ACS6501 Foundation of Robotics
System Modelling and Simulation Assignment

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1 Part A: Task 1

Question: Derive a linear approximation mathematical model of the pendulum system (in the question paper) to describe the relationship between the torque developed on the mass M and the angle theta between the rod and the vertical plane. State clearly your assumptions and your modelling methodology.

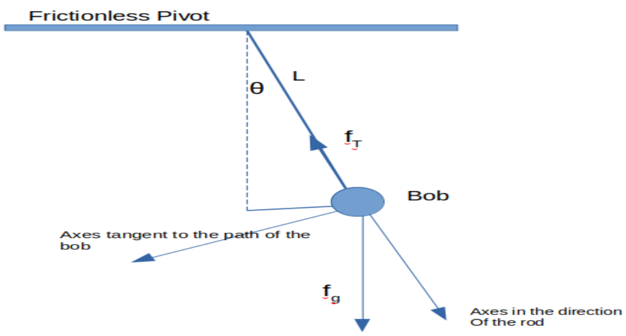


Figure 1: Simple Pendulum

In order to derive the mathematical model of the given system I am assuming that the centre of mass of the bob is in the centre of the bob and the mass of the string by which it is hanging is negligible and without friction. $\Theta(t)$ is the angle at time t. By Newton's second law of motion we know that

$$\sum F = M \cdot a(t) \tag{1}$$

where $\sum F$ is the sum of all forces acting on the mass M, M is the constant mass, and $a(t)$ is acceleration of mass at time t. Now focusing on the term $\sum F$, it is observed that two forces acts on the mass at the end of the bob.

$$\sum \vec{F} = [tension] + [gravity] \tag{2}$$

$$\sum \vec{F} = \vec{f_T} + \vec{f_g} \tag{3}$$

The magnitude of $\vec{f_g} = M.g$. Since the direction is not specified in terms of components. Now $\vec{f_g}$ has two components.

$$\vec{f_g} = \vec{f_h} + \vec{f_i} \tag{4}$$

where $\vec{f_h}$ is the component of the gravity in the direction of the axes of chord tension, $\vec{f_g}$ is the component of the gravity perpendicular to the axes of the tension. These two forces are perpendicular to each other. Since the motion of the mass takes an arc-shaped path and the string/rod is not extensible with a constant length, the total force acting in the direction of the rod/string is zero. Therefore,

$$\vec{f_h} + \vec{f_T} = 0 \tag{5}$$

So,

$$\left|\sin\left(-\Theta\left(t\right)\right)\right| = \frac{\left\|\vec{f_i}\right\|}{\left\|\vec{f_g}\right\|} \tag{6}$$

Now the length of the vector $\vec{f_i}$ will be

$$\left\|\vec{f_i}\right\| = \left\|\vec{f_g}\right\| \cdot \left|\sin -\Theta\left(t\right)\right| = M \cdot g \left|\sin -\Theta\left(t\right)\right| \tag{7}$$

Let $s(t)$ be the arc-length of the segment between equilibrium position of the bob with mass M and the position of the bob at time t. Hence,

$$x\left(t\right) = L \cdot \Theta\left(t\right) \tag{8}$$

$$a\left(t\right) = \ddot{x}\left(t\right) \tag{9}$$

Also,

$$\dot{x}\left(t\right) = L \cdot \dot{\Theta}\left(t\right) \tag{10}$$

According newton's second law

$$\begin{aligned} \sum F &= M \cdot a\left(t\right) \\ \Rightarrow -M \cdot g \cdot \sin\left(\Theta\left(t\right)\right) &= M \cdot L \cdot \ddot{\Theta}\left(t\right) \\ \Rightarrow -g \cdot \sin\left(\Theta\left(t\right)\right) &= L \cdot \ddot{\Theta}\left(t\right) \\ \Rightarrow \ddot{\Theta}\left(t\right) &= -\frac{g}{L} \cdot \sin\left(\Theta\left(t\right)\right) \end{aligned} \tag{11}$$

