

Numbers, Sequences and Series

Lecture Notes

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Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

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Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

Readings

We will study the set of real numbers \mathbb{R} , and then sequences and series in \mathbb{R} . I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

! You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to know in order to excel in the final exam.

1 Introduction

The first aim of this lecture notes is to rigorously introduce the set of **real numbers**, which is denoted by \mathbb{R} . But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol \in denotes that m and n belong to \mathbb{N} . For example $3 + 7$ results in 10.

Question 1.1

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m? \tag{1.1}$$

Of course to invert (1.1) we can just perform a **subtraction**, implying that

$$x = m - n.$$

But there is a catch. In general x does not need to be in \mathbb{N} . For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to \mathbb{N} . Therefore the answer to Question 1.1 is **NO**.

To make sure that we can always invert the sum, we need to **extend** the set \mathbb{N} . This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : n \in \mathbb{N}\},$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The sum can be extended to \mathbb{Z} , by defining

$$(-n) + (-m) := -(m + n) \quad (1.2)$$

for all $m, n \in \mathbb{N}$. Now every element of \mathbb{Z} possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0.$$

Can we characterize m explicitly? Of course! Seeing the definition at (1.2), we simply have

$$m = -n.$$

On the set \mathbb{Z} we can also define the operation of **multiplication**, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm or $n \cdot m$. For example $7 \cdot 2 = 14$ and $1 \cdot (-1) = -1$.

Question 1.2

Can the multiplication in \mathbb{Z} be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m? \quad (1.3)$$

To invert (1.3) if $n \neq 0$, we can just perform a **division**, to obtain

$$x = \frac{m}{n}.$$

But again there is a catch. Indeed taking $n = 2$ and $m = 1$ yields $x = 1/2$, which does not belong to \mathbb{Z} . The answer to Question 1.2 is therefore **NO**.

Thus, in order to invert the multiplication, we need to **extend** the set of integers \mathbb{Z} . This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

We then extend the operations of sum and multiplication to \mathbb{Q} by defining

$$\frac{m}{n} + \frac{p}{q} := \frac{mq + np}{nq}$$

and

$$\frac{m}{n} \cdot \frac{p}{q} := \frac{mp}{nq}$$

Now the multiplication is invertible in \mathbb{Q} . Specifically, each non-zero element has an inverse: the inverse of m/n is given by n/m .

To summarize, we have extended \mathbb{N} to \mathbb{Z} , and \mathbb{Z} to \mathbb{Q} . By construction we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Moreover **sum** and **product** are **invertible** in \mathbb{Q} . Now we are happy right? So and so.

Question 1.3

Can we draw the set \mathbb{Q} ?

It is clear how to draw \mathbb{Z} , as seen below.

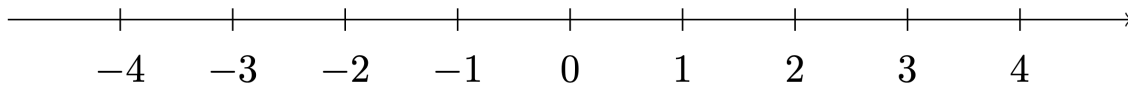


Figure 1.1: Representation of integers \mathbb{Z}

However \mathbb{Q} is much **larger** than the set \mathbb{Z} represented by the ticks in Figure 1.1. What do we mean by **larger**? For example, consider $0 \in \mathbb{Q}$.

Question 1.4

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, you can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n}.$$

For example think about the case of the numbers

$$\frac{1}{n} \text{ for } n \in \mathbb{N}, n \neq 0.$$

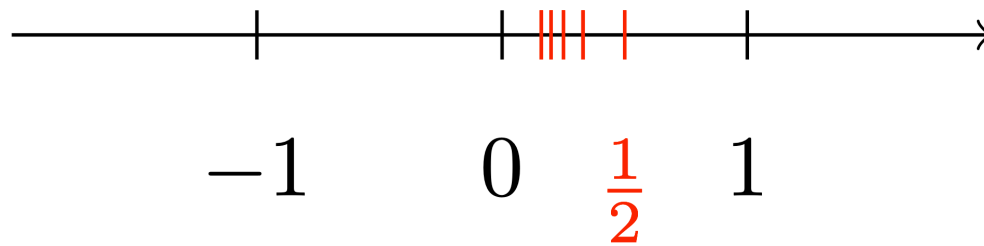


Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Such numbers get arbitrarily close to 0, as depicted below.

