

# **Numbers, Sequences and Series**

## **Lecture Notes**

Dr. Silvio Fanzon

20 Sep 2023

# Table of contents

<b>Welcome</b>	<b>3</b>
Readings . . . . .	3
<b>1 Introduction</b>	<b>4</b>
<b>2 Preliminaries</b>	<b>13</b>
2.1 Sets . . . . .	13
2.2 Logic . . . . .	14
2.3 Operations on sets . . . . .	15
2.3.1 Union and intersection . . . . .	15
2.3.2 Inclusion and equality . . . . .	16
2.3.3 Infinite unions and intersections . . . . .	18
2.3.4 Complement . . . . .	19
2.3.5 Product of sets . . . . .	20
2.4 Equivalence and order relation . . . . .	21
2.5 Intervals . . . . .	23
2.6 Absolute value or Modulus . . . . .	24
2.7 Triangle inequality . . . . .	29
2.8 Proofs involving $\varepsilon$ . . . . .	33
2.9 Induction . . . . .	33
2.10 Example: Approximating $\sqrt{2}$ . . . . .	33
<b>3 Real Numbers</b>	<b>34</b>
<b>4 Sequences</b>	<b>35</b>
<b>5 Series</b>	<b>36</b>
<b>License</b>	<b>37</b>
Reuse . . . . .	37
Citation . . . . .	37
<b>References</b>	<b>38</b>

# Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

[S.Fanzon@hull.ac.uk](mailto:S.Fanzon@hull.ac.uk)

Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

[canvas.hull.ac.uk/courses/67551](https://canvas.hull.ac.uk/courses/67551)

and on the **Course Webpage** hosted on my website

[silviofanzon.com/blog/2023/NSS](https://silviofanzon.com/blog/2023/NSS)

## Readings

We will study the set of real numbers  $\mathbb{R}$ , and then sequences and series in  $\mathbb{R}$ . I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

**!** You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to know in order to excel in the final exam.

# 1 Introduction

The first aim of this lecture notes is to rigorously introduce the set of **real numbers**, which is denoted by  $\mathbb{R}$ . But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for  $n, m \in \mathbb{N}$ . Here the symbol  $\in$  denotes that  $m$  and  $n$  belong to  $\mathbb{N}$ . For example  $3 + 7$  results in 10.

## Question 1.1

Can the sum be inverted? That is, given any  $n, m \in \mathbb{N}$ , can you always find  $x \in \mathbb{N}$  such that

$$n + x = m? \tag{1.1}$$

Of course to invert (1.1) we can just perform a **subtraction**, implying that

$$x = m - n.$$

But there is a catch. In general  $x$  does not need to be in  $\mathbb{N}$ . For example, take  $n = 10$  and  $m = 1$ . Then  $x = -9$ , which does not belong to  $\mathbb{N}$ . Therefore the answer to Question 1.1 is **NO**.

To make sure that we can always invert the sum, we need to **extend** the set  $\mathbb{N}$ . This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : n \in \mathbb{N}\},$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The sum can be extended to  $\mathbb{Z}$ , by defining

$$(-n) + (-m) := -(m + n) \quad (1.2)$$

for all  $m, n \in \mathbb{N}$ . Now every element of  $\mathbb{Z}$  possesses an **inverse**, that is, for each  $n \in \mathbb{Z}$ , there exists  $m \in \mathbb{Z}$ , such that

$$n + m = 0.$$

Can we characterize  $m$  explicitly? Of course! Seeing the definition at (1.2), we simply have

$$m = -n.$$

On the set  $\mathbb{Z}$  we can also define the operation of **multiplication**, in the usual way we learnt in school. For  $n, m \in \mathbb{Z}$ , we denote the multiplication by  $nm$  or  $n \cdot m$ . For example  $7 \cdot 2 = 14$  and  $1 \cdot (-1) = -1$ .

### Question 1.2

Can the multiplication in  $\mathbb{Z}$  be inverted? That is, given any  $n, m \in \mathbb{Z}$ , can you always find  $x \in \mathbb{Z}$  such that

$$nx = m? \quad (1.3)$$

To invert (1.3) if  $n \neq 0$ , we can just perform a **division**, to obtain

$$x = \frac{m}{n}.$$

But again there is a catch. Indeed taking  $n = 2$  and  $m = 1$  yields  $x = 1/2$ , which does not belong to  $\mathbb{Z}$ . The answer to Question 1.2 is therefore **NO**.

Thus, in order to invert the multiplication, we need to **extend** the set of integers  $\mathbb{Z}$ . This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

We then extend the operations of sum and multiplication to  $\mathbb{Q}$  by defining

$$\frac{m}{n} + \frac{p}{q} := \frac{mq + np}{nq}$$

and

$$\frac{m}{n} \cdot \frac{p}{q} := \frac{mp}{nq}$$

Now the multiplication is invertible in  $\mathbb{Q}$ . Specifically, each non-zero element has an inverse: the inverse of  $m/n$  is given by  $n/m$ .

To summarize, we have extended  $\mathbb{N}$  to  $\mathbb{Z}$ , and  $\mathbb{Z}$  to  $\mathbb{Q}$ . By construction we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Moreover **sum** and **product** are **invertible** in  $\mathbb{Q}$ . Now we are happy right? So and so.

### Question 1.3

Can we draw the set  $\mathbb{Q}$ ?

It is clear how to draw  $\mathbb{Z}$ , as seen below.

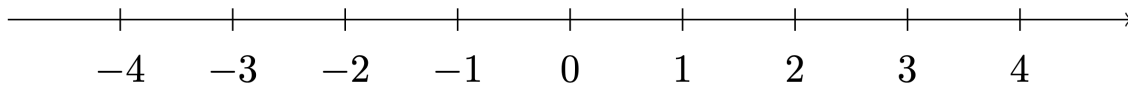


Figure 1.1: Representation of integers  $\mathbb{Z}$

However  $\mathbb{Q}$  is much **larger** than the set  $\mathbb{Z}$  represented by the ticks in Figure 1.1. What do we mean by **larger**? For example, consider  $0 \in \mathbb{Q}$ .

