Numbers, Sequences and Series

Lecture Notes

Dr. Silvio Fanzon

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Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

S.Fanzon@hull.ac.uk

Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

Readings

We will study the set of real numbers $\mathbb{R}$, and then sequences and series in $\mathbb{R}$. I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

> **!** You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to known in order to excel in the final exam.

1 Introduction

The first aim of this lecture notes is to rigorously introduce the set of **real numbers**, which is denoted by $\mathbb{R}$. But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol $\in$ denotes that m and n belong to $\mathbb{N}$. For example $3 + 7$ results in 10.

Question 1.1

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m\,? \tag{1.1}$$

Of course to invert (1.1) we can just perform a **subtraction**, implying that

$$x = m - n\,.$$

But there is a catch. In general x does not need to be in $\mathbb{N}$. For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to $\mathbb{N}$. Therefore the answer to Question 1.1 is **NO**.

To make sure that we can always invert the sum, we need to **extend** the set $\mathbb{N}$. This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : \; n \in \mathbb{N}\}\,,$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}\,.$$

The sum can be extended to $\mathbb{Z}$, by defining

$$(-n) + (-m) := -(m + n) \tag{1.2}$$

for all $m, n \in \mathbb{N}$. Now every element of $\mathbb{Z}$ possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0\,.$$

Can we characterize m explicitly? Of course! Seeing the definition at (1.2), we simply have

$$m = -n\,.$$

On the set $\mathbb{Z}$ we can also define the operation of **multiplication**, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm or $n \cdot m$. For example $7 \cdot 2 = 14$ and $1 \cdot (-1) = -1$.

Question 1.2

Can the multiplication in $\mathbb{Z}$ be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m\,? \tag{1.3}$$

To invert (1.3) if $n \neq 0$, we can just perform a **division**, to obtain

$$x = \frac{m}{n}\,.$$

But again there is a catch. Indeed taking $n = 2$ and $m = 1$ yields $x = 1/2$, which does not belong to $\mathbb{Z}$. The answer to Question 1.2 is therefore **NO**.

Thus, in order to invert the multiplication, we need to **extend** the set of integers $\mathbb{Z}$. This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} \,:\, m, n \in \mathbb{Z},\, n \neq 0 \right\}\,.$$

We then extend the operations of sum and multiplication to $\mathbb{Q}$ by defining

$$\frac{m}{n} + \frac{p}{q} := \frac{mq + np}{nq}$$

and

$$\frac{m}{n} \cdot \frac{p}{q} := \frac{mp}{nq}$$

Now the multiplication is invertible in $\mathbb{Q}$. Specifically, each non-zero element has an inverse: the inverse of m/n is given by n/m.

To summarize, we have extended $\mathbb{N}$ to $\mathbb{Z}$, and $\mathbb{Z}$ to $\mathbb{Q}$. By construction we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}\,.$$

Moreover **sum** and **product** are **invertible** in $\mathbb{Q}$. Now we are happy right? So and so.

Question 1.3

Can we draw the set $\mathbb{Q}$?

It is clear how to draw $\mathbb{Z}$, as seen below.

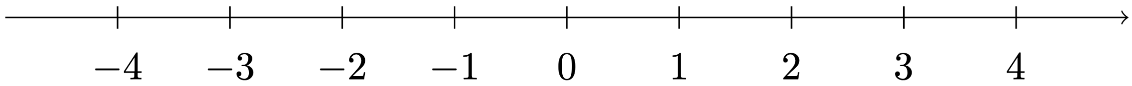

Figure 1.1: Representation of integers $\mathbb{Z}$

However $\mathbb{Q}$ is much **larger** than the set $\mathbb{Z}$ represented by the ticks in Figure 1.1. What do we mean by **larger**? For example, consider $0 \in \mathbb{Q}$.

Question 1.4

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, you can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n} \,.$$

For example think about the case of the numbers

$$\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N},\, n \neq 0 \,.$$

Such numbers get arbitrarily close to 0, as depicted below.

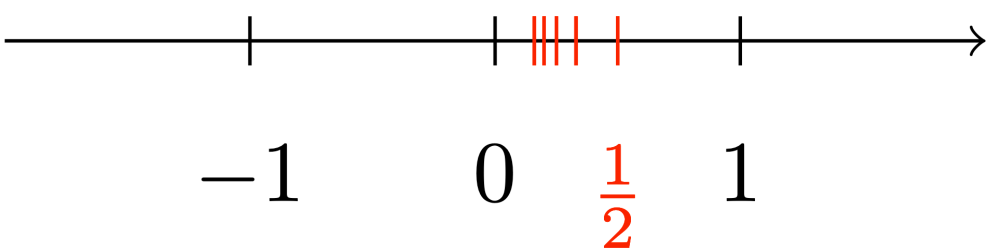

Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0, 1]$. In other words, we might conjecture the following.

Conjecture 1.5

Maybe $\mathbb{Q}$ can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.5 is **false**, as shown by the Theorem below.

Theorem 1.6

The number $\sqrt{2}$ does not belong to $\mathbb{Q}$.

Theorem 1.6 is the reason why $\sqrt{2}$ is called an **irrational number**. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048\ldots$$

and the situation is as in the picture below.

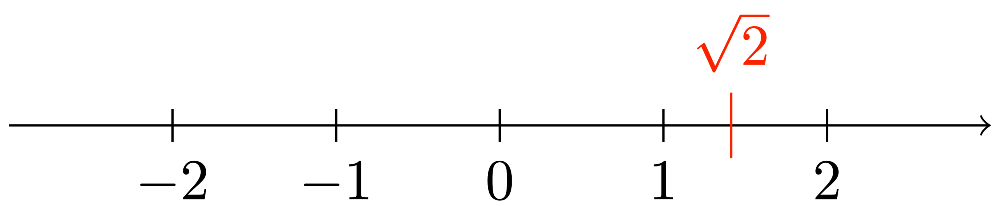

Figure 1.3: Representing $\sqrt{2}$ on the numbers line.

We can therefore see that Conjecture 1.5 is **false**, and $\mathbb{Q}$ is not a line: indeed $\mathbb{Q}$ has a **gap** at $\sqrt{2}$. Let us see why Theorem 1.6 is true.

Proof: Proof of Theorem 1.6

We prove that

$$\sqrt{2} \notin \mathbb{Q}$$

by **contradiction**.
Wait, what does this mean? Proving the claim by contradiction means assuming that the claim is **false**. This means we **assume** that

$$\sqrt{2} \in \mathbb{Q}\,. \tag{1.4}$$

From this assumption we then start deducing other statements, hoping to encounter a statement which is **`FALSE`**. But if (1.4) leads to a false statement, then it must be that (1.4) is **`FALSE`**. Thus the contrary of (1.4) must hold, meaning that

$$\sqrt{2} \notin \mathbb{Q}$$

as we wanted to show. This would conclude the proof.
Now we need to actually show that (1.4) will lead to a contradiction. Since this is our first proof, let us take it slowly, step-by-step.

1. Assuming (1.4) just means that there exists $q \in \mathbb{Q}$ such that
$$q = \sqrt{2}\,. \tag{1.5}$$
2. Since $q \in \mathbb{Q}$, by definition we have
$$q = \frac{m}{n}$$
for some $m, n \in \mathbb{N}$ with $n \neq 0$.
3. Recalling (1.5), we then have
$$\frac{m}{n} = \sqrt{2}\,.$$
4. We can square the above equation to get
$$\frac{m^2}{n^2} = 2\,. \tag{1.6}$$
5. **Withouth loss of generality**, we can **assume** that m and n have no common factors.

   Wait. What does Step 5 mean? You will encounter the sentence *withouth loss of generality* many times in mathematics. It is often abbreviated in **`WLOG`**. WLOG means that the assumption that follows is chosen arbitrarily, but does not affect the validity of the proof in general.
   For example in our case we can assume that m and n have no common factor. This is because if m and n had common factors, then it would mean
$$m = a\tilde{m}\,, \quad n = a\tilde{n}$$
   for some $a \in \mathbb{N}$ with $a \neq 0$. Then
$$\frac{m}{n} = \frac{a\tilde{m}}{a\tilde{n}} = \frac{\tilde{m}}{\tilde{n}}\,.$$
   Therefore by (1.6)
$$\frac{\tilde{m}^2}{\tilde{n}^2} = 2\,.$$
   The proof can now proceed in the same way we would have proceeded from Step 4, but in addition we have the hypothesis that $\tilde{m}$ and $\tilde{n}$ have no common factors.
6. Equation (1.6) implies
$$m^2 = 2n^2\,. \tag{1.7}$$
Therefore the integer m^2 is an even number.

Why is m^2 even? As you already know, **even** numbers are

$$0, 2, 4, 6, 8, 10, 12, \dots$$

All these numbers have in common that they can be divided by 2, and so they can be written as

$$2p$$

for some $p \in \mathbb{N}$. For example 52 is even, because

$$52 = 2 \cdot 26\,.$$

Instead, **odd** numbers are

$$1, 3, 5, 7, 8, 9, 11, \dots$$

These can be all written as

$$2p + 1$$

for some $p \in \mathbb{N}$. For example 53 is odd, because

$$52 = 2 \cdot 26 + 1\,.$$

7. Thus m is an even number, and so there exists $p \in \mathbb{N}$ such that

$$m = 2p\,. \tag{1.8}$$

Why is (1.8) true? Let us see what happens if we take the square of an even number $m = 2p$

$$m^2 = (2p)^2 = 4p^2 = 2(2p^2) = 2q\,.$$

Thus $m^2 = 2q$ for some $q \in \mathbb{N}$, and so m^2 is an even number. If instead m is odd, then $m = 2p + 1$ and

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

showing that also m^2 is odd.
This justifies Step 7: Indeed we know that m^2 is an even number from Step 6. If m was odd, then m^2 would be odd. Hence m must be even as well.

8. If we substitute (1.8) in (1.7) we get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2\,. \tag{1.9}$$

9. We now make a series of observations:

- Equation (1.9) says that n^2 is even.
- Step 6 says that m^2 is even.
- Therefore n and m are also even.

- Hence n and m have 2 as common factor.
- But Step 5 says that n and m have no common factors.
- `CONTRADICTION`

10. Our reasoning has run into a **contradiction**, starting from assumption (1.4), which says that

$$\sqrt{2} \in \mathbb{Q}\,.$$

Hence the above must be **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

Seeing that $\sqrt{2} \notin \mathbb{Q}$, we might be tempted to just fill in the gap by adding $\sqrt{2}$ to $\mathbb{Q}$. However, with analogous proof to Theorem 1.6, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number p. As there are infinite prime numbers, this means that $\mathbb{Q}$ has infinite gaps. Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : \; p \; \text{ prime}\}\,.$$

However even this is not enough, as we would still have numbers which are not contained in $\tilde{Q}$, for example

$$\sqrt{2} + \sqrt{3},\, \pi,\, \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}\,.$$

Remark 1.7

Proving that

$$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

is relatively easy, and will be left as an **exercise**. Instead, proving that

$$\pi \notin \mathbb{Q}$$

is way more complicated. There are several proof of the fact, all requiring mathematics which is more advanced of the one presented in this course. For some proofs, see this Wikipedia page.

The reality of things is that to **complete** $\mathbb{Q}$ and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in $\mathbb{Q}$. Such extension of $\mathbb{Q}$ will be called $\mathbb{R}$, the set of **real numbers**. The inclusions will therefore be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}\,.$$

The set $\mathbb{R}$ is not at all trivial to construct. In fact, at first we will not construct it, but just do the following:

- We will assume that $\mathbb{R}$ **exists** and satisfies some basic **axioms**.
- One of the axioms is that $\mathbb{R}$ fills **all** the **gaps** that $\mathbb{Q}$ has. Therefore $\mathbb{R}$ can be thought as a **continuous** line.
- We will study the **properties** of $\mathbb{R}$ which descend from such **axioms**.

