Q1:
(i). Do you agree with Jimmy or Ronald? Justify using statistical arguments (e.g. equations) with an example or counter-example.
(ii). Using the 2012 donation data above, explain in detail how you would verify the Democrats' claims. Give a formal statistical argument and present your conclusions. Also argue whether you can prove the DEMs to be right or wrong.
(iii). Group the 50 states in ones that you think support the DEM claim and the states that do not (inside each group list them in alphabetical order). Give a formal statistical argument for your results and present your conclusions clearly.
(iv). What is the drawback of your solution to Q1.iii? Can you quantify the problem with your approach?
(v). Suppose we reversed the roles and the GOP claimed that GOP donations are on average smaller than DEM donations. Separate donations by state and show for which states you can support this GOP.
(vi). Now consider the donations per candidate. We are trying to test the claim that DEM candidates in average raise less money than GOP candidates. Give formal statistical arguments (e.g. equations) and present your conclusions.
A:
(i). Given that no data is provided, a solid conclusion can hardly be drown. That being said, I am more intend to agree with Ronald's at this point, that the the number obtained from an article proves nothing.

1. DEM's argument is that DEM gets smaller donations over the years (a hypothesis) while the evidence Jimmy provided is the number for 2012 only (incomplete data). We can not draw a solid conclusion without testing hypothesises. We will design a hypothesis testing in the following part

2. Media is not necessarily trustworthy. To verify the empirical mean we have conducted the computation ourselves based on the data:

$$empirical\ mean:\ \mu^{(DEM)} = \frac{\sum_{i=1}^{N} x_i^{(DEM)}}{N}$$

   Similarly,

$$empirical\ mean:\ \mu^{(GOP)} = \frac{\sum_{i=1}^{N} x_i^{(GOP)}}{N}$$

   If we do not exclude negative samples:
   $\mu_{DEM} = 1887$ and $\mu_{GOP} = 2064$, which match the numbers from the article.
   However, if we exclude the negative samples in the computation we obtain:
   $\mu_{DEM} = 1980$ and $\mu_{GOP} = 2150$, which are different from the numbers from the article.

(ii). We conduct hypothesis testing from two samples (democratic donation pool and republican donation pool). Denote x_1 is drawn from republican pool, and x_2 is drawn from democratic pool. The procedures to conduct the hypothesis testing would be:

1. Compute the empirical standard error:

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

   where

$$s_i^2 = \frac{1}{n_i} \sum_{k=1}^{n_i} (x_k^{(i)} - \bar{x}^{(i)})^2$$

   and $\bar{x}_i$ are the corresponding empirical class mean.

2. The degrees of freedom would then be:

$$DF = \left\lfloor \frac{(\sigma_1^2/n_1 + \sigma_2^2/n_2)^2}{(\sigma_1^2/n_1)^2/(n_1-1) + (\sigma_2^2/n_2)^2/(n_2-1)} \right\rfloor$$

3. Compute test statistic (t-score, also known as Welsh's t)

$$t_d = \frac{(\bar{x}_1 - \bar{x}_2) - d}{SE}$$

4. Our **Null hypothesis** is: $\mu_1 \geq \mu_2$, accordingly: $\mu_1 - \mu_2 \geq 0$. The **Alternative Hypothesis** would than be: $\mu_1 - \mu_2 < 0$

5. The p value can then be computed as:

$$\int_{d \geq 0} P[T_{DF} > t_d] \partial d$$

The empirical variance of the GOP is $\sigma_1^2 = \sigma_{GOP}^2 = 101468387$.
The empirical variance of the DEM is $\sigma_2^2 = \sigma_{DEM}^2 = 26660303$.
Use the equation of empirical standard error:

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$SE = 36.0479$$

Then compute the test statistic (t-score) as:

$$t_d = \frac{(\bar{x}_1 - \bar{x}_2) - d}{SE}$$

$$t_d = 4.712$$

We assume the $\bar{x}_i$ follows the Gaussian Distribution. To verify our assumption, we plot the histogram of random 200 sub-samples for each class and make visual observation (seems this is sufficient for now).

As this is a one-tail t-test, we compute the probability to reject the Null Hypothesis as:

$$\int_{d \geq 0} P[T_{DF} > t_d] \partial d$$

As illustrated in the Figure 1 and Figure 2, we conclude the mean of the sample pool follows Gaussian distribution. (This can also be backed up by Central Limit Theorem). Now assuming the mean of samples follow Gaussian distribution,

$$\int_{d \geq 0} P[T_{DF} > 4.712] \partial d = \Phi(-4.712) < 0.01\%$$

The probability that Democratic party is wrong is only less than 0.01%, hence does not demonstrate any statistical significance to reject null hypothesis. Hence we will likely to accept null hypothesis. In this case, based on the sampled data we should support DEM's claim that on average DEM receive less donation amount.

(iii). For this part basically we are repeating the procedures done in previous part. Only this time we consider **one state at a time**. **Repeat procedures 1-5 from part (ii)**. Similarly we set $x_1^{(state)}$ to be drawn from GOP samples and $x_2^{(state)}$ to be drawn from DEM samples. Our Null hypothesis still is $\mu_1 > \mu_2$. In order to determine if we support DEM's argument or not, we need to set a threshold value for t_d, (as t_d

