Derivation of the Equation of Motion

Newton's Second Law applies only in a non-accelerating frame of reference. Since the earth rotates, it is an accelerating frame of reference and thus Newton's Laws do not hold for time scales more than a few minutes. To apply Newton's Law of Motion, we need to consider the effects of a platform that rotates with a constant angular velocity. To start, consider a vector in an inertial (non-accelerating) system:

$$\vec{A} = Ax\vec{i} + Ay\,\vec{j} + Az\,\vec{k}$$

Let's also consider the same vector in a rotating (accelerating) system:

$$\vec{A} = Ax'\vec{i}\,' + Ay'\vec{j}\,' + Az'\vec{k}\,'$$

For the inertial frame of reference, take the time derivative:

$$\frac{d\vec{A}}{dt} = \frac{dAx}{dt}\vec{i} + \frac{dAy}{dt}\vec{j} + \frac{dAz}{dt}\vec{k} \quad \textit{note that } \frac{d\vec{i}}{dt} = 0, \textit{ etc.}$$

For the rotating frame of reference, also take the time derivative:

$$\frac{d\vec{A}}{dt} = \underbrace{\frac{dAx'}{dt}\vec{i}\,' + \frac{dAy'}{dt}\vec{j}\,' + \frac{dAz'}{dt}\vec{k}\,'}_{\text{Time rate of change of A with respect to rotating system}} + \underbrace{Ax'\frac{d\vec{i}\,'}{dt} + Ay'\frac{d\vec{j}\,'}{dt} + Az'\frac{d\vec{k}\,'}{dt}}_{\text{Time rate of change of rotating system}}$$

Note terms such as $\frac{d\vec{i}\,'}{dt}$ relate to change of the unit vector relative to the rotating earth with time. Such a change represents the effect of rotation of the relative coordinate system and is not zero. It can be shown that $\frac{d\vec{i}\,'}{dt} = \vec{\Omega}\times\vec{i}\,'$, etc. Thus, above equation can be written:

$$\frac{d\vec{A}}{dt} = \frac{dAx'}{dt}\vec{i}\,' + \frac{dAy'}{dt}\vec{j}\,' + \frac{dAz'}{dt}\vec{k}\,' + Ax'\left(\vec{\Omega}\times\vec{i}\,'\right) + Ay'\left(\vec{\Omega}\times\vec{j}\,'\right) + Az'\left(\vec{\Omega}\times\vec{k}\,'\right), \text{ or,}$$

$$\frac{d\vec{A}}{dt} = \frac{dAx'}{dt}\vec{i}\,' + \frac{dAy'}{dt}\vec{j}\,' + \frac{dAz'}{dt}\vec{k}\,' + \left(\vec{\Omega}\times Ax'\vec{i}\,'\right) + \left(\vec{\Omega}\times Ay'\vec{j}\,'\right) + \left(\vec{\Omega}\times Az'\vec{k}\,'\right), \text{ and}$$

$$\frac{d\vec{A}}{dt}=\frac{dAx'}{dt}\vec{i}\,'+\frac{dAy'}{dt}\vec{j}\,'+\frac{dAz'}{dt}\vec{k}\,'+\left(\vec{\Omega}\times\vec{A}\right),\text{ since }\vec{A}=Ax'\vec{i}\,'+Ay'\vec{j}\,'+Az'\vec{k}\,'$$

We can write this equation as:

$$\frac{d\vec{A}}{dt}=\left(\frac{d\vec{A}}{dt}\right)_r+\left(\vec{\Omega}\times\vec{A}\right), \tag{1}$$

where the subscript r refers to the *relative frame of reference,* which in our case is the earth.

This equation states that the total time derivative of vector A with time can be written in terms of the total time derivative of vector A *relative to the rotating system* plus the cross product of the angular velocity vector and vector A. We can apply (1) to the position vector $\boldsymbol{r}$ (we can think of r as the distance vector from the center of the earth to a point on the earth) to get the following:

$$\frac{d\vec{r}}{dt}=\left(\frac{d\vec{r}}{dt}\right)_r+\left(\vec{\Omega}\times\vec{r}\right)$$

The first term on the left-hand side is the velocity in an absolute or non-accelerating frame of reference. The first term on the right-hand side is the velocity vector with respect to the relative frame of reference, the earth. The second term on the right-hand side is the velocity of a point on the earth's surface. We can thus write the above equation as:

$$\vec{V}_a=\vec{V}_r+\vec{V}_e\ , \tag{2}$$

For our application, V_r will be the velocity of the air relative to the earth, or simply the vector wind.

If we once again apply (1) but this time to the velocity with respect to the non-accelerating coordinate system (or the absolute velocity V_a), we get:

$$\frac{d\vec{V}_a}{dt} = \left(\frac{d\vec{V}_a}{dt}\right)_r + \left(\vec{\Omega}\times\vec{V}_a\right)$$

The equation states that the time rate of change of the absolute velocity vector (also known as the absolute acceleration) can be expressed in terms of the time rate of change of the velocity vector with respect to the rotating earth plus the term omega cross the absolute velocity. From (2) we can rewrite the above equation as:

$$\frac{d\vec{V}_a}{dt} = \left(\frac{d\left(\vec{V}_r+\vec{V}_e\right)}{dt}\right)_r + \left(\vec{\Omega}\times\left(\vec{V}_r+\vec{V}_e\right)\right), \text{ or}$$

$$\frac{d\vec{V}_a}{dt} = \left(\frac{d\vec{V}_r}{dt}\right)_r + \left(\frac{d\left(\vec{\Omega}\times\vec{r}\right)}{dt}\right)_r + \vec{\Omega}\times\vec{V}_r + \left(\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right)$$

The second term on the right-hand side must be evaluated using the product rule to get:

$$\frac{d\vec{V}_a}{dt} = \left(\frac{d\vec{V}_r}{dt}\right)_r + \left(\frac{d\vec{\Omega}}{dt}\times\vec{r}\right)_r + \left(\vec{\Omega}\times\frac{d\vec{r}}{dt}\right)_r + \vec{\Omega}\times\vec{V}_r + \left(\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right)$$

Since Ω is constant, the new second term disappears. Noting that $\frac{d\vec{r}}{dt}$ is simply the velocity, we can write the above equation as:

$$\frac{d\vec{V}_a}{dt} = \left(\frac{d\vec{V}_r}{dt}\right)_r + \vec{\Omega}\times\vec{V}_r + \vec{\Omega}\times\vec{V}_r + \left(\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right), \text{ or finally}$$

$$\frac{d\vec{V}_a}{dt} = \left(\frac{d\vec{V}_r}{dt}\right)_r + 2\vec{\Omega}\times\vec{V}_r + \left(\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right)$$

We can interpret the first term on the right-hand side as the change of the velocity vector with respect to the earth or, more simply for atmospheric motions, the change of the wind vector with time. The second right-hand side term is known as the Coriolis acceleration and the third right-hand side term is the centripetal acceleration.

We can thus apply Newton's Second Law of Motion to relative motions by simply substituting all three right-hand side terms for the absolute acceleration term $\frac{d\vec{V}_a}{dt}$,

$$\vec{F} = m\,\frac{d\vec{V}_a}{dt} \;\; or \;\; F = m\left[\left(\frac{d\vec{V}_r}{dt}\right)_r + 2\vec{\Omega}\times\vec{V}_r + \left(\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right)\right]$$