Decomposition of Graphs: Exploring Graphs

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Graph Algorithms
Data Structures and Algorithms

Learning Objectives

- Implement the explore algorithm.
- Figure out whether or not one vertex of a graph is reachable from another.

Outline

Motivation

You're playing a video game and want to make sure that you've found everything in a level before moving on.
How do you ensure that you accomplish this?

Examples

This notion of exploring a graph has many applications:

- Finding routes
- Ensuring connectivity
- Solving puzzles and mazes

Paths

We want to know what is reachable from a given vertex.

Definition

A path in a graph G is a sequence of vertices $v_0, v_1, \ldots, v_n$ so that for all i, (v_i, v_{i+1}) is an edge of G.

Formal Description

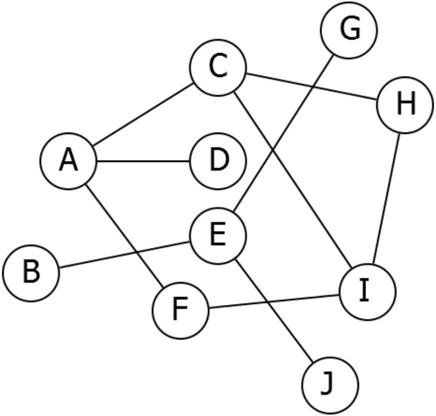
Reachability

Input: Graph G and vertex s

Output: The collection of vertices v of G so that there is a path from s to v.

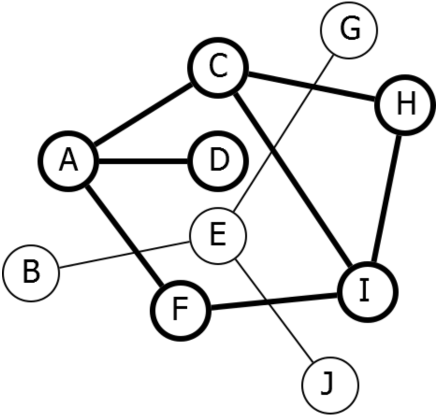
Problem

Which vertices are reachable from A?



Solution

$A, C, D, F, H, I.$

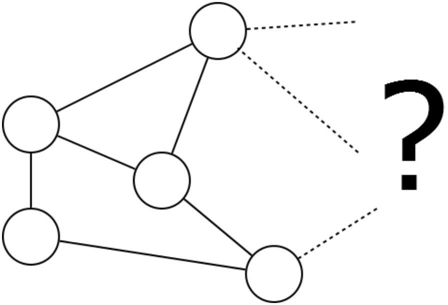


Outline

1. Problem Discussion
2. Ideas
3. Explore
4. Correctness
5. DFS

Basic Idea

We want to make sure that we have explored every edge leaving every vertex we have found.



Pseudocode

Component(s)

$DiscoveredNodes \leftarrow \{s\}$

while there is an edge e leaving $DiscoveredNodes$ that has not been explored:

    add vertex at other end of e to $DiscoveredNodes$

return $DiscoveredNodes$

Formal Specification

We need to do some work to handle the bookkeeping for this algorithm.

- How do we keep track of which edges/vertices we have dealt with?
- What order do we explore new edges in?

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Visit Markers

To keep track of vertices found:

Give each vertex boolean `visited`(v).

Unprocessed Vertices

Keep a list of vertices with edges left to check.
This will end up getting hidden in the program stack.

Depth First Ordering

We will explore new edges in Depth First order. We will follow a long path forward, only backtracking when we hit a dead end.

Explore

```
Explore(v)

visited(v) ← true
for (v, w) ∈ E:
  if not visited(w):
    Explore(w)
```

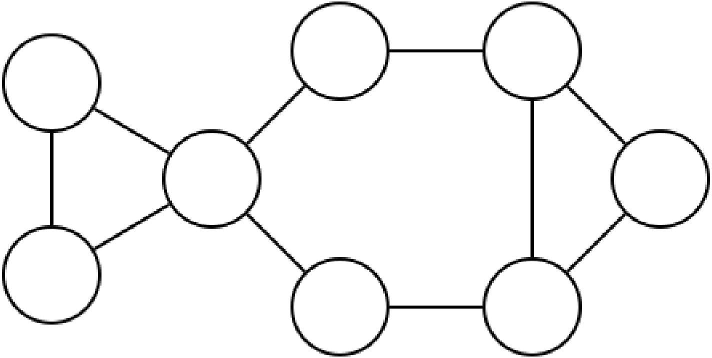
Explore

Explore(v)

```
visited(v) ← true
for (v, w) ∈ E:
    if not visited(w):
        Explore(w)
```

Need adjacency list representation!

