Computer Problem Set 1.2

Simulation of the Brownian motion

The present problem set is attached to Chapter 4 of the lectures notes. All implementations should be run with the value $T = 1$. For a positive integer n, we denote $\Delta T := 2^{-n}T$, $t_i^n := i\,\Delta T$, $i = 0, \ldots, 2^n$. Our objective is to simulate a discretization of a Brownian motion W, and to study some properties.

1. Forward simulation of $\{W_{t_1^n}, \ldots, W_{t_n^n}\}$.
   (a) Justify that $W_{t_i^n} = W_{t_{i-1}^n} + Z_i\sqrt{\Delta T}$ where $(Z_i)_{1\le i\le 2^n}$ is an iid family of $\mathcal{N}(0,1)$ random variables.
   (b) Draw a sample of 1000 copies of the discretized Brownian motion $\{W_{t_1^n}, \ldots, W_{t_n^n}\}$.
   (c) Compute the corresponding sample mean and variance of W_T, and the sample covariance of $(W_T, W_{T/2})$. Comment the results by varying the value of n.

2. Backward simulation of $\{W_{t_1^n}, \ldots, W_{t_n^n}\}$.
   (a) For $0 \le s_1 < s_2$, we recall that the pair (W_{s_1}, W_{s_2}) is a centered Gaussian vector with variance matrix $\begin{pmatrix} s_1 & s_1 \\ s_1 & s_2 \end{pmatrix}$, and we verify therefore that $W_{s_1}|W_{s_2}$ is also Gaussian with characteristics
   $$\mathbb{E}\big[W_{s_1}|W_{s_2}\big] = \frac{s_1}{s_2}W_{s_2} \quad \text{and} \quad \mathbb{V}ar\big[W_{s_1}|W_{s_2}\big] = s_1\Big(1 - \frac{s_1}{s_2}\Big).$$
   With $\bar{s} := \frac{s_1+s_2}{2}$, justify that $W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2)$ has a Gaussian distribution with conditional mean $\bar{x} := \frac{x_1+x_2}{2}$ and conditional variance $\frac{s_2-s_1}{4}$.
   (b) Justify that the conditional distribution of $W_{\bar{s}}\big|\big(W_{s_1} = x_1, W_{s_2} = x_2, (W_u)_{u\notin[s_1,s_2]}\big)$ is $\mathcal{N}(\bar{x}, \frac{s_2-s_1}{4})$.
   (c) Use the last property to simulate backward the discretized Brownian motion: start by drawing copies of W_T, then $W_{T/2} = W_{t_1^1}$, then $W_{T/4} = W_{t_1^2}$ and $W_{3T/4} = W_{t_3^2}$, etc...
   (d) Compute the corresponding sample mean and variance of W_T, and the sample covariance of $(W_T, W_{T/2})$. Comment the results by varying the value of n.

3. Using successively the forward and backward simulated samples, compute an approximation of $\mathrm{QV}^n(W)_T$, the quadratic variation of the Brownian motion along the partition $(t_i^n)_i$. Provide two graphs displaying the departure from the limit T as a function of $n \in \{10, \ldots, 20\}$.