ATSC5011 Physical Meteorology II

4 Credit Hours, 11:00 to 11:50 MWF (EN6085) and 9:00 to 12:00 T (STEM 220)
Jeff Snider, EN6019, jsnider@uwyo.edu, x2637
Class Website - http://www.atmos.uwyo.edu/~jsnider/atsc5011/

ATSC5011 examines the dynamical behavior of clouds and the behavior of hydrometeors within clouds. Topics include cloud thermodynamics and dynamics, cloud radiative transfer through clouds, the nucleation of water and ice, diffusion growth of droplets and ice crystals, and the development of rain and snow via collection.

Cloud Thermodynamics – Topic: Saturated flow and precipitation over a barrier
Reading – Chapter 1 Lamb and Verlinde, Sections 1, 1.1 and 1.7 Curry and Webster; Section 4.14 Iribarne and Godson

Cloud Thermodynamics – Topic: Adiabatic liquid water content – Rigorous treatment for deep clouds

Cloud Thermodynamics – Topic: Adiabatic liquid water content – Approximate treatment for shallow clouds
Albrecht et al. (Geophys. Res. Lett., 17, 89-92, 1990)

Cloud Dynamics – Topic: Convective cloud downdraft
Reading – pp. 48 – 54 Rogers and Yau

Cloud Dynamics – Topic: Radiatively driven circulations within a cloud-topped boundary layer
Reading – Sections 6.3 and 6.4, Lamb and Verlinde

Cloud Radiative Properties – Topic: Shortwave radiative transfer through clouds
Reading – Sections 7.4.4, 13.3, 13.4, Petty

Midterm Exam

Mie Theory – Topic: Sizing calibration of an optical particle counter
Reading – Sections 12.3, 12.4, 12.5.1, Petty

Activation of Cloud Droplets
Reading – Section 7.1, Lamb and Verlinde

Droplet Growth by Condensation
Reading – Chapter 7, Rogers and Yau

Crystal Growth by Deposition in Mixed-phase Clouds
Reading - Chapter 9, Rogers and Yau

Rain in Nonfreezing Clouds
Reading - Section 7.2, Lamb and Verlinde

Accretion Growth and Hail
Reading – Selected sections of Chapters 12, 13, Rogers and Yau

Final Exam

Textbooks –

Physics and Chemistry of Clouds, D.Lamb and J.Verlinde, Cambridge University Press, 2011
Atmospheric Thermodynamics, Second Edition, Iribarne and Godson, Reidel, 1981
A Short Course in Cloud Physics, Third Edition, R.R.Rogers and M.K.Yau, Elsevier, 1989
Thermodynamics of Atmospheres and Oceans, J.Curry and P.Webster, Academic Press, 1999
A First Course in Atmospheric Radiation, Second Edition, G.Petty, Sundog Publishing, 2006

Grading

Midterm Exam 20%, Final Exam 20%, Homework 30%, STEM 220 Assignments 30%

Rules –

1) During lecture, put your cell phone away.

2) Homework is an essential part of this class. While I encourage you to study collaboratively, I will insist that your homework be unique. When I grade the homework I will look to see that your solution is unique. If it is not, I will return the homework ungraded.

3) All homework should begin with a statement of the problem; if this is not present, I will return the homework ungraded.

4) Class material (handouts, notes, and etc.) should be organized into a notebook. This material should be brought to class.

A note on academic integrity and plagiarism

Academic integrity is the pursuit of scholarly activity in an open, honest and responsible manner. Academic integrity is a basic guiding principle for all academic activity at the University of Wyoming, and all students are expected to act in accordance with this principle. Consistent with this expectation, all students should act with personal integrity, respect other students' dignity, rights and property, and help create and maintain an environment in which all can succeed through the fruits of their efforts. Academic integrity includes a commitment not to engage in or tolerate acts of plagiarism, falsification, misrepresentation, or deception. Such acts of dishonesty violate the fundamental ethical principles of the academic community and compromise the worth of work completed by others. Evidence of plagiarism may result in expulsion from the course (with an F grade) as well as dismissal or suspension from the University of Wyoming (Unireg #030-1970).

wet-bulb temperature

1.Isobaric wet-bulb temperature: the temperature an air parcel would have if cooled adiabatically to saturation at constant pressure by evaporation of water into it, all latent heat being supplied by the parcel.

2.Adiabatic wet-bulb temperature (or pseudo wet-bulb temperature): the temperature an air parcel would have if cooled adiabatically to saturation and then compressed adiabatically to the original pressure in a moist-adiabatic process.

This is the wet-bulb temperature as read off the thermodynamic diagram and is always less than the isobaric wet-bulb temperature, usually by a fraction of a degree centigrade.

3.The temperature read from the wet-bulb thermometer.

dewpoint

(Or dewpoint temperature.) The temperature to which a given air parcel must be cooled at constant pressure and constant water vapor content in order for saturation to occur.

When this temperature is below 0°C, it is sometimes called the frost point. The dewpoint may alternatively be defined as the temperature at which the saturation vapor pressure of the parcel is equal to the actual vapor pressure of the contained water vapor. Isobaric heating or cooling of an air parcel does not alter the value of that parcel's dewpoint, as long as no vapor is added or removed. Therefore, the dewpoint is a conservative property of air with respect to such processes. However, the dewpoint is nonconservative with respect to vertical adiabatic motions of air in the atmosphere. The dewpoint of ascending moist air decreases at a rate only about one-fifth as great as the dry-adiabatic lapse rate. The dewpoint can be measured directly by several kinds of dewpoint hygrometers or it can be deduced indirectly from psychrometers or devices that measure the water vapor density or mixing ratio.

See dewpoint formula.

Saturating and saturated thermodynamic steps - see Problem 1 for more detail

(when thinking about these steps we envision a starting state with coordinates $\{e,T\}$ in the graph below)

Unsaturated ascent; mixing ratio constant and unsaturated entropy constant. This step defines the lifted condensation level

Adiabatic and isobaric humidification; enthalpy constant. This step defines the wet bulb temperature

Isobaric Cooling; mixing ratio constant and pressure constant. This step defines the dew point temperature

Saturated Ascent; total mixing ratio constant and saturated entropy constant. This step defines the adiabatic cloud temperature and the adiabatic liquid water content
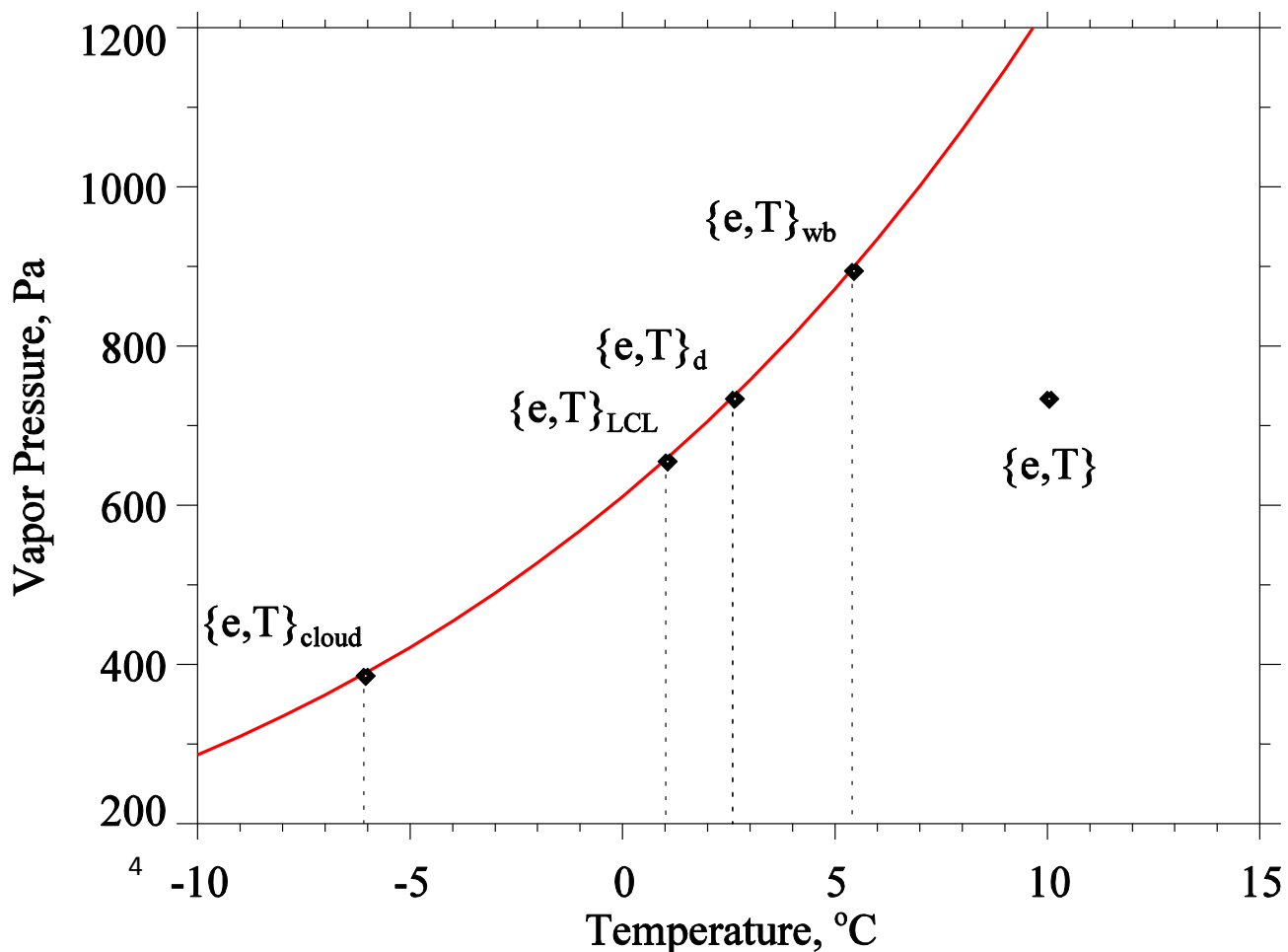
Saturated flow and precipitation over a barrier - see Problem 2 for more detail
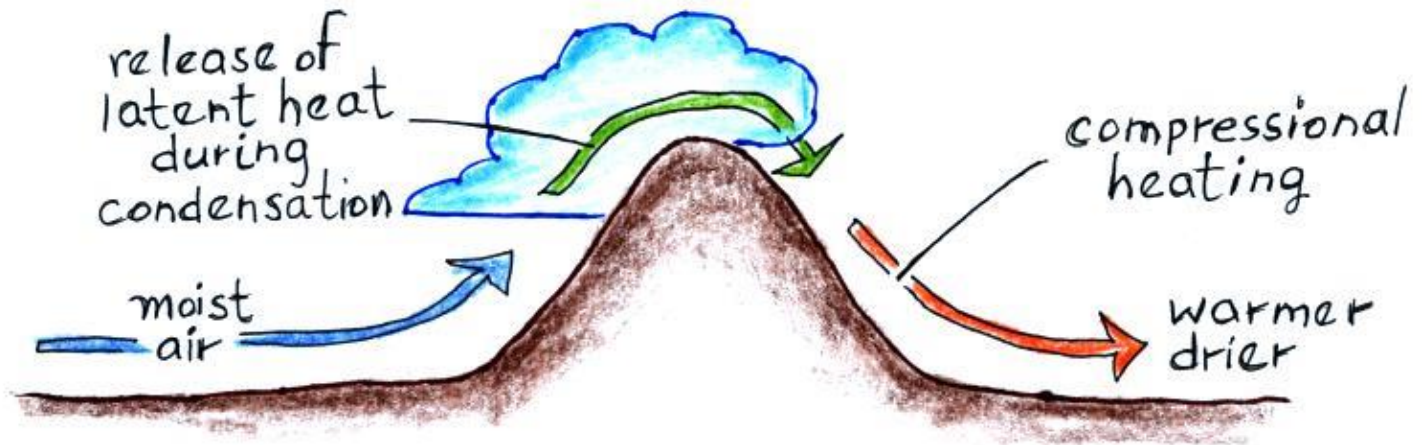
State 1 to state 2

Unsaturated adiabatic ascent to the LCL; unsaturated entropy and vapor mixing ratio are constant

$$\varphi_1 = C_{pd} \cdot ln(T_2) - R_d \cdot ln(P_2 - e_2) + w_2 \cdot \left(C_{pv} \cdot ln(T_2) - R_v \cdot ln(e_2)\right)$$

$$w_1 = \varepsilon \cdot e_s(T_2)/\left(P_2 - e_s(T_2)\right)$$

State 2 to state 3

Saturated adiabatic ascent from LCL to a prescribed cloud state pressure; saturated entropy and total mixing ratio are constant

$$\varphi_2 = \left(C_{pd} + w_{tot,3} \cdot C_\ell\right) \cdot ln(T_3) - R_d \cdot ln(P_3 - e_s(T_3)) + L_v(T_3) \cdot w_3/T_3$$

$$w_3 = \varepsilon \cdot e_s(T_3)/\left(P_3 - e_s(T_3)\right)$$

$$w_{2,tot} = w_3 + w_{\ell,3}$$

Note: P_3 is prescribed

State 3 to state 4

Condensate (cloud liquid water) is removed from the parcel as precipitation. We approximate this process using a prescribed precipitation efficiency.