Now, considering the torque of the pendulum, the force providing the restoring torque is the component of the mass of the pendulum bob that acts along the length of the arc. Therefore, the torque is given as

$$\ddot{\Theta}\left(t\right) = -\frac{g}{L} \cdot \sin\left(\Theta\left(t\right)\right) \tag{12}$$

Multiplying both sides by $M \cdot L^2$

$$\begin{aligned} M \cdot L^2 \ddot{\Theta}\left(t\right) &= -L \cdot M \cdot g \cdot \sin\left(\Theta\left(t\right)\right) \\ \Rightarrow I \cdot \ddot{\Theta}\left(t\right) &= -L \cdot M \cdot g \cdot \sin\left(\Theta\left(t\right)\right) \\ \Rightarrow I \cdot \alpha &= -L \cdot M \cdot g \cdot \sin\left(\Theta\left(t\right)\right) \\ \Rightarrow \tau &= -L \cdot M \cdot g \cdot \sin\left(\Theta\left(t\right)\right) \end{aligned} \tag{13}$$

Where I is the moment of inertia and α is angular acceleration, and τ is torque.

2 Part A: Task 2

Question: For the mechanical system (simplified car suspension) shown in Figure 2 (a). Derive the mathematical model of the system in the time domain to describe the displacement associated with the three masses M1 (wheel), M2 (chassis) and M3 (car seat). b. Express the mathematical model of the system in the Laplace Domain

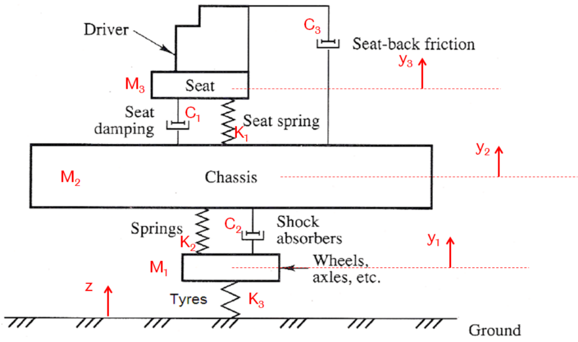


Figure 2: Simple Car Chassis

2.1 Answer (a)

The model in Figure 2 consists of three masses $M1, M2, M3$ which are mass of the wheel, mass of the chassis and mass of the seat, respectively. The seat spring, damping, friction coefficients are K_1, C_1, C_3. Chassis spring and shock absorb coefficients are $K_1, C2$. The wheel ground shock absorb coefficient is K_3.

By newton's second law we know

$$\sum F = M \cdot a \tag{14}$$

By D'Alembert's principal we know

$$\sum F = 0 \tag{15}$$

Which is the sum of all the forces internal and external acting on the body is zero when the body is in equilibrium state.

Here I assume that the mass of the seat includes the driver's mass, and $C3, C1$ are working parallel under $M3$.

$$F_{e_3} = K_1(y_2 - y_3) \tag{16}$$

Where F_{e_3} is the elastic force generated by the spring K_1 and y_2 is the displacement of the chassis, and y_3 is the displacement of the seat.

$$F_{i_3} = M_3 \cdot a_3 = M_3 \cdot \ddot{y}_3 \tag{17}$$

Where F_{i_3} is force of inertia generated by mass M_3.

$$F_{d_3} = c_1\left(\dot{y}_2 - \dot{y}_3\right) + c_3\left(\dot{y}_2 - \dot{y}_3\right) \tag{18}$$

Where F_{d_3} is the damping force generated by the c_1 and c_3. Now, by D'Alembert's principal we get

$$M_3\ddot{y}_3 = k_1\left(y_2 - y_3\right) + c_1\left(\dot{y}_2 - \dot{y}_3\right) + c_3\left(\dot{y}_2 - \dot{y}_3\right) \tag{19}$$

Similarly for mass M_2 and M_1 by D'Alembert's principal we get

$$M_2\ddot{y}_2 = k_2\left(y_1 - y_2\right) + c_2\left(\dot{y}_1 - \dot{y}_2\right) - c_1\left(\dot{y}_2 - \dot{y}_3\right) - c_3\left(\dot{y}_2 - \dot{y}_3\right) - k_1\left(y_2 - y_3\right) \tag{20}$$

$$M_1\ddot{y}_1 = k_3\left(z - y_1\right) - k_2\left(y_1 - y_2\right) - c_2\left(\dot{y}_1 - \dot{y}_2\right) \tag{21}$$

Where y_1 is the displacement of the wheel and z is the ground reaction given by the height of the road profile.