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0, 1]$. In other words, we might conjecture the following.

Conjecture 1.5

Maybe \mathbb{Q} can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.5 is **false**, as shown by the Theorem below.

Theorem 1.6

The number $\sqrt{2}$ does not belong to \mathbb{Q} .

Theorem 1.6 is the reason why $\sqrt{2}$ is called an **irrational number**. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048 \dots$$

and the situation is as in the picture below.

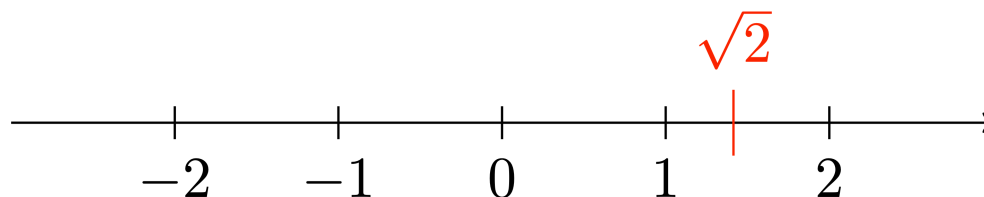


Figure 1.3: Representing $\sqrt{2}$ on the numbers line.

We can therefore see that Conjecture 1.5 is **false**, and \mathbb{Q} is not a line: indeed \mathbb{Q} has a **gap** at $\sqrt{2}$. Let us see why Theorem 1.6 is true.

Proof: Proof of Theorem 1.6

We prove that

$$\sqrt{2} \notin \mathbb{Q}$$

by **contradiction**.

Wait, what does this mean? Proving the claim by contradiction means assuming that the claim is **false**. This means we **assume** that

$$\sqrt{2} \in \mathbb{Q}. \quad (1.4)$$

From this assumption we then start deducing other statements, hoping to encounter a statement which is **FALSE**. But if (1.4) leads to a false statement, then it must be that (1.4) is **FALSE**. Thus the contrary of (1.4) must hold, meaning that

$$\sqrt{2} \notin \mathbb{Q}$$

as we wanted to show. This would conclude the proof.

Now we need to actually show that (1.4) will lead to a contradiction. Since this is our first proof, let us take it slowly, step-by-step.

1. Assuming (1.4) just means that there exists $q \in \mathbb{Q}$ such that

$$q = \sqrt{2}. \quad (1.5)$$

2. Since $q \in \mathbb{Q}$, by definition we have

$$q = \frac{m}{n}$$

for some $m, n \in \mathbb{N}$ with $n \neq 0$.

3. Recalling (1.5), we then have

$$\frac{m}{n} = \sqrt{2}.$$

4. We can square the above equation to get

$$\frac{m^2}{n^2} = 2. \quad (1.6)$$

5. **Withouth loss of generality**, we can **assume** that m and n have no common factors.

Wait. What does Step 5 mean? You will encounter the sentence *withouth loss of generality* many times in mathematics. It is often abbreviated in **WLOG**. WLOG means that the assumption that follows is chosen arbitrarily, but does not affect the validity of the proof in general.

For example in our case we can assume that m and n have no common factor. This is because if m and n had common factors, then it would mean

$$m = a\tilde{m}, \quad n = a\tilde{n}$$

for some $a \in \mathbb{N}$ with $a \neq 0$. Then

$$\frac{m}{n} = \frac{a\tilde{m}}{a\tilde{n}} = \frac{\tilde{m}}{\tilde{n}}.$$

Therefore by (1.6)

$$\frac{\tilde{m}^2}{\tilde{n}^2} = 2.$$

The proof can now proceed in the same way we would have proceeded from Step 4, but in addition we have the hypothesis that \tilde{m} and \tilde{n} have no common factors.

