Differential Geometry

Lecture Notes, T1 2023

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2023-01-09

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Preface

Welcome to the Lecture Notes of **Differential Geometry 600727** for T1 2023 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

S.Fanzon@hull.ac.uk

Up to date informations about the course and homework will be published on the course webpage

silviofanzon.com/blog/2023/Differential-Geometry

A **pdf** version of the notes is available to download on the top-right.

References

We will study curves and surfaces in $\mathbb{R}^3$. I will follow mainly the textbook by Pressley [6]. Other references that inspired these notes are the books by do Carmo [2], O'Neill [5] and Bär [1].

I will assume some knowledge from Analysis and Linear Algebra. A good place to revise these topics are the books by Zorich [7, 8]. In addition, it can be helpful to plot curves and surfaces to aid visualization. I will do this with Python 3. I recommend installation through Anaconda or Miniconda. The actual coding can then be done through, for example, Jupyter Notebook. Good references for scientific Python programming are [3, 4].

> **! Important**
>
> You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to known in order to excel in the final exam.

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    author = {Fanzon, Silvio},
    title = {Lecture Notes on Differential Geometry},
    url = {https://silviofanzon.quarto.pub/2023-differential-geometry/},
    year = {2023}}
```

1 Curves

1.1 Plotting curves with Python

1.1.1 Plotting 2D curves

Suppose we want to plot the parabola $y = t^2$ for t in the interval $[-3, 3]$. In our language, this is the two-dimensional curve

$$\gamma(t) = (t, t^2), \quad t \in [-3, 3].$$

The two Python libraries we use to plot γ are **numpy** and **matplotlib**. In short, **numpy** handles multi-dimensional arrays and matrices, and can perform high-level mathematical functions on them. For any question you may have about numpy, answers can be found in the searchable documentation available here. Instead **matplotlib** is a plotting library, with documentation here. Python libraries need to be imported every time you want to use them. In our case we will import:

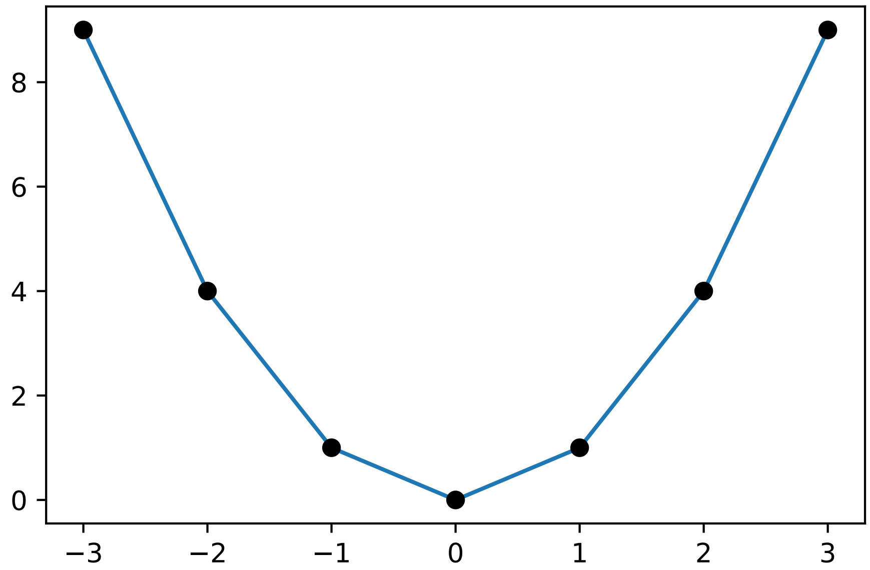
```
import numpy as np
import matplotlib.pyplot as plt
```

The above imports **numpy** and the module **pyplot** from **matplotlib**, and renames them to `np` and `plt`, respectively. These shorthands are standard in the literature, and they make code much more readable. The module for plotting 2D graphs is called `plot(x,y)` and is contained in `plt`. As the syntax suggests, `plot` takes as arguments two arrays $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_n]$. As output it produces a graph which is the linear interpolation of the points (x_i, y_i) in $\mathbb{R}^2$, that is, consecutive points (x_i, y_i) and (x_{i+1}, y_{i+1}) are connected by a segment. Using `plot`, we can graph the curve $\gamma(t) = (t, t^2)$ like so:

