

# **Differential Geometry**

## **Revision Guide**

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# Revision Guide

Revision Guide for the Exam of the module **Differential Geometry 661955** 2024/25 at the University of Hull. If you have any question or find any typo, please email me at

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Full length Lecture Notes of the module available at

[silviofanzon.com/2024-Differential-Geometry-Notes](https://silviofanzon.com/2024-Differential-Geometry-Notes)

## Recommended revision strategy

Make sure you are very comfortable with:

1. The Definitions, Theorems, Proofs, and Examples contained in this Revision Guide
2. The Homework questions
3. The 2022/23 and 2023/24 Exam Papers questions.
4. The Checklist below

## Checklist

You should be comfortable with the following topics/taks:

### Curves

- Regularity of curves
- Computing the length of a curve
- Computing arc-length function and arc-length reparametrization
- Calculating the curvature and torsion of unit-speed curves from the definitions
- Calculating the curvature and torsion of (possibly not unit-speed) curves from the formulae
- Calculating the Frenet frame of a unit-speed curve from the definitions
- Calculating the Frenet frame of a (possibly not unit-speed) unit-speed curve from the formulas
- Applying the Fundamental Theorem of Space Curves to determine if two curves coincide, up to a rigid motion
- Proving that a curve is contained in a plane, and computing the equation of such plane
- Proving that a curve is part of a circle

### Topology:

- Proving that a given collection of sets is a topology
- Proving that a given set is open / closed
- Proving that a given topology is discrete
- Comparing two topologies, and determining which one is finer
- Studying convergent sequences in topological space
- Proving that a given set with a distance function is a metric space
- Studying the topology induced by the metric
- Studying convergent sequences in metric space
- Proving that a topological space is Hausdorff
- Proving that a given function between topological spaces is continuous
- Studying the subspace topology of a given subset of a topological space

- Showing that a given topological space is connected / path-connected
- Proving that two given topological spaces are not homeomorphic, by making use of connectedness arguments

### Surfaces:

- Regularity of surface charts
- Computing reparametrizations of surface charts
- Calculating the standard unit normal of a surface chart
- Given a surface chart, compute a basis and the equation of the tangent plane
- Calculating the differential of a smooth function between surfaces
- Proving that a given level surface is regular, and computing its tangent plane
- Proving that a given surface is ruled
- Calculating the first fundamental form of a surface chart
- Proving that a given map is a local isometry / conformal
- Prove that a given parametrization is conformal
- Calculating length and angles of curves on surfaces
- Calculating the second fundamental form of a surface chart
- Calculating the matrix of the Weingarten map, the principal curvatures and vectors of a surface chart
- Calculating Gaussian and mean curvature of a surface chart
- Calculating normal and geodesic curvature of a unit-speed curve on a surface
- Calculating the normal and geodesic curvature of a (possibly not unit-speed) curve on a surface from the formulae
- Classifying surface points as elliptic, parabolic, hyperbolic, planar, umbilical

# 1 Curves

## Definition 1.1: Length of a curve

The **length** of the curve  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  is

$$L(\gamma) = \int_a^b \|\dot{\gamma}(u)\| \, du.$$

## Example 1.2: Length of the Helix

**Question.** Compute the length of the Helix

$$\gamma(t) = (R \cos(t), R \sin(t), Ht), \quad t \in (0, 2\pi).$$

**Solution.** We compute

$$\begin{aligned} \dot{\gamma}(t) &= (-R \sin(t), R \cos(t), H) & \|\dot{\gamma}(t)\| &= \sqrt{R^2 + H^2} \\ L(\gamma) &= \int_0^{2\pi} \|\dot{\gamma}(u)\| \, du = 2\pi \sqrt{R^2 + H^2} \end{aligned}$$

## Definition 1.3: Arc-Length of a curve

The **arc-length** along  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  from  $t_0$  to  $t$  is

$$s : (a, b) \rightarrow \mathbb{R}, \quad s(t) = \int_{t_0}^t \|\dot{\gamma}(u)\| \, du.$$

## Example 1.4: Arc-length of Logarithmic Spiral

**Question.** Compute the arc-length of

$$\gamma(t) = (e^{kt} \cos(t), e^{kt} \sin(t), 0).$$

**Solution.** The arc-length starting from  $t_0$  is

$$\begin{aligned} \dot{\gamma}(t) &= e^{kt}(k \cos(t) - \sin(t), k \sin(t) + \cos(t), 0) \\ \|\dot{\gamma}(t)\|^2 &= (k^2 + 1)e^{2kt} \\ s(t) &= \int_{t_0}^t \|\dot{\gamma}(\tau)\| \, d\tau = \frac{\sqrt{k^2 + 1}}{k} (e^{kt} - e^{kt_0}). \end{aligned}$$

## Definition 1.5: Unit-speed curve

A curve  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  is **unit-speed** if

$$\|\dot{\gamma}(t)\| = 1, \quad \forall t \in (a, b).$$

## Proposition 1.6

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be unit-speed. Then

$$\dot{\gamma} \cdot \ddot{\gamma} = 0, \quad \forall t \in (a, b).$$

## Proof

Since  $\gamma$  is unit-speed, we have  $\dot{\gamma} \cdot \dot{\gamma} = 1$ . Differentiating both sides, we get the thesis:

$$0 = \frac{d}{dt}(\dot{\gamma} \cdot \dot{\gamma}) = \ddot{\gamma} \cdot \dot{\gamma} + \dot{\gamma} \cdot \ddot{\gamma} = 2\dot{\gamma} \cdot \ddot{\gamma}.$$

## Definition 1.7: Reparametrization

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$ . A **reparametrization** of  $\gamma$  is a curve  $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \rightarrow \mathbb{R}^3$  such that

$$\tilde{\gamma}(t) = \gamma(\phi(t)), \quad \forall t \in (\tilde{a}, \tilde{b}),$$

for  $\phi : (\tilde{a}, \tilde{b}) \rightarrow (a, b)$  diffeomorphism. We call both  $\phi$  and  $\phi^{-1}$  **reparametrization maps**.

## Definition 1.8: Unit-speed reparametrization

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$ . A **unit-speed reparametrization** of  $\gamma$  is a reparametrization  $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \rightarrow \mathbb{R}^3$  which is unit-speed, that is,

$$\|\dot{\tilde{\gamma}}(t)\| = 1, \quad \forall t \in (\tilde{a}, \tilde{b}).$$

## Definition 1.9: Regular curve

A curve  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  is **regular** if

$$\|\dot{\gamma}(t)\| \neq 0, \quad \forall t \in (a, b)$$

## Theorem 1.10: Existence of unit-speed reparametrization

Let  $\gamma$  be a curve. They are equivalent:

1.  $\gamma$  is regular,
2.  $\gamma$  admits unit-speed reparametrization.

## Theorem 1.11: Characterization of unit-speed reparametrizations

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be a regular curve. Let  $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \rightarrow \mathbb{R}^3$  be a reparametrization of  $\gamma$ , that is,

$$\gamma(t) = \tilde{\gamma}(\phi(t)), \quad \forall t \in (a, b)$$

for some diffeomorphism  $\phi : (a, b) \rightarrow (\tilde{a}, \tilde{b})$ . We have

1. If  $\tilde{\gamma}$  is unit-speed, there exists  $c \in \mathbb{R}$  such that

$$\phi(t) = \pm s(t) + c, \quad \forall t \in (a, b). \quad (1.1)$$

2. If  $\phi$  is given by (1.1), then  $\tilde{\gamma}$  is unit-speed.

**Definition 1.12:** Arc-length reparametrization

Let  $\gamma$  be regular. The **arc-length reparametrization** of  $\gamma$  is

$$\tilde{\gamma} = \gamma \circ s^{-1},$$

with  $s^{-1}$  inverse of the arc-length function of  $\gamma$ .

**Example 1.13:** Reparametrization by arc-length

**Question.** Consider the curve

$$\gamma(t) = (5 \cos(t), 5 \sin(t), 12t).$$

Prove that  $\gamma$  is regular, and reparametrize it by arc-length.

**Solution.**  $\gamma$  is regular because

$$\dot{\gamma}(t) = (-5 \sin(t), 5 \cos(t), 12), \quad \|\dot{\gamma}(t)\| = 13 \neq 0$$

The arc-length of  $\gamma$  starting from  $t_0 = 0$ , and its inverse, are

$$s(t) = \int_0^t \|\dot{\gamma}(u)\| du = 13t, \quad t(s) = \frac{s}{13}.$$

The arc-length reparametrization of  $\gamma$  is

$$\tilde{\gamma}(s) = \gamma(t(s)) = \left(5 \cos\left(\frac{s}{13}\right), 5 \sin\left(\frac{s}{13}\right), \frac{12}{13}s\right).$$

## 1.1 Curvature

**Definition 1.14:** Curvature of unit-speed curve

The **curvature** of a unit-speed curve  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  is

$$\kappa(t) = \|\ddot{\gamma}(t)\|.$$

**Example 1.15:** Curvature of the Circle

**Question.** Compute the curvature of the circle of radius  $R > 0$

$$\gamma(t) = \left(x_0 + R \cos\left(\frac{t}{R}\right), y_0 + \sin\left(\frac{t}{R}\right), 0\right).$$

**Solution.** First, check that  $\gamma$  is unit-speed:

$$\dot{\gamma}(t) = \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right), 0\right), \quad \|\dot{\gamma}(t)\| = 1$$