6. Equation (1.6) implies

$$m^2 = 2n^2. \tag{1.7}$$

Therefore the integer m^2 is an even number.

Why is m^2 even? As you already know, **even** numbers are

$$0, 2, 4, 6, 8, 10, 12, \dots$$

All these numbers have in common that they can be divided by 2, and so they can be written as

$$2p$$

for some $p \in \mathbb{N}$. For example 52 is even, because

$$52 = 2 \cdot 26.$$

Instead, **odd** numbers are

$$1, 3, 5, 7, 9, 11, \dots$$

These can be all written as

$$2p + 1$$

for some $p \in \mathbb{N}$. For example 53 is odd, because

$$53 = 2 \cdot 26 + 1.$$

7. Thus m is an even number, and so there exists $p \in \mathbb{N}$ such that

$$m = 2p. \quad (1.8)$$

Why is (1.8) true? Let us see what happens if we take the square of an even number $m = 2p$

$$m^2 = (2p)^2 = 4p^2 = 2(2p^2) = 2q.$$

Thus $m^2 = 2q$ for some $q \in \mathbb{N}$, and so m^2 is an even number. If instead m is odd, then $m = 2p + 1$ and

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

showing that also m^2 is odd.

This justifies Step 7: Indeed we know that m^2 is an even number from Step 6. If m was odd, then m^2 would be odd. Hence m must be even as well.

8. If we substitute (1.8) in (1.7) we get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2. \quad (1.9)$$

9. We now make a series of observations:

- Equation (1.9) says that n^2 is even.
- Step 6 says that m^2 is even.
- Therefore n and m are also even.

- Hence n and m have 2 as common factor.
- But Step 5 says that n and m have no common factors.
- **CONTRADICTION**

10. Our reasoning has run into a **contradiction**, starting from assumption (1.4), which says that

$$\sqrt{2} \in \mathbb{Q}.$$

Hence the above must be **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

Seeing that $\sqrt{2} \notin \mathbb{Q}$, we might be tempted to just fill in the gap by adding $\sqrt{2}$ to \mathbb{Q} . However, with analogous proof to Theorem 1.6, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number p . As there are infinite prime numbers, this means that \mathbb{Q} has infinite gaps. Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : p \text{ prime}\}.$$

However even this is not enough, as we would still have numbers which are not contained in $\tilde{\mathbb{Q}}$, for example

$$\sqrt{2} + \sqrt{3}, \pi, \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}.$$

Remark 1.7

Proving that

$$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

is relatively easy, and will be left as an **exercise**. Instead, proving that

$$\pi \notin \mathbb{Q}$$

is way more complicated. There are several proof of the fact, all requiring mathematics which is more advanced of the one presented in this course. For some proofs, see this [Wikipedia page](#).

The reality of things is that to **complete** \mathbb{Q} and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in \mathbb{Q} . Such extension of \mathbb{Q} will be called \mathbb{R} , the set of **real numbers**. The inclusions will therefore be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The set \mathbb{R} is not at all trivial to construct. In fact, at first we will not construct it, but just do the following:

- We will assume that \mathbb{R} **exists** and satisfies some basic **axioms**.
- One of the axioms is that \mathbb{R} fills **all** the **gaps** that \mathbb{Q} has. Therefore \mathbb{R} can be thought as a **continuous** line.
- We will study the **properties** of \mathbb{R} which descend from such **axioms**.

For example one of the properties of \mathbb{R} will be the following:

Theorem 1.8: We will prove this in the future

\mathbb{R} contains all the square roots. This means that for every $x \in \mathbb{R}$ with $x \geq 0$, we have

$$\sqrt{x} \in \mathbb{R}.$$

At the end of this chapter we will provide a concrete **model** for the real numbers \mathbb{R} , to prove once and for all that such set indeed exists.