$$w_{\ell,4} = w_{\ell,3} \cdot (1 - PE)$$

Note: PE is prescribed

State 4 to state 5

Saturated adiabatic descent from the cloud state to the LCL; saturated entropy and total mixing ratio are constant

$$\varphi_4 = \left(C_{pd} + w_{tot,5} \cdot C_{\ell}\right) \cdot ln(T_5) - R_d \cdot ln(P_5 - e_s(T_5)) + L_v(T_5) \cdot w_5 / T_5$$

$$w_{4,tot} = \varepsilon \cdot e_s(T_5) / \left(P_5 - e_s(T_5)\right)$$

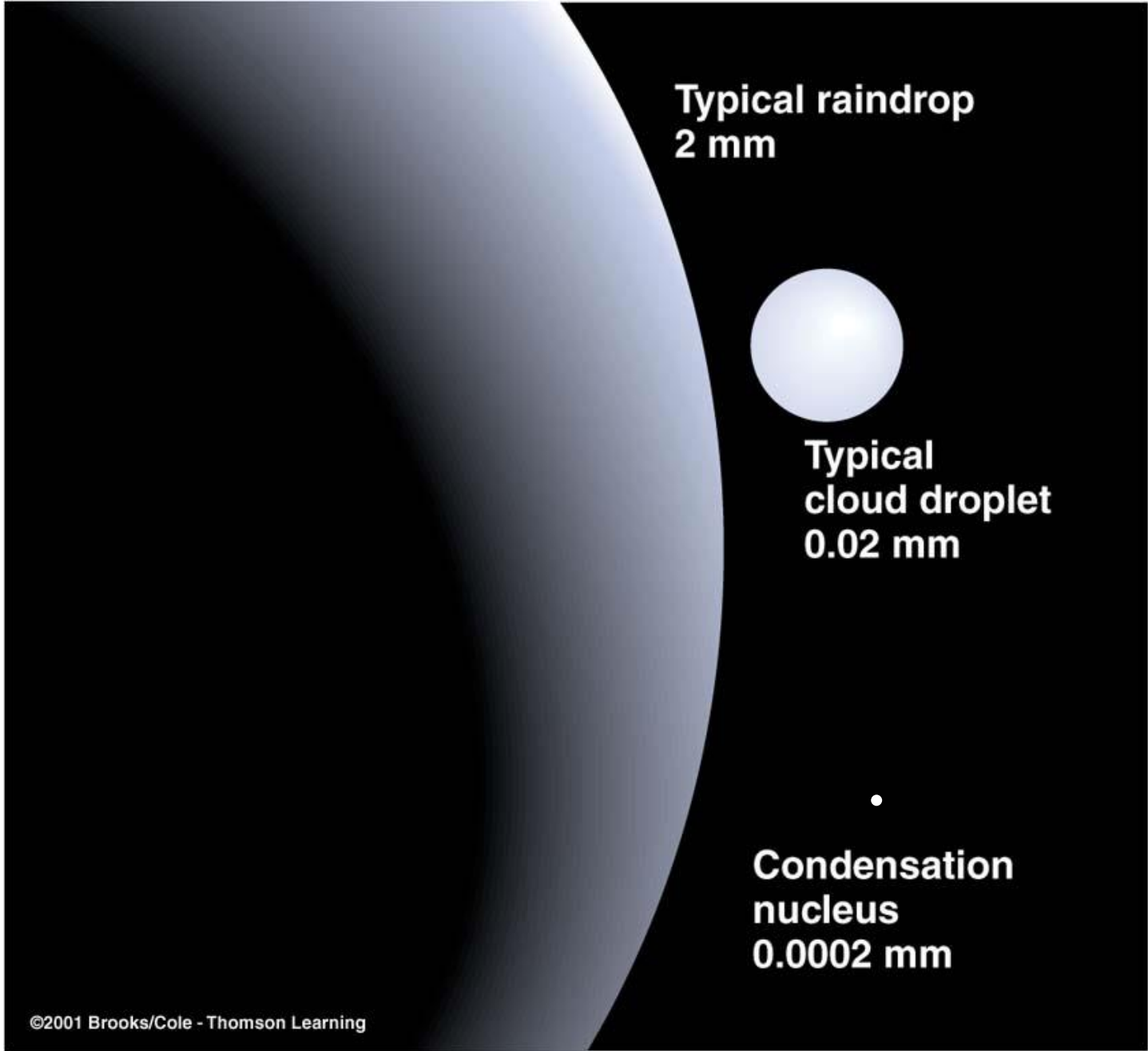Note: In IDL, guess with a pressure larger than P_4, otherwise NEWTON diverges from solution

State 5 to state 6

Unsaturated adiabatic descent to the original pressure ($P_6 = P_1$); unsaturated entropy and vapor mixing ratio are constant

$$\varphi_5 = C_{pd} \cdot ln(T_6) - R_d \cdot ln(P_6 - e_6) + w_6 \cdot \left(C_{pv} \cdot ln(T_6) - R_v \cdot ln(e_6)\right)$$

$$w_5 = \varepsilon \cdot e_6 / \left(P_6 - e_6\right)$$

Note: P_6 is prescribed

Typical raindrop
2 mm

Typical
cloud droplet
0.02 mm

Condensation
nucleus
0.0002 mm

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Adiabatic Liquid Water Amount in Shallow Clouds

Typically, liquid amount within clouds is less than that anticipated theoretically. Theory describing the liquid amount envisions a closed air parcel that ascends adiabatically above its lifted condensation level. Improved understanding of processes that lower the liquid water amount, relative to the adiabatic prediction, have developed from analyses of measured and theoretical (adiabatic) cloud properties (Paluch, 1979).

This note describes a particular form of adiabatic theory and presents a simple equation that produces a robust approximation for the adiabatic liquid water mixing ratio (w_ℓ) and the adiabatic liquid water content (LWC) of shallow clouds (e.g., stratus and stratocumulus).

The theory is based on a formulation of the lapse rate of liquid water mixing ratio (Γ_ℓ) (Albrecht et al., 1990)

$$\Gamma_\ell = \frac{(\varepsilon + w_s) \cdot w_s \cdot L_v}{R_d \cdot T_{cb}^2} \cdot \Gamma_s - \frac{w_s \cdot P_{cb}}{(P_{cb} - e_s) \cdot H} \qquad . \tag{1}$$

Here, w_s is saturation vapor mixing ratio, L_v is latent heat of evaporation, Γ_s is the saturated adiabatic (temperature) lapse rate, e_s is saturation vapor pressure, and H is the atmospheric scale height.

Throughout this note the cloud base pressure and temperature are symbolized as P_{cb} and T_{cb}, and the scale height is calculated using cloud base temperature, the gravitational constant (g), and the dry air gas constant (R_d)

$$H = \frac{R_d \cdot T_{cb}}{g} \tag{2}$$

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Equations 1 and 2, and the gas law for the dry air component, define the lapse rate of liquid water content (Γ_{lwc})

$$\Gamma_{lwc} = \frac{P_{cb} - e_s}{R_d \cdot T_{cb}} \cdot \Gamma_\ell \quad . \tag{3}$$

The Smithsonian Tables and atmospheric physics textbooks provide formulas for the saturated adiabatic temperature lapse rate (Γ_s) seen in Equation 1 (List, 1951; Iribarne and Godson, 1981; Bohren and Albrecht, 1998; Curry and Webster, 1999). The formulation used here comes from the Curry and Webster (1999), and the relevant page of that textbook is attached to the final page of this reading assignment.

$$\Gamma_s = \Gamma_d \cdot \frac{1 + \dfrac{L_v \cdot w_s}{R_d \cdot T_{cb}}}{1 + \dfrac{\varepsilon \cdot L_v^2 \cdot w_s}{c_{pd} \cdot R_d \cdot T_{cb}^2}} \quad . \tag{4}$$

In ATSC5011 we formulate the temperature-dependent latent heat of evaporation using Kirchhoff's Law

$$L_v = L_o + (c_\ell - c_{pv}) \cdot (T_o - T_{cb}) \tag{5}$$

and we formulate the saturated vapor pressure as

$$e_s = e_{s,o} \cdot exp\left(\frac{(L_o + (c_\ell - c_{pv}) \cdot T_o) \cdot (1/T_o - 1/T_{cb}) - (c_\ell - c_{pv}) \cdot ln(T_{cb}/T_o)}{R_v} \right) \tag{6}$$

with $e_o = 610.7$ Pa.

Assuming a relatively shallow cloud, and thus assuming that the cloud-base value of Γ_{lwc} can be applied to the whole cloud layer, the vertical profile of the adiabatic LWC is described by Equation 7

$$LWC(z) = \Gamma_{lwc} \cdot (z - z_{cb}). \tag{7}$$

Here z_{cb} is the cloud base altitude.

Researchers who have means of estimating both the thermodynamic state at cloud base (P_{cb} and T_{cb}) and cloud thickness $(z_{ct} - z_{cb})$ use Equation 7 to derive an adiabatic liquid water path (LWP; Albrecht et al. (1990))

$$LWP = \int_{z_{cb}}^{z_{ct}} LWC(\,z\,) \cdot dz = \frac{1}{2} \cdot \Gamma_{lwc} \cdot (\,z_{ct} - z_{cb}\,)^2 . \tag{8}$$

Adiabatic cloud properties, derived for a range of P_{cb} and T_{cb}, assuming a 500 m thick cloud, are presented in Table 1.

References –

Albrecht, B.A., C.W.Fairall, D.W.Thomson and A.B.White, Surface-based remote sensing of the observed and the adiabatic liquid water content of stratocumulus clouds, Geophys. Res. Lett., 17, 89-92, 1990

Bohren, C.F., and B.A.Albrecht, *Atmospheric Thermodynamics*, 402 pages, Oxford University Press, New York, USA, 1998

Curry, J.A. and P.J.Webster, *The Thermodynamics of Atmospheres and Oceans*, 467 pages, Academic Press, San Diego, USA, 1999

Iribarne, J.V. and W.L.Godson, Atmospheric Thermodynamics, 259 pages, Reidel Publishing Company, Dordrecht, Holland, 1981

List, R.J. (ed.), *Smithsonian Meteorological Tables*, 527 pages, Smithsonian Institution, Washington, 1951

Paluch, I., The entrainment mechanism is Colorado cumuli, J.Atmos.Sci., 36, 2467-2478, 1979

Table 1 – Adiabatic cloud properties derived using Equations 1-8. Calculations of the cloud-top adiabatic water content and the liquid water path use $z_{ct} - z_{cb} = 500$ m

Liquid Water Content Lapse Rate, g m^{-3} km^{-1} (Equation 3)	P_{cb}=900 hPa	P_{cb}=800 hPa	P_{cb}=700 hPa
T_{cb}= 10 ^{o}C	2.10	1.95	1.80
T_{cb}= 0 ^{o}C	1.61	1.53	1.43
T_{cb}= -10 ^{o}C	1.09	1.05	1.01
Saturated Adiabatic (temperature) Lapse Rate, K km^{-1} (Equation 4)	P_{cb}=900 hPa	P_{cb}=800 hPa	P_{cb}=700 hPa
T_{cb}= 10 ^{o}C	5.09	4.87	4.61
T_{cb}= 0 ^{o}C	6.27	6.03	5.76
T_{cb}= -10 ^{o}C	7.47	7.27	7.03
Liquid Water Content at Cloud Top, g m^{-3} (Equation 7)	P_{cb}=900 hPa	P_{cb}=800 hPa	P_{cb}=700 hPa
T_{cb}= 10 ^{o}C	1.048	0.977	0.899
T_{cb}= 0 ^{o}C	0.805	0.764	0.717
T_{cb}= -10 ^{o}C	0.544	0.525	0.504
Liquid Water Path, g m^{-2} (Equation 8)	P_{cb}=900 hPa	P_{cb}=800 hPa	P_{cb}=700 hPa
T_{cb}= 10 ^{o}C	262	244	225
T_{cb}= 0 ^{o}C	201	191	179
T_{cb}= -10 ^{o}C	136	131	126

Results of Prob03
Integration of First Law (Exact)
Linear Approximation
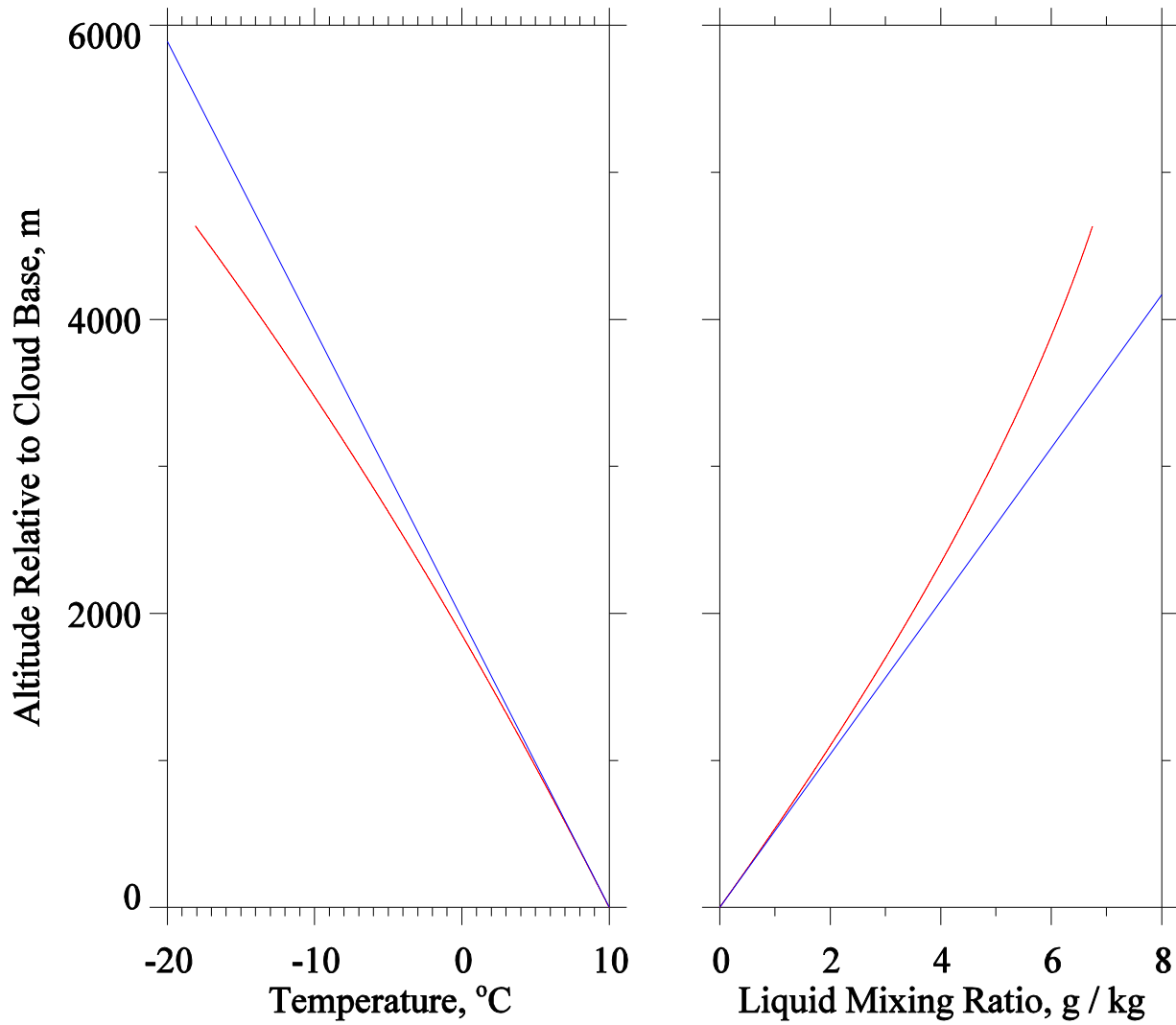
Conclusion : Linear theory works well for shallow clouds (depth < 1.5 km) (see Problem 3 for more detail).
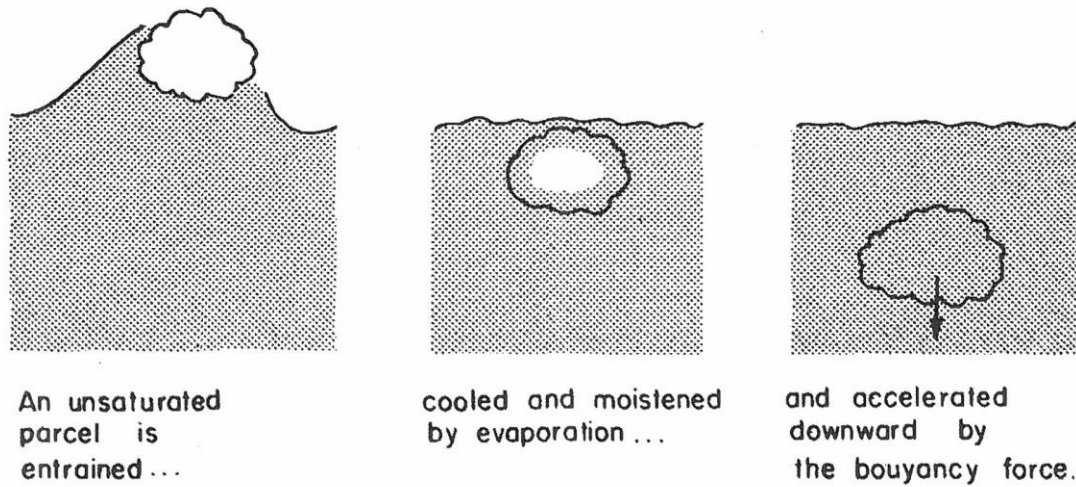
Entrainment/Mixing

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An unsaturated parcel is entrained ...

cooled and moistened by evaporation ...

and accelerated downward by the bouyancy force.