Now, compute second derivative and curvature

$$\ddot{\gamma}(t) = \left(-\frac{1}{R} \cos\left(\frac{t}{R}\right), -\frac{1}{R} \sin\left(\frac{t}{R}\right), 0\right),$$

$$\kappa(t) = \|\ddot{\gamma}(t)\| = \frac{1}{R}.$$

**Definition 1.16:** Curvature of regular curve

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be a regular curve and  $\tilde{\gamma}$  be a unit-speed reparametrization of  $\gamma$ , with  $\gamma = \tilde{\gamma} \circ \phi$  and  $\phi : (a, b) \rightarrow (\tilde{a}, \tilde{b})$ . Let  $\tilde{\kappa} : (\tilde{a}, \tilde{b}) \rightarrow \mathbb{R}$  be the curvature of  $\tilde{\gamma}$ . The **curvature** of  $\gamma$  is

$$\kappa(t) = \tilde{\kappa}(\phi(t)).$$

**Remark 1.17:** Computing curvature of regular  $\gamma$ 

1. Compute the arc-length  $s(t)$  of  $\gamma$  and its inverse  $t(s)$ .
2. Compute the arc-length reparametrization

$$\tilde{\gamma}(s) = \gamma(t(s)).$$

3. Compute the curvature of  $\tilde{\gamma}$

$$\tilde{\kappa}(s) = \|\ddot{\tilde{\gamma}}(s)\|.$$

4. The curvature of  $\gamma$  is

$$\kappa(t) = \tilde{\kappa}(s(t)).$$

**Definition 1.18:** Hyperbolic functions

$$\begin{aligned} \cosh(t) &= \frac{e^t + e^{-t}}{2} & \sinh(t) &= \frac{e^t - e^{-t}}{2} \\ \tanh(t) &= \frac{\sinh(t)}{\cosh(t)} & \coth(t) &= \frac{\cosh(t)}{\sinh(t)} \\ \operatorname{sech}(t) &= \frac{1}{\cosh(t)} & \operatorname{csch}(t) &= \frac{1}{\sinh(t)} \end{aligned}$$

**Theorem 1.19:** Properties of Hyperbolic Functions

$$\begin{aligned} \cosh^2(t) - \sinh^2(t) &= 1 & \operatorname{sech}^2(t) + \tanh^2(t) &= 1 \\ \sinh(t)' &= \cosh(t) & \cosh(t)' &= \sinh(t) \\ \tanh(t)' &= \operatorname{sech}^2(t) & \operatorname{sech}(t)' &= -\operatorname{sech}(t) \tanh(t) \end{aligned}$$

**Example 1.20:** Curvature of the Catenary

**Question.** Consider the Catenary curve

$$\gamma(t) = (t, \cosh(t)), \quad t \in \mathbb{R}.$$

1. Prove that  $\gamma$  is regular.
2. Compute the arc-length reparametrization of  $\gamma$ .
3. Compute the curvature of  $\tilde{\gamma}$ .
4. Compute the curvature of  $\gamma$ .

**Solution.**

1.  $\gamma$  is regular because

$$\dot{\gamma}(t) = (1, \sinh(t))$$

$$\|\dot{\gamma}\| = \sqrt{1 + \sinh^2(t)} = \cosh(t) \geq 1$$

2. The arc-length of  $\gamma$  starting at  $t_0 = 0$  is

$$s(t) = \int_0^t \|\dot{\gamma}(u)\| du = \int_0^t \cosh(u) du = \sinh(t)$$

where we used that  $\sinh(0) = 0$ . Moreover,

$$\begin{aligned} s = \sinh(t) &\iff s = \frac{e^t - e^{-t}}{2} \\ &\iff e^{2t} - 2se^t - 1 = 0 \end{aligned}$$

Substitute  $y = e^t$  to obtain

$$\begin{aligned} e^{2t} - 2se^t - 1 = 0 &\iff y^2 - 2sy - 1 = 0 \\ &\iff y_{\pm} = s \pm \sqrt{1 + s^2}. \end{aligned}$$

Notice that

$$y_+ = s + \sqrt{1 + s^2} \geq s + \sqrt{s^2} = s + |s| \geq 0$$

by definition of absolute value. Therefore,

$$e^t = y_+ = s + \sqrt{1 + s^2} \implies t(s) = \log(s + \sqrt{1 + s^2})$$

The arc-length reparametrization of  $\gamma$  is

$$\tilde{\gamma}(s) = \gamma(t(s)) = (\log(s + \sqrt{1 + s^2}), \sqrt{1 + s^2})$$

3. Compute the curvature of  $\tilde{\gamma}$

$$\begin{aligned} \dot{\tilde{\gamma}}(s) &= \left( \frac{1}{\sqrt{1 + s^2}}, \frac{s}{\sqrt{1 + s^2}} \right) \\ \ddot{\tilde{\gamma}}(s) &= \left( -\frac{s}{(1 + s^2)^{3/2}}, \frac{1}{(1 + s^2)^{3/2}} \right) \\ \tilde{\kappa}(s) &= \|\ddot{\tilde{\gamma}}(s)\| = \frac{1}{1 + s^2} \end{aligned}$$

4. Recalling that  $s(t) = \sinh(t)$ , the curvature of  $\gamma$  is

$$\kappa(t) = \tilde{\kappa}(s(t)) = \frac{1}{1 + \sinh^2(t)} = \frac{1}{\cosh^2(t)}.$$

### Definition 1.21: Vector product

The **vector product** of two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$  is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

### Theorem 1.22: Geometric Properties of vector product

Let  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$  be linearly independent. Then

- $\mathbf{u} \times \mathbf{v}$  is orthogonal to the plane spanned by  $\mathbf{u}, \mathbf{v}$
- $\|\mathbf{u} \times \mathbf{v}\|$  is the area of the parallelogram with sides  $\mathbf{u}, \mathbf{v}$
- The triple  $(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})$  is a positive basis of  $\mathbb{R}^3$

### Theorem 1.23

For all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$  it holds:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

### Theorem 1.24

Let  $\gamma, \eta : (a, b) \rightarrow \mathbb{R}^3$ . Then, the curve  $\gamma \times \eta$  is smooth, and

$$\frac{d}{dt}(\gamma \times \eta) = \dot{\gamma} \times \eta + \gamma \times \dot{\eta}.$$

### Theorem 1.25: Curvature formula

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be regular. The curvature of  $\gamma$  is

$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}.$$

### Example 1.26: Curvature of the Helix

**Question.** Consider the Helix of radius  $R > 0$  and rise  $H$ ,

$$\gamma(t) = (R \cos(t), R \sin(t), Ht).$$

1. Prove that  $\gamma$  is regular.
2. Compute the curvature of  $\gamma$ .

**Solution.**

1.  $\gamma$  is regular because

$$\dot{\gamma}(t) = (-R \sin(t), R \cos(t), H)$$

$$\|\dot{\gamma}(t)\| = \sqrt{R^2 + H^2} \geq R > 0$$

2. Compute the curvature using the formula:

$$\ddot{\gamma}(t) = (-R \cos(t), -R \sin(t), 0)$$

$$\dot{\gamma} \times \ddot{\gamma} = (RH \sin(t), -RH \cos(t), R^2)$$

$$\|\dot{\gamma} \times \ddot{\gamma}\| = R\sqrt{R^2 + H^2}$$

$$\kappa(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3} = \frac{R}{R^2 + H^2}$$

### Example 1.27: Calculation of curvature

**Question.** Define the curve

$$\gamma(t) = \left( \frac{8}{5} \cos(t), 1 - 2 \sin(t), \frac{6}{5} \cos(t) \right).$$

1. Prove that  $\gamma$  is regular.
2. Compute the curvature of  $\gamma$ .

**Solution.**

1.  $\gamma$  is regular because

$$\dot{\gamma} = \left( -\frac{8}{5} \sin(t), -2 \cos(t), -\frac{6}{5} \sin(t) \right), \quad \|\dot{\gamma}\| = 2 \neq 0.$$

2. Compute the curvature using the formula:

$$\ddot{\gamma} = \left( -\frac{8}{5} \cos(t), 2 \sin(t), -\frac{6}{5} \cos(t) \right) \quad \|\dot{\gamma} \times \ddot{\gamma}\| = 4$$

$$\dot{\gamma} \times \ddot{\gamma} = \left( -\frac{12}{5}, 0, \frac{16}{5} \right) \quad \kappa = \frac{1}{2}.$$

**Example 1.28:** Different curves, same curvature**Question** Let  $\gamma$  be a circle

$$\gamma(t) = (2 \cos(t), 2 \sin(t), 0),$$

and  $\eta$  be a helix of radius  $S > 0$  and rise  $H > 0$ 

$$\eta(t) = (S \cos(t), S \sin(t), Ht).$$

Find  $S$  and  $H$  such that  $\gamma$  and  $\eta$  have the same curvature.**Solution.** Curvatures of  $\gamma$  and  $\eta$  were already computed:

$$\kappa^\gamma = \frac{1}{2}, \quad \kappa^\eta = \frac{S}{S^2 + H^2}.$$

Imposing that  $\kappa^\gamma = \kappa^\eta$ , we get

$$\frac{1}{2} = \frac{S}{S^2 + H^2} \implies H^2 = 2S - S^2.$$

Choosing  $S = 1$  and  $H = 1$  yields  $\kappa^\gamma = \kappa^\eta$ .

## 1.2 Frenet frame and torsion

**Definition 1.29:** Frenet frame of unit-speed curveLet  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be unit-speed, with  $\kappa \neq 0$ .

1. The
- tangent vector**
- to
- $\gamma$
- at
- $\gamma(t)$
- is

$$\mathbf{t}(t) = \dot{\gamma}(t).$$

2. The
- principal normal vector**
- to
- $\gamma$
- at
- $\gamma(t)$
- is

$$\mathbf{n}(t) = \frac{\ddot{\gamma}(t)}{\kappa(t)}.$$

3. The
- binormal vector**
- to
- $\gamma$
- at
- $\gamma(t)$
- is

$$\mathbf{b}(t) = \dot{\gamma}(t) \times \mathbf{n}(t).$$

4. The
- Frenet frame**
- of
- $\gamma$
- at
- $\gamma(t)$
- is the triple

$$\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}.$$

**Theorem 1.30:** Frenet frame is orthonormal basisLet  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be unit-speed, with  $\kappa \neq 0$ . The Frenet frame

$$\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$$

is a positive orthonormal basis of  $\mathbb{R}^3$  for each  $t \in (a, b)$ .**Definition 1.31:** Torsion of unit-speed curve with  $\kappa \neq 0$ Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be unit-speed, with  $\kappa \neq 0$ . The **torsion** of  $\gamma$  is the unique scalar  $\tau(t)$  such that

$$\dot{\mathbf{b}}(t) = -\tau(t)\mathbf{n}(t).$$

In particular,

$$\tau(t) = -\dot{\mathbf{b}}(t) \cdot \mathbf{n}(t).$$

**Definition 1.32:** Torsion of regular curve with  $\kappa \neq 0$ Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be a regular curve with  $\kappa \neq 0$ . Let  $\tilde{\gamma}$  be a unit-speed reparametrization of  $\gamma$  with  $\gamma = \tilde{\gamma} \circ \phi$  and  $\phi : (a, b) \rightarrow (\tilde{a}, \tilde{b})$ . Let  $\tilde{\tau} : (\tilde{a}, \tilde{b}) \rightarrow \mathbb{R}$  be the torsion of  $\tilde{\gamma}$ . The **torsion** of  $\gamma$  is

$$\tau(t) = \tilde{\tau}(\phi(t)).$$

**Example 1.33:** Curvature and torsion of Helix with Frenet frame**Question.** Consider the Helix of radius  $R > 0$  and rise  $H$ 

$$\gamma(t) = (R \cos(t), R \sin(t), tH), \quad t \in \mathbb{R}.$$

1. Compute the arc-length reparametrization  $\tilde{\gamma}$  of  $\gamma$ .
2. Compute Frenet frame, curvature and torsion of  $\tilde{\gamma}$ .
3. Compute curvature and torsion  $\gamma$ .