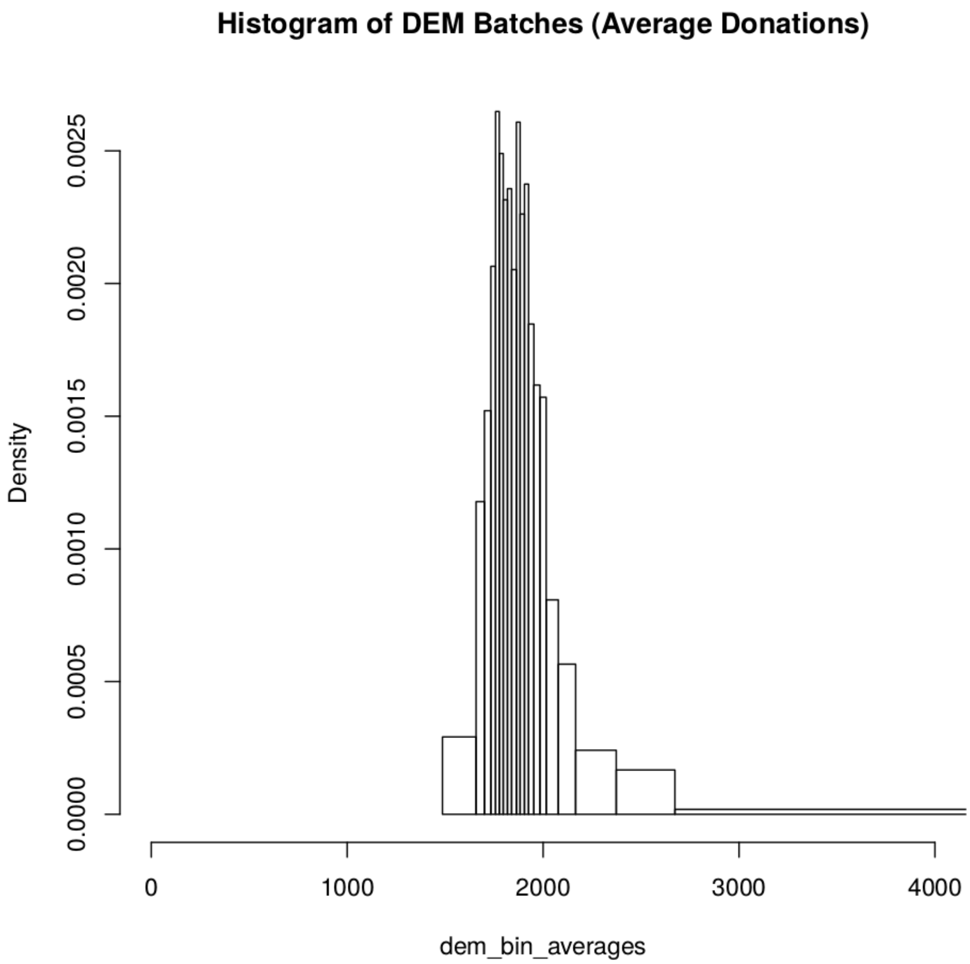

Figure 1: The 200-sample mean distribution of Democratic party.

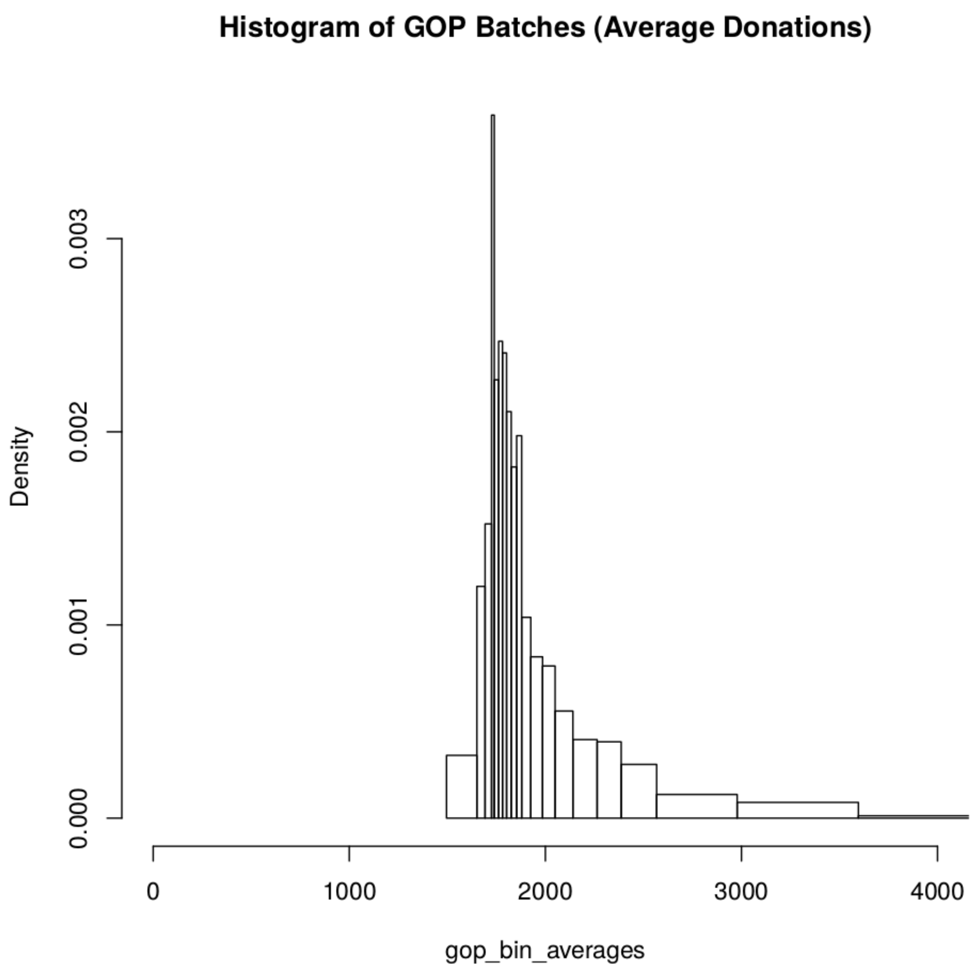

Figure 2: The 200-sample mean distribution of Grand Old Party.

will determine the p value). **We assume the statistical significance to reject DEM's claim, is** 5%. Hence we set our $t_d = 1.65$ according to the $\Phi(t)$ table for Gaussian distribution ($\Phi(-1.65) \approx 0.05$).
Results and Conclusion Based on our tests for each state (setting statistical significance to be 5%), in the following state the DEM's claim is supported:
AL, DE, IA, IN, KS, KY, MA, ME, MT, NC, NM, OH, OR, TX, UT
The DEM's is rejected in the following state:
AK, AR, AZ, CA, CO, CT, FL, GA, HI, ID, IL, LA, MD, MI, MN, MO, MS, ND, NE, NH, NJ, NV, NY, OK, PA, RI, SC, SD, TN, VA, WA, WI, WV.

(iv). There are several drawbacks that I can name of:

1. Dividing the data by state results in states with very few samples (e.g. 6 samples for GOP in DE and 2 samples for DEM in KS). There are high probabilities there exists missing data.
2. Adjusting the significance probability for rejection NUll hypothesis (for example, change from 5% to 10%) will modify our conclusion. (which is essentially adjusting the threshold we set on t_d).

The 1st bullet comes from the way data was collected hence there is nothing we can add. For the 2nd bullet, **instead of only giving a 'YES' or 'NO' answer** on if we support DEM's claim, we can quantify the decision based on t_d and claim we are x% sure that DEM's is right in state X and we are y% sure that DEM's is right in state Y. **Now the classification problem has been turned into a regression problem and hence the problem is now quantified**.