FIG. 4. A schematic illustration of the "parcel" interpretation of CIFKU. The stippled area represents cloudy air.

Consequences of Entrainment/Mixing

a) Entrained parcels are cooled evaporatively and are accelerated downward by the buoyancy force

$$\frac{dU}{dt} = \frac{\rho_{env} - \rho_{parcel}}{\rho_{parcel}} g \qquad \text{or} \qquad \frac{dU}{dt} = \frac{\rho' - \rho}{\rho} g$$

b) Clouds affected by entrainment/mixing contain a liquid amount less than the adiabatic liquid amount

c) Generally, clouds affected by entrainment/mixing contain smaller droplets, compared to the adiabatic prediction

d) Generally, clouds affected by entrainment/mixing produce precipitation slower than clouds unaffected by entrainment/mixing

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Convective Downdraft Model (see Problem 4 for more detail)

Here are the Equations from Rogers and Yau (originally from Squires (1958))

$$\frac{d}{dt}\left(T_p - T\right) = -E \cdot \left(T_p - T\right) - E\frac{L\mu}{C_p} - \left(\Gamma_d - \gamma_c\right)\frac{dz}{dt} \tag{4.21}$$

$$\frac{dU_p}{dt} = \left(\frac{T_p - T}{T}\right)g + \mu g - E \cdot \left(U_p - U\right) \tag{4.22}$$

$$\frac{dw_p}{dt} = -E \cdot \left(w_p - w - \mu\right) \tag{4.23}$$

If there is no cloud, and we take care to distinguish "Atmosphere" from "Parcel", we have this:

$$\frac{d}{dt}\left(T_p - T_{Atm}\right) = -E \cdot \left(T_p - T_{Atm}\right) - \left(\Gamma_d - \Gamma_{Atm}\right) \cdot U_p \tag{1-no cloud}$$

$$\frac{dU_p}{dt} = \left(\frac{T_p - T_{Atm}}{T_{Atm}}\right)g - E \cdot \left(U_p - U_{Atm}\right) \tag{2 - no cloud}$$

$$\frac{dw_p}{dt} = -E \cdot \left(w_p - w_{Atm}\right) \tag{3 - no cloud}$$

$$\frac{dz_p}{dt} = U_p \tag{4 - no cloud}$$

Convective Downdraft Model (Continued)

Here are the Equations from Rogers and Yau Equations modified using notation from ATSC5011

$$\frac{d}{dt}\left(T_p{-}T_c\right) = -E \cdot \left(T_p - T_c\right) - E \cdot \frac{L_v}{C_{pd}} \cdot w_{\ell c} - \left(\Gamma_d - \Gamma_s\right) \cdot U_p \qquad (1)$$

$$\frac{dU_p}{dt} = \left(\frac{T_p - T_c}{T_c}\right)g + w_{\ell c} \cdot g - E \cdot \left(U_p - U_c\right) \qquad (2)$$

$$\frac{dw_p}{dt} = -E \cdot \left(w_p - w_c - w_{\ell c}\right) \qquad (3)$$

$$\frac{dz_p}{dt} = U_p \qquad (4)$$

Variable	Equation 1 T-difference Equation	Equation 2 Parcel Equation of Motion	Equation 3 Parcel Water Budget	Equation 4 Parcel Kinematics
T_p	X	X		
U_p	X	X		X
w_p			X	
T_c	X	X		
U_c		X		
$w_{\ell c}$	X	X	X	
w_c		X	X	
E	X	X	X	
P_c				
Γ_s	X			
z_p				X

In the following notation the subscript "c" indicates a cloud property and subscript "p" indicates a parcel property.

Prognostic variables (i.e., the variables derived by numerical integration):

$$T_p\text{--}T_c \qquad U_p \qquad w_p \qquad z_p$$

Prescribed (static) properties of the cloud:

$$U_c \qquad P_{cb} \qquad z_{cb} \qquad z_{ct} \qquad T_{cb} \qquad E$$

Cloud Profiles – Linear approximations

$$T_c(z) = T_{cb} - \Gamma_s \cdot (z - z_{cb})$$

$$w_{\ell c}(z) = \Gamma_\ell \cdot (z - z_{cb})$$

$$w_c(z) = w_s(T_{cb}, P_{cb}) - \Gamma_\ell \cdot (z - z_{cb})$$

Pressure/altitude transformation: This is applied to both the cloud and the parcel:

$$P(z) = P_{cb} \cdot \exp\left(-(z - z_{cb})/H\right)$$

Parcel initialization:

$$T_p(t = 0) = T_{ct} - 1\,K \qquad\qquad \text{(parcel is negatively buoyant at start; i.e., at cloud top)}$$

$$w_p(t = 0) = 0.5 \cdot w_s\left(T_p(t = 0), P_{ct}\right) \qquad \text{(parcel is subsaturated at start; i.e., at cloud top the RH is 50\%)}$$

$$U_p(t = 0) = 0. \qquad\qquad \text{(parcel is stationary at start; i.e., at cloud top)}$$

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Vertical Profiles of temperature, relative humidity and total mixing ratio. In all three panels, the cloud is the dotted line and the parcel is the solid line. The parcel is initialized at z=2500 m, 1 K cooler than the cloud.
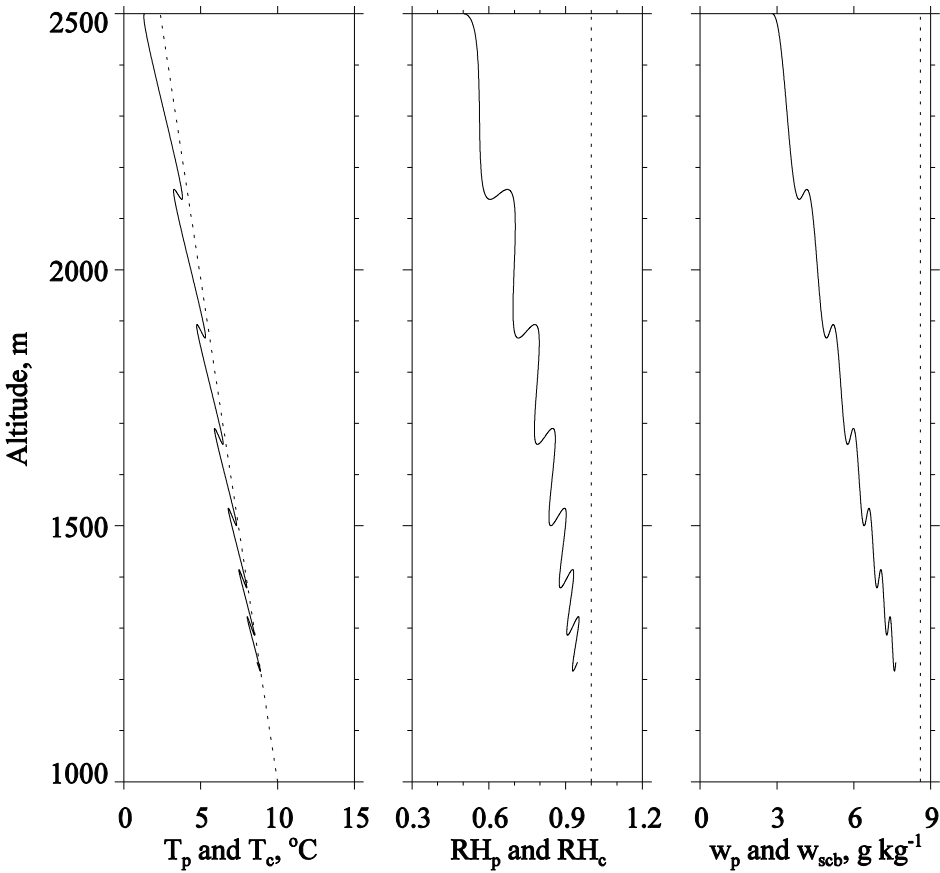
Time-series of parcel altitude, parcel velocity and parcel acceleration:
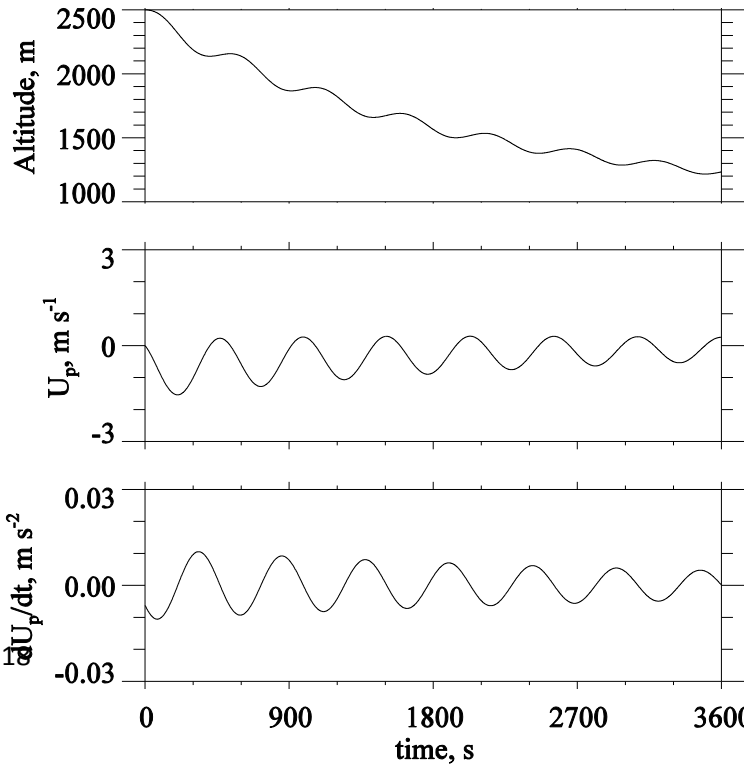
Cloud Microphysics is more than thermodynamics

1) Buoyancy – Parcel vertical velocity is connected to environmental and parcel properties (density and everything that affects density) via the vertical equation of motion:

$$\frac{dU}{dt} = g\,\frac{\rho' - \rho}{\rho} \qquad\qquad (1)$$

For some problems we write Equation 1 this way:

$$U\frac{dU}{dz} = g\,\frac{\rho' - \rho}{\rho} \qquad\qquad (2)$$

These dynamical equations add additional variables – buoyancy, vertical velocity, and time - to the thermodynamic view of clouds.

2) Effects of Entrainment/mixing

     a) Evaporative Cooling – produces negative buoyancy, downdrafts
     b) Condensate dilution – Depletes the liquid amount relative to the adiabatic prediction
     c) Cloud droplet size is decreased
     d) Suppression of precipitation development (i.e., suppression of the conversion of droplets to drops)
     e) In cold clouds, downward transport of ice particles from cloud top toward cloud base

3) Precipitation also depletes liquid amount relative to the adiabatic prediction; that process (precipitation formation) is time dependent

Thermodynamics and Time-dependency (Kinetics) in the Atmospheric Sciences

Thermodynamics describes whether or not a system can, given enough time, transition from one state to another; whereas kinetics addresses the question of how fast such a transition takes place.

Applications of thermodynamics, previously discussed in ATSC5011, are the following: 1) the change of temperature and pressure during reversible/saturated/adiabatic ascent, and 2) the change of temperature and pressure during a reversible/saturated/adiabatic descent. Implicit in these formulations are the laws of thermodynamics, and the assumption that the system (parcel) is closed.

We have also seen that entrainment leads to cloud properties which are somewhat inconsistent with thermodynamics. That is, entrainment/mixing leads to clouds that are colder and contain less liquid compared to the prediction of thermodynamics. We have described entrainment/mixing using a time-dependent model framework. We have also seen that the formation of precipitation, and its fallout, depletes the amount of liquid within a cloud. Liquid water depletion is important for precipitation measured at the ground, for consideration of cloud life cycle, and for atmospheric energetics. The precipitation process is also time dependent.

Nucleation is another topic which requires consideration of time dependence.

Cloud Physics Sensors-

LIGHT SCATTERING AS A MEASURE OF PARTICLE SIZE IN AEROSOLS[1]

The Production of Monodisperse Aerosols

DAVID SINCLAIR[2] AND VICTOR K. LA MER

Department of Chemistry, Columbia University, New York, New York

INTRODUCTION

The measurement of colloid particle size by observation of the scattered light is based primarily on the electromagnetic theory developed by Gustav Mie (16) in 1908. Previously, Rayleigh (18) had derived his well-known equations for scattering by small particles of radius much less than the wave length of the incident light. Since that time, additional investigations, both theoretical and experimental, have been made by Debye (3), Love (15), Blumer (1), Jobst (7), Engelhard and Friess (4), Ray (17), Bromwich (2), Stratton and Houghton (22), Ruedy (19), La Mer and Barnes (10), Johnson and La Mer (8), La Mer and Kenyon (12; see also 9, 11), Van de Hulst (23), and others. The complete theory is given in a compact form by Stratton (21). The pertinent equations of the Mie theory follow.

Consider a plane wave of light incident on any isotropic dielectric sphere of radius, r, small compared to the wave length. The total light energy scattered per second per unit intensity of illumination (unit energy per unit cross-sectional area) is:

$$S = 24\pi^3 \left| \frac{m^2 - 1}{m^2 + 2} \right|^2 \frac{V^2}{\lambda^4} \tag{1}$$

Here V is the volume of the sphere equal to $4\pi r^3/3$, λ is the wave length of the light in the surrounding medium, and m is the real or complex refractive index of the particle relative to the real refractive index of the medium (see appendix A). S is defined as the scattering cross-section of the sphere. Equation 1 is in the form given by Rayleigh (18) and it applies only when $r << \lambda$.

For spheres of any size the general form of equation 1 is:

$$S = \frac{\lambda^2}{2\pi} \sum_{n=1}^{\infty} (2n + 1) \left(|a_n|^2 + |p_n|^2 \right) \tag{2}$$

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See Problem 5 for more detail.

