

Computer Problem Set 2.3

Profit and Loss of the Black-Scholes hedging

The present problem set refers to Section 9.2.6 of the lectures notes. Consider a European call option, with strike $K > 0$ and maturity $T > 0$. Then, given a volatility parameter $\Sigma > 0$, the Profit and Loss induced by the Black-Scholes hedging is defined by

$$P\&L_T(\Sigma) := X_T^{\Delta^{BS}} - (S_T - K)^+ \quad \text{where} \quad \Delta^{BS} := \mathbf{N}(\mathbf{d}_+(S_0, K, \Sigma^2 T)),$$

and $\mathbf{d}_+(s, k, v) := \frac{\ln(s/k)}{\sqrt{v}} + \frac{1}{2}\sqrt{v}$. Assume that the underlying risky asset price is defined by the following stochastic volatility model:

$$S_t := S_0 e^{-\frac{1}{2} \int_0^t \sigma_u^2 du + \int_0^t \sigma_u dW_u^{(1)}}, \quad d\sigma_t = \lambda(c - \sigma_t)dt + \gamma dW_t^{(2)},$$

where $W = (W^{(1)}, W^{(2)})$ is a Brownian motion in \mathbb{R}^2 , and $S_0, \sigma_0, \lambda, c, \gamma$ are given parameters. We recall that an explicit expression for the Ornstein-Uhlenbeck process σ is available by applying Itô's formula to $\sigma_t e^{\lambda t}$, see Section 8.1. In this context, it is then shown that the Profit and Loss reduces to

$$P\&L_T(\Sigma) = \frac{1}{2} \int_0^T e^{r(T-u)} (\Sigma^2 - \sigma_u^2) S_u^2 \Gamma^{BS}(u, S_u, \Sigma) du,$$

where $\Gamma^{BS}(t, s, \Sigma) := \partial_{ss}^2 \text{BS}(t, s, \Sigma)$.

1. build a program which produces a sample of $M = 1000$ copies of the discrete path $\{\sigma_{t_i^n}, i = 0, \dots, n\}$, $t_i^n := iT/n$. Display some representative paths, and comment.
2. build a program which produces a sample of $M = 1000$ copies $\{S_{t_i^n}^n, i = 0, \dots, n\}$ of an appropriate discretization of S . Display some representative paths, and produce the corresponding histogramme.
3. Using the parameters values $S_0 = 100$, $T = 2$, $\sigma_0 = 0.2$, $\lambda = 2$, $c = 0.4$, $\gamma = 0.3$, and $r = 0.02$, provide $M = 1000$ copies of an appropriate discretization of $P\&L_T(\Sigma)$ for $\Sigma = \sigma_0$, $K \in \{50 + j, j = 5, \dots, 150\}$, and $n \in \{50, 60, \dots, 100\}$. Compute the corresponding sample mean and variance.