```
# Code for plotting gamma

import numpy as np
import matplotlib.pyplot as plt

t = np.array([-3,-2,-1,0,1,2,3])
f = t**2
plt.plot(t,f)
plt.plot(t,f,"ko")
plt.show()
```



Let us comment the above code. The variable `t` is a numpy array containing the ordered values

$$t = [-3, -2, -1, 0, 1, 2, 3] \,. \tag{1.1}$$

This array is then squared entry-by-entry via the operation $t**2$ and saved in the new numpy array `f`, that is,

$$f = [9, 4, 1, 0, 1, 4, 9] \,.$$

The arrays `t` and `f` are then passed to `plot(t,f)`, which produces the above linear interpolation, with `t` on the *x-axis* and `f` on the *y-axis*. The command `plot(t,f,'ko')` instead plots a black dot at each point (t_i, f_i). The latter is clearly not needed to obtain a plot, and it was only included to highlight the fact that `plot` is actually producing a linear interpolation between points. Finally `plt.show()` displays the figure in the user window[1].

Of course one can refine the plot so that it resembles the continuous curve $\gamma(t) = (t, t^2)$ that we all have in mind. This is achieved by generating a numpy array `t` with a finer stepsize, invoking the function `np.linspace(a,b,n)`. Such call will return a numpy array which contains `n` evenly spaced points, starts at `a`, and ends in `b`. For example `np.linspace(-3,3,7)` returns our original array `t` at Equation 2.1, as shown below

```
# Displaying output of np.linspace

import numpy as np

t = np.linspace(-3,3, 7)
print("t =", t)
```

```
t = [-3. -2. -1.  0.  1.  2.  3.]
```

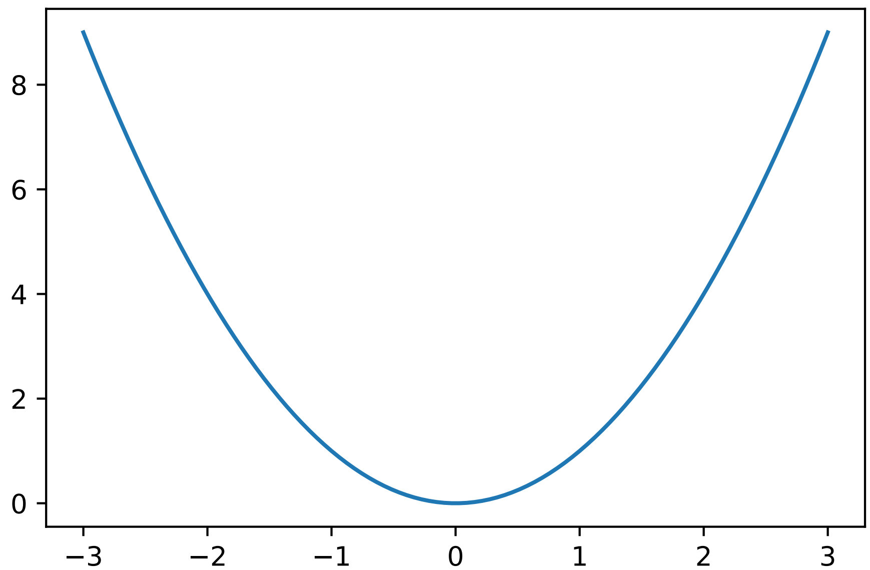
In order to have a more refined plot of γ, we just need to increase n.

[1]The command `plt.show()` can be omitted if working in Jupyter Notebook, as it is called by default.

```
# Plotting gamma with finer step-size

import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(-3,3, 100)
f = t**2
plt.plot(t,f)
plt.show()
```



Let us now plot something more interesting, such as the two-dimensional curve known as the Fermat's spiral

$$\gamma(t) = (\sqrt{t}\cos(t), \sqrt{t}\sin(t)) \quad \text{for} \quad t \in [0, 50]\,.$$

Clearly we need to modify the above code. The variable `t` will still be a numpy array produced by `linspace`. We then need to introduce the arrays `x` and `y` which ecode the first and second components of γ, respectively.

