mathbb{C}^{kn}))$$

Now it is easy to see how we can generate all the integers, by selecting $p, q$ such that the difference of the projected subspaces are integers numbers and we can easily generate the integers. We already have an isomorphisms between $K_0(M_n(\mathbb{C}))$ and $\mathbb{Z}$ as suggested before, first note that the image of $K_0(Tr)$ is a subgroup of $\mathbb{Z}$ since it is closed under addition

$$K_0(Tr)(g) + K_0(Tr)(a) = Tr(p-q) + Tr(p'-q') \text{ if } g = [p]_0 - [q]_0 \text{ and } a = [p']_0 - [q']_0$$
$$\text{and } Tr(p-q) + Tr(p'-q')$$

is again an integer. The inverse for $K_0(Tr)([p]_0 - [q]_0)$ is given by $K_0(Tr)([q']_0 - [p']_0)$ where

$$dim(p(\mathbb{C}^{kn})) = dim(p'(\mathbb{C}^{kn})) \text{ and } dim(q(\mathbb{C}^{kn})) = dim(q'(\mathbb{C}^{kn}))$$

Now a subgroup of $\mathbb{Z}$ is equal to $\mathbb{Z}$ iff 1 is in that subgroup. We have 1 by picking $e \in P_\infty(M_n(\mathbb{C}))$ such that it projects to a 1 dimensional subspace then

$$K_0(Tr)[e]_0 = Tr(e) = 1$$

This shows that $K_0(Tr)$ is surjective, now to see its injective

$$K_0(Tr)([p]_0 - [q]_0) = K_0(Tr)([p']_0 - [q']_0)$$
$$Tr(p-q) = Tr(p'-q')$$
$$Tr(p) - Tr(q) = Tr(p') - Tr(q')$$

This then implies that

$$[p]_D - [q]_D = [p']_D - [q']_D$$
$$\gamma([p]_D) - \gamma([q]_D) = \gamma([p']_D) - \gamma([q']_D) \text{ we applied the } \gamma \text{ function then use that its linear}$$

$$[p]_0 - [q]_0 = [p']_0 - [q']_0$$

So it is also injective and thus an isomorphism.

😎

**Definition 0.11** (Contractible Space).
A space $X$ is contractible if the identity map $id_X : X \to X$ is homotopic to a constant map so $id_X \sim_h x_0$ for some $x_0 \in X$. In this case we say that $id_X$ is null-homotopic.

**Theorem 0.12.** Let $X$ be contractible, compact Hausdorff space, Then $K_0(C(X))$ is isomorphic to the integers. The isomorphism is given by the map $\dim : K_0(C(X)) \to \mathbb{Z}$

$$dim([p]_0) = Tr(p(x)),\ p \in P_\infty(C(X))$$

Where $x \in X$ and Tr is the standard trace on $M_n(\mathbb{C})$

**Proof.** Since $X$ is contractible there exist a function $g : [0,1] \times X \to X$ such that for all $x \in X$, $g(1,x) = x$ and $g(0,x) = x_0$ for some $x_0 \in X$. Define a $^*-$homomorphism for all $t \in [0,1]$ by $\phi_t : C(X) \to C(X)$ by $\phi_t(f)(x) = f(g(t,x))$ then we have that

$$\phi_0(f) = f(x_0) \text{ and } \phi_1(f) = f(x) = id_{C(X)}$$

since $\phi_t(f)$ is continuous for all $f \in C(X)$, we have a that $\phi_0 \sim_h id_{C(X)}$, so $C(X)$ is contractible as well.

Now define 2 more $^*-$homoorphisms by $\varphi : C(X) \to \mathbb{C}$ by $\varphi(f) = f(x_0)$ and $\psi : \mathbb{C} \to C(X)$ by $\psi(a) = a$ then note that

$$\varphi \circ \psi = id_{\mathbb{C}} \text{ and } \psi \circ \varphi = \phi_0 \sim_h id_{C(X)}$$

Thus $C(X) \sim_h \mathbb{C}$

$$\begin{tikzcd} & K_0(\mathbb{C}) \arrow[dr, "K_0(Tr)"] & \\ K_0(C(X)) \arrow[ur, "K_0(\varphi)"] \arrow[rr, "\dim"] & & \mathbb{Z} \end{tikzcd}$$

This diagram is commutative and because $C(x) \sim_h \mathbb{C}$ by theorem 0.10 $K_0(C(X))$ and $K_0(\mathbb{C})$ are isomorphic and as discussed before $K_0(\mathbb{C})$ and $\mathbb{Z}$ are isomorphic thus we can conclude that $K_0(C(X)) \cong \mathbb{Z}$ 😎