Differential Geometry

Lecture Notes, T1 2023

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Welcome

These are the Lecture Notes of **Differential Geometry 600727** for T1 2023 at the University of Hull. We will study curves and surfaces in $\mathbb{R}^3$. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

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Up to date informations about the course and homework will be published on the **`course webpage`**

silviofanzon.com/blog/2023/Differential-Geometry

A **pdf** version of the notes is available to download on the top-right.

References

The main textbook of the course is Pressley [5]. Other useful references are the books by do Carmo [2] and Bär [1]. I will assume some knowledge from Analysis and Linear Algebra. A good place to revise these topics are the books by Zorich [6, 7].

Visualization

It is important to visualize the geometrical objects and concepts we are going to talk about in this course. I will show basic Python code to plot curves and surfaces. This part of the course is **`not required`** for the final examination. If you want to have fun plotting with Pyhton, I recommend installation through Anaconda or Miniconda. The actual coding can then be done through Jupyter Notebook. Good references for scientific Python programming are [3, 4].
If you do not want to mess around with Python, do not despair. You can still visualize pretty much everything we will do in this course using the excellent online 3D grapher tool CalcPlot3D. To understand how it works, please refer to the help manual or to the short video introduction.

! Important

You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to known in order to excel in the final exam.

1 Curves

2 Plotting curves

2.1 Curves in 2D

Suppose we want to plot the parabola $y = t^2$ for t in the interval $[-3, 3]$. In our language, this is the two-dimensional curve

$$\gamma(t) = (t, t^2), \quad t \in [-3, 3].$$

The two Python libraries we use to plot γ are **numpy** and **matplotlib**. In short, **numpy** handles multi-dimensional arrays and matrices, and can perform high-level mathematical functions on them. For any question you may have about numpy, answers can be found in the searchable documentation available here. Instead **matplotlib** is a plotting library, with documentation here. Python libraries need to be imported every time you want to use them. In our case we will import:

```
import numpy as np
import matplotlib.pyplot as plt
```

The above imports **numpy** and the module **pyplot** from **matplotlib**, and renames them to `np` and `plt`, respectively. These shorthands are standard in the literature, and they make code much more readable.

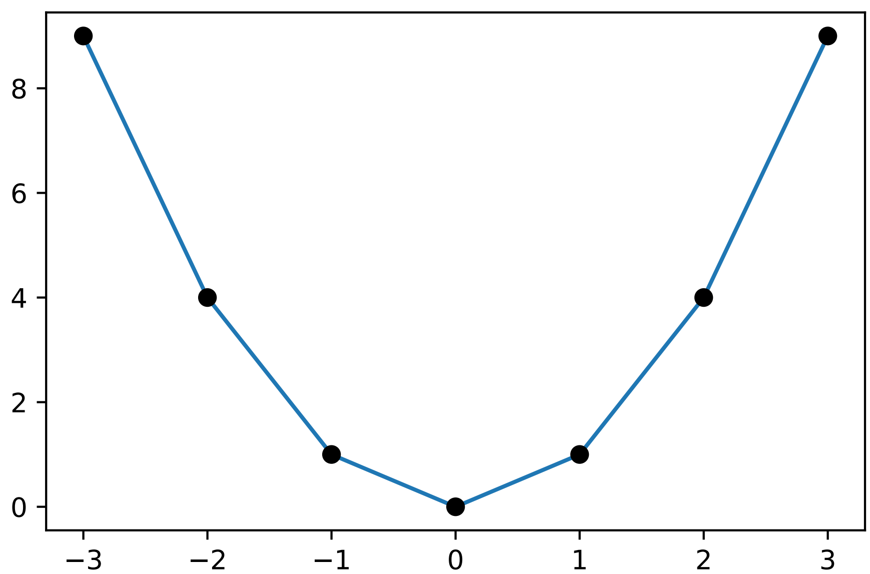
The function for plotting 2D graphs is called `plot(x,y)` and is contained in `plt`. As the syntax suggests, `plot` takes as arguments two arrays

$$x = [x_1, \dots, x_n], \quad y = [y_1, \dots, y_n].$$

As output it produces a graph which is the linear interpolation of the points (x_i, y_i) in $\mathbb{R}^2$, that is, consecutive points (x_i, y_i) and (x_{i+1}, y_{i+1}) are connected by a segment. Using `plot`, we can graph the curve $\gamma(t) = (t, t^2)$ like so:

```
# Code for plotting gamma

import numpy as np
import matplotlib.pyplot as plt

# Generating array t
t = np.array([-3,-2,-1,0,1,2,3])
```

```
# Computing array f
f = t**2

# Plotting the curve
plt.plot(t,f)

# Plotting dots
plt.plot(t,f,"ko")

# Showing the plot
plt.show()
```



Let us comment the above code. The variable `t` is a numpy array containing the ordered values

$$t = [-3, -2, -1, 0, 1, 2, 3]\,. \tag{2.1}$$

This array is then squared entry-by-entry via the operation $t**2$ and saved in the new numpy array `f`, that is,

$$f = [9, 4, 1, 0, 1, 4, 9]\,.$$

The arrays `t` and `f` are then passed to `plot(t,f)`, which produces the above linear interpolation, with `t` on the *x-axis* and `f` on the *y-axis*. The command `plot(t,f,'ko')` instead plots a black dot at each point (t_i, f_i). The latter is clearly not needed to obtain a plot, and it was only included to highlight the fact that `plot` is actually producing a linear interpolation between

points. Finally `plt.show()` displays the figure in the user window[1].
Of course one can refine the plot so that it resembles the continuous curve $\gamma(t) = (t, t^2)$ that we all have in mind. This is achieved by generating a numpy array `t` with a finer stepsize, invoking the function `np.linspace(a,b,n)`. Such call will return a numpy array which contains `n` evenly spaced points, starts at `a`, and ends in `b`. For example `np.linspace(-3,3,7)` returns our original array `t` at Equation 2.1, as shown below

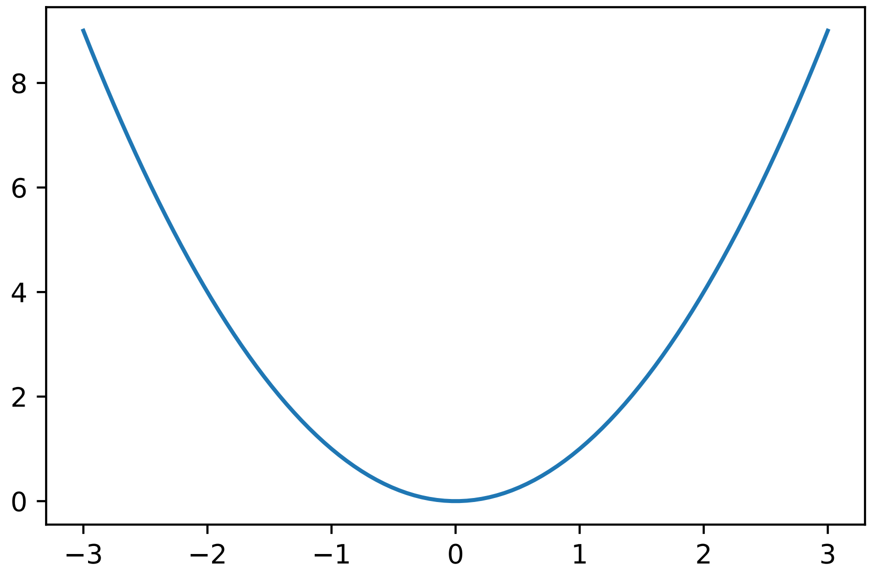
```
# Displaying output of np.linspace

import numpy as np

# Generates array t by dividing interval
# (-3,3) in 7 parts
t = np.linspace(-3,3, 7)

# Prints array t
print("t =", t)
```

```
t = [-3. -2. -1.  0.  1.  2.  3.]
```

In order to have a more refined plot of γ, we just need to increase n.

```
# Plotting gamma with finer step-size

import numpy as np
import matplotlib.pyplot as plt

# Generates array t by dividing interval
# (-3,3) in 100 parts
t = np.linspace(-3,3, 100)

# Computes f
f = t**2

# Plotting
plt.plot(t,f)
plt.show()
```

[1]The command `plt.show()` can be omitted if working in Jupyter Notebook, as it is called by default.