Example



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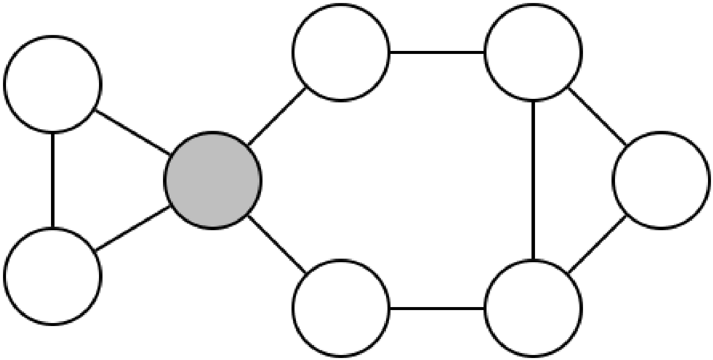

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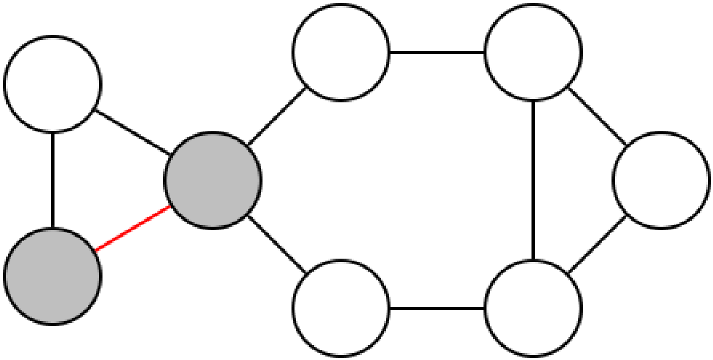

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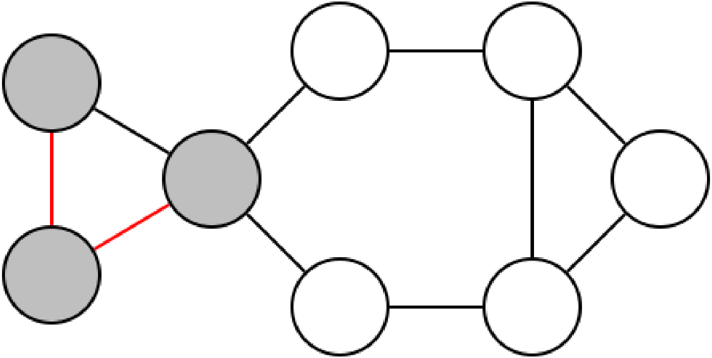

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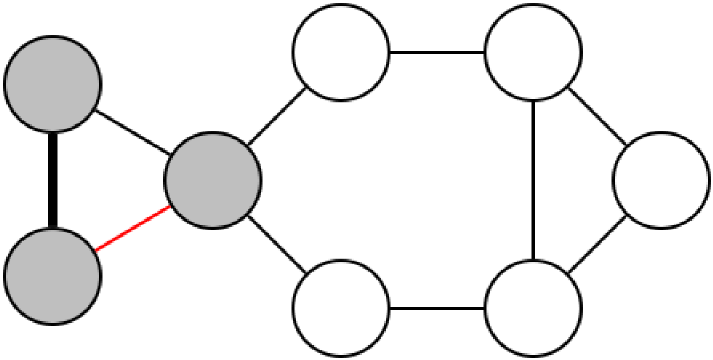

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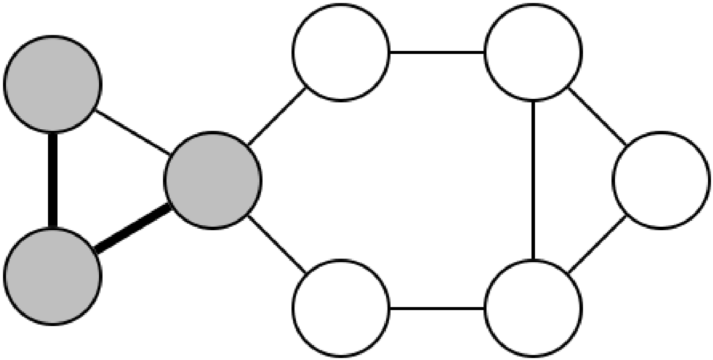

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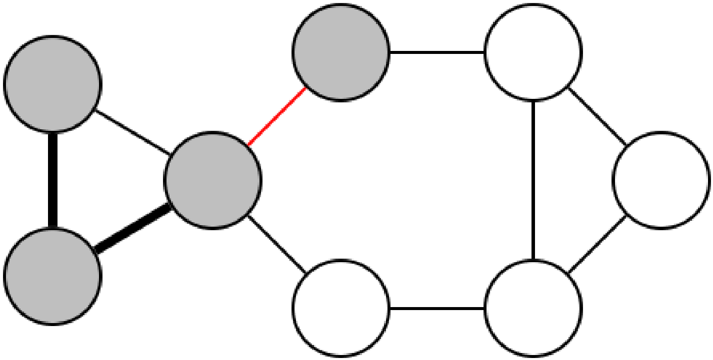