**Solution.**

1. The arc-length of
- $\gamma$
- starting at
- $t_0 = 0$
- , and its inverse, are

$$\dot{\gamma}(t) = (-R \sin(t), R \cos(t), H)$$

$$\|\dot{\gamma}\| = \rho, \quad \rho := \sqrt{R^2 + H^2}$$

$$s(t) = \int_0^t \|\dot{\gamma}(u)\| du = \rho t, \quad t(s) = \frac{s}{\rho}.$$

The arc-length reparametrization  $\tilde{\gamma}$  of  $\gamma$  is

$$\tilde{\gamma}(s) = \gamma(t(s)) = \left( R \cos\left(\frac{s}{\rho}\right), R \sin\left(\frac{s}{\rho}\right), \frac{Hs}{\rho} \right).$$

2. Compute the tangent vector to
- $\tilde{\gamma}$
- and its derivative

$$\tilde{\mathbf{t}}(s) = \dot{\tilde{\gamma}} = \frac{1}{\rho} \left( -R \sin\left(\frac{s}{\rho}\right), R \cos\left(\frac{s}{\rho}\right), H \right)$$

$$\dot{\tilde{\mathbf{t}}}(s) = \frac{R}{\rho^2} \left( -\cos\left(\frac{s}{\rho}\right), -\sin\left(\frac{s}{\rho}\right), 0 \right)$$

The curvature of  $\tilde{\gamma}$  is

$$\tilde{\kappa}(s) = \|\ddot{\tilde{\gamma}}(s)\| = \|\dot{\tilde{\mathbf{t}}}(s)\| = \frac{R}{R^2 + H^2}.$$

The principal normal vector and binormal are

$$\tilde{\mathbf{n}}(s) = \frac{\dot{\tilde{\mathbf{t}}}}{\tilde{\kappa}} = \left( -\cos\left(\frac{s}{\rho}\right), -\sin\left(\frac{s}{\rho}\right), 0 \right)$$

$$\tilde{\mathbf{b}}(s) = \tilde{\mathbf{t}} \times \tilde{\mathbf{n}} = \frac{1}{\rho} \left( H \sin\left(\frac{s}{\rho}\right), -H \cos\left(\frac{s}{\rho}\right), R \right).$$

We are left to compute the torsion of  $\tilde{\gamma}$ :

$$\dot{\tilde{\mathbf{b}}}(s) = \frac{H}{\rho^2} \left( \cos\left(\frac{s}{\rho}\right), \sin\left(\frac{s}{\rho}\right), 0 \right)$$

$$\dot{\tilde{\mathbf{b}}}(s) \cdot \tilde{\mathbf{n}}(s) = -\frac{H}{\rho^2}$$

$$\tilde{\tau}(s) = -\dot{\tilde{\mathbf{b}}}(s) \cdot \tilde{\mathbf{n}}(s) = \frac{H}{\rho^2} = \frac{H}{R^2 + H^2}$$

3. The curvature and torsion of
- $\gamma$
- are

$$\kappa(t) = \tilde{\kappa}(s(t)) = \frac{R}{R^2 + H^2}$$

$$\tau(t) = \tilde{\tau}(s(t)) = \frac{H}{R^2 + H^2}$$

**Theorem 1.34:** Torsion formula

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be regular, with  $\kappa \neq 0$ . The torsion of  $\gamma$  is

$$\tau(t) = \frac{(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \dddot{\gamma}(t)}{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2}.$$

**Example 1.35:** Torsion of the Helix with formula

**Question.** Consider the Helix of radius  $R > 0$  and rise  $H > 0$

$$\gamma(t) = (R \cos(t), R \sin(t), Ht), \quad t \in \mathbb{R}.$$

1. Prove that  $\gamma$  is regular with non-vanishing curvature.
2. Compute the torsion of  $\gamma$ .

**Solution.**

1.  $\gamma$  is regular with non-vanishing curvature, since

$$\|\dot{\gamma}(t)\| = \sqrt{R^2 + H^2} \geq R > 0, \quad \kappa = \frac{R}{R^2 + H^2} > 0.$$

2. We compute the torsion using the formula:

$$\begin{aligned} \dot{\gamma}(t) &= (-R \sin(t), R \cos(t), H) \\ \ddot{\gamma}(t) &= (-R \cos(t), -R \sin(t), 0) \\ \ddot{\gamma}(t) &= (R \sin(t), -R \cos(t), 0) \\ \dot{\gamma} \times \ddot{\gamma} &= (RH \sin(t), -RH \cos(t), R^2) \\ \|\dot{\gamma} \times \ddot{\gamma}\| &= R\sqrt{R^2 + H^2} \\ (\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma} &= R^2 H \\ \tau(t) &= \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \frac{H}{R^2 + H^2} \end{aligned}$$

**Example 1.36:** Calculation of torsion

**Question.** Compute the torsion of the curve

$$\gamma(t) = \left( \frac{8}{5} \cos(t), 1 - 2 \sin(t), \frac{6}{5} \cos(t) \right).$$

**Solution.** Resuming calculations from Example 1.27,

$$\begin{aligned} \ddot{\gamma} &= \left( \frac{8}{5} \sin(t), 2 \cos(t), \frac{6}{5} \sin(t) \right) \\ (\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma} &= \frac{96}{25} \sin(t) - \frac{96}{25} \sin(t) = 0 \\ \tau(t) &= \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = 0 \end{aligned}$$

**Theorem 1.37:** General Frenet frame formulas

The Frenet frame of a regular curve  $\gamma$  is

$$\mathbf{t} = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \quad \mathbf{b} = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t} = \frac{(\dot{\gamma} \times \ddot{\gamma}) \times \dot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\| \|\dot{\gamma}\|}.$$

**Example 1.38:** Twisted cubic

**Question.** Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  be the *twisted cubic*

$$\gamma(t) = (t, t^2, t^3).$$

1. Is  $\gamma$  regular/unit-speed? Justify your answer.
2. Compute the curvature and torsion of  $\gamma$ .
3. Compute the Frenet frame of  $\gamma$ .

**Solution.**

1.  $\gamma$  is regular, but not-unit speed, because

$$\begin{aligned} \dot{\gamma}(t) &= (1, 2t, 3t^2) \\ \|\dot{\gamma}(t)\| &= \sqrt{1 + 4t^2 + 9t^4} \geq 1 \quad \|\dot{\gamma}(1)\| = \sqrt{14} \neq 1 \end{aligned}$$

2. Compute the following quantities

$$\begin{aligned} \ddot{\gamma} &= (0, 2, 6t) & \|\dot{\gamma} \times \ddot{\gamma}\| &= 2\sqrt{1 + 9t^2 + 9t^4} \\ \ddot{\gamma} &= (0, 0, 6) & (\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma} &= 12 \\ \dot{\gamma} \times \ddot{\gamma} &= (6t^2, -6t, 2) \end{aligned}$$

Compute curvature and torsion using the formulas:

$$\begin{aligned} \kappa(t) &= \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}} \\ \tau(t) &= \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \frac{3}{1 + 9t^2 + 9t^4}. \end{aligned}$$

3. By the Frenet frame formulas and the above calculations,

$$\begin{aligned} \mathbf{t} &= \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} (1, 2t, 3t^2) \\ \mathbf{b} &= \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|} = \frac{1}{\sqrt{1 + 9t^2 + 9t^4}} (3t^2, -3t, 1) \\ \mathbf{n} &= \mathbf{b} \times \mathbf{t} = \frac{(-9t^3 - 2t, 1 - 9t^4, 6t^3 + 3t)}{\sqrt{1 + 9t^2 + 9t^4} \sqrt{1 + 4t^2 + 9t^4}} \end{aligned}$$

**1.3 Frenet-Serret equations****Theorem 1.39:** Frenet-Serret equations

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be unit-speed with  $\kappa \neq 0$ . The Frenet frame of  $\gamma$  solves the **Frenet-Serret** equations

$$\dot{\mathbf{t}} = \kappa \mathbf{n}, \quad \dot{\mathbf{n}} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \dot{\mathbf{b}} = -\tau \mathbf{n}.$$

**Definition 1.40:** Rigid motion

A **rigid motion** of  $\mathbb{R}^3$  is a map  $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of the form

$$M(\mathbf{v}) = R\mathbf{v} + \mathbf{p}, \quad \mathbf{v} \in \mathbb{R}^3,$$

where  $\mathbf{p} \in \mathbb{R}^3$ , and  $R \in \text{SO}(3)$  **rotation matrix**,

$$\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} : R^T R = I, \det(R) = 1\}.$$



**Theorem 1.41:** Fundamental Theorem of Space Curves

Let  $\kappa, \tau : (a, b) \rightarrow \mathbb{R}$  be smooth, with  $\kappa > 0$ . Then:

1. There exists a unit-speed curve  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  with curvature  $\kappa(t)$  and torsion  $\tau(t)$ .
2. Suppose that  $\tilde{\gamma} : (a, b) \rightarrow \mathbb{R}^3$  is a unit-speed curve whose curvature  $\tilde{\kappa}$  and torsion  $\tilde{\tau}$  satisfy

$$\tilde{\kappa}(t) = \kappa(t), \quad \tilde{\tau}(t) = \tau(t), \quad \forall t \in (a, b).$$

