Vorticity Equation:

$$\frac{\partial}{\partial x}(\, y\ equation\ of\ motion) - \frac{\partial}{\partial y}(x\ equation\ of\ motion)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial y} - fu\right)$$

$$-$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + fv\right)$$

$$\left(\frac{\partial^2 v}{\partial x\partial t} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + w\frac{\partial^2 v}{\partial x\partial z} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z}\right)$$
$$-\left(\frac{\partial^2 u}{\partial y\partial t} + u\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + w\frac{\partial^2 u}{\partial y\partial z} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{-1}{\rho}\frac{\partial^2 p}{\partial y\partial x} + \frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - f\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} + \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x} - f\frac{\partial v}{\partial y} - v\frac{\partial f}{\partial y}\right)$$

Note: $\frac{-1}{\rho}\frac{\partial^2 p}{\partial y\partial x} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} = 0,$

and, $\frac{\partial^2 v}{\partial x\partial t} - \frac{\partial^2 u}{\partial y\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial \zeta}{\partial t}$

$$u\frac{\partial^2 v}{\partial x^2} - u\frac{\partial^2 u}{\partial y\partial x} = u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) - u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = u\frac{\partial \zeta}{\partial x}$$

$$v\frac{\partial^2 v}{\partial y\partial x} - v\frac{\partial^2 u}{\partial y^2} = v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) - v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = v\frac{\partial \zeta}{\partial y}$$

Combining above terms: $u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = V \cdot \nabla_H(\zeta)$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial x}\zeta$$

$$\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial v}{\partial y}\zeta$$

Combining above terms: $-(\nabla \cdot \mathrm{V}_H)(\zeta)$, after moving to right-hand side

Also, $$-f\frac{\partial u}{\partial x} - f\frac{\partial v}{\partial y} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -f(\nabla \cdot \mathrm{V}_H)$$

Can combine above terms: $-(\zeta + f)(\nabla \cdot \mathrm{V}_H)$

$$w\frac{\partial^2 v}{\partial x \partial z} - w\frac{\partial^2 u}{\partial y \partial z} = w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = w\frac{\partial \zeta}{\partial z}$$

Putting all terms together:

$$\frac{\partial \zeta}{\partial t} + V \cdot \nabla_H(\zeta) + w\frac{\partial \zeta}{\partial z} = -(\zeta + f)(\nabla \cdot \mathrm{V}_H) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) +$$

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) - v\frac{\partial f}{\partial y}$$

Moving last term to left-hand side and recognizing that f varies only with y, can rewrite the vorticity equation as:

$$\frac{\partial \zeta}{\partial t} + V \cdot \nabla_H(\zeta + f) + w\frac{\partial \zeta}{\partial z} = -(\zeta + f)(\nabla \cdot \mathrm{V}_H)$$
$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

$\frac{\partial \zeta}{\partial t}$	local rate of change of relative vorticity
$V \cdot \nabla_H(\zeta + f)$	horizontal advection of absolute vorticity
$w\frac{\partial \zeta}{\partial z}$	vertical advection of relative vorticity
$-(\zeta + f)(\nabla \cdot \mathrm{V}_H)$	divergence term
$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$	tilting or twisting term (older texts - tipping)
$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$	solenoid term

Vorticity Equation

$$\frac{\partial \zeta}{\partial t} \; + \; u\frac{\partial \zeta}{\partial x} \; + \; v\frac{\partial (\zeta + \mathrm{f})}{\partial y} \; + \; w\frac{\partial \zeta}{\partial z}$$

A B B C

$$= \; -(\zeta + \mathrm{f})\left(\frac{\partial \mathrm{u}}{\partial x} + \frac{\partial \mathrm{v}}{\partial y}\right) - \left(\frac{\partial \mathrm{w}}{\partial x}\frac{\partial \mathrm{v}}{\partial z} - \frac{\partial \mathrm{w}}{\partial y}\frac{\partial \mathrm{u}}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

D E F

Scale Analysis

$U \sim 10$ m s^{-1} , $W \sim 10^{-2}$ m s^{-1} , $L \sim 10^{6}$ m , $H \sim 10^{4}$ m , $\delta p \sim 10$ hPa

$\rho \sim 1$ kg m^{-3} , $\tau = L/U \sim 10^{5}$ s , $\delta\rho/\rho \sim 10^{-2}$, $f = 10^{-4}$ s^{-1} , $\beta \sim 10^{-11}$ m^{-1} s^{-1}

A Local change of relative vorticity with time (10^{-10} s^{-2})

B Horizontal advection of absolute vorticity (10^{-10} s^{-2})

C Vertical advection of relative vorticity (10^{-11} s^{-2})

D Divergence term (10^{-9} s^{-2})

E Tilting or Twisting term (10^{-11} s^{-2})

F Solenoid term (10^{-11} s^{-2})