We now want to plot a parametric curve $\gamma \colon (a, b) \to \mathbb{R}^2$ with

$$\gamma(t) = (x(t), y(t)) \,.$$

Clearly we need to modify the above code. The variable `t` will still be a numpy array produced by `linspace`. We then need to introduce the arrays `x` and `y` which ecode the first and second components of γ, respectively.

```
import numpy as np
import matplotlib.pyplot as plt

# Divides time interval (a,b) in n parts
# and saves output to numpy array t
t = np.linspace(a, b, n)

# Computes gamma from given functions x(y) and y(t)
x = x(t)
y = y(t)

# Plots the curve
plt.plot(x,y)

# Shows the plot
plt.show()
```

We use the above code to plot the 2D curve known as the Fermat's spiral

$$\gamma(t) = (\sqrt{t}\cos(t), \sqrt{t}\sin(t)) \quad \text{for} \quad t \in [0, 50] \,. \tag{2.2}$$

```
# Plotting Fermat's spiral

import numpy as np
import matplotlib.pyplot as plt

# Divides time interval (0,50) in 500 parts
t = np.linspace(0, 50, 500)

# Computes Fermat's Spiral
x = np.sqrt(t) * np.cos(t)
y = np.sqrt(t) * np.sin(t)

# Plots the Spiral
plt.plot(x,y)
plt.show()
```

Before displaying the output of the above code, a few comments are in order. The array `t` has size 500, due to the behavior of `linspace`. You can also fact check this information by printing `np.size(t)`, which is the numpy function that returns the size of an array. We then use the numpy function `np.sqrt` to compute the square root of the array `t`. The outcome is still an array with the same size of `t`, that is,

$$t = [t_1, \dots, t_n] \quad \implies \quad \sqrt{t} = [\sqrt{t_1}, \dots, \sqrt{t_n}] \,.$$

Similary, the call `np.cos(t)` returns the array

$$\cos(t) = [\cos(t_1), \dots, \cos(t_n)] \,.$$

The two arrays `np.sqrt(t)` and `np.cos(t)` are then multiplied, term-by-term, and saved in the array `x`. The array `y` is computed similarly. The command `plt.plot(x,y)` then yields the graph of the Fermat's spiral:

The above plots can be styled a bit. For example we can give a title to the plot, label the axes, plot the spiral by means of green dots, and add a plot legend, as coded below:

```
# Adding some style

import numpy as np
import matplotlib.pyplot as plt

# Computing Spiral
t = np.linspace(0, 50, 500)
x = np.sqrt(t) * np.cos(t)
y = np.sqrt(t) * np.sin(t)
```

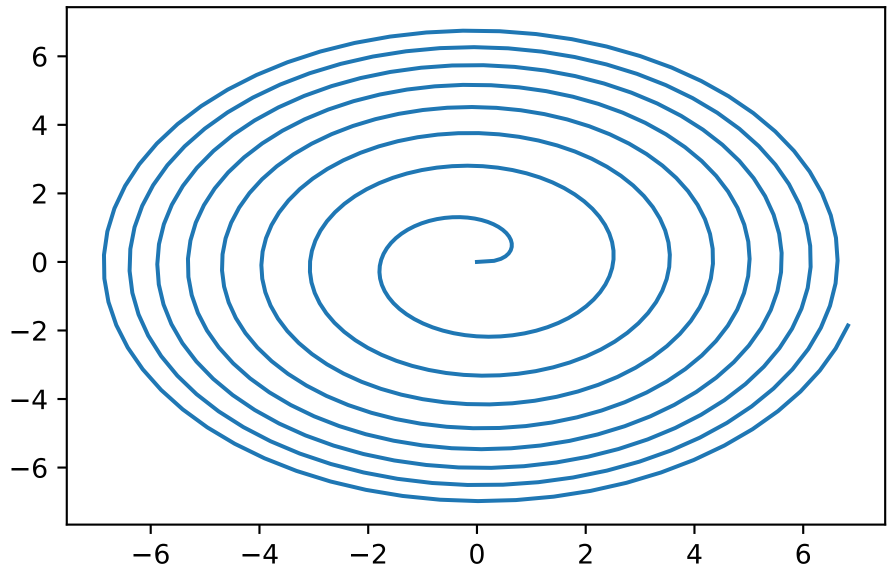


Figure 2.1: Fermat's spiral

```
# Generating figure
plt.figure(1, figsize = (5,5))

# Plotting the Spiral with some options
plt.plot(x, y, '--', color = 'deeppink', linewidth = 1.5, label = 'Spiral')

# Adding grid
plt.grid(True, color = 'lightgray')

# Adding title
plt.title("Fermat's spiral for t between 0 and 50")

# Adding axes labels
plt.xlabel("x-axis", fontsize = 15)
plt.ylabel("y-axis", fontsize = 15)

# Showing plot legend
plt.legend()

# Show the plot
plt.show()
```

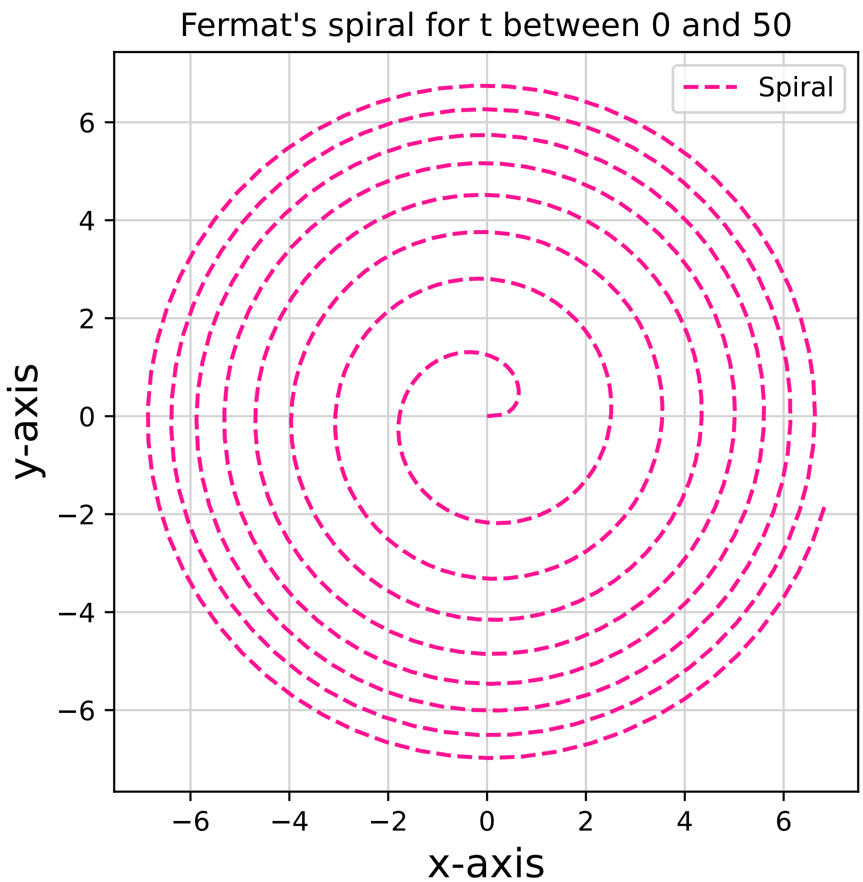


Figure 2.2: Adding a bit of style

Let us go over the novel part of the above code:

- `plt.figure()`: This command generates a figure object. If you are planning on plotting just one figure at a time, then this command is optional: a figure object is generated implicitly when calling `plt.plot`. Otherwise, if working with `n` figures, you need to generate a figure object with `plt.figure(i)` for each `i` between `1` and `n`. The number `i` uniquely identifies the i-th figure: whenever you call `plt.figure(i)`, Python knows that the next commands will refer to the i-th figure. In our case we only have one figure, so we have used the identifier `1`. The second argument `figsize = (a,b)` in `plt.figure()` specifies the size of `figure 1` in inches. In this case we generated a figure 5 by 5 inches.
- `plt.plot`: This is plotting the arrays `x` and `y`, as usual. However we are adding a few aestethic touches: the curve is plotted in *dashed* style with `--`, in *deep pink* color and with a line width of 1.5. Finally this plot is labelled *Spiral*.
- `plt.grid`: This enables a grid in *light gray* color.
- `plt.title`: This gives a title to the figure, displayed on top.
- `plt.xlabel` and `plt.ylabel`: These assign labels to the axes, with font size 15 points.