Shortwave Radiative Transfer through Shallow Clouds

The first thing is to define $n(r)$, the particle size distribution function. We do this indirectly by defining the group $n(r) \cdot dr$. This represents the concentration of particles in a narrow size interval between r and $r + dr$. In the MKS system, the dimension of $n(r) \cdot dr$ is per cubic meter (m^{-3}). Hence, the dimension of $n(r)$ is m^{-3} m^{-1}.

Cloud Liquid Water Content:

$$LWC = \frac{4}{3} \cdot \pi \cdot \rho_\ell \cdot \int r^3 n(r) dr \qquad \text{Microphysical Definition}$$

$$LWC \approx \frac{4}{3} \cdot \pi \cdot \rho_\ell \cdot r^3 N \qquad \text{Monodisperse Assumption}$$

Cloud Scattering Coefficient:

$$\beta_{scat} = \pi \int Q_{scat}(r, \lambda) \cdot r^2 n(r) dr \qquad \text{Scattering Theory Definition}$$
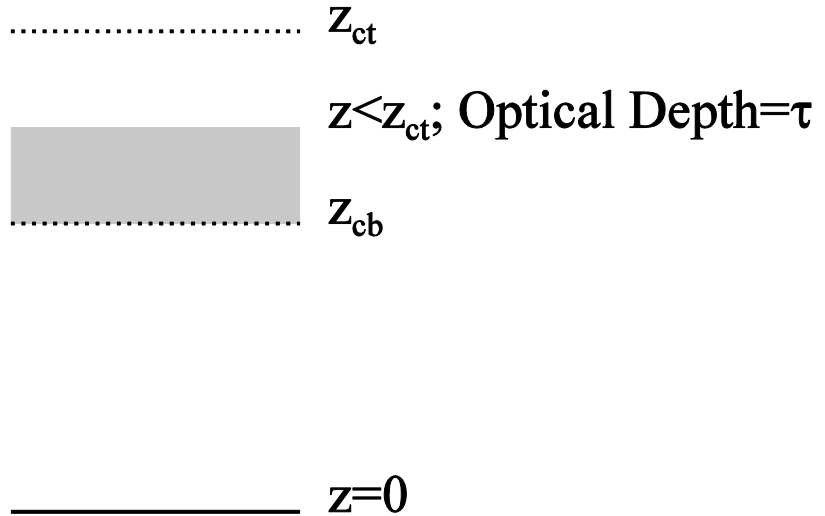
$$\beta_{scat} \approx 2 \cdot \pi \cdot \int r^2 n(r) dr \qquad r \gg \lambda \text{ Assumption}$$

$$\beta_{scat} \approx 2 \cdot \pi \cdot r^2 N \qquad \text{Monodisperse Assumption}$$

http://glossary.ametsoc.org/wiki/Extinction_coefficient

Cloud Scattering Optical Depth:

$$\tau = \int_{z_{cb}}^{z<z_{ct}} \beta_{scat} \cdot dz$$

.... z_{ct}

$z<z_{ct}$; Optical Depth=τ

.... z_{cb}

_____ $z=0$

Cloud Scattering Optical Thickness:

$$\tau_c = \int_{z_{cb}}^{z_{ct}} \beta_{scat} \cdot dz$$

.... $z=z_{ct}$; Optical Thickness=τ_c

.... z_{cb}

_____ $z=0$
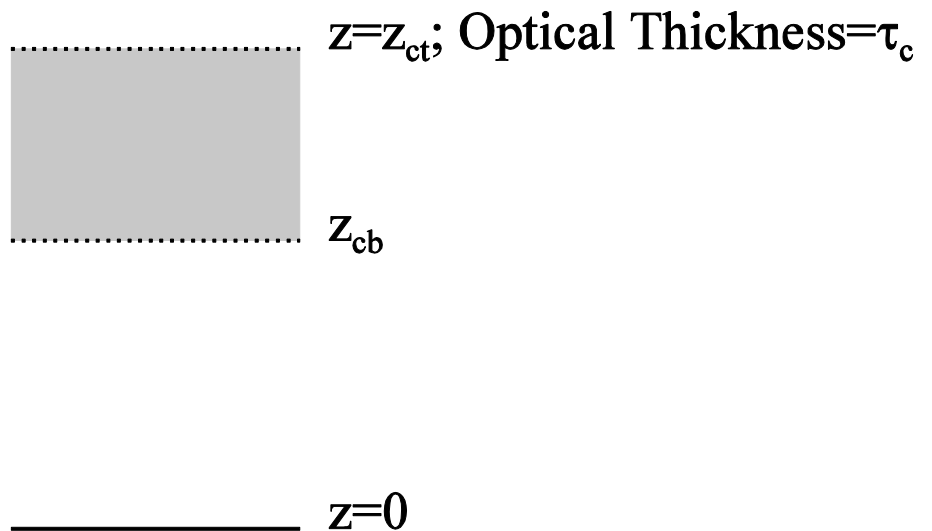
http://glossary.ametsoc.org/wiki/Optical_thickness

Here are differential equations, similar to those presented in Petty, describing radiant fluxes (upwelling and downwelling) within a cloud layer. It is assumed that the cloud droplets attenuate by scattering only and that the droplets are large relative to the wavelength of the radiant energy ($r \gg \lambda$). Both assumptions are reasonable for clouds illuminated by solar radiation.

$$\frac{dF_d}{d\tau} - (F_d - F_u) \cdot (1 - f) = 0 \qquad\qquad (1)$$

$$\frac{dF_u}{d\tau} - (F_d - F_u) \cdot (1 - f) = 0 \qquad\qquad (2)$$

(f is defined on the next page)

Here are boundary conditions:

$$F_d(\tau = \tau_c) = F_s \qquad\qquad\qquad \text{Bdry. Cond. \#1}$$

(Downwelling flux at cloud-top is the solar flux)

$$F_u(\tau = 0) = 0 \qquad\qquad\qquad \text{Bdry. Cond. \#2}$$

(At cloud base, the upwelling flux is zero; this implies that the Earth surface is 100% absorbing)

In a homework assignment you will derive the cloud reflectance as

$$A = \frac{(1-f) \cdot \tau_C}{1+(1-f) \cdot \tau_C}$$

In the previous equations, f is the fraction of scattered energy that propagates into the forward direction. The f can be defined in terms of another quantity (g)

$$f = \frac{1+g}{2}$$

In the previous equation, g is the mean cosine of the scattering angle. Commonly, g is referred to as the asymmetry parameter.

$$g = \frac{1}{4\pi} \cdot \int_0^{2\pi} \int_0^{\pi} cos(\Theta) \cdot p(cos(\Theta)) \cdot sin(\Theta) \cdot d\Theta \cdot d\varphi$$

http://glossary.ametsoc.org/wiki/Asymmetry_factor

In the previous equation, $p(cos(\Theta))$ is the phase function describing the angular distribution of scattered energy.

Here are integral solutions to the differential equations describing the upwelling and downwelling fluxes. In these formulas the vertical coordinate is the cloud scattering optical depth τ, the cloud scattering optical thickness is τ_c (see above for context), and F_s is the solar flux at cloud top.

$$F_d(\tau) = F_s \cdot \frac{(1-f)}{1+(1-f)\cdot\tau_c} \cdot (\tau - \tau_c) + F_s$$

$$F_u(\tau) = F_s \cdot \frac{(1-f)}{1+(1-f)\cdot\tau_c} \cdot \tau$$

Putting it all together:
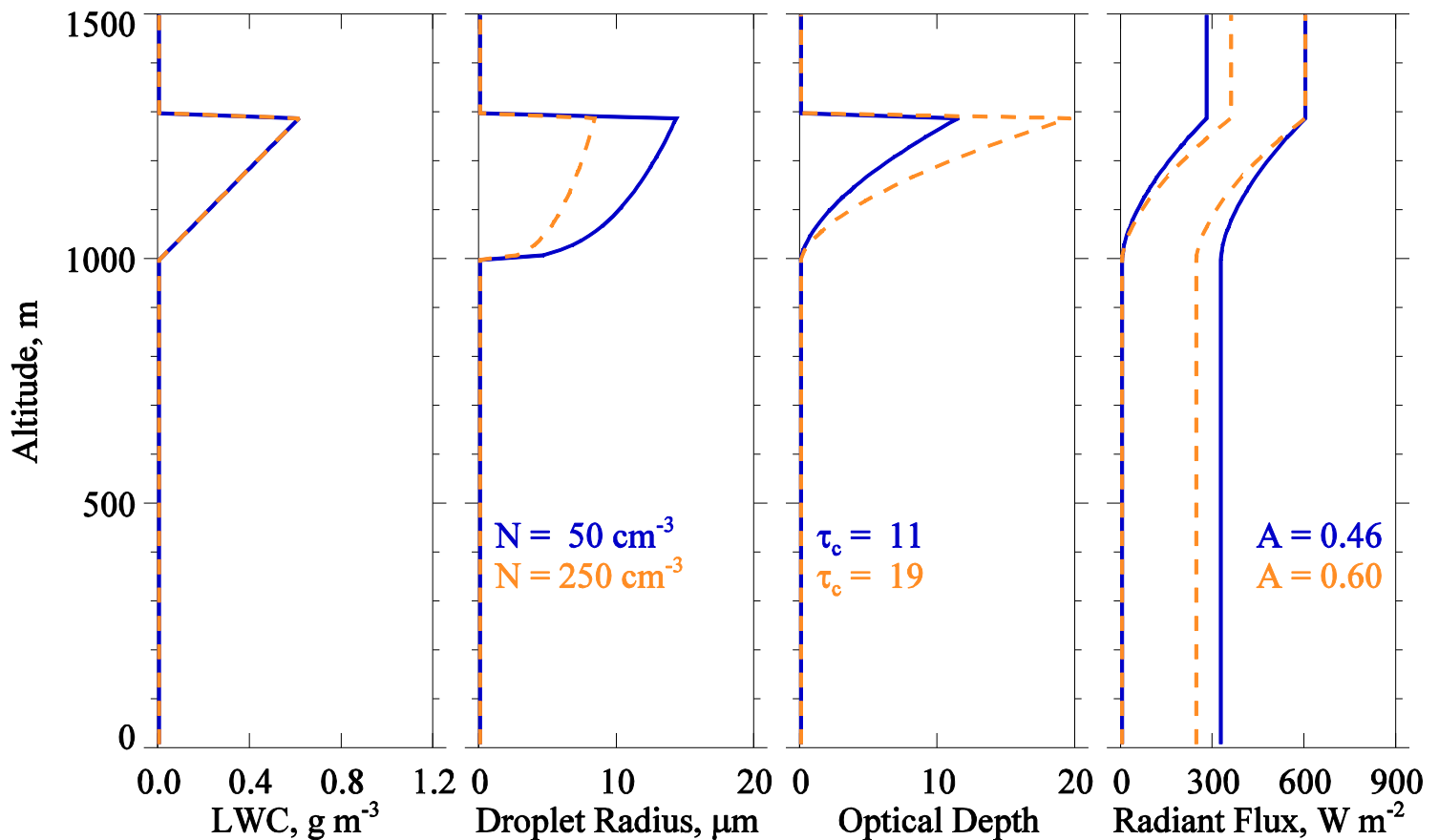
In this graph there are two cloud scenarios. The scenarios are the same thermodynamically, but they differ microphysically. The distinction is that one cloud has fewer droplets ($N=50$ cm^{-3}) compared to the other ($N=250$ cm^{-3}). The cloud with larger N has smaller droplets, larger optical depth, larger optical thickness, larger upwelling flux at cloud top, and is thus more reflective (larger albedo). See Problem 6 for more detail.

Effect of Longwave Radiative Cooling and Precipitation on CAPE – See Problem 7 for more detail

Start	End	Process	Comment
0	1	Unsaturated Adiabatic Ascent	See how Γ_d is defined
1	2	Saturated Adiabatic Ascent (Condensational warming accompanies ascent/expansion/cooling)	See how Γ_s and Γ_l are defined
2	3	2 ^{o}C of Cloud-top Diabatic Cooling (No precipitation is removed in this model run)	This step is isobaric
3	4	Saturated Adiabatic Descent (Evaporational cooling accompanies descent/compression/warming)	See how Γ_s and Γ_l are defined
4	5	Unsaturated Adiabatic Descent	See how Γ_d is defined
5	0	Air warms diabatically	This step is isobaric

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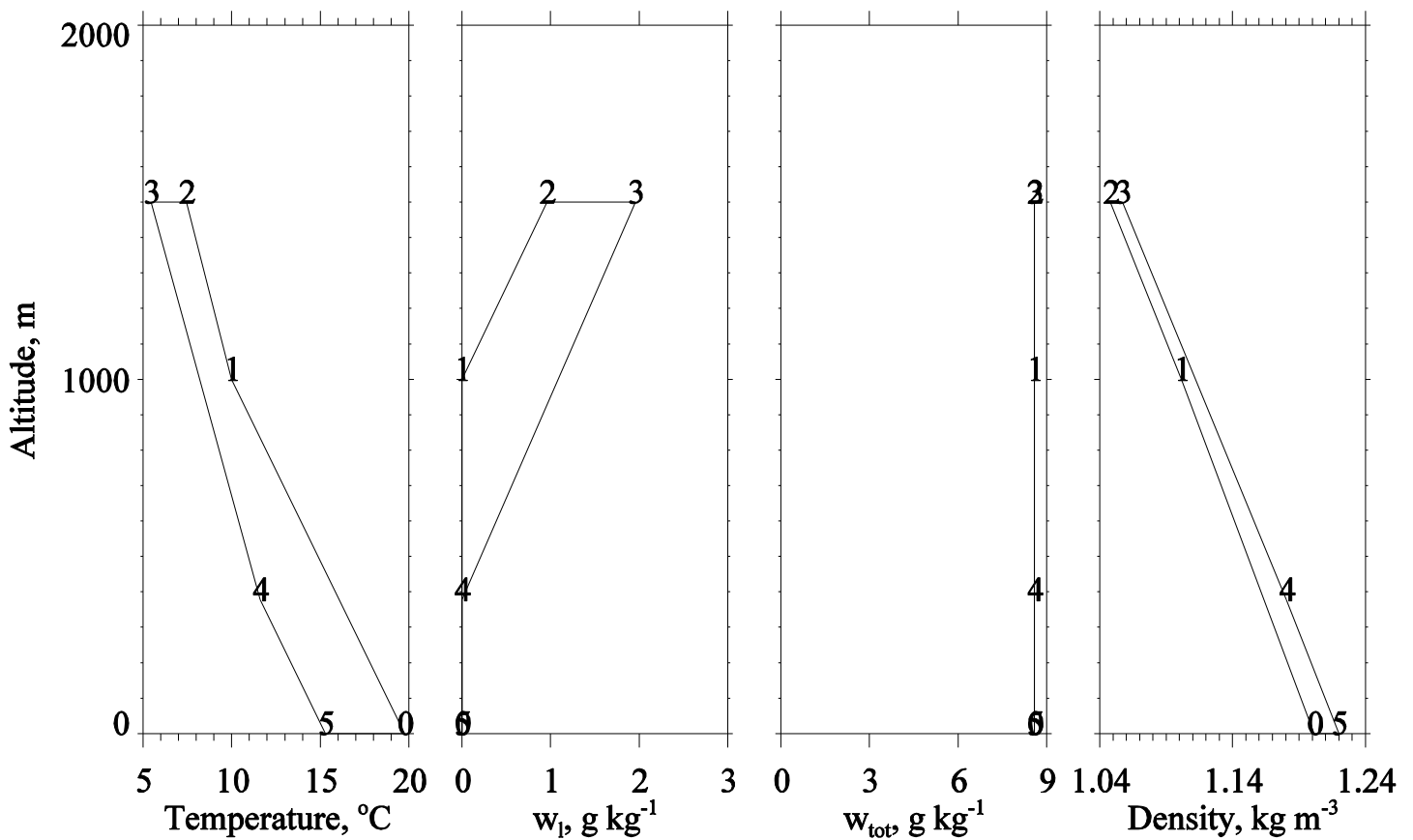
Nucleation-

