Numbers, Sequences and Series

Lecture Notes

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Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

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Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

Readings

We will study the set of real numbers $\mathbb{R}$, and then sequences and series in $\mathbb{R}$. I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

> ! You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to known in order to excel in the final exam.

1 Introduction

The first aim of this lecture notes is to rigorously introduce the set of **real numbers**, which is denoted by $\mathbb{R}$. But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol $\in$ denotes that m and n belong to $\mathbb{N}$. For example $3 + 7$ results in 10.

Question 1.1

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m\,? \tag{1.1}$$

Of course to invert (1.1) we can just perform a **subtraction**, implying that

$$x = m - n\,.$$

But there is a catch. In general x does not need to be in $\mathbb{N}$. For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to $\mathbb{N}$. Therefore the answer to Question 1.1 is **NO**.

To make sure that we can always invert the sum, we need to **extend** the set $\mathbb{N}$. This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : \, n \in \mathbb{N}\}\,,$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}\,.$$

The sum can be extended to $\mathbb{Z}$, by defining

$$(-n) + (-m) := -(m + n) \tag{1.2}$$

for all $m, n \in \mathbb{N}$. Now every element of $\mathbb{Z}$ possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0\,.$$

Can we characterize m explicitly? Of course! Seeing the definition at (1.2), we simply have

$$m = -n\,.$$

On the set $\mathbb{Z}$ we can also define the operation of **multiplication**, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm or $n \cdot m$. For example $7 \cdot 2 = 14$ and $1 \cdot (-1) = -1$.

Question 1.2

Can the multiplication in $\mathbb{Z}$ be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m\,? \tag{1.3}$$

To invert (1.3) if $n \neq 0$, we can just perform a **division**, to obtain

$$x = \frac{m}{n}\,.$$

But again there is a catch. Indeed taking $n = 2$ and $m = 1$ yields $x = 1/2$, which does not belong to $\mathbb{Z}$. The answer to Question 1.2 is therefore **NO**.

Thus, in order to invert the multiplication, we need to **extend** the set of integers $\mathbb{Z}$. This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} \,:\, m, n \in \mathbb{Z},\, n \neq 0 \right\}\,.$$

We then extend the operations of sum and multiplication to $\mathbb{Q}$ by defining

$$\frac{m}{n} + \frac{p}{q} := \frac{mq + np}{nq}$$

and

$$\frac{m}{n} \cdot \frac{p}{q} := \frac{mp}{nq}$$

Now the multiplication is invertible in $\mathbb{Q}$. Specifically, each non-zero element has an inverse: the inverse of m/n is given by n/m.

To summarize, we have extended $\mathbb{N}$ to $\mathbb{Z}$, and $\mathbb{Z}$ to $\mathbb{Q}$. By construction we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}\,.$$

Moreover **sum** and **product** are **invertible** in $\mathbb{Q}$. Now we are happy right? So and so.

Question 1.3

Can we draw the set $\mathbb{Q}$?

It is clear how to draw $\mathbb{Z}$, as seen below.

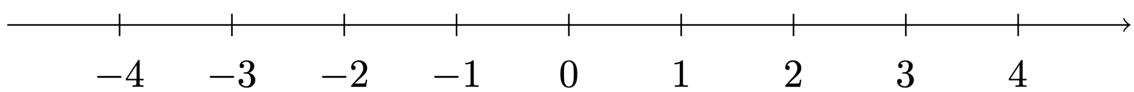

Figure 1.1: Representation of integers $\mathbb{Z}$

However $\mathbb{Q}$ is much **larger** than the set $\mathbb{Z}$ represented by the ticks in Figure 1.1. What do we mean by **larger**? For example, consider $0 \in \mathbb{Q}$.

Question 1.4

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, you can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n}\,.$$

For example think about the case of the numbers

$$\frac{1}{n} \;\text{ for }\; n \in \mathbb{N},\, n \neq 0\,.$$

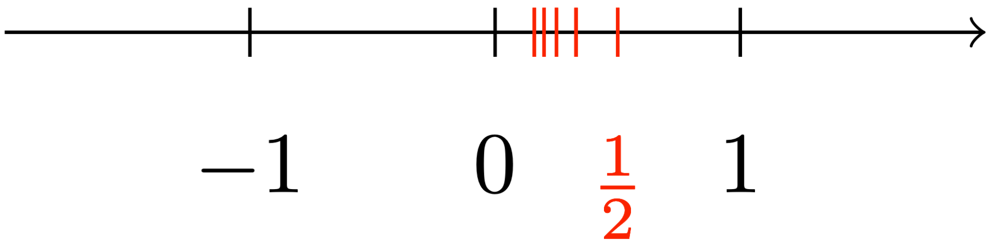

Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Such numbers get arbitrarily close to 0, as depicted below.