2.2 Answer (b)

Now, I will be applying Laplace transform on the previously derived differential equations of equation number 19, 20 and 21.

$$\begin{aligned}
M_3\ddot{y_3} &= k_1\left(y_2-y_3\right)+c_1\left(\dot{y_2}-\dot{y_3}\right)+c_3\left(\dot{y_2}-\dot{y_3}\right)\\
\Rightarrow M_3\ddot{y_3} &= k_1\cdot y_2-k_1\cdot y_3+(c_1+c_3)\dot{y_2}-(c_1+c_3)\dot{y_3}\\
\Rightarrow \ddot{y_3} &= -\frac{(c_1+c_3)}{M_3}\dot{y_3}+\frac{(c_1+c_3)}{M_3}\dot{y_2}-\frac{k_1}{M_3}y_3+\frac{k_1}{M_3}y_2\\
\Rightarrow s^2Y_3\left(s\right)-sY_3\left(0\right)-\dot{Y_3}\left(0\right) &= -\frac{(c_1+c_3)}{M_3}\left[sY_3\left(s\right)-Y_3\left(0\right)\right]+\frac{(c_1+c_3)}{M_3}\left[sY_2\left(s\right)-Y_2\left(0\right)\right]\\
&\quad -\frac{k_1}{M_3}Y_3\left(s\right)+\frac{k_1}{M_3}Y_2\left(s\right)
\end{aligned} \tag{22}$$

Considering the initial condition $Y_3(0)=0$, $\dot{Y_3}(0)=0$, and $Y_2(0)=0$, the previous equation will become

$$\begin{aligned}
\left[s^2+\frac{c_1+c_3}{M_3}s+\frac{k_1}{M_3}\right]\cdot Y_3\left(s\right) &= \left[\frac{c_1+c_3}{M_3}s+\frac{k_1}{M_3}\right]\cdot Y_2\left(s\right)\\
\Rightarrow Y_3(s) &= \frac{\left[\frac{c_1+c_3}{M_3}s+\frac{k_1}{M_3}\right]}{\left[s^2+\frac{c_1+c_3}{M_3}s+\frac{k_1}{M_3}\right]}Y_2\left(s\right)
\end{aligned} \tag{23}$$

After applying Laplace transform on equation 20,

$$M_2\ddot{y_2}=k_2\left(y_1-y_2\right)+c_2\left(\dot{y_1}-\dot{y_2}\right)-c_1\left(\dot{y_2}-\dot{y_3}\right)-c_3\left(\dot{y_2}-\dot{y_3}\right)-k_1\left(y_2-y_3\right)$$

$$\Rightarrow \ddot{y_2}+\frac{k_2}{M_2}y_2+\frac{c_2}{M_2}\dot{y_2}+\frac{(c_1+c_3)}{M_2}\dot{y_2}+\frac{k_1}{M_2}y_2=\frac{k_2}{M_2}y_1+\frac{c_2}{M_2}\dot{y_1}+\frac{(c_1+c_3)}{M_2}\dot{y_3}+\frac{k_1}{M_2}y_3 \tag{24}$$

After transformation and expansion

$$\begin{aligned}
&\left[s^2Y_2\left(s\right)-sY_2\left(0\right)-\dot{Y_2}\left(0\right)\right]+\frac{k_2}{M_2}Y_2(s)+\frac{c_2}{M_2}\left[sY_2(s)-Y_2(0)\right]\\
&\quad+\frac{(c_1+c_3)}{M_2}\left[sY_2(s)-Y_2(0)\right]+\frac{k_1}{M_2}Y_2(s)=\frac{c_2}{M_2}\left[sY_1(s)-Y_1(0)\right]+\frac{k_2}{M_2}Y_1(s)\\
&\quad+\frac{(c_1+c_3)}{M_2}\left[sY_3(s)-Y_3(0)\right]\\
&\quad+\frac{k_1}{M_2}Y_3(s)
\end{aligned} \tag{25}$$