There exists a rigid motion  $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that

$$\tilde{\gamma}(t) = M(\gamma(t)), \quad \forall t \in (a, b).$$

**Example 1.42:** Application of FTSC

**Question.** Consider the curve

$$\gamma(t) = (\sqrt{3}t - \sin(t), \sqrt{3}\sin(t) + t, 2\cos(t)).$$

1. Calculate the curvature and torsion of  $\gamma$ .
2. The helix of radius  $R$  and rise  $H$  is parametrized by

$$\eta(t) = (R\cos(t), R\sin(t), Ht).$$

Recall that  $\eta$  has curvature and torsion

$$\kappa^\eta = \frac{R}{R^2 + H^2}, \quad \tau^\eta = \frac{H}{R^2 + H^2}.$$

Prove that there exist a rigid motion  $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that

$$\gamma(t) = M(\eta(t)), \quad \forall t \in \mathbb{R}. \quad (1.2)$$

**Solution.**

1. Compute curvature and torsion with the formulas

$$\dot{\gamma}(t) = (\sqrt{3} - \cos(t), \sqrt{3}\cos(t) + 1, -2\sin(t))$$

$$\ddot{\gamma}(t) = (\sin(t), -\sqrt{3}\sin(t), -2\cos(t))$$

$$\ddot{\gamma}(t) = (\cos(t), -\sqrt{3}\cos(t), 2\sin(t))$$

$$\dot{\gamma}(t) \times \ddot{\gamma}(t) = (-2(\sqrt{3} + \cos(t)), 2(\sqrt{3}\cos(t) - 1), -4\sin(t))$$

$$\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|^2 = 32$$

$$\|\dot{\gamma}(t)\|^2 = 8$$

$$(\dot{\gamma}(t) \times \ddot{\gamma}(t)) \cdot \ddot{\gamma}(t) = -8$$

$$\kappa(t) = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{\sqrt{32}}{8^{3/2}} = \frac{1}{4}$$

$$\tau(t) = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = \frac{-8}{32} = -\frac{1}{4}.$$

2. Equating  $\kappa = \kappa^\eta$  and  $\tau = \tau^\eta$ , we obtain

$$\frac{R}{R^2 + H^2} = \frac{1}{4}, \quad \frac{H}{R^2 + H^2} = -\frac{1}{4}$$

Rearranging both equalities we get

$$R^2 + H^2 = 4R, \quad R^2 + H^2 = -4H,$$

from which we find the relation  $R = -H$ . Substituting into  $R^2 + H^2 = -4H$ , we get

$$H = -2, \quad R = -H = 2.$$

For these values of  $R$  and  $H$  we have  $\kappa = \kappa^\eta$  and  $\tau = \tau^\eta$ . By the FTSC, there exists a rigid motion  $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  satisfying (1.2).

**Theorem 1.43:** Curves contained in a plane - Part I

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be regular with  $\kappa \neq 0$ . They are equivalent:

1. The torsion of  $\gamma$  satisfies

$$\tau(t) = 0, \quad \forall t \in (a, b).$$

2.  $\gamma$  is contained in a plane: There exists a vector  $\mathbf{P} \in \mathbb{R}^3$  and a scalar  $d \in \mathbb{R}$  such that

$$\gamma(t) \cdot \mathbf{P} = d, \quad \forall t \in (a, b).$$

**Theorem 1.44:** Curves contained in a plane - Part II

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be regular, with  $\kappa \neq 0$  and  $\tau = 0$ . Then, the binormal  $\mathbf{b}$  is a constant vector, and  $\gamma$  is contained in the plane of equation

$$(\mathbf{x} - \gamma(t_0)) \cdot \mathbf{b} = 0.$$

**Example 1.45:** A planar curve

**Question.** Consider the curve

$$\gamma(t) = (t, 2t, t^4), \quad t > 0.$$

1. Prove that  $\gamma$  is regular.
2. Compute the curvature and torsion of  $\gamma$ .
3. Prove that  $\gamma$  is contained in a plane. Compute the equation of such plane.

**Solution.**

1.  $\gamma$  is regular because  $\dot{\gamma}(t) = (1, 2, 4t^3) \neq \mathbf{0}$ .

2. Compute the following quantities

$$\|\dot{\gamma}\| = \sqrt{5 + 16t^4}$$

$$\dot{\gamma} \times \ddot{\gamma} = 12(2t^2, -t^2, 0)$$

$$\ddot{\gamma} = 12(0, 0, t^2)$$

$$\|\dot{\gamma} \times \ddot{\gamma}\| = 12\sqrt{5}t^2$$

$$\ddot{\gamma} = 24(0, 0, t)$$

$$(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma} = 0$$

Compute curvature and torsion with the formulas

$$\kappa(t) = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{12\sqrt{5}t^2}{\sqrt{5 + 16t^4}^3}$$

$$\tau(t) = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} = 0.$$

3.  $\gamma$  lies in a plane because  $\tau = 0$ . The binormal is

$$\mathbf{b} = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|} = \frac{1}{\sqrt{5}}(2, -1, 0).$$

At  $t_0 = 0$  we have  $\gamma(0) = \mathbf{0}$ . The equation of the plane containing  $\gamma$  is then  $\mathbf{x} \cdot \mathbf{b} = 0$ , which reads

$$\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y = 0 \quad \implies \quad 2x - y = 0.$$

**Theorem 1.46:** Curves contained in a circle

Let  $\gamma : (a, b) \rightarrow \mathbb{R}^3$  be unit-speed. They are equivalent:

1.  $\gamma$  is contained in a circle of radius  $R > 0$ .
2. There exists  $R > 0$  such that

$$\kappa(t) = \frac{1}{R}, \quad \tau(t) = 0, \quad \forall t \in (a, b).$$

**Example 1.47:** A curve contained in a circle

**Question.** Consider the curve

$$\gamma(t) = \left( \frac{4}{5} \cos(t), 1 - \sin(t), -\frac{3}{5} \cos(t) \right).$$

1. Prove that  $\gamma$  is unit-speed.
2. Compute Frenet frame, curvature and torsion of  $\gamma$ .
3. Prove that  $\gamma$  is part of a circle.

**Solution.**

1.  $\gamma$  is unit-speed because

$$\begin{aligned} \dot{\gamma}(t) &= \left( -\frac{4}{5} \sin(t), -\cos(t), \frac{3}{5} \sin(t) \right) \\ \|\dot{\gamma}(t)\|^2 &= \frac{16}{25} \sin^2(t) + \cos^2(t) + \frac{9}{25} \sin^2(t) = 1 \end{aligned}$$

2. As  $\gamma$  is unit-speed, the tangent vector is  $\mathbf{t}(t) = \dot{\gamma}(t)$ . The curvature, normal, binormal and torsion are

$$\begin{aligned} \mathbf{t}(t) &= \left( -\frac{4}{5} \cos(t), \sin(t), \frac{3}{5} \cos(t) \right) \\ \kappa(t) &= \|\dot{\mathbf{t}}(t)\| = \frac{16}{25} \cos^2(t) + \sin^2(t) + \frac{9}{25} \cos^2(t) = 1 \\ \mathbf{n}(t) &= \frac{1}{\kappa(t)} \ddot{\gamma}(t) = \left( -\frac{4}{5} \cos(t), \sin(t), \frac{3}{5} \cos(t) \right) \\ \mathbf{b}(t) &= \dot{\gamma}(t) \times \mathbf{n}(t) = \left( -\frac{3}{5}, 0, -\frac{4}{5} \right) \\ \dot{\mathbf{b}} &= \mathbf{0} \\ \tau &= -\dot{\mathbf{b}} \cdot \mathbf{n} = 0 \end{aligned}$$

3. The curvature of  $\gamma$  is constant and the torsion is zero. Therefore  $\gamma$  is contained in a circle of radius

$$R = \frac{1}{\kappa} = 1.$$

## 2 Topology

### Definition 2.1: Topological space

Let  $X$  be a set and  $\mathcal{T}$  a collection of subsets of  $X$ . We say that  $\mathcal{T}$  is a **topology** on  $X$  if the following 3 properties hold:

- (A1) The sets  $\emptyset, X$  belong to  $\mathcal{T}$ ,
- (A2) If  $\{A_i\}_{i \in I}$  is an arbitrary family of elements of  $\mathcal{T}$ , then

$$\bigcup_{i \in I} A_i \in \mathcal{T}.$$

- (A3) If  $A, B \in \mathcal{T}$  then  $A \cap B \in \mathcal{T}$ .

Further, we say:

- The pair  $(X, \mathcal{T})$  is a **topological space**.
- The elements of  $X$  are called **points**.
- The sets in the topology  $\mathcal{T}$  are called **open sets**.

### Definition 2.2: Trivial topology

Let  $X$  be a set. The **trivial topology** on  $X$  is the collection of sets

$$\mathcal{T}_{\text{trivial}} := \{\emptyset, X\}.$$

### Definition 2.3: Discrete topology

Let  $X$  be a set. The **discrete topology** on  $X$  is the collection of all subsets of  $X$

$$\mathcal{T}_{\text{discrete}} := \{A : A \subseteq X\}.$$

### Definition 2.4: Open set of $\mathbb{R}^n$

Let  $A \subseteq \mathbb{R}^n$ . We say that the set  $A$  is **open** if it holds:

$$\forall \mathbf{x} \in A, \exists r > 0 \text{ s.t. } B_r(\mathbf{x}) \subseteq A, \quad (2.1)$$

where  $B_r(\mathbf{x})$  is the ball of radius  $r > 0$  centered at  $\mathbf{x}$

$$B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| < r\},$$

and the **Euclidean norm** of  $\mathbf{x} \in \mathbb{R}^n$  is defined by

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^n x_i^2}.$$

### Definition 2.5: Euclidean topology of $\mathbb{R}^n$

The **Euclidean topology** on  $\mathbb{R}^n$  is the collection of sets

$$\mathcal{T}_{\text{euclid}} := \{A : A \subseteq \mathbb{R}^n, A \text{ is open}\}.$$

### Proof: $\mathcal{T}_{\text{euclid}}$ is a topology on $\mathbb{R}^n$

To prove  $\mathcal{T}_{\text{euclid}}$  is a topology on  $\mathbb{R}^n$ , we need to check the axioms:

- (A1) We have  $\emptyset, \mathbb{R}^n \in \mathcal{T}_{\text{euclid}}$ : Indeed  $\emptyset$  is open because there is no point  $\mathbf{x}$  for which (2.1) needs to be checked. Moreover,  $\mathbb{R}^n$  is open because (2.1) holds with any radius  $r > 0$ .
- (A2) Let  $A_i \in \mathcal{T}_{\text{euclid}}$  for all  $i \in I$ . Define the union  $A = \bigcup_i A_i$ . We need to check that  $A$  is open. Let  $\mathbf{x} \in A$ . By definition of union, there exists an index  $i_0 \in I$  such that  $\mathbf{x} \in A_{i_0}$ . Since  $A_{i_0}$  is open, by (2.1) there exists  $r > 0$  such that  $B_r(\mathbf{x}) \subseteq A_{i_0}$ . As  $A_{i_0} \subseteq A$ , we conclude that  $B_r(\mathbf{x}) \subseteq A$ , so that  $A \in \mathcal{T}_{\text{euclid}}$ .
- (A3) Let  $A, B \in \mathcal{T}_{\text{euclid}}$ . We need to check that  $A \cap B$  is open. Let  $\mathbf{x} \in A \cap B$ . Therefore  $\mathbf{x} \in A$  and  $\mathbf{x} \in B$ . Since  $A$  and  $B$  are open, by (2.1) there exist  $r_1, r_2 > 0$  such that  $B_{r_1}(\mathbf{x}) \subseteq A$  and  $B_{r_2}(\mathbf{x}) \subseteq B$ . Set  $r := \min\{r_1, r_2\}$ . Then

$$B_r(\mathbf{x}) \subseteq B_{r_1}(\mathbf{x}) \subseteq A, \quad B_r(\mathbf{x}) \subseteq B_{r_2}(\mathbf{x}) \subseteq B,$$

Hence  $B_r(\mathbf{x}) \subseteq A \cap B$ , showing that  $A \cap B \in \mathcal{T}_{\text{euclid}}$ .

This proves that  $\mathcal{T}_{\text{euclid}}$  is a topology on  $\mathbb{R}^n$ .

### Proposition 2.6: $B_r(\mathbf{x})$ is an open set of $\mathcal{T}_{\text{euclid}}$

Let  $\mathbb{R}^n$  be equipped with the Euclidean topology  $\mathcal{T}_{\text{euclid}}$ . Let  $r > 0$  and  $\mathbf{x} \in \mathbb{R}^n$ . Then  $B_r(\mathbf{x}) \in \mathcal{T}_{\text{euclid}}$ .

### Definition 2.7: Closed set

Let  $(X, \mathcal{T})$  be a topological space. A set  $C \subseteq X$  is **closed** if

$$C^c \in \mathcal{T},$$

where  $C^c := X \setminus C$  is the complement of  $C$  in  $X$ .

### Definition 2.8: Comparing topologies

Let  $X$  be a set and let  $\mathcal{T}_1, \mathcal{T}_2$  be topologies on  $X$ .

1.  $\mathcal{T}_1$  is **finer** than  $\mathcal{T}_2$  if  $\mathcal{T}_2 \subseteq \mathcal{T}_1$ .
2.  $\mathcal{T}_1$  is **strictly finer** than  $\mathcal{T}_2$  if  $\mathcal{T}_2 \subsetneq \mathcal{T}_1$ .
3.  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are the **same** topology if  $\mathcal{T}_1 = \mathcal{T}_2$ .

### Example 2.9: Comparing $\mathcal{T}_{\text{trivial}}$ and $\mathcal{T}_{\text{discrete}}$

Let  $X$  be a set. Then  $\mathcal{T}_{\text{trivial}} \subseteq \mathcal{T}_{\text{discrete}}$ .

### Example 2.10: Cofinite topology on $\mathbb{R}$

**Question.** The **cofinite topology** on  $\mathbb{R}$  is the collection of sets

$$\mathcal{T}_{\text{cofinite}} := \{U \subseteq \mathbb{R} : U^c \text{ is finite, or } U^c = \mathbb{R}\}.$$

1. Prove that  $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$  is a topological space.
2. Prove that  $\mathcal{T}_{\text{cofinite}} \subseteq \mathcal{T}_{\text{euclid}}$ .
3. Prove that  $\mathcal{T}_{\text{cofinite}} \neq \mathcal{T}_{\text{euclid}}$ .

**Solution. Part 1.** Show that the topology properties are satisfied:  
(A1) We have  $\emptyset \in \mathcal{T}_{\text{cofinite}}$ , since  $\emptyset^c = \mathbb{R}$ . We have  $\mathbb{R} \in \mathcal{T}_{\text{cofinite}}$  because  $\mathbb{R}^c = \emptyset$  is finite.

(A2) Let  $U_i \in \mathcal{T}_{\text{cofinite}}$  for all  $i \in I$ , and define  $U := \bigcup_{i \in I} U_i$ . By the De Morgan's laws we have

$$U^c = (\cup_{i \in I} U_i)^c = \cap_{i \in I} U_i^c.$$

We have two cases:

1. There exists  $i_0 \in I$  such that  $U_{i_0}^c$  is finite. Then

$$U^c = \cap_{i \in I} U_i^c \subset U_{i_0}^c,$$

and therefore  $U^c$  is finite, showing that  $U \in \mathcal{T}_{\text{cofinite}}$ .

2. None of the sets  $U_i^c$  is finite. Therefore  $U_i^c = \mathbb{R}$  for all  $i \in I$ , from which we deduce

$$U^c = \cap_{i \in I} U_i^c = \mathbb{R} \implies U \in \mathcal{T}_{\text{cofinite}}.$$

In both cases, we have  $U \in \mathcal{T}_{\text{cofinite}}$ , so that (A2) holds.

(A3) Let  $U, V \in \mathcal{T}_{\text{cofinite}}$ . Set  $A = U \cap V$ . Then

$$A^c = U^c \cup V^c.$$

We have 2 possibilities:

1.  $U^c, V^c$  finite: Then  $A^c$  is finite, and  $A \in \mathcal{T}_{\text{cofinite}}$ .
2.  $U^c = \mathbb{R}$  or  $V^c = \mathbb{R}$ : Then  $A^c = \mathbb{R}$ , and  $A \in \mathcal{T}_{\text{cofinite}}$ .

In all cases, we have shown that  $A \in \mathcal{T}_{\text{cofinite}}$ , so that (A3) holds.

**Part 2.** Let  $U \in \mathcal{T}_{\text{cofinite}}$ . We have two cases:

- $U^c$  is finite. Then  $U^c = \{x_1, \dots, x_n\}$  for some points  $x_i \in \mathbb{R}$ . Up to relabeling the points, we can assume that  $x_i < x_j$  when  $i < j$ . Therefore,

$$U = \{x_1, \dots, x_n\}^c = \bigcup_{i=0}^n (x_i, x_{i+1}), \quad x_0 := -\infty, \quad x_{n+1} := \infty.$$

The sets  $(x_i, x_{i+1})$  are open in  $\mathcal{T}_{\text{euclid}}$ , and therefore  $U \in \mathcal{T}_{\text{euclid}}$ .

- $U^c = \mathbb{R}$ . Then  $U = \emptyset$ , which belongs to  $\mathcal{T}_{\text{euclid}}$  by (A1).

In both cases,  $U \in \mathcal{T}_{\text{euclid}}$ . Therefore  $\mathcal{T}_{\text{cofinite}} \subseteq \mathcal{T}_{\text{euclid}}$ .

**Part 3.** consider the interval  $U = (0, 1)$ . Then  $U \in \mathcal{T}_{\text{euclid}}$ . However  $U^c$  is neither  $\mathbb{R}$ , nor finite. Thus  $U \notin \mathcal{T}_{\text{cofinite}}$ .

$\mathcal{T}$ , if the following property holds:

$$\forall U \in \mathcal{T} \text{ s.t. } x_0 \in U, \exists N = N(U) \in \mathbb{N} \text{ s.t. } x_n \in U, \forall n \geq N. \quad (2.2)$$

The convergence of  $x_n$  to  $x_0$  is denoted by  $x_n \rightarrow x_0$ .

**Proposition 2.12:** Convergent sequences in  $\mathcal{T}_{\text{trivial}}$

Let  $X$  be equipped with  $\mathcal{T}_{\text{trivial}}$ . Let  $\{x_n\} \subseteq X$ ,  $x_0 \in X$ . Then  $x_n \rightarrow x_0$ .

**Proof**

To show that  $x_n \rightarrow x_0$  we need to check that (2.2) holds. Let  $U \in \mathcal{T}_{\text{trivial}}$  with  $x_0 \in U$ . We have two cases:

- $U = \emptyset$ : There is nothing to prove, since  $x_0$  cannot be in  $U$ .
- $U = X$ : Take  $N = 1$ . Since  $U = X$ , we have  $x_n \in U$  for all  $n \geq 1$ .

Thus (2.2) holds for all the sets  $U \in \mathcal{T}_{\text{trivial}}$ , showing that  $x_n \rightarrow x_0$ .

**Warning**

Proposition 2.12 shows the topological limit may **not be unique!**

**Proposition 2.13:** Convergent sequences in  $\mathcal{T}_{\text{discrete}}$

Let  $X$  be equipped with  $\mathcal{T}_{\text{discrete}}$ . Let  $\{x_n\} \subseteq X$ ,  $x_0 \in X$ . They are equivalent:

1.  $x_n \rightarrow x_0$  in the topology  $\mathcal{T}_{\text{discrete}}$ .
2.  $\{x_n\}$  is eventually constant:  $\exists N \in \mathbb{N}$  s.t.  $x_n = x_0, \forall n \geq N$

**Proof**

*Part 1.* Assume that  $x_n \rightarrow x_0$ . Let  $U = \{x_0\}$ . Then  $U \in \mathcal{T}_{\text{discrete}}$ . Since  $x_n \rightarrow x_0$ , by (2.2) there exists  $N \in \mathbb{N}$  such that

$$x_n \in U, \quad \forall n \geq N.$$

As  $U = \{x_0\}$ , we infer  $x_n = x_0$  for all  $n \geq N$ . Hence  $x_n$  is eventually constant.