For example one of the properties of $\mathbb{R}$ will be the following:

Theorem 1.8: We will prove this in the future

$\mathbb{R}$ contains all the square roots. This means that for every $x \in \mathbb{R}$ with $x \geq 0$, we have

$$\sqrt{x} \in \mathbb{R}\,.$$

At the end of this chapter we will provide a concrete **model** for the real numbers $\mathbb{R}$, to prove once and for all that such set indeed exists.

Theorem 1.9: We will prove this in the future

There exists a set $\mathbb{R}$, called the set of real numbers, which has the following properties:

- $\mathbb{R}$ extends $\mathbb{Q}$, that is,
$$\mathbb{Q} \subset \mathbb{R}\,.$$
- $\mathbb{R}$ satisfies certain **axioms**.
- $\mathbb{R}$ fills **all** the **gaps** that $\mathbb{Q}$ has. In particular $\mathbb{R}$ can be represented by a **continuous** line.

2 Preliminaries

Before introducing $\mathbb{R}$ we want to make sure that we cover all the basics needed for the task.

2.1 Sets

A sets is a **collection** of objects. These objects are called **elements** of the set. For example in the previous section we mentioned the following sets:

- $\mathbb{N}$ the set of natural numbers
- $\mathbb{Z}$ the set of integers
- $\mathbb{Q}$ the set of rational numbers
- $\mathbb{R}$ the set of real numbers

Given an arbitrary set A, we write

$$x \in A$$

if the element x belongs to the set A. If an element x is not contained in A, we say that

$$x \notin A\,.$$

Remark 2.1

A set can contain all sorts of elements. For example the students in a classroom can be modelled by a set S. The elements of the set are the students. For example

$$S = \{\text{Alice, Olivia, Jake, Sahab}\}$$

In this case we have

$$\text{Alice} \in S$$

but instead

$$\text{Silvio} \notin S\,.$$

2.2 Logic

In this section we introduce some basic logic symbols. Suppose that you are given two statements, say α and β. The formula

$$\alpha \implies \beta$$

means that α **implies** β. In other words, if α is true then also β is true.
The formula

$$\alpha \impliedby \beta$$

means that α is implied by β: if β is true then also α is true.
When we write

$$\alpha \iff \beta \tag{2.1}$$

we mean that α and β are **equivalent**. Note that (2.1) is equivalent to

$$\alpha \implies \beta \;\text{ and }\; \beta \implies \alpha \,.$$

Such equivalence is very useful in proofs.

Example 2.2

We have that

$$x > 0 \implies x > -100 \,,$$

and

$$\text{contradiction} \quad \impliedby \sqrt{2} \in \mathbb{Q} \,.$$

Concerning $\iff$ we have

$$x^2 < 2 \iff -\sqrt{2} < x < \sqrt{2} \,.$$

We now introduce logic **quantifiers**. These are

- $\forall$ which reads **for all**
- $\exists$ which reads **exists**
- $\exists!$ which reads **exists unique**
- $\nexists$ which reads **does not exists**

These work in the following way. Suppose that you are given a statement $\alpha(x)$ which depends on the point $x \in \mathbb{R}$. Then we say

- $\alpha(x)$ is satisfied for all $x \in A$ with A some collection of numbers. This translates to the symbols

$$\alpha(x) \;\text{ is true }\; \forall \, x \in A \,,$$

- There exists some x in $\mathbb{R}$ such that $\alpha(x)$ is satisfied: in symbols

$$\exists \, x \in \mathbb{R} \;\text{ such that }\; \alpha(x) \;\text{ is true,}$$

- There exists a unique x_0 in $\mathbb{R}$ such that $\alpha(x)$ is satisfied: in symbols

$$\exists!\, x_0 \in \mathbb{R} \text{ such that } \alpha(x_0) \text{ is true,}$$

- $\alpha(x)$ is never satisfied:

$$\nexists\, x \in \mathbb{R} \text{ such that } \alpha(x) \text{ is true.}$$

Example 2.3

Let us make concrete examples:

- The expression x^2 is always non-negative. Thus we can say

$$x^2 \geq 0 \text{ for all } x \in \mathbb{R}\,.$$

- The equation $x^2 = 1$ has two solutions $x = 1$ and $x = -1$. Therefore we can say

$$\exists\, x \in \mathbb{R} \text{such that } x^2 = 1\,.$$

- The equation $x^3 = 1$ has a unique solution $x = 1$. Thus

$$\exists!\, x \in \mathbb{R} \text{such that } x^3 = 1\,.$$

- We know that the equation $x^2 = 2$ has no solutions in $\mathbb{Q}$. Then

$$\nexists\, x \in \mathbb{Q} \text{such that } x^2 = 2\,.$$

2.3 Operations on sets

2.3.1 Union and intersection

For two sets A and B we define their **union** as the set

$$A \cup B := \{x : \, x \in A \text{ or } x \in B\}\,.$$

The **intersection** of A and B is defined by

$$A \cap B := \{x : \, x \in A \text{ and } x \in B\}\,.$$

We denote the **empty set** by the symbol $\emptyset$. Two sets are **disjoint** if

$$A \cap B = \emptyset\,.$$

Example 2.4

Define the subset of rational numbers

$$S := \left\{ x \in \mathbb{Q} : \ 0 < x < \frac{5}{2} \right\} .$$

Then we have

$$\mathbb{N} \cap S = \{1, 2\} .$$

We can also define the sets of **even** and **odd** numbers by

$$E := \{2n : \ n \in \mathbb{N}\} , \tag{2.2}$$
$$O := \{2n + 1 : \ n \in \mathbb{N}\} . \tag{2.3}$$

Then we have

$$\mathbb{N} \cap E = E , \ \mathbb{N} \cap O = O , \tag{2.4}$$
$$O \cup E = \mathbb{N} , \ O \cap D = \emptyset . \tag{2.5}$$

2.3.2 Inclusion and equality

Given two sets A and B, we say that A is **contained** in B if all the elements of A are also contained in B. This will be denoted with the **inclusion** symbol $\subset$, that is,

$$A \subset B .$$

In this case we say that

- A is a **subset** of B,
- B is a **superset** of A.

The inclusion $A \subset B$ is equivalent to the implication:

$$x \in A \implies x \in B$$

for all $x \in A$. The symbol $\implies$ reads **implies**, and denotes the fact that the first condition implies the second.

Example 2.5

Given two sets A and B we always have

$$(A \cap B) \subset A\,, \;\; (A \cap B) \subset B\,, \tag{2.6}$$
$$A \subset (A \cup B)\,, \;\; B \subset (A \cup B)\,. \tag{2.7}$$

We say that two sets A and B are equal if they contain the **same** elements. We denote equality by the symbol

$$A = B\,.$$

Example 2.6

The sets

$$A = \{1, 2, 3\}$$

and

$$B = \{3, 1, 2\}$$

are equal. This is because they contain exactly the same elements: **order** does not matter when talking about sets.

Proposition 2.7

Let A and B be sets. Then

$$A = B$$

if and only if

$$A \subset B \;\text{ and }\; B \subset A\,.$$

Proof

The proof is almost trivial. However it is a good exercise in basic logic, so let us do it.

1. First implication $\implies$:
   Suppose that $A = B$. Let us show that $A \subset B$. Since $A = B$, this means that all the elements of A are also contained in B. Therefore if we take $x \in A$ we have

   $$x \in A \quad \implies \quad x \in B\,.$$

   This shows $A \subset B$. The proof of $B \subset A$ is similar.

2. Second implication $\impliedby$:
   Suppose that $A \subset B$ and $B \subset A$. We need to show $A = B$, that is, A and B have the same

elements. To this end let $x \in A$. Since $A \subset B$ then we have $x \in B$. Thus B contains all the elements of A. Since we are also assuming $B \subset A$, this means that A contains all the elements of B. Hence A and B contain the same elements, and $A = B$.

The above proposition is very useful when we need to **prove** that two sets are equal: rather than showing directly that $A = B$, we can prove that $A \subset B$ and $B \subset A$.

2.3.3 Infinite unions and intersections

Suppose given a set Ω, and a family of sets $A_n \subset \Omega$, where $n \in \mathbb{N}$. Then we can define the **infinte union**

$$\bigcup_{n\in\mathbb{N}} A_n := \{x \in \Omega : \, x \in A_n \, \text{ for at least one } \, n \in \mathbb{N}\} \,.$$

The **infinte intersection** is defined as

$$\bigcap_{n\in\mathbb{N}} A_n := \{x \in \Omega : \, x \in A_n \, \text{ for all } \, n \in \mathbb{N}\} \,.$$

Example 2.8

Let the ambient set be $\Omega := \mathbb{N}$ and define the family A_n by

$$A_1 := \{1, 2, 3, 4, \dots\} \tag{2.8}$$
$$A_2 := \{2, 3, 4, 5, \dots\} \tag{2.9}$$
$$A_3 := \{3, 4, 5, 6, \dots\} \tag{2.10}$$
$$\dots\dots \tag{2.11}$$
$$A_n := \{n, n+1, n+2, n+3, \dots\} \,, \tag{2.12}$$

for arbitrary $n \in \mathbb{N}$. Then

$$\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N} \,. \tag{2.13}$$

The above equality can be easily proven using Proposition 2.7. Indeed, assume that $m \in \cup_n A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subset \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows

$$\bigcup_{n\in\mathbb{N}} A_n \subset \mathbb{N} \,.$$

Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index n, $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \cup_{n\in\mathbb{N}} A_n$, showing that

$$\mathbb{N} \subset \bigcup_{n\in\mathbb{N}} A_n \,.$$

Hence we conclude (2.13) by Proposition 2.7.
We also have that

$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset \,. \tag{2.14}$$

We prove the above by **contradiction**. Indeed, suppose that (2.14) is false, i.e.,

$$\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset \,.$$

This means there exists some $m \in \mathbb{N}$ such that $m \in \cap_{n\in\mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (2.14) holds.

2.3.4 Complement

Suppose that A and B are subsets of a larger set Ω. The **complement** of A with respect to B is the set of elements of B which do not belong to A, that is

$$B \setminus A := \{x \in \Omega : \; x \in B \; \text{ and } \; x \notin A\} \,.$$

In particular, the complement of A with respect to Ω is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : \; x \notin A\} \,.$$

Remark 2.9

Suppose that $A \subset \Omega$. Then A and A^c form a **partition** of Ω, in the sense that

$$A \cup A^c = \Omega \quad \text{and} \quad A \cap A^c = \emptyset \,.$$

Example 2.10

Suppose $A, B \subset \Omega$. Then

$$A \subset B \iff B^c \subset A^c \,.$$

Let us prove the above claim:

- First implication $\implies$:
  Suppose that $A \subset B$. We need to show that $B^c \subset A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subset B$, hence $x \in B$. But we had assumed $x \in B$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subset A^c$.

- Second implication $\Longleftarrow$:
  Essentially the same proof, hence we omit it.

We conclude by stating the De Morgan's Laws. The proof will be left as an exercise.

Proposition 2.11: De Morgan's Laws

Suppose $A, B \subset \Omega$. Then
$$(A \cap B)^c = A^c \cup B^c$$
and
$$(A \cup B)^c = A^c \cap B^c \,.$$

2.3.5 Power set

Let Ω be an arbitray set. We define the **power set** of Ω as
$$\mathcal{P}(\Omega) := \{A : \; A \subset \Omega\} \,,$$
that is, the power set of Ω is the set of all subsets of Ω.

Remark 2.12

It holds that:

1. $\mathcal{P}(\Omega)$ is always non-empty, since we have that
$$\emptyset \in \mathcal{P}(\Omega) \,, \quad \Omega \in \mathcal{P}(\Omega) \,.$$
2. Given $A, B \in \mathcal{P}(\Omega)$, then the sets
$$A \cup B, \quad A \cap B, \quad A^c, \quad B \setminus A$$
are all elements of $\mathcal{P}(\Omega)$.
3. Suppose Ω is **discrete** and **finite**, that is,
$$\Omega = \{x_1, \dots, x_m\}$$
for some $m \in \mathbb{N}$. Then $\mathcal{P}(\Omega)$ contains 2^m elements. This is because for each $x_i \in \Omega$ we have just two choices: either include x_i in a subset, or do not include x_i in a subset.