Considering the initial condition $Y_2(0)=0$, $\dot{Y_2}(0)=0$, $Y_1(0)=0$,$Y_3(0)=0$,

$$\left[s^2+\frac{(c_1+c_2+c_3)\,s}{M_2}+\frac{k_1+k_2}{M_2}\right]Y_2(s)=\left[\frac{sc_2+k_2}{M_2}\right]Y_1(s)+\left[\frac{s(c_1+c_3)+k_1}{M_2}\right]Y_3(s)$$

$$\Rightarrow Y_2(s)=\frac{\left[\frac{sc_2+k_2}{M_2}\right]}{\left[s^2+\frac{(c_1+c_2+c_3)s}{M_2}+\frac{k_1+k_2}{M_2}\right]}Y_1(s)+\frac{\left[\frac{s(c_1+c_3)+k_1}{M_2}\right]}{\left[s^2+\frac{(c_1+c_2+c_3)s}{M_2}+\frac{k_1+k_2}{M_2}\right]}Y_3(s) \tag{26}$$

After applying Laplace transform on equation 21,

$$\begin{aligned} M_1\ddot{y_1} &= k_3\left(z-y_1\right)-k_2\left(y_1-y_2\right)-c_2\left(\dot{y_1}-\dot{y_2}\right)\\ \Rightarrow \ddot{y_1} &= \frac{k_3}{M_1}z-\frac{k_3}{M_1}y_1-\frac{k_2}{M_1}y_1+\frac{k_2}{M_1}y_2-\frac{c_2}{M_1}\dot{y_1}+\frac{c_2}{M_1}\dot{y_2} \\ \Rightarrow \ddot{y_1}+\frac{k_3}{M_1}y_1+\frac{k_2}{M_1}y_1+\frac{c_2}{M_1}\dot{y_1} &= \frac{k_3}{M_1}z+\frac{k_2}{M_1}y_2+\frac{c_2}{M_1}\dot{y_2} \end{aligned} \tag{27}$$

After transformation and expansion

$$s^2Y_1(s)-sY_1(0)-\dot{Y_1}(0)+\frac{k_3}{M_1}Y_1(s)+\frac{k_2}{M_1}Y_1(s)+\frac{c_2}{M_1}\left[sY_1(s)-Y_1(0)\right]=\frac{k_3}{M_1}z(s)+\frac{k_2}{M_1}Y_2(s) + \frac{c_2}{M_1}\left[sY_2(s)-Y_2(0)\right] \tag{28}$$

Considering the initial condition $Y_1(0)=0$, $\dot{Y_1}(0)=0$, $Y_2(0)=0$,

$$Y_1(s)=\frac{\left[\frac{c_2 s}{M_1}+\frac{k_2}{M_1}\right]}{\left[s_2+\frac{c_2}{M_1}s+\frac{k_2+k_3}{M_1}\right]}Y_2(s)+\frac{\frac{k_3}{M_1}}{\left[s_2+\frac{c_2}{M_1}s+\frac{k_2+k_3}{M_1}\right]}z(s) \tag{29}$$

3 Part B: Task 1

Task 1 :What is a signal flow graph, and how does it compare against the block diagram representation? For the system modelled in Part A, Task 2 derive the signal flow graph (working methodology and diagram).

3.0.1 Signal Flow Graph:

A signal flow graph or SFG is a graphical depiction of a set of algebric equations which consists of nodes and branches, where a node is a point that presents either a variable or a signal. A branch in a SGF is a line segment that connects two nodes, which has both gain and direction.

3.0.2 Comparison between Signal Flow Graph and Block Diagram

Topic	Block Diagram	Signal Flow Graph
Aplpication to Linear time invarient (LTI) systems	Applicable to LTI systems	Applicable to LTI systems
Representation	Each element is represented by block	Each element is represented by node
Summing points	summing points and takeoff points are separate	not used
Self loop	Self loops do not exist	Self loop exists
Time consumption	Time consuming method	Requires less time than block diagram
Feedback path	Feedback paths exist	Feedback paths exist.