Water Clusters – The Dimer

$$\theta = 49.9°$$

$$\phi = 0°$$

$$\phi = 90°$$

$$\chi = 180°$$

$$R = 2.952 \, \text{Å}$$

$$\theta = 48.5°$$
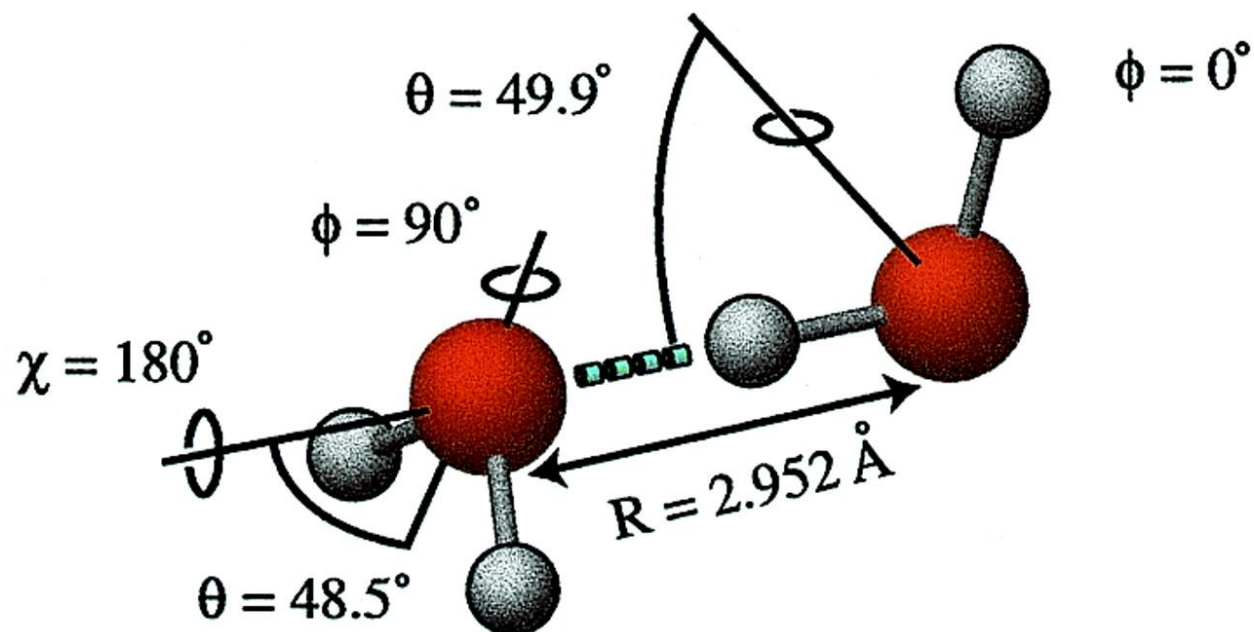
Water Clusters – Some Octamers

Maeda and Ohno

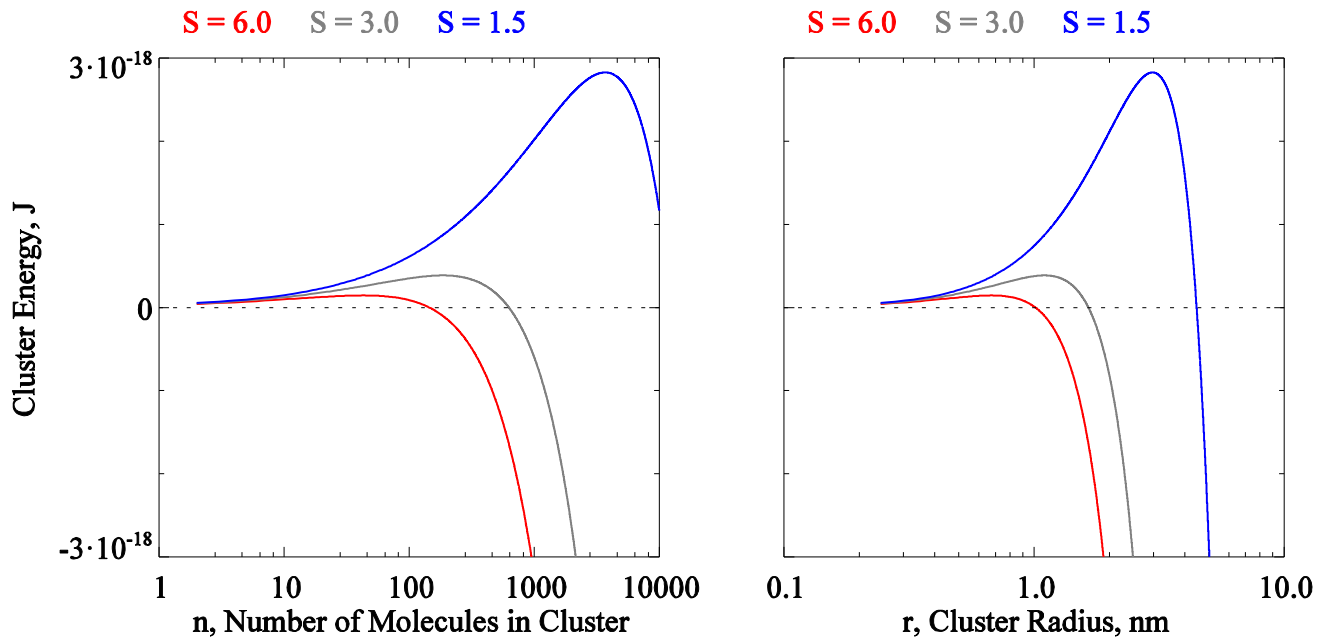EQ1 (D2d) EQ2 (S4) EQ3 (C2) EQ4 (Ci)

EQ5 (C1) EQ6 (C1) EQ7 (C1) EQ8 (C1)

30

Cluster Energy versus Droplet Size Predicted by Homogenous Condensation Theory

The left graph shows cluster energy, predicted by homogeneous condensation nucleation theory, versus number of molecules in the cluster. The right graph shows cluster energy, predicted by homogeneous condensation nucleation theory, versus cluster radius. We conclude the following:

1) Large environmental saturation (S) implies small critical radius and small maximum cluster energy

2) Small environmental saturation (S) large critical radius and large maximum cluster energy

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Nucleation Rate – According to Homogenous Condensation Theory
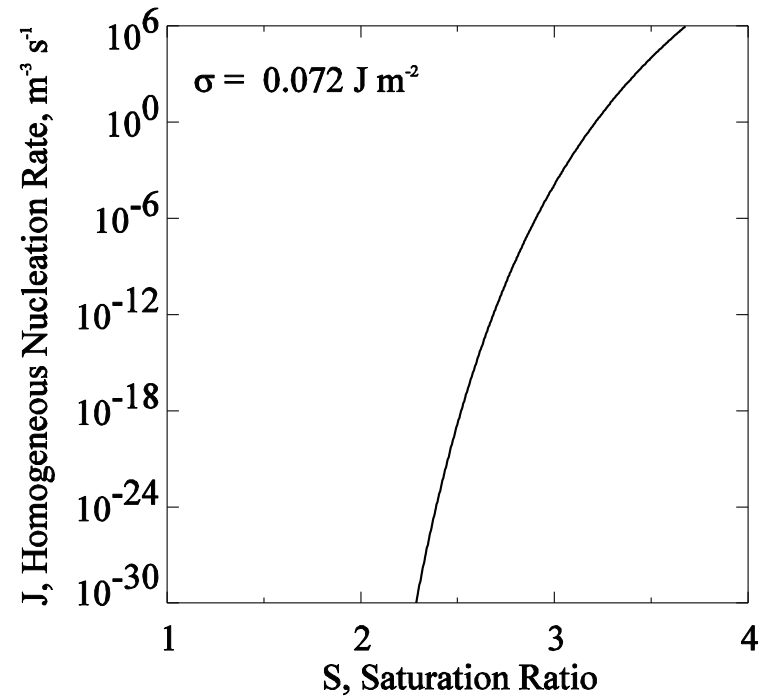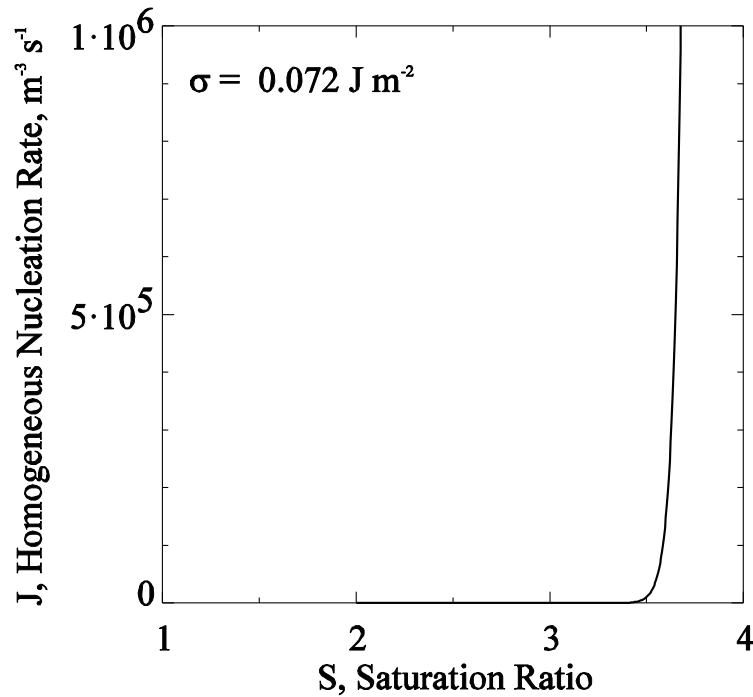
These graphs help us to understand why homogeneous condensation nucleation *cannot* explain the appearance of the cloud droplets in atmospheric clouds. There are two reasons for this. First, assuming an environmental saturation ratio typical of a cloud updraft (S=1.01) the nucleation rate predicted by the theory is way too slow. Put another way, for the theory to explain the nucleation rate within atmosphere clouds (typically ~ 10^6 m^{-3} s^{-1}) the atmospheric saturation ratio would need to be ~ 3.7. Second, observations of cloud base height, for example with a ceilometer or lidar, demonstrate that droplets appear at S=1 (or there abouts), and not at the value S=3.7 necessitated by homogeneous theory
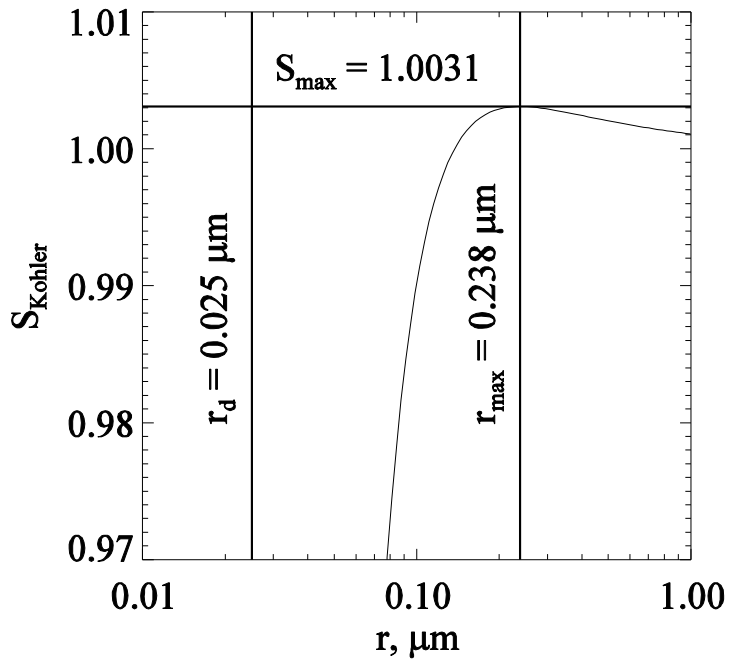
Kelvin Theory-
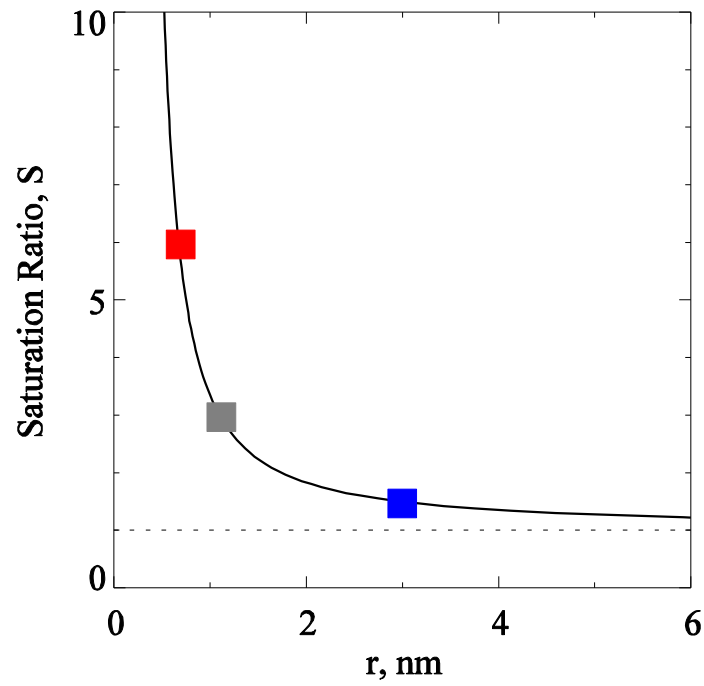
This graph shows the saturation ratio, over a curved droplet, versus the droplet radius. The points correspond to the values of r evaluated at the maximum of the cluster energy curves (two slides back).

