

Numbers, Sequences and Series

Lecture Notes

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Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

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Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

Readings

We will study the set of real numbers \mathbb{R} , and then sequences and series in \mathbb{R} . I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

! You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to know in order to excel in the final exam.

1 Introduction

The first aim of this lecture notes is to rigorously introduce the set of **real numbers**, which is denoted by \mathbb{R} . But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol \in denotes that m and n belong to \mathbb{N} . For example $3 + 7$ results in 10.

Question 1.1

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m? \tag{1.1}$$

Of course to invert (1.1) we can just perform a **subtraction**, implying that

$$x = m - n.$$

But there is a catch. In general x does not need to be in \mathbb{N} . For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to \mathbb{N} . Therefore the answer to Question 1.1 is **NO**.

To make sure that we can always invert the sum, we need to **extend** the set \mathbb{N} . This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : n \in \mathbb{N}\},$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The sum can be extended to \mathbb{Z} , by defining

$$(-n) + (-m) := -(m + n) \tag{1.2}$$

for all $m, n \in \mathbb{N}$. Now every element of \mathbb{Z} possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0.$$

Can we characterize m explicitly? Of course! Seeing the definition at (1.2), we simply have

$$m = -n.$$

On the set \mathbb{Z} we can also define the operation of **multiplication**, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm or $n \cdot m$. For example $7 \cdot 2 = 14$ and $1 \cdot (-1) = -1$.

Question 1.2

Can the multiplication in \mathbb{Z} be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m? \tag{1.3}$$

To invert (1.3) if $n \neq 0$, we can just perform a **division**, to obtain

$$x = \frac{m}{n}.$$

But again there is a catch. Indeed taking $n = 2$ and $m = 1$ yields $x = 1/2$, which does not belong to \mathbb{Z} . The answer to Question 1.2 is therefore **NO**.

Thus, in order to invert the multiplication, we need to **extend** the set of integers \mathbb{Z} . This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

We then extend the operations of sum and multiplication to \mathbb{Q} by defining

$$\frac{m}{n} + \frac{p}{q} := \frac{mq + np}{nq}$$

and

$$\frac{m}{n} \cdot \frac{p}{q} := \frac{mp}{nq}$$

Now the multiplication is invertible in \mathbb{Q} . Specifically, each non-zero element has an inverse: the inverse of m/n is given by n/m .

To summarize, we have extended \mathbb{N} to \mathbb{Z} , and \mathbb{Z} to \mathbb{Q} . By construction we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Moreover **sum** and **product** are **invertible** in \mathbb{Q} . Now we are happy right? So and so.

Question 1.3

Can we draw the set \mathbb{Q} ?

It is clear how to draw \mathbb{Z} , as seen below.

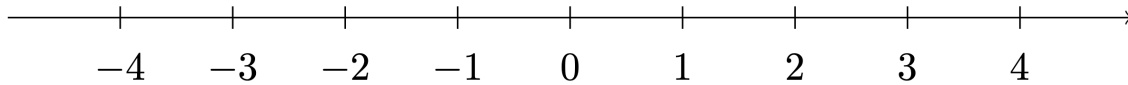


Figure 1.1: Representation of integers \mathbb{Z}

However \mathbb{Q} is much **larger** than the set \mathbb{Z} represented by the ticks in Figure 1.1. What do we mean by **larger**? For example, consider $0 \in \mathbb{Q}$.

Question 1.4

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, you can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n}.$$

For example think about the case of the numbers

$$\frac{1}{n} \text{ for } n \in \mathbb{N}, n \neq 0.$$

Such numbers get arbitrarily close to 0, as depicted below.

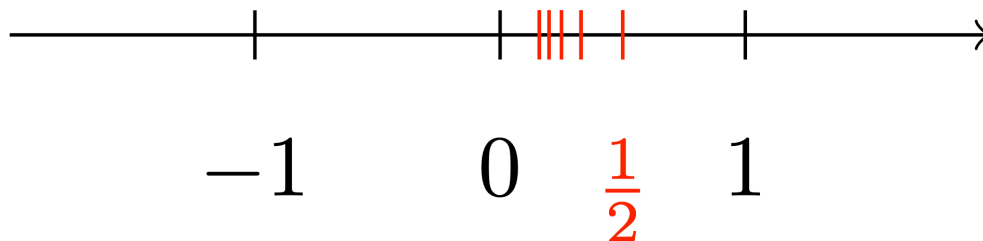


Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0, 1]$. In other words, we might conjecture the following.

Conjecture 1.5

Maybe \mathbb{Q} can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.5 is **false**, as shown by the Theorem below.

Theorem 1.6

The number $\sqrt{2}$ does not belong to \mathbb{Q} .

Theorem 1.6 is the reason why $\sqrt{2}$ is called an **irrational number**. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048 \dots$$

and the situation is as in the picture below.

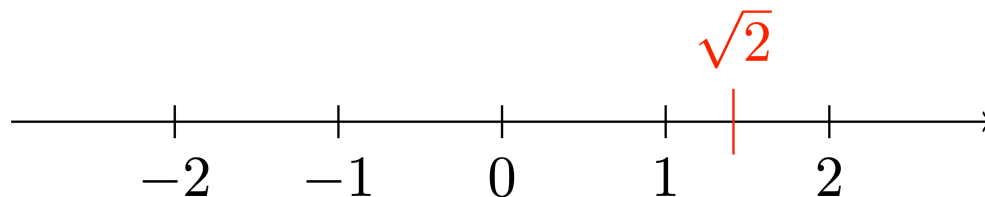


Figure 1.3: Representing $\sqrt{2}$ on the numbers line.

We can therefore see that Conjecture 1.5 is **false**, and \mathbb{Q} is not a line: indeed \mathbb{Q} has a **gap** at $\sqrt{2}$. Let us see why Theorem 1.6 is true.

Proof: Proof of Theorem 1.6

We prove that

$$\sqrt{2} \notin \mathbb{Q}$$

by **contradiction**.

Wait, what does this mean? Proving the claim by contradiction means assuming that the claim is **false**. This means we **assume** that

$$\sqrt{2} \in \mathbb{Q}. \quad (1.4)$$

From this assumption we then start deducing other statements, hoping to encounter a statement which is **FALSE**. But if (1.4) leads to a false statement, then it must be that (1.4) is **FALSE**. Thus the contrary of (1.4) must hold, meaning that

$$\sqrt{2} \notin \mathbb{Q}$$

as we wanted to show. This would conclude the proof.

Now we need to actually show that (1.4) will lead to a contradiction. Since this is our first proof, let us take it slowly, step-by-step.

1. Assuming (1.4) just means that there exists $q \in \mathbb{Q}$ such that

$$q = \sqrt{2}. \quad (1.5)$$

2. Since $q \in \mathbb{Q}$, by definition we have

$$q = \frac{m}{n}$$

for some $m, n \in \mathbb{N}$ with $n \neq 0$.