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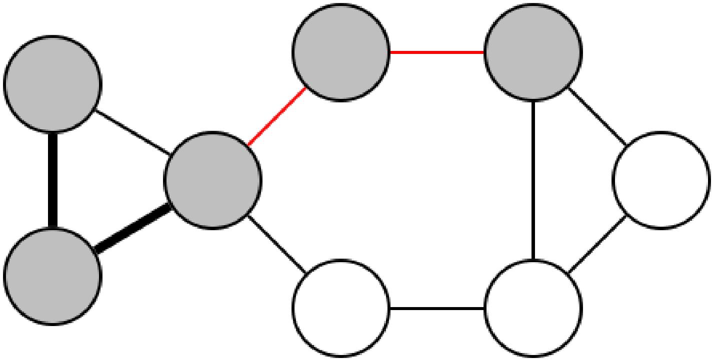

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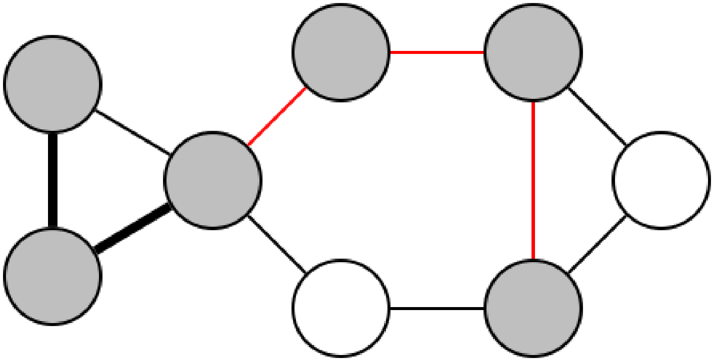

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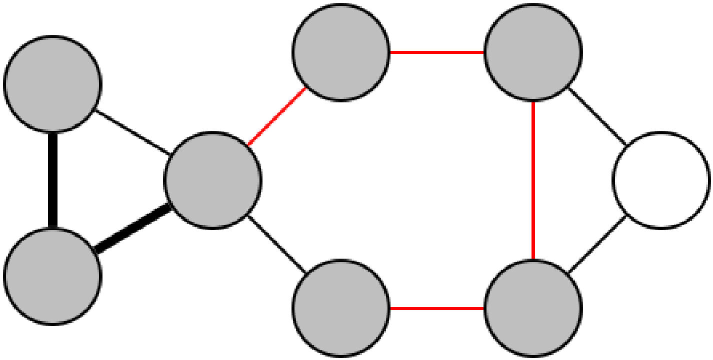

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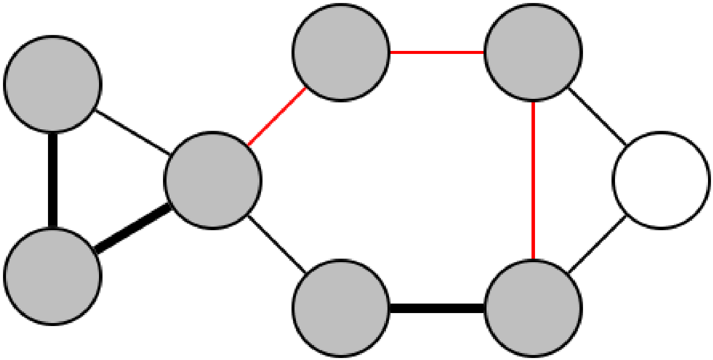

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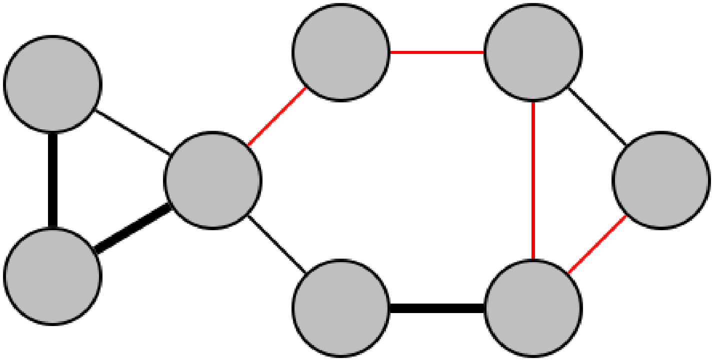