$$S_{Kelvin} = \frac{e_r}{e_s} = exp\left(\frac{2 \cdot \sigma}{R_v \cdot T \cdot \rho_\ell \cdot r} \right)$$
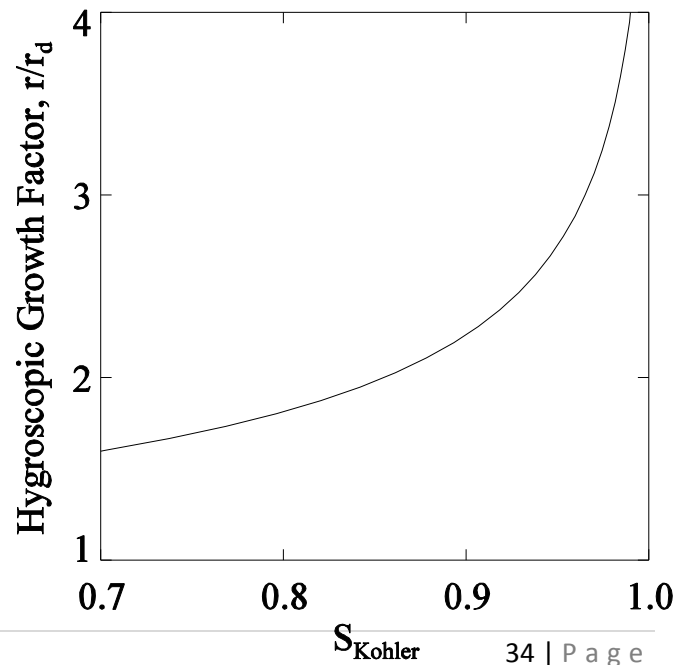
Summary of the Köhler Equation (see Problem 8)

$$S_{Kohler} = 1. + \frac{a}{r} - \frac{b}{r^3} \qquad a = \frac{2\sigma}{R_v T \rho_\ell} \qquad b = \frac{i \cdot M_1 \cdot \rho_2 \cdot r_d^3}{M_2 \cdot \rho_\ell}$$

$$r_{max} = \sqrt{3b/a}$$

$$S_{max} = 1 + \sqrt{4a^3/(27b)}$$

Summary of the Köhler Equation

Like the Kelvin Equation, the Köhler Equation describes the saturation ratio over a droplet. The situation is different for the Köhler Equation, because it includes the solute effect. The latter makes it possible for droplets to exist at environmental saturations less than $S_{env} = 1$. We saw pictures of haze droplets, and evaluated their size, in Problem 8.

In the case of the Köhler Equation, there are two scenarios. These are defined by comparing the saturation ratio, at the maximum of the Köhler curve, to the ambient saturation ratio (S_{env}).
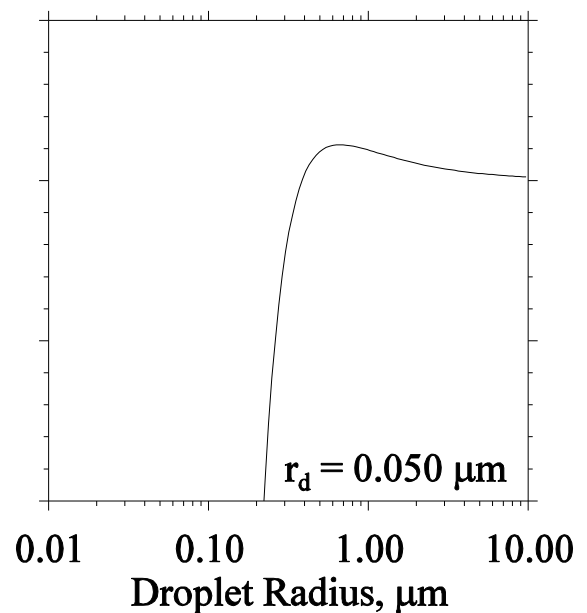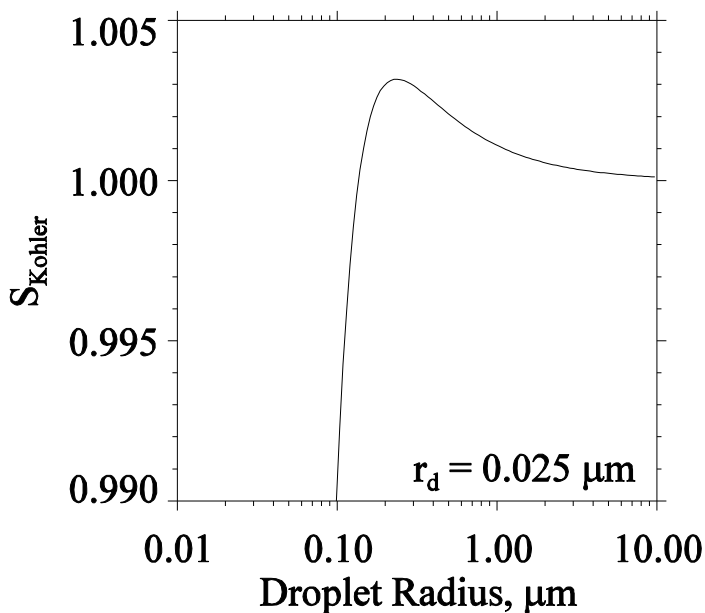
$S_{env} < S_{Kohler,max}$
Haze Droplet
"Unactivated"

$S_{env} > S_{Kohler,max}$
Cloud Droplet
"Activated"

Note: We will see that S_{env} is time-dependent and increases to a maximum, slightly greater than one, a few tens of meters above the lifted condensation level.

Diffusion of Sensible Heat and Diffusion of Mass (see Problem 9 for more detail) -

In class we solved the steady-state diffusion equation in spherical coordinates. Angular symmetry was assumed. Two equations were derived, one for the transport of water vapor and one for the transport of sensible heat. The analysis considered the diffusion equation, the concept of steady state, the Laplacian operator, Fick's law and the gradient operator.

We have the following:

A) Rate of vapor mass transfer *to* the droplet

$$\frac{dm}{dt} = 4 \cdot \pi \cdot r \cdot D \cdot (\rho_{v,env} - \rho_{v,r})$$
(1)

B) Rate of sensible heat transfer *from* the droplet

$$\frac{dQ}{dt} = 4 \cdot \pi \cdot r \cdot K \cdot (T_r - T)$$
(2)

In Equation 2 we have used "T" to represent the environmental temperature. Further, we note that the vapor state, at the droplet surface, is "saturated." It follows that $\rho_{v,r}$ depends (implicitly) on the droplet temperature (T_r). It also follows that (1) and (2) are coupled, both physically and mathematically.
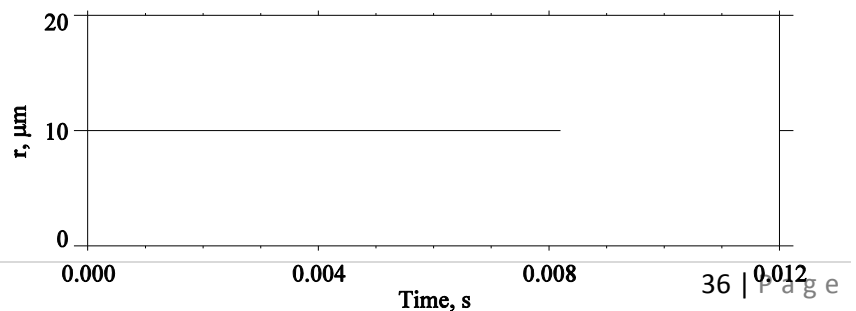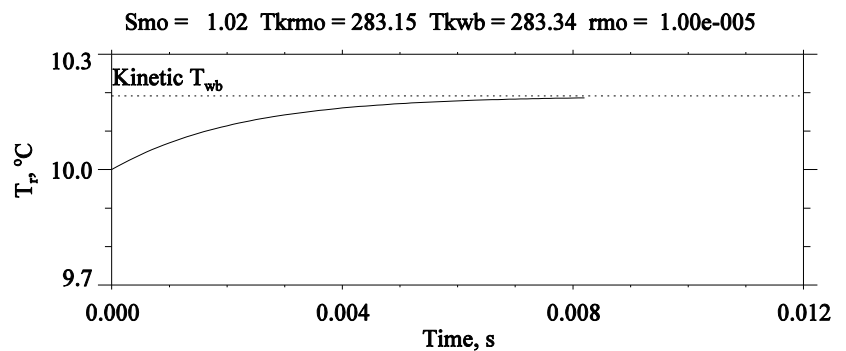
Time to Establish a Steady-state Energy Budget (see Problem 10 for more detail) -

Prognostic equation for droplet temperature

$$\frac{dT_r}{dt} = \frac{3 \cdot L_v(T_r) \cdot D}{C_\ell \cdot \rho_\ell \cdot r^2} \cdot (\rho_{v,env} - \rho_{v,s}(T_r)) - \frac{3 \cdot K}{C_\ell \cdot \rho_\ell \cdot r^2} \cdot (T_r - T)$$

Prognostic equation for droplet radius

$$\frac{dr}{dt} = \frac{D}{\rho_\ell \cdot r} \cdot (\rho_{v,env} - \rho_{v,s}(T_r))$$

Smo = 1.02 Tkrmo = 283.15 Tkwb = 283.34 rmo = 1.00e-005

Derivation of the Maxwell Mason Growth Equation -

In the following, Equations 1 and 2 (previous page) are manipulated to arrive at the Maxwell-Mason growth equation. The derivation starts with a statement of steady-state energy balance:

0= condensational heating rate – sensible heat loss rate

Mathematically, the steady-state energy balance looks like this:

$$0. = L_v \cdot 4 \cdot \pi \cdot r \cdot D \cdot (\rho_{v,env} - \rho_{v,r}) - 4 \cdot \pi \cdot r \cdot K \cdot (T_r - T) \qquad (3a)$$

Rearranging Equation 3a, we have:

$$(T - T_r) + \frac{D \cdot L_v}{K} \cdot (\rho_{v,env} - \rho_{v,r}) = 0. \qquad (3b)$$

Starting with Equation 3b, we define the *environmental* saturation ratio

$$S_{env} = \frac{\rho_{v,env}}{\rho_{v,s}(T)} \qquad (4)$$

By combining (3b) and (4), recalling the definition $S_{Kohler} = \left(1 + \dfrac{a}{r} - \dfrac{b}{r^3}\right)$, and noting that

$$S_{Kohler} = \frac{\rho_{v,r}}{\rho_{v,s}(T_r)} \text{ , we have}$$

$$(T - T_r) + \frac{D \cdot L_v \cdot \rho_{v,s}(T)}{K} \cdot \left(S_{env} - S_{Kohler} \cdot \frac{\rho_{v,s}(T_r)}{\rho_{v,s}(T)}\right) = 0. \qquad (5)$$

The Maxwell-Mason derivation requires a Taylor series simplification of the Clausius-Clapeyron Equation. Here is a statement of the Clausius-Clapeyron Equation relevant to Equation 5:

$$ln\left(\frac{e_s(T_r)}{e_s(T)}\right) = \frac{L_v}{R_v} \cdot \left(\frac{1}{T} - \frac{1}{T_r}\right) = \frac{L_v}{R_v} \cdot \left(\frac{T_r - T}{T_r \cdot T}\right)$$

$$ln\left(\frac{\rho_{v,s}(T_r) \cdot R_v \cdot T_r}{\rho_{v,s}(T) \cdot R_v \cdot T}\right) = \frac{L_v}{R_v} \cdot \left(\frac{T_r - T}{T_r \cdot T}\right) \qquad \text{(ideal gas equation)}$$

$$\ln\left(\frac{\rho_{v,s}(T_r)\cdot T_r}{\rho_{v,s}(T)\cdot T}\right) = \frac{L_v}{R_v}\cdot\left(\frac{T_r - T}{T^2}\right)$$

(approximation of rhs)

$$\ln\left(\frac{\rho_{v,s}(T_r)}{\rho_{v,s}(T)}\right) = \frac{L_v}{R_v}\cdot\left(\frac{T_r - T}{T^2}\right)$$

(approximation of lhs)

$$\frac{\rho_{v,s}(T_r)}{\rho_{v,s}(T)} - 1 = \frac{L_v}{R_v}\cdot\left(\frac{T_r - T}{T^2}\right)$$

(Taylor series expansion of lhs)

$$\frac{\rho_{v,s}(T_r)}{\rho_{v,s}(T)} = \frac{L_v}{R_v}\cdot\left(\frac{T_r - T}{T^2}\right) + 1$$

(6)

Combining (6) and (5), we have this:

$$(T - T_r) + \frac{D\cdot L_v\cdot\rho_{v,s}(T)}{K}\cdot\left(S_{env} - S_{Kohler}\cdot\left(\frac{L_v}{R_v}\cdot\left(\frac{T_r - T}{T^2}\right) + 1\right)\right) = 0.$$

(7)

Rearranging, and simplifying one of the terms with $\rho_{v,s}(T)\cdot S_{Kohler} \approx \rho_{v,s}(T)$, we get an expression for the droplet temperature:

$$(T_r - T) = \frac{D\cdot L_v\cdot\rho_{v,s}(T)\cdot(S_{env} - S_{Kohler})}{K\cdot\left(1 + \frac{L_v^2\cdot D\cdot\rho_{v,s}(T)}{K\cdot R_v\cdot T^2}\right)}$$

(8)

Combining (8), (3b) and (1), we have

$$\frac{dm}{dt} = \frac{4\cdot\pi\cdot r\cdot D\cdot\rho_{v,s}(T)\cdot(S_{env} - S_{Kohler})}{\left(1 + \frac{L_v^2\cdot D\cdot\rho_{v,s}(T)}{K\cdot R_v\cdot T^2}\right)}$$

(9)

Equation 9 describes the droplet's mass growth rate. We note, in contrast to Equation 1, Equation 9 does not depend on T_r.

Cloud droplets are spheres, hence

$$\frac{dm}{dt} = 4 \cdot \pi \cdot \rho_\ell \cdot r^2 \cdot \frac{dr}{dt} \qquad\qquad (10)$$

In Equation 10, ρ_ℓ is the mass density of bulk liquid water (1000 kg/m^3).

Combining (9) and (10)

$$r\frac{dr}{dt} = \frac{1}{\rho_\ell} \cdot \frac{(S_{env} - S_{Kohler})}{\left(\dfrac{1}{D \cdot \rho_{v,s}(T)} + \dfrac{L_v^2}{K \cdot R_v \cdot T^2} \right)} \qquad \text{(Maxwell-Mason Equation)}$$

Extension of Maxwell-Mason Equation to **Cloud Models:**

According to the Maxwell-Mason Equation, $r\dfrac{dr}{dt}$ depends on the following:

1) environmental pressure (via the pressure-dependence of the vapor diffusivity, D),

2) environmental temperature (T) (also, D, $\rho_{v,s}$, K and L_v are temperature dependent),

3) environmental saturation ratio (S_{env}), and

4) droplet radius (via S_{Kohler}).