(v). If instead, that GOP claim their average donation is smaller than those of DEM's, similarly as steps 1-5 in (ii), we calculate the empirical standard error of two parties for each state and then the t_d score for each state can easily be obtained. Luckily we have calculated the t_d score for each state in previous part (iv), instead of setting $t_d > 1.65$ to determine which state support DEM's claim, we now set $t_d < -1.65$ to support GOP's claim that GOP has smaller average donations. (setting 5% to be statistical significant)
The state that support GOP's claim is: **CO, FL, GA, ID, IL, MD, MN, NJ, PA, SC, WV** The state that reject GOP's claim is: **AK, AL, AR, AZ, CA, CT, DE, HI, IA, IN, KS, KY, LA, MA, ME, MI, MO, MS, MT, NC, ND, NE, NH, NM, NV, NY, OH, OK, OR, RI, SD, TN, TX, UT, VA, WI**.

(vi). The major distance in part (vi). is that individual donations have been merged into donations for each candidate in each state.

$$x_i^{(j)} = \sum_{k=1}^{n} x_{\{state:\ i,\ candidate:\ j\}}$$

For each state, the average value would be:

$$\bar{x}_i = \bar{x}_i^{(j)}$$

Similarly we can obtain empirical variance for each state and the size of candidates in state i.
The t_d score for each state can then be obtained accordingly as part (ii) step 1–5 and the procedures in step (iii). We start repeat the tasks from (i)–(iii).
Repeat (i): The empirical mean of the GOP is 288690 where the empirical mean of the DEM is 233741.
Repeat (ii): Following the steps 1–5 in (ii) we obtain the t_d score where:

$$t_d = 2.21$$

As we use 5% to denote the statistical significance, there is no significant evidence to reject DEM's claim. Hence we accept DEM's claim that DEM collect less donations by candidates.
Repeat (iii): Repeating the procedures in (iii) by comparing t_d value to threshold, the following state did not reject DEM's claim: **AR, HI, IL, ND, NH, NM, OK, SD, TN, TX, WA**
The following state reject DEM's claim: **AK, AL, AZ, CA, CO, CT, DE, FL, GA, IA, ID, IN, KS, KY, LA, MA, MD, ME, MI, MN, MS, MT, NC, NE, NJ, NV, NY, OH, OR, PA, RI, SC, UT, VA, WI, WV**.

Q2:
(i) Test if population1 b .csv" has the same average donation as population2 b .csv" assuming donations are independent and identically distributed Bernoulli random variables. What is the average donation in each population?
(ii) Test if population3 b .csv" has the same average donation as population4 b .csv" assuming donations are independent and identically distributed Bernoulli random variables. What is the average donation in each population?
(iii) Test if population1 p .csv" has the same average donation as population2 p .csv" assuming donations are independent and identically distributed from an unknown random variable with finte variance. What is the average donation in each population?
(iv) Test if population3 p .csv" has the same average donation as population4 p .csv" assuming donations are independent and identically distributed from an unknown random variable with finite variance. What is the average donation in each population?
A:

Before we start testing hypothesis, we first want to test that the batch mean of each population is a Gaussian distribution. Based on our observation of batch mean histogram, we concluded that each batch mean follows **Gaussian** distribution. We attach the histogram of batch means for each population as shown in Figure 3 and Figure 4.
(i). Assuming Bernoulli distribution, the Maximum likelihood estimator is actually the empirical mean, proof as following:

$$\mathbf{L}(\theta|x) = \theta^{x_1}(1-\theta)(1-x_1)\cdots\theta^{x_n}(1-\theta)^{1-x_n} = \theta^{x_1+\cdots+x_n}(1-\theta)^{n-(x_1+\cdots+x_n)}$$

$$\ln\mathbf{L}(\theta|x) = \ln\theta\sum_{i=1}^{n} x_i + \ln(1-\theta)(n-\sum_{i=1}^{n} x_i)$$

$$\ln\mathbf{L}(\theta|x) = n\bar{x}\ln\theta + n(1-\bar{x})\ln(1-\theta)$$

$$\frac{\partial}{\partial\theta}\ln\mathbf{L}(\theta|x) = n(\bar{x}/\theta - (1-\bar{x}/(1-\theta)))$$

$$\rightarrow \hat{\theta}(x) = \bar{x}$$

To test the hypothesis that if population a has the same mean of population b, we follow the steps as below:

1. Compute the empirical standard error:

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

   where

$$s_i^2 = \frac{1}{n_i}\sum_{k=1}^{n_i}(x_k^{(i)} - \bar{x}^{(i)})^2$$

   and $\bar{x}_i$ are the corresponding empirical class mean.

2. The degrees of freedom would then be:

$$DF = \left\lfloor \frac{(\sigma_1^2/n_1 + \sigma_2^2/n_2)^2}{(\sigma_1^2/n_1)^2/(n_1-1) + (\sigma_2^2/n_2)^2/(n_2-1)} \right\rfloor$$