Example 2.13

Define the set

$$\Omega = \{x, y, z\}\,.$$

Then $\mathcal{P}(\Omega)$ has $2^3 = 8$ elements. These are

- $\emptyset$
- $\{x\}$
- $\{y\}$
- $\{z\}$
- $\{x, y\}$
- $\{x, z\}$
- $\{y, z\}$
- $\{x, y, z\}$

We therefore write

$$\begin{aligned} \mathcal{P}(\Omega) = \{\emptyset, \{x\}, \,\{y\}, \,\{z\}, \{x, y\} \quad & (2.15) \\ \{x, z\}, \,\{y, z\}, \,\{x, y, z\}\}\,. \quad & (2.16) \end{aligned}$$

2.3.6 Product of sets

Suppose A and B are two sets. The **product** of A and B is the set of pairs

$$A \times B := \{(a, b) : \; a \in A, \, b \in B\}\,.$$

By definition two elements in $A \times B$ are the same, in symbols

$$(a, b) = (\tilde{a}, \tilde{b})$$

if and only if they are equal component-by-componenent, that is

$$a = \tilde{a}\,, \qquad b = \tilde{b}\,.$$

2.4 Equivalence relation

Suppose A is a set. A **binary relation** R on A is a subset

$$R \subset A \times A\,.$$

Definition 2.14: Equivalence relation

A binary relation R is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has
$$(x,x) \in R\,,$$
This is saying that all the elements in A must be related to themselves

2. **Symmetric**: We have
$$(x,y) \in R \implies (y,x) \in R$$
If x is related to y, then y is related to x

3. **Transitive**: We have
$$(x,y) \in R\,, \ (y,z) \in R \implies (x,z) \in R$$
If x is related to y, and y is related to z, then x must be related to z

If $(x,y) \in R$ we write
$$x \sim y$$
and we say that x and y are **equivalent**.

Definition 2.15: Equivalence classes

Suppose R is an **equivalence relation** on A. The **equivalence class** of an element $x \in A$ is the set
$$[x] := \{y \in A : \ y \sim x\}\,.$$
The set of equivalence classes of elements of A with respect to the equivalence relation R is denoted by
$$A/R := \{[x] : \ x \in A\}\,.$$

Let us immediately clarify the above definitions by considering the prototypical example of equivalence relation: the **equality**.

Example 2.16: Equality is an equivalence relation

Consider the set of natural numbers $\mathbb{N}$. The equality defines a **binary relation** on $\mathbb{N} \times \mathbb{N}$, via
$$R := \{(x,y) \in \mathbb{N} \times \mathbb{N} : \ x = y\}\,.$$
Let us check that R is an **equivalence relation**:

1. Reflexive: It holds, since $x = x$ for all $x \in \mathbb{N}$,

2. Symmetric: Again $x = y$ if and only if $y = x$,

3. Transitive: If $x = y$ and $y = z$ then $x = z$.

The class of equivalence of $x \in \mathbb{N}$ is given by

$$[x] = \{x\},$$

that is, this relation is quite trivial, given that each element of $\mathbb{N}$ can only be related to itself. The quotient space is then

$$\mathbb{N}/R = \{[x] : x \in \mathbb{N}\} = \{\{x\} : x \in \mathbb{N}\}.$$

Example 2.17

Suppose that R is a binary relation on the set $\mathbb{Q}$ of rational numbers defined by

$$x \sim y \iff x - y \in \mathbb{Z}.$$

Then R is an equivalence relation on $\mathbb{Q}$. Indeed:

1. Reflexive: Let $x \in \mathbb{Q}$. Then $x - x = 0$ and $0 \in \mathbb{Z}$. Thus $x \sim x$.
2. Symmetric: If $x \sim y$ then $x - y \in \mathbb{Z}$. But then also

$$-(x-y) = y - x \in \mathbb{Z}$$

   and so $y \sim x$.
3. Transitive: Suppose $x \sim y$ and $y \sim z$. Then

$$x - y \in \mathbb{Z} \text{ and } y - z \in \mathbb{Z}.$$

   Thus we have

$$x - z = (x-y) + (y-z) \in \mathbb{Z}$$

   showing that $x \sim z$. This shows that R is an equivalence relation on $\mathbb{Q}$.

Now note that

$$y \sim x \iff y - x \in \mathbb{Z}$$

and the above is equivalent to

$$\exists\, n \in \mathbb{Z} \text{ s.t. } y - x = n$$

which again is equivalent to

$$\exists\, n \in \mathbb{Z} \text{ s.t. } y = x + n.$$

Therefore all the elements of $\mathbb{Q}$ related to x by R are of the form

$$x + n, \forall\, n \in \mathbb{Z}.$$

The equivalence classes with respect to R are then

$$[x] = \{x + n \,:\, n \in \mathbb{Z}\}\,.$$

Each equivalence class has exactly one element in $[0, 1) \cap \mathbb{Q}$, meaning that:

$$\forall x \in \mathbb{Q}\,,\ \exists!\, q \in \mathbb{Q} \ \text{ s.t } \ 0 \leq q < 1 \ \text{ and } \ q \in [x]\,.$$

Therefore

$$\mathbb{Q}/R = \{[x] \,:\, x \in \mathbb{Q}\} = \{q \in \mathbb{Q} \,:\, 0 \leq q < 1\}\,.$$

2.5 Order relation

Similarly, we define **order relations**.

Definition 2.18: Partial order

A binary relation R on A is called a **partial order** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has
$$(x, x) \in R\,,$$
2. **Transitive**: We have
$$(x, y) \in R\,,\ (y, z) \in R \implies (x, z) \in R$$
3. **Antisymmetric**: We have
$$(x, y) \in R \ \text{ and } \ (y, x) \in R \implies x = y$$
This is the only new condition with respect to the definition of equivalence relation, and it replaces symmetry.

Definition 2.19: Total order

A binary relation R on A is called a **total order relation** if it satisfies the following properties:

1. **Partial order**: R is a partial order on A.
2. **Total**: For each $x, y \in A$ we have
$$(x, y) \in R \ \text{ or } \ (y, x) \in R\,.$$
This is saying that all elements in A are related.

An example of partial order is the operation of **set inclusion**.

Example 2.20: Set inclusion is a partial order

Consider an arbitrary non-empty set Ω and consider its **power set**

$$\mathcal{P}(\Omega) = \{A : \, A \subset \Omega\} \, .$$

The inclusion defines **binary relation** on $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$, via

$$R := \{(A, B) \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) : \, A \subset B\} \, .$$

Let us check that R is an **order relation**:

1. Reflexive: It holds, since $A \subset A$ for all $A \in \mathcal{P}(\Omega)$,
2. Transitive: If $A \subset B$ and $B \subset C$ then by definition of inclusion $A \leq C$.
3. Antisymmetric: If $A \subset B$ and $B \subset A$, then $A = B$ by Proposition 2.7.

Therefore R is a **partial order** on $\mathcal{P}(\Omega)$. Note that in general R is **not** a total order. For example if we consider

$$\Omega = \{x, y\} \, ,$$

then we have

$$\mathcal{P}(\Omega) = \{\emptyset, \, \{x\}, \, \{y\}, \, \{x, y\}\} \, .$$

If we pick $A = \{x\}$ and $B = \{y\}$ then $A \cap B = \emptyset$, meaning that

$$(A, B) \notin R \, , \quad (B, A) \notin R \, ,$$

showing that R is not a total order.

A very important example of total order is the **inequality** on $\mathbb{Q}$.

Example 2.21: Inequality is a total order

Consider the set of rationals $\mathbb{Q}$. The usual inequality defines a **binary relation** on $\mathbb{Q} \times \mathbb{Q}$, via

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : \, x \leq y\} \, .$$

Let us check that R is an **order relation**:

1. Reflexive: It holds, since $x \leq x$ for all $x \in \mathbb{Q}$,
2. Transitive: If $x \leq y$ and $y \leq z$ then $x \leq z$.

3. Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$.

Finally, we halso have that R is a **total order** on $\mathbb{Q}$, since for all $x, y \in \mathbb{Q}$ we have

$$x \leq y \quad \text{or} \quad y \leq x \,.$$

Notation 2.22

If Ω is a set and R is a total order on Ω, we write

$$(x, y) \in R \iff x \leq y \,.$$

Therefore the symbol $\leq$ will always denote a total order relation.

2.6 Intervals

In this section we assume to have available the set $\mathbb{R}$ of **real numbers**, which we recall is an extension of $\mathbb{Q}$. We now introduce the concept of **interval**.

Definition 2.23

Let $a, b \in \mathbb{R}$ with $a < b$. We define the **open interval** (a, b) as the set

$$(a, b) := \{x \in \mathbb{R} : \; a < x < b\} \,.$$

We define the **close interval** $[a, b]$ as the set

$$[a, b] := \{x \in \mathbb{R} : \; a \leq x \leq b\} \,.$$

In general we also define the intervals

$$[a, b) := \{x \in \mathbb{R} : \; a \leq x < b\} \,, \tag{2.17}$$
$$(a, b] := \{x \in \mathbb{R} : \; a \leq x \leq b\} \,, \tag{2.18}$$
$$(a, \infty) := \{x \in \mathbb{R} : \; x > a\} \,, \tag{2.19}$$
$$[a, \infty) := \{x \in \mathbb{R} : \; x \geq a\} \,, \tag{2.20}$$
$$(-\infty, b) := \{x \in \mathbb{R} : \; x < b\} \,, \tag{2.21}$$
$$(-\infty, b] := \{x \in \mathbb{R} : \; x \leq b\} \,. \tag{2.22}$$

Some of the above intervals are depicted in Figure 2.1, Figure 2.2, Figure 2.3, Figure 2.4 below.

Figure 2.1: Interval (a, b)

Figure 2.2: Interval $[a, b]$

2.7 Functions

Definition 2.24: Functions

Assume given two sets A and B. A **function** from A to B is a rule which associates **at each** element $x \in A$ a **single** element $y \in B$. Notations:

- We write $f : A \to B$ to indicate such rule,
- For $x \in A$, we denote by $f(x)$ the element $y \in B$ associated with x by f.

In addition:

- The set A is called the **domain** of f,
- The **range** of f is the set

$$\{y \in B : \ y = f(x) \ \text{ for some } \ x \in A\} \subset B\,.$$

Warning

We want to stress the importance of the first two sentences in Definition 2.24. Assume that $f : A \to B$ is a function. Then:

- To each element $x \in A$ we can **only** associate **one** element $f(x) \in B$,
- Every element $x \in A$ has to be associated to an element $f(x)$ in B.

Example 2.25

Assume given the two sets

$$A = \{a_1, a_2\}\,, \quad B = \{b_1, b_2, b_3\}\,.$$

Figure 2.3: Interval (a, ∞)

Figure 2.4: Interval $(-\infty, b]$

Let us see a few examples:

- Define $f : A \to B$ by setting

$$f(a_1) = b_1 \,, \quad f(a_2) = b_1 \,.$$

  In this way f is a function, with domain A and range

$$f(A) = \{b_1\} \subset B \,.$$

- Define $g : A \to B$ by setting

$$g(a_1) = b_2 \,, \quad g(a_1) = b_3 \,, \quad g(a_2) = b_3$$

  Then g is **NOT** a function, since the element a_1 has two elements associated.
- Define $h : A \to B$ by setting

$$h(a_1) = b_1 \,.$$

  Then g is **NOT** a function, since the element a_2 has no element associated.