In cloud models, S_{env} is prognosed using an equation that we develop next. What we are going to see is this:

$$\frac{d(S_{env})}{dt} = f(\text{ State, Updraft, and Condensation Rate })$$

When we couple the Maxwell-Mason Equation with a prognostic equation for S_{env}, we have a system of differential equations that describe the growth of a population of droplets. More detail will be provided after we develop the S_{env} prognostic equation.

The Prognostic S_{env} Equation – Take 1

Here we develop a prognostic equation for the environmental saturation ratio (S_{env}). To simplify the notation, we will symbolize S_{env} as S. The equation we develop, combined with the Maxwell-Mason growth equation, and other prognostic relationships, allows us to predict the time evolution of a population of droplets within an ascending cloud parcel.

Before we start let's think about how humidity varies within an air parcel as it approaches the LCL from below. According to thermodynamics the vapor mixing ratio is constant ($dw/dz = 0$) and saturation increases ($dS/dz > 0$). Also according to thermodynamics, $dS/dz > 0$ switches to constant saturation ($S = 1$) at the LCL. In fact, at altitudes above the LCL, S increases beyond $S = 1$. This result is not anticipated by thermodynamics.

Our derivation of the prognostic S equation starts with definitions for the vapor mixing ratio (w) and the saturation ratio (S). After an approximation, these definitions are combined in Equation 1c.

$$w = \frac{\varepsilon \cdot e}{p - e} \cong \frac{\varepsilon \cdot e}{p} \tag{1a}$$

$$S = \frac{e}{e_s} \tag{1b}$$

$$w = \frac{\varepsilon \cdot S \cdot e_s}{p} \tag{1c}$$

By taking the time-derivative of Equation 1c, and applying the chain rule, the time-derivative of S can be related to dw/dt, de_s/dt and dp/dt

$$\frac{1}{\varepsilon} \cdot \frac{dw}{dt} = \frac{e_s}{p} \cdot \frac{dS}{dt} + \frac{S}{p} \cdot \frac{de_s}{dt} - \frac{S \cdot e_s}{p^2} \cdot \frac{dp}{dt}. \tag{2}$$

Our objective is to formulate dS/dt as a function of vertical velocity (U) and the time derivative of vapor mixing ratio (dw/dt). We begin by acknowledging the temperature-dependence of e_s. This comes from the differential form of the Clausius-Clapeyron equation:

$$\frac{de_s}{dt} = \frac{de_s}{dT} \cdot \frac{dT}{dt} = \frac{L_v \cdot e_s}{R_v \cdot T^2} \cdot \frac{dT}{dt} \tag{3}$$

Next, we start with the differential form of the hydrostatic equation, and develop a relationship between dp/dt and U.

$$\frac{dp}{dz} = -g \cdot \rho \tag{4a}$$

$$dp = -g \cdot \rho \cdot dz \tag{4b}$$

$$\frac{dp}{dt} = -g \cdot \rho \cdot \frac{dz}{dt} = -g \cdot \rho \cdot U \tag{4c}$$

$$\frac{1}{p} \cdot \frac{dp}{dt} = -g \cdot \frac{\rho}{p} \cdot U = -g \cdot \frac{1}{R_{mix} \cdot T} \cdot U \tag{4d}$$

In the last step (Equation 4d), the gas law for the air-vapor mixture is applied. We note that R_{mix}, the gas constant for an air-vapor mixture, depends on w (see earlier ATSC5011 notes). Approximating R_{mix} as R_d, we derive Equation 4e

$$\frac{1}{p} \cdot \frac{dp}{dt} \cong -\frac{g \cdot U}{R_d \cdot T} \tag{4e}$$

Equations 2, 3, and 4e are not sufficient for our objective of describing dS/dt as a function of U and dw/dt. The final ingredient comes from consideration of the first law of thermodynamics. We start with the differential and extensive form of the first law

$$dH = \delta Q + V \cdot dp. \tag{5a}$$

The process we are envisioning is adiabatic ($\delta Q = 0$). Further, we acknowledge that the enthalpy differential has contributions from both sensible and latent effects

$$\left(c_{pd} \cdot m_d + c_{pv} \cdot m_v + c_\ell \cdot m_\ell \right) \cdot dT + L_v \cdot dm_v = V \cdot dp. \tag{5b}$$

In Equation 5b the m's are the masses of the three components (dry air, vapor and liquid) and the term in parenthesis is the parcel's overall heat capacity. If we divide Equation 5b by $(m_d + m_v + m_\ell)$, and on the lhs we ignore terms with m_v or m_ℓ, the approximate result is Equation 5c.

$$c_{pd} \cdot dT + L_v \cdot dw \cong v \cdot dp \tag{5c}$$

On the rhs of Equation 5c, v is the ratio of the parcel's volume and mass, and we note that the reciprocal of v is the parcel density.

In the next two steps we manipulate Equation 5c, making the differentials into time-derivatives,

$$c_{pd} \cdot \frac{dT}{dt} + L_v \cdot \frac{dw}{dt} = v \cdot \frac{dp}{dt} \tag{5d}$$

and we also solve for the temperature-time derivative, and substitute Equation 4c for dp/dt. At that point we acknowledge that the volume-to-mass ratio (v) is the reciprocal of density (ρ) and we end up with a relation between the temperature-time derivative, U and dw/dt.

$$\frac{dT}{dt} = -\frac{g}{c_p} \cdot U - \frac{L_v}{c_p} \cdot \frac{dw}{dt} \tag{5e}$$

Now we can express dS/dt in terms of U and dw/dt. To do that we put Equations 3, 4e and 5e into Equation 2

$$\frac{1}{\varepsilon} \cdot \frac{dw}{dt} = \frac{e_s}{p} \cdot \frac{dS}{dt} - \frac{S}{p} \cdot \frac{L_v \cdot e_s}{R_v \cdot T^2} \cdot \left(\frac{g}{c_{pd}} \cdot U + \frac{L_v}{c_{pd}} \cdot \frac{dw}{dt} \right) + \frac{S \cdot e_s}{p} \cdot \left(\frac{g \cdot U}{R_d \cdot T} \right)$$

With some algebra we get Equation 6

$$\frac{dS}{dt} = S \cdot \left(\frac{L_v \cdot g}{c_{pd} \cdot R_v \cdot T^2} - \frac{g}{R_d \cdot T} \right) \cdot U + \left(\frac{p}{\varepsilon \cdot e_s} + \frac{S \cdot L_v^2}{c_{pd} \cdot R_v \cdot T^2} \right) \cdot \frac{dw}{dt} \tag{6}$$

We note that if parcel motion is not accompanied by condensation, i.e. if an air parcel is subsaturated and thus $S < 1$, the vapor mixing ratio prognostic (dw/dt) is zero. Parcel ascent or descent below the LCL, are two examples of this. For parcels that are saturated ($S = 1$), or supersaturated ($S > 1$), we can set $dw/dt = -dw_\ell/dt$, implying w_{tot} conservation. In either of these cases (i.e., $S = 1$ or $S > 1$) we can substitute $S = 1$ on the rhs of Equation 6. The latter substitution is valid because an upper-limit for S is approximately 1.02. In the saturated or supersaturated cases the prognostic S equation is:

$$\frac{dS}{dt} = \left(\frac{L_v \cdot g}{c_{pd} \cdot R_v \cdot T^2} - \frac{g}{R_d \cdot T} \right) \cdot U - \left(\frac{p}{\varepsilon \cdot e_s} + \frac{L_v^2}{c_{pd} \cdot R_v \cdot T^2} \right) \cdot \frac{dw_\ell}{dt} \tag{7}$$

.........S source Term………   ……… S Sink Term…………

Now we can understand why S increases beyond $S = 1$ within an air parcel that ascends through the LCL. Examples are shown below. At the time when $S = 1$ (i.e. when the air parcel is at the LCL) the droplets are haze droplets (unactivated) and these have small radii (~ 0.2 µm; p. 34). A haze droplet takes up vapor slowly because its size is constrained by the Köhler curve. This is evident on p.34 where we see that the left branch of the Köhler curve is almost vertical. In general, the haze droplets are unable to take up vapor at the rate it is being supplied by parcel expansion, and consequently the saturation "overshoots" $S = 1$. With ascent beyond $S = 1$ some of the haze droplets activate; the growth history of these activated droplets is now described by the Maxwell-Mason Equation. However, since the freshly-activated droplets are relatively small (radius ~ 0.5 µm; p. 34), and because their mass growth rate is proportional to their radius (Maxwell Mason Equation p. 36), these still cannot take up vapor at the rate it is being provided by the expansion. Consequently, S continues to increase but at a slower rate. As soon as the activated droplets grow a bit larger (radius ~ 1 µm) they can diminish the vapor in excess of saturation and thus draw down the saturation toward $S = 1$.
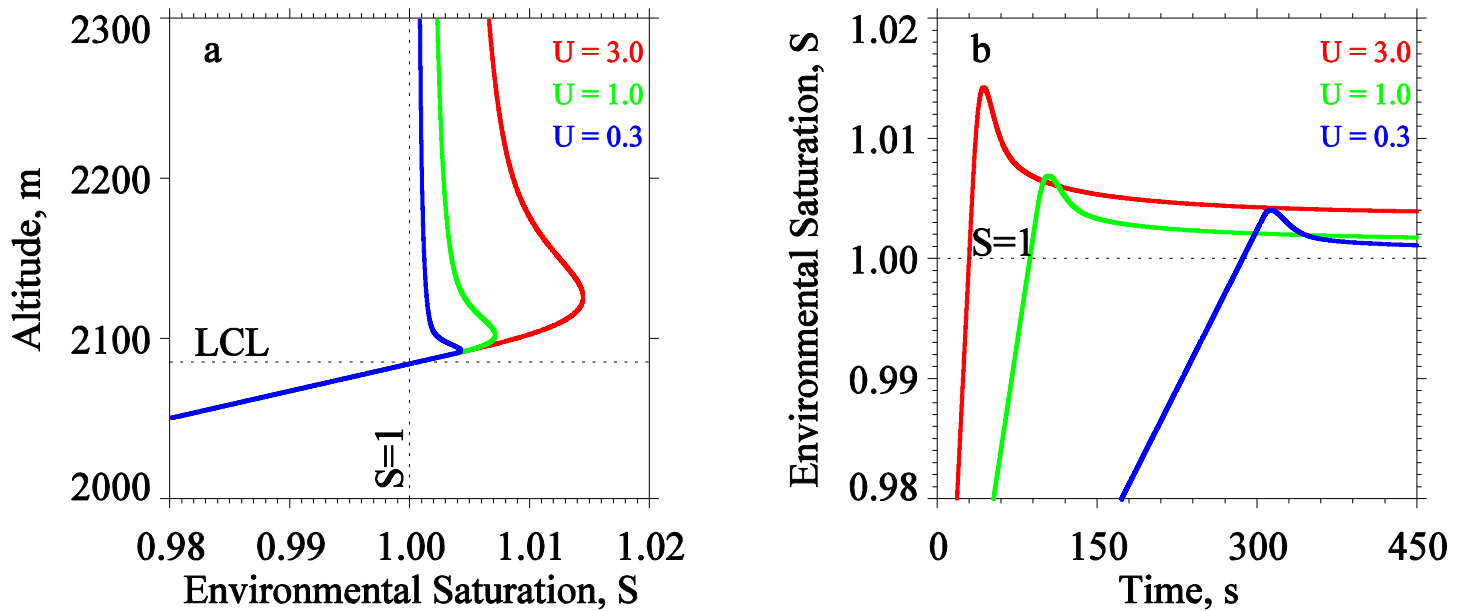
Summary

1) Below the LCL the source term is the only term contributing to the tendency and saturation increases with increasing time. The lines in Figure b show this result for three updraft velocities (3.0, 1.0 and 0.3 m s^{-1}). 2) As an air parcel passes through the LCL there is some uptake of vapor by the haze droplets but this is small; therefore source > sink and dS / dt continues to increase. 3) Only once the haze droplets activate, and thus grow in accord with the Maxwell Mason Equation, can the sink term dominate the source term; beyond that point the dS / dt turns negative and S relaxes toward $S = 1$.

The Prognostic S_{env} Equation – Take 2

In problem session 11 we use a form of the prognostic S equation different for Equation 7. This is obtained by combining Equations 2, 3 and 4e with the w_{tot} conservation assumption and by substituting $S = 1$ in two places on the rhs of Equation 2.

$$\frac{dS}{dt} = -\frac{g}{R_d \cdot T} \cdot U \quad -\frac{1}{w} \cdot \frac{dw_\ell}{dt} \quad -\frac{L_v}{R_v \cdot T^2} \cdot \frac{dT}{dt} \tag{8}$$

We also apply a prognostic equation for environmental temperature. This is obtained by combining Equation 5e with the w_{tot} conservation assumption.

$$\frac{dT}{dt} = -\frac{g}{c_{pd}} \cdot U \quad +\frac{L_v}{c_{pd}} \cdot \frac{dw_\ell}{dt} \tag{9}$$

Finally, we have prognostics for altitude, pressure, liquid water and droplet size. The last of these depends on whether a droplet is unactivated (haze droplet) or activated (cloud droplet). The complete set of ordinary differential equations is shown on the following page. In problem 11 the set of prognostic equations is integrated using the IDL-built function RK4. We will refer to this as the "condensational parcel model."