### Question 1.4

What is the number  $x \in \mathbb{Q}$  which is closest to 0?

There is no right answer to the above question, since whichever rational number  $m/n$  you consider, you can always squeeze the rational number  $m/(2n)$  in between:

$$0 < \frac{m}{2n} < \frac{m}{n}.$$

For example think about the case of the numbers

$$\frac{1}{n} \text{ for } n \in \mathbb{N}, n \neq 0.$$

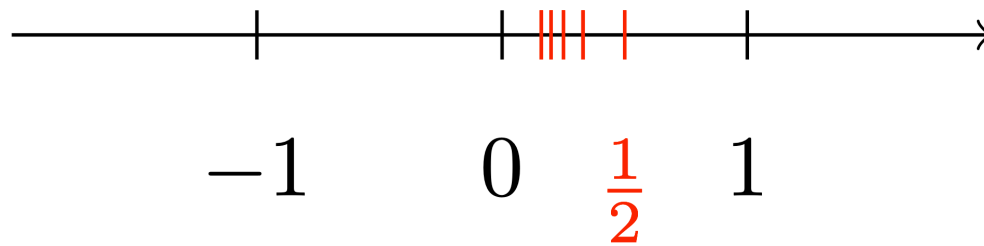


Figure 1.2: Fractions  $\frac{1}{n}$  can get arbitrarily close to 0

Such numbers get arbitrarily close to 0, as depicted below.

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval  $[0, 1]$ . In other words, we might conjecture the following.

### Conjecture 1.5

Maybe  $\mathbb{Q}$  can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.5 is **false**, as shown by the Theorem below.

### Theorem 1.6

The number  $\sqrt{2}$  does not belong to  $\mathbb{Q}$ .

Theorem 1.6 is the reason why  $\sqrt{2}$  is called an **irrational number**. For reference, a few digits of  $\sqrt{2}$  are given by

$$\sqrt{2} = 1.414213562373095048 \dots$$

and the situation is as in the picture below.

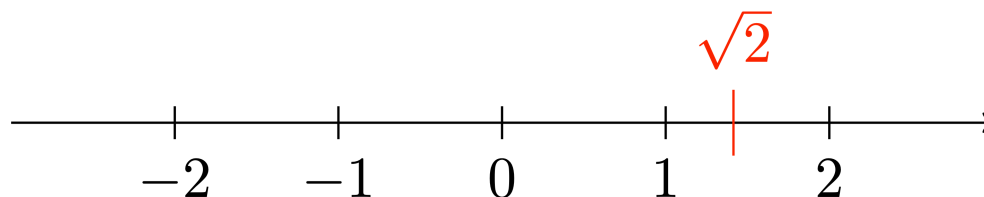


Figure 1.3: Representing  $\sqrt{2}$  on the numbers line.

We can therefore see that Conjecture 1.5 is **false**, and  $\mathbb{Q}$  is not a line: indeed  $\mathbb{Q}$  has a **gap** at  $\sqrt{2}$ . Let us see why Theorem 1.6 is true.

**Proof:** Proof of Theorem 1.6

We prove that

$$\sqrt{2} \notin \mathbb{Q}$$

by **contradiction**.

Wait, what does this mean? Proving the claim by contradiction means assuming that the claim is **false**. This means we **assume** that

$$\sqrt{2} \in \mathbb{Q}. \quad (1.4)$$

From this assumption we then start deducing other statements, hoping to encounter a statement which is **FALSE**. But if (1.4) leads to a false statement, then it must be that (1.4) is **FALSE**. Thus the contrary of (1.4) must hold, meaning that

$$\sqrt{2} \notin \mathbb{Q}$$

as we wanted to show. This would conclude the proof.

Now we need to actually show that (1.4) will lead to a contradiction. Since this is our first proof, let us take it slowly, step-by-step.

1. Assuming (1.4) just means that there exists  $q \in \mathbb{Q}$  such that

$$q = \sqrt{2}. \quad (1.5)$$

2. Since  $q \in \mathbb{Q}$ , by definition we have

$$q = \frac{m}{n}$$

for some  $m, n \in \mathbb{N}$  with  $n \neq 0$ .

3. Recalling (1.5), we then have

$$\frac{m}{n} = \sqrt{2}.$$

4. We can square the above equation to get

$$\frac{m^2}{n^2} = 2. \quad (1.6)$$

5. **Withouth loss of generality**, we can **assume** that  $m$  and  $n$  have no common factors.



Wait. What does Step 5 mean? You will encounter the sentence *withouth loss of generality* many times in mathematics. It is often abbreviated in **WLOG**. WLOG means that the assumption that follows is chosen arbitrarily, but does not affect the validity of the proof in general.

For example in our case we can assume that  $m$  and  $n$  have no common factor. This is because if  $m$  and  $n$  had common factors, then it would mean

$$m = a\tilde{m}, \quad n = a\tilde{n}$$

for some  $a \in \mathbb{N}$  with  $a \neq 0$ . Then

$$\frac{m}{n} = \frac{a\tilde{m}}{a\tilde{n}} = \frac{\tilde{m}}{\tilde{n}}.$$

Therefore by (1.6)

$$\frac{\tilde{m}^2}{\tilde{n}^2} = 2.$$

The proof can now proceed in the same way we would have proceeded from Step 4, but in addition we have the hypothesis that  $\tilde{m}$  and  $\tilde{n}$  have no common factors.

6. Equation (1.6) implies

$$m^2 = 2n^2. \tag{1.7}$$

Therefore the integer  $m^2$  is an even number.