Theorem 1.9: We will prove this in the future

There exists a set \mathbb{R} , called the set of real numbers, which has the following properties:

- \mathbb{R} extends \mathbb{Q} , that is,

$$\mathbb{Q} \subset \mathbb{R}.$$

- \mathbb{R} satisfies certain **axioms**.
- \mathbb{R} fills **all** the **gaps** that \mathbb{Q} has. In particular \mathbb{R} can be represented by a **continuous** line.

2 Preliminaries

Before introducing \mathbb{R} we want to make sure that we cover all the basics needed for the task.

2.1 Sets

A set is a **collection** of objects. These objects are called **elements** of the set. For example in the previous section we mentioned the following sets:

- \mathbb{N} the set of natural numbers
- \mathbb{Z} the set of integers
- \mathbb{Q} the set of rational numbers
- \mathbb{R} the set of real numbers

Given an arbitrary set A , we write

$$x \in A$$

if the element x belongs to the set A . If an element x is not contained in A , we say that

$$x \notin A.$$

Remark 2.1

A set can contain all sorts of elements. For example the students in a classroom can be modelled by a set S . The elements of the set are the students. For example

$$S = \{\text{Alice, Olivia, Jake, Sahab}\}$$

In this case we have

$$\text{Alice} \in S$$

but instead

Silvio $\notin S$.

2.1.1 Union and intersection

For two sets A and B we define their **union** as the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

The **intersection** of A and B is defined by

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

We denote the **empty set** by the symbol \emptyset . Two sets are **disjoint** if

$$A \cap B = \emptyset.$$

Example 2.2

Define the subset of rational numbers

$$S := \left\{x \in \mathbb{Q} : 0 < x < \frac{5}{2}\right\}.$$

Then we have

$$\mathbb{N} \cap S = \{1, 2\}.$$

We can also define the sets of **even** and **odd** numbers by

$$E := \{2n : n \in \mathbb{N}\}, \tag{2.1}$$

$$O := \{2n + 1 : n \in \mathbb{N}\}. \tag{2.2}$$

Then we have

$$\mathbb{N} \cap E = E, \mathbb{N} \cap O = O, \tag{2.3}$$

$$O \cup E = \mathbb{N}, O \cap D = \emptyset. \tag{2.4}$$

2.1.2 Inclusion and equality

Given two sets A and B , we say that A is **contained** in B if all the elements of A are also contained in B . This will be denoted with the **inclusion** symbol \subset , that is,

$$A \subset B.$$

In this case we say that

- A is a **subset** of B ,
- B is a **superset** of A .

The inclusion $A \subset B$ is equivalent to the implication:

$$x \in A \implies x \in B$$

for all $x \in A$. The symbol \implies reads **implies**, and denotes the fact that the first condition implies the second.

Example 2.3

Given two sets A and B we always have

$$(A \cap B) \subset A, (A \cap B) \subset B, \quad (2.5)$$

$$A \subset (A \cup B), B \subset (A \cup B). \quad (2.6)$$

We say that two sets A and B are equal if they contain the **same** elements. We denote equality by the symbol

$$A = B.$$

Example 2.4

The sets

$$A = \{1, 2, 3\}$$

and

$$B = \{3, 1, 2\}$$

are equal. This is because they contain exactly the same elements: **order** does not matter when talking about sets.

Proposition 2.5

Let A and B be sets. Then

$$A = B$$

if and only if

$$A \subset B \text{ and } B \subset A.$$

Before proceeding with the proof, let us clarify what the expression **if and only if** mean. Suppose that you are given two statements, say α and β . Then

$$\alpha \iff \beta$$

is equivalent to the following conditions:

- $\alpha \implies \beta$, that is, if α is true then also β is true,
- $\beta \implies \alpha$, that is, if β is true then also α is true.

Proof

The proof is almost trivial. However it is a good exercise in basic logic, so let us do it.

1. First implication \implies :

Suppose that $A = B$. Let us show that $A \subset B$. Since $A = B$, this means that all the elements of A are also contained in B . Therefore if we take $x \in A$ we have

$$x \in A \implies x \in B.$$

This shows $A \subset B$. The proof of $B \subset A$ is similar.