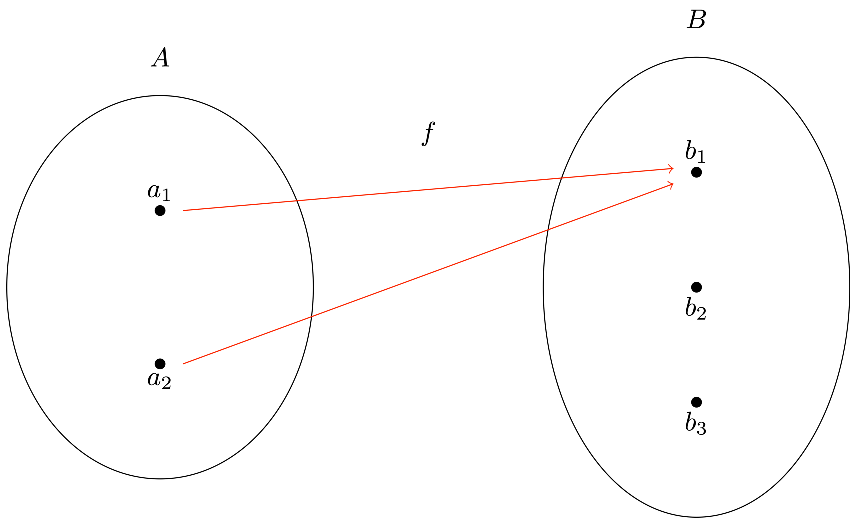

Figure 2.5: Schematic picture of the function f

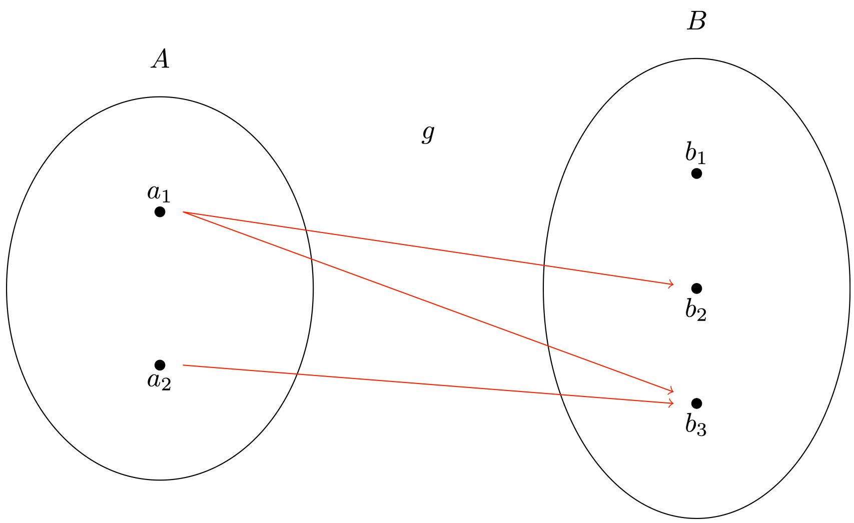

Figure 2.6: Schematic picture of the function g

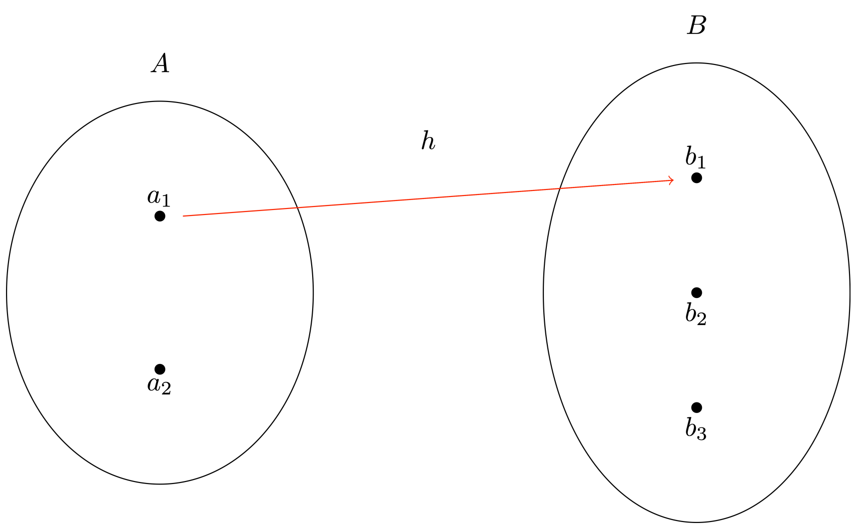

Figure 2.7: Schematic picture of the function h

Example 2.26

Let us make two examples of functions on $\mathbb{R}$:

- Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = x^2 .$$
Note that the domain of f is given by $\mathbb{R}$, while the range is
$$f(\mathbb{R}) = [0, \infty) .$$
- Define $g : \mathbb{R} \to \mathbb{R}$ as the logarithm:
$$g(x) = \log(x) .$$
This time the domain is $(0, \infty)$, while the range is $g(\mathbb{R}) = \mathbb{R}$.

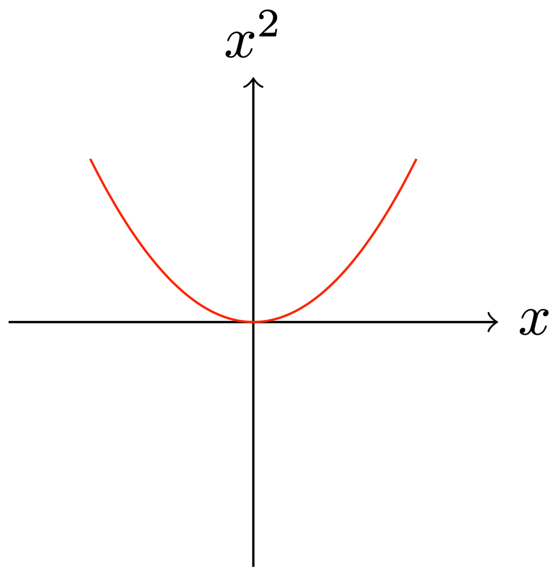

Figure 2.8: Plot of function $f(x) = x^2$

2.8 Absolute value or Modulus

In this section we assume to have available the set $\mathbb{R}$ of **real numbers**, which we recall is an extension of $\mathbb{Q}$.

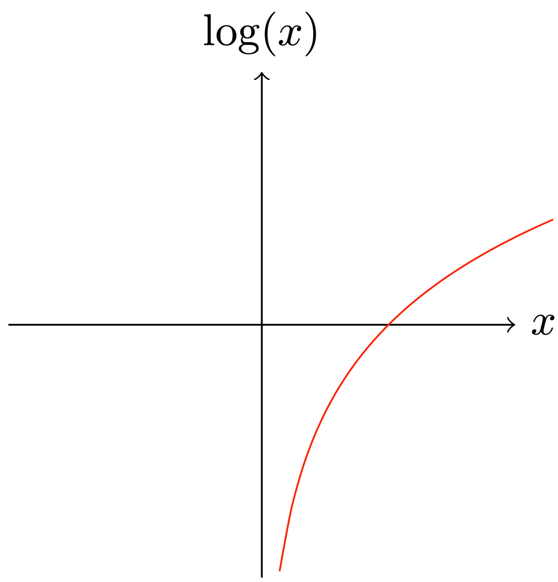

Figure 2.9: Plot of function $g(x) = \log(x)$

Definition 2.27: Absolute value

For $x \in \mathbb{R}$ we define its **absolute value** as the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 2.28

By definition one has $|x| = x$ if $x \geq 0$. For example

$$|\pi| = \pi \,, \; \left|\sqrt{2}\right| = \sqrt{2} \,, \; |0| = 0 \,.$$

Instead $|x| = -x$ if $x < 0$. For example

$$|-\pi| = \pi \,, \; \left|-\sqrt{2}\right| = \sqrt{2} \,, \; |-10| = 10 \,.$$

Let us also make the following basic remark, whose proof will be left as an exercise.

Remark 2.29

For all $x \in \mathbb{R}$ one has

$$|x| \geq 0 \,.$$

Moreover

$$|x| = 0 \iff x = 0 \,.$$

Another basic remark (proof by exercise).

Remark 2.30

For all $x \in \mathbb{R}$ one has

$$|x| = |-x| \,.$$

We can use the definition of abosulte value to define the **absolute value function**. This is a map

$$f : \mathbb{R} \to \mathbb{R} \,, \quad f(x) := |x| \,.$$

You might be familiar with the graph associated to f, as seen below.

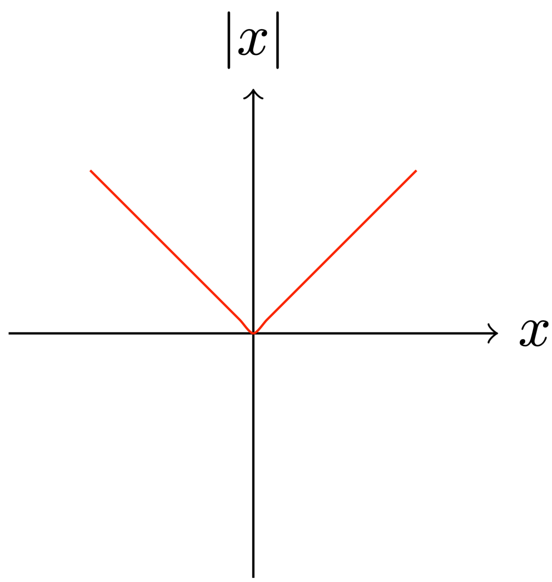

Figure 2.10: Plot of the absolute value function $f(x) = |x|$

It is also useful to understand the absolute value in a geometric way.

Remark 2.31: Geometric interpretation of $|x|$

A number $x \in \mathbb{R}$ can be represented with a point on the real line $\mathbb{R}$.The non-negative number $|x|$ represents the **distance** of x from the origin 0. Notice that this works for both positive and negative numbers x_1 and x_2 respectively, as shown in Figure 2.11 below.

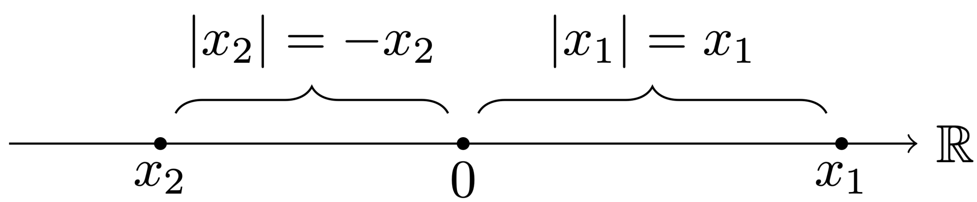

Figure 2.11: Geometric interpretation of $|x|$

Remark 2.32: Geometric interpretation of $|x-y|$

If $x, y \in \mathbb{R}$ then the number $|x-y|$ represents the distance between x and y on the real line, as shown in Figure 2.12 below. Note that by Remark 2.30 we have

$$|x-y| = |y-x| \, .$$

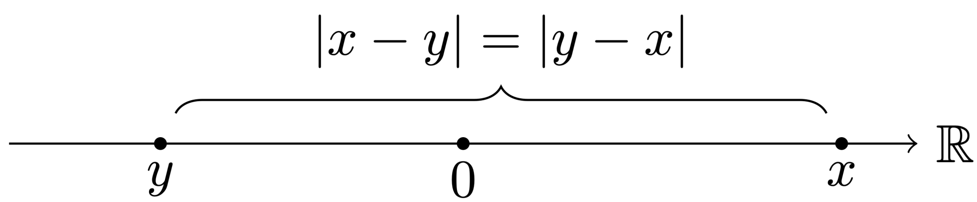

Figure 2.12: Geometric interpretation of $|x-y|$

In the next Lemma we show a fundamental equivalence regarding the absolute value.

Lemma 2.33

Let $x, y \in \mathbb{R}$. Then

$$|x| \leq y \iff -y \leq x \leq y \, .$$

The geometric meaning of the above statement is clear: the distance of x from the origin is less than y, in formulae

$$|x| \leq y \, ,$$

if and only if x belongs to the interval $[-y, y]$, in formulae

$$-y \leq x \leq y \, .$$

A sketch of this explanation is seen in Figure 2.13 below.

