

Numbers, Sequences and Series

Lecture Notes

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Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

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Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

Readings

We will study the set of real numbers \mathbb{R} , and then sequences and series in \mathbb{R} . I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

! You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to know in order to excel in the final exam.

1 Introduction

The first aim of this lecture notes is to rigorously introduce the set of **real numbers**, which is denoted by \mathbb{R} . But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol \in denotes that m and n belong to \mathbb{N} . For example $3 + 7$ results in 10.

Question 1.1

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m? \tag{1.1}$$

Of course to invert (1.1) we can just perform a **subtraction**, implying that

$$x = m - n.$$

But there is a catch. In general x does not need to be in \mathbb{N} . For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to \mathbb{N} . Therefore the answer to Question 1.1 is **NO**.

To make sure that we can always invert the sum, we need to **extend** the set \mathbb{N} . This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : n \in \mathbb{N}\},$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The sum can be extended to \mathbb{Z} , by defining

$$(-n) + (-m) := -(m + n) \quad (1.2)$$

for all $m, n \in \mathbb{N}$. Now every element of \mathbb{Z} possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0.$$

Can we characterize m explicitly? Of course! Seeing the definition at (1.2), we simply have

$$m = -n.$$

On the set \mathbb{Z} we can also define the operation of **multiplication**, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm or $n \cdot m$. For example $7 \cdot 2 = 14$ and $1 \cdot (-1) = -1$.

Question 1.2

Can the multiplication in \mathbb{Z} be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m? \quad (1.3)$$

To invert (1.3) if $n \neq 0$, we can just perform a **division**, to obtain

$$x = \frac{m}{n}.$$

But again there is a catch. Indeed taking $n = 2$ and $m = 1$ yields $x = 1/2$, which does not belong to \mathbb{Z} . The answer to Question 1.2 is therefore **NO**.

Thus, in order to invert the multiplication, we need to **extend** the set of integers \mathbb{Z} . This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

We then extend the operations of sum and multiplication to \mathbb{Q} by defining

$$\frac{m}{n} + \frac{p}{q} := \frac{mq + np}{nq}$$

and

$$\frac{m}{n} \cdot \frac{p}{q} := \frac{mp}{nq}$$

Now the multiplication is invertible in \mathbb{Q} . Specifically, each non-zero element has an inverse: the inverse of m/n is given by n/m .

To summarize, we have extended \mathbb{N} to \mathbb{Z} , and \mathbb{Z} to \mathbb{Q} . By construction we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Moreover **sum** and **product** are **invertible** in \mathbb{Q} . Now we are happy right? So and so.

Question 1.3

Can we draw the set \mathbb{Q} ?

It is clear how to draw \mathbb{Z} , as seen below.

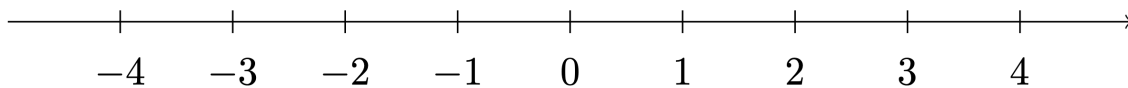


Figure 1.1: Representation of integers \mathbb{Z}

However \mathbb{Q} is much **larger** than the set \mathbb{Z} represented by the ticks in Figure 1.1. What do we mean by **larger**? For example, consider $0 \in \mathbb{Q}$.

Question 1.4

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, you can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n}.$$

For example think about the case of the numbers

$$\frac{1}{n} \text{ for } n \in \mathbb{N}, n \neq 0.$$

Such numbers get arbitrarily close to 0, as depicted below.

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0, 1]$. In other words, we might conjecture the following.

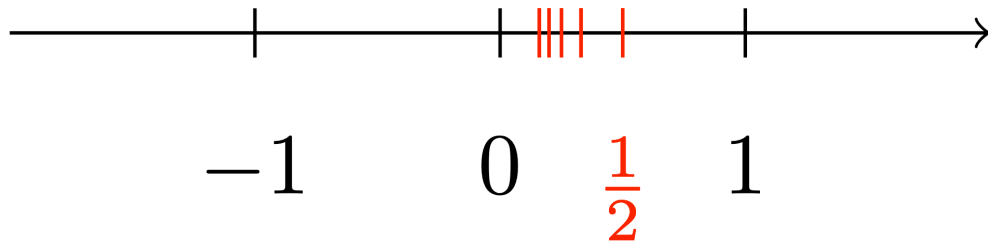


Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Conjecture 1.5

Maybe \mathbb{Q} can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.5 is **false**, as shown by the Theorem below.