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0,1]$. In other words, we might conjecture the following.

Conjecture 1.5

Maybe $\mathbb{Q}$ can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.5 is **false**, as shown by the Theorem below.

Theorem 1.6

The number $\sqrt{2}$ does not belong to $\mathbb{Q}$.

Theorem 1.6 is the reason why $\sqrt{2}$ is called an **irrational number**. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048\ldots$$

and the situation is as in the picture below.

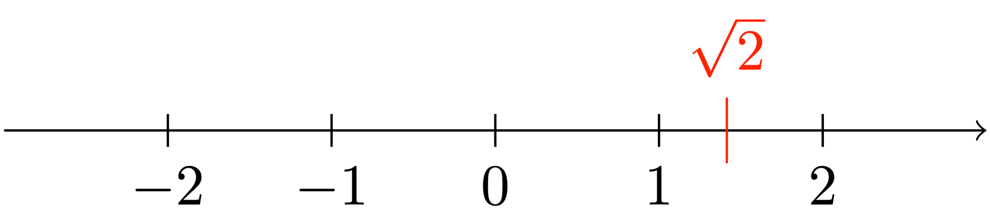

Figure 1.3: Representing $\sqrt{2}$ on the numbers line.

We can therefore see that Conjecture 1.5 is **false**, and $\mathbb{Q}$ is not a line: indeed $\mathbb{Q}$ has a **gap** at $\sqrt{2}$. Let us see why Theorem 1.6 is true.

Proof: Proof of Theorem 1.6

We prove that

$$\sqrt{2} \notin \mathbb{Q}$$

by **contradiction**.
Wait, what does this mean? Proving the claim by contradiction means assuming that the claim is **false**. This means we **assume** that

$$\sqrt{2} \in \mathbb{Q}\,. \tag{1.4}$$

From this assumption we then start deducing other statements, hoping to encounter a statement which is **`FALSE`**. But if (1.4) leads to a false statement, then it must be that (1.4) is **`FALSE`**. Thus the contrary of (1.4) must hold, meaning that

$$\sqrt{2} \notin \mathbb{Q}$$

as we wanted to show. This would conclude the proof.
Now we need to actually show that (1.4) will lead to a contradiction. Since this is our first proof, let us take it slowly, step-by-step.

1. Assuming (1.4) just means that there exists $q \in \mathbb{Q}$ such that
$$q = \sqrt{2}\,. \tag{1.5}$$
2. Since $q \in \mathbb{Q}$, by definition we have
$$q = \frac{m}{n}$$
for some $m, n \in \mathbb{N}$ with $n \neq 0$.
3. Recalling (1.5), we then have
$$\frac{m}{n} = \sqrt{2}\,.$$
4. We can square the above equation to get
$$\frac{m^2}{n^2} = 2\,. \tag{1.6}$$
5. **Withouth loss of generality**, we can **assume** that m and n have no common factors.

Wait. What does Step 5 mean? You will encounter the sentence *withouth loss of generality* many times in mathematics. It is often abbreviated in **WLOG**. WLOG means that the assumption that follows is chosen arbitrarily, but does not affect the validity of the proof in general.
For example in our case we can assume that m and n have no common factor. This is because if m and n had common factors, then it would mean

$$m = a\tilde{m} \,, \quad n = a\tilde{n}$$

for some $a \in \mathbb{N}$ with $a \neq 0$. Then

$$\frac{m}{n} = \frac{a\tilde{m}}{a\tilde{n}} = \frac{\tilde{m}}{\tilde{n}} \,.$$

Therefore by (1.6)

$$\frac{\tilde{m}^2}{\tilde{n}^2} = 2 \,.$$

The proof can now proceed in the same way we would have proceeded from Step 4, but in addition we have the hypothesis that $\tilde{m}$ and $\tilde{n}$ have no common factors.