Why is  $m^2$  even? As you already know, **even** numbers are

$$0, 2, 4, 6, 8, 10, 12, \dots$$

All these numbers have in common that they can be divided by 2, and so they can be written as

$$2p$$

for some  $p \in \mathbb{N}$ . For example 52 is even, because

$$52 = 2 \cdot 26.$$

Instead, **odd** numbers are

$$1, 3, 5, 7, 9, 11, \dots$$

These can be all written as

$$2p + 1$$

for some  $p \in \mathbb{N}$ . For example 53 is odd, because

$$53 = 2 \cdot 26 + 1.$$

7. Thus  $m$  is an even number, and so there exists  $p \in \mathbb{N}$  such that

$$m = 2p. \quad (1.8)$$

Why is (1.8) true? Let us see what happens if we take the square of an even number  $m = 2p$

$$m^2 = (2p)^2 = 4p^2 = 2(2p^2) = 2q.$$

Thus  $m^2 = 2q$  for some  $q \in \mathbb{N}$ , and so  $m^2$  is an even number. If instead  $m$  is odd, then  $m = 2p + 1$  and

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

showing that also  $m^2$  is odd.

This justifies Step 7: Indeed we know that  $m^2$  is an even number from Step 6. If  $m$  was odd, then  $m^2$  would be odd. Hence  $m$  must be even as well.

8. If we substitute (1.8) in (1.7) we get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2. \quad (1.9)$$

9. We now make a series of observations:

- Equation (1.9) says that  $n^2$  is even.
- Step 6 says that  $m^2$  is even.
- Therefore  $n$  and  $m$  are also even.
  
- Hence  $n$  and  $m$  have 2 as common factor.
- But Step 5 says that  $n$  and  $m$  have no common factors.
- **CONTRADICTION**

10. Our reasoning has run into a **contradiction**, starting from assumption (1.4), which says that

$$\sqrt{2} \in \mathbb{Q}.$$

Hence the above must be **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

Seeing that  $\sqrt{2} \notin \mathbb{Q}$ , we might be tempted to just fill in the gap by adding  $\sqrt{2}$  to  $\mathbb{Q}$ . However, with analogous proof to Theorem 1.6, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number  $p$ . As there are infinite prime numbers, this means that  $\mathbb{Q}$  has infinite gaps. Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : p \text{ prime}\}.$$

However even this is not enough, as we would still have numbers which are not contained in  $\tilde{\mathbb{Q}}$ , for example

$$\sqrt{2} + \sqrt{3}, \pi, \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}.$$

### Remark 1.7

Proving that

$$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

is relatively easy, and will be left as an **exercise**. Instead, proving that

$$\pi \notin \mathbb{Q}$$

is way more complicated. There are several proof of the fact, all requiring mathematics which is more advanced of the one presented in this course. For some proofs, see this [Wikipedia page](#).

The reality of things is that to **complete**  $\mathbb{Q}$  and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in  $\mathbb{Q}$ . Such extension of  $\mathbb{Q}$  will be called  $\mathbb{R}$ , the set of **real numbers**. The inclusions will therefore be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The set  $\mathbb{R}$  is not at all trivial to construct. In fact, at first we will not construct it, but just do the following:

- We will assume that  $\mathbb{R}$  **exists** and satisfies some basic **axioms**.
- One of the axioms is that  $\mathbb{R}$  fills **all** the **gaps** that  $\mathbb{Q}$  has. Therefore  $\mathbb{R}$  can be thought as a **continuous** line.
- We will study the **properties** of  $\mathbb{R}$  which descend from such **axioms**.

For example one of the properties of  $\mathbb{R}$  will be the following:

**Theorem 1.8:** We will prove this in the future

$\mathbb{R}$  contains all the square roots. This means that for every  $x \in \mathbb{R}$  with  $x \geq 0$ , we have

$$\sqrt{x} \in \mathbb{R}.$$

At the end of this chapter we will provide a concrete **model** for the real numbers  $\mathbb{R}$ , to prove once and for all that such set indeed exists.

**Theorem 1.9:** We will prove this in the future

There exists a set  $\mathbb{R}$ , called the set of real numbers, which has the following properties:

- $\mathbb{R}$  extends  $\mathbb{Q}$ , that is,

$$\mathbb{Q} \subset \mathbb{R}.$$

- $\mathbb{R}$  satisfies certain **axioms**.
- $\mathbb{R}$  fills **all** the **gaps** that  $\mathbb{Q}$  has. In particular  $\mathbb{R}$  can be represented by a **continuous** line.

## 2 Preliminaries

Before introducing  $\mathbb{R}$  we want to make sure that we cover all the basics needed for the task.

### 2.1 Sets

A set is a **collection** of objects. These objects are called **elements** of the set. For example in the previous section we mentioned the following sets:

- $\mathbb{N}$  the set of natural numbers
- $\mathbb{Z}$  the set of integers
- $\mathbb{Q}$  the set of rational numbers
- $\mathbb{R}$  the set of real numbers

Given an arbitrary set  $A$ , we write

$$x \in A$$

if the element  $x$  belongs to the set  $A$ . If an element  $x$  is not contained in  $A$ , we say that

$$x \notin A.$$

#### Remark 2.1

A set can contain all sorts of elements. For example the students in a classroom can be modelled by a set  $S$ . The elements of the set are the students. For example

$$S = \{\text{Alice, Olivia, Jake, Sahab}\}$$

In this case we have

$$\text{Alice} \in S$$

but instead

Silvio  $\notin S$ .

## 2.2 Logic

In this section we introduce some basic logic symbols. Suppose that you are given two statements, say  $\alpha$  and  $\beta$ . The formula

$$\alpha \implies \beta$$

means that  $\alpha$  **implies**  $\beta$ . In other words, if  $\alpha$  is true then also  $\beta$  is true.

The formula

$$\alpha \impliedby \beta$$

means that  $\alpha$  is implied by  $\beta$ : if  $\beta$  is true then also  $\alpha$  is true.