```
# Plotting Fermat's spiral

import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(0,50, 500)
x = np.sqrt(t) * np.cos(t)
y = np.sqrt(t) * np.sin(t)

plt.plot(x,y)
plt.show()
```

Before displaying the output of the above code, a few comments are in order. The array `t` has size 500, due to the behavior of `linspace`. You can also fact check this information by printing `np.size(t)`, which is the numpy function that returns the size of an array. We then use the numpy function `np.sqrt` to compute the square root of the array `t`. The outcome is still an array with the same size of `t`, that is,

$$t = [t_1, \dots, t_n] \quad \implies \quad \sqrt{t} = [\sqrt{t_1}, \dots, \sqrt{t_n}] \,.$$

Similary, the call `np.cos(t)` returns the array

$$\cos(t) = [\cos(t_1), \dots, \cos(t_n)] \,.$$

The two arrays `np.sqrt(t)` and `np.cos(t)` are then multiplied, term-by-term, and saved in the array `x`. The array `y` is computed similarly. The command `plt.plot(x,y)` then yields the graph of the Fermat's spiral:

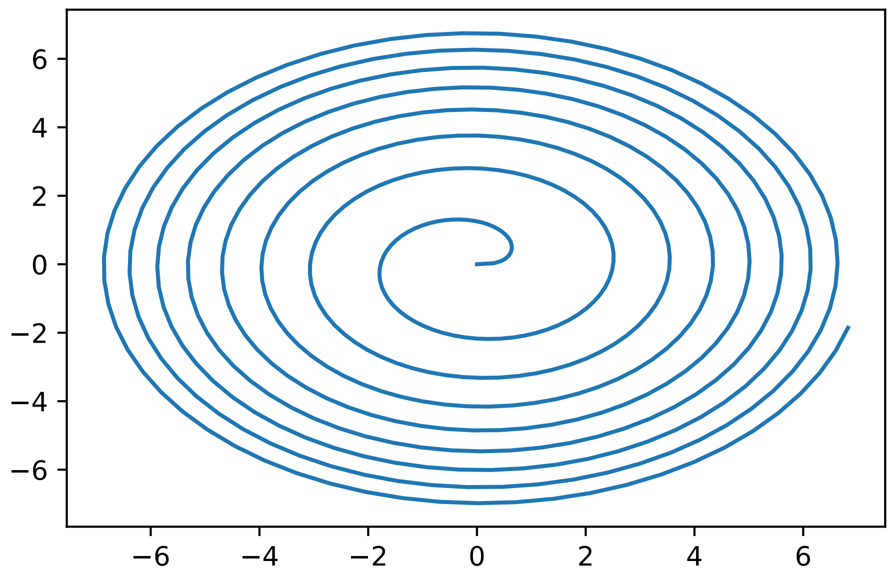


Figure 1.1: Fermat's spiral

The above plots can be styled a bit. For example we can give a title to the plot, label the axes, plot the spiral by means of green dots, and add a plot legend, as coded below:

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(0,50, 500)
x = np.sqrt(t) * np.cos(t)
y = np.sqrt(t) * np.sin(t)

plt.figure(1, figsize = (5,5))

plt.plot(x, y, "--", color="deeppink", linewidth=1.5, label="Spiral")
plt.grid(True, color="lightgray")
```

```
plt.title("Fermat's spiral for t between 0 and 50")
plt.xlabel("x-axis", fontsize = 15)
plt.ylabel("y-axis", fontsize = 15)
plt.legend()
plt.show()
```