3.0.3 Signal flow graph derivation

The system equations derived for *Part A Task 2* will be used here to derive the SFG. Equation 23, 26, 29. Here, I will represent these equations in simplified form to represent them in the SFG graphs easily. Therefore Equation 23, 26, 29 will become

$$Y_3(s) = F_{23}Y_2\,(s) \tag{30}$$

$$Y_2(s) = F_{12} \cdot Y_1(s) + F_{32}Y_3(s) \tag{31}$$

$$Y_1(s) = F_{21}Y_2(s) + F_{z1}z(s) \tag{32}$$

where $\frac{\left[\frac{c_1+c_3}{M_3}s+\frac{k_1}{M_3}\right]}{\left[s^2+\frac{c_1+c_3}{M_3}s+\frac{k_1}{M_3}\right]}$ is F_{23}, $\frac{\left[\frac{sc_2+k_2}{M_2}\right]}{\left[s^2+\frac{(c_1+c_2+c_3)s}{M_2}+\frac{k_1+k_2}{M_2}\right]}$ is F_{12}, $\frac{\left[\frac{s(c_1+c_3)+k_1}{M_2}\right]}{\left[s^2+\frac{(c_1+c_2+c_3)s}{M_2}+\frac{k_1+k_2}{M_2}\right]}$ is F_{32}, $\frac{\left[\frac{c_2 s}{M_1}+\frac{k_2}{M_1}\right]}{\left[s2+\frac{c_2}{M_1}s+\frac{k_2+k_3}{M_1}\right]}$ is F_{21} and $\frac{\frac{k_3}{M_1}}{\left[s2+\frac{c_2}{M_1}s+\frac{k_2+k_3}{M_1}\right]}$ is F_{z1}. From the algebric equations the SFG is given below

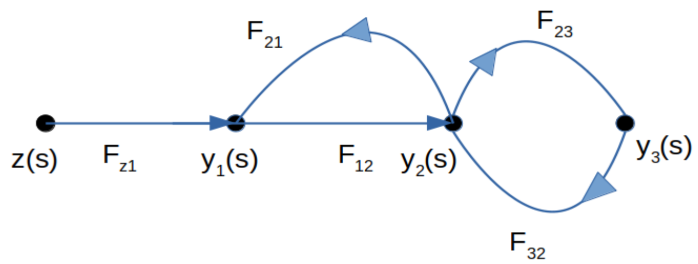


Figure 3: Signal Flow Graph from Equation 30,31, and 32

4 Part B: Task 2

Using the module's handout "Practical simulation experiments using SIMULINK", complete the simulation exercise Positioning System and answer all relevant questions/tasks.

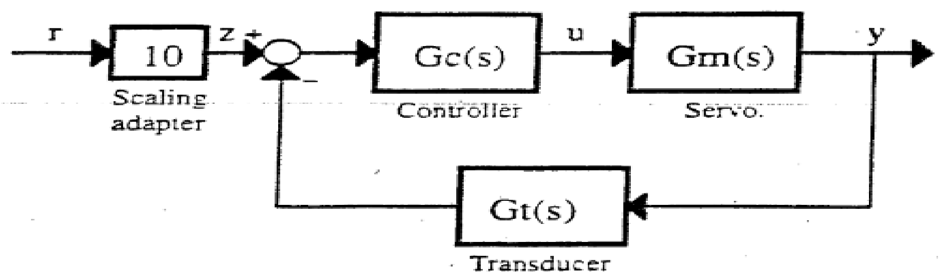


Figure 4: Block diagram of the given displacement control system.