When we write

$$\alpha \iff \beta \tag{2.1}$$

we mean that  $\alpha$  and  $\beta$  are **equivalent**. Note that (2.1) is equivalent to

$$\alpha \implies \beta \text{ and } \beta \implies \alpha.$$

Such equivalence is very useful in proofs.

### Example 2.2

We have that

$$x > 0 \implies x > -100,$$

and

$$\text{contradiction} \impliedby \sqrt{2} \in \mathbb{Q}.$$

Concerning  $\iff$  we have

$$x^2 < 2 \iff -\sqrt{2} < x < \sqrt{2}.$$

We now introduce logic **quantifiers**. These are

- $\forall$  which reads **for all**
- $\exists$  which reads **exists**
- $\exists!$  which reads **exists unique**
- $\nexists$  which reads **does not exists**

These work in the following way. Suppose that you are given a statement  $\alpha(x)$  which depends on the point  $x \in \mathbb{R}$ . Then we say

- $\alpha(x)$  is satisfied for all  $x \in A$  with  $A$  some collection of numbers. This translates to the symbols

$$\alpha(x) \text{ is true } \forall x \in A,$$

- There exists some  $x$  in  $\mathbb{R}$  such that  $\alpha(x)$  is satisfied: in symbols

$$\exists x \in \mathbb{R} \text{ such that } \alpha(x) \text{ is true,}$$

- There exists a unique  $x_0$  in  $\mathbb{R}$  such that  $\alpha(x)$  is satisfied: in symbols

$$\exists! x_0 \in \mathbb{R} \text{ such that } \alpha(x_0) \text{ is true,}$$

- $\alpha(x)$  is never satisfied:

$$\nexists x \in \mathbb{R} \text{ such that } \alpha(x) \text{ is true.}$$

### Example 2.3

Let us make concrete examples:

- The expression  $x^2$  is always non-negative. Thus we can say

$$x^2 \geq 0 \text{ for all } x \in \mathbb{R}.$$

- The equation  $x^2 = 1$  has two solutions  $x = 1$  and  $x = -1$ . Therefore we can say

$$\exists x \in \mathbb{R} \text{ such that } x^2 = 1.$$

- The equation  $x^3 = 1$  has a unique solution  $x = 1$ . Thus

$$\exists! x \in \mathbb{R} \text{ such that } x^3 = 1.$$

- We know that the equation  $x^2 = 2$  has no solutions in  $\mathbb{Q}$ . Then

$$\nexists x \in \mathbb{Q} \text{ such that } x^2 = 2.$$

## 2.3 Operations on sets

### 2.3.1 Union and intersection

For two sets  $A$  and  $B$  we define their **union** as the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

The **intersection** of  $A$  and  $B$  is defined by

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

We denote the **empty set** by the symbol  $\emptyset$ . Two sets are **disjoint** if

$$A \cap B = \emptyset.$$

### Example 2.4

Define the subset of rational numbers

$$S := \left\{x \in \mathbb{Q} : 0 < x < \frac{5}{2}\right\}.$$

Then we have

$$\mathbb{N} \cap S = \{1, 2\}.$$

We can also define the sets of **even** and **odd** numbers by

$$E := \{2n : n \in \mathbb{N}\}, \quad (2.2)$$

$$O := \{2n + 1 : n \in \mathbb{N}\}. \quad (2.3)$$

Then we have

$$\mathbb{N} \cap E = E, \quad \mathbb{N} \cap O = O, \quad (2.4)$$

$$O \cup E = \mathbb{N}, \quad O \cap D = \emptyset. \quad (2.5)$$

## 2.3.2 Inclusion and equality

Given two sets  $A$  and  $B$ , we say that  $A$  is **contained** in  $B$  if all the elements of  $A$  are also contained in  $B$ . This will be denoted with the **inclusion** symbol  $\subset$ , that is,

$$A \subset B.$$

In this case we say that

- $A$  is a **subset** of  $B$ ,
- $B$  is a **superset** of  $A$ .



The inclusion  $A \subset B$  is equivalent to the implication:

$$x \in A \implies x \in B$$

for all  $x \in A$ . The symbol  $\implies$  reads **implies**, and denotes the fact that the first condition implies the second.

### Example 2.5

Given two sets  $A$  and  $B$  we always have

$$(A \cap B) \subset A, (A \cap B) \subset B, \quad (2.6)$$

$$A \subset (A \cup B), B \subset (A \cup B). \quad (2.7)$$

We say that two sets  $A$  and  $B$  are equal if they contain the **same** elements. We denote equality by the symbol

$$A = B.$$

### Example 2.6

The sets

$$A = \{1, 2, 3\}$$

and

$$B = \{3, 1, 2\}$$

are equal. This is because they contain exactly the same elements: **order** does not matter when talking about sets.

### Proposition 2.7

Let  $A$  and  $B$  be sets. Then

$$A = B$$

**if and only if**

$$A \subset B \text{ and } B \subset A.$$

**Proof**

The proof is almost trivial. However it is a good exercise in basic logic, so let us do it.

1. First implication  $\Rightarrow$  :

Suppose that  $A = B$ . Let us show that  $A \subset B$ . Since  $A = B$ , this means that all the elements of  $A$  are also contained in  $B$ . Therefore if we take  $x \in A$  we have

$$x \in A \Rightarrow x \in B.$$

This shows  $A \subset B$ . The proof of  $B \subset A$  is similar.

2. Second implication  $\Leftarrow$  :

Suppose that  $A \subset B$  and  $B \subset A$ . We need to show  $A = B$ , that is,  $A$  and  $B$  have the same elements. To this end let  $x \in A$ . Since  $A \subset B$  then we have  $x \in B$ . Thus  $B$  contains all the elements of  $A$ . Since we are also assuming  $B \subset A$ , this means that  $A$  contains all the elements of  $B$ . Hence  $A$  and  $B$  contain the same elements, and  $A = B$ .