Summary

A system of ordinary differential equations is now developed to describe the time-evolution of a population of droplets within an ascending cloud parcel. The prognostic variables are environmental saturation (S), environmental temperature (T), parcel altitude (z), pressure (p), liquid water mixing ratio (w_ℓ) and droplet size (r).

```
function derivative_haze_droplet, t, x
common state, state
dz_dt      = !U
dp_dt      = -!g*state.rhoa*!U
dS_dr      = -!kohler_a/x[2]^2 + 3.*!kohler_b/x[2]^4      ; Skohler versus r derivative
dwl_dt     = 0d ; intentionally ignoring this term
dT_dt      = -(!g/!Cpd)*!U + (lv(state.Tenv)/!Cpd)*dwl_dt
dS_dt      = -!g*!U/(!Rd*x[5]) - (1./state.w)*dwl_dt - (lv(state.Tenv)/(!Rv*x[5]^2))*dT_dt
return, [dz_dt, $
        dp_dt, $
        (1/dS_dr)*dS_dt, $
        dwl_dt, $
        dS_dt, $
        dT_dt]
end
;
;
function derivative_cloud_droplet, t, x
common state, state
dz_dt      = !U
dp_dt      = -!g*state.rhoa*!U
dr_dt      = (1/x[2])*(1/!rhol)*state.G*(x[4] - 1. - !kohler_a/x[2] + !kohler_b/x[2]^3)
dwl_dt     = 4*!pi*!rhol*!nmo*x[2]^2*dr_dt
dT_dt      = -(!g/!Cpd)*!U + (lv(state.Tenv)/!Cpd)*dwl_dt
dS_dt      = -!g*!U/(!Rd*x[5]) - (1./state.w)*dwl_dt - (lv(state.Tenv)/(!Rv*x[5]^2))*dT_dt
return, [dz_dt, $
        dp_dt, $
        dr_dt, $
        dwl_dt, $
        dS_dt, $
        dT_dt]
end
```
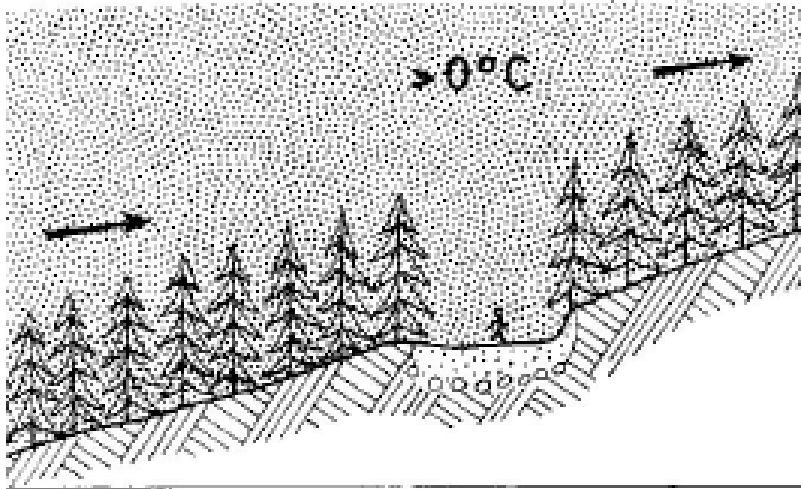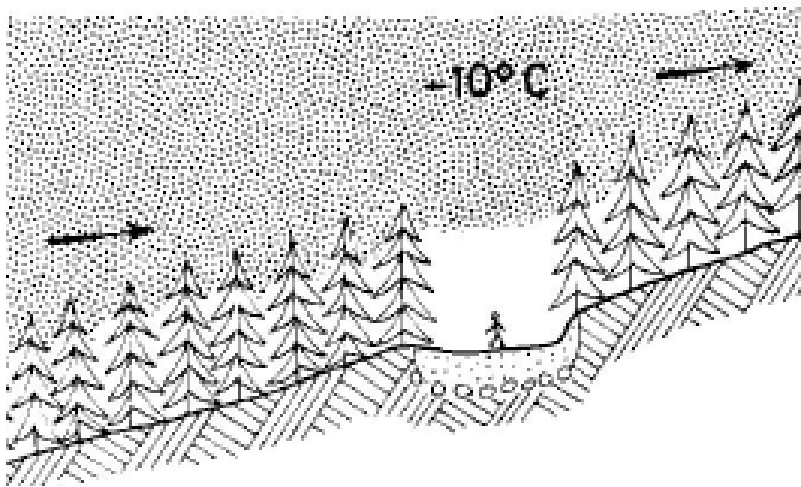
What's going on here?

Bergeron, 1935:
From field observations he concluded about the importance of mixed-phase (supercooled) clouds for precipitation
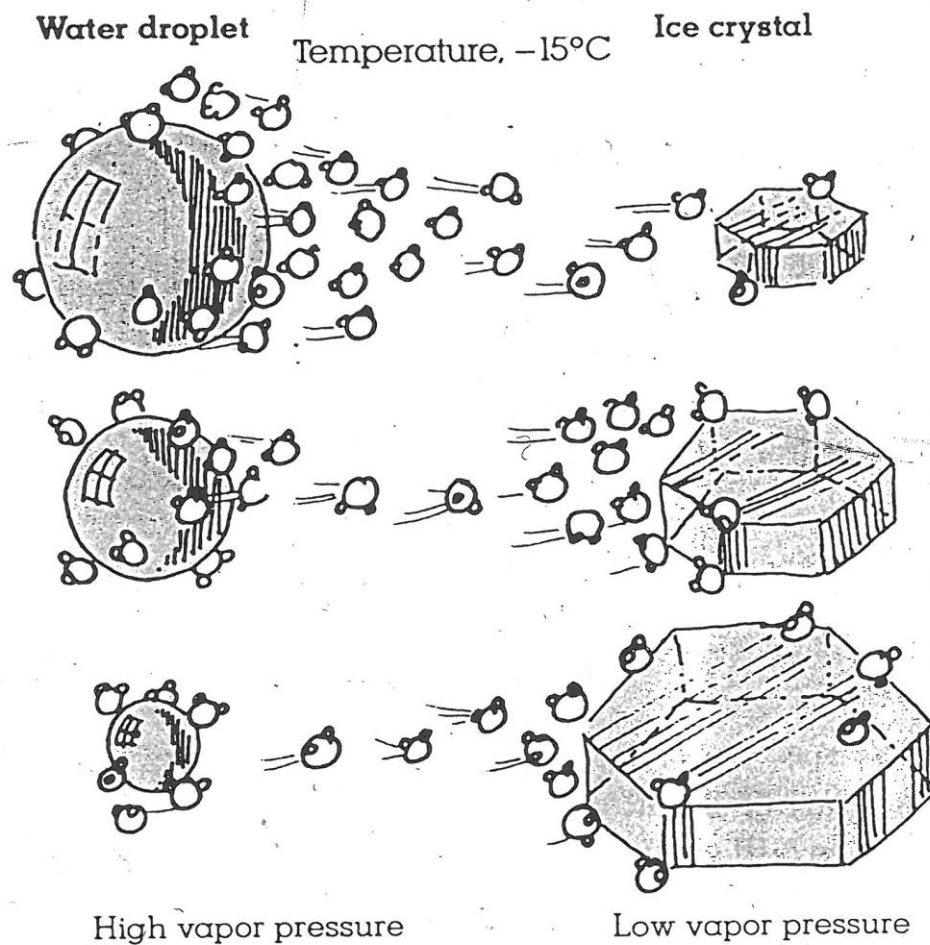

Environmental Saturation Ratio in a "Warm Cloud", from the Perspective of the Cloud Droplets:

$$S = \frac{e}{e_s} = \frac{\rho_v}{\rho_{v,s}}$$

Environmental Saturation Ratio in a Mixed-phase Cloud, from the Perspective of an Ice Crystal:

$$S_i = \frac{e_s}{e_i} = \frac{\rho_{v,s}}{\rho_{v,i}}$$

Water droplet

Temperature, −15°C

Ice crystal

High vapor pressure

Low vapor pressure

FIGURE 9.6   The ice-crystal process. The greater vapor pressure around the liquid droplets causes the ice crystals to grow by diffusion as water vapor molecules move from the liquid droplets toward the ice crystals. The ice crystals absorb the water vapor and grow larger, while the water droplets grow smaller.
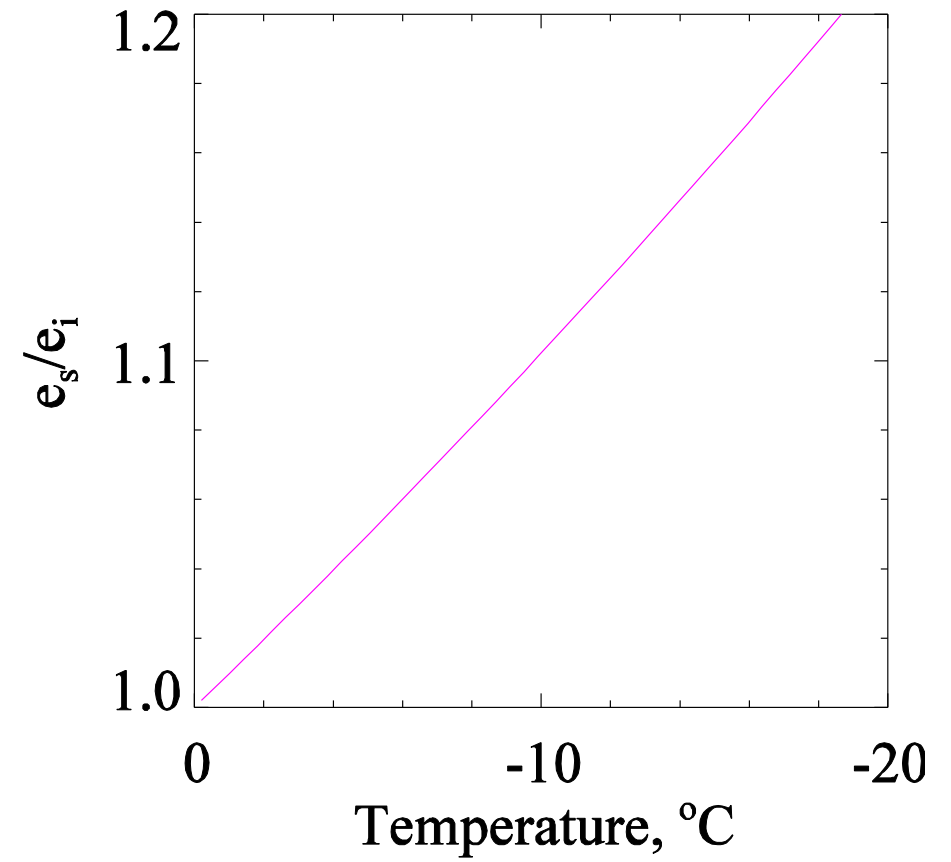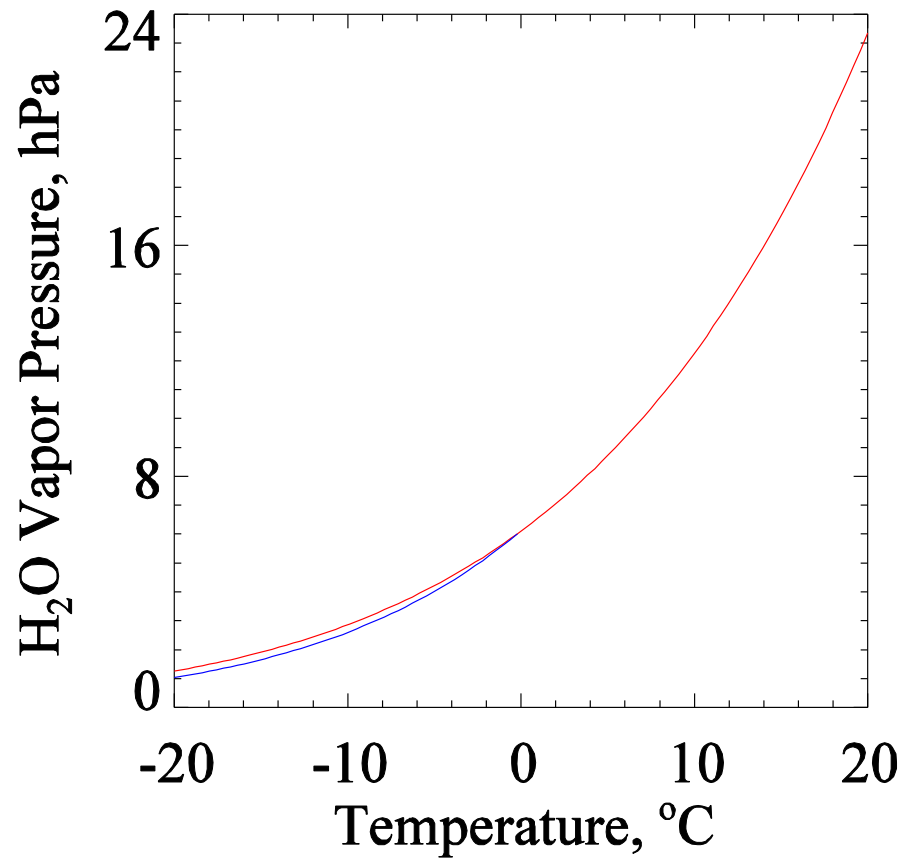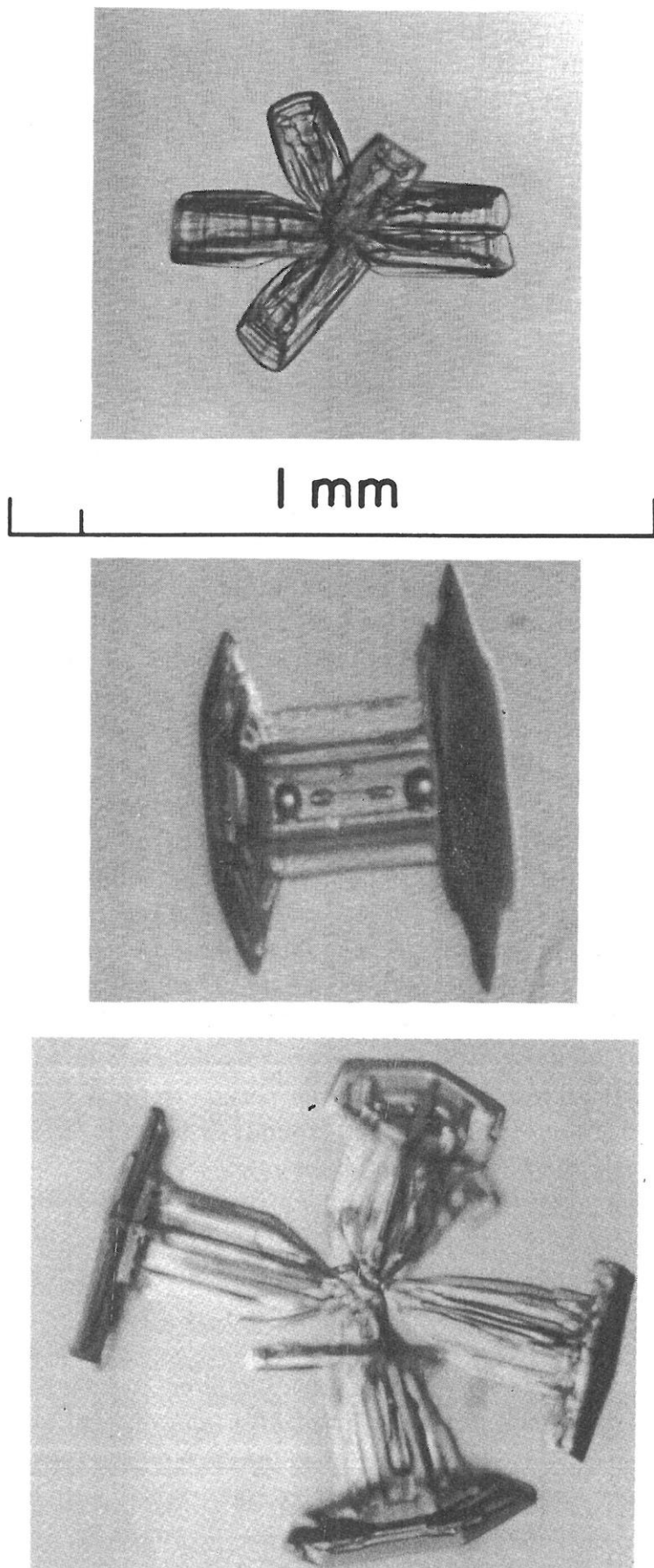
FIG. 6.6. Examples of complex ice crystal habits. (a) Bullet rosette (C2a); (b) capped column (CP1a); (c) bullet rosette with plates (CP2a) (courtesy of C.A. Knight).