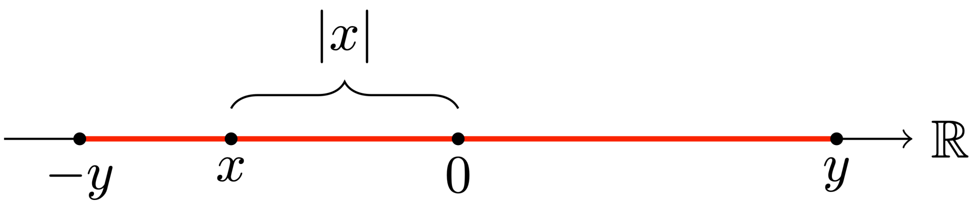

Figure 2.13: Geometric meaning of Lemma 2.33

Proof: Proof of Lemma 2.33

We divide the proof in steps.

- Step 1: implication $\implies$
  Suppose first that
  $$|x| \leq y \,. \tag{2.23}$$
  Recalling that the absolute value is non-negative, from (2.23) we deduce that $0 \leq |x| \leq y$. In particular it holds
  $$y \geq 0 \,. \tag{2.24}$$
  We make separate arguments for the cases $x \geq 0$ and $x < 0$:
  - Case 1: $x \geq 0$. From (2.23), (2.24) and from $x \geq 0$ we have
    $$-y \leq 0 \leq x = |x| \leq y$$
    which shows
    $$-y \leq x \leq y \,.$$
  - Case 2: $x < 0$. From (2.23), (2.24) and from $x < 0$ we have
    $$-y \leq 0 < -x = |x| \leq y$$
    which shows
    $$-y \leq -x \leq y \,.$$
    Multiplying the above inequalities by -1 yields
    $$-y \leq x \leq y \,.$$
- Step 2: implication $\impliedby$
  Suppose now that
  $$-y \leq x \leq y \,. \tag{2.25}$$
  We make separate arguments for the cases $x \geq 0$ and $x < 0$:

– Case 1: $x \geq 0$. Since $x \geq 0$, from (2.25) we get

$$|x| = x \leq y$$

showing that

$$|x| \leq y \,.$$

– Case 2: $x < 0$. Since $x < 0$, from (2.25) we have

$$-y \leq x = -|x| \,.$$

Multiplying the above inequality by -1 yields

$$|x| \leq y \,.$$

With basically the same arguments, one can also show the following.

Lemma 2.34

Let $x, y \in \mathbb{R}$. Then

$$|x| < y \iff -y < x < y \,.$$

2.9 Triangle inequality

The triangle inequality relates the absolute value to the sum operation. It is a very important inequality, which we will use a lot in the future.

Theorem 2.35: Triangle inequality

For every $x, y \in \mathbb{R}$ we have

$$||x| - |y|| \leq |x - y| \leq |x| + |y| \,. \tag{2.26}$$

Before proceeding with the proof, let us discuss the geometric meaning of the triangle inequality.

Remark 2.36: Geometric meaning of triangle inequality

The notion of absolute value can be extended also to vectors in the plane. Suppose that x and y are two vectors in the plane, as in Figure 2.14 below. Then $|x|$ and $|y|$ can be interpreted as the

lengths of these vectors.
Using the rule of sum of vectors, we can draw $x + y$, as shown in Figure 2.15 below. From the picture it is evident that

$$|x + y| \leq |x| + |y| \,, \tag{2.27}$$

that is, *the length of each side of a triangle does not exceed the sum of the lengths of the two remaining sides*. Note that (2.27) is exactly the second inequality in (2.26). This is why (2.26) is called triangle inequality.

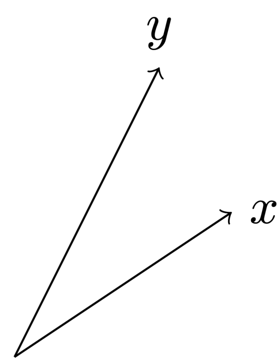

Figure 2.14: Vectors x and y

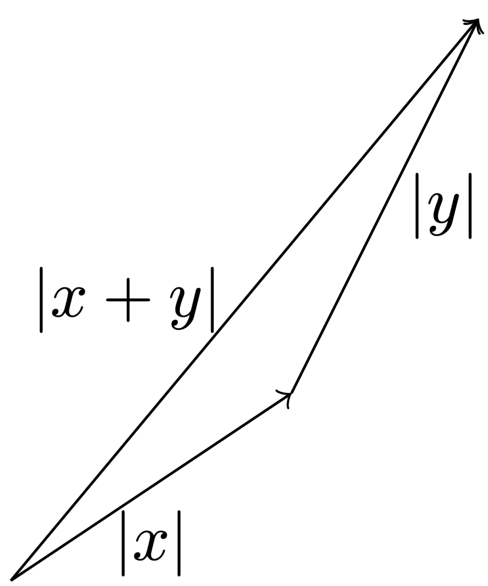

Figure 2.15: Summing the vectors x and y. The triangle inequality relates the length of $x + y$ to the length of x and y

Proof: Proof of Theorem 2.32

Assume that $x, y \in \mathbb{R}$. We prove the two inequalities in (2.26) individually.

- Proof of the second inequality in (2.26):
  Trivially we have
  $$|x| \leq |x| \,.$$
  Therefore we can apply Lemma 2.33 and infer
  $$-|x| \leq x \leq |x| \,. \tag{2.28}$$
  Similarly we have that $|y| \leq |y|$, and so Lemma 2.33 implies
  $$-|y| \leq y \leq |y| \,. \tag{2.29}$$
  Summing (2.28) and (2.29) we get
  $$-(|x| + |y|) \leq x + y \leq |x| + |y| \,.$$
  We can now again apply Lemma 2.33 to get
  $$|x + y| \leq |x| + |y| \,, \tag{2.30}$$
  which is the second inequality in (2.26).

- Proof of the second inequality in (2.26):
  Note that the trivial identity
  $$x = x + y - y$$
  always holds. We then have
  $$\begin{aligned} |x| &= |x + y - y| && (2.31)\\ &= |(x + y) + (-y)| && (2.32)\\ &= |a + b| && (2.33) \end{aligned}$$
  with $a = x + y$ and $b = -y$. We can now apply (2.30) to a and b to obtain
  $$\begin{aligned} |x| &= |a + b| && (2.34)\\ &\leq |a| + |b| && (2.35)\\ &= |x + y| + |-y| && (2.36)\\ &= |x + y| + |y| && (2.37) \end{aligned}$$
  Therefore
  $$|x| - |y| \leq |x + y| \,. \tag{2.38}$$
  We can now swap x and y in (2.38) to get
  $$|y| - |x| \leq |x + y| \,.$$

By rearranging the above inequality we obtain

$$-|x+y| \leq |x| - |y| \,. \tag{2.39}$$

Putting together (2.38) and (2.39) yields

$$-|x+y| \leq |x| - |y| \leq |x+y| \,.$$

By Lemma 2.33 the above is equivalent to

$$||x| - |y|| \leq |x+y| \,,$$

which is the first inequality in (2.26).

An immediate consequence of the triangle inequality are the following inequalities, which are left as an exercise.

Remark 2.37

For any $x, y \in \mathbb{R}$ it holds

$$||x| - |y|| \leq |x-y| \leq |x| + |y| \,.$$

Moreover for any $x, y, z \in \mathbb{R}$ it holds

$$|x-y| \leq |x-z| + |z-y| \,.$$

2.10 Proofs in Mathematics

In this section we carry out the proof of a seemingly trivial statement, to get used to the process. In a proof one needs to show that

$$\alpha \implies \beta \tag{2.40}$$

where

- α is a given set of assumptions, or **hypothesis**
- β is a conclusion, or **thesis**

To show (2.40) we need to convince ourselves that β follows by assuming α.

Common strategies to prove (2.40) are:

- **Contradiction**: Assume that the thesis is **false**, and hope to reach a contradiction: that is, prove that

$$\neg\beta \implies \text{contradiction}$$

where $\neg\beta$ is the **negation** of β. For example we already proved by contradiction that

$$\text{Definition of } \mathbb{Q} \implies \sqrt{2} \notin \mathbb{Q} \,,$$

In the above statement

$$\alpha \rightsquigarrow \text{Definition of } \mathbb{Q} \,.$$
$$\beta \rightsquigarrow \sqrt{2} \notin \mathbb{Q} \,.$$

Therefore

$$\neg\beta \rightsquigarrow \sqrt{2} \in \mathbb{Q} \,.$$

- **Direct**: Sometimes proofs will also need **direct** arguments, meaning that one need to show directly that (2.40).
- **Contrapositive**: The statement is equivalent to (2.40)

$$\neg\beta \implies \neg\alpha \,. \tag{2.41}$$

Thus, instead of proving (2.40) one could instead show (2.41). The statement (2.41) is called the **contrapositive** of (2.40).

Let us make an example.

Proposition 2.38

Two real numbers a, b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a-b| < \varepsilon$.

Before proceeding with the proof, note that the above stetement is just saying that:

Two numbers are equal if and only if they are **arbitrarily** close

By *arbitrarily close* we mean that they are *as close as you want the to be.*

Proof: of Proposition 2.38

Let us first rephrase the statement using mathematical symbols:

Let $a, b \in \mathbb{R}$. Then it holds:

$$a = b \iff |a-b| < \varepsilon \,, \ \forall\, \varepsilon > 0 \,.$$

Setting

$$\alpha \rightsquigarrow \ a = b \tag{2.42}$$
$$\beta \rightsquigarrow \ |a-b| < \varepsilon \,, \ \forall\, \varepsilon > 0 \tag{2.43}$$

the statement is equivalent to

$$\alpha \iff \beta\,.$$

To show the above, it is sufficient to show that

$$\alpha \implies \beta$$

and

$$\beta \implies \alpha\,.$$

Step 1. Proof that $\alpha \implies \beta$:
This proof can be carried out by a **direct** argument. Since we are assuming α, this means

$$a = b\,.$$

We want to see that β holds. Therefore fix an arbitrary $\varepsilon > 0$. This means that ε can be **any** positive number. Clearly

$$|a - b| = |0| = 0 < \varepsilon$$

since $a = b$, $|0| = 0$, and $\varepsilon > 0$. The above shows that

$$|a - b| < \varepsilon\,.$$

As $\varepsilon > 0$ was arbitrary, we have just proven that

$$|a - b| < \varepsilon\,, \ \forall\, \varepsilon > 0\,,$$

meaning that β holds and the proof is concluded.
Step 2. Proof that $\beta \implies \alpha$:
Let us prove this implication by showing the **contrapositive**

$$\neg\alpha \implies \neg\beta\,.$$

So let us assume $\neg\alpha$ is true. This means that

$$a \neq b\,.$$

We have to see that $\neg\beta$ holds. But $\neg\beta$ means that

$$\exists\, \varepsilon_0 > 0 \ \text{ s.t } \ |a - b| \geq \varepsilon_0\,.$$

The above is satisfied by choosing

$$\varepsilon_0 := |a - b|\,,$$

since $\varepsilon_0 > 0$ given that $a \neq b$,

2.11 Induction

Another technique for carrying out proofs is **induction**, which we take as an axiom.

Axiom 2.39: Principle of Induction

Let S be a subset of $\mathbb{N}$. Suppose that

1. We have $1 \in S$, and
2. Whenever $n \in S$, then $(n+1) \in S$.

Then we have
$$S = \mathbb{N}\,.$$

Important

The above is an **axiom**, meaning that we do not prove it, but rather we just **assume it holds**.