- `plt.legend()`: This plots the legend, with all the labels assigned in the `plt.plot` call. In this case the only label is *Spiral*.

> **Matplotlib styles**
>
> There are countless plot types and options you can specify in **matplotlib**, see for example the Matplotlib Gallery. Of course there is no need to remember every single command: a quick Google search can do wonders.

> **Generating arrays**
>
> There are several ways of generating evenly spaced arrays in Python. For example the function `np.arange(a,b,s)` returns an array with values within the half-open interval $[a, b)$, with spacing between values given by `s`. For example

```python
import numpy as np

t = np.arange(0,1, 0.2)
print("t =",t)
```

```
t = [0.  0.2 0.4 0.6 0.8]
```

2.2 Implicit curves 2D

A curve γ in $\mathbb{R}^2$ can also be defined as the set of points $(x, y) \in \mathbb{R}^2$ satisfying

$$f(x, y) = 0$$

for some given $f \colon \mathbb{R}^2 \to \mathbb{R}$. For example let us plot the curve γ implicitly defined by

$$f(x, y) = (3x^2 - y^2)^2 \; y^2 - (x^2 + y^2)^4$$

for $-1 \leq x, y \leq 1$. First, we need a way to generate a grid in $\mathbb{R}^2$ so that we can evaluate f on such grid. To illustrate how to do this, let us generate a grid of spacing 1 in the 2D square $[0, 4]^2$. The goal is to obtain the 5 x 5 matrix of coordinates

$$A = \begin{pmatrix} (0,0) & (1,0) & (2,0) & (3,0) & (4,0) \\ (0,1) & (1,1) & (2,1) & (3,1) & (4,1) \\ (0,2) & (1,2) & (2,2) & (2,3) & (2,4) \\ (0,3) & (1,3) & (2,3) & (3,3) & (3,4) \\ (0,4) & (1,4) & (2,4) & (3,4) & (4,4) \end{pmatrix}$$

which corresponds to the grid of points

To achieve this, first generate `x` and `y` coordinates using

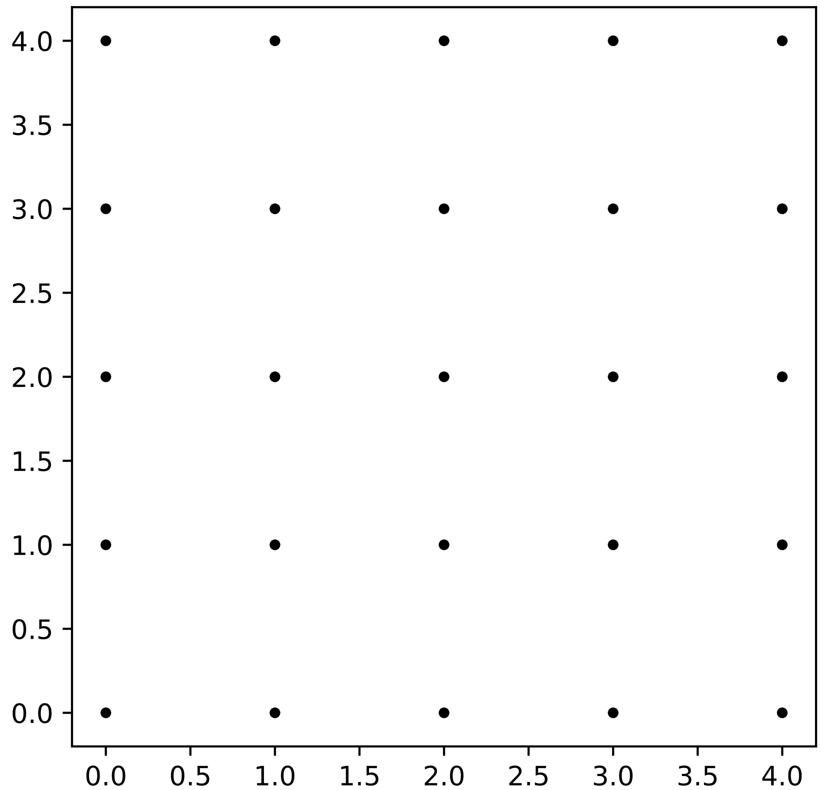


Figure 2.3: The 5 x 5 grid corresponding to the matrix A

```
x = np.linspace(0, 4, 5)
y = np.linspace(0, 4, 5)
```

This generates coordinates

$$x = [0, 1, 2, 3, 4]\,, \quad y = [0, 1, 2, 3, 4]\,.$$

We then need to obtain two matrices X and Y: one for the x coordinates in A, and one for the y coordinates in A. This can be achieved with the code

```
X[0,0] = 0
X[0,1] = 1
X[0,2] = 2
X[0,3] = 3
X[0,4] = 4
X[1,0] = 0
X[1,1] = 1
...
x[4,3] = 3
x[4,4] = 4
```

and similarly for Y. The output would be the two matrices X and Y

$$X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} , \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}$$

If now we plot X against Y via the command

```
plt.plot(X, Y, 'k.')
```

we obtain Figure 2.3. In the above command the style `'k.'` represents black dots. This procedure would be impossible with large vectors. Thankfully there is a function in numpy doing exactly what we need: `np.meshgrid`.

```
# Demonstrating np.meshgrid

import numpy as np

# Generating x and y coordinates
xlist = np.linspace(0, 4, 5)
ylist = np.linspace(0, 4, 5)

# Generating grid X, Y
X, Y = np.meshgrid(xlist, ylist)

# Printing the matrices X and Y
# np.array2string is only needed to align outputs
print('X =', np.array2string(X, prefix='X= '))
print('\n')
print('Y =', np.array2string(Y, prefix='Y= '))
```

```
X = [[0. 1. 2. 3. 4.]
    [0. 1. 2. 3. 4.]
    [0. 1. 2. 3. 4.]
    [0. 1. 2. 3. 4.]
    [0. 1. 2. 3. 4.]]


Y = [[0. 0. 0. 0. 0.]
    [1. 1. 1. 1. 1.]
    [2. 2. 2. 2. 2.]
```

```
[3. 3. 3. 3. 3.]
[4. 4. 4. 4. 4.]]
```

Now that we have our grid, we can evaluate the function f on it. This is simply done with the command

```
Z =((3*(X**2) - Y**2)**2)*(Y**2) - (X**2 + Y**2)**4
```

This will return the matrix Z containing the values $f(x_i, y_i)$ for all (x_i, y_i) in the grid $[X, Y]$. We are now interested in plotting the points in the grid $[X, Y]$ for which Z is zero. This is achieved with the command

```
plt.contour(X, Y, Z, [0])
```

Putting the above observations together, we have the code for plotting the curve $f = 0$ for $-1 \le x, y \le 1$.

```
# Plotting f=0

import numpy as np
import matplotlib.pyplot as plt

# Generates coordinates and grid
xlist = np.linspace(-1, 1, 5000)
ylist = np.linspace(-1, 1, 5000)
X, Y = np.meshgrid(xlist, ylist)

# Computes f
Z =((3*(X**2) - Y**2)**2)*(Y**2) - (X**2 + Y**2)**4

# Creates figure object
plt.figure(figsize = (5.5,5.5))

# Plots level set Z = 0
plt.contour(X, Y, Z, [0])

# Set axes labels
plt.xlabel("x-axis", fontsize = 15)
plt.ylabel("y-axis", fontsize = 15)

# Shows plot
plt.show()
```