The above proposition is very useful when we need to **prove** that two sets are equal: rather than showing directly that  $A = B$ , we can prove that  $A \subset B$  and  $B \subset A$ .

### 2.3.3 Infinite unions and intersections

Suppose given a set  $\Omega$ , and a family of sets  $A_n \subset \Omega$ , where  $n \in \mathbb{N}$ . Then we can define the **infinte union**

$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for at least one } n \in \mathbb{N}\}.$$

The **infinte intersection** is defined as

$$\bigcap_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

**Example 2.8**

Let the ambient set be  $\Omega := \mathbb{N}$  and define the family  $A_n$  by

$$A_1 := \{1, 2, 3, 4, \dots\} \quad (2.8)$$

$$A_2 := \{2, 3, 4, 5, \dots\} \quad (2.9)$$

$$A_3 := \{3, 4, 5, 6, \dots\} \quad (2.10)$$

$$\dots \dots \quad (2.11)$$

$$A_n := \{n, n+1, n+2, n+3, \dots\}, \quad (2.12)$$

for arbitrary  $n \in \mathbb{N}$ . Then

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}. \quad (2.13)$$

The above equality can be easily proven using Proposition 2.7. Indeed, assume that  $m \in \bigcup_n A_n$ . Then  $m \in A_n$  for at least one  $n \in \mathbb{N}$ . Since  $A_n \subset \mathbb{N}$ , we conclude that  $m \in \mathbb{N}$ . This shows

$$\bigcup_{n \in \mathbb{N}} A_n \subset \mathbb{N}.$$

Conversely, suppose that  $m \in \mathbb{N}$ . By definition  $m \in A_m$ . Hence there exists at least one index  $n$ ,  $n = m$  in this case, such that  $m \in A_n$ . Then by definition  $m \in \bigcup_{n \in \mathbb{N}} A_n$ , showing that

$$\mathbb{N} \subset \bigcup_{n \in \mathbb{N}} A_n.$$

Hence we conclude (2.13) by Proposition 2.7.

We also have that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset. \quad (2.14)$$

We prove the above by **contradiction**. Indeed, suppose that (2.14) is false, i.e.,

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset.$$

This means there exists some  $m \in \mathbb{N}$  such that  $m \in \bigcap_{n \in \mathbb{N}} A_n$ . Hence, by definition,  $m \in A_n$  for all  $n \in \mathbb{N}$ . However  $m \notin A_{m+1}$ , yielding a contradiction. Thus (2.14) holds.

**2.3.4 Complement**

Suppose that  $A$  and  $B$  are subsets of a larger set  $\Omega$ . The **complement** of  $A$  with respect to  $B$  is the set of elements of  $B$  which do not belong to  $A$ , that is

$$B \setminus A := \{x \in \Omega : x \in B \text{ and } x \notin A\}.$$

In particular, the complement of  $A$  with respect to  $\Omega$  is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : x \notin A\}.$$

### Remark 2.9

Suppose that  $A \subset \Omega$ . Then  $A$  and  $A^c$  form a **partition** of  $\Omega$ , in the sense that

$$A \cup A^c = \Omega \quad \text{and} \quad A \cap A^c = \emptyset.$$

### Example 2.10

Suppose  $A, B \subset \Omega$ . Then

$$A \subset B \iff B^c \subset A^c.$$

Let us prove the above claim:

- First implication  $\implies$  :  
Suppose that  $A \subset B$ . We need to show that  $B^c \subset A^c$ . Hence, assume  $x \in B^c$ . By definition this means that  $x \notin B$ . Now notice that we cannot have that  $x \in A$ . Indeed, assume  $x \in A$ . By assumption we have  $A \subset B$ , hence  $x \in B$ . But we had assumed  $x \in B^c$ , contradiction. Therefore it must be that  $x \notin A$ . Thus  $B^c \subset A^c$ .
- Second implication  $\impliedby$  :  
Essentially the same proof, hence we omit it.

We conclude by stating the De Morgan's Laws. The proof will be left as an exercise.

### Proposition 2.11: De Morgan's Laws

Suppose  $A, B \subset \Omega$ . Then

$$(A \cap B)^c = A^c \cup B^c$$

and

$$(A \cup B)^c = A^c \cap B^c.$$

## 2.3.5 Product of sets

Suppose  $A$  and  $B$  are two sets. The **product** of  $A$  and  $B$  is the set of pairs

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

By definition two elements in  $A \times B$  are the same, in symbols

$$(a, b) = (\tilde{a}, \tilde{b})$$

if and only if they are equal component-by-component, that is

$$a = \tilde{a}, \quad b = \tilde{b}.$$

## 2.4 Equivalence and order relation

Suppose  $A$  is a set. A **binary relation**  $R$  on  $A$  is a subset

$$R \subset A \times A.$$

### Definition 2.12: Equivalence relation

A binary relation  $R$  is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive:** For each  $x \in A$  one has

$$(x, x) \in R,$$

This is saying that all the elements in  $A$  must be related to themselves

2. **Symmetric:** We have

$$(x, y) \in R \implies (y, x) \in R$$

If  $x$  is related to  $y$ , then  $y$  is related to  $x$

3. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

If  $x$  is related to  $y$ , and  $y$  is related to  $z$ , then  $x$  must be related to  $z$

If  $(x, y) \in R$  we write

$$x \sim y$$

and we say that  $x$  and  $y$  are **equivalent**.

**Definition 2.13:** Equivalence classes

Suppose  $R$  is an **equivalence relation** on  $A$ . The **equivalence class** of an element  $x \in A$  is the set

$$[x] := \{y \in A : y \sim x\}.$$

The set of equivalence classes of elements of  $A$  with respect to the equivalence relation  $R$  is denoted by

$$A/R := \{[x] : x \in A\}.$$

Let us immediately clarify the above definitions by considering the prototypical example of equivalence relation: the **equality**.