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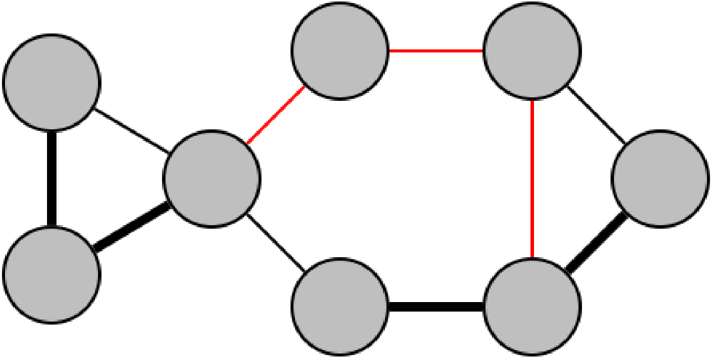

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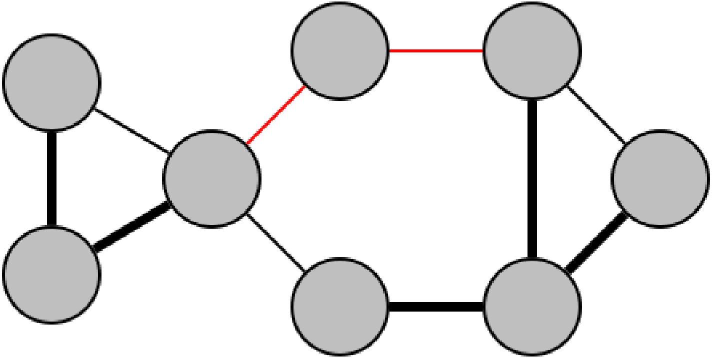

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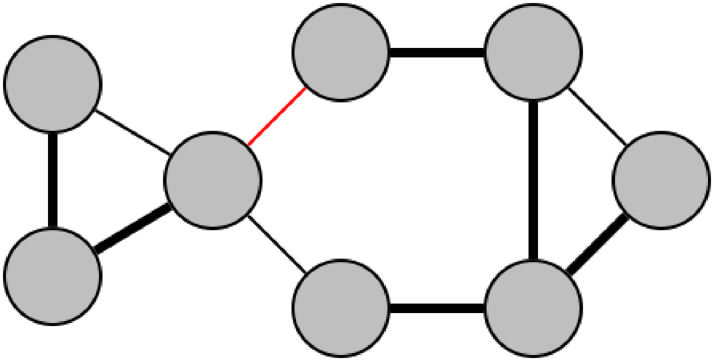

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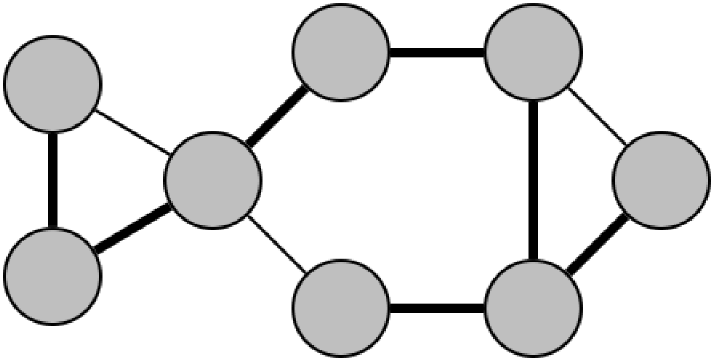


Outline

Result

Theorem

If all vertices start unvisited, `Explore`(v) marks as visited exactly the vertices reachable from v.

Proof

Proof.

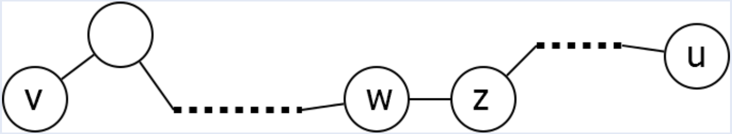
- Only explores things reachable from v.
- w not marked as visited unless explored.
- If w explored, all neighbors explored.

$\square$

Proof (continued)

Proof.

- u reachable from v by path.
- Assume w furthest along path explored.



- Must explore next item.