6. Equation (1.6) implies

$$m^2 = 2n^2 \,. \tag{1.7}$$

Therefore the integer m^2 is an even number.

Why is m^2 even? As you already know, **even** numbers are

$$0, 2, 4, 6, 8, 10, 12, \dots$$

All these numbers have in common that they can be divided by 2, and so they can be written as

$$2p$$

for some $p \in \mathbb{N}$. For example 52 is even, because

$$52 = 2 \cdot 26 \,.$$

Instead, **odd** numbers are

$$1, 3, 5, 7, 8, 9, 11, \dots$$

These can be all written as

$$2p + 1$$

for some $p \in \mathbb{N}$. For example 53 is odd, because

$$52 = 2 \cdot 26 + 1 \,.$$

7. Thus m is an even number, and so there exists $p \in \mathbb{N}$ such that

$$m = 2p\,. \tag{1.8}$$

Why is (1.8) true? Let us see what happens if we take the square of an even number $m = 2p$

$$m^2 = (2p)^2 = 4p^2 = 2(2p^2) = 2q\,.$$

Thus $m^2 = 2q$ for some $q \in \mathbb{N}$, and so m^2 is an even number. If instead m is odd, then $m = 2p + 1$ and

$$m^2 = (2p+1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

showing that also m^2 is odd.
This justifies Step 7: Indeed we know that m^2 is an even number from Step 6. If m was odd, then m^2 would be odd. Hence m must be even as well.

8. If we substitute (1.8) in (1.7) we get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2\,. \tag{1.9}$$

9. We now make a series of observations:
   - Equation (1.9) says that n^2 is even.
   - Step 6 says that m^2 is even.
   - Therefore n and m are also even.

   - Hence n and m have 2 as common factor.
   - But Step 5 says that n and m have no common factors.
   - `CONTRADICTION`

10. Our reasoning has run into a **contradiction**, starting from assumption (1.4), which says that

$$\sqrt{2} \in \mathbb{Q}\,.$$

Hence the above must be **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

Seeing that $\sqrt{2} \notin \mathbb{Q}$, we might be tempted to just fill in the gap by adding $\sqrt{2}$ to $\mathbb{Q}$. However, with analogous proof to Theorem 1.6, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number p. As there are infinite prime numbers, this means that $\mathbb{Q}$ has infinite gaps. Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : \, p \text{ prime}\} .$$

However even this is not enough, as we would still have numbers which are not contained in $\tilde{Q}$, for example

$$\sqrt{2} + \sqrt{3}, \, \pi, \, \pi + \sqrt{2} \notin \tilde{\mathbb{Q}} .$$

Remark 1.7

Proving that

$$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

is relatively easy, and will be left as an **exercise**. Instead, proving that

$$\pi \notin \mathbb{Q}$$

is way more complicated. There are several proof of the fact, all requiring mathematics which is more advanced of the one presented in this course. For some proofs, see this Wikipedia page.

The reality of things is that to **complete** $\mathbb{Q}$ and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in $\mathbb{Q}$. Such extension of $\mathbb{Q}$ will be called $\mathbb{R}$, the set of **real numbers**. The inclusions will therefore be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} .$$

The set $\mathbb{R}$ is not at all trivial to construct. In fact, at first we will not construct it, but just do the following:

- We will assume that $\mathbb{R}$ **exists** and satisfies some basic **axioms**.
- One of the axioms is that $\mathbb{R}$ fills **all** the **gaps** that $\mathbb{Q}$ has. Therefore $\mathbb{R}$ can be thought as a **continuous** line.
- We will study the **properties** of $\mathbb{R}$ which descend from such **axioms**.

For example one of the properties of $\mathbb{R}$ will be the following:

Theorem 1.8: We will prove this in the future

$\mathbb{R}$ contains all the square roots. This means that for every $x \in \mathbb{R}$ with $x \geq 0$, we have

$$\sqrt{x} \in \mathbb{R}\,.$$

At the end of this chapter we will provide a concrete **model** for the real numbers $\mathbb{R}$, to prove once and for all that such set indeed exists.