3. Recalling (1.5), we then have

$$\frac{m}{n} = \sqrt{2}.$$

4. We can square the above equation to get

$$\frac{m^2}{n^2} = 2. \quad (1.6)$$

5. **Withouth loss of generality**, we can **assume** that m and n have no common factors.

Wait. What does Step 5 mean? You will encounter the sentence *withouth loss of generality* many times in mathematics. It is often abbreviated in **WLOG**. WLOG means that the assumption that follows is chosen arbitrarily, but does not affect the validity of the proof in general.

For example in our case we can assume that m and n have no common factor. This is because if m and n had common factors, then it would mean

$$m = a\tilde{m}, \quad n = a\tilde{n}$$

for some $a \in \mathbb{N}$ with $a \neq 0$. Then

$$\frac{m}{n} = \frac{a\tilde{m}}{a\tilde{n}} = \frac{\tilde{m}}{\tilde{n}}.$$

Therefore by (1.6)

$$\frac{\tilde{m}^2}{\tilde{n}^2} = 2.$$

The proof can now proceed in the same way we would have proceeded from Step 4, but in addition we have the hypothesis that \tilde{m} and \tilde{n} have no common factors.

6. Equation (1.6) implies

$$m^2 = 2n^2. \quad (1.7)$$

Therefore the integer m^2 is an even number.

Why is m^2 even? As you already know, **even** numbers are

$$0, 2, 4, 6, 8, 10, 12, \dots$$

All these numbers have in common that they can be divided by 2, and so they can be written as

$$2p$$

for some $p \in \mathbb{N}$. For example 52 is even, because

$$52 = 2 \cdot 26.$$

Instead, **odd** numbers are

$$1, 3, 5, 7, 9, 11, \dots$$

These can be all written as

$$2p + 1$$

for some $p \in \mathbb{N}$. For example 53 is odd, because

$$53 = 2 \cdot 26 + 1.$$

7. Thus m is an even number, and so there exists $p \in \mathbb{N}$ such that

$$m = 2p. \tag{1.8}$$

Why is (1.8) true? Let us see what happens if we take the square of an even number $m = 2p$

$$m^2 = (2p)^2 = 4p^2 = 2(2p^2) = 2q.$$

Thus $m^2 = 2q$ for some $q \in \mathbb{N}$, and so m^2 is an even number. If instead m is odd, then $m = 2p + 1$ and

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

showing that also m^2 is odd.

This justifies Step 7: Indeed we know that m^2 is an even number from Step 6. If m was odd, then m^2 would be odd. Hence m must be even as well.

8. If we substitute (1.8) in (1.7) we get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2. \tag{1.9}$$

9. We now make a series of observations:

- Equation (1.9) says that n^2 is even.
- Step 6 says that m^2 is even.
- Therefore n and m are also even.
- Hence n and m have 2 as common factor.
- But Step 5 says that n and m have no common factors.
- **CONTRADICTION**

10. Our reasoning has run into a **contradiction**, starting from assumption (1.4), which says that

$$\sqrt{2} \in \mathbb{Q}.$$

Hence the above must be **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

Seeing that $\sqrt{2} \notin \mathbb{Q}$, we might be tempted to just fill in the gap by adding $\sqrt{2}$ to \mathbb{Q} . However, with analogous proof to Theorem 1.6, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number p . As there are infinite prime numbers, this means that \mathbb{Q} has infinite gaps. Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : p \text{ prime}\}.$$

However even this is not enough, as we would still have numbers which are not contained in $\tilde{\mathbb{Q}}$, for example

$$\sqrt{2} + \sqrt{3}, \pi, \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}.$$

Remark 1.7

Proving that

$$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

is relatively easy, and will be left as an **exercise**. Instead, proving that

$$\pi \notin \mathbb{Q}$$

is way more complicated. There are several proof of the fact, all requiring mathematics which is more advanced of the one presented in this course. For some proofs, see this [Wikipedia page](#).

The reality of things is that to **complete** \mathbb{Q} and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in \mathbb{Q} . Such extension of \mathbb{Q} will be called \mathbb{R} , the set of **real numbers**. The inclusions will therefore be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The set \mathbb{R} is not at all trivial to construct. In fact, at first we will not construct it, but just do the following:

- We will assume that \mathbb{R} **exists** and satisfies some basic **axioms**.
- One of the axioms is that \mathbb{R} fills **all** the **gaps** that \mathbb{Q} has. Therefore \mathbb{R} can be thought as a **continuous** line.
- We will study the **properties** of \mathbb{R} which descend from such **axioms**.

For example one of the properties of \mathbb{R} will be the following:

Theorem 1.8: We will prove this in the future

\mathbb{R} contains all the square roots. This means that for every $x \in \mathbb{R}$ with $x \geq 0$, we have

$$\sqrt{x} \in \mathbb{R}.$$

At the end of this chapter we will provide a concrete **model** for the real numbers \mathbb{R} , to prove once and for all that such set indeed exists.

Theorem 1.9: We will prove this in the future

There exists a set \mathbb{R} , called the set of real numbers, which has the following properties:

- \mathbb{R} extends \mathbb{Q} , that is,

$$\mathbb{Q} \subset \mathbb{R}.$$

- \mathbb{R} satisfies certain **axioms**.
- \mathbb{R} fills **all** the **gaps** that \mathbb{Q} has. In particular \mathbb{R} can be represented by a **continuous** line.

2 Preliminaries

Before introducing \mathbb{R} we want to make sure that we cover all the basics needed for the task.

2.1 Sets

A set is a **collection** of objects. These objects are called **elements** of the set. For example in the previous section we mentioned the following sets:

- \mathbb{N} the set of natural numbers
- \mathbb{Z} the set of integers
- \mathbb{Q} the set of rational numbers
- \mathbb{R} the set of real numbers

Given an arbitrary set A , we write

$$x \in A$$

if the element x belongs to the set A . If an element x is not contained in A , we say that

$$x \notin A.$$

Remark 2.1

A set can contain all sorts of elements. For example the students in a classroom can be modelled by a set S . The elements of the set are the students. For example

$$S = \{\text{Alice, Olivia, Jake, Sahab}\}$$

In this case we have

$$\text{Alice} \in S$$

but instead

$$\text{Silvio} \notin S.$$

2.2 Logic

In this section we introduce some basic logic symbols. Suppose that you are given two statements, say α and β . The formula

$$\alpha \implies \beta$$

means that α **implies** β . In other words, if α is true then also β is true.

The formula

$$\alpha \impliedby \beta$$

means that α is implied by β : if β is true then also α is true.

When we write

$$\alpha \iff \beta \tag{2.1}$$

we mean that α and β are **equivalent**. Note that (2.1) is equivalent to

$$\alpha \implies \beta \text{ and } \beta \implies \alpha.$$

Such equivalence is very useful in proofs.