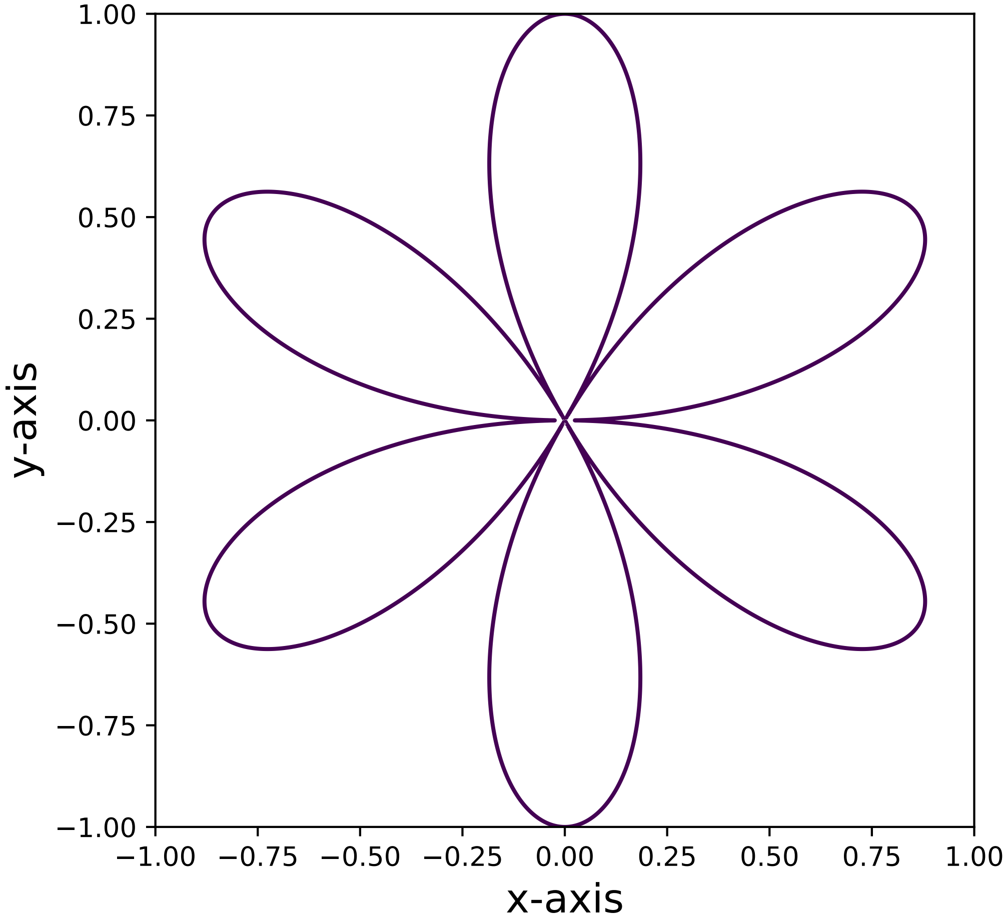


Figure 2.4: Plot of the curve defined by f=0

2.3 Curves in 3D

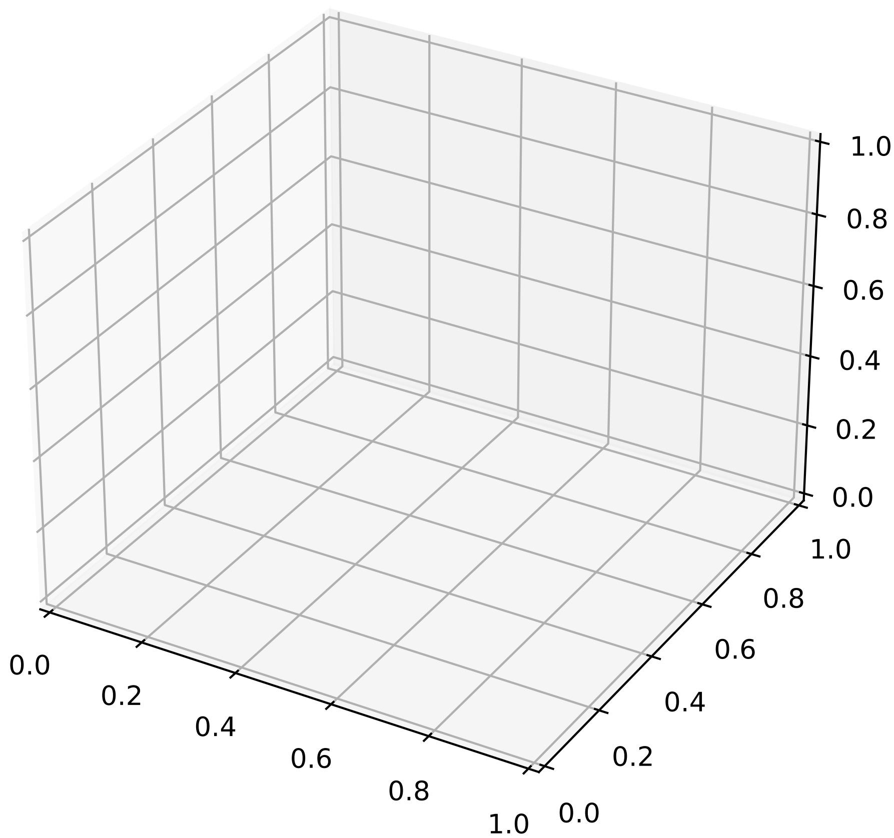
Plotting in 3D with matplotlib requires the `mplot3d` toolkit, see here for documentation. Therefore our first lines will always be

```
# Packages for 3D plots

import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d
```