Remark 2.40

It would be possible to prove the Principle of Induction starting from elementary axioms for $\mathbb{N}$, called the **Peano Axioms**, see the Wikipedia page.
However, in justifying basic principles of mathematics, one at some point needs to draw a line. This means that something which looks elementary needs to be assumed to hold, in order to have a starting point for proving deeper statements.
In the case of the Principle of Induction, the intuition is clear:

> The Principle of Induction is just describing the **domino effect**: *If one tile falls, then the next one will fall as well.* Therefore if the *first tile falls, all the tiles will fall.*

It seems reasonable to assume such evident principle.

The Principle of Induction can be used to prove statements which depend on some index $n \in \mathbb{N}$. Precisely, the following statement holds.

Corollary 2.41: Principle of Inducion - Alternative formulation

Let $\alpha(n)$ be a statement which depends on $n \in \mathbb{N}$. Suppose that

1. $\alpha(1)$ is true, and

2. Whenever $\alpha(n)$ is true, then $\alpha(n+1)$ is true.

Then $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Proof

Define the set

$$S := \{n \in \mathbb{N} \ \text{s.t} \ \alpha(n) \text{ is true}\} .$$

Then

1. We have $1 \in S$, since $\alpha(1)$ is true.
2. If $n \in S$ then $\alpha(n)$ is true. By assumption this implies that $\alpha(n+1)$ is true. Therefore $(n+1) \in S$.

Therefore S satisfies the assumptions of the Induction Principle and we conclude that

$$S = \mathbb{N} .$$

By definition this means that $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Example 2.42: Formula for summing first n natural numbers

Using the Principle of Induction we can prove that

$$1 + 2 + 3 + ... + (n-1) + n = \frac{n(n+1)}{2} \tag{2.44}$$

holds for all $n \in \mathbb{N}$.
Proof. To be really precise, consider the statement

$$\alpha(n) := \text{ the above formula is true for } \ n .$$

In order to apply induction, we need to show that

1. $\alpha(1)$ is true,
2. If $\alpha(n)$ is true then $\alpha(n+1)$ is true.

Let us proceed: 1. It is immediate to check that (2.44) holds for $n = 1$. 2. Suppose (2.44) holds for

n. Then

$$1 + ... + n + (n+1) = \frac{n(n+1)}{2} + (n+1) \tag{2.45}$$

$$= \frac{n(n+1) + 2(n+1)}{2} \tag{2.46}$$

$$= \frac{(n+1)(n+2)}{n} \tag{2.47}$$

where in the first equality we used that (2.44) holds for n. We then have

$$1 + ... + n + (n+1) = \frac{(n+1)(n+2)}{n},$$

which shows that (2.44) holds for $n+1$.
By the Principle of Induction we then conclude that $\alpha(n)$ is true for all $n \in \mathbb{N}$, which means that (2.44) holds for all $n \in \mathbb{N}$.

Example 2.43: Statements about sequences of numbers

Suppose you are given a collection of numbers

$$\{x_n \text{ s.t } n \in \mathbb{N}\}.$$

Such collection of numbers is called **sequence**. Assume that

$$x_1 = 1$$

and that

$$x_{n+1} := \frac{x_n}{2} + 1.$$

A sequence defined as above is called **recurrence sequence**. Using the above rule we can compute all the terms of x_n: for example

$$x_2 = \frac{x_1}{2} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$$

and

$$x_3 = \frac{x_2}{2} + 1 = \frac{3}{4} + 1 = \frac{7}{4}.$$

By computing these terms we notice that the sequence might be increasing. Indeed we can prove by induction that

$$x_{n+1} \geq x_n \tag{2.48}$$

for all $n \in \mathbb{N}$.
Proof. By induction:

1. We have seen that $x_1 = 1$ and $x_2 = 3/2$. Thus
$$x_2 \geq x_1 \,.$$
2. Suppose now that
$$x_{n+1} \geq x_n \,.$$
We need to prove that
$$x_{n+2} \geq x_{n+1} \,.$$
Indeed, we can multiply the inequality $x_{n+1} \geq x_n$ by $1/2$ and add 1 to get
$$\frac{x_{n+1}}{2} + 1 \geq \frac{x_n}{2} + 1 \,.$$
The above is equivalent, by definition, to $x_{n+2} \geq x_{n+1}$.

Therefore the assumptions of the Induction Principle are satisfied, and (2.48) follows.

3 Real Numbers

In this chapter we introduce the sysyem of Real Numbers $\mathbb{R}$.

3.1 Fields

In order to introduce $\mathbb{R}$, we need the concepts of **binary operation** and **field**. We proceed in a general setting, starting from a set K.

Definition 3.1: Binary operation

A binary operation on a set K is a function

$$\circ : K \times K \to K$$

which maps the ordered pair (x, y) into $x \circ y$.

Notation 3.2

There are two main binary operations we are interested in:

- **Addition**: denoted by $+$. The addition, or **sum** of $x, y \in K$ is denoted by

$$x + y \,.$$

- **Multiplication**: denoted by $\cdot$. The multiplication, or **product** of $x, y \in K$ is denoted by

$$x \cdot y \quad \text{or} \quad xy \,.$$

Example 3.3: of binary operation

Let $K = \{0, 1\}$. We can for example define the operations of sum and product according to the tables

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

The above mean that

$$0+0=1+1=0\,, \quad 0+1=1+0=1\,,$$

$$0\cdot 0=0\cdot 1=1\cdot 0=0\,, \quad 1\cdot 1=1\,.$$

This is just one option. Note that we could not have defined

$$1+1=2\,,$$

since $2 \notin K$.

Binary operations take ordered pairs of elements of K as input. Therefore the operation

$$x \circ y \circ z$$

does not make sense, since we do not know which one between

$$x \circ y \quad \text{or} \quad y \circ z$$

has to be performed first. Also one could have that the outcome of an operation depends on order:

$$x \circ y \neq y \circ x\,.$$

This motivates the following definition.

Definition 3.4

Let K be a set and $\circ : K \times K \to K$ be a binary operation on K. We say that:

1. $\circ$ is **commutative** if $\forall\, x, y \in K$
$$x \circ y = y \circ x\,.$$
2. $\circ$ is **associative** if $\forall\, x, y, z \in K$
$$(x \circ y) \circ z = x \circ (y \circ z)\,.$$
3. An element $e \in K$ is called **neutral element** of $\circ$ if $\forall\, x \in K$
$$x \circ e = e \circ x = e\,.$$
4. Let e be a neutral element of $\circ$ and let $x \in K$. An element $y \in K$ is called an **inverse** of x with respect to $\circ$ if
$$x \circ y = y \circ x = e\,.$$

Example 3.5

Let K with $+$ and $\cdot$ be as in Example 3.1. For the sum we have

- $+$ is commutative, since
$$0+1=1+0=0\,.$$
- $+$ is associative, since for example
$$(0+1)+1=1+1=0\,,\quad 0+(1+1)=0+0=0\,,$$
and therefore
$$(0+1)+1=0+(1+1)\,.$$
In general one can show that $+$ is associative by checking all the other permutations.
- The neutral element of $+$ is 0, since
$$0+0=0\,,\quad 1+0=0+1=1\,.$$
- Every element has an inverse. Indeed, the inverse of 0 is 0, since
$$0+0=0\,,$$
while the inverse of 1 is 1, since
$$1+1=1+1=0\,.$$

For the product we have:

- $\cdot$ is commutative, since
$$1\cdot 0=0\cdot 1=0\,.$$
- $\cdot$ is associative, since for example
$$(0\cdot 1)\cdot 1=0\cdot 1=0\,,\quad 0\cdot(1\cdot 1)=0\cdot 1=0\,,$$
and therefore
$$(0\cdot 1)\cdot 1=0\cdot(1\cdot 1)\,.$$
By checking all the other permutations one can show that $\cdot$ is associative.
- The neutral element of $\cdot$ is 1, since
$$0\cdot 1=1\cdot 0=0\,,\quad 1\cdot 1=1\,.$$
- The element 0 has no inverse, since
$$0\cdot 0=0\cdot 1=1\cdot 0=0\,,$$
and thus we never obtain the neutral element 1. The inverse of 1 is given by 1, since
$$1\cdot 1=1\,.$$

Example 3.6

Let $K = \{0, 1\}$ be a set with binary relation $\circ$ defined by the table

$\circ$	0	1
0	1	1
1	0	0

In this case $\circ$ is not commutative since

$$0 \circ 1 = 1\,, \quad 1 \circ 0 = 0$$

and therefore

$$0 \circ 1 \neq 1 \circ 0\,.$$

Moreover $\circ$ is not associative, since

$$(0 \circ 1) \circ 1 = 1 \circ 1 = 0\,,$$

while

$$0(\circ 1 \circ 1) = 0 \circ 0 = 1\,.$$

We are ready to define fields.

Definition 3.7: Field

Let K be a set with binary operations $+$ and $\cdot$. We call the triple $(K, +, \cdot)$ a **field** if:

1. The addition $+$ satisfies:
   - (A1) $+$ is **commutative** and **associative**.
   - (A2) There exists a **neutral element** in K for $+$. We call this element 0.
   - (A3) For every $x \in K$ there exists an inverse $y \in K$ with respect to $+$. We call this element the **additive inverse** of x and denote it by $y = -x$.
2. The multiplication $\cdot$ satisfies:
   - (M1) $\cdot$ is **commutative** and **associative**.
   - (M2) There exists a **neutral element** in K for $\cdot$. We call this element 1.
   - (M3) For every $x \in K$ with $x \neq 0$ there exists an inverse $y \in K$ with respect to $\cdot$. We call this element the **multiplciative inverse** of x and denote it by $y = x^{-1}$.
3. $+$ and $\cdot$ are related by the **distributive property**:
   - (AM) For all $x, y, z \in K$ we have

$$x \cdot (y + z) = (x \cdot y) + (y \cdot z)\,.$$

Example 3.8

Let K with $+$ and $\cdot$ be as in Example 3.1. We can show that $(K, +, \cdot)$ is a field. Indeed we have already shown in Example 3.5 that:

- (A1) and (M1) hold,
- (A2) holds with neutral element 0,
- (M2) holds with neutral element 1,
- (A3) every element has an additive inverse, with

$$-0 = 0\,, \quad -1 = 1\,,$$

- (M3) every element which is not 0 a multiplicative inverse, with

$$1^{-1} = 1\,.$$

- (AM) this is left to show: Indeed, for all $y, z \in K$ we have

$$0 \cdot (y + z) = 0\,, \quad (0 \cdot y) + (0 \cdot z) = 0 + 0 = 0\,,$$

  and also

$$1 \cdot (y + z) = y + z\,, \quad (1 \cdot y) + (1 \cdot z) = y + z\,.$$

Definition 3.9: Subtraction and division

Let $(K, +, \cdot)$ be a field. We define:

- **Subtraction** as the operation $-$ defined by

$$x - y := x + (-y)\,, \quad \forall\, x, y \in K\,,$$

  where $-y$ is the additive inverse of y.
- **Division** as the operation $/$ defined by

$$x/y := x \cdot y^{-1}\,, \quad \forall\, x, y \in K\,,\ y \neq 0\,,$$

  where y^{-1} is the multiplicative inverse of y.

Proposition 3.10: Uniqueness of neutral elements and inverses

Let $(K, +, \cdot)$ be a field. Then

1. There is a unique element in K with the property of 0,
2. There is a unique element in K with the property of 1,
3. For all $x \in K$ there is a unique additive inverse $-x$,
4. For all $x \in K$, $x \neq 0$, there is a unique multiplicative inverse x^{-1}.