Example 2.2

We have that

$$x > 0 \implies x > -100,$$

and

$$\text{contradiction} \impliedby \sqrt{2} \in \mathbb{Q}.$$

Concerning \iff we have

$$x^2 < 2 \iff -\sqrt{2} < x < \sqrt{2}.$$

We now introduce logic **quantifiers**. These are

- \forall which reads **for all**
- \exists which reads **exists**
- $\exists!$ which reads **exists unique**
- \nexists which reads **does not exists**

These work in the following way. Suppose that you are given a statement $\alpha(x)$ which depends on the point $x \in \mathbb{R}$. Then we say

- $\alpha(x)$ is satisfied for all $x \in A$ with A some collection of numbers. This translates to the symbols

$$\alpha(x) \text{ is true } \forall x \in A,$$

- There exists some x in \mathbb{R} such that $\alpha(x)$ is satisfied: in symbols

$$\exists x \in \mathbb{R} \text{ such that } \alpha(x) \text{ is true,}$$

- There exists a unique x_0 in \mathbb{R} such that $\alpha(x)$ is satisfied: in symbols

$$\exists! x_0 \in \mathbb{R} \text{ such that } \alpha(x_0) \text{ is true,}$$

- $\alpha(x)$ is never satisfied:

$$\nexists x \in \mathbb{R} \text{ such that } \alpha(x) \text{ is true.}$$

Example 2.3

Let us make concrete examples:

- The expression x^2 is always non-negative. Thus we can say

$$x^2 \geq 0 \text{ for all } x \in \mathbb{R}.$$

- The equation $x^2 = 1$ has two solutions $x = 1$ and $x = -1$. Therefore we can say

$$\exists x \in \mathbb{R} \text{ such that } x^2 = 1.$$

- The equation $x^3 = 1$ has a unique solution $x = 1$. Thus

$$\exists! x \in \mathbb{R} \text{ such that } x^3 = 1.$$

- We know that the equation $x^2 = 2$ has no solutions in \mathbb{Q} . Then

$$\nexists x \in \mathbb{Q} \text{ such that } x^2 = 2.$$

2.3 Operations on sets

2.3.1 Union and intersection

For two sets A and B we define their **union** as the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

The **intersection** of A and B is defined by

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

We denote the **empty set** by the symbol \emptyset . Two sets are **disjoint** if

$$A \cap B = \emptyset.$$

Example 2.4

Define the subset of rational numbers

$$S := \left\{ x \in \mathbb{Q} : 0 < x < \frac{5}{2} \right\}.$$

Then we have

$$\mathbb{N} \cap S = \{1, 2\}.$$

We can also define the sets of **even** and **odd** numbers by

$$E := \{2n : n \in \mathbb{N}\}, \quad (2.2)$$

$$O := \{2n + 1 : n \in \mathbb{N}\}. \quad (2.3)$$

Then we have

$$\mathbb{N} \cap E = E, \quad \mathbb{N} \cap O = O, \quad (2.4)$$

$$O \cup E = \mathbb{N}, \quad O \cap D = \emptyset. \quad (2.5)$$

2.3.2 Inclusion and equality

Given two sets A and B , we say that A is **contained** in B if all the elements of A are also contained in B . This will be denoted with the **inclusion** symbol \subset , that is,

$$A \subset B.$$

In this case we say that

- A is a **subset** of B ,
- B is a **superset** of A .

The inclusion $A \subset B$ is equivalent to the implication:

$$x \in A \implies x \in B$$

for all $x \in A$. The symbol \implies reads **implies**, and denotes the fact that the first condition implies the second.

Example 2.5

Given two sets A and B we always have

$$(A \cap B) \subset A, (A \cap B) \subset B, \quad (2.6)$$

$$A \subset (A \cup B), B \subset (A \cup B). \quad (2.7)$$

We say that two sets A and B are equal if they contain the **same** elements. We denote equality by the symbol

$$A = B.$$

Example 2.6

The sets

$$A = \{1, 2, 3\}$$

and

$$B = \{3, 1, 2\}$$

are equal. This is because they contain exactly the same elements: **order** does not matter when talking about sets.

Proposition 2.7

Let A and B be sets. Then

$$A = B$$

if and only if

$$A \subset B \text{ and } B \subset A.$$

Proof

The proof is almost trivial. However it is a good exercise in basic logic, so let us do it.

1. First implication \Rightarrow :

Suppose that $A = B$. Let us show that $A \subset B$. Since $A = B$, this means that all the elements of A are also contained in B . Therefore if we take $x \in A$ we have

$$x \in A \Rightarrow x \in B.$$

This shows $A \subset B$. The proof of $B \subset A$ is similar.

2. Second implication \Leftarrow :

Suppose that $A \subset B$ and $B \subset A$. We need to show $A = B$, that is, A and B have the same

elements. To this end let $x \in A$. Since $A \subset B$ then we have $x \in B$. Thus B contains all the elements of A . Since we are also assuming $B \subset A$, this means that A contains all the elements of B . Hence A and B contain the same elements, and $A = B$.

The above proposition is very useful when we need to **prove** that two sets are equal: rather than showing directly that $A = B$, we can prove that $A \subset B$ and $B \subset A$.

2.3.3 Infinite unions and intersections

Suppose given a set Ω , and a family of sets $A_n \subset \Omega$, where $n \in \mathbb{N}$. Then we can define the **infinite union**

$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for at least one } n \in \mathbb{N}\}.$$

The **infinite intersection** is defined as

$$\bigcap_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

Example 2.8

Let the ambient set be $\Omega := \mathbb{N}$ and define the family A_n by

$$A_1 := \{1, 2, 3, 4, \dots\} \quad (2.8)$$

$$A_2 := \{2, 3, 4, 5, \dots\} \quad (2.9)$$

$$A_3 := \{3, 4, 5, 6, \dots\} \quad (2.10)$$

$$\dots \dots \quad (2.11)$$

$$A_n := \{n, n+1, n+2, n+3, \dots\}, \quad (2.12)$$

for arbitrary $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}. \quad (2.13)$$

The above equality can be easily proven using Proposition 2.7. Indeed, assume that $m \in \bigcup_n A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subset \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows

$$\bigcup_{n \in \mathbb{N}} A_n \subset \mathbb{N}.$$

Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index n , $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \bigcup_{n \in \mathbb{N}} A_n$, showing that

$$\mathbb{N} \subset \bigcup_{n \in \mathbb{N}} A_n.$$

Hence we conclude (2.13) by Proposition 2.7.

We also have that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset. \quad (2.14)$$

We prove the above by **contradiction**. Indeed, suppose that (2.14) is false, i.e.,

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset.$$

This means there exists some $m \in \mathbb{N}$ such that $m \in \bigcap_{n \in \mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (2.14) holds.

2.3.4 Complement

Suppose that A and B are subsets of a larger set Ω . The **complement** of A with respect to B is the set of elements of B which do not belong to A , that is

$$B \setminus A := \{x \in \Omega : x \in B \text{ and } x \notin A\}.$$

In particular, the complement of A with respect to Ω is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : x \notin A\}.$$

Remark 2.9

Suppose that $A \subset \Omega$. Then A and A^c form a **partition** of Ω , in the sense that

$$A \cup A^c = \Omega \quad \text{and} \quad A \cap A^c = \emptyset.$$

Example 2.10

Suppose $A, B \subset \Omega$. Then

$$A \subset B \iff B^c \subset A^c.$$

Let us prove the above claim:

- First implication \implies :
Suppose that $A \subset B$. We need to show that $B^c \subset A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subset B$, hence $x \in B$. But we had assumed $x \in B^c$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subset A^c$.