We can now generate empty 3D axes

```
# Generates and plots empty 3D axes
```

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

# Creates figure object
fig = plt.figure(figsize = (6,6))

# Creates 3D axes object
ax = plt.axes(projection = '3d')

# Shows the plot
plt.show()
```

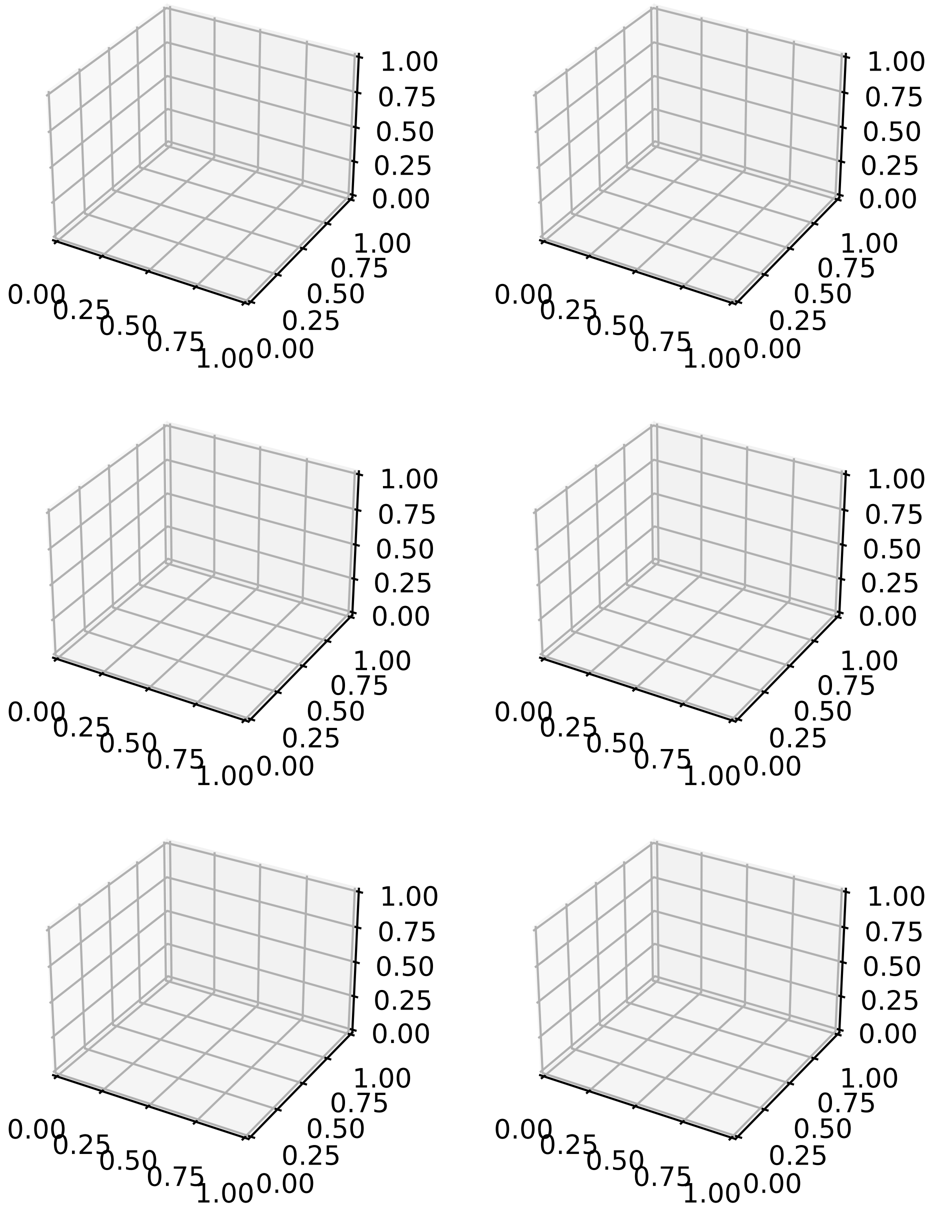


In the above code `fig` is a figure object, while `ax` is an axes object. In practice, the figure object contains the axes objects, and the actual plot information will be contained in axes. If you want multiple plots in the figure container, you should use the command

```
ax = fig.add_subplot(nrows = m, ncols = n, pos = k)
```

This generates an axes object `ax` in position `k` with respect to a `m x n` grid of plots in the container figure. For example we can create a 3 x 2 grid of empty 3D axes as follows

```
# Generates 3 x 2 empty 3D axes

import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

# Creates container figure object
fig = plt.figure(figsize = (6,8))

# Creates 6 empty 3D axes objects
ax1 = fig.add_subplot(3, 2, 1, projection = '3d')
ax2 = fig.add_subplot(3, 2, 2, projection = '3d')
ax3 = fig.add_subplot(3, 2, 3, projection = '3d')
ax4 = fig.add_subplot(3, 2, 4, projection = '3d')
ax5 = fig.add_subplot(3, 2, 5, projection = '3d')
ax6 = fig.add_subplot(3, 2, 6, projection = '3d')

# Shows the plot
plt.show()
```



We are now ready to plot a 3D parametric curve $\gamma\colon (a,b) \to \mathbb{R}^3$ of the form

$$\gamma(t) = (x(t), y(t), z(t))$$

with the code

```
# Code to plot 3D curve

import numpy as np
import matplotlib.pyplot as plt
```

```
from mpl_toolkits import mplot3d

# Generates figure and 3D axes
fig = plt.figure(figsize = (size1,size2))
ax = plt.axes(projection = '3d')

# Plots grid
ax.grid(True)

# Divides time interval (a,b)
# into n parts and saves them in array t
t = np.linspace(a, b, n)

# Computes the curve gamma on array t
# for given functions x(t), y(t), z(t)
x = x(t)
y = y(t)
z = z(t)

# Plots gamma
ax.plot3D(x, y, z)

# Setting title for plot
ax.set_title('3D Plot of gamma')

# Setting axes labels
ax.set_xlabel('x', labelpad = 'p')
ax.set_ylabel('y', labelpad = 'p')
ax.set_zlabel('z', labelpad = 'p')

# Shows the plot
plt.show()
```

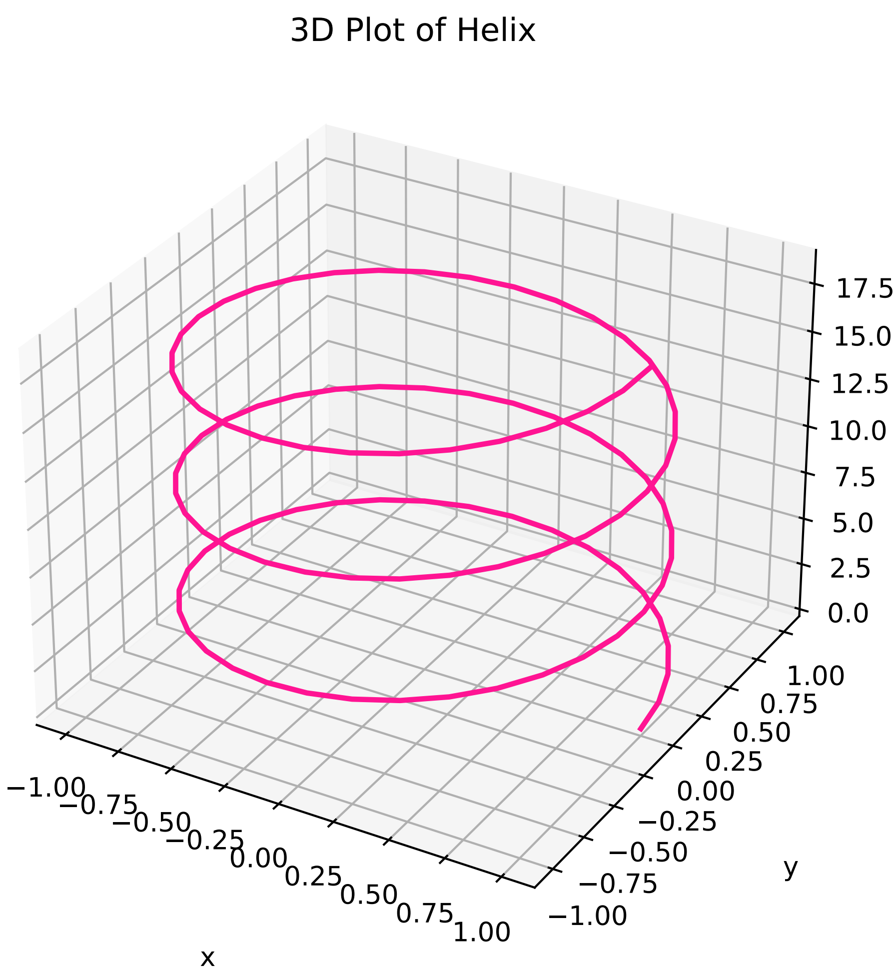
For example we can use the above code to plot the Helix

$$x(t) = \cos(t)\,, \quad y(t) = \sin(t)\,, \quad z(t) = t$$

for $t \in [0, 6\pi]$.