$\square$

Outline

Reach all Vertices

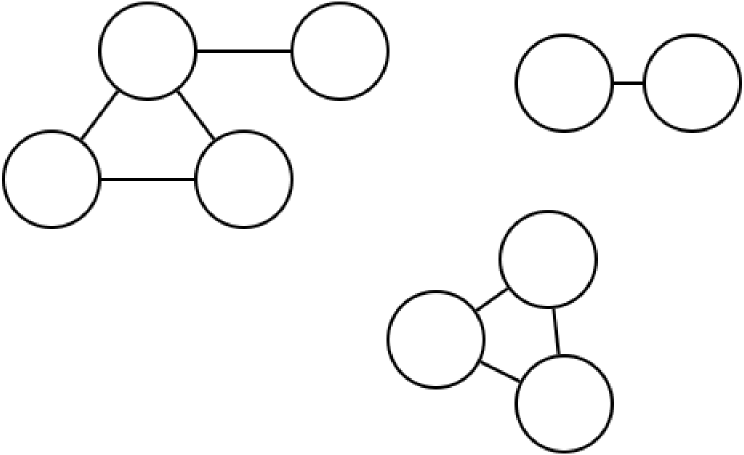
Sometimes you want to find all vertices of G, not just those reachable from v.

DFS

```
DFS(G)
```

```
for all v ∈ V:      mark v unvisited
for v ∈ V:
  if not visited(v):
    Explore(v)
```

Example



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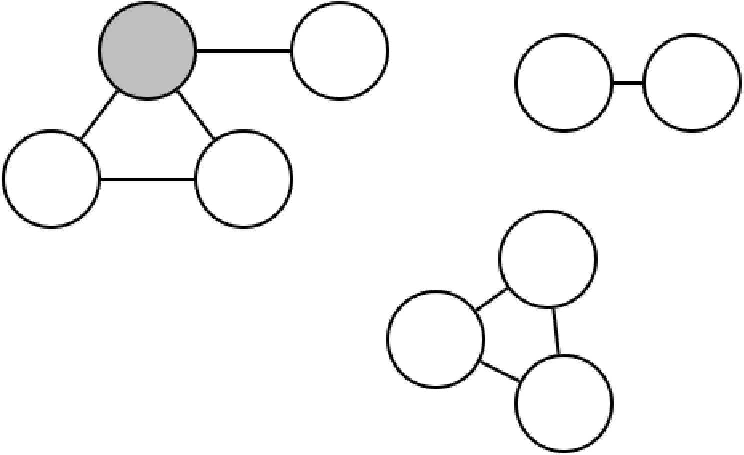

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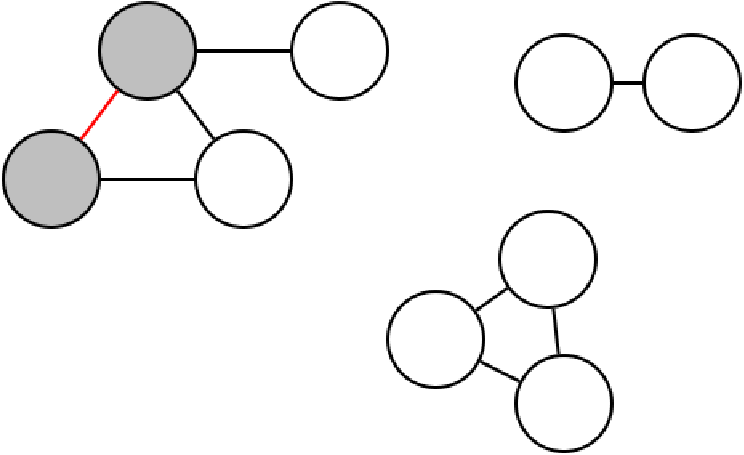

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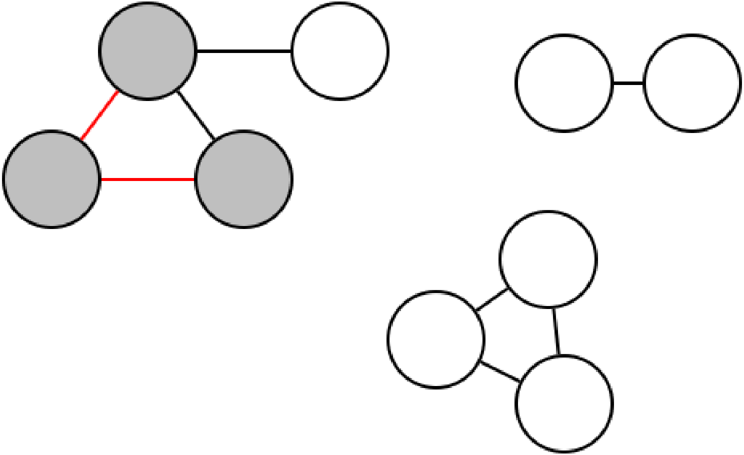