- Second implication \Leftarrow :
Essentially the same proof, hence we omit it.

We conclude by stating the De Morgan's Laws. The proof will be left as an exercise.

Proposition 2.11: De Morgan's Laws

Suppose $A, B \subset \Omega$. Then

$$(A \cap B)^c = A^c \cup B^c$$

and

$$(A \cup B)^c = A^c \cap B^c.$$

2.3.5 Product of sets

Suppose A and B are two sets. The **product** of A and B is the set of pairs

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

By definition two elements in $A \times B$ are the same, in symbols

$$(a, b) = (\tilde{a}, \tilde{b})$$

if and only if they are equal component-by-component, that is

$$a = \tilde{a}, \quad b = \tilde{b}.$$

2.4 Equivalence relation

Suppose A is a set. A **binary relation** R on A is a subset

$$R \subset A \times A.$$

Definition 2.12: Equivalence relation

A binary relation R is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive:** For each $x \in A$ one has

$$(x, x) \in R,$$

This is saying that all the elements in A must be related to themselves

2. **Symmetric:** We have

$$(x, y) \in R \implies (y, x) \in R$$

If x is related to y , then y is related to x

3. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

If x is related to y , and y is related to z , then x must be related to z

If $(x, y) \in R$ we write

$$x \sim y$$

and we say that x and y are **equivalent**.

Definition 2.13: Equivalence classes

Suppose R is an **equivalence relation** on A . The **equivalence class** of an element $x \in A$ is the set

$$[x] := \{y \in A : y \sim x\}.$$

The set of equivalence classes of elements of A with respect to the equivalence relation R is denoted by

$$A/R := \{[x] : x \in A\}.$$

Let us immediately clarify the above definitions by considering the prototypical example of equivalence relation: the **equality**.

Example 2.14: Equality is an equivalence relation

Consider the set of natural numbers \mathbb{N} . The equality defines a **binary relation** on $\mathbb{N} \times \mathbb{N}$, via

$$R := \{(x, y) \in \mathbb{N} \times \mathbb{N} : x = y\}.$$

Let us check that R is an **equivalence relation**:

1. Reflexive: It holds, since $x = x$ for all $x \in \mathbb{N}$,
2. Symmetric: Again $x = y$ if and only if $y = x$,

3. Transitive: If $x = y$ and $y = z$ then $x = z$.

The class of equivalence of $x \in \mathbb{N}$ is given by

$$[x] = \{x\},$$

that is, this relation is quite trivial, given that each element of \mathbb{N} can only be related to itself. The quotient space is then

$$\mathbb{N}/R = \{[x] : x \in \mathbb{N}\} = \{\{x\} : x \in \mathbb{N}\}.$$

Example 2.15

Suppose that R is a binary relation on the set \mathbb{Q} of rational numbers defined by

$$x \sim y \iff x - y \in \mathbb{Z}.$$

Then R is an equivalence relation on \mathbb{Q} . Indeed:

1. Reflexive: Let $x \in \mathbb{Q}$. Then $x - x = 0$ and $0 \in \mathbb{Z}$. Thus $x \sim x$.
2. Symmetric: If $x \sim y$ then $x - y \in \mathbb{Z}$. But then also

$$-(x - y) = y - x \in \mathbb{Z}$$

and so $y \sim x$.

3. Transitive: Suppose $x \sim y$ and $y \sim z$. Then

$$x - y \in \mathbb{Z} \text{ and } y - z \in \mathbb{Z}.$$

Thus we have

$$x - z = (x - y) + (y - z) \in \mathbb{Z}$$

showing that $x \sim z$. This shows that R is an equivalence relation on \mathbb{Q} .

Now note that

$$y \sim x \iff y - x \in \mathbb{Z}$$

and the above is equivalent to

$$\exists n \in \mathbb{Z} \text{ s.t. } y - x = n$$

which again is equivalent to

$$\exists n \in \mathbb{Z} \text{ s.t. } y = x + n.$$

Therefore all the elements of \mathbb{Q} related to x by R are of the form

$$x + n, \forall n \in \mathbb{Z}.$$

The equivalence classes with respect to R are then

$$[x] = \{x + n : n \in \mathbb{Z}\}.$$

Each equivalence class has exactly one element in $[0, 1) \cap \mathbb{Q}$, meaning that:

$$\forall x \in \mathbb{Q}, \exists! q \in \mathbb{Q} \text{ s.t. } 0 \leq q < 1 \text{ and } q \in [x].$$

Therefore

$$\mathbb{Q}/R = \{[x] : x \in \mathbb{Q}\} = \{q \in \mathbb{Q} : 0 \leq q < 1\}.$$

2.5 Order relation

Similarly, we define **order relations**.

Definition 2.16: Order relation

A binary relation R is called an **order relation** if it satisfies the following properties:

1. **Reflexive:** For each $x \in A$ one has

$$(x, x) \in R,$$

2. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

3. **Antisymmetric:** We have

$$(x, y) \in R \text{ and } (y, x) \in R \implies x = y$$

This is the only new condition with respect to the definition of equivalence relation.

The prototypical example of order relation is the **inequality** relation.

Example 2.17: Inequality is an order relation

Consider the set of integers \mathbb{Z} . The inequality defines a **binary relation** on $\mathbb{Z} \times \mathbb{Z}$, via

$$R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \leq y\}.$$

Let us check that R is an **order relation**:

1. Reflexive: It holds, since $x \leq x$ for all $x \in \mathbb{Z}$,
2. Transitive: If $x \leq y$ and $y \leq z$ then $x \leq z$.

3. Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$.

2.6 Intervals

In this section we assume to have available the set \mathbb{R} of **real numbers**, which we recall is an extension of \mathbb{Q} . We now introduce the concept of **interval**.

Definition 2.18

Let $a, b \in \mathbb{R}$ with $a < b$. We define the **open interval** (a, b) as the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

We define the **close interval** $[a, b]$ as the set

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

In general we also define the intervals

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}, \quad (2.15)$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}, \quad (2.16)$$

$$(a, \infty) := \{x \in \mathbb{R} : x > a\}, \quad (2.17)$$

$$[a, \infty) := \{x \in \mathbb{R} : x \geq a\}, \quad (2.18)$$

$$(-\infty, b) := \{x \in \mathbb{R} : x < b\}, \quad (2.19)$$

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}. \quad (2.20)$$

Some of the above intervals are depicted in Figure 2.1, Figure 2.2, Figure 2.3, Figure 2.4 below.



Figure 2.1: Interval (a, b)



Figure 2.2: Interval $[a, b]$

Figure 2.3: Interval (a, ∞) Figure 2.4: Interval $(-\infty, b]$

2.7 Absolute value or Modulus

In this section we assume to have available the set \mathbb{R} of **real numbers**, which we recall is an extension of \mathbb{Q} .