3. Compute test statistic (t-score, also known as Welsh's t)

$$t_d = \frac{(\bar{x}_1 - \bar{x}_2) - d}{SE}$$

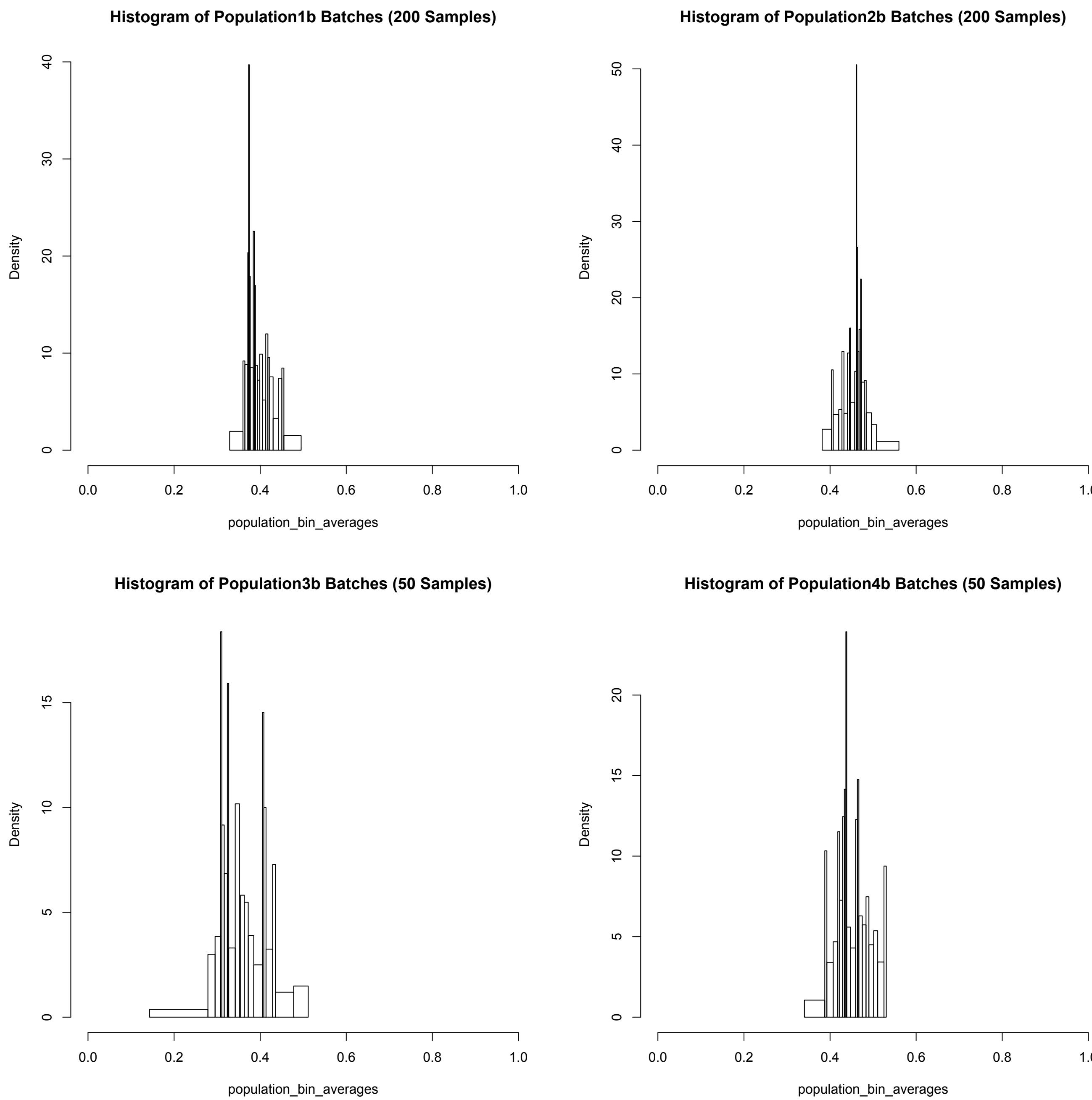

Figure 3: The histogram of batch mean for each population ib.

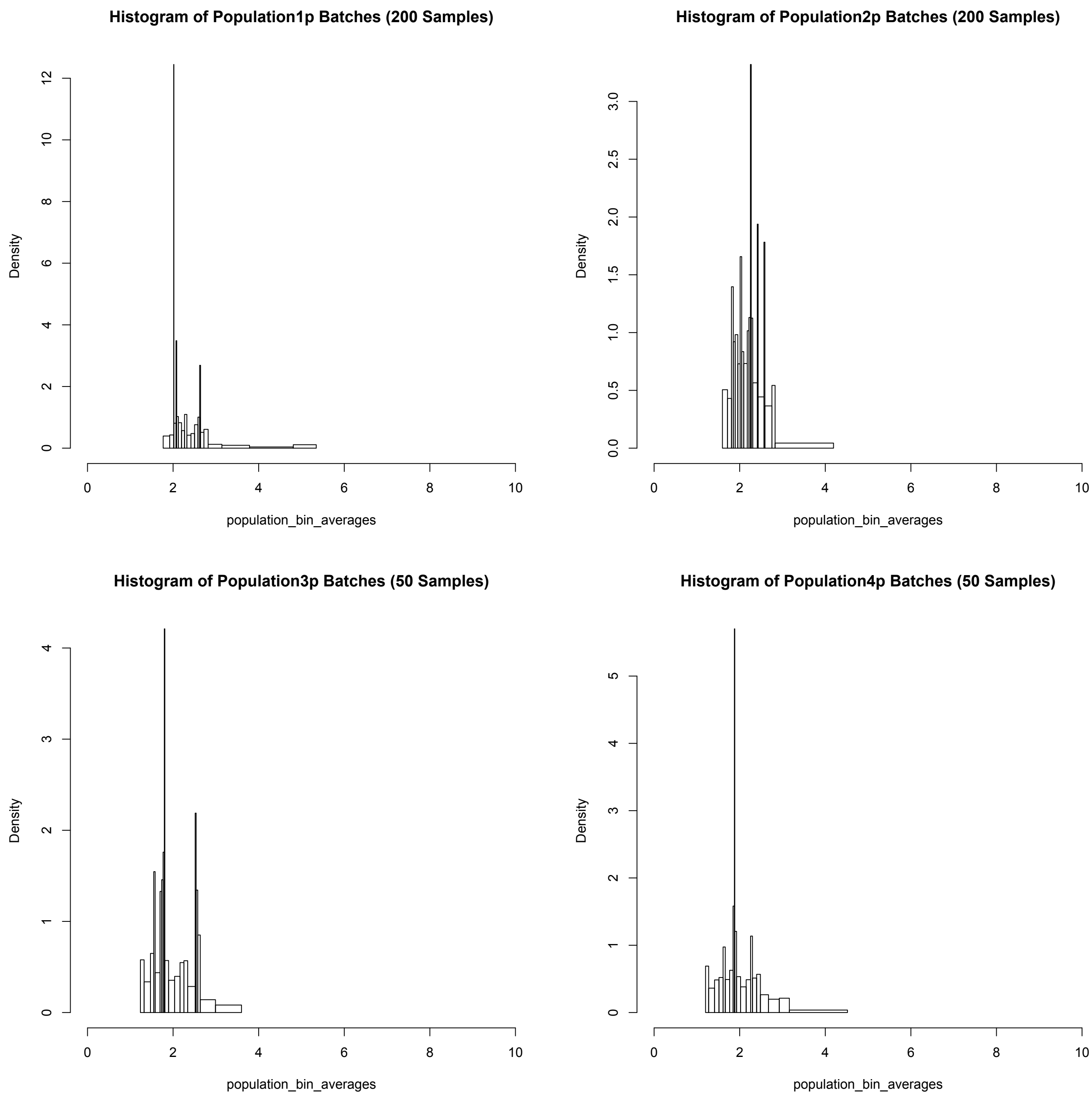

Figure 4: The histogram of batch mean for each population ip.