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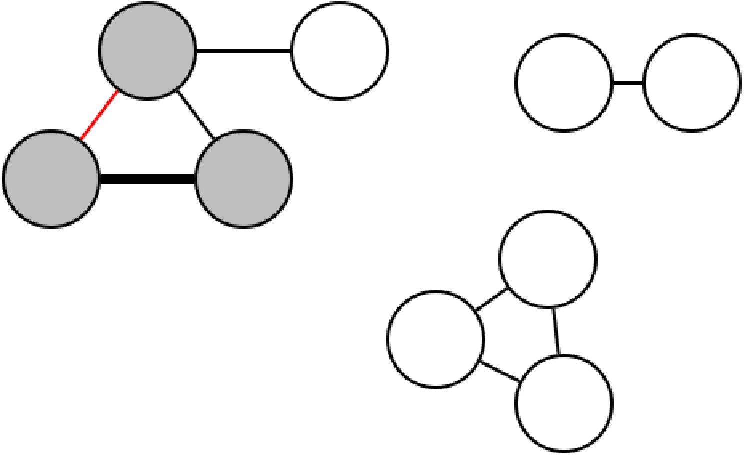

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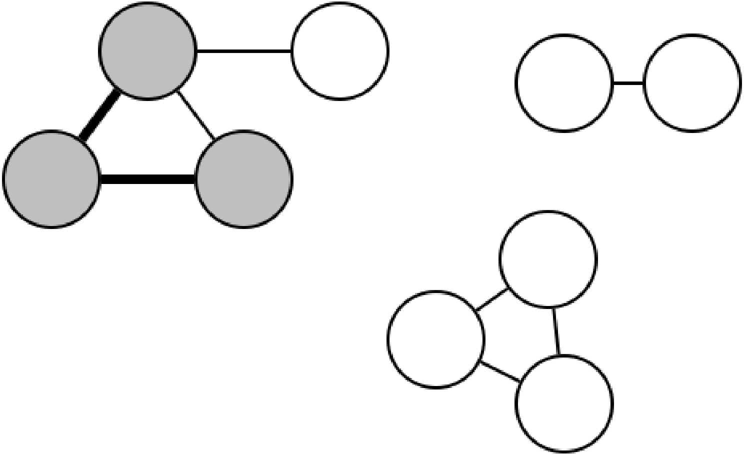

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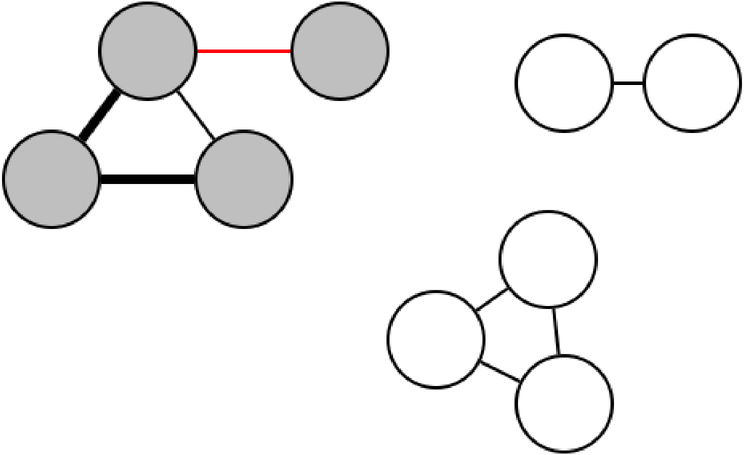

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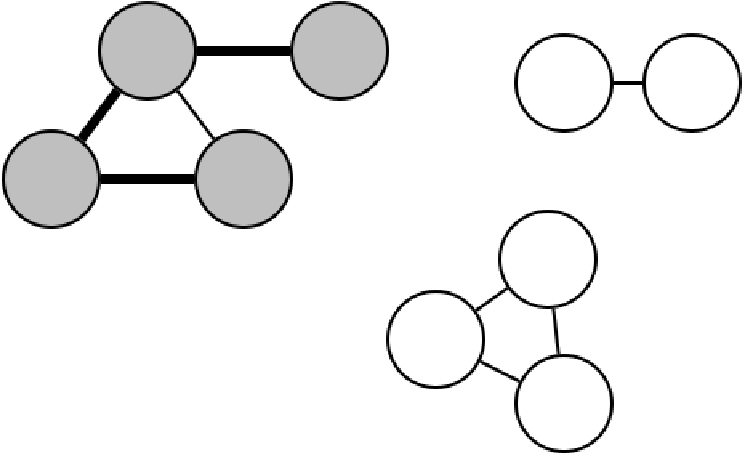