Theorem 1.6

The number $\sqrt{2}$ does not belong to \mathbb{Q} .

Theorem 1.6 is the reason why $\sqrt{2}$ is called an **irrational number**. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048 \dots$$

and the situation is as in the picture below.

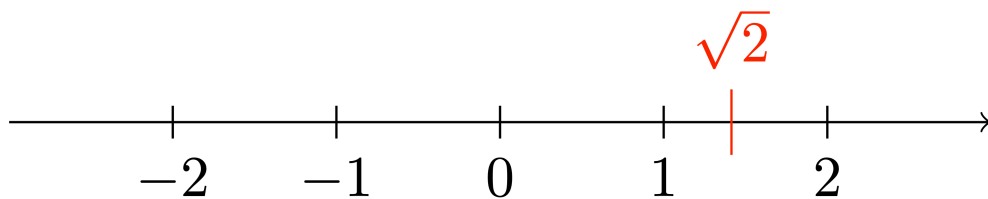


Figure 1.3: Representing $\sqrt{2}$ on the numbers line.

We can therefore see that Conjecture 1.5 is **false**, and \mathbb{Q} is not a line: indeed \mathbb{Q} has a **gap** at $\sqrt{2}$. Let us see why Theorem 1.6 is true.

Proof: Proof of Theorem 1.6

We prove that

$$\sqrt{2} \notin \mathbb{Q}$$

by **contradiction**.

Wait, what does this mean? Proving the claim by contradiction means assuming that the claim is **false**. This means we **assume** that

$$\sqrt{2} \in \mathbb{Q}. \quad (1.4)$$

From this assumption we then start deducing other statements, hoping to encounter a statement which is **FALSE**. But if (1.4) leads to a false statement, then it must be that (1.4) is **FALSE**. Thus the contrary of (1.4) must hold, meaning that

$$\sqrt{2} \notin \mathbb{Q}$$

as we wanted to show. This would conclude the proof.

Now we need to actually show that (1.4) will lead to a contradiction. Since this is our first proof, let us take it slowly, step-by-step.

1. Assuming (1.4) just means that there exists $q \in \mathbb{Q}$ such that

$$q = \sqrt{2}. \quad (1.5)$$

2. Since $q \in \mathbb{Q}$, by definition we have

$$q = \frac{m}{n}$$

for some $m, n \in \mathbb{N}$ with $n \neq 0$.

3. Recalling (1.5), we then have

$$\frac{m}{n} = \sqrt{2}.$$

4. We can square the above equation to get

$$\frac{m^2}{n^2} = 2. \quad (1.6)$$

5. **Withouth loss of generality**, we can **assume** that m and n have no common factors.

Wait. What does Step 5 mean? You will encounter the sentence *withouth loss of generality* many times in mathematics. It is often abbreviated in **WLOG**. WLOG means that the assumption that follows is chosen arbitrarily, but does not affect the validity of the proof in general.

For example in our case we can assume that m and n have no common factor. This is because if m and n had common factors, then it would mean

$$m = a\tilde{m}, \quad n = a\tilde{n}$$

for some $a \in \mathbb{N}$ with $a \neq 0$. Then

$$\frac{m}{n} = \frac{a\tilde{m}}{a\tilde{n}} = \frac{\tilde{m}}{\tilde{n}}.$$

Therefore by (1.6)

$$\frac{\tilde{m}^2}{\tilde{n}^2} = 2.$$

The proof can now proceed in the same way we would have proceeded from Step 4, but in addition we have the hypothesis that \tilde{m} and \tilde{n} have no common factors.

6. Equation (1.6) implies

$$m^2 = 2n^2. \tag{1.7}$$

Therefore the integer m^2 is an even number.

Why is m^2 even? As you already know, **even** numbers are

$$0, 2, 4, 6, 8, 10, 12, \dots$$

All these numbers have in common that they can be divided by 2, and so they can be written as

$$2p$$

for some $p \in \mathbb{N}$. For example 52 is even, because

$$52 = 2 \cdot 26.$$

Instead, **odd** numbers are

$$1, 3, 5, 7, 9, 11, \dots$$

These can be all written as

$$2p + 1$$

for some $p \in \mathbb{N}$. For example 53 is odd, because

$$53 = 2 \cdot 26 + 1.$$

7. Thus m is an even number, and so there exists $p \in \mathbb{N}$ such that

$$m = 2p. \tag{1.8}$$

Why is (1.8) true? Let us see what happens if we take the square of an even number $m = 2p$

$$m^2 = (2p)^2 = 4p^2 = 2(2p^2) = 2q.$$

Thus $m^2 = 2q$ for some $q \in \mathbb{N}$, and so m^2 is an even number. If instead m is odd, then $m = 2p + 1$ and

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

showing that also m^2 is odd.