*Part 2.* Assume that  $x_n$  is eventually equal to  $x_0$ , that is, there exists  $N \in \mathbb{N}$  such that

$$x_n = x_0, \quad \forall n \geq N. \quad (2.3)$$

Let  $U \in \mathcal{T}$  be an open set such that  $x_0 \in U$ . By (2.3) we have that

$$x_n \in U, \quad \forall n \geq N.$$

Since  $U$  was arbitrary, we conclude that  $x_n \rightarrow x_0$ .

**Definition 2.14:** Classical convergence in  $\mathbb{R}^n$

Let  $\{\mathbf{x}_n\} \subseteq \mathbb{R}^n$  and  $\mathbf{x}_0 \in \mathbb{R}^n$ . We say that  $\mathbf{x}_n$  converges  $\mathbf{x}_0$  in the classical sense if  $\|\mathbf{x}_n - \mathbf{x}_0\| \rightarrow 0$ , that is,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \|\mathbf{x}_n - \mathbf{x}_0\| < \varepsilon, \forall n \geq N.$$

## 2.1 Sequences

**Definition 2.11:** Convergent sequence

Let  $(X, \mathcal{T})$  be a topological space. Consider a sequence  $\{x_n\} \subseteq X$  and a point  $x \in X$ . We say that  $x_n$  converges to  $x_0$  in the topology

**Proposition 2.15:** Convergent sequences in  $\mathcal{T}_{\text{euclid}}$ 

Let  $\mathbb{R}^n$  be equipped with  $\mathcal{T}_{\text{euclid}}$ . Let  $\{x_n\} \subseteq \mathbb{R}^n$ ,  $x_0 \in \mathbb{R}^n$ . They are equivalent:

1.  $x_n \rightarrow x_0$  in the topology  $\mathcal{T}_{\text{euclid}}$ .
2.  $x_n \rightarrow x_0$  in the classical sense.

## 2.2 Metric spaces

**Definition 2.16:** Distance and Metric space

Let  $X$  be a set. A **distance** on  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  such that, for all  $x, y, z \in X$  they hold:

- (M1) Positivity:  $d(x, y) \geq 0$  and  $d(x, y) = 0 \iff x = y$
- (M2) Symmetry:  $d(x, y) = d(y, x)$
- (M3) Triangle Inequality:  $d(x, z) \leq d(x, y) + d(y, z)$

The pair  $(X, d)$  is called a **metric space**.

**Definition 2.17:** Euclidean distance on  $\mathbb{R}^n$ 

The **Euclidean distance** over  $\mathbb{R}^n$  is defined by

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

**Proposition 2.18**

Let  $d$  be the Euclidean distance on  $\mathbb{R}^n$ . Then  $(\mathbb{R}^n, d)$  is a metric space.

**Definition 2.19:** Topology induced by the metric

Let  $(X, d)$  be a metric space. The set  $A \subseteq X$  is **open** if it holds

$$\forall x \in U, \exists r \in \mathbb{R}, r > 0 \text{ s.t. } B_r(x) \subseteq U,$$

where  $B_r(x)$  is the ball centered at  $x$  of radius  $r$ , defined by

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

The topology **induced by the metric**  $d$  is the collection of sets

$$\mathcal{T}_d = \{U : U \subseteq X, U \text{ open}\}.$$

**Remark 2.20:** Topology induced by Euclidean distance

Consider the metric space  $(\mathbb{R}^n, d)$  with  $d$  the Euclidean distance. Then

$$\mathcal{T}_d = \mathcal{T}_{\text{euclid}},$$

where  $\mathcal{T}_{\text{euclid}}$  is the Euclidean topology on  $\mathbb{R}^n$ .

**Example 2.21:** Discrete distance

**Question.** Let  $X$  be a set. The **discrete distance** is the function

$d : X \times X \rightarrow \mathbb{R}$  defined by

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

1. Prove that  $(X, d)$  is a metric space.
2. Prove that  $\mathcal{T}_d = \mathcal{T}_{\text{discrete}}$ .

**Solution.** See Question 3 in Homework 3.

**Proposition 2.22:** Convergence in metric space

Suppose  $(X, d)$  is a metric space and  $\mathcal{T}_d$  the topology induced by  $d$ . Let  $\{x_n\} \subseteq X$  and  $x_0 \in X$ . They are equivalent:

1.  $x_n \rightarrow x_0$  with respect to the topology  $\mathcal{T}_d$ .
2.  $d(x_n, x_0) \rightarrow 0$  in  $\mathbb{R}$ .
3. For all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$x_n \in B_\varepsilon(x_0), \quad \forall n \geq N.$$

## 2.3 Hausdorff spaces

**Definition 2.23:** Hausdorff space

We say that a topological space  $(X, \mathcal{T})$  is **Hausdorff** if for every  $x, y \in X$  with  $x \neq y$ , there exist  $U, V \in \mathcal{T}$  such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

**Proposition 2.24**

Let  $(X, d)$  be a metric space,  $\mathcal{T}_d$  the topology induced by  $d$ . Then  $(X, \mathcal{T}_d)$  is a Hausdorff space.

**Proof**

Let  $x, y \in X$  with  $x \neq y$ . Define

$$U := B_\varepsilon(x), \quad V := B_\varepsilon(y), \quad \varepsilon := \frac{1}{2} d(x, y).$$

By Proposition 2.24, we know that  $U, V \in \mathcal{T}_d$ . Moreover  $x \in U$ ,  $y \in V$ . We are left to show that  $U \cap V = \emptyset$ . Suppose by contradiction that  $U \cap V \neq \emptyset$  and let  $z \in U \cap V$ . Therefore

$$d(x, z) < \varepsilon, \quad d(y, z) < \varepsilon.$$

By triangle inequality we have

$$d(x, y) \leq d(x, z) + d(y, z) < \varepsilon + \varepsilon = d(x, y),$$

where in the last inequality we used the definition of  $\varepsilon$ . This is a contradiction. Therefore  $U \cap V = \emptyset$  and  $(X, \mathcal{T}_d)$  is Hausdorff.

**Definition 2.25:** Metrizable space

Let  $(X, \mathcal{T})$  be a topological space. We say that the topology  $\mathcal{T}$  is

**metrizable** if there exists a metric  $d$  on  $X$  such that

$$\mathcal{T} = \mathcal{T}_d,$$

with  $\mathcal{T}_d$  the topology induced by  $d$ .

### Corollary 2.26

Let  $(X, \mathcal{T})$  be a metrizable space. Then  $X$  is Hausdorff.

### Example 2.27: $(X, \mathcal{T}_{\text{trivial}})$ is not Hausdorff

**Question.** Let  $X$  be equipped with the trivial topology  $\mathcal{T}_{\text{trivial}}$ . Then  $X$  is not Hausdorff.

**Solution.** Assume by contradiction  $(X, \mathcal{T}_{\text{trivial}})$  is Hausdorff and let  $x, y \in X$  with  $x \neq y$ . Then, there exist  $U, V \in \mathcal{T}_{\text{trivial}}$  such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

In particular  $U \neq \emptyset$  and  $V \neq \emptyset$ . Since  $\mathcal{T} = \{\emptyset, X\}$ , we conclude that

$$U = V = X \implies U \cap V = X \neq \emptyset.$$

This is a contradiction, and thus  $(X, \mathcal{T}_{\text{trivial}})$  is not Hausdorff.

### Example 2.28: $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$ is not Hausdorff

**Question.** Consider the cofinite topology on  $\mathbb{R}$

$$\mathcal{T}_{\text{cofinite}} = \{U \subseteq \mathbb{R} : U^c \text{ is finite, or } U^c = \mathbb{R}\}.$$

Prove that  $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$  is not Hausdorff.

**Solution.** Assume by contradiction  $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$  is Hausdorff and let  $x, y \in \mathbb{R}$  with  $x \neq y$ . Then, there exist  $U, V \in \mathcal{T}_{\text{cofinite}}$  such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Taking the complement of  $U \cap V = \emptyset$ , we infer

$$\mathbb{R} = (U \cap V)^c = U^c \cup V^c. \quad (2.4)$$

There are two possibilities:

1.  $U^c$  and  $V^c$  are finite. Then  $U^c \cup V^c$  is finite, so that (2.4) is a contradiction.
2. Either  $U^c = \mathbb{R}$  or  $V^c = \mathbb{R}$ . If  $U^c = \mathbb{R}$ , then  $U = \emptyset$ . This is a contradiction, since  $x \in U$ . If  $V^c = \mathbb{R}$ , then  $V = \emptyset$ . This is a contradiction, since  $y \in V$ .

Hence  $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$  is not Hausdorff.

### Example 2.29: Lower-limit topology on $\mathbb{R}$ is not Hausdorff

**Question.** The **lower-limit topology** on  $\mathbb{R}$  is the collection of sets

$$\mathcal{T}_{\text{LL}} = \{\emptyset, \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}.$$

1. Prove that  $(\mathbb{R}, \mathcal{T}_{\text{LL}})$  is a topological space.
2. Prove that  $(\mathbb{R}, \mathcal{T}_{\text{LL}})$  is not Hausdorff.

**Solution. Part 1.** We show that  $(\mathbb{R}, \mathcal{T}_{\text{LL}})$  is a topological space by verifying the axioms:

(A1) By definition  $\emptyset, \mathbb{R} \in \mathcal{T}_{\text{LL}}$ .

(A2) Let  $A_i \in \mathcal{T}_{\text{LL}}$  for all  $i \in I$ . We have 2 cases:

- If  $A_i = \emptyset$  for all  $i$ , then  $\cup_i A_i = \emptyset \in \mathcal{T}_{\text{LL}}$ .
- At least one of the sets  $A_i$  is non-empty. As empty-sets do not contribute to the union, we can discard them. Therefore,  $A_i = (-\infty, a_i)$  with  $a_i \in \mathbb{R} \cup \{\infty\}$ . Define:

$$a := \sup_{i \in I} a_i, \quad A := (-\infty, a).$$

Then  $A \in \mathcal{T}$  and:

$$A = \cup_{i \in I} A_i.$$

To prove this, let  $x \in A$ . Then  $x < a$ , so there exists  $i_0 \in I$  such that  $x < a_{i_0}$ . Thus,  $x \in A_{i_0}$ , showing  $A \subseteq \cup_{i \in I} A_i$ . Conversely, if  $x \in \cup_{i \in I} A_i$ , then  $x \in A_{i_0}$  for some  $i_0 \in I$ , implying  $x < a_{i_0} \leq a$ . Thus,  $x \in A$ , proving  $\cup_{i \in I} A_i \subseteq A$ .