Definition 2.19: Absolute value

For $x \in \mathbb{R}$ we define its **absolute value** as the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 2.20

By definition one has $|x| = x$ if $x \geq 0$. For example

$$|\pi| = \pi, \quad |\sqrt{2}| = \sqrt{2}, \quad |0| = 0.$$

Instead $|x| = -x$ if $x < 0$. For example

$$|-\pi| = \pi, \quad |-\sqrt{2}| = \sqrt{2}, \quad |-10| = 10.$$

Let us also make the following basic remark, whose proof will be left as an exercise.

Remark 2.21

For all $x \in \mathbb{R}$ one has

$$|x| \geq 0.$$

Moreover

$$|x| = 0 \iff x = 0.$$

Another basic remark (proof by exercise).

Remark 2.22

For all $x \in \mathbb{R}$ one has

$$|x| = |-x|.$$

You might be familiar with the graph associated to the absolute value function:

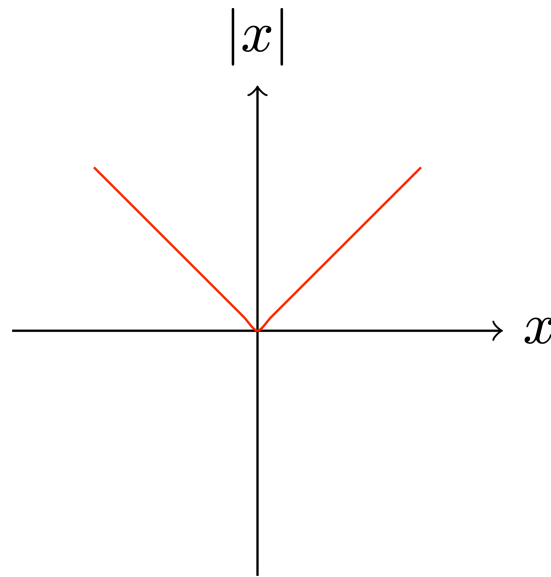


Figure 2.5: Plot of the absolute value function $f(x) = |x|$

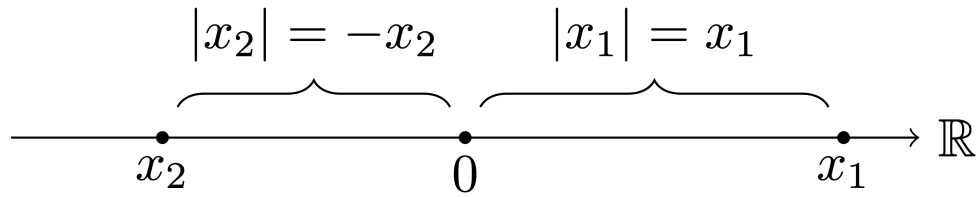
However in these Lecture Notes we are not dealing with functions, so it is better to think about the absolute value in a geometric way.

Remark 2.23: Geometric interpretation of $|x|$

A number $x \in \mathbb{R}$ can be represented with a point on the real line \mathbb{R} . The non-negative number $|x|$ represents the **distance** of x from the origin 0. Notice that this works for both positive and negative numbers x_1 and x_2 respectively, as shown in Figure 2.6 below.

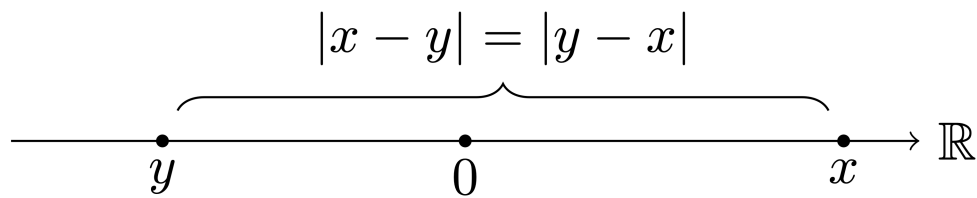
Remark 2.24: Geometric interpretation of $|x - y|$

If $x, y \in \mathbb{R}$ then the number $|x - y|$ represents the distance between x and y on the real line, as

Figure 2.6: Geometric interpretation of $|x|$

shown in Figure 2.7 below. Note that by Remark 2.22 we have

$$|x - y| = |y - x|.$$

Figure 2.7: Geometric interpretation of $|x - y|$

In the next Lemma we show a fundamental equivalence regarding the absolute value.

Lemma 2.25

Let $x, y \in \mathbb{R}$. Then

$$|x| \leq y \iff -y \leq x \leq y.$$

The geometric meaning of the above statement is clear: the distance of x from the origin is less than y , in formulae

$$|x| \leq y,$$

if and only if x belongs to the interval $[-y, y]$, in formulae

$$-y \leq x \leq y.$$

A sketch of this explanation is seen in Figure 2.8 below.

Proof: Proof of Lemma 2.25

We divide the proof in steps.

- Step 1: implication \implies

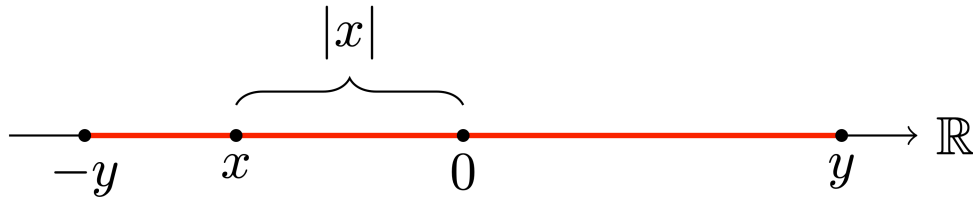


Figure 2.8: Geometric meaning of Lemma 2.25

Suppose first that

$$|x| \leq y. \quad (2.21)$$

Recalling that the absolute value is non-negative, from (2.21) we deduce that $0 \leq |x| \leq y$. In particular it holds

$$y \geq 0. \quad (2.22)$$

We make separate arguments for the cases $x \geq 0$ and $x < 0$:

- Case 1: $x \geq 0$. From (2.21), (2.22) and from $x \geq 0$ we have

$$-y \leq 0 \leq x = |x| \leq y$$

which shows

$$-y \leq x \leq y.$$

- Case 2: $x < 0$. From (2.21), (2.22) and from $x < 0$ we have

$$-y \leq 0 < -x = |x| \leq y$$

which shows

$$-y \leq -x \leq y.$$

Multiplying the above inequalities by -1 yields

$$-y \leq x \leq y.$$

- Step 2: implication \Leftarrow
Suppose now that

$$-y \leq x \leq y. \quad (2.23)$$

We make separate arguments for the cases $x \geq 0$ and $x < 0$:

- Case 1: $x \geq 0$. Since $x \geq 0$, from (2.23) we get

$$|x| = x \leq y$$

showing that

$$|x| \leq y.$$

– Case 2: $x < 0$. Since $x < 0$, from (2.23) we have

$$-y \leq x = -|x|.$$

Multiplying the above inequality by -1 yields

$$|x| \leq y.$$

With basically the same arguments, one can also show the following.