4. Our **Null hypothesis** is: $\mu_1 = \mu_2$, accordingly: $\mu_1 - \mu_2 = 0$. The **Alternative Hypothesis** would than be: $\mu_1 - \mu_2 \neq 0$

5. The p value can then be computed using two tails:

$$p = P[T_{DF} < -|t_d|] + P[T_{DF} > |t_d|]$$

Note that the major difference between the hypothesis testing in this problem w.r.t to Problem 1. is that we use two tails in P2 as we were using one tail in P1. Similarly, we use 5% as statistical significance to reject the Null hypothesis. The threshold to reject Null hypothesis would then be

$$Threshold: \ t_d = \pm 2.81$$

Results:
The empirical mean for population1b is: 0.4008, the empirical variance for population1b is: 0.24.
The empirical mean for population2b is: 0.4557, the empirical variance for population2b is: 0.25.
The t_d between population1b and population2b is:

$$t_d = -7.86$$

$$|t_d| > 2.81$$

Hence we are fairly certain that the population1b and population2b has same average.

(ii). Similarly as in part (i):
The empirical mean for population3b is: 0.36, the empirical variance for population3b is: 0.2304.
The empirical mean for population4b is: 0.45, the empirical variance for population4b is: 0.2475.
The t_d between population3b and population4b is:

$$t_d = -4.11$$

$$|t_d| > 2.81$$

Hence we are fairly certain that the population3b and population4b has same average.

(iii). Similarly as in part (i):
The empirical mean for population1p is: 2.647, the empirical variance for population1p is: 129.81.
The empirical mean for population2p is: 2.245, the empirical variance for population2p is: 37.62.
The t_d between population1p and population2p is:

$$t_d = 3.10$$

$$|t_d| > 2.81$$

Hence we are fairly certain that the population1p and population2p has same average.

(iv). Similarly as in part (i):
The empirical mean for population3p is: 2.20, the empirical variance for population3p is: 16.097.
The empirical mean for population4p is: 2.57, the empirical variance for population4p is: 87.754.
The t_d between population3p and population4p is:

$$t_d = -1.546$$

$$|t_d| < 2.81$$

Hence we reject the clam that the population3p and population4p has same average.

Q3:
(i). Implement UCB1 with 4 arms. Compute the total donations, for populations ib.
(ii). Implement UCB1 with 4 arms. Compute the total donations, for population ip.
(iii). Redo (i) and (ii) using ϵ -greedy. Is it better? Give formal statistical argument.
(iv). Can you explain the difference between (i) and (ii) using results in Q2? Give formal statistical argument and present your conclusion clearly.
(v). Redo (i) using Thompson sampling.
A:

Note: Assume we have K arms, we will initialize both UCBI and ϵ-greedy by pulling each arm once at the beginning of the algorithm.

(i). The UCBI algorithm is focused on selecting an arm to play. We assume $X_1, \cdots, X_{n_i}$ be i.i.d. rewards from arm i with distribution bounded in [0,1], then for any $\epsilon \in (0, 1)$:

$$P[\sum_{k=1}^{n_i} X_k \leq n_i \mathbb{E}[X-1] - \epsilon] \leq \exp(-\frac{2\epsilon^2}{n_i})$$

While t is the total armed played at this time $n_i(t)$ is the times arm i has been played so far. We can set $\epsilon = \sqrt{2n_i(t)\log t}$. We will play the arm i with the largest:

$$\frac{1}{n_i(t)} \sum_{k=1}^{n_i(t)} X_k + \sqrt{\frac{2\log t}{n_i(t)}}$$

The total reward is defined by:

$$R_t = \sum_{h=1}^{t} X_{m'(h,\pi_h)}^{(\pi_h)}$$

where $m'(h, \pi_h) = \sum_{m=1}^{h} \mathbf{1}\{\pi_m = \pi_h\}$. The total reward:

$$R_t = 417$$

The $n_i(t)$ distribution along with time t can be obtained as in Figure 5.

(ii). The total reward for ip family:

$$R_t = 3063.2$$

The $n_i(t)$ distribution along with time t can be obtained as in Figure 6.

(iii).
ϵ- greedy: The ϵ- greedy can be implemented following the steps as below:

1. Assume rewards in [0,1], at time t we need to obtain ϵ_t so that we will pull best arm so far with probability $1 - \epsilon_t$, and pull an random arm with probability ϵ_t.
2. $\epsilon_t = min(\frac{12}{\Delta^2 t}, 1)$ where $\Delta = min_{i:\Delta_i>0}\Delta_i$, to be more specific, $\Delta_i = mu^* - \mu_i = max\{\mu_i\} - \mu_i$ where μ_i is the average awards so far from arm i.

The total rewards for ib family is:

$$R_t = 418$$

The $n_i(t)$ distribution along with time t can be obtained as in Figure 7.

The total rewards for ip family is:

$$R_t = 2113.09$$

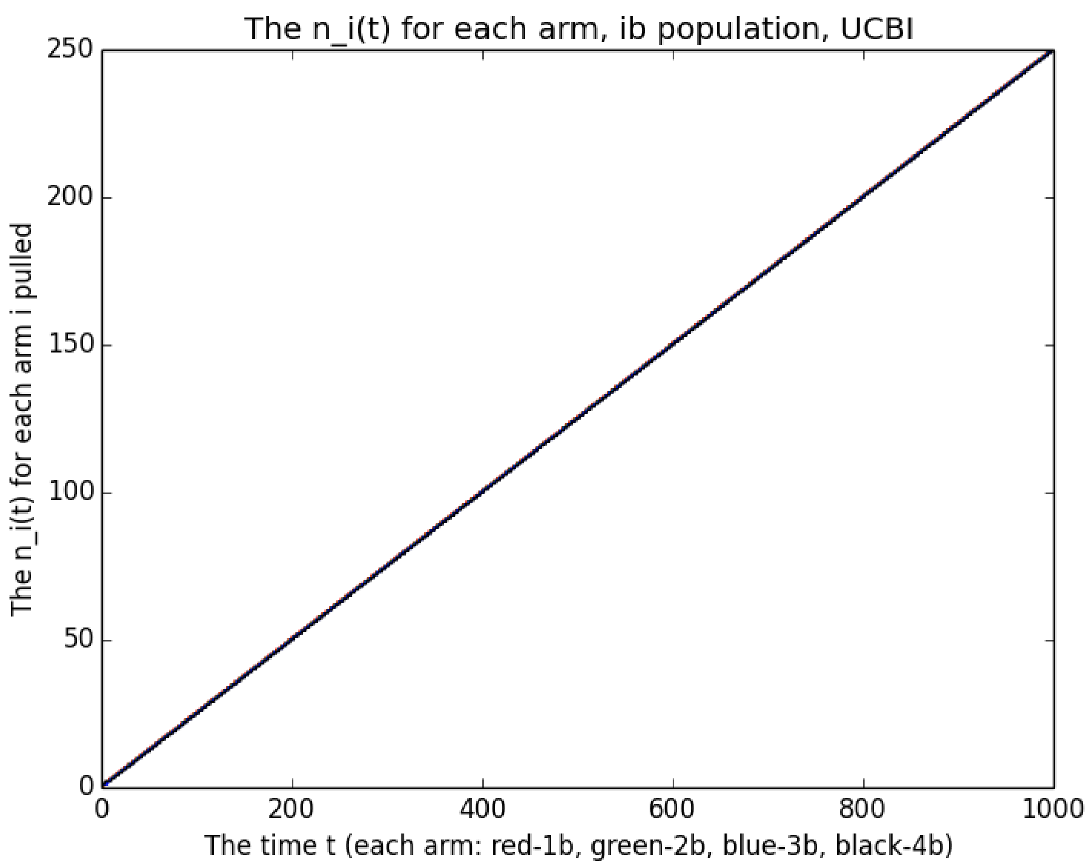

Figure 5: UCBI tested on ib populations.