(A3) Let  $A, B \in \mathcal{T}_{\text{LL}}$ . We have 3 cases:

- $A = \emptyset$  or  $B = \emptyset$ . Then  $A \cap B = \emptyset \in \mathcal{T}_{\text{LL}}$ .
- $A \neq \emptyset$  and  $B \neq \emptyset$ . Therefore,  $A = (-\infty, a)$  and  $B = (-\infty, b)$  with  $a, b \in \mathbb{R} \cup \{\infty\}$ . Define

$$U := A \cap B, \quad z := \min\{a, b\}.$$

Then  $U = (-\infty, z) \in \mathcal{T}_{\text{LL}}$ .

Thus,  $(\mathbb{R}, \mathcal{T}_{\text{LL}})$  is a topological space.

**Part 2.** To show  $(\mathbb{R}, \mathcal{T}_{\text{LL}})$  is not Hausdorff, assume otherwise. Let  $x, y \in \mathbb{R}$  with  $x \neq y$ . Then there exist  $U, V \in \mathcal{T}_{\text{LL}}$  such that:

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

As  $U, V$  are non-empty, by definition of  $\mathcal{T}_{\text{LL}}$ , there exist  $a, b \in \mathbb{R} \cup \{\infty\}$  such that  $U = (-\infty, a)$  and  $V = (-\infty, b)$ . Define:

$$z := \min\{a, b\}, \quad Z := U \cap V = (-\infty, z).$$

Hence  $Z \neq \emptyset$ , contradicting  $U \cap V = \emptyset$ . Thus,  $(\mathbb{R}, \mathcal{T}_{\text{LL}})$  is not Hausdorff.

### Proposition 2.30: Uniqueness of limit in Hausdorff spaces

Let  $(X, \mathcal{T})$  be a Hausdorff space. If a sequence  $\{x_n\} \subseteq X$  converges, then the limit is unique.

## 2.4 Continuity



**Definition 2.31:** Images and Pre-images

Let  $X, Y$  be sets and  $f : X \rightarrow Y$  be a function.

1. Let  $U \subseteq X$ . The image of  $U$  under  $f$  is the subset of  $Y$  defined by
 
$$f(U) := \{y \in Y : \exists x \in X \text{ s.t. } y = f(x)\} = \{f(x) : x \in X\}.$$
2. Let  $V \subseteq Y$ . The pre-image of  $V$  under  $f$  is the subset of  $X$  defined by

$$f^{-1}(V) := \{x \in X : f(x) \in V\}.$$

**Warning**

The notation  $f^{-1}(V)$  does not mean that we are inverting  $f$ . In fact, the pre-image is defined for all functions.

**Definition 2.32:** Continuous function

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. Let  $f : X \rightarrow Y$  be a function.

1. Let  $x_0 \in X$ . We say that  $f$  is continuous at  $x_0$  if it holds:

$$\forall V \in \mathcal{T}_Y \text{ s.t. } f(x_0) \in V, \exists U \in \mathcal{T}_X \text{ s.t. } x_0 \in U, f(U) \subseteq V.$$

2. We say that  $f$  is continuous from  $(X, \mathcal{T}_X)$  to  $(Y, \mathcal{T}_Y)$  if  $f$  is continuous at each point  $x_0 \in X$ .

**Proposition 2.33**

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. Let  $f : X \rightarrow Y$  be a function. They are equivalent:

1.  $f$  is continuous from  $(X, \mathcal{T}_X)$  to  $(Y, \mathcal{T}_Y)$ .
2. It holds:  $f^{-1}(V) \in \mathcal{T}_X$  for all  $V \in \mathcal{T}_Y$ .

**Example 2.34**

**Question.** Let  $X$  be a set and  $\mathcal{T}_1, \mathcal{T}_2$  be topologies on  $X$ . Define the identity map

$$\text{Id}_X : (X, \mathcal{T}_1) \rightarrow (X, \mathcal{T}_2), \quad \text{Id}_X(x) := x.$$

Prove that they are equivalent:

1.  $\text{Id}_X$  is continuous from  $(X, \mathcal{T}_1)$  to  $(X, \mathcal{T}_2)$ .
2.  $\mathcal{T}_1$  is finer than  $\mathcal{T}_2$ , that is,  $\mathcal{T}_2 \subseteq \mathcal{T}_1$ .

**Solution.**  $\text{Id}_X$  is continuous if and only if

$$\text{Id}_X^{-1}(V) \in \mathcal{T}_1, \quad \forall V \in \mathcal{T}_2.$$

But  $\text{Id}_X^{-1}(V) = V$ , so that the above reads

$$V \in \mathcal{T}_1, \quad \forall V \in \mathcal{T}_2,$$

which is equivalent to  $\mathcal{T}_2 \subseteq \mathcal{T}_1$ .

**Definition 2.35:** Continuity in the classical sense

Let  $f : \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ . We say that  $f$  is continuous at  $\mathbf{x}_0$  if it holds:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon \text{ if } \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

**Proposition 2.36**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and suppose  $\mathbb{R}^n, \mathbb{R}^m$  are equipped with the Euclidean topology. Let  $\mathbf{x}_0 \in \mathbb{R}^n$ . They are equivalent:

1.  $f$  is continuous at  $\mathbf{x}_0$  in the topological sense.
2.  $f$  is continuous at  $\mathbf{x}_0$  in the classical sense.

**Proposition 2.37**

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Denote by  $\mathcal{T}_X$  and  $\mathcal{T}_Y$  the topologies induced by the metrics. Let  $f : X \rightarrow Y$  and  $x_0 \in X$ . They are equivalent:

1.  $f$  is continuous at  $x_0$  in the topological sense.
2. It holds:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.}$$

$$d_Y(f(x), f(x_0)) < \varepsilon \text{ if } d_X(x, x_0) < \delta.$$

**Example 2.38**

**Question.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be a topological space. Suppose that  $\mathcal{T}_Y$  is the trivial topology, that is,

$$\mathcal{T}_Y = \{\emptyset, Y\}.$$

Prove that every function  $f : X \rightarrow Y$  is continuous.

**Solution.**  $f$  is continuous if  $f^{-1}(V) \in \mathcal{T}_X$  for all  $V \in \mathcal{T}_Y$ . We have two cases:

- $V = \emptyset$ : Then  $f^{-1}(V) = f^{-1}(\emptyset) = \emptyset \in \mathcal{T}_X$ .
- $V = Y$ : Then  $f^{-1}(V) = f^{-1}(Y) = X \in \mathcal{T}_X$ .

Therefore  $f$  is continuous.

**Example 2.39**

**Question.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. Suppose that  $\mathcal{T}_Y$  is the discrete topology, that is,

$$\mathcal{T}_Y = \{V \text{ s.t. } V \subseteq Y\}.$$

Let  $f : X \rightarrow Y$ . Prove that they are equivalent:

1.  $f$  is continuous from  $X$  to  $Y$ .
2.  $f^{-1}(\{y\}) \in \mathcal{T}_X$  for all  $y \in Y$ .

**Solution.** Suppose that  $f$  is continuous. Then

$$f^{-1}(V) \in \mathcal{T}_X, \quad \forall V \in \mathcal{T}_Y.$$

As  $V = \{y\} \in \mathcal{T}_Y$ , we conclude that  $f^{-1}(\{y\}) \in \mathcal{T}_X$ .

Conversely, assume that  $f^{-1}(\{y\}) \in \mathcal{T}_X$  for all  $y \in Y$ . Let  $V \in \mathcal{T}_Y$ . Trivially, we have  $V = \cup_{y \in V} \{y\}$ . Therefore

$$f^{-1}(V) = f^{-1}\left(\bigcup_{y \in V} \{y\}\right) = \bigcup_{y \in V} f^{-1}(\{y\}).$$

As  $f^{-1}(\{y\}) \in \mathcal{T}_X$  for all  $y \in Y$ , by property (A2) we conclude that  $f^{-1}(V) \in \mathcal{T}_X$ . Therefore  $f$  is continuous.

### Proposition 2.40: Continuity of compositions

Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z)$  be topological spaces. Assume  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous. Then  $(g \circ f) : X \rightarrow Z$  is continuous.

### Definition 2.41: Homeomorphism

Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological space. A function  $f : X \rightarrow Y$  is called an **homeomorphism** if they hold:

1.  $f$  is continuous.
2.  $f$  admits continuous inverse  $f^{-1} : Y \rightarrow X$ .

## 2.5 Subspace topology

### Definition 2.42: Subspace topology

Let  $(X, \mathcal{T})$  be a topological space and  $Y \subseteq X$  a subset. Define the family of sets

$$\begin{aligned}\mathcal{S} &:= \{A \subseteq Y : \exists U \in \mathcal{T} \text{ s.t. } A = U \cap Y\} \\ &= \{U \cap Y, U \in \mathcal{T}\}.\end{aligned}$$

The family  $\mathcal{S}$  is the **subspace topology** on  $Y$  induced by the inclusion  $Y \subseteq X$ .

### Proposition 2.43

Let  $(X, \mathcal{T})$  be a topological space and  $Y \in \mathcal{T}$ . Let  $A \subseteq Y$ . Then

$$A \in \mathcal{S} \iff A \in \mathcal{T}.$$

### Warning

Let  $(X, \mathcal{T})$  be a topological space,  $A \subseteq Y \subseteq X$ . In general we could have

$$A \in \mathcal{S} \text{ and } A \notin \mathcal{T}.$$

**Example.** Let  $X = \mathbb{R}$  with  $\mathcal{T}_{\text{euclid}}$ . Consider the subset  $Y = [0, 2)$ , and equip  $Y$  with the subspace topology  $\mathcal{S}$ . Let  $A = [0, 1)$ . Then  $A \notin \mathcal{T}_{\text{euclid}}$  but  $A \in \mathcal{S}$ , since

$$A = (-1, 1) \cap Y, \quad (-1, 1) \in \mathcal{T}_{\text{euclid}}.$$

### Example 2.44

**Question.** Let  $X = \mathbb{R}$  be equipped with  $\mathcal{T}_{\text{euclid}}$ . Let  $\mathcal{S}$  be the subspace topology on  $\mathbb{Z}$ . Prove that

$$\mathcal{S} = \mathcal{T}_{\text{discrete}}.$$

**Solution.** To prove that  $\mathcal{S} = \mathcal{T}_{\text{discrete}}$ , we need to show that all the subsets of  $\mathbb{Z}$  are open in  $\mathcal{S}$ .