**Example 2.14:** Equality is an equivalence relation

Consider the set of natural numbers  $\mathbb{N}$ . The equality defines a **binary relation** on  $\mathbb{N} \times \mathbb{N}$ , via

$$R := \{(x, y) \in \mathbb{N} \times \mathbb{N} : x = y\}.$$

Let us check that  $R$  is an **equivalence relation**:

1. Reflexive: It holds, since  $x = x$  for all  $x \in \mathbb{N}$ ,
2. Symmetric: Again  $x = y$  if and only if  $y = x$ ,
3. Transitive: If  $x = y$  and  $y = z$  then  $x = z$ .

The class of equivalence of  $x \in \mathbb{N}$  is given by

$$[x] = \{x\},$$

that is, this relation is quite trivial, given that each element of  $\mathbb{N}$  can only be related to itself.

Similarly, we define **order relations**.

**Definition 2.15:** Order relation

A binary relation  $R$  is called an **order relation** if it satisfies the following properties:

1. **Reflexive**: For each  $x \in A$  one has

$$(x, x) \in R,$$

2. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

3. **Antisymmetric:** We have

$$(x, y) \in R \text{ and } (y, x) \in R \implies x = y$$

This is the only new condition with respect to the definition of equivalence relation.

The prototypical example of order relation is the **inequality** relation.

**Example 2.16:** Inequality is an order relation

Consider the set of integers  $\mathbb{Z}$ . The inequality defines a **binary relation** on  $\mathbb{Z} \times \mathbb{Z}$ , via

$$R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \leq y\}.$$

Let us check that  $R$  is an **order relation**:

1. Reflexive: It holds, since  $x \leq x$  for all  $x \in \mathbb{Z}$ ,
2. Transitive: If  $x \leq y$  and  $y \leq z$  then  $x \leq z$ .
3. Antisymmetric: If  $x \leq y$  and  $y \leq x$  then  $x = y$ .

## 2.5 Intervals

In this section we assume to have available the set  $\mathbb{R}$  of **real numbers**, which we recall is an extension of  $\mathbb{Q}$ . We now introduce the concept of **interval**.

**Definition 2.17**

Let  $a, b \in \mathbb{R}$  with  $a < b$ . We define the **open interval**  $(a, b)$  as the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

We define the **close interval**  $[a, b]$  as the set

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

In general we also define the intervals

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}, \quad (2.15)$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}, \quad (2.16)$$

$$(a, \infty) := \{x \in \mathbb{R} : x > a\}, \quad (2.17)$$

$$[a, \infty) := \{x \in \mathbb{R} : x \geq a\}, \quad (2.18)$$

$$(-\infty, b) := \{x \in \mathbb{R} : x < b\}, \quad (2.19)$$

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}. \quad (2.20)$$

Some of the above intervals are depicted in Figure 2.1, Figure 2.2, Figure 2.3, Figure 2.4 below.



Figure 2.1: Interval  $(a, b)$



Figure 2.2: Interval  $[a, b]$



Figure 2.3: Interval  $(a, \infty)$

## 2.6 Absolute value or Modulus

In this section we assume to have available the set  $\mathbb{R}$  of **real numbers**, which we recall is an extension of  $\mathbb{Q}$ .



Figure 2.4: Interval  $(-\infty, b]$ **Definition 2.18:** Absolute value

For  $x \in \mathbb{R}$  we define its **absolute value** as the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

**Example 2.19**

By definition one has  $|x| = x$  if  $x \geq 0$ . For example

$$|\pi| = \pi, \quad |\sqrt{2}| = \sqrt{2}, \quad |0| = 0.$$

Instead  $|x| = -x$  if  $x < 0$ . For example

$$|-\pi| = \pi, \quad |-\sqrt{2}| = \sqrt{2}, \quad |-10| = 10.$$

Let us also make the following basic remark, whose proof will be left as an exercise.

**Remark 2.20**

For all  $x \in \mathbb{R}$  one has

$$|x| \geq 0.$$

Moreover

$$|x| = 0 \iff x = 0.$$

Another basic remark (proof by exercise).

**Remark 2.21**

For all  $x \in \mathbb{R}$  one has

$$|x| = |-x|.$$

You might be familiar with the graph associated to the absolute value function:

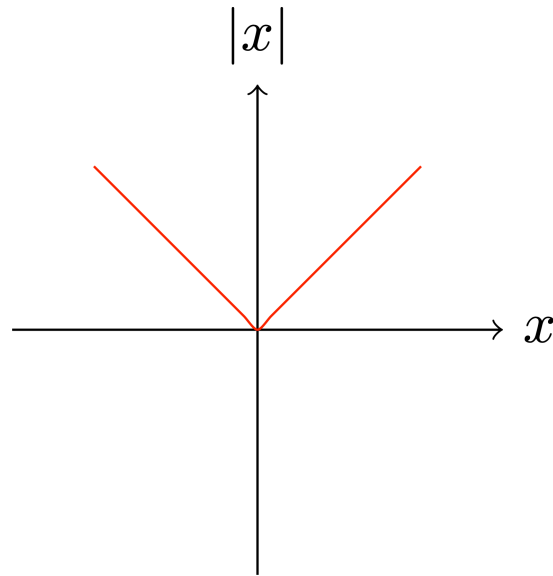


Figure 2.5: Plot of the absolute value function  $f(x) = |x|$

However in these Lecture Notes we are not dealing with functions, so it is better to think about the absolute value in a geometric way.

**Remark 2.22:** Geometric interpretation of  $|x|$

A number  $x \in \mathbb{R}$  can be represented with a point on the real line  $\mathbb{R}$ . The non-negative number  $|x|$  represents the **distance** of  $x$  from the origin 0. Notice that this works for both positive and negative numbers  $x_1$  and  $x_2$  respectively, as shown in Figure 2.6 below.