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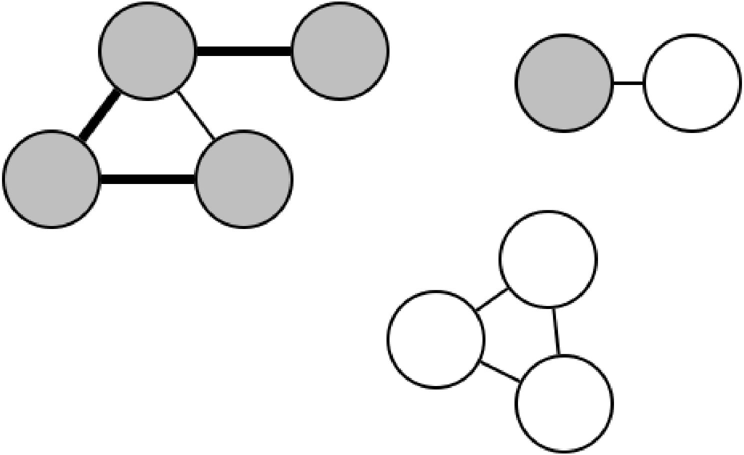

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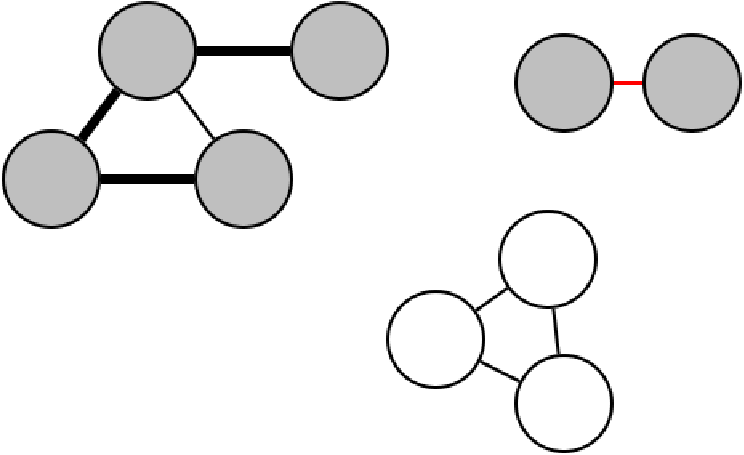

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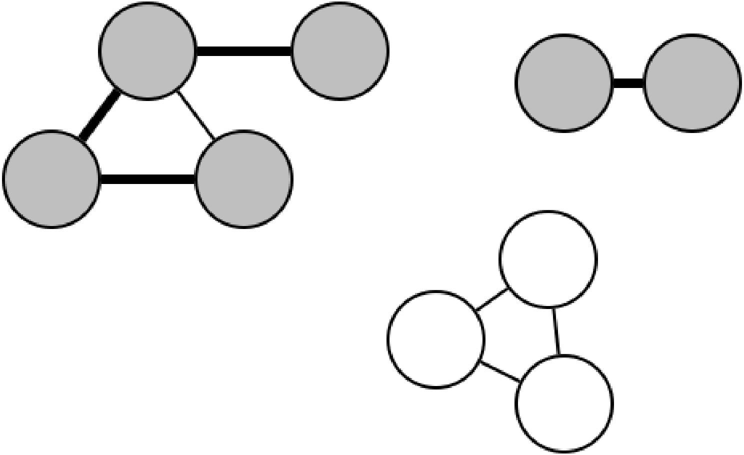

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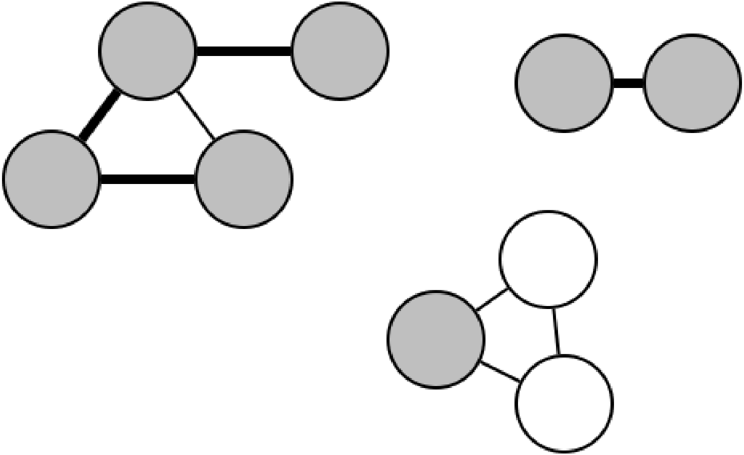