```
# Plotting 3D Helix

import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d
```

```
# Generates figure and 3D axes
fig = plt.figure(figsize = (6,6))
ax = plt.axes(projection = '3d')

# Plots grid
ax.grid(True)

# Divides time interval (0,6pi) in 100 parts
t = np.linspace(0, 6*np.pi, 100)

# Computes Helix
x = np.cos(t)
y = np.sin(t)
z = t

# Plots Helix - We added some styling
ax.plot3D(x, y, z, color = "deeppink", linewidth = 2)

# Setting title for plot
ax.set_title('3D Plot of Helix')

# Setting axes labels
ax.set_xlabel('x', labelpad = 20)
ax.set_ylabel('y', labelpad = 20)
ax.set_zlabel('z', labelpad = 20)

# Shows the plot
plt.show()
```



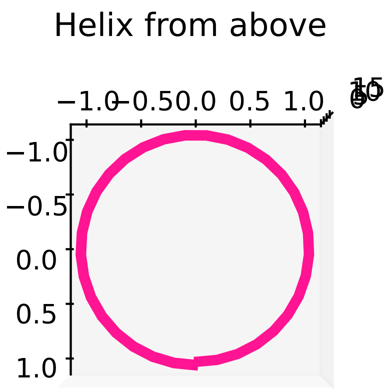
We can also change the viewing angle for a 3D plot store in `ax`. This is done via

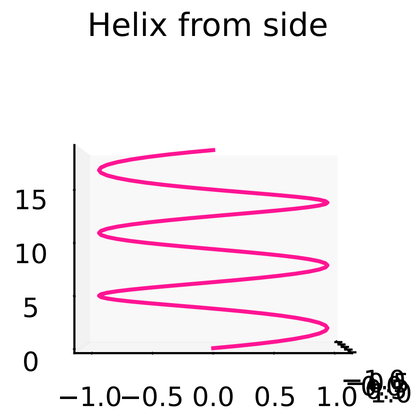
```
ax.view_init(elev = e, azim = a)
```

which displays the 3D axes with an elevation angle `elev` of `e` degrees and an azimuthal angle `azim` of `a` degrees. In other words, the 3D plot will be rotated by `e` degrees above the xy-plane and by `a` degrees around the z-axis. For example, let us plot the helix with 2 viewing angles. Note that we generate 2 sets of axes with the `add_subplot` command discussed above.

```
# Plotting 3D Helix

import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

# Generates figure object
fig = plt.figure(figsize = (6,6))
```

```
# Generates 2 sets of 3D axes
ax1 = fig.add_subplot(1, 2, 1, projection = '3d')
ax2 = fig.add_subplot(1, 2, 2, projection = '3d')

# We will not show a grid this time
ax1.grid(False)
ax2.grid(False)

# Divides time interval (0,6pi) in 100 parts
t = np.linspace(0, 6*np.pi, 100)

# Computes Helix
x = np.cos(t)
y = np.sin(t)
z = t

# Plots Helix on both axes
ax1.plot3D(x, y, z, color = "deeppink", linewidth = 1.5)
ax2.plot3D(x, y, z, color = "deeppink", linewidth = 1.5)

# Setting title for plots
ax1.set_title('Helix from above')
ax2.set_title('Helix from side')

# Changing viewing angle of ax1
# View from above has elev = 90 and azim = 0
ax1.view_init(elev = 90, azim = 0)

# Changing viewing angle of ax2
# View from side has elev = 0 and azim = 0
ax2.view_init(elev = 0, azim = 0)

# Shows the plot
plt.show()
```





2.4 Interactive plots

`Matplotlib` produces beautiful static plots; however it lacks built in interactivity. For this reason I would also like to show you how to plot curves with `Plotly`, a very popular Python graphic library which has built in interactivity. Documentation for `Plotly` and lots of examples can be found here.

2.4.1 2D Plots

We start by plotting the Fermat's Spiral as defined at Equation 2.2. This is achieved

link

```
# Plotting 2D curve with Plotly

import numpy as np
import plotly.graph_objects as go

# Generates coordinates u and v by dividing
# the interval (0,2pi) in 100 parts
u = np.linspace(0, 2*np.pi, 100)
v = np.linspace(0, 2*np.pi, 100)

# Generates grid [U,V] from the coordinates u, v
U, V = np.meshgrid(u, v)

# Computes the torus on grid [U,V]
```

```
# with radii r = 1 and R = 2
R = 2
r = 1

x = (R + r * np.cos(U)) * np.cos(V)
y = (R + r * np.cos(U)) * np.sin(V)
z = r * np.sin(U)



fig = go.Figure()
data = go.Surface(x=x,y=y,z=z,showscale=False, colorscale='teal')
fig.add_trace(data)
fig.update_layout(title_text="Torus with Plotly")
fig.show()
```

```
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```

```
Unable to display output for mime type(s): text/html
```

Documentation of the function `Surface` can be found here.

The above code generates an image that cannot be rendered in pdf. To see the output, checkout the digital version of these notes.

3 Surfaces

4 Plotting surfaces

4.1 Plots with Matplotlib

I will take for granted all the commands explained in Chapter 2. Suppose we want to plot a surface S which is defined by the parametric equations

$$x = x(u, v)\,, \quad y = y(u, v)\,, \quad z = z(u, v)$$

for $u \in (a, b)$ and $v \in (c, d)$. This can be done via the function called `plot_surface` contained in the mplot3d Toolkit. This function works as follows: first we generate a mesh-grid $[U, V]$ from the coordinates (u, v) via the command

```
[U, V] = np.meshgrid(u, v)
```

Then we compute the parametric surface on the mesh

```
x = x (U, V)
y = y (U, V)
z = z (U, V)
```

Finally we can plot the surface with the command

```
plt.plot_surface(x, y, z)
```

The complete code looks as follows.

```
# Plotting surface S

# Importing numpy, matplotlib and mplot3d
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

# Generates figure object of size m x n
fig = plt.figure(figsize = (m,n))

# Generates 3D axes
```

```
ax = plt.axes(projection = '3d')

# Shows axes grid
ax.grid(True)

# Generates coordinates u and v
# by dividing the interval (a,b) in n parts
# and the interval (c,d) in m parts
u = np.linspace(a, b, m)
v = np.linspace(c, d, n)

# Generates grid [U,V] from the coordinates u, v
U, V = np.meshgrid(u, v)

# Computes S given the functions x, y, z
# on the grid [U,V]
x = x(U,V)
y = y(U,V)
z = z(U,V)

# Plots the surface S
ax.plot_surface(x, y, z)

# Setting plot title
ax.set_title('The surface S')

# Setting axes labels
ax.set_xlabel('x', labelpad=10)
ax.set_ylabel('y', labelpad=10)
ax.set_zlabel('z', labelpad=10)

# Setting viewing angle
ax.view_init(elev = e, azim = a)

# Showing the plot
plt.show()
```

For example let us plot a cone described parametrically by:

$$x = u\cos(v)\,, \quad y = u\sin(v)\,, \quad z = u$$

for $u \in (0,1)$ and $v \in (0,2\pi)$. We adapt the above code:

```python
# Plotting a cone

# Importing numpy, matplotlib and mplot3d
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

# Generates figure object of size 6 x 6
fig = plt.figure(figsize = (6,6))

# Generates 3D axes
ax = plt.axes(projection = '3d')

# Shows axes grid
ax.grid(True)

# Generates coordinates u and v by dividing
# the intervals (0,1) and (0,2pi) in 100 parts
u = np.linspace(0, 1, 100)
v = np.linspace(0, 2*np.pi, 100)

# Generates grid [U,V] from the coordinates u, v
U, V = np.meshgrid(u, v)

# Computes the surface on grid [U,V]
x = U * np.cos(V)
y = U * np.sin(V)
z = U

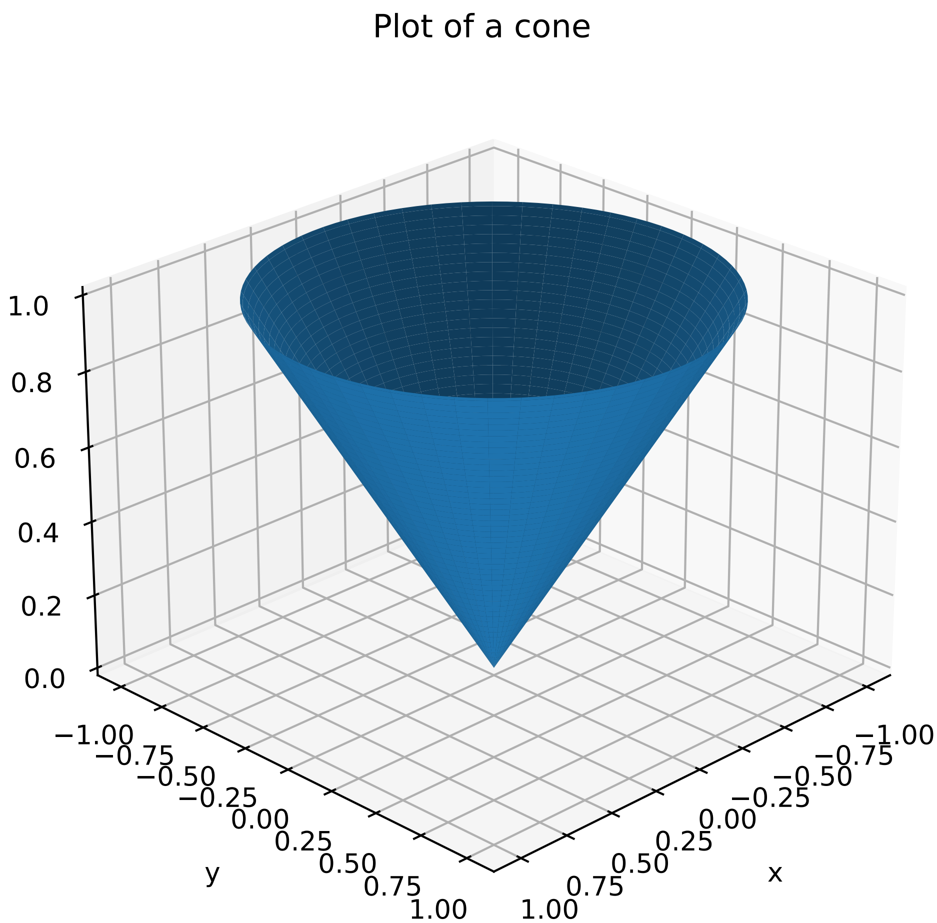
# Plots the cone
ax.plot_surface(x, y, z)

# Setting plot title
ax.set_title('Plot of a cone')

# Setting axes labels
ax.set_xlabel('x', labelpad=10)
ax.set_ylabel('y', labelpad=10)
ax.set_zlabel('z', labelpad=10)

# Setting viewing angle
ax.view_init(elev = 25, azim = 45)
```

```
# Showing the plot
plt.show()
```



As discussed in Chapter 2, we can have multiple plots in the same figure. For example let us plot the torus viewed from 2 angles. The parametric equations are:

$$x = (R + r\cos(u))\cos(v)$$
$$y = (R + r\cos(u))\sin(v)$$
$$z = r\sin(u)$$

for $u, v \in (0, 2\pi)$ and with

- R distance from the center of the tube to the center of the torus
- r radius of the tube

```
# Plotting torus seen from 2 angles

# Importing numpy, matplotlib and mplot3d
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

# Generates figure object of size 6 x 10
fig = plt.figure(figsize = (6,10))

# Generates 2 sets of 3D axes
ax1 = fig.add_subplot(2, 1, 1, projection = '3d')
ax2 = fig.add_subplot(2, 1, 2, projection = '3d')

# Shows axes grid
ax1.grid(True)
ax2.grid(True)

# Generates coordinates u and v by dividing
# the interval (0,2pi) in 100 parts
u = np.linspace(0, 2*np.pi, 100)
v = np.linspace(0, 2*np.pi, 100)

# Generates grid [U,V] from the coordinates u, v
U, V = np.meshgrid(u, v)

# Computes the torus on grid [U,V]
# with radii r = 1 and R = 2
R = 2
r = 1

x = (R + r * np.cos(U)) * np.cos(V)
y = (R + r * np.cos(U)) * np.sin(V)
z = r * np.sin(U)

# Plots the torus on both axes
ax1.plot_surface(x, y, z, rstride = 5, cstride = 5, color = 'dimgray', edgecolors =

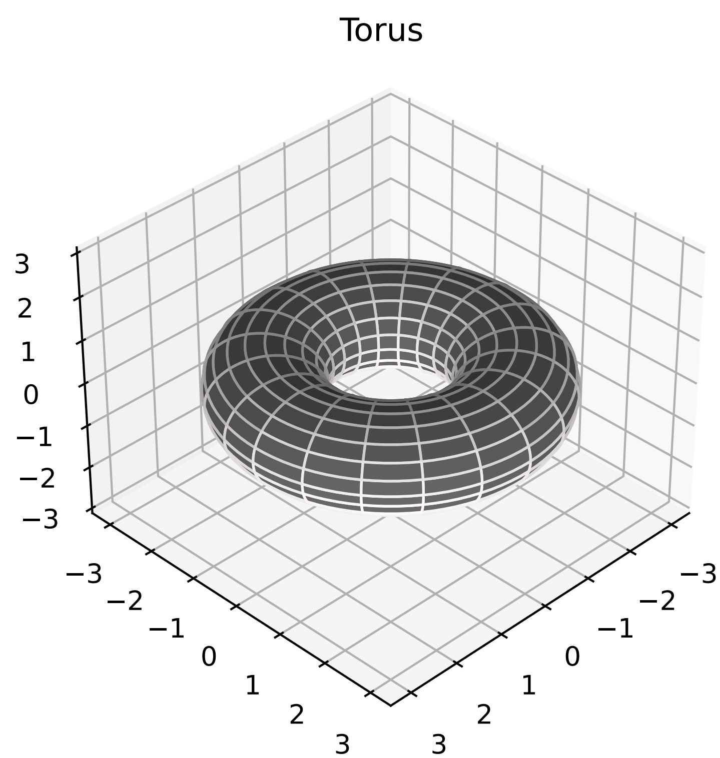
ax2.plot_surface(x, y, z, rstride = 5, cstride = 5, color = 'dimgray', edgecolors =

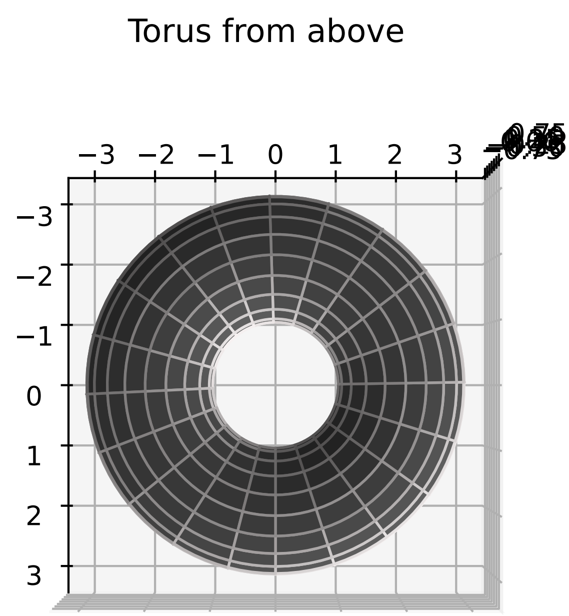
# Setting plot titles
ax1.set_title('Torus')
ax2.set_title('Torus from above')
```

```
# Setting range for z axis in ax1
ax1.set_zlim(-3,3)

# Setting viewing angles
ax1.view_init(elev = 35, azim = 45)
ax2.view_init(elev = 90, azim = 0)

# Showing the plot
plt.show()
```