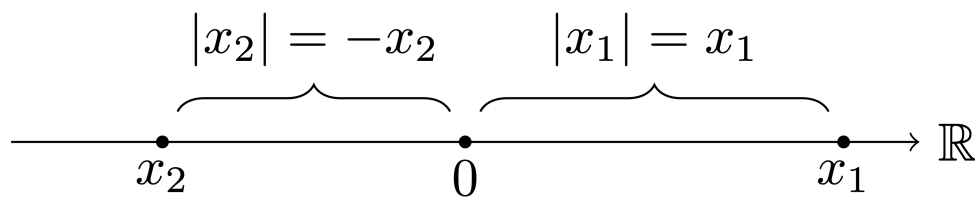


Figure 2.6: Geometric interpretation of  $|x|$

**Remark 2.23:** Geometric interpretation of  $|x - y|$ 

If  $x, y \in \mathbb{R}$  then the number  $|x - y|$  represents the distance between  $x$  and  $y$  on the real line, as shown in Figure 2.7 below. Note that by Remark 2.21 we have

$$|x - y| = |y - x|.$$

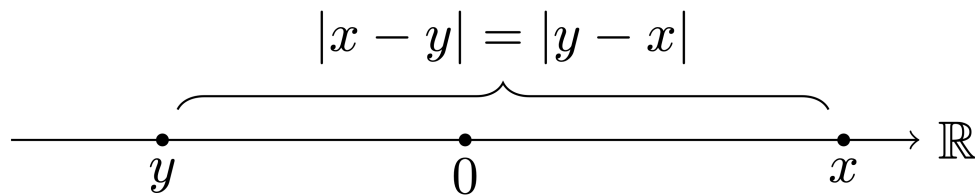


Figure 2.7: Geometric interpretation of  $|x - y|$

In the next Lemma we show a fundamental equivalence regarding the absolute value.

**Lemma 2.24**

Let  $x, y \in \mathbb{R}$ . Then

$$|x| \leq y \iff -y \leq x \leq y.$$

The geometric meaning of the above statement is clear: the distance of  $x$  from the origin is less than  $y$ , in formulae

$$|x| \leq y,$$

if and only if  $x$  belongs to the interval  $[-y, y]$ , in formulae

$$-y \leq x \leq y.$$

A sketch of this explanation is seen in Figure 2.8 below.

**Proof:** Proof of Lemma 2.24

We divide the proof in steps.

- Step 1: implication  $\implies$

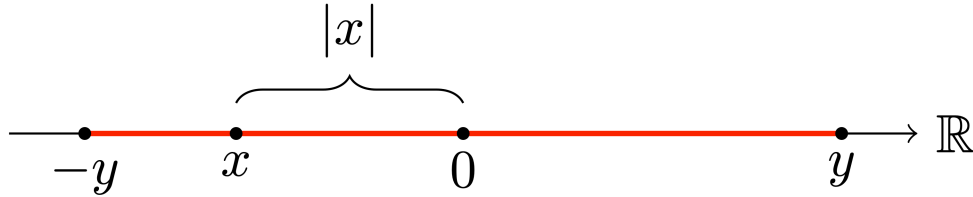


Figure 2.8: Geometric meaning of Lemma 2.24

Suppose first that

$$|x| \leq y. \quad (2.21)$$

Recalling that the absolute value is non-negative, from (2.21) we deduce that  $0 \leq |x| \leq y$ . In particular it holds

$$y \geq 0. \quad (2.22)$$

We make separate arguments for the cases  $x \geq 0$  and  $x < 0$ :

- Case 1:  $x \geq 0$ . From (2.21), (2.22) and from  $x \geq 0$  we have

$$-y \leq 0 \leq x = |x| \leq y$$

which shows

$$-y \leq x \leq y.$$

- Case 2:  $x < 0$ . From (2.21), (2.22) and from  $x < 0$  we have

$$-y \leq 0 < -x = |x| \leq y$$

which shows

$$-y \leq -x \leq y.$$

Multiplying the above inequalities by  $-1$  yields

$$-y \leq x \leq y.$$

- Step 2: implication  $\Leftarrow$   
Suppose now that

$$-y \leq x \leq y. \quad (2.23)$$

We make separate arguments for the cases  $x \geq 0$  and  $x < 0$ :

- Case 1:  $x \geq 0$ . Since  $x \geq 0$ , from (2.23) we get

$$|x| = x \leq y$$

showing that

$$|x| \leq y.$$

- Case 2:  $x < 0$ . Since  $x < 0$ , from (2.23) we have

$$-y \leq x = -|x|.$$

Multiplying the above inequality by  $-1$  yields

$$|x| \leq y.$$

With basically the same arguments, one can also show the following.

### Lemma 2.25

Let  $x, y \in \mathbb{R}$ . Then

$$|x| < y \iff -y < x < y.$$

## 2.7 Triangle inequality

The triangle inequality relates the absolute value to the sum operation. It is a very important inequality, which we will use a lot in the future.

### Theorem 2.26: Triangle inequality

For every  $x, y \in \mathbb{R}$  we have

$$||x| - |y|| \leq |x - y| \leq |x| + |y|. \quad (2.24)$$

Before proceeding with the proof, let us discuss the geometric meaning of the triangle inequality.

**Remark 2.27:** Geometric meaning of triangle inequality

The notion of absolute value can be extended also to vectors in the plane. Suppose that  $x$  and  $y$  are two vectors in the plane, as in Figure 2.9 below. Then  $|x|$  and  $|y|$  can be interpreted as the **lengths** of these vectors.

Using the rule of sum of vectors, we can draw  $x + y$ , as shown in Figure 2.10 below. From the picture it is evident that

$$|x + y| \leq |x| + |y|, \quad (2.25)$$

that is, *the length of each side of a triangle does not exceed the sum of the lengths of the two remaining sides*. Note that (2.25) is exactly the second inequality in (2.24). This is why (2.24) is called triangle inequality.