The schematic diagram of a simplified positioning system is shown in the Part B Task 2 assignment. The figure represents the angular displacement of the DC motor, which is translated into linear movement. The block diagram is given in Figure 4, and the state equations are given below.

$$G_c(s) = \frac{\frac{1}{RC}}{s + \frac{1}{PC}} \tag{33}$$

$$G_m(s) = k_m \cdot k_s \cdot \frac{1}{s} \tag{34}$$

$$G_t(s) = \frac{E}{d} \tag{35}$$

where the values of the variables are given as $R = 10K\Omega$, $C = 0.33\mu F$,$k_s = 52.5rad/sV$,$k_m = 0.005m/rad$, $E = 10V$,$d = 1metere$. The simulink model is shown in Figure 5. After the task 1 experiment I added saturation and backlash blocks for further experiments.

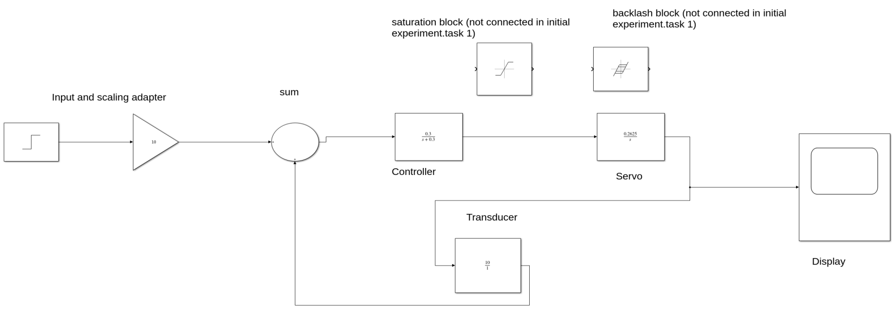
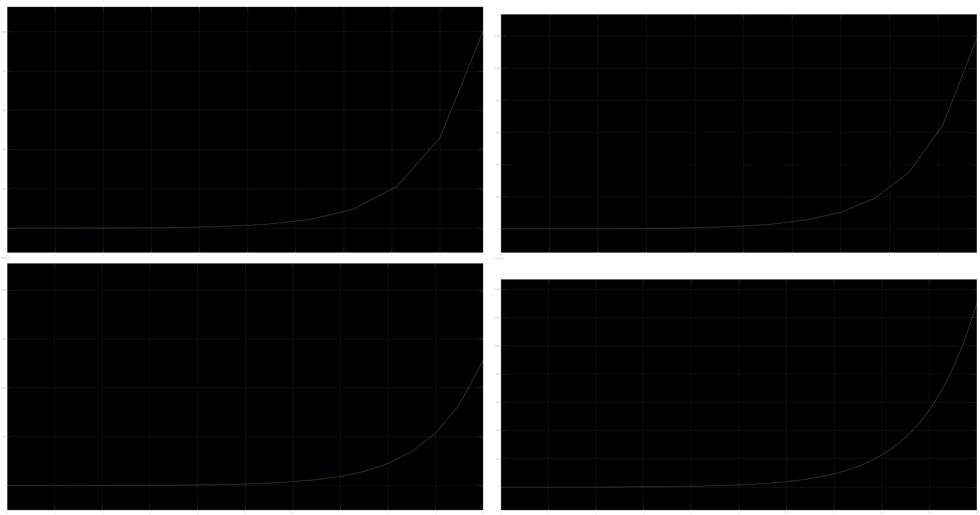


Figure 5: Simulink model of the control system

Task 1:

The goal of this task is to understand the behavior of step size (the starting point is 0.5). I observed smoothness behavior in the samples while gradually increasing and decreasing the step size. The less step size causes finer and smoother graphical representation.