This justifies Step 7: Indeed we know that m^2 is an even number from Step 6. If m was odd, then m^2 would be odd. Hence m must be even as well.

8. If we substitute (1.8) in (1.7) we get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2. \tag{1.9}$$

9. We now make a series of observations:

- Equation (1.9) says that n^2 is even.
- Step 6 says that m^2 is even.
- Therefore n and m are also even.

- Hence n and m have 2 as common factor.
- But Step 5 says that n and m have no common factors.
- **CONTRADICTION**

10. Our reasoning has run into a **contradiction**, starting from assumption (1.4), which says that

$$\sqrt{2} \in \mathbb{Q}.$$

Hence the above must be **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

Seeing that $\sqrt{2} \notin \mathbb{Q}$, we might be tempted to just fill in the gap by adding $\sqrt{2}$ to \mathbb{Q} . However,

with analogous proof to Theorem 1.6, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number p . As there are infinite prime numbers, this means that \mathbb{Q} has infinite gaps. Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : p \text{ prime}\}.$$

However even this is not enough, as we would still have numbers which are not contained in $\tilde{\mathbb{Q}}$, for example

$$\sqrt{2} + \sqrt{3}, \pi, \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}.$$

Remark 1.7

Proving that

$$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

is relatively easy, and will be left as an **exercise**. Instead, proving that

$$\pi \notin \mathbb{Q}$$

is way more complicated. There are several proof of the fact, all requiring mathematics which is more advanced of the one presented in this course. For some proofs, see this [Wikipedia page](#).

The reality of things is that to **complete** \mathbb{Q} and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in \mathbb{Q} . Such extension of \mathbb{Q} will be called \mathbb{R} , the set of **real numbers**. The inclusions will therefore be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The set \mathbb{R} is not at all trivial to construct. In fact, at first we will not construct it, but just do the following:

- We will assume that \mathbb{R} **exists** and satisfies some basic **axioms**.
- One of the axioms is that \mathbb{R} fills **all** the **gaps** that \mathbb{Q} has. Therefore \mathbb{R} can be thought as a **continuous** line.
- We will study the **properties** of \mathbb{R} which descend from such **axioms**.

For example one of the properties of \mathbb{R} will be the following:

Theorem 1.8: We will prove this in the future

\mathbb{R} contains all the square roots. This means that for every $x \in \mathbb{R}$ with $x \geq 0$, we have

$$\sqrt{x} \in \mathbb{R}.$$

At the end of this chapter we will provide a concrete **model** for the real numbers \mathbb{R} , to prove once and for all that such set indeed exists.

Theorem 1.9: We will prove this in the future

There exists a set \mathbb{R} , called the set of real numbers, which has the following properties:

- \mathbb{R} extends \mathbb{Q} , that is,

$$\mathbb{Q} \subset \mathbb{R}.$$

- \mathbb{R} satisfies certain **axioms**.
- \mathbb{R} fills **all** the **gaps** that \mathbb{Q} has. In particular \mathbb{R} can be represented by a **continuous** line.

2 Preliminaries

Before introducing \mathbb{R} we want to make sure that we cover all the basics needed for the task.

2.1 Sets

A set is a **collection** of objects. These objects are called **elements** of the set. For example in the previous section we mentioned the following sets: \mathbb{N} the set of natural numbers \mathbb{Z} the set of integers \mathbb{Q} the set of rational numbers \mathbb{R} the set of real numbers Given an arbitrary set A , we write

$$x \in A$$

if the element x belongs to the set (collection) A . If an element x is not contained in A , we say that

$$x \notin A.$$

Remark 2.1

A set can contain all sorts of elements. For example the students in a classroom can be modelled by a set S . The elements of the set are the students. For example

$$S = \{\text{Alice, Olivia, Jake, Sahab}\}$$

In this case we have

$$\text{Alice} \in S$$

but instead

$$\text{Silvio} \notin S.$$

If x is not an element of A , then we write $x \notin A$. Given two sets A and B , the union is written $A \cup B$ and is defined by asserting that $x \in A \cup B$ provided that $x \in A$ or $x \in B$ (or potentially both). The intersection $A \cap B$ is the set defined by the rule $x \in A \cap B$ provided $x \in A$ and $x \in B$.

3 Real Numbers: axioms

Coming soon

4 Sequences

Coming soon

5 Series

Coming soon

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