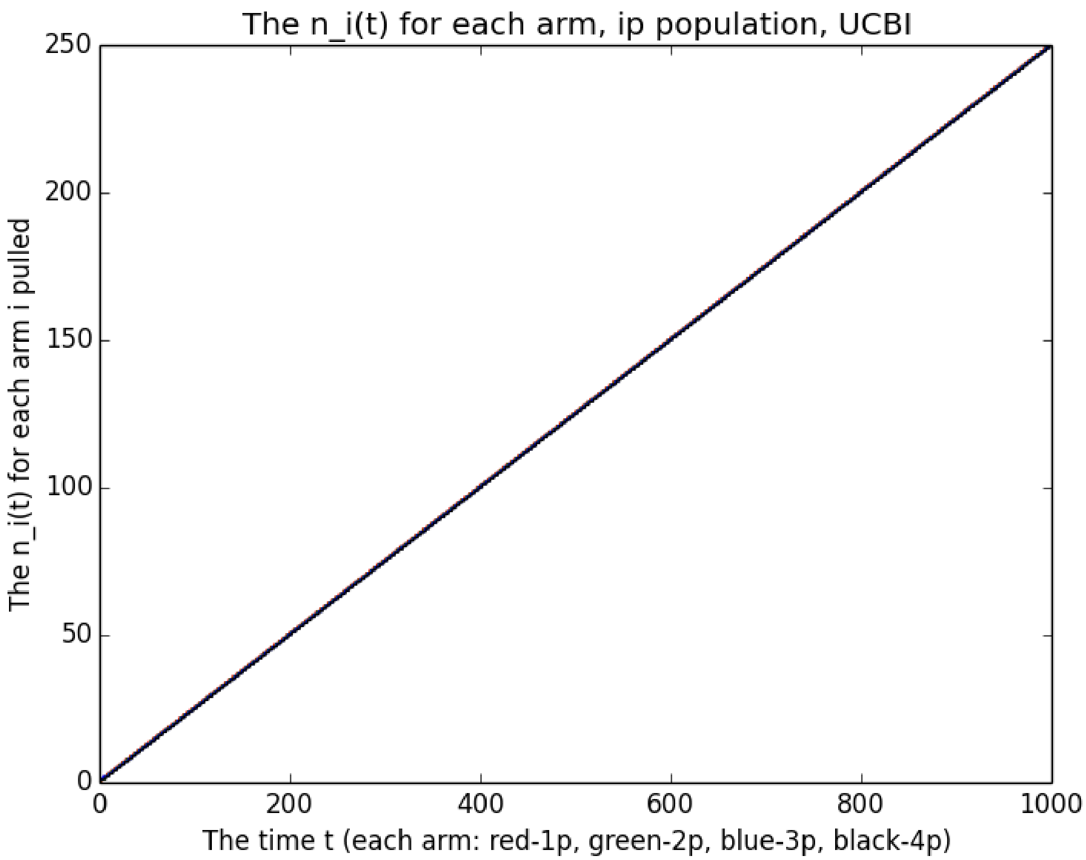

Figure 6: UCBI tested on ip populations.

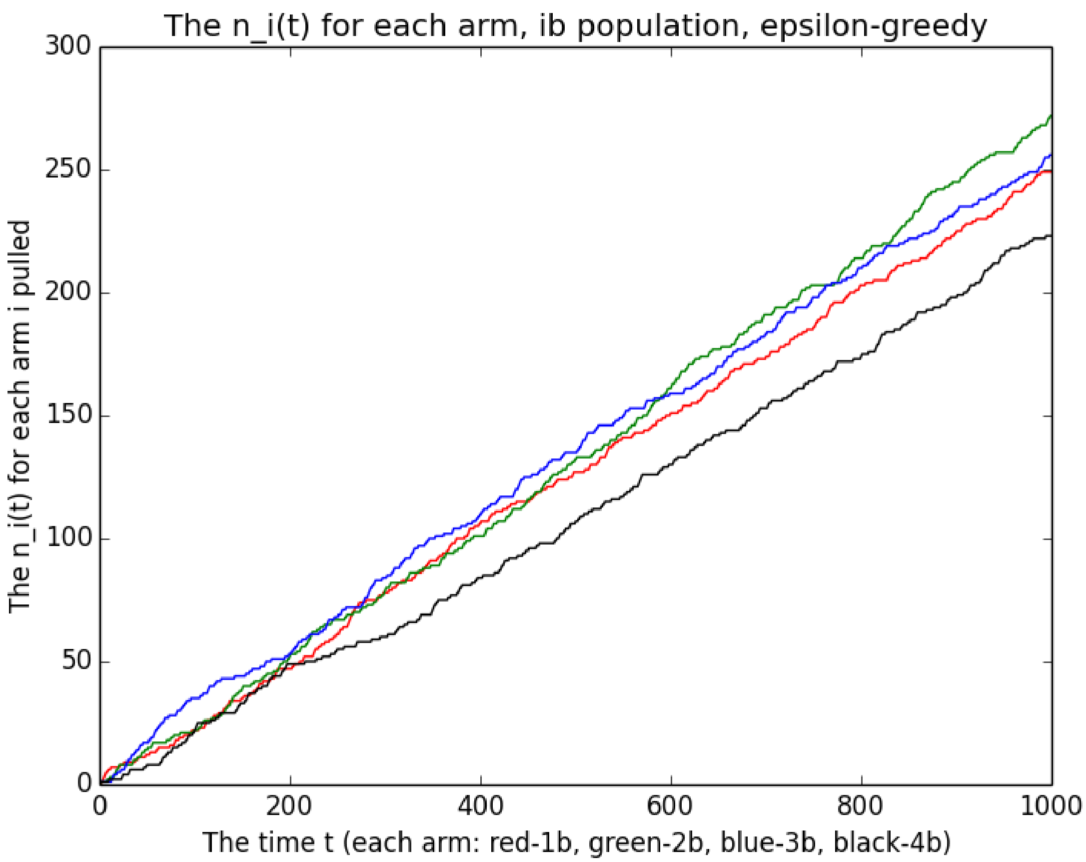

Figure 7: ϵ-greedy tested on ib populations.