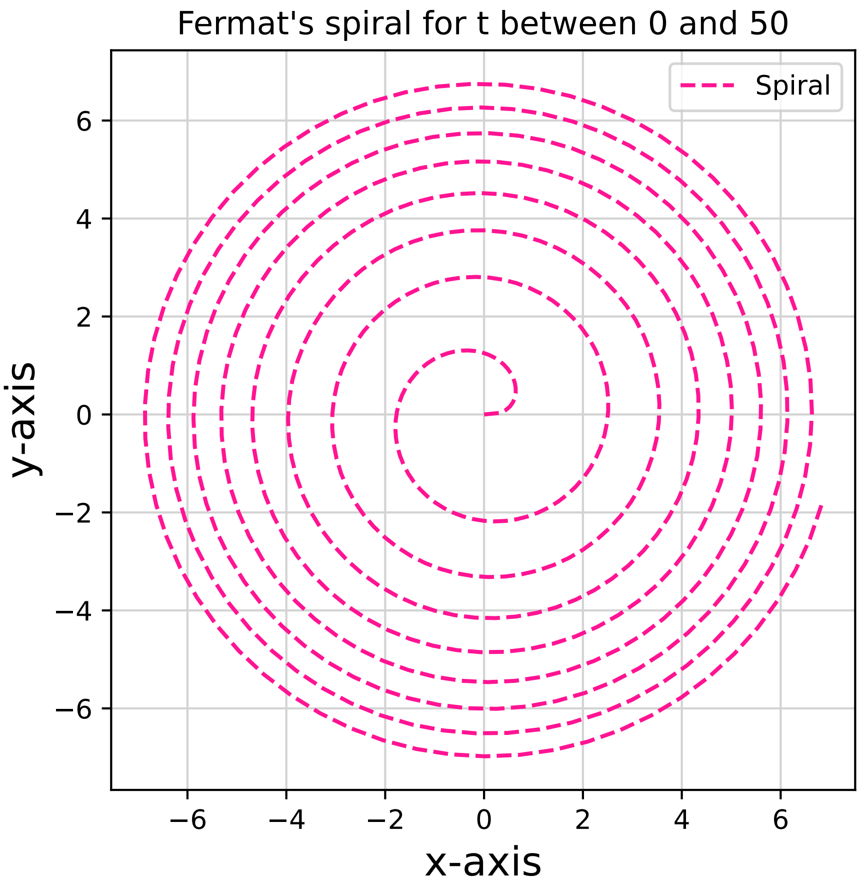


Figure 1.2: Adding a bit of style

Let us go over the novel part of the above code:

- `plt.figure()`: This command generates a figure object. If you are planning on plotting just one figure at a time, then this command is optional: a figure object is generated implicitly when calling `plt.plot`. Otherwise, if working with `n` figures, you need to generate a figure object with `plt.figure(i)` for each `i` between `1` and `n`. The number `i` uniquely identifies the i-th figure: whenever you call `plt.figure(i)`, Python knows that the next commands will refer to the i-th figure. In our case we only have one figure, so we have used the identifier `1`. The second argument `figsize = (a,b)` in `plt.figure()` specifies the size of figure `1` in inches. In this case we generated a figure 5 by 5 inches.
- `plt.plot`: This is plotting the arrays `x` and `y`, as usual. However we are adding a few aestethic touches: the curve is plotted in *dashed* style with `--`, in *deep pink* color and with a line width of 1.5. Finally this plot is labelled *Spiral*.

- `plt.grid`: This enables a grid in *light gray* color.
- `plt.title`: This gives a title to the figure, displayed on top.
- `plt.xlabel` and `plt.ylabel`: These assign labels to the axes, with font size 15 points.
- `plt.legend()`: This plots the legend, with all the labels assigned in the `plt.plot` call. In this case the only label is *Spiral*.

Matplotlib styles

There are countless plot types and options you can specify in **matplotlib**, see for example the Matplotlib Gallery. Of course there is no need to remember every single command: a quick Google search can do wonders.

Generating arrays

There are several ways of generating evenly spaced arrays in Python. For example the function `np.arange(a,b,s)` returns an array with values within the half-open interval $[a, b)$, with spacing between values given by `s`. For example

```
import numpy as np

t = np.arange(0,1, 0.2)
print("t =",t)
```

```
t = [0.  0.2 0.4 0.6 0.8]
```

1.1.2 Plotting implicitly defined curves in 2D

A curve γ in $\mathbb{R}^2$ can also be defined as the set of points $(x, y) \in \mathbb{R}^2$ satisfying

$$f(x, y) = 0$$

for some given $f\colon \mathbb{R}^2 \to \mathbb{R}$. For example let us plot the curve γ implicitly defined by

$$f(x, y) = (3x^2 - y^2)^2\ y^2 - (x^2 + y^2)^4$$

for $-1 \leq x, y \leq 1$. First, we need a way to generate a grid in $\mathbb{R}^2$ so that we can evaluate f on such grid. To illustrate how to do this, let us generate a grid of spacing 1 in the 2D square $[0, 4]^2$. The goal is to obtain the 5 x 5 matrix of coordinates

$$A = \begin{pmatrix} (0,4) & (1,4) & (2,4) & (3,4) & (4,4) \\ (0,3) & (1,3) & (2,3) & (3,3) & (3,4) \\ (0,2) & (1,2) & (2,2) & (2,3) & (2,4) \\ (0,1) & (1,1) & (2,1) & (3,1) & (4,1) \\ (0,0) & (1,0) & (2,0) & (3,0) & (4,0) \end{pmatrix}$$

which corresponds to the grid of points

To achieve this, first generate `x` and `y` coordinates using `linspace(0,4, 5)`. We would then obtain coordinates