Proof

1. Suppose that $0 \in K$ and $\tilde{0} \in K$ are both neutral element of $+$, that is, they both satisfy (A2). Then
$$0 + \tilde{0} = 0$$
since $\tilde{0}$ is a neutral element for $+$. Moreover
$$\tilde{0} + 0 = \tilde{0}$$
since 0 is a neutral element for $+$. By commutativity of $+$, see property (A1), we have
$$0 = 0 + \tilde{0} = \tilde{0} + 0 = \tilde{0}\,,$$
showing that $0 = \tilde{0}$. Hence the neutral element for $+$ is unique.
2. Exercise.
3. Let $x \in K$ and suppose that $y, \tilde{y} \in K$ are both additive inverses of x, that is, they both satisfy (A3). Therefore
$$x + y = 0$$
since y is an additive inverse of x and
$$x + \tilde{y} = 0$$
since $\tilde{y}$ is an additive inverse of x. Therefore we can use commutativity and associativity and of $+$, see property (A1), and the fact that 0 is the neutral element of $+$, to infer
$$\begin{aligned} y = y + 0 &= y + (x + \tilde{y}) && (3.1)\\ &= (y + x) + \tilde{y} = (x + y) + \tilde{y} && (3.2)\\ &= 0 + \tilde{y} = \tilde{y}\,, && (3.3) \end{aligned}$$
concluding that $y = \tilde{y}$. Thus there is a unique additive inverse of x, and
$$y = \tilde{y} = -x\,,$$
with $-x$ the element from property (A3).
4. Exercise.

Using the properties of field we can also show that the usual properties of sum, subtraction, multiplication and division still hold in any field. We list such properties in the following proposition.

Proposition 3.11: Properties of field operations

Let $(K, +, \cdot)$ be a field. Then for all $x, y, z \in K$,

- $x + y = x + z \quad \Longrightarrow \quad y = z$
- $x \cdot y = x \cdot z$ and $x \neq 0 \quad \Longrightarrow \quad y = z$

- $-0 = 0$
- $1^{-1} = 1$
- $x \cdot 0 = 0$
- $-1 \cdot x = -x$
- $-(-x) = x$
- $(x^{-1})^{-1} = x$ if $x \neq 0$
- $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$

The above properties can be all proven with elementary use of the field properties (A1)-(A3), (M1)-(M3) and (AM). This is an exercise in patience, and is left to the reader.

Let us conclude with examining the sets of numbers introduced in Chapter 1.

Theorem 3.12

Consider the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ with the usual operations $+$ and $\cdot$. We have:

- $(\mathbb{N}, +, \cdot)$ is **not a field**:
  It satisfies properties (A1), (A2), (M1), (M2), (AM) of fields. It is missing properties (A3) and (M3), the additive and multiplicative inverse properties, respectively.
- $(\mathbb{Z}, +, \cdot)$ is **not a field**:
  It satisfies properties (A1), (A2), (A3), (M1), (M2), (AM) of fields. Thus it is only missing (M3), the multiplicative inverse property.
- $(\mathbb{Q}, +, \cdot)$ **is a field**.

The proof is omitted.

3.2 Cut Property

We have just introduced the notion of **field**, and noted that the set of rational numbers with the usual operations

$$(\mathbb{Q}, +, \cdot)$$

is a **field**.

We now need to address the key issue we proved in Chapter 1, that is, that

$$\sqrt{2} \notin \mathbb{Q} .$$

This means that $\mathbb{Q}$ has **gaps**, and cannot be represented as a **continuous** line. The rigorous definition of **lack of gaps** needs the concept of **cut** of a set.

Definition 3.13: Partition of a set

Let S be a set. The pair (A, B) is a **partition** of S if

$$A, B \subset S\,, \quad A \neq \emptyset\,, \quad B \neq \emptyset\,,$$

and

$$S = A \cup B\,, \quad A \cap B = \emptyset\,.$$

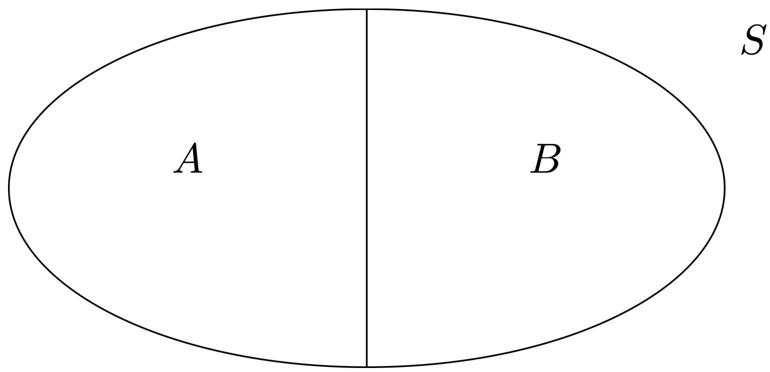

Figure 3.1: Schematic picture of partition

Definition 3.14: Cut of a set

Let S be a set with a total order relation $\leq$. The pair (A, B) is a **cut** of S if

1. (A, B) is a **partition** of S,
2. We have
$$a \leq b\,, \quad \forall\, a \in A\,,\ \forall\, b \in B\,.$$

The **cut** of a set is often called **Dedekind cut**, named after **Richard Dedekind**, who used cuts to give an explicit construction of the real numbers $\mathbb{R}$, see Wikipedia page.

Definition 3.15: Cut property

Let S be a set with a total order relation $\leq$. We say that S has the **cut property** if for every cut (A, B) of S there exists some $s \in S$ such that

$$a \leq s \leq b\,, \quad \forall\, a \in A\,,\ \forall\, b \in B\,.$$

We call s the **separator** of the cut (A, B).

Example 3.16

Let $S = \mathbb{Q}$ and consider the sets

$$A = (-\infty, s] \cap \mathbb{Q}\,, \quad B = (0, \infty) \cap \mathbb{Q}\,.$$

for some $s \in \mathbb{Q}$. Then the pair (A, B) is a cut of $\mathbb{Q}$, and s is the separator.

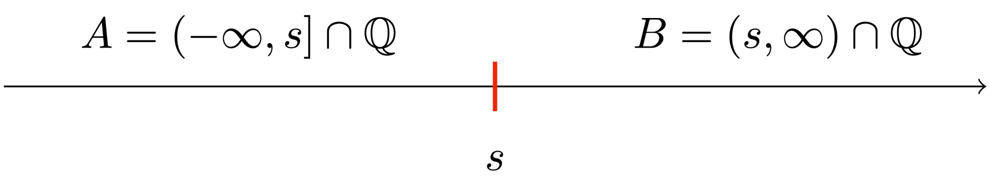

Figure 3.2: (A, B) is a cut of $\mathbb{Q}$ with separator s.

Question 3.17

Do all cuts of $\mathbb{Q}$ have a **separator**? In other words, does $\mathbb{Q}$ have the **cut property**?

The answer to the above question is **NO**. For example the pair

$$A = (-\infty, \sqrt{2}) \cap \mathbb{Q}\,, \quad B = (\sqrt{2}, \infty) \cap \mathbb{Q}\,. \tag{3.4}$$

is a cut of $\mathbb{Q}$, since $\sqrt{2} \notin \mathbb{Q}$. However what is the separator? It should be $s = \sqrt{2}$, given that clearly

$$a \leq \sqrt{2} \leq b\,, \quad \forall\, a \in A\,,\ \forall\, b \in B\,.$$

However $\sqrt{2} \notin \mathbb{Q}$, so we are **NOT ALLOWED** to take it as separator. Indeed, we can show that (A, B) defined as in (3.4) has no separator.

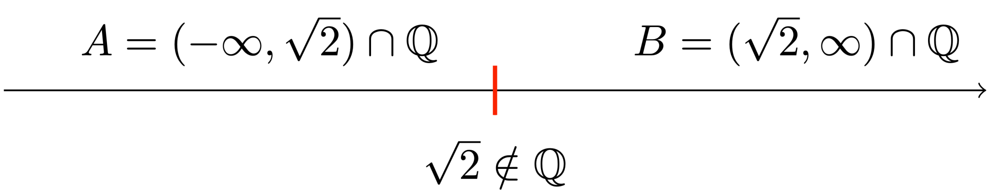

Figure 3.3: (A, B) is a cut of $\mathbb{Q}$ which has no separator

Theorem 3.18: $\mathbb{Q}$ does not have the cut property.

$\mathbb{Q}$ does not have the cut property. More explicitly, there exist a cut (A, B) of $\mathbb{Q}$ which has no separator.

Proof

Let A and B be the sets defined in (3.4). It is useful to rewrite A and B in the form

$$A = A_1 \cup A_2 \,,$$

where

$$A_1 = \{q \in \mathbb{Q} : \; q < 0\} \,,$$
$$A_2 = \{q \in \mathbb{Q} : \; q \geq 0 \,, \; q^2 < 2\} \,,$$

and

$$B = \{q \in \mathbb{Q} : \; q > 0, \; q^2 > 2\} \,.$$

Step 1. (A, B) is a cut of $\mathbb{Q}$:
We need to prove the following:

1. (A, B) is a partition of $\mathbb{Q}$. This is because $A, B \subset \mathbb{Q}$ with $A \neq \emptyset$ and $B \neq \emptyset$. Moreover $A \cap B = \emptyset$ and
$$A \cup B = \mathbb{Q} \,,$$
given that $\sqrt{2} \notin \mathbb{Q}$, and so there is no element $q \in \mathbb{Q}$ such that $q^2 = 2$.
2. It holds
$$a \leq b \,, \quad \forall a \in A \,, \; \forall \, b \in B \,.$$
Indeed, suppose that $a \in A$ and $b \in B$. We have two cases:
   - $a \in A_1$: Therefore $a < 0$. In particular
   $$a < 0 < b \,,$$
   given that $b > 0$ for all $b \in B$. Thus $a < b$.
   - $a \in A_2$: Therefore $a \geq 0$ and $a^2 < 2$. In particular
   $$a^2 < 2 < b^2 \,,$$
   since $b^2 > 2$ for all $b \in B$. In particular
   $$a^2 < b^2 \,.$$
   Since $b > 0$ for all $b \in B$, from the above inequality we infer $a < b$, concluding.

Step 2. (A, B) has no separator:
Suppose by contradiction that (A, B) admits a separator

$$L \in \mathbb{Q} \,.$$

By definition this means

$$a \leq L \leq b \,, \quad \forall a \in A \,, \; \forall \, b \in B \,. \tag{3.5}$$

Since
$$L \in \mathbb{Q}\,, \quad \mathbb{Q} = A \cup B\,, \quad A \cap B = \emptyset\,,$$
then either $L \in A$ or $L \in B$. We will see that both these possibilities lead to a contradiction:
Case 1: $L \in A$.
By (3.5) we know that
$$a \leq L\,, \quad \forall\, a \in A\,. \tag{3.6}$$
In particular the above implies
$$L \geq 0 \tag{3.7}$$
since $0 \in A$. Therefore we must have $L \in A_2$, that is,
$$L \geq 0 \;\text{ and }\; L^2 < 2\,. \tag{3.8}$$
Set
$$\tilde{L} := L + \frac{1}{n}$$
for $n \in \mathbb{N}$, $n \neq 0$ to be chosen later. Clearly we have
$$\tilde{L} \in \mathbb{Q} \;\text{ and }\; L < \tilde{L}\,. \tag{3.9}$$
From (3.7) and (3.9) we have also
$$\tilde{L} > 0\,. \tag{3.10}$$
We now want to show that there is a choice of n such that $\tilde{L}^2 < 2$, which will lead to a contradiction. Indeed, we can estimate
$$\begin{aligned} \tilde{L}^2 &= \left(L + \frac{1}{n}\right)^2 && (3.11)\\ &= L^2 + \frac{1}{n^2} + 2\frac{L}{n} && (3.12)\\ &< L^2 + \frac{1}{n} + 2\frac{L}{n} \qquad \left(\text{using } \frac{1}{n} < \frac{1}{n^2}\right) && (3.13)\\ &= L^2 + \frac{2L+1}{n}\,. && (3.14) \end{aligned}$$
If we now impose that
$$L^2 + \frac{2L+1}{n} < 2\,,$$
we can rearrange the above and obtain
$$n(2 - L^2) > 2L + 1\,.$$
Now note that $L^2 < 2$ by assumption (3.8). Thus we can divived by $(2 - L^2)$ and obtain
$$n > \frac{2L+1}{2 - L^2}\,.$$

