Numbers, Sequences and Series

Lecture Notes, T1 2023/24

Silvio Fanzon

3 Sep 2023

Table of contents

Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

S.Fanzon@hull.ac.uk

Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

Readings

We will study the set of real numbers $\mathbb{R}$, and then sequences and series in $\mathbb{R}$. I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

> **!** You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to known in order to excel in the final exam.

1 Numbers

1.1 Introduction

The aim of this chapter is to rigorously introduce the set of real numbers $\mathbb{R}$. But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol $\in$ denotes that m and n belong to $\mathbb{N}$. For example $3 + 7$ results in 10.

Question

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m\,?$$

The answer to the above question is clearly **no**. For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to $\mathbb{N}$. We therefore need to **extend** the set $\mathbb{N}$ in order to invert the sum. This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : \ n \in \mathbb{N}\}\,,$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}\,.$$

The sum can be extended to $\mathbb{Z}$, by defining

$$(-n) + (-m) := -(m + n) \tag{1.1}$$

for all $m, n \in \mathbb{N}$. Now every element of $\mathbb{Z}$ possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0\,.$$

What is m for a given n? Seeing the definition at (1.1), we simply have

$$m = -n\,.$$

We can also multiply integers, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm.

Question

Can the multiplication in $\mathbb{Z}$ be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m\,?$$

The answer is of course **no**. Just take $n = 2$ and $m = 1$. The answer should be $1/2$, but $1/2$ does not belong to $\mathbb{Z}$. Thus, in order to invert the multiplication, we need to **extend** the set of integers. This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} \, : \, m, n \in \mathbb{Z}, \, n \neq 0 \right\}\,.$$

In $\mathbb{Q}$ the multiplication is invertible, and each non-zero element has an inverse: the inverse of m/n is given by n/m. Moreover by construction

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}\,.$$

Question

How can we draw the set $\mathbb{Q}$?

It is clear how to draw $\mathbb{Z}$, as seen below.

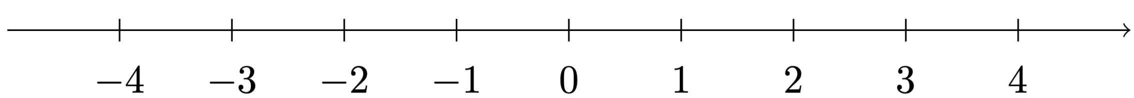

Figure 1.1: Representation of integers

However $\mathbb{Q}$ is much **denser** than the elements of $\mathbb{Z}$ represented in Figure 1.1. For example, consider $0 \in \mathbb{Q}$.

Question

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, we can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n} .$$

For example think about the case of the numbers $1/n$ for $n \in \mathbb{N}$ and $n \neq 0$. Such numbers get arbitrarily close to 0, as depicted below.

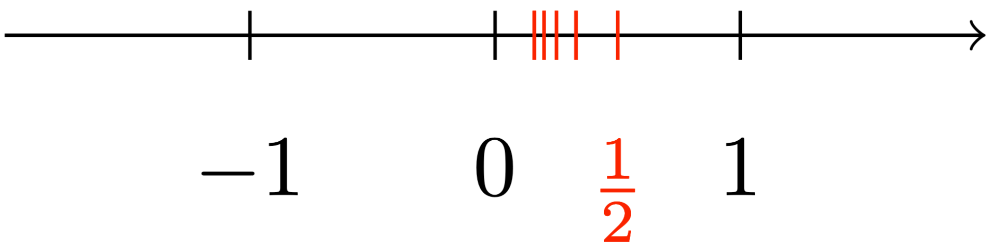

Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0, 1]$. In other words, we might conjecture the following.

Conjecture 1.1

Maybe $\mathbb{Q}$ can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.1 is **false**, as shown by the Theorem below.

Theorem 1.2

The number $\sqrt{2}$ does not belong to $\mathbb{Q}$.

The above theorem is the reason why $\sqrt{2}$ is called an irrational number. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048 \ldots$$

and the situation is as in the picture below.

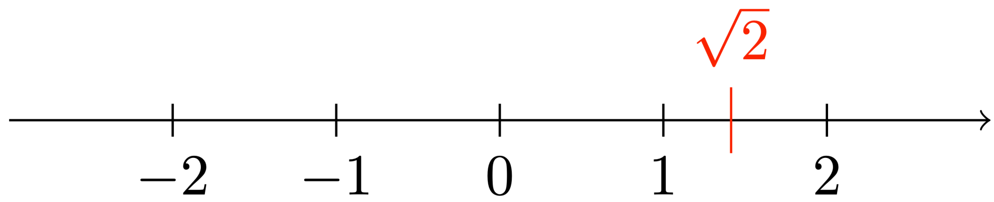

Figure 1.3: Representing $\sqrt{2}$

We can therefore see that Conjecture 1.1 is **false**, and $\mathbb{Q}$ is not a line, given that it has a **gap** at $\sqrt{2}$. Let us see why Theorem 1.2 is true.

Proof: Proof of Theorem 1.2

We prove that $\sqrt{2}$ does not belong to $\mathbb{Q}$ by **contradiction**. This means assuming the existence of $q \in \mathbb{Q}$ such that $q = \sqrt{2}$. In other words

$$q^2 = 2\,. \tag{1.2}$$

We will show that (1.2) leads to a contradiction. Thus (1.2) must be false, and so

$$\sqrt{2} \notin \mathbb{Q}\,,$$

concluding the thesis.

Seeing the above, we might be tempted to just fill in the gap by adding $\sqrt{2}$ to $\mathbb{Q}$. However, with analogous proof to Theorem 1.2, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number p. Thus $\mathbb{Q}$ has infinite gaps (recall that there are infinite prime numbers). Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : \ p \ \text{ prime}\}\,.$$

However even this is not enough, as we would still have numbers which are not contained in $\tilde{Q}$, for example

$$\sqrt{2} + \sqrt{3},\ \pi,\ \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}\,.$$

The reality of things is that to **complete** $\mathbb{Q}$ and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in $\mathbb{Q}$. Such extension of $\mathbb{Q}$ will be called $\mathbb{R}$, the set of **real numbers**. The inclusions will be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}\,.$$

The set $\mathbb{R}$ is not at all trivial to construct. In fact, at first we will just assume its **existence** and study its **properties**. We will then provide a concrete **model** for the real numbers $\mathbb{R}$, to prove once and for all that such set indeed exists.

Theorem 1.3: To keep in mind for the next lessons

There exists a set $\mathbb{R}$, called the set of real numbers, which extends $\mathbb{Q}$ by filling in all the gaps.

2 Sequences

3 Series

License

Reuse

Citation

For attribution, please cite this work as:

> Fanzon, Silvio. (2023). *Lecture Notes on Numbers, Sequences and Series.* https://www.silviofanzon.com/2023-NSS-Notes/

BibTex citation:

```
@electronic{Fanzon-NSS-2023,
    author = {Fanzon, Silvio},
    title = {Lecture Notes on Numbers, Sequences and Series},
    url = {https://www.silviofanzon.com/2023-NSS-Notes/},
    year = {2023}}
```

References

[1] S. Abbott. *Understanding Analysis*. Second Edition. Springer, 2015.

[2] R. G. Bartle and D. R. Sherbert. *Introduction to Real Analysis*. Fourth Edition. Wiley, 2011.

[3] W. Rudin. *Principles of Mathematical Analysis*. Third Edition. McGraw Hill, 1976.