Lemma 2.26

Let $x, y \in \mathbb{R}$. Then

$$|x| < y \iff -y < x < y.$$

2.8 Triangle inequality

The triangle inequality relates the absolute value to the sum operation. It is a very important inequality, which we will use a lot in the future.

Theorem 2.27: Triangle inequality

For every $x, y \in \mathbb{R}$ we have

$$||x| - |y|| \leq |x - y| \leq |x| + |y|. \quad (2.24)$$

Before proceeding with the proof, let us discuss the geometric meaning of the triangle inequality.

Remark 2.28: Geometric meaning of triangle inequality

The notion of absolute value can be extended also to vectors in the plane. Suppose that x and y are two vectors in the plane, as in Figure 2.9 below. Then $|x|$ and $|y|$ can be interpreted as the **lengths** of these vectors.

Using the rule of sum of vectors, we can draw $x + y$, as shown in Figure 2.10 below. From the picture it is evident that

$$|x + y| \leq |x| + |y|, \quad (2.25)$$

that is, *the length of each side of a triangle does not exceed the sum of the lengths of the two remaining sides*. Note that (2.25) is exactly the second inequality in (2.24). This is why (2.24) is

called triangle inequality.

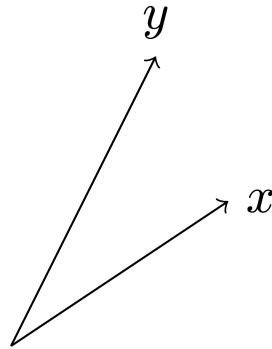


Figure 2.9: Vectors x and y

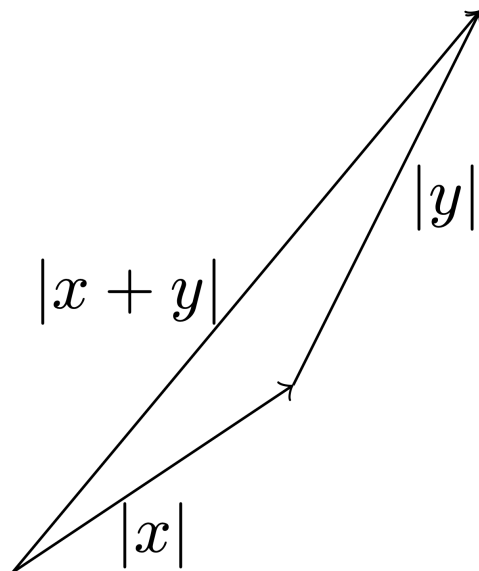


Figure 2.10: Summing the vectors x and y . The triangle inequality relates the length of $x + y$ to the length of x and y

Proof: Proof of Theorem 2.27

Assume that $x, y \in \mathbb{R}$. We prove the two inequalities in (2.24) individually.

- Proof of the second inequality in (2.24):

Trivially we have

$$|x| \leq |x|.$$

Therefore we can apply Lemma 2.25 and infer

$$-|x| \leq x \leq |x|. \quad (2.26)$$

Similarly we have that $|y| \leq |y|$, and so Lemma 2.25 implies

$$-|y| \leq y \leq |y|. \quad (2.27)$$

Summing (2.26) and (2.27) we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

We can now again apply Lemma 2.25 to get

$$|x + y| \leq |x| + |y|, \quad (2.28)$$

which is the second inequality in (2.24).

- Proof of the second inequality in (2.24):

Note that the trivial identity

$$x = x + y - y$$

always holds. We then have

$$|x| = |x + y - y| \quad (2.29)$$

$$= |(x + y) + (-y)| \quad (2.30)$$

$$= |a + b| \quad (2.31)$$

with $a = x + y$ and $b = -y$. We can now apply (2.28) to a and b to obtain

$$|x| = |a + b| \quad (2.32)$$

$$\leq |a| + |b| \quad (2.33)$$

$$= |x + y| + |-y| \quad (2.34)$$

$$= |x + y| + |y| \quad (2.35)$$

Therefore

$$|x| - |y| \leq |x + y|. \quad (2.36)$$

We can now swap x and y in (2.36) to get

$$|y| - |x| \leq |x + y|.$$

By rearranging the above inequality we obtain

$$-|x + y| \leq |x| - |y|. \quad (2.37)$$

Putting together (2.36) and (2.37) yields

$$-|x + y| \leq |x| - |y| \leq |x + y|.$$

By Lemma 2.25 the above is equivalent to

$$||x| - |y|| \leq |x + y|,$$

which is the first inequality in (2.24).

An immediate consequence of the triangle inequality are the following inequalities, which are left as an exercise.

Remark 2.29

For any $x, y \in \mathbb{R}$ it holds

$$||x| - |y|| \leq |x - y| \leq |x| + |y|.$$

Moreover for any $x, y, z \in \mathbb{R}$ it holds

$$|x - y| \leq |x - z| + |z - y|.$$

2.9 Proofs in Mathematics

In this section we carry out the proof of a seemingly trivial statement, to get used to the process. In a proof one needs to show that

$$\alpha \implies \beta \tag{2.38}$$

where

- α is a given set of assumptions, or **hypothesis**
- β is a conclusion, or **thesis**

To show (2.38) we need to convince ourselves that β follows by assuming α .

Common strategies to prove (2.38) are:

- **Contradiction:** Assume that the thesis is **false**, and hope to reach a contradiction: that is, prove that

$$\neg\beta \implies \text{contradiction}$$

where $\neg\beta$ is the **negation** of β . For example we already proved by contradiction that

$$\text{Definition of } \mathbb{Q} \implies \sqrt{2} \notin \mathbb{Q},$$

In the above statement

$$\alpha \rightsquigarrow \text{Definition of } \mathbb{Q}.$$

$$\beta \rightsquigarrow \sqrt{2} \notin \mathbb{Q}.$$

Therefore

$$\neg\beta \rightsquigarrow \sqrt{2} \in \mathbb{Q}.$$

- **Direct:** Sometimes proofs will also need **direct** arguments, meaning that one need to show directly that (2.38).

- **Contrapositive:** The statement is equivalent to (2.38)

$$\neg\beta \implies \neg\alpha. \quad (2.39)$$

Thus, instead of proving (2.38) one could instead show (2.39). The statement (2.39) is called the **contrapositive** of (2.38).

Let us make an example.

Proposition 2.30

Two real numbers a, b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a - b| < \varepsilon$.

Before proceeding with the proof, note that the above statement is just saying that:

Two numbers are equal if and only if they are **arbitrarily** close

By *arbitrarily close* we mean that they are *as close as you want them to be*.