2. Second implication \impliedby :

Suppose that $A \subset B$ and $B \subset A$. We need to show $A = B$, that is, A and B have the same elements. To this end let $x \in A$. Since $A \subset B$ then we have $x \in B$. Thus B contains all the elements of A . Since we are also assuming $B \subset A$, this means that A contains all the elements of B . Hence A and B contain the same elements, and $A = B$.

The above proposition is very useful when we need to **prove** that two sets are equal: rather than showing directly that $A = B$, we can prove that $A \subset B$ and $B \subset A$.

2.1.3 Infinite unions and intersections

Suppose given a set Ω , and a family of sets $A_n \subset \Omega$, where $n \in \mathbb{N}$. Then we can define the **infinte union**

$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for at least one } n \in \mathbb{N}\}.$$

The **infinte intersection** is defined as

$$\bigcap_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

Example 2.6

Let the ambient set be $\Omega := \mathbb{N}$ and define the family A_n by

$$A_1 := \{1, 2, 3, 4, \dots\} \quad (2.7)$$

$$A_2 := \{2, 3, 4, 5, \dots\} \quad (2.8)$$

$$A_3 := \{3, 4, 5, 6, \dots\} \quad (2.9)$$

$$\dots \dots \quad (2.10)$$

$$A_n := \{n, n+1, n+2, n+3, \dots\}, \quad (2.11)$$

for arbitrary $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}. \quad (2.12)$$

The above equality can be easily proven using Proposition 2.5. Indeed, assume that $m \in \bigcup_n A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subset \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows

$$\bigcup_{n \in \mathbb{N}} A_n \subset \mathbb{N}.$$

Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index n , $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \bigcup_{n \in \mathbb{N}} A_n$, showing that

$$\mathbb{N} \subset \bigcup_{n \in \mathbb{N}} A_n.$$

Hence we conclude (2.12) by Proposition 2.5.

We also have that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset. \quad (2.13)$$

We prove the above by **contradiction**. Indeed, suppose that (2.13) is false, i.e.,

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset.$$

This means there exists some $m \in \mathbb{N}$ such that $m \in \bigcap_{n \in \mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (2.13) holds.

2.1.4 Complement

Suppose that A and B are subsets of a larger set Ω . The **complement** of A with respect to B is the set of elements of B which do not belong to A , that is

$$B \setminus A := \{x \in \Omega : x \in B \text{ and } x \notin A\}.$$

In particular, the complement of A with respect to Ω is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : x \notin A\}.$$

Remark 2.7

Suppose that $A \subset \Omega$. Then A and A^c form a **partition** of Ω , in the sense that

$$A \cup A^c = \Omega \text{ and } A \cap A^c = \emptyset.$$

Example 2.8

Suppose $A, B \subset \Omega$. Then

$$A \subset B \iff B^c \subset A^c.$$

Let us prove the above claim:

- First implication \implies :
Suppose that $A \subset B$. We need to show that $B^c \subset A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subset B$, hence $x \in B$. But we had assumed $x \in B^c$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subset A^c$.
- Second implication \impliedby :
Essentially the same proof, hence we omit it.

We conclude by stating the De Morgan's Laws. The proof will be left as an exercise.

Proposition 2.9: De Morgan's Laws

Suppose $A, B \subset \Omega$. Then

$$(A \cap B)^c = A^c \cup B^c$$

and

$$(A \cup B)^c = A^c \cap B^c.$$

2.1.5 Product of sets

Suppose A and B are two sets. The **product** of A and B is the set of pairs

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

By definition two elements in $A \times B$ are the same, in symbols

$$(a, b) = (\tilde{a}, \tilde{b})$$

if and only if they are equal component-by-component, that is

$$a = \tilde{a}, \quad b = \tilde{b}.$$

2.2 Equivalence and order relation

Suppose A is a set. A **binary relation** R on A is a subset

$$R \subset A \times A.$$

Definition 2.10: Equivalence relation

A binary relation R is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive:** For each $x \in A$ one has

$$(x, x) \in R,$$

This is saying that all the elements in A must be related to themselves

2. **Symmetric:** We have

$$(x, y) \in R \implies (y, x) \in R$$

If x is related to y , then y is related to x

3. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

If x is related to y , and y is related to z , then x must be related to z

If $(x, y) \in R$ we write

$$x \sim y$$

and we say that x and y are **equivalent**.