$$x = [0, 1, 2, 3, 4]\,, \quad y = [0, 1, 2, 3, 4]\,.$$

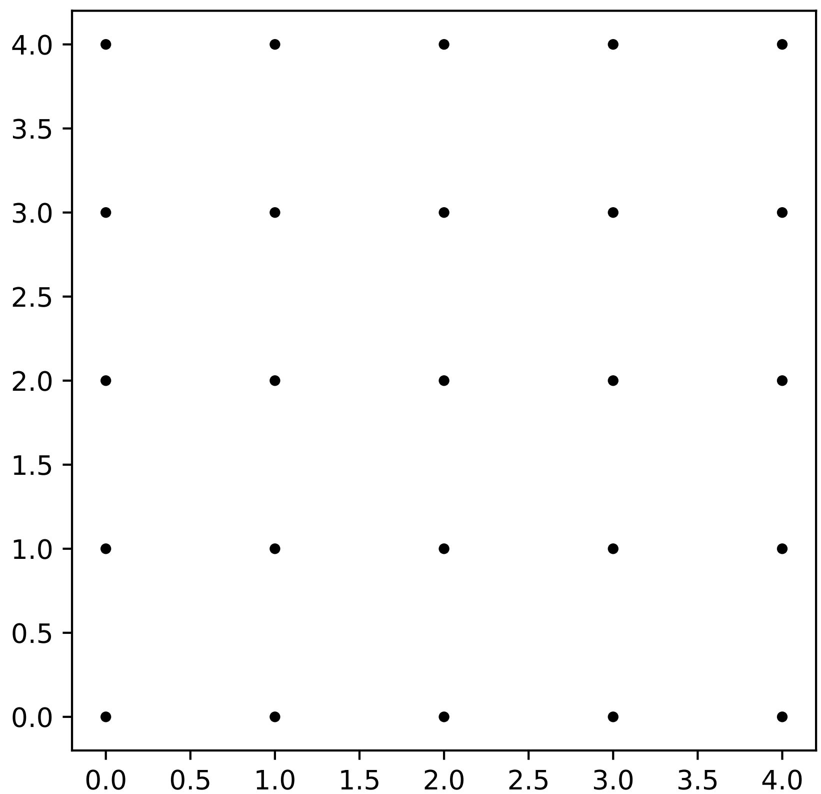


Figure 1.3: The 5 x 5 grid corresponding to the matrix A

We then need to obtain two matrices X and Y: one for the x coordinates in A, and one for the y coordinates in A. Thus

$$X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If we plot X against Y via the command

```
plt.plot(X,Y, marker='.', color='k', linestyle='none')
```

we obtain Figure 1.3. This is all very tedious. Thankfully there exists a function in numpy doing exactly what we need: `np.meshgrid`.

```
# Plotting f=0

import numpy as np
import matplotlib.pyplot as plt

xlist = np.linspace(-1, 1, 5000)
ylist = np.linspace(-1, 1, 5000)
X, Y = np.meshgrid(xlist, ylist)

Z =((3*(X**2) - Y**2)**2)*(Y**2) - (X**2 + Y**2)**4

plt.figure(figsize = (5.5,5.5))
plt.contour(X, Y, Z, [0])
plt.xlabel("x-axis", fontsize = 15)
plt.ylabel("y-axis", fontsize = 15)
plt.show()
```