Proof: of Proposition 2.30

Let us first rephrase the statement using mathematical symbols:

Let $a, b \in \mathbb{R}$. Then it holds:

$$a = b \iff |a - b| < \varepsilon, \forall \varepsilon > 0.$$

Setting

$$\alpha \rightsquigarrow a = b \quad (2.40)$$

$$\beta \rightsquigarrow |a - b| < \varepsilon, \forall \varepsilon > 0 \quad (2.41)$$

the statement is equivalent to

$$\alpha \iff \beta.$$

To show the above, it is sufficient to show that

$$\alpha \implies \beta$$

and

$$\beta \implies \alpha.$$

Step 1. Proof that $\alpha \implies \beta$:

This proof can be carried out by a **direct** argument. Since we are assuming α , this means

$$a = b.$$

We want to see that β holds. Therefore fix an arbitrary $\varepsilon > 0$. This means that ε can be **any** positive number. Clearly

$$|a - b| = |0| = 0 < \varepsilon$$

since $a = b$, $|0| = 0$, and $\varepsilon > 0$. The above shows that

$$|a - b| < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we have just proven that

$$|a - b| < \varepsilon, \quad \forall \varepsilon > 0,$$

meaning that β holds and the proof is concluded.

Step 2. Proof that $\beta \implies \alpha$:

Let us prove this implication by showing the **contrapositive**

$$\neg\alpha \implies \neg\beta.$$

So let us assume $\neg\alpha$ is true. This means that

$$a \neq b.$$

We have to see that $\neg\beta$ holds. But $\neg\beta$ means that

$$\exists \varepsilon_0 > 0 \text{ s.t. } |a - b| \geq \varepsilon_0.$$

The above is satisfied by choosing

$$\varepsilon_0 := |a - b|,$$

since $\varepsilon_0 > 0$ given that $a \neq b$,

2.10 Induction

Another technique for carrying out proofs is **induction**, which we take as an axiom.

Axiom 2.31: Principle of Induction

Let S be a subset of \mathbb{N} . Suppose that

1. We have $1 \in S$, and
2. Whenever $n \in S$, then $(n + 1) \in S$.

Then we have

$$S = \mathbb{N}.$$

Important

The above is an **axiom**, meaning that we do not prove it, but rather we just **assume it holds**.

Remark 2.32

It would be possible to prove the Principle of Induction starting from elementary axioms for \mathbb{N} , called the **Peano Axioms**, see the [Wikipedia page](#).

However, in justifying basic principles of mathematics, one at some point needs to draw a line. This means that something which looks elementary needs to be assumed to hold, in order to have a starting point for proving deeper statements.

In the case of the Principle of Induction, the intuition is clear:

The Principle of Induction is just describing the **domino effect**: *If one tile falls, then the next one will fall as well. Therefore if the first tile falls, all the tiles will fall.*

It seems reasonable to assume such evident principle.

The Principle of Induction can be used to prove statements which depend on some index $n \in \mathbb{N}$. Precisely, the following statement holds.

Corollary 2.33: Principle of Induction - Alternative formulation

Let $\alpha(n)$ be a statement which depends on $n \in \mathbb{N}$. Suppose that

1. $\alpha(1)$ is true, and
2. Whenever $\alpha(n)$ is true, then $\alpha(n+1)$ is true.

Then $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Proof

Define the set

$$S := \{n \in \mathbb{N} \text{ s.t. } \alpha(n) \text{ is true}\}.$$

Then

1. We have $1 \in S$, since $\alpha(1)$ is true.
2. If $n \in S$ then $\alpha(n)$ is true. By assumption this implies that $\alpha(n+1)$ is true. Therefore $(n+1) \in S$.

Therefore S satisfies the assumptions of the Induction Principle and we conclude that

$$S = \mathbb{N}.$$

By definition this means that $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Example 2.34: Formula for summing first n natural numbers

Using the Principle of Induction we can prove that

$$1 + 2 + 3 + \dots + (n-1) + n = \frac{n(n+1)}{2} \quad (2.42)$$

holds for all $n \in \mathbb{N}$.

Proof. To be really precise, consider the statement

$$\alpha(n) := \text{the above formula is true for } n.$$

In order to apply induction, we need to show that

1. $\alpha(1)$ is true,
2. If $\alpha(n)$ is true then $\alpha(n+1)$ is true.

Let us proceed: 1. It is immediate to check that (2.42) holds for $n = 1$. 2. Suppose (2.42) holds for n . Then

$$1 + \dots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) \quad (2.43)$$

$$= \frac{n(n+1) + 2(n+1)}{2} \quad (2.44)$$

$$= \frac{(n+1)(n+2)}{n+1} \quad (2.45)$$

where in the first equality we used that (2.42) holds for n . We then have

$$1 + \dots + n + (n+1) = \frac{(n+1)(n+2)}{n+1},$$

which shows that (2.42) holds for $n+1$.

By the Principle of Induction we then conclude that $\alpha(n)$ is true for all $n \in \mathbb{N}$, which means that (2.42) holds for all $n \in \mathbb{N}$.

Example 2.35: Statements about sequences of numbers

Suppose you are given a collection of numbers

$$\{x_n \text{ s.t. } n \in \mathbb{N}\}.$$

Such collection of numbers is called **sequence**. Assume that

$$x_1 = 1$$

and that

$$x_{n+1} := \frac{x_n}{2} + 1.$$

A sequence defined as above is called **recurrence sequence**. Using the above rule we can compute all the terms of x_n : for example

$$x_2 = \frac{x_1}{2} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$$

and

$$x_3 = \frac{x_2}{2} + 1 = \frac{3}{4} + 1 = \frac{7}{4}.$$

By computing these terms we notice that the sequence might be increasing. Indeed we can prove by induction that

$$x_{n+1} \geq x_n \tag{2.46}$$

for all $n \in \mathbb{N}$.

Proof. By induction:

1. We have seen that $x_1 = 1$ and $x_2 = 3/2$. Thus

$$x_2 \geq x_1.$$

2. Suppose now that

$$x_{n+1} \geq x_n.$$

We need to prove that

$$x_{n+2} \geq x_{n+1}.$$

Indeed, we can multiply the inequality $x_{n+1} \geq x_n$ by $1/2$ and add 1 to get

$$\frac{x_{n+1}}{2} + 1 \geq \frac{x_n}{2} + 1.$$

The above is equivalent, by definition, to $x_{n+2} \geq x_{n+1}$.

Therefore the assumptions of the Induction Principle are satisfied, and (2.46) follows.

3 Real Numbers

In Chapter 1 we proved that

$$\sqrt{2} \notin \mathbb{Q},$$

where \mathbb{Q} is the set of rational numbers. Therefore \mathbb{Q} has a **gap**, and cannot be represented as a **continuous** line. Let us make this concept more precise, by giving a mathematical definition of **gap**.

3.1 Cut Property

The rigorous definition of **gap** is given in terms of **cuts** of a set.

Definition 3.1: Cut of a set

Suppose that

4 Sequences

Coming soon

4.1 Example: Heron's Method

The first explicit algorithm for approximating

$$\sqrt{x}$$

for $x > 0$ is known **Heron's method**, after the first-century Greek mathematician [Heron of Alexandria](#) who described the method in his AD 60 work *Metrica*, see reference to [Wikipedia page](#).

