

Numbers, Sequences and Series

Lecture Notes, T1 2023/24

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Welcome

These are the Lecture Notes of **Numbers, Sequences & Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

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Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

References

We will study the set of real numbers \mathbb{R} , and then sequences and series in \mathbb{R} . I will follow mainly the textbook by Bartle and Sherbert [2]. Other references that inspired these notes are the books by Abbott [1] and Rudin [3].

! You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to know in order to excel in the final exam.

1 Numbers

1.1 Introduction

The aim of this chapter is to rigorously introduce the set of real numbers \mathbb{R} . But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \dots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol \in denotes that m and n belong to \mathbb{N} . For example $3 + 7$ results in 10.

Question

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m?$$

The answer to the above question is clearly **no**. For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to \mathbb{N} . We therefore need to **extend** the set \mathbb{N} in order to invert the sum. This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n : n \in \mathbb{N}\},$$

that is, the set

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The sum can be extended to \mathbb{Z} , by defining

$$(-n) + (-m) := -(m + n) \tag{1.1}$$

for all $m, n \in \mathbb{N}$. Now every element of \mathbb{Z} possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0.$$

What is m for a given n ? Seeing the definition at (1.1), we simply have

$$m = -n.$$

We can also multiply integers, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm .

Question

Can the multiplication in \mathbb{Z} be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m?$$

The answer is of course **no**. Just take $n = 2$ and $m = 1$. The answer should be $1/2$, but $1/2$ does not belong to \mathbb{Z} . Thus, in order to invert the multiplication, we need to **extend** the set of integers. This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

In \mathbb{Q} the multiplication is invertible, and each non-zero element has an inverse: the inverse of m/n is given by n/m . Moreover by construction

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Question

How can we draw the set \mathbb{Q} ?

It is clear how to draw \mathbb{Z} , as seen below.

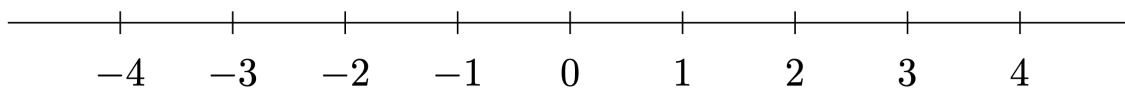


Figure 1.1: Representation of integers

However \mathbb{Q} is much **denser** than the elements of \mathbb{Z} represented in Figure 1.1. For example, consider $0 \in \mathbb{Q}$.

Question

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, we can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n}.$$

For example think about the case of the numbers $1/n$ for $n \in \mathbb{N}$ and $n \neq 0$. Such numbers get arbitrarily close to 0, as depicted below.

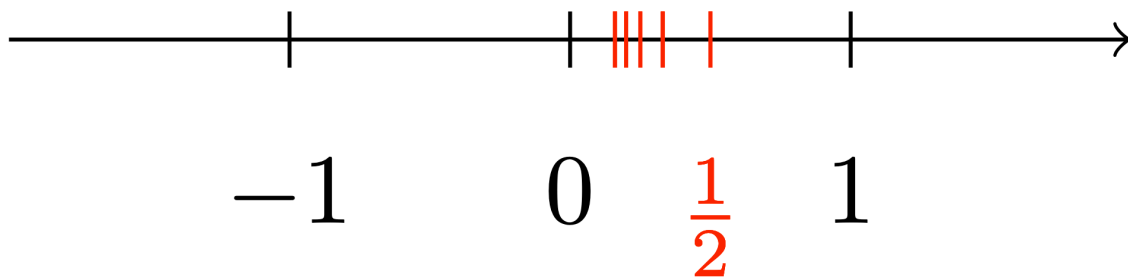


Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0, 1]$. In other words, we might conjecture the following.

Conjecture 1.1

Maybe \mathbb{Q} can be represented by a continuous line.

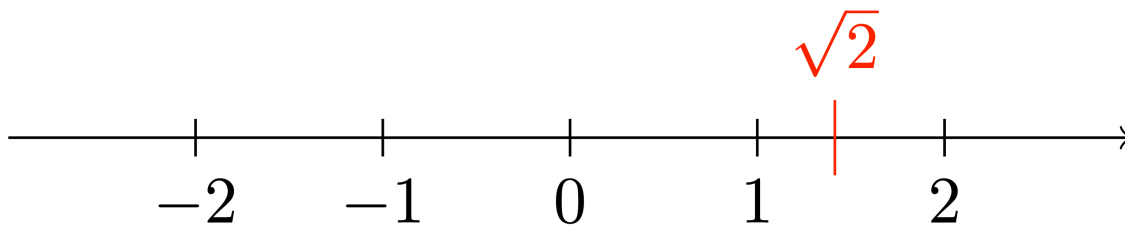
Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.1 is **false**, as shown by the Theorem below.

Theorem 1.2

The number $\sqrt{2}$ does not belong to \mathbb{Q} .

The above theorem is the reason why $\sqrt{2}$ is called an irrational number. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048 \dots$$

Figure 1.3: Representing $\sqrt{2}$

and the situation is as in the picture below.

We can therefore see that Conjecture 1.1 is **false**, and \mathbb{Q} is not a line, given that it has a hole at $\sqrt{2}$. Let us see why Theorem 1.2 is true.

Proof: Proof of Theorem 1.2

We prove that $\sqrt{2}$ does not belong to \mathbb{Q} by **contradiction**. This means assuming the existence of $q \in \mathbb{Q}$ such that $q = \sqrt{2}$. In other words

$$q^2 = 2. \quad (1.2)$$

We will show that (1.2) leads to a contradiction. Thus (1.2) must be false, and so

$$\sqrt{2} \notin \mathbb{Q},$$

concluding the thesis.

Seeing the above, we might be tempted to just extend \mathbb{Q} to include $\sqrt{2}$. But then we also have $\sqrt{p} \notin \mathbb{Q}$ for each prime number p . Then we might attempt

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : p \text{ prime}\}.$$

However even this is not enough, as we would have, for example

$$\sqrt{2} + \sqrt{3}, \pi, \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}.$$

The reality of things is that to **complete** \mathbb{Q} and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in \mathbb{Q} . Such extension of \mathbb{Q} will be called \mathbb{R} , the set of **real numbers**. The inclusions will be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The set \mathbb{R} is not at all trivial to construct. In fact, at first we will just assume its **existence** and study its **properties**. We will then provide a concrete **model** for the real numbers \mathbb{R} , to prove once and for all that such set indeed exists.

2 Sequences

3 Series

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References

- [1] S. Abbott. *Understanding Analysis*. Second Edition. Springer, 2015.
- [2] R. G. Bartle and D. R. Sherbert. *Introduction to Real Analysis*. Fourth Edition. Wiley, 2011.
- [3] W. Rudin. *Principles of Mathematical Analysis*. Third Edition. McGraw Hill, 1976.