ECE661: Computer Vision (Fall 2014)

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1 Problem 1

The 3D vector $\mathcal{X} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \in \mathcal{R}^3$ is the homogeneous coordinate representation of a physical 2D point $(x, y) \in \mathcal{R}^2$. Thus, all the points in the representational space $\mathcal{R}^3$ that are the homogeneous coordinates of the origin in the physical space are: $k \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Note that: From the lecture notes it is defined that for any $k \in \mathcal{R},\ k \neq 0,\ k \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ is the same physical point in $\mathcal{R}^2$ as that corresponding to $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

2 Problem 2

All points at infinity in the physical plane $\mathcal{R}^2$ are **NOT** the same.

Justification: By definition in lecture notes, the intersection of the two parallel lines $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c' \end{pmatrix}$ is at point: $\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$, which is a point at infinity in $\mathcal{R}^2$, along a specific direction controlled by the pair $\begin{pmatrix} a \\ b \end{pmatrix}$. Hence, all points at infinity of physical plane $\mathcal{R}^2$ are not the same, but they are grouped as **Ideal Points**

3 Problem 3

$$l_1 = x_1 \times x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$l_2 = y_1 \times y_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ 7 \end{pmatrix}$$

Thus the intersection of l_1 and l_2 is at: $l_1 \times l_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 7 \\ -7 \\ 7 \end{pmatrix} = \begin{pmatrix} -7 \\ -7 \\ 0 \end{pmatrix}$

Done in three steps!

Now substitute y_1 and y_2 with $y_1' = (3,\ 4)$ and $y_2' = (-3,\ -4)$

$l_2' = y_1' \times y_2' = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix}$

Thus the intersection of l_1 and l_2' is at: $l_1 \times l_2' = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 8 \\ -6 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

It will still take three steps.

4 Problem 4

From lecture notes, every point on the degenerate conic C is either on line l or on line m, $C = lm^T + ml^T$. Assume that $l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$, $m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$

Easily: $lm^T = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \cdot (m_1,\ m_2,\ m_3) = \begin{bmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{bmatrix}$

Similarly: $ml^T = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \cdot (l_1,\ l_2,\ l_3) = \begin{bmatrix} m_1l_1 & m_1l_2 & m_1l_3 \\ m_2l_1 & m_2l_2 & m_2l_3 \\ m_3l_1 & m_3l_2 & m_3l_3 \end{bmatrix}$

$rank(lm^T) = 1$ because every column is a constant times the first column. E.g. the second column is $\frac{m_2}{m_1}$ times the first column. So all columns are linearly dependent. **Thus,** $rank(lm^T) = rank(ml^T) = 1$.

Now Assume that rank of a degenerate conic matrix could exceed 2: $rank(C) \geq 3$. Then this implies $rank(C) = rank(lm^T + ml^T) \geq 3$.

But according to matrix rank axioms, $rank(A + B) \leq rank(A) + rank(B)*$, we have $rank(C) = rank(lm^T + ml^T) \leq rank(lm^T) + rank(ml^T) = 2 \leq 3$. **CONTRADICTION TO ASSUMPTION!**

Thus, the matrix rank of degenerate conic can never exceed 2.

*`Proof of` $rank(A + B) \leq rank(A) + rank(B)$:

The rank of a matrix can be defined as the dimensions of the space that its column vectors span. Let $(a^1, a^2, ..., a^n)$ denote the column vectors of A, $(b^1, b^2, ..., b^n)$ denote the column vectors of B, and the columns vectors for (A + B) are linear combinations of a^i, b^i so the dimension of the space spanned by them can not be greater than the sum of spaces spanned by A and B.

Reference: `http://statweb.stanford.edu/~candes/acm104/Hw/hw2sol.pdf`

5 Problem 5

As our conic is a circle of radius 1 that is centered at the coordinates $(5,\ 5)$:

$$(x-5)^2 + (y-5)^2 = 1$$

$$x^2 - 10x + 25 + y^2 - 10y + 25 = 1$$

$$x^2 - 10x + y^2 - 10y + 49 = 0$$

Write the equation in standard form for conic $x^T C x = 0$:

$$(x,\ y,\ 1) \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0,\ while\ C = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{bmatrix}$$

As x is the origin point of the $\mathcal{R}^2$ physical plane, $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $y - axis$ is denoted as:

$y - axis = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$.

The polar line is: $polar\ line = Cx = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -5 \\ -5 & -5 & 49 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ 49 \end{pmatrix}$

The intercept of polar line with y-axis is: $\begin{pmatrix} -5 \\ -5 \\ 49 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -49 \\ -5 \end{pmatrix} = k \cdot \begin{pmatrix} 0 \\ \frac{49}{5} \\ 1 \end{pmatrix}, k = -5$