Theorem 1.9: We will prove this in the future

There exists a set $\mathbb{R}$, called the set of real numbers, which has the following properties:

- $\mathbb{R}$ extends $\mathbb{Q}$, that is,
$$\mathbb{Q} \subset \mathbb{R}\,.$$
- $\mathbb{R}$ satisfies certain **axioms**.
- $\mathbb{R}$ fills **all** the **gaps** that $\mathbb{Q}$ has. In particular $\mathbb{R}$ can be represented by a **continuous** line.

2 Preliminaries

Before introducing $\mathbb{R}$ we want to make sure that we cover all the basics needed for the task.

2.1 Sets

A sets is a **collection** of objects. These objects are called **elements** of the set. For example in the previous section we mentioned the following sets:

- $\mathbb{N}$ the set of natural numbers
- $\mathbb{Z}$ the set of integers
- $\mathbb{Q}$ the set of rational numbers
- $\mathbb{R}$ the set of real numbers

Given an arbitrary set A, we write

$$x \in A$$

if the element x belongs to the set A. If an element x is not contained in A, we say that

$$x \notin A\,.$$

Remark 2.1

A set can contain all sorts of elements. For example the students in a classroom can be modelled by a set S. The elements of the set are the students. For example

$$S = \{\text{Alice, Olivia, Jake, Sahab}\}$$

In this case we have

$$\text{Alice} \in S$$

but instead

$$\text{Silvio} \notin S\,.$$

2.1.1 Union and intersection

For two sets A and B we define their **union** as the set

$$A \cup B := \{x : \; x \in A \;\text{ or }\; x \in B\}\,.$$

The **intersection** of A and B is defined by

$$A \cap B := \{x : \; x \in A \;\text{ and }\; x \in B\}\,.$$

We denote the **empty set** by the symbol $\emptyset$. Two sets are **disjoint** if

$$A \cap B = \emptyset\,.$$

Example 2.2

Define the subset of rational numbers

$$S := \left\{ x \in \mathbb{Q} : \; 0 < x < \frac{5}{2} \right\}\,.$$

Then we have

$$\mathbb{N} \cap S = \{1, 2\}\,.$$

We can also define the sets of **even** and **odd** numbers by

$$E := \{2n : \; n \in \mathbb{N}\}\,, \tag{2.1}$$

$$O := \{2n + 1 : \; n \in \mathbb{N}\}\,. \tag{2.2}$$

Then we have

$$\mathbb{N} \cap E = E\,, \; \mathbb{N} \cap O = O\,, \tag{2.3}$$

$$O \cup E = \mathbb{N}\,, \; O \cap D = \emptyset\,. \tag{2.4}$$

2.1.2 Inclusion and equality

Given two sets A and B, we say that A is **contained** in B if all the elements of A are also contained in B. This will be denoted with the **inclusion** symbol $\subset$, that is,

$$A \subset B\,.$$

In this case we say that

- A is a **subset** of B,
- B is a **superset** of A.

The inclusion $A \subset B$ is equivalent to the implication:

$$x \in A \implies x \in B$$

for all $x \in A$. The symbol $\implies$ reads **implies**, and denotes the fact that the first condition implies the second.

Example 2.3

Given two sets A and B we always have

$$(A \cap B) \subset A\,, \;\; (A \cap B) \subset B\,, \tag{2.5}$$
$$A \subset (A \cup B)\,, \;\; B \subset (A \cup B)\,. \tag{2.6}$$

We say that two sets A and B are equal if they contain the **same** elements. We denote equality by the symbol

$$A = B\,.$$

Example 2.4

The sets

$$A = \{1, 2, 3\}$$

and

$$B = \{3, 1, 2\}$$

are equal. This is because they contain exactly the same elements: **order** does not matter when talking about sets.

Proposition 2.5

Let A and B be sets. Then

$$A = B$$

if and only if

$$A \subset B \quad \text{and} \quad B \subset A \, .$$

Before proceeding with the proof, let us clarify what the expression **if and only if** mean. Suppose that you are given two statements, say α and β. Then

$$\alpha \iff \beta$$

is equivalent to the following conditions:

- $\alpha \implies \beta$, that is, if α is true then also β is true,
- $\beta \implies \alpha$, that is, if β is true then also α is true.

Proof

The proof is almost trivial. However it is a good exercise in basic logic, so let us do it.