Notice that we have added some customization to the `plot_surface` command. Namely, we have set the color of the figure with `color = 'dimgray'` and of the edges with `edgecolors = 'snow'`. Moreover the commands `rstride` and `cstride` set the number of *wires* you see in the plot. More precisely, they set by how much the data in the mesh $[U, V]$ is downsampled in each direction, where rstride sets the row direction, and cstride sets the column direction. On the torus this is a bit difficult to visualize, due to the fact that $[U, V]$ represents angular coordinates. To appreciate the effect, we can plot for example the paraboiloid

$$
\begin{aligned}
x &= u \\
y &= v \\
z &= -u^2 - v^2
\end{aligned}
$$

for $u, v \in [-1, 1]$.

```
# Showing the effect of rstride and cstride

# Importing numpy, matplotlib and mplot3d
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

# Generates figure object of size 10 x 10
fig = plt.figure(figsize = (7,7))

# Generates 2 sets of 3D axes
ax1 = fig.add_subplot(2, 2, 1, projection = '3d')
ax2 = fig.add_subplot(2, 2, 2, projection = '3d')
ax3 = fig.add_subplot(2, 2, 3, projection = '3d')
ax4 = fig.add_subplot(2, 2, 4, projection = '3d')

# Generates coordinates u and v by dividing
# the interval (-1,1) in 100 parts
u = np.linspace(-1, 1, 100)
v = np.linspace(-1, 1, 100)

# Generates grid [U,V] from the coordinates u, v
U, V = np.meshgrid(u, v)

# Computes the paraboloid on grid [U,V]
x = U
y = V
z = - U**2 - V**2

# Plots the paraboloid on the 4 axes
# but with different stride settings
ax1.plot_surface(x, y, z, rstride = 5, cstride = 5, color = 'dimgray', edgecolors =
```

```
ax2.plot_surface(x, y, z, rstride = 5, cstride = 20, color = 'dimgray', edgecolors =

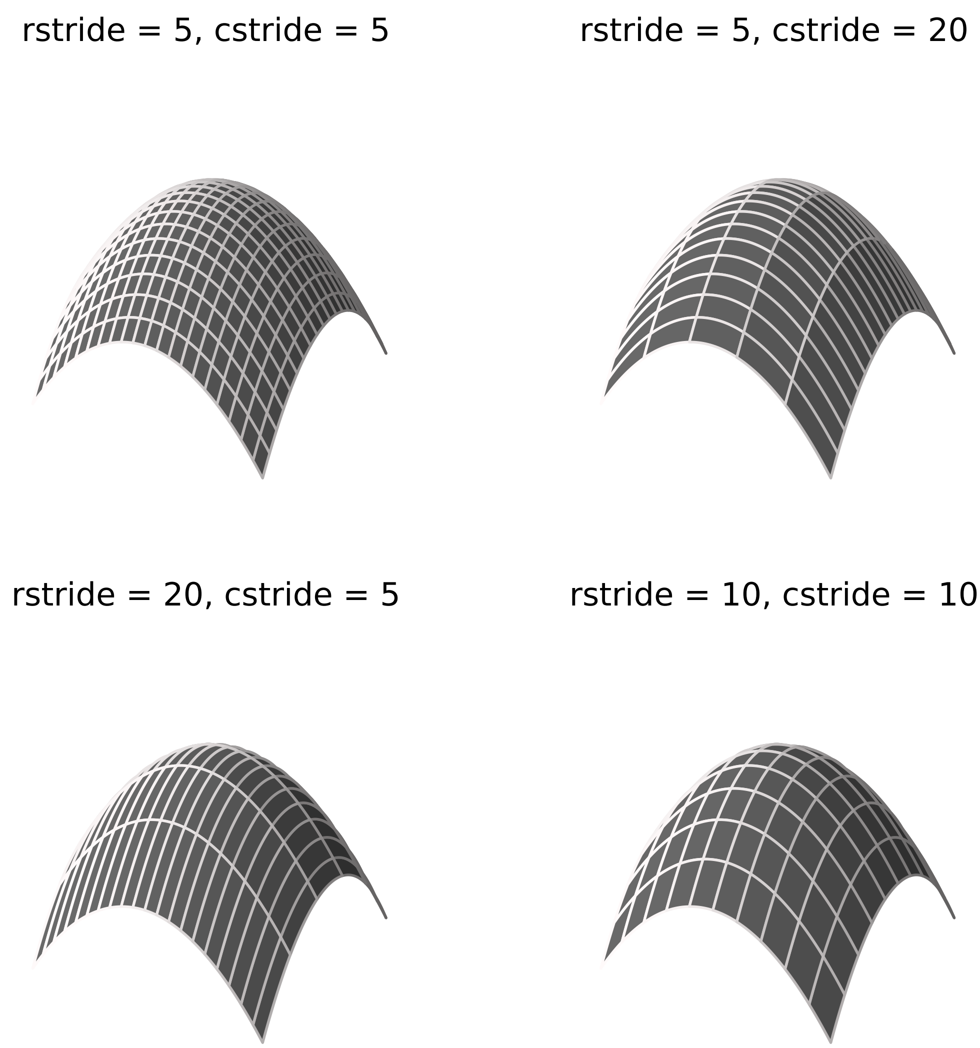
ax3.plot_surface(x, y, z, rstride = 20, cstride = 5, color = 'dimgray', edgecolors =

ax4.plot_surface(x, y, z, rstride = 10, cstride = 10, color = 'dimgray', edgecolors

# Setting plot titles
ax1.set_title('rstride = 5, cstride = 5')
ax2.set_title('rstride = 5, cstride = 20')
ax3.set_title('rstride = 20, cstride = 5')
ax4.set_title('rstride = 10, cstride = 10')

# We do not plot axes, to get cleaner pictures
ax1.axis('off')
ax2.axis('off')
ax3.axis('off')
ax4.axis('off')

# Showing the plot
plt.show()
```



In this case our mesh is `100 x 100`, since `u` and `v` both have 100 components. Therefore setting `rstride` and `cstride` to 5 implies that each row and column of the mesh is sampled one time every 5 elements, for a total of

$$100/5 = 20$$

samples in each direction. This is why in the first picture you see a `20 x 20` grid. If instead one sets `rstride` and `cstride` to 10, then each row and column of the mesh is sampled one time every 10 elements, for a total of

$$100/10 = 10$$

samples in each direction. This is why in the fourth figure you see a `10x10` grid.

4.2 Plots with Plotly

As done in Section 2.4, we now see how to use `Plotly` to generate an interactive 3D plot of a surface. This can be done by means of functions contained in the `Plotly` module `graph_objects`, usually imported as `go`. Specifically, we will use the function `go.Surface`. The code will look similar to the one used to plot surfaces with `matplotlib`:

- generate meshgrid on which to compute the parametric surface,
- store such surface in the numpy array `[x,y,z]`,
- pass the array `[x,y,z]` to `go.Surface` to produce the plot.

The full code is below.

```
# Plotting a Torus with Plotly

# Import "numpy" and the "graph_objects" module from Plotly
import numpy as np
import plotly.graph_objects as go

# Generates coordinates u and v by dividing
# the interval (0,2pi) in 100 parts
u = np.linspace(0, 2*np.pi, 100)
v = np.linspace(0, 2*np.pi, 100)

# Generates grid [U,V] from the coordinates u, v
U, V = np.meshgrid(u, v)

# Computes the torus on grid [U,V]
# with radii r = 1 and R = 2
R = 2
r = 1

x = (R + r * np.cos(U)) * np.cos(V)
y = (R + r * np.cos(U)) * np.sin(V)
z = r * np.sin(U)

# Generate and empty figure object with Plotly
# and saves it to the variable called "fig"
fig = go.Figure()

# Plot the torus with go.Surface and store it
# in the variable "data". We also do now show the
# plot scale, and set the color map to "teal"
```

```
data = go.Surface(x = x , y = y, z = z, showscale = False, colorscale='teal')

# Add the plot stored in "data" to the figure "fig"
# This is done with the command add_trace
fig.add_trace(data)

# Set the title of the figure in "fig"
fig.update_layout(title_text="Plotting a Torus with Plotly")

# Show the figure
fig.show()
```

```
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```

```
Unable to display output for mime type(s): text/html
```

The above code generates an image that cannot be rendered in pdf. To see the output, please refer to the digital version of these notes. To further customize your plots, you can check out the documentation of `go.Surface` at this link.

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    author = {Fanzon, Silvio},
    title = {Lecture Notes on Differential Geometry},
    url = {https://silviofanzon.quarto.pub/2023-differential-geometry/},
    year = {2023}}
```

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