Let us see what is the idea of the algorithm:

- Suppose that a_1 is an approximation of \sqrt{x} from above, that is,

$$\sqrt{x} < a_1. \tag{4.1}$$

- Multiplying (4.1) by \sqrt{x}/a_1 we obtain

$$\frac{x}{a_1} < \sqrt{x}, \tag{4.2}$$

obtaining an approximation of \sqrt{x} from below.

- Therefore, putting together the above inequalities,

$$\frac{x}{a_1} < \sqrt{x} < a_1 \tag{4.3}$$

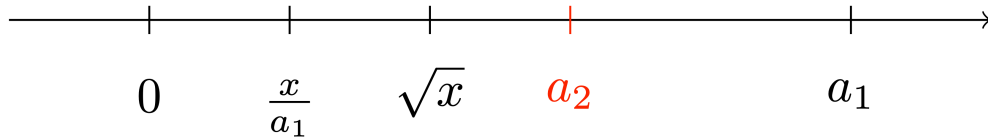
- If we take the average of the points x/a_1 and a_1 , it is reasonable to think that we find a better approximation of \sqrt{x} . Thus our next approximation is

$$a_2 := \frac{1}{2} \left(a_1 + \frac{x}{a_1} \right),$$

see figure below.

Iterating, we define by recurrence the sequence

$$a_{n+1} := \frac{1}{2} \left(a_n + \frac{x}{a_n} \right)$$

Figure 4.1: Heron's Algorithm for approximating \sqrt{x}

for all $n \in \mathbb{N}$, where the initial guess a_1 has to satisfy (4.1). The aim of the section is to show that

$$\lim_{n \rightarrow \infty} a_n = \sqrt{x}. \quad (4.4)$$

We start by showing that (4.3) holds for all $n \in \mathbb{N}$.

Proposition 4.1

We have

$$\frac{x}{a_n} < \sqrt{x} < a_n \quad (4.5)$$

for all $n \in \mathbb{N}$.

Proof

We prove it by induction:

1. By (4.1) and (4.2) we know that (4.5) holds for $n = 1$.
2. Suppose now that (4.5) holds for n . Then

$$a_{n+1} - \sqrt{x} = \frac{1}{2} \left(a_n + \frac{x}{a_n} \right) - \sqrt{x} \quad (4.6)$$

$$= \frac{1}{2a_n} (a_n^2 + x - 2a_n\sqrt{x}) \quad (4.7)$$

$$= \frac{1}{2a_n} (a_n - \sqrt{x})^2 > 0, \quad (4.8)$$

since we are assuming that $a_n > \sqrt{x}$. Therefore

$$\sqrt{x} < a_{n+1}. \quad (4.9)$$

Multiplying the above by \sqrt{x}/a_{n+1} we get

$$\frac{x}{a_{n+1}} < \sqrt{x}. \quad (4.10)$$

Inequalities (4.9) and (4.10) show that (4.5) holds for $n + 1$.

Therefore we conclude (4.5) by the Principle of Induction.

We are now ready to prove error estimates, that is, estimating how far away a_n is from \sqrt{x} .

Proposition 4.2: Error estimate

For all $n \in \mathbb{N}$ we have

$$a_{n+1} - \sqrt{x} < \frac{1}{2}(a_n - \sqrt{x}). \quad (4.11)$$

Proof

By Proposition 4.1 we know that

$$\frac{x}{a_n} < \sqrt{x}$$

for all $n \in \mathbb{N}$. Therefore

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{x}{a_n} \right) \quad (4.12)$$

$$< \frac{1}{2} (a_n + \sqrt{x}). \quad (4.13)$$

Subtracting \sqrt{x} from both members in the above inequality we get the thesis.

Inequality (4.11) is saying that the error halves at each step. Therefore we can prove that after n steps the error is exponentially lower, as detailed in the following proposition.

Proposition 4.3

For all $n \in \mathbb{N}$ we have

$$a_{n+1} - \sqrt{x} < \frac{1}{2^n} (a_1 - \sqrt{x}) \quad (4.14)$$

Proof

We prove (4.14) by induction:

1. For $n = 1$ we have that (4.14) is satisfied, since it coincides with (4.11) for $n = 1$.

2. Suppose that (4.14) holds for n . By (4.11) with n replaced by $n + 1$ we have

$$a_{n+2} - \sqrt{x} < \frac{1}{2}(a_{n+1} - \sqrt{x}) \quad (4.15)$$

$$< \frac{1}{2} \cdot \frac{1}{2^n}(a_1 - \sqrt{x}) \quad (4.16)$$

$$= \frac{1}{2^{n+1}}(a_1 - \sqrt{x}) \quad (4.17)$$

where in the second inequality we used the induction hypothesis (4.14). Hence (4.14) holds for $n + 1$.

By invoking the Induction Principle we conclude the proof.

Let us comment estimate (4.14). Denote the error at step n by

$$e_n := a_n - \sqrt{x}.$$

The initial error e_1 depends on how far the initial guess is from \sqrt{x} . The estimate in (4.14) is telling us that e_n is a fraction of e_1 , and actually

$$\lim_{n \rightarrow \infty} e_n = 0$$

exponentially fast. From this fact we are finally able to prove (4.4).

Theorem 4.4: Convergence of Heron's Algorithm

We have that

$$\lim_{n \rightarrow \infty} a_n = \sqrt{x}.$$

Proof

By Proposition 4.3 we have that

$$a_{n+1} - \sqrt{x} < \frac{1}{2^n}(a_1 - \sqrt{x})$$

Moreover Proposition 4.1 tells us that

$$\sqrt{x} < a_{n+1}.$$

Putting together the two inequalities above we infer

$$\sqrt{x} < a_{n+1} < \sqrt{x} + \frac{1}{2^n}(a_1 - \sqrt{x}). \quad (4.18)$$

Now note that

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0.$$

Therefore the RHS of (4.18) converges to \sqrt{x} as $n \rightarrow \infty$. Applying the Squeeze Theorem to (4.18) we conclude that $a_n \rightarrow \sqrt{x}$ as $n \rightarrow \infty$.

4.1.1 Coding the Algorithm

Heron's Algorithm can be easily coded in Python. For example, see the function below:

```
# x is the number for which to compute sqrt(x)
# guess is the point a_1
# a_1 must be strictly larger than sqrt(x)
# n is the number of iterations
# the function returns a_{n+1}

def herons_algorithm(x, guess, n):
    for i in range(n):
        guess = (guess + x / guess) / 2.0
    return guess
```

For example let us use the Algorithm to compute $\sqrt{2}$ after 3 iterations. For initial guess we take $a_1 = 2$.

```
# Calculate sqrt(2) with 3 iterations and guess 2
sqrt_2 = herons_algorithm(2, 2, 3)

print(f"The sqrt(2) is approximately {sqrt_2}")
```

The `sqrt(2)` is approximately 1.4142156862745097

That is a pretty good approximation in just 3 iterations!

4.2 Fibonacci Sequence

5 Series

Coming soon

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