Definition 2.11: Equivalence classes

Suppose R is an **equivalence relation** on A . The **equivalence class** of an element $x \in A$ is the set

$$[x] := \{y \in A : y \sim x\}.$$

The set of equivalence classes of elements of A with respect to the equivalence relation R is denoted by

$$A/R := \{[x] : x \in A\}.$$

Let us immediately clarify the above definitions by considering the prototypical example of equivalence relation: the **equality**.

Example 2.12: Equality is an equivalence relation

Consider the set of natural numbers \mathbb{N} . The equality defines a **binary relation** on $\mathbb{N} \times \mathbb{N}$, via

$$R := \{(x, y) \in \mathbb{N} \times \mathbb{N} : x = y\}.$$

Let us check that R is an **equivalence relation**:

1. Reflexive: It holds, since $x = x$ for all $x \in \mathbb{N}$,
2. Symmetric: Again $x = y$ if and only if $y = x$,
3. Transitive: If $x = y$ and $y = z$ then $x = z$.

The class of equivalence of $x \in \mathbb{N}$ is given by

$$[x] = \{x\},$$

that is, this relation is quite trivial, given that each element of \mathbb{N} can only be related to itself.

Similarly, we define **order relations**.

Definition 2.13: Order relation

A binary relation R is called an **order relation** if it satisfies the following properties:

1. **Reflexive:** For each $x \in A$ one has

$$(x, x) \in R,$$

2. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

3. **Antisymmetric:** We have

$$(x, y) \in R \text{ and } (y, x) \in R \implies x = y$$

This is the only new condition with respect to the definition of equivalence relation.

The prototypical example of order relation is the **inequality** relation.

Example 2.14: Inequality is an order relation

Consider the set of integers \mathbb{Z} . The inequality defines a **binary relation** on $\mathbb{Z} \times \mathbb{Z}$, via

$$R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \leq y\}.$$

Let us check that R is an **order relation**:

1. Reflexive: It holds, since $x \leq x$ for all $x \in \mathbb{Z}$,
2. Transitive: If $x \leq y$ and $y \leq z$ then $x \leq z$.
3. Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$.

2.3 Absolute value or Modulus

In this section we assume to have available the set \mathbb{R} of **real numbers**, which we recall is an extension of \mathbb{Q} .

Definition 2.15: Absolute value

For $x \in \mathbb{R}$ we define its **absolute value** as the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 2.16

By definition one has $|x| = x$ if $x \geq 0$. For example

$$|\pi| = \pi, \quad |\sqrt{2}| = \sqrt{2}, \quad |0| = 0.$$

Instead $|x| = -x$ if $x < 0$. For example

$$|-\pi| = \pi, \quad |-\sqrt{2}| = \sqrt{2}, \quad |-10| = 10.$$

Let us also make the following basic remark, whose proof will be left as an exercise.

Remark 2.17

For all $x \in \mathbb{R}$ one has

$$|x| \geq 0.$$

Moreover

$$|x| = 0 \iff x = 0.$$

Another basic remark (proof by exercise).

Remark 2.18

For all $x \in \mathbb{R}$ one has

$$|x| = |-x|.$$

You might be familiar with the graph associated to the absolute value function:

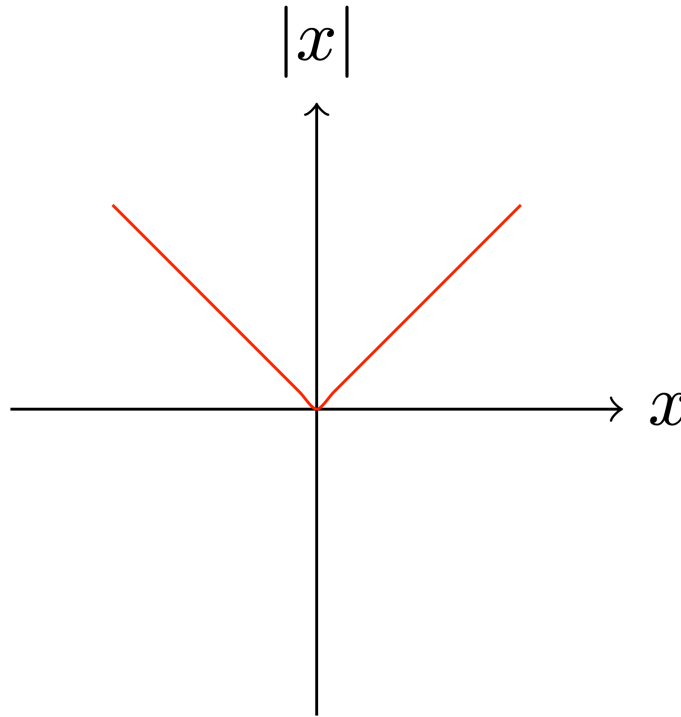
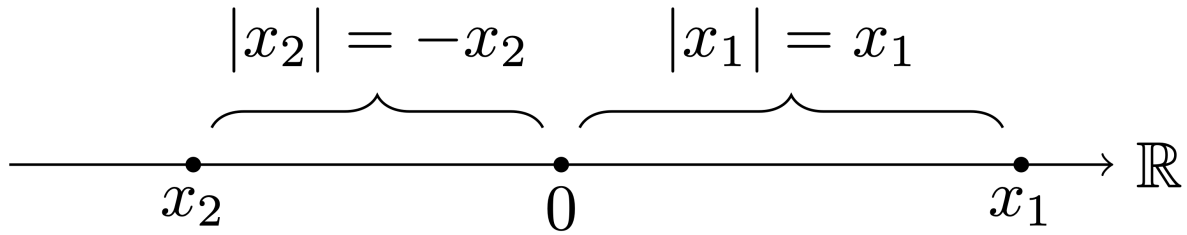


Figure 2.1: Plot of the absolute value function $f(x) = |x|$

However in these Lecture Notes we are not dealing with functions, so it is better to think about the absolute value in a geometric way.

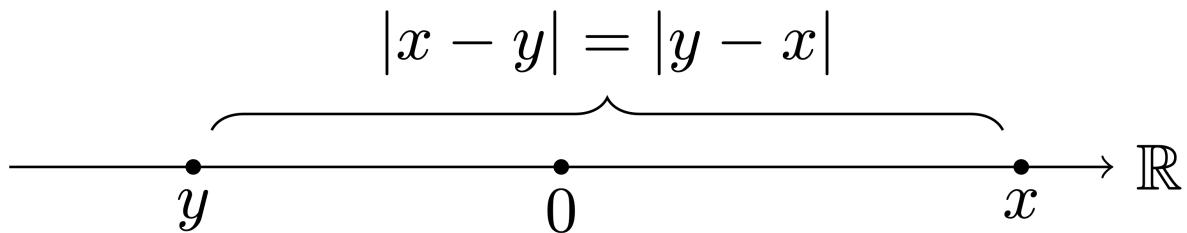
Remark 2.19: Geometric interpretation of $|x|$

A number $x \in \mathbb{R}$ can be represented with a point on the real line \mathbb{R} . The non-negative number $|x|$ represents the **distance** of x from the origin 0. Notice that this works for both positive and negative numbers x_1 and x_2 respectively, as shown in Figure 2.2 below.

Figure 2.2: Geometric interpretation of $|x|$ **Remark 2.20:** Geometric interpretation of $|x - y|$

If $x, y \in \mathbb{R}$ then the number $|x - y|$ represents the distance between x and y on the real line, as shown in Figure 2.3 below. Note that by Remark 2.18 we have

$$|x - y| = |y - x|.$$

Figure 2.3: Geometric interpretation of $|x - y|$

3 Real Numbers

Coming soon

4 Sequences

Coming soon

5 Series

Coming soon

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