Computer Problem Set 3.3
Two-factor gaussian yield curve

Let $B = (B^1, B^2)$ be a Brownian motion in $\mathbb{R}^2$ under the risk-neutral measure $\mathbb{Q}$, and $x_i, \lambda_i, \theta_i, \sigma_i$ be some positive parameters. We assume that the instantaneous interest rate is defined by

$$r_t := X_t^{(1)} + X_t^{(2)} \text{ where } dX_t^{(i)} = \lambda_i(\theta_i - X_t^{(i)})dt + \sigma_i dB_t^i,\ X_0^{(i)} = x_i,\ i = 1, 2.$$

The no-arbitrage price of the $T-$maturity zero-coupon bond is given by $P_0(T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_t dt}\right]$, inducing the yields to maturity $R_0(T) := -\frac{\ln P_0(T)}{T}$:

$$R_0(T) \ = \ \rho_1(T) + \rho_2(T), \quad \text{for all} \ \ T \geq 0,$$

$$\text{with } \rho_i(T) = \theta_i + (x_i - \theta_i)\frac{\Lambda_i(T)}{T} - \frac{\sigma_i^2}{2T}\int_0^T \Lambda_i(t)^2 dt, \text{ and } \Lambda_i(t) := \frac{1 - e^{-\lambda_i t}}{\lambda_i}.$$

1. Fix some maturity $T > 0$, a number of time steps n, and set $h := T/n$, $t_j = jh$, $j = 0, \ldots, n$. By Itô's formula, we have $X_{t_j}^{(i)} = \theta_i + (X_{t_{j-1}}^{(i)} - \theta_i)e^{-\lambda_i h} + \sigma \int_{t_{j-1}}^{t_j} e^{-\lambda_i(t_j - s)} dB_s^i$, $j = 1, \ldots, n$, $i = 1, 2$.
   (a) Build a program which simulates M trajectories of the instantaneous rate $\{r_{t_j}, j = 0, \ldots, n\}$.
   (b) Build a program which computes a Monte Carlo approximation $\hat{P}_0^{n,M}(t_j)$ of the zero-coupon bond approximate value $P_0^n(t_j) := \mathbb{E}^{\mathbb{Q}}\left[e^{-h\sum_{k=1}^j r_{t_k}}\right]$, and the corresponding Monte Carlo approximation of the yields to maturity $\hat{R}_0^{n,M}(t_j)$.
   (c) Consider the parameters values $\lambda_1 = 1$, $\lambda_2 = 0.1$, $n := 1500$, $T := 15$, $M = 10^4$, and $\theta_i = 0.05$, $\sigma_i = 0.05$, $x_i = 0.02$, $i = 1, 2$. Plot the yield curve $\{R_0(t_j), j = 0, \ldots, n\}$, and compare to the corresponding Monte Carlo approximation $\left\{\hat{R}_0^{n,M}(t_j), j = 0, \ldots, n\right\}$.

2. Under the parameters values of Question 1c, comment the effect of x_i on the Monte Carlo approximation, by
   (a) fixing x_2 and varying x_1 in $\{0.01, 0.02, 0.05\}$,
   (b) fixing x_1 and varying x_2 in $\{0.01, 0.02, 0.05\}$.

3. Under the parameters values of Question 1c, comment the effect of θ_i on the Monte Carlo approximation, by
   (a) fixing θ_2 and varying θ_1 in $\{0.01, 0.05, 0.1\}$,
   (b) fixing θ_1 and varying θ_2 in $\{0.01, 0.05, 0.1\}$.

4. Under the parameters values of Question 1c, comment the effect of σ_i on the Monte Carlo approximation by
   (a) fixing σ_2 and varying σ_1 in $\{0.01, 0.05, 0.1\}$,
   (b) fixing σ_1 and varying σ_2 in $\{0.01, 0.05, 0.1\}$.