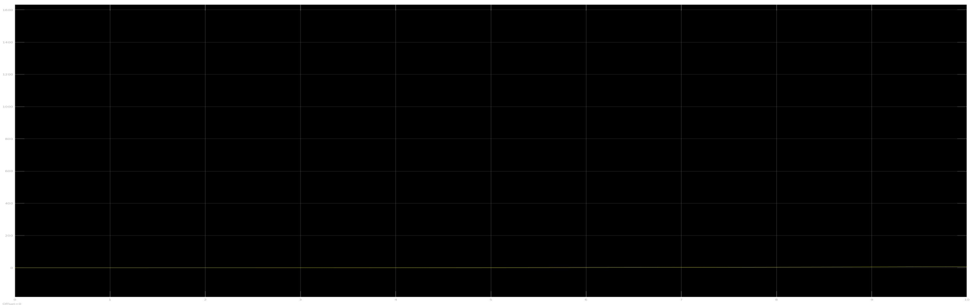


Step size experiment Task 1: Top left Step size 0.9, Top Right Step size 0.7, Bottom left Step size 0.5, Bottom right step size 0.1

The displacement curve gets gradually smoother with smaller step size. The more the step size the samples are taken in higher duration, resulting less smooth graph.

Task 2:

The goal of task 2 is to find the appropriate value of P to optimise the system to reduce overshooting and quicker response. I have increased the P value 10 times, and the overshooting behavior increased significantly. So I started to reduce the value gradually and I noticed a reduction in overshooting. At P value 1Ω the graph almost overlapped with the X axis with almost no overshooting. Hence, I found that $P = 1\Omega$ is a optimised value.



P adjustment experiment Task 2: I have observed that with 1kilo ohm P value the graph almost coincided with x-axis.

Task 3:

As part of task 3, I added the saturation function in my model as instructed in the task. I found out that the graph behaved similarly when I ran the model with $P = 1K$. So, I came to the conclusion that saturation clips the input signal and limits it to be bound between an upper and lower signal (Figure 6.

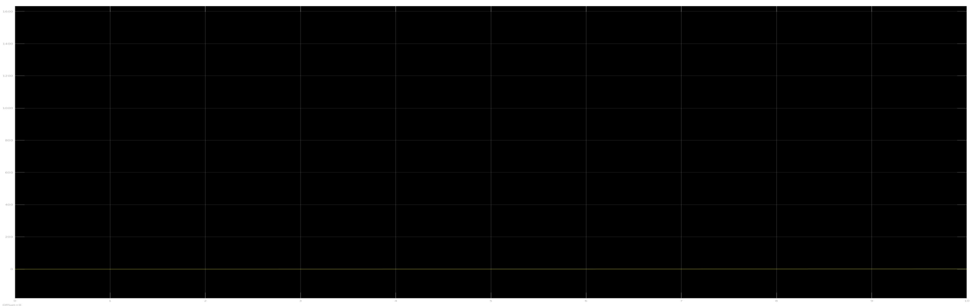


Figure 6: Adding Saturation to the command signal. No overshooting.

Task 4: As task 4 I applied backlash with 0.02. It engages with the input signal depending on if it is within the deadband. The output graph is shown in figure 7.

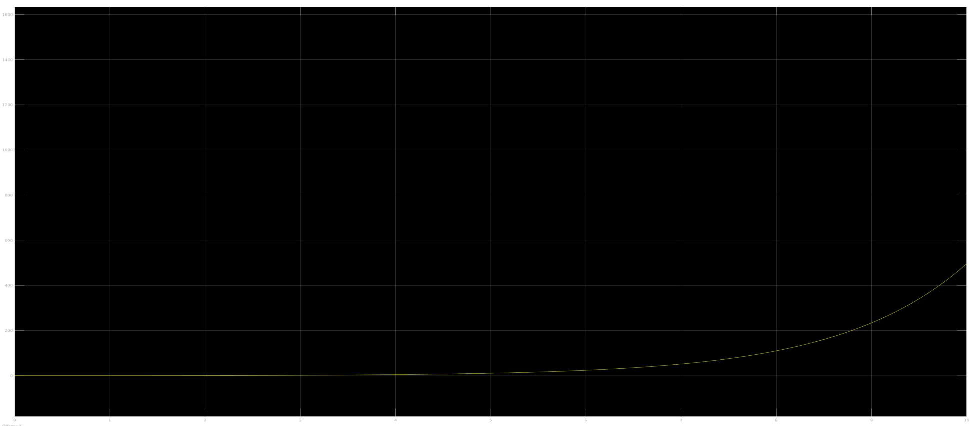


Figure 7: Effect of 0.02 backlash in the gear