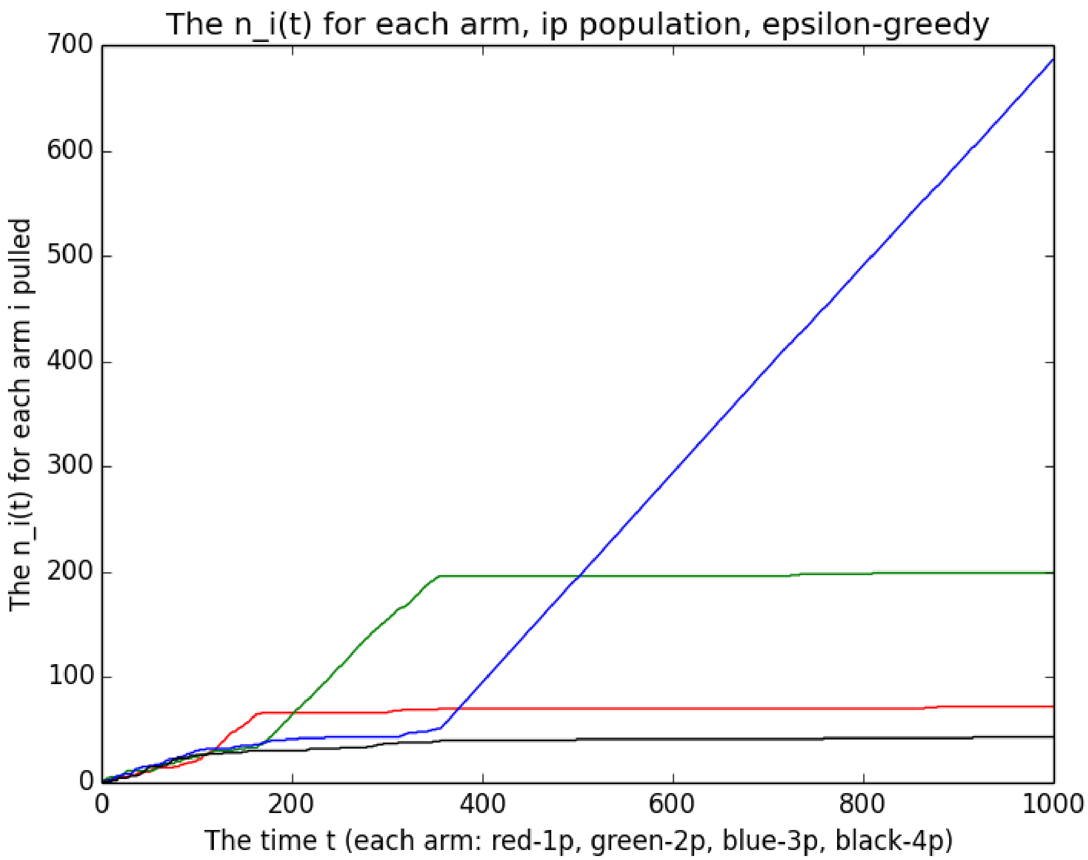

Figure 8: ϵ-greedy tested on ip populations.

The $n_i(t)$ distribution along with time t can be obtained as in Figure 8.
Is ϵ-greedy better than UCBI? We can form a hypothesis testing following the steps we have been through numerous times in Problem 1 (ii).
The t_d score for population ib across epsilon-greedy and UCBI is 0.1098. Hence there demonstrate a statistical significance to reject Null hypothesis, which is ϵ-greedy is better than UCBI.
The t_d score for population ip across epsilon-greedy and UCBI is -1.544. Hence there demonstrate a very strong statistical significance to reject Null hypothesis (highly biased against Null hypothesis), which is ϵ-greedy is better than UCBI.

(iv). We have observed that in (i) and (ii) we have the tendency to equally sample from each arm. **Our hypothesis testing in P2 did not reject the hypothesis that different populations have same average, for most of the case**. Hence, it is reasonable to see arms equally pulled through the entire processes. We have also noticed that, when data has large variance, UCBI is able to obtain an average reward, that is higher than any averages from each arm i, where ϵ-greedy failed to generate similar results. However, if each arm has same average and similar empirical variance, we would be able to obtain similar level of reward using both UCBI and ϵ-greedy.

(v). Thompson Algorithm (for Bernoulli rewards)
Prior arm i: $\mu_i\ Beta(\alpha, \beta)$.

1. For every i, draw $\hat{\mu}_i\ Beta(S_i + \alpha, F_i + \beta)$
2. Choose arm $I_t = argmax_i\{\hat{\mu}_i\ \}$ and get reward Y_t
3. $S_{I_t} = S_{I_t} + Y_t$
4. $F_{I_t} = F_{I_t} + (1 - Y_t)$

(a) With Beta(1,1) prior per arm, the $n_i(t)$ distribution is as shown in Figure 9. The Total reward

$$R_t = 449$$

(b) With Beta(100,100) prior per arm, the $n_i(t)$ distribution is as shown in Figure 10. The Total reward

$$R_t = 437$$

(c) It will probability work better if there are significant difference between different arm's branches (e.g. different average and variance).

(d) If the prior is $Beta(\epsilon, \epsilon), \epsilon \approx 0$, **then the Thompson degenerate into UCBI**, the $n_i(t)$ distribution is as shown in Figure 10. The Total reward

