Computer Problem Set 3.1

Monte Carlo approximation of the Greeks

In the context of the Black-Scholes model with constant interest rate r, constant volatility parameter $\sigma > 0$, and Brownian motion B under the risk-neutral measure:

$$dS_t \quad := \quad S_t\big(r\,dt + \sigma dB_t\big),$$

we are interested in the numerical approximation of the no arbitrage price of a binary option defined by the payoff $\mathbf{1}_{\{S_T \le K\}}$ at the maturity date T, for some $K > 0$, together with the corresponding optimal hedging strategy

$$C_0 := e^{-rT}\mathbb{E}\big[\mathbf{1}_{\{S_T\le K\}}\big] \quad \text{and} \quad \Delta_0 := e^{-rT}\frac{\partial}{\partial S_0}\mathbb{E}\big[\mathbf{1}_{\{S_T\le K\}}\big].$$

In terms of the cdf of the $\mathcal{N}(0,1)$ distribution, direct calculation leads to

$$C_0 := e^{-rT}\mathbf{N}\big(-d_-(X_0,\sigma^2T)\big),\ \ \Delta_0 := \frac{-e^{-rT}}{S_0\sqrt{\sigma^2T}}\mathbf{N}'\big(-d_-(X_0,\sigma^2T)\big),\ \ \text{with}\ X_0 := \frac{S_0}{Ke^{-rT}},$$

and $d_-(x,v) := \frac{\ln(x)}{\sqrt{v}} - \frac{1}{2}\sqrt{v}$.

1. We first focus on the Monte Carlo approximation of C_0.
   (a) Build a program which returns C_0 and Δ_0 for given values of r, σ, S_0, T, K.
   (b) Build a program which returns a Monte Carlo approximation C_0^M of C_0 based on M copies of B_T.
   (c) Discuss the numerical results using the parameters values $r = 0.02$, $\sigma_0 = 0.4$, $S_0 = 100$, $T = 0.9$, $K = 80 + i$, $i = 0, \ldots, 40$, for various samples sizes.
2. We denote $C_0^M(S_0)$ to emphasize the dependence of this function on S_0.
   (a) Build a program which returns the centered finite-differences approximation of Δ_0 and the corresponding 5% confidence interval:
   $$\Delta_0^{M,\varepsilon} \quad := \quad \frac{C_0^M(S_0+\varepsilon) - C_0^M(S_0-\varepsilon)}{2\varepsilon}.$$
   (b) Using the parameters values of Question 1c, discuss numerically the choice of the parameters ε and M.
3. By writing the price C_0 as an integral with respect to the distribution of S_T, we obtain the following representation:
   $$\Delta_0 \quad = \quad e^{-rT}\mathbb{E}\big[\mathbf{1}_{\{S_T\le K\}}\frac{B_T}{S_0\sigma T}\big]$$
   (a) Build a program which returns a Monte Carlo approximation $\hat{\Delta}_0^M$ of Δ_0 based on the last representation.
   (b) Using the parameters values of Question 1c, compare the performances of the approximation Δ_0^M and $\hat{\Delta}_0^M$.