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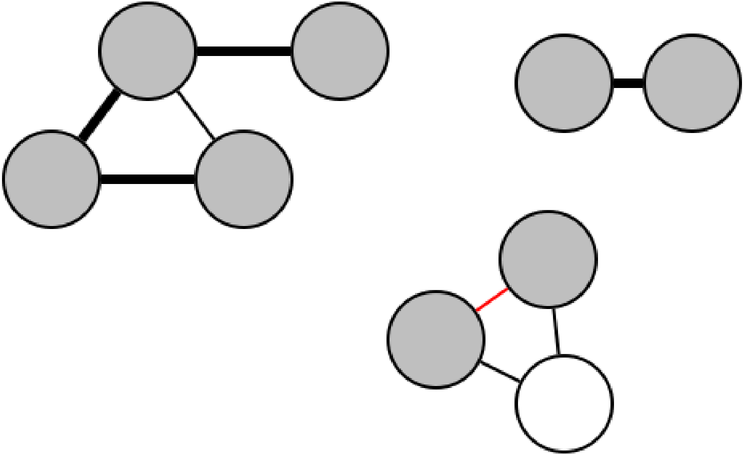

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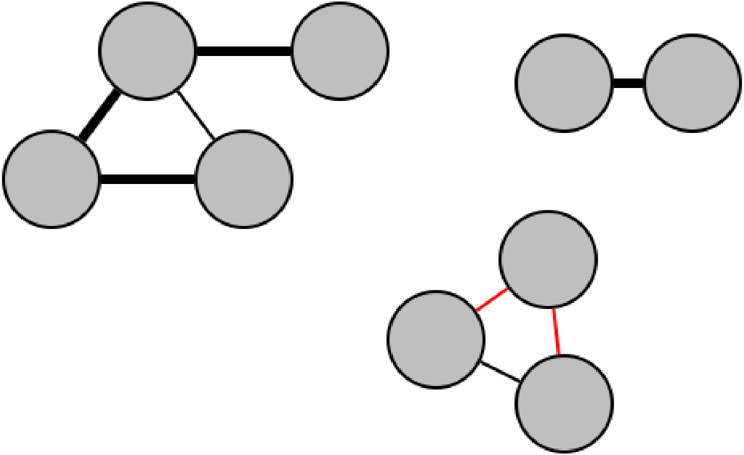

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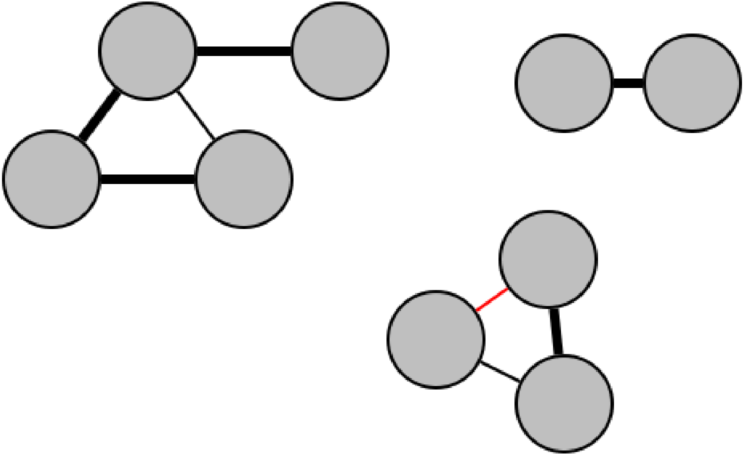

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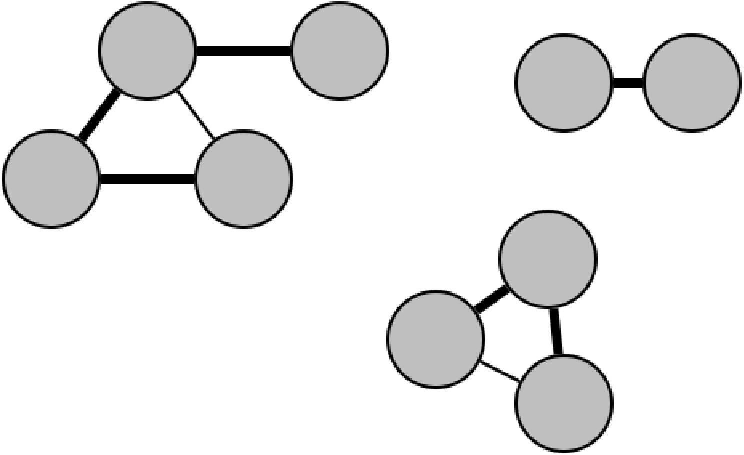

Runtime

Number of calls to `explore`:

- Each explored vertex is marked visited.
- No vertex is explored after visited once.
- Each vertex is explored exactly once.

Runtime

Checking for neighbors:

- Each vertex checks each neighbor.
- Total number of neighbors over all vertices is $O(|E|)$.

Runtime

Total runtime:

- $O(1)$ work per vertex.
- $O(1)$ work per edge.
- Total $O(|V| + |E|)$.

Next Time

- More on reachability in graphs.
- Application of `DFS`.