$$R_t = 450$$

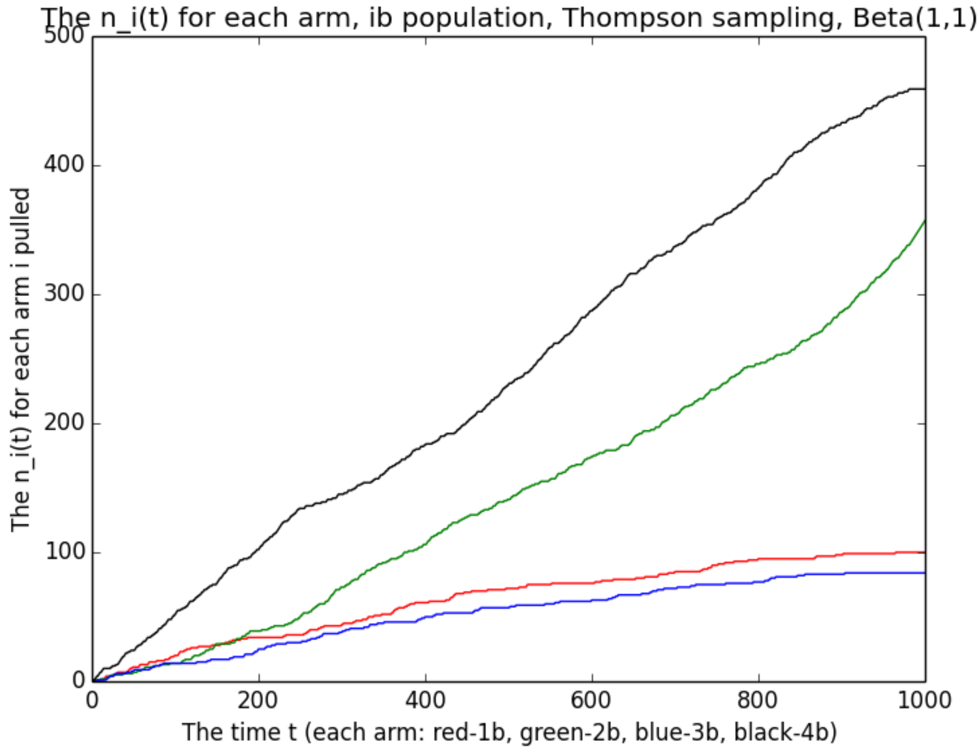

Figure 9: The Thompson Sampling with Beta(1,1)

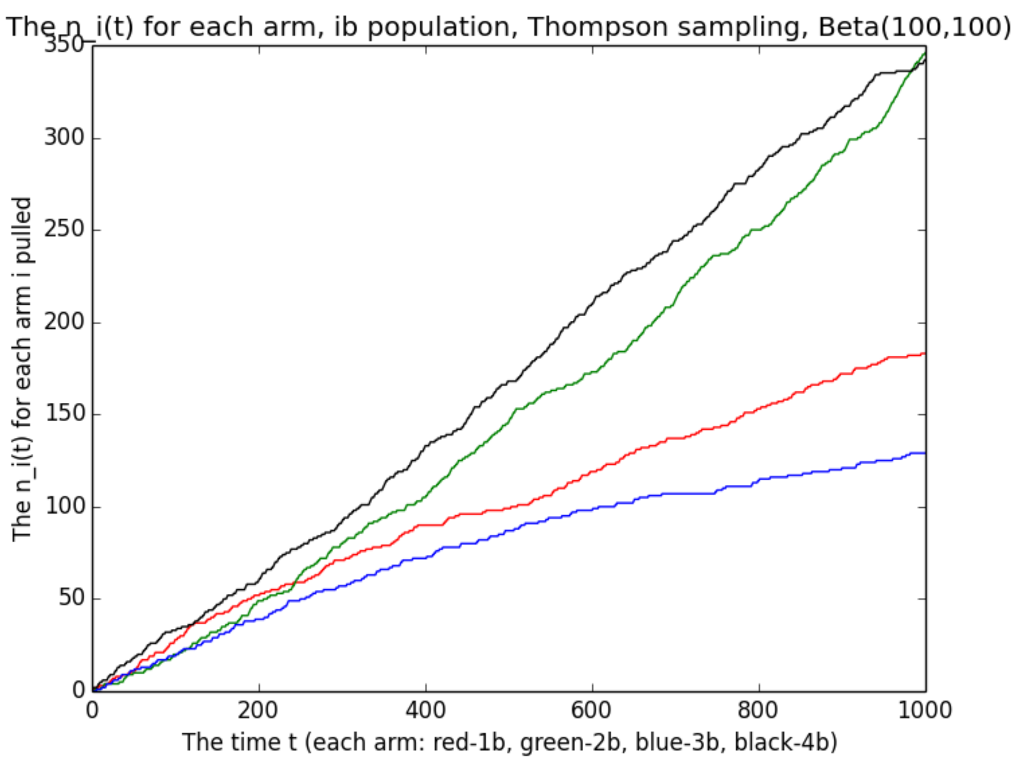

Figure 10: The Thompson Sampling with Beta(100,100)

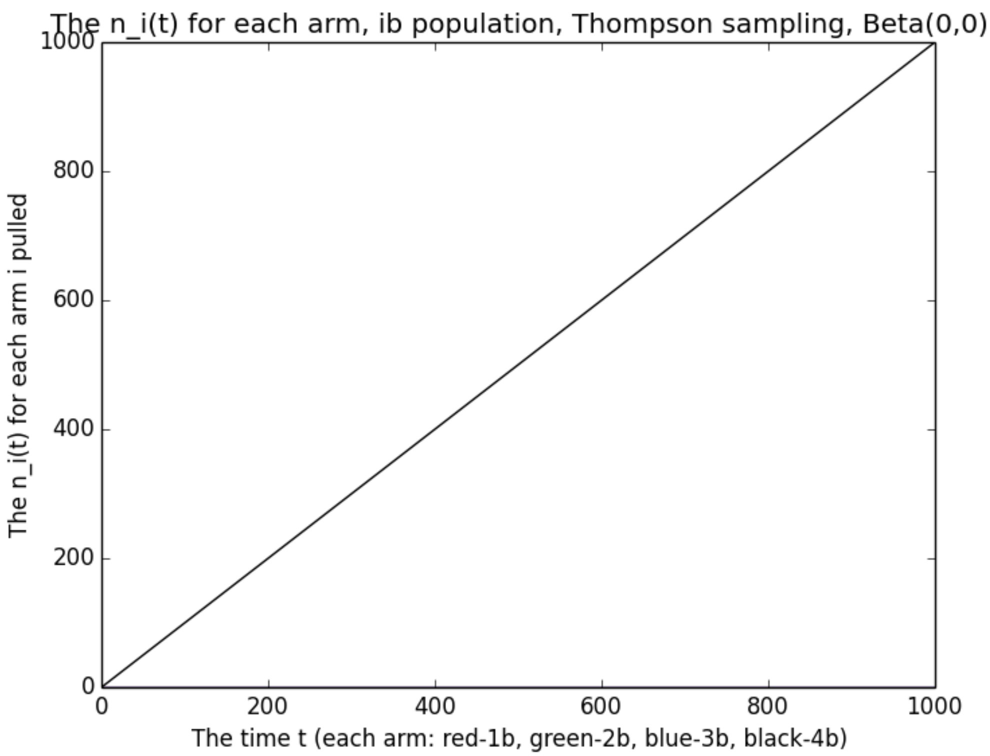

Figure 11: The Thompson Sampling with Beta(ϵ,ϵ)