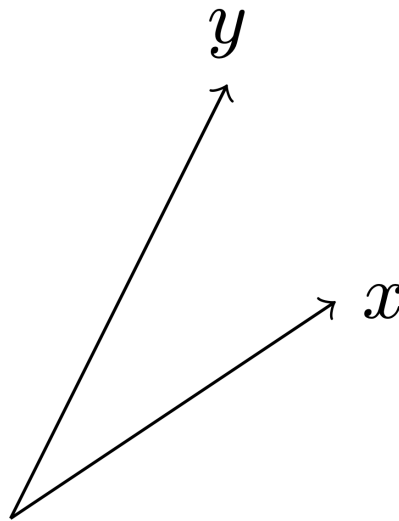


Figure 2.9: Vectors  $x$  and  $y$

**Proof:** Proof of Theorem 2.26

Assume that  $x, y \in \mathbb{R}$ . We prove the two inequalities in (2.24) individually.

- Proof of the second inequality in (2.24):  
Trivially we have

$$|x| \leq |x|.$$

Therefore we can apply Lemma 2.24 and infer

$$-|x| \leq x \leq |x|. \quad (2.26)$$

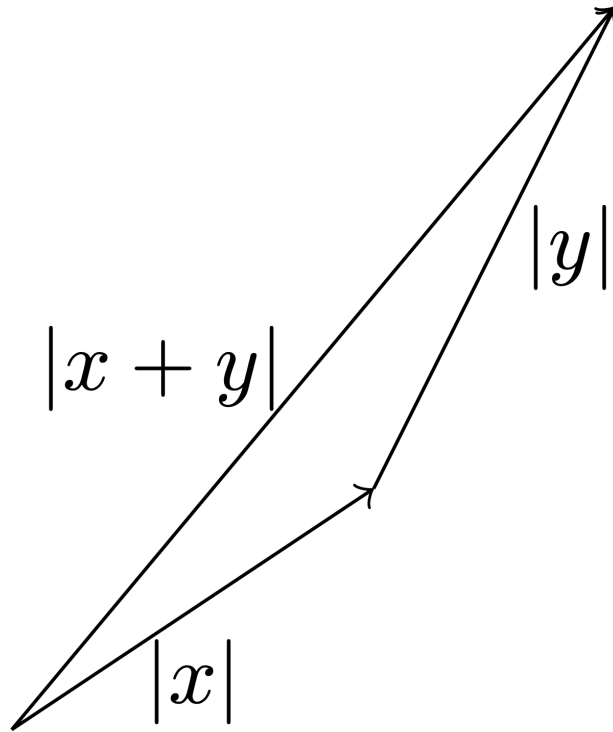


Figure 2.10: Summing the vectors  $x$  and  $y$ . The triangle inequality relates the length of  $x + y$  to the length of  $x$  and  $y$

Similarly we have that  $|y| \leq |y|$ , and so Lemma 2.24 implies

$$-|y| \leq y \leq |y|. \quad (2.27)$$

Summing (2.26) and (2.27) we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

We can now again apply Lemma 2.24 to get

$$|x + y| \leq |x| + |y|, \quad (2.28)$$

which is the second inequality in (2.24).

- Proof of the second inequality in (2.24):  
Note that the trivial identity

$$x = x + y - y$$

always holds. We then have

$$|x| = |x + y - y| \quad (2.29)$$

$$= |(x + y) + (-y)| \quad (2.30)$$

$$= |a + b| \quad (2.31)$$

with  $a = x + y$  and  $b = -y$ . We can now apply (2.28) to  $a$  and  $b$  to obtain

$$|x| = |a + b| \quad (2.32)$$

$$\leq |a| + |b| \quad (2.33)$$

$$= |x + y| + |-y| \quad (2.34)$$

$$= |x + y| + |y| \quad (2.35)$$

Therefore

$$|x| - |y| \leq |x + y|. \quad (2.36)$$

We can now swap  $x$  and  $y$  in (2.36) to get

$$|y| - |x| \leq |x + y|.$$

By rearranging the above inequality we obtain

$$-|x + y| \leq |x| - |y|. \quad (2.37)$$

Putting together (2.36) and (2.37) yields

$$-|x + y| \leq |x| - |y| \leq |x + y|.$$

By Lemma 2.24 the above is equivalent to

$$||x| - |y|| \leq |x + y|,$$

which is the first inequality in (2.24).

An immediate consequence of the triangle inequality are the following inequalities, which are left as an exercise.



**Remark 2.28**

For any  $x, y \in \mathbb{R}$  it holds

$$||x| - |y|| \leq |x - y| \leq |x| + |y|.$$

Moreover for any  $x, y, z \in \mathbb{R}$  it holds

$$|x - y| \leq |x - z| + |z - y|.$$

## 2.8 Proofs involving $\varepsilon$

## 2.9 Induction

## 2.10 Example: Approximating $\sqrt{2}$

# 3 Real Numbers

Coming soon

# 4 Sequences

Coming soon

# 5 Series

Coming soon

# License

## Reuse

This work is licensed under a [Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License](https://creativecommons.org/licenses/by-nc-nd/4.0/)



## Citation

For attribution, please cite this work as:

Fanzon, Silvio. (2023). *Lecture Notes on Numbers, Sequences and Series*.  
<https://www.silviofanzon.com/2023-NSS-Notes/>

BibTex citation:

```
@electronic{Fanzon-NSS-2023,  
  author = {Fanzon, Silvio},  
  title = {Lecture Notes on Numbers, Sequences and Series},  
  url = {https://www.silviofanzon.com/2023-NSS-Notes/},  
  year = {2023}}
```

# References

- [1] S. Abbott. *Understanding Analysis*. Second Edition. Springer, 2015.
- [2] R. G. Bartle and D. R. Sherbert. *Introduction to Real Analysis*. Fourth Edition. Wiley, 2011.
- [3] W. Rudin. *Principles of Mathematical Analysis*. Third Edition. McGraw Hill, 1976.