1. First implication $\implies$:
   Suppose that $A = B$. Let us show that $A \subset B$. Since $A = B$, this means that all the elements of A are also contained in B. Therefore if we take $x \in A$ we have

   $$x \in A \implies x \in B \, .$$

   This shows $A \subset B$. The proof of $B \subset A$ is similar.

2. Second implication $\impliedby$:
   Suppose that $A \subset B$ and $B \subset A$. We need to show $A = B$, that is, A and B have the same elements. To this end let $x \in A$. Since $A \subset B$ then we have $x \in B$. Thus B contains all the elements of A. Since we are also assuming $B \subset A$, this means that A contains all the elements of B. Hence A and B contain the same elements, and $A = B$.

The above proposition is very useful when we need to **prove** that two sets are equal: rather than showing directly that $A = B$, we can prove that $A \subset B$ and $B \subset A$.

2.1.3 Infinite unions and intersections

Suppose given a set Ω, and a family of sets $A_n \subset \Omega$, where $n \in \mathbb{N}$. Then we can define the **infinte union**

$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega : \; x \in A_n \; \text{ for at least one } \; n \in \mathbb{N}\} \, .$$

The **infinte intersection** is defined as

$$\bigcap_{n\in\mathbb{N}} A_n := \{x \in \Omega : \; x \in A_n \quad \text{for all} \quad n \in \mathbb{N}\} \,.$$

Example 2.6

Let the ambient set be $\Omega := \mathbb{N}$ and define the family A_n by

$$A_1 := \{1, 2, 3, 4, \dots\} \tag{2.7}$$
$$A_2 := \{2, 3, 4, 5, \dots\} \tag{2.8}$$
$$A_3 := \{3, 4, 5, 6, \dots\} \tag{2.9}$$
$$\dots\dots \tag{2.10}$$
$$A_n := \{n, n+1, n+2, n+3, \dots\} \,, \tag{2.11}$$

for arbitrary $n \in \mathbb{N}$. Then

$$\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N} \,. \tag{2.12}$$

The above equality can be easily proven using Proposition 2.5. Indeed, assume that $m \in \cup_n A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subset \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows

$$\bigcup_{n\in\mathbb{N}} A_n \subset \mathbb{N} \,.$$

Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index n, $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \cup_{n\in\mathbb{N}} A_n$, showing that

$$\mathbb{N} \subset \bigcup_{n\in\mathbb{N}} A_n \,.$$

Hence we conclude (2.12) by Proposition 2.5.
We also have that

$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset \,. \tag{2.13}$$

We prove the above by **contradiction**. Indeed, suppose that (2.13) is false, i.e.,

$$\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset \,.$$

This means there exists some $m \in \mathbb{N}$ such that $m \in \cap_{n\in\mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (2.13) holds.

2.1.4 Complement

Suppose that A and B are subsets of a larger set Ω. The **complement** of A with respect to B is the set of elements of B which do not belong to A, that is

$$B \setminus A := \{x \in \Omega : \, x \in B \, \text{ and } \, x \notin A\} \,.$$

In particular, the complement of A with respect to Ω is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : \, x \notin A\} \,.$$

Remark 2.7

Suppose that $A \subset \Omega$. Then A and A^c form a **partition** of Ω, in the sense that

$$A \cup A^c = \Omega \quad \text{and} \quad A \cap A^c = \emptyset \,.$$

Example 2.8

Suppose $A, B \subset \Omega$. Then

$$A \subset B \iff B^c \subset A^c \,.$$

Let us prove the above claim:

- First implication $\implies$:
  Suppose that $A \subset B$. We need to show that $B^c \subset A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subset B$, hence $x \in B$. But we had assumed $x \in B$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subset A^c$.

- Second implication $\impliedby$:
  Essentially the same proof, hence we omit it.

We conclude by stating the De Morgan's Laws, which we do not prove.

Proposition 2.9: De Morgan's Laws

Suppose $A, B \subset \Omega$. Then

$$(A \cap B)^c = A^c \cup B^c$$

and

$$(A \cup B)^c = A^c \cap B^c \,.$$

2.1.5 Cartesian product

Suppose A and B are two sets. We

3 Real Numbers

Coming soon

4 Sequences

Coming soon

5 Series

Coming soon

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