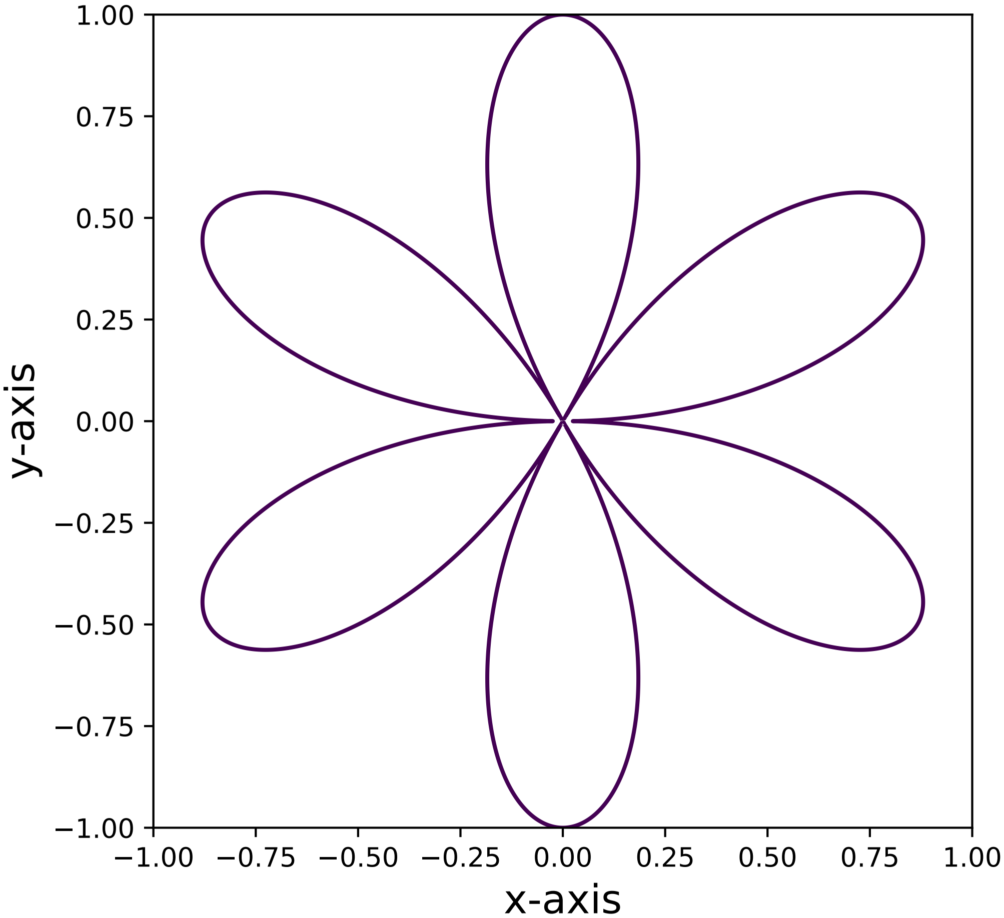


Figure 1.4: Plot of the curve defined by f=0

1.1.3 Plotting 3D curves

If you have understood how to plot 2D curves in Section 1.1.1, then plotting 3D curves will not be much more difficult.

2 Surfaces

2.1 Plotting surfaces with Python

Suppose we want to plot the parabola $y = t^2$ for t in the interval $[-3, 3]$. In our language, this is the two-dimensional curve

$$\gamma(t) = (t, t^2) , \quad t \in [-3, 3] .$$

The two Python libraries we use to plot γ are **numpy** and **matplotlib**. In short, **numpy** handles multi-dimensional arrays and matrices, and can perform high-level mathematical functions on them. For any question you may have about numpy, answers can be found in the searchable documentation available here. Instead **matplotlib** is a plotting library, with documentation here. Python libraries need to be imported every time you want to use them. In our case we will import:

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plt.plot(t,f,"ko")
plt.show()
```

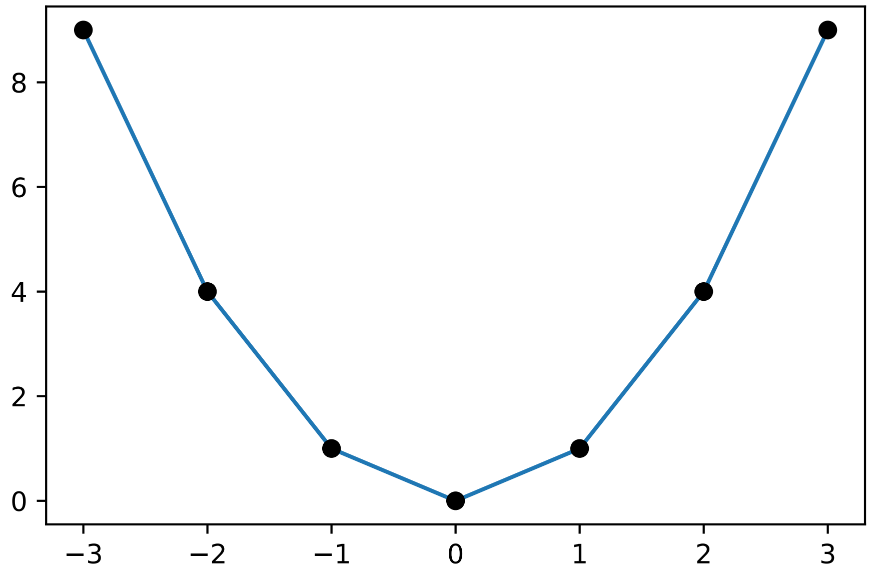

Let us comment the above code. The variable `t` is a numpy array containing the ordered values

$$t = [-3, -2, -1, 0, 1, 2, 3] . \tag{2.1}$$

This array is then squared entry-by-entry via the operation $t**2$ and saved in the new numpy array `f`, that is,

$$f = [9, 4, 1, 0, 1, 4, 9] .$$

The arrays `t` and `f` are then passed to `plot(t,f)`, which produces the above linear interpolation, with `t` on the *x-axis* and `f` on the *y-axis*. The command `plot(t,f,'ko')` instead plots a black dot at each point (t_i, f_i). The latter is clearly not needed to obtain a plot, and it was only included to highlight the fact that `plot` is actually producing a linear interpolation between points. Finally `plt.show()` displays the figure in the user window[1].

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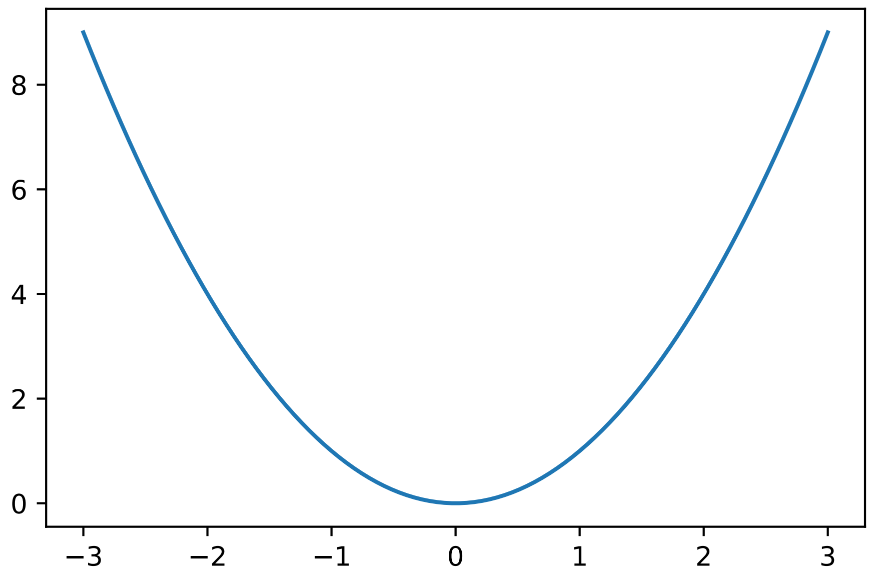

Let us now plot something more interesting, such as the two-dimensional curve known as the Fermat's spiral

$$\gamma(t) = (\sqrt{t}\cos(t), \sqrt{t}\sin(t)) \quad \text{for} \quad t \in [0, 50]\,.$$

Clearly we need to modify the above code. The variable `t` will still be a numpy array produced by `linspace`. We then need to introduce the arrays `x` and `y` which ecode the first and second components of γ, respectively.

```
# Plotting Fermat's spiral

import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(0,50, 500)
x = np.sqrt(t) * np.cos(t)
y = np.sqrt(t) * np.sin(t)

plt.plot(x,y)
plt.show()
```

Before displaying the output of the above code, a few comments are in order. The array `t` has size 500, due to the behavior of `linspace`. You can also fact check this information by printing `np.size(t)`, which is the numpy function that returns the size of an array. We then use the numpy function `np.sqrt` to compute the square root of the array `t`. The outcome is still an array with the same size of `t`, that is,

$$t = [t_1, \dots, t_n] \quad \Longrightarrow \quad \sqrt{t} = [\sqrt{t_1}, \dots, \sqrt{t_n}]\,.$$

Similary, the call `np.cos(t)` returns the array

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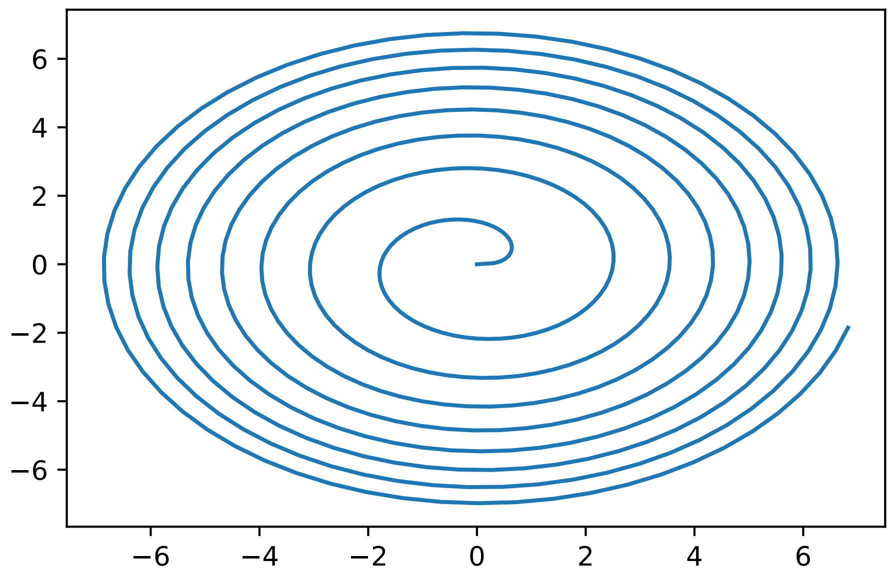

Figure 2.1: Fermat's spiral

The above plots can be styled a bit. For example we can give a title to the plot, label the axes, plot the spiral by means of green dots, and add a plot legend, as coded below:

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plt.plot(x, y, "--", color="deeppink", linewidth=1.5, label="Spiral")
plt.grid(True, color="lightgray")
```

```
plt.title("Fermat's spiral for t between 0 and 50")
plt.xlabel("x-axis", fontsize = 15)
plt.ylabel("y-axis", fontsize = 15)
plt.legend()
plt.show()
```