Therefore we have just shown that

$$n > \frac{2L+1}{2-L^2} \quad \Longrightarrow \quad \tilde{L}^2 < 2\,.$$

Together with (3.10) this implies $\tilde{L} \in A$. Therefore we have

$$\tilde{L} \leq L$$

by (3.6). On the other hand it also holds

$$\tilde{L} > L$$

by (3.9), and therefore we have a contradiction. Thus $L \notin A$.
Case 2: $L \in B$.
As $L \in B$, we have by definition

$$L > 0\,, \quad L^2 > 2\,. \tag{3.15}$$

Moreover since L is a separator, see (3.5), in particular

$$L \leq b\,, \ \forall\, b \in B\,. \tag{3.16}$$

Define now

$$\tilde{L} := L - \frac{1}{n}$$

with $n \in \mathbb{N}$, $n \neq 0$ to be chosen later. Clearly we have

$$\tilde{L} < L\,. \tag{3.17}$$

We now show that n can be chosen so that $\tilde{L} \in B$. Indeed

$$\begin{align}
\tilde{L}^2 &= \left(L - \frac{1}{n}\right)^2 \tag{3.18}\\
&= L^2 + \frac{1}{n^2} - 2\frac{L}{n} \tag{3.19}\\
&> L^2 - \frac{1}{n^2} - 2\frac{L}{n} \qquad \left(\text{using } \frac{1}{n^2} > -\frac{1}{n^2}\right) \tag{3.20}\\
&> L^2 - \frac{1}{n} - 2\frac{L}{n} \qquad \left(\text{using } -\frac{1}{n^2} > -\frac{1}{n}\right) \tag{3.21}\\
&= L^2 - \frac{1+2L}{n}\,. \tag{3.22}
\end{align}$$

Now we impose

$$L^2 - \frac{1+2L}{n} > 2$$

which is equivalent to

$$n(L^2-2) > 1 + 2L\,.$$

Since we are assuming $L \in B$, then $L^2 > 2$, see (3.15). Therefore we can divide by $(L^2 - 2)$ and get

$$n > \frac{1+2L}{L^2-2}.$$

In total, we have just shown that

$$n > \frac{1+2L}{L^2-2} \quad \Longrightarrow \quad \tilde{L}^2 > 2\,,$$

proving that $\tilde{L} \in B$. Therefore by (3.16) we get

$$L \leq \tilde{L}\,.$$

This contradicts (3.17).
Conclusion:
We have seen that assuming that (A, B) has a separator $L \in \mathbb{Q}$ leads to a contradiction. Thus the cut (A, B) has no separator.

Remark 3.19

The above proof can be summarized by saying that the set

$$A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$$

does not admit a **largest** element in $\mathbb{Q}$, and that the set

$$B = (\sqrt{2}, \infty) \cap \mathbb{Q}$$

does not admit a **lowest element** in $\mathbb{Q}$. We will make this more precise.

3.3 Axioms of Real Numbers

We now have all the key elements to introduce the Real Numbers $\mathbb{R}$. These ingredients are

- Definition of **field**
- Definition of **total order**
- The **Cut Property**

The definition of $\mathbb{R}$ is given in an axiomatic way.

Definition 3.20: System of Real Numbers $\mathbb{R}$

A system of Real Numbers is a set $\mathbb{R}$ satisfying the following properties:

1. There is a relation $\leq$ of **total order** on $\mathbb{R}$, that is, they hold
   - (A1) Reflexivity: $\forall\, a \in \mathbb{R}$
   $$a \leq a\,,$$
   - (A2) Antisymmetry: It holds
   $$a \leq b \, \text{ and } \, b \leq a \quad \Longrightarrow \quad a = b\,,$$
   - (A3) Transitivity: It holds
   $$a \leq b \, \text{ and } \, b \leq c \quad \Longrightarrow \quad a = c\,,$$
   - (A4) Total order: $\forall\, a, b \in \mathbb{R}$ we have
   $$a \leq b \, \text{ or } \, b \leq a\,.$$
2. There is an operation $+$ of **sum** on $\mathbb{R}$ which associates to pairs $a, b \in \mathbb{R}$ the number $(a+b) \in \mathbb{R}$. The sum satisfies:
   - (B1) Commutativity: $\forall\, a, b \in \mathbb{R}$
   $$a + b = b + a$$
   - (B2) Associativity: $\forall\, a, b, c \in \mathbb{R}$
   $$(a + b) + c = a + (b + c)\,,$$
   - (B3) Additive Identity: $\exists\, 0 \in \mathbb{R}$ s.t
   $$a + 0 = 0 + a = a\,, \ \ \forall\, a \in \mathbb{R}\,,$$
   - (B4) Additive Inverse: $\forall$

4 Sequences

Coming soon

4.1 Example: Heron's Method

The first explicit algorithm for approximating

$$\sqrt{x}$$

for $x > 0$ is known **Heron's method**, after the first-century Greek mathematician Heron of Alexandria who described the method in his AD 60 work Metrica, see reference to Wikipedia page.

Let us see what is the idea of the algorithm:

- Suppose that a_1 is an approximation of $\sqrt{x}$ from above, that is,

$$\sqrt{x} < a_1 \,. \tag{4.1}$$

- Multiplying (4.1) by $\sqrt{x}/a_1$ we obtain

$$\frac{x}{a_1} < \sqrt{x} \,, \tag{4.2}$$

  obtaining an approximation of $\sqrt{x}$ from below.
- Therefore, putting together the above inequalities,

$$\frac{x}{a_1} < \sqrt{x} < a_1 \tag{4.3}$$

- If we take the average of the points x/a_1 and a_1, it is reasonable to think that we find a better approximation of $\sqrt{x}$. Thus our next approximation is

$$a_2 := \frac{1}{2}\left(a_1 + \frac{x}{a_1}\right) \,,$$

  see figure below.

Iterating, we define by recurrence the sequence

$$a_{n+1} := \frac{1}{2}\left(a_n + \frac{x}{a_n}\right)$$

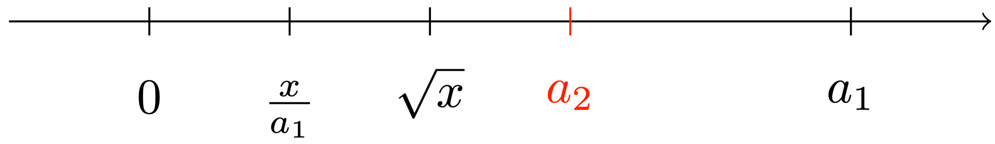

Figure 4.1: Heron's Algorithm for approximating $\sqrt{x}$

for all $n \in \mathbb{N}$, where the initial guess a_1 has to satisfy (4.1). The aim of the section is to show that

$$\lim_{n \to \infty} a_n = \sqrt{x} \,. \tag{4.4}$$

We start by showing that (4.3) holds for all $n \in \mathbb{N}$.

Proposition 4.1

We have

$$\frac{x}{a_n} < \sqrt{x} < a_n \tag{4.5}$$

for all $n \in \mathbb{N}$.

Proof

We prove it by induction:

1. By (4.1) and (4.2) we know that (4.5) holds for $n = 1$.
2. Suppose now that (4.5) holds for n. Then

$$a_{n+1} - \sqrt{x} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) - \sqrt{x} \tag{4.6}$$

$$= \frac{1}{2a_n}(a_n^2 + x - 2a_n\sqrt{x}) \tag{4.7}$$

$$= \frac{1}{2a_n}(a_n - \sqrt{x})^2 > 0 \,, \tag{4.8}$$

   since we are assuming that $a_n > \sqrt{x}$. Therefore

$$\sqrt{x} < a_{n+1} \,. \tag{4.9}$$

   Multiplying the above by $\sqrt{x}/a_{n+1}$ we get

$$\frac{x}{a_{n+1}} < \sqrt{x} \,. \tag{4.10}$$

   Inequalities (4.9) and (4.10) show that (4.5) holds for $n+1$.

Therefore we conclude (4.5) by the Principle of Induction.

We are now ready to prove error estimates, that is, estimating how far away a_n is from $\sqrt{x}$.

Proposition 4.2: Error estimate

For all $n \in \mathbb{N}$ we have

$$a_{n+1} - \sqrt{x} < \frac{1}{2}(a_n - \sqrt{x}) \,. \tag{4.11}$$

Proof

By Proposition 4.1 we know that

$$\frac{x}{a_n} < \sqrt{x}$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} a_{n+1} &= \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) & (4.12) \\ &< \frac{1}{2}\left(a_n + \sqrt{x}\right) \,. & (4.13) \end{aligned}$$

Subtracting $\sqrt{x}$ from both members in the above inequality we get the thesis.

Inequality (4.11) is saying that the error halves at each step. Therefore we can prove that after n steps the error is exponentially lower, as detailed in the following proposition.

Proposition 4.3

For all $n \in \mathbb{N}$ we have

$$a_{n+1} - \sqrt{x} < \frac{1}{2^n}(a_1 - \sqrt{x}) \tag{4.14}$$

Proof

We prove (4.14) by induction:

1. For $n = 1$ we have that (4.14) is satisfied, since it coincides with (4.11) for $n = 1$.

2. Suppose that (4.14) holds for n. By (4.11) with n replaced by $n+1$ we have

$$\begin{aligned} a_{n+2} - \sqrt{x} &< \frac{1}{2}(a_{n+1} - \sqrt{x}) && (4.15) \\ &< \frac{1}{2} \cdot \frac{1}{2^n}(a_1 - \sqrt{x}) && (4.16) \\ &= \frac{1}{2^{n+1}}(a_1 - \sqrt{x}) && (4.17) \end{aligned}$$

where in the second inequality we used the induction hypothesis (4.14). Hence (4.14) holds for $n+1$.

By invoking the Induction Principle we conclude the proof.

Let us comment estimate (4.14). Denote the error at step n by

$$e_n := a_n - \sqrt{x}\,.$$

The initial error e_1 depends on how far the initial guess is from $\sqrt{x}$. The estimate in (4.14) is telling us that e_n is a fraction of e_1, and actually

$$\lim_{n\to\infty} e_n = 0$$

exponentially fast. From this fact we are finally able to prove (4.4).

Theorem 4.4: Convergence of Heron's Algorithm

We have that

$$\lim_{n\to\infty} a_n = \sqrt{x}\,.$$

Proof

By Proposition 4.3 we have that

$$a_{n+1} - \sqrt{x} < \frac{1}{2^n}(a_1 - \sqrt{x})$$

Moreover Proposition 4.1 tells us that

$$\sqrt{x} < a_{n+1}\,.$$

Putting together the two inequalities above we infer

$$\sqrt{x} < a_{n+1} < \sqrt{x} + \frac{1}{2^n}(a_1 - \sqrt{x})\,. \tag{4.18}$$

Now note that

$$\lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0 \,.$$

Therefore the RHS of (4.18) converges to $\sqrt{x}$ as $n \to \infty$. Applying the Squeeze Theorem to (4.18) we conclude that $a_n \to \sqrt{x}$ as $n \to \infty$.

4.1.1 Coding the Algorithm

Heron's Algorithm can be easily coded in Python. For example, see the function below:

```
# x is the number for which to compute sqrt(x)
# guess is the point a_1
# a_1 must be strictly larger than sqrt(x)
# n is the number of iterations
# the function returns a_{n+1}

def herons_algorithm(x, guess, n):
    for i in range(n):
        guess = (guess + x / guess) / 2.0
    return guess
```

For example let us use the Algorithm to compute $\sqrt{2}$ after 3 iterations. For initial guess we take $a_1 = 2$.

```
# Calculate sqrt(2) with 3 iterations and guess 2
sqrt_2 = herons_algorithm(2, 2, 3)

print(f"The sqrt(2) is approximately {sqrt_2}")
```

```
The sqrt(2) is approximately 1.4142156862745097
```

That is a pretty good approximation in just 3 iterations!

4.2 Fibonacci Sequence

5 Series

Coming soon

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    year = {2023}}
```

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