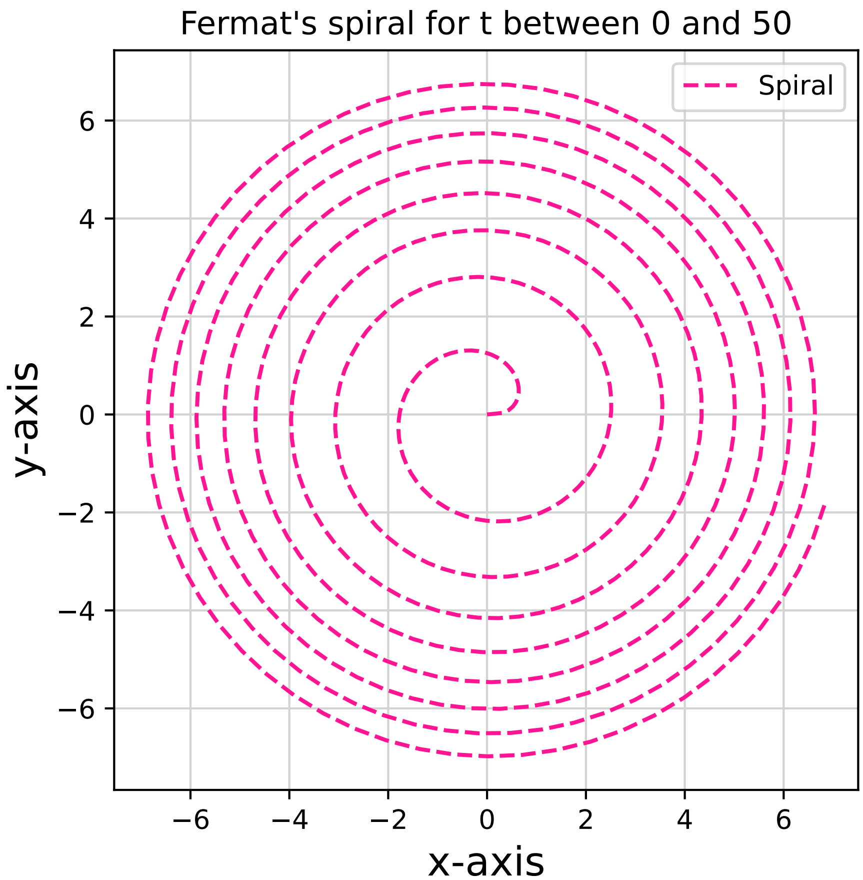

Figure 2.2: Adding a bit of style

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- `plt.grid`: This enables a grid in *light gray* color.
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t = np.arange(0,1, 0.2)
print("t =",t)
```

```
t = [0.  0.2 0.4 0.6 0.8]
```

References

[1] C. Bär. *Elementary Differential Geometry*. Cambridge University Press, 2010.

[2] M. P. do Carmo. *Differential Geometry of Curves and Surfaces*. Second Edition. Dover Books on Mathematics, 2017.

[3] R. Johansson. *Numerical Python. Scientific Computing and Data Science Applications with Numpy, SciPy and Matplotlib*. Second Edition. Apress, 2019.

[4] Q. Kong, T. Siauw, and A. Bayen. *Python Programming and Numerical Methods*. Academic Press, 2020.

[5] B. O'Neill. *Elementary Differential Geometry*. Second Edition. Academic Press, 2006.

[6] A. Pressley. *Elementary Differential Geometry*. Second Edition. Springer, 2010.

[7] V. A. Zorich. *Mathematical Analysis I*. Second Edition. Springer, 2015.

[8] V. A. Zorich. *Mathematical Analysis II*. Second Edition. Springer, 2016.