1. Let  $z \in \mathbb{Z}$  be arbitrary. Notice that

$$\{z\} = (z - 1, z + 1) \cap \mathbb{Z}$$

and  $(z - 1, z + 1) \in \mathcal{T}_{\text{euclid}}$ . Thus  $\{z\} \in \mathcal{S}$ .

2. Let now  $A \subseteq \mathbb{Z}$  be an arbitrary subset. Trivially,

$$A = \cup_{z \in A} \{z\}.$$

As  $\{z\} \in \mathcal{S}$ , we infer that  $A \in \mathcal{S}$  by (A2).

## 2.6 Connectedness

### Definition 2.45: Connected space

Let  $(X, \mathcal{T})$  be a topological space. We say that:

1.  $X$  is **connected** if the only subsets of  $X$  which are both open and closed are  $\emptyset$  and  $X$ .
2.  $X$  is **disconnected** if it is not connected.

### Definition 2.46: Proper subset

Let  $X$  be a set. A subset  $A \subseteq X$  is **proper** if  $A \neq \emptyset$  and  $A \neq X$ .

### Proposition 2.47: Equivalent definition for connectedness

Let  $(X, \mathcal{T})$  be a topological space. They are equivalent:

1.  $X$  is disconnected.
2.  $X$  is the disjoint union of two proper open subsets.
3.  $X$  is the disjoint union of two proper closed subsets.

### Example 2.48

**Question.** Consider the set  $X = \{0, 1\}$  with the subspace topology induced by the inclusion  $X \subseteq \mathbb{R}$ , where  $\mathbb{R}$  is equipped with the Euclidean topology  $\mathcal{T}_{\text{euclid}}$ . Prove that  $X$  is disconnected.

**Solution.** Note that

$$X = \{0\} \cup \{1\}, \quad \{0\} \cap \{1\} = \emptyset.$$

The set  $\{0\}$  is open for the subspace topology, since

$$\{0\} = X \cap (-1, 1), \quad (-1, 1) \in \mathcal{T}_{\text{euclid}}.$$

Similarly, also  $\{1\}$  is open for the subspace topology, since

$$\{1\} = X \cap (0, 2), \quad (0, 2) \in \mathcal{T}_{\text{euclid}}.$$

Since  $\{0\}$  and  $\{1\}$  are proper subsets of  $X$ , we conclude that  $X$  is disconnected.

### Example 2.49

**Question.** Let  $\mathbb{R}$  be equipped with  $\mathcal{T}_{\text{euclid}}$ , and let  $p \in \mathbb{R}$ . Prove that the set  $X = \mathbb{R} \setminus \{p\}$  is disconnected.

**Solution.** Define the sets

$$A = (-\infty, p), \quad B = (p, \infty).$$



$A$  and  $B$  are proper subsets of  $X$ . Moreover

$$X = A \cup B, \quad A \cap B = \emptyset.$$

Finally,  $A, B$  are open for the subspace topology on  $X$ , since they are open in  $(\mathbb{R}, \mathcal{T}_{\text{euclid}})$ . Therefore  $X$  is disconnected.

### Theorem 2.50

Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological spaces. Suppose that  $f : X \rightarrow Y$  is continuous and let  $f(X) \subseteq Y$  be equipped with the subspace topology. If  $X$  is connected, then  $f(X)$  is connected.

### Theorem 2.51: Connectedness is topological invariant

Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be homeomorphic topological spaces. Then

$$X \text{ is connected} \iff Y \text{ is connected}$$

### Example 2.52

**Question.** Define the one dimensional unit circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Prove that  $\mathbb{S}^1$  and  $[0, 1]$  are not homeomorphic.

**Solution.** Suppose by contradiction that there exists a homeomorphism

$$f : [0, 1] \rightarrow \mathbb{S}^1.$$

The restriction of  $f$  to  $[0, 1] \setminus \{\frac{1}{2}\}$  defines a homeomorphism

$$g : ([0, 1] \setminus \{\frac{1}{2}\}) \rightarrow (\mathbb{S}^1 \setminus \{\mathbf{p}\}), \quad \mathbf{p} := f\left(\frac{1}{2}\right).$$

The set  $[0, 1] \setminus \{\frac{1}{2}\}$  is disconnected, since

$$[0, 1] \setminus \{1/2\} = [0, 1/2) \cup (1/2, 1]$$

with  $[0, 1/2)$  and  $(1/2, 1]$  open for the subset topology, non-empty and disjoint. Therefore, using that  $g$  is a homeomorphism, we conclude that also  $\mathbb{S}^1 \setminus \{\mathbf{p}\}$  is disconnected. Let  $\theta_0 \in [0, 2\pi)$  be the unique angle such that

$$\mathbf{p} = (\cos(\theta_0), \sin(\theta_0)).$$

Thus  $\mathbb{S}^1 \setminus \{\mathbf{p}\}$  is parametrized by

$$\gamma(t) := (\cos(t), \sin(t)), \quad t \in (\theta_0, \theta_0 + 2\pi).$$

Since  $\gamma$  is continuous and  $(\theta_0, \theta_0 + 2\pi)$  is connected, by Theorem 2.50, we conclude that  $\mathbb{S}^1 \setminus \{\mathbf{p}\}$  is connected. Contradiction.

### Definition 2.53: Interval

A subset  $I \subseteq \mathbb{R}$  is an interval if it holds:

$$\forall a, b \in I, x \in \mathbb{R} \text{ s.t. } a < x < b \implies x \in I.$$

### Theorem 2.54: Intervals are connected

Let  $\mathbb{R}$  be equipped with the Euclidean topology and let  $I \subseteq \mathbb{R}$ . They are equivalent:

1.  $I$  is connected.
2.  $I$  is an interval.

### Theorem 2.55: Intermediate Value Theorem

Let  $(X, \mathcal{T})$  be a connected topological space. Suppose that  $f : X \rightarrow \mathbb{R}$  is continuous. Suppose that  $a, b \in X$  are such that  $f(a) < f(b)$ . It holds:

$$\forall c \in \mathbb{R} \text{ s.t. } f(a) < c < f(b), \exists \xi \in X \text{ s.t. } f(\xi) = c.$$

### Example 2.56: Intervals are connected - Alternative proof

**Question.** Prove the following statements.

1. Let  $(X, \mathcal{T})$  be a disconnected topological space. Prove that there exists a function  $f : X \rightarrow \{0, 1\}$  which is continuous and surjective.
2. Consider  $\mathbb{R}$  equipped with the Euclidean topology. Let  $I \subseteq \mathbb{R}$  be an interval. Use point (1), and the Intermediate Value Theorem in  $\mathbb{R}$  (see statement below), to show that  $I$  is connected.

*Intermediate Value Theorem in  $\mathbb{R}$ :* Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, and  $f(a) < f(b)$ . Let  $c \in \mathbb{R}$  be such that  $f(a) < c < f(b)$ . Then, there exists  $\xi \in [a, b]$  such that  $f(\xi) = c$ .

**Solution. Part 1.** Since  $X$  is disconnected, there exist  $A, B \in \mathcal{T}$  proper and such that

$$X = A \cup B, \quad A \cap B = \emptyset.$$

Define  $f : X \rightarrow \{0, 1\}$  by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

Since  $A$  and  $B$  are non-empty, it follows that  $f$  is surjective. Moreover  $f$  is continuous: Indeed suppose  $U \subseteq \mathbb{R}$  is open. We have 4 cases:

- $0, 1 \notin U$ . Then  $f^{-1}(U) = \emptyset \in \mathcal{T}$ .
- $0 \in U, 1 \notin U$ . Then  $f^{-1}(U) = A \in \mathcal{T}$ .
- $0 \notin U, 1 \in U$ . Then  $f^{-1}(U) = B \in \mathcal{T}$ .
- $0, 1 \in U$ . Then  $f^{-1}(U) = X \in \mathcal{T}$ .

Then  $f^{-1}(U) \in \mathcal{T}$  for all  $U \subseteq \mathbb{R}$  open, showing that  $f$  is continuous.

**Part 2.** Let  $I \subseteq \mathbb{R}$  be an interval. Suppose by contradiction  $I$  is disconnected. By Point (1), there exists a map  $f : I \rightarrow \{0, 1\}$  which is continuous and surjective. As  $f$  is surjective, there exist  $a, b \in I$  such that

$$f(a) = 0, \quad f(b) = 1.$$

Since  $f$  is continuous, and  $f(a) = 0 < 1 = f(b)$ , by the *Intermediate Value Theorem in  $\mathbb{R}$* , there exists  $\xi \in [a, b]$  such that  $f(\xi) = 1/2$ . As  $I$  is an interval,  $a, b \in I$ , and  $a \leq \xi \leq b$ , it follows that  $\xi \in I$ . This is a contradiction, since  $f$  maps  $I$  into  $\{0, 1\}$ , and  $f(\xi) = 1/2 \notin \{0, 1\}$ . Therefore  $I$  is connected.

## 2.7 Path-connectedness

### Definition 2.57: Path-connectedness

Let  $(X, \mathcal{T})$  be a topological space. We say that  $X$  is **path-connected** if for every  $x, y \in X$  there exist  $a, b \in \mathbb{R}$  with  $a < b$ , and a continuous function

$$\alpha : [a, b] \rightarrow X \quad \text{s.t.} \quad \alpha(a) = x, \quad \alpha(b) = y.$$

### Theorem 2.58: Path-connectedness implies connectedness

Let  $(X, \mathcal{T})$  be a path-connected topological space. Then  $X$  is connected.

### Example 2.59

**Question.** Let  $A \subseteq \mathbb{R}^n$  be convex. Show that  $A$  is path-connected, and hence connected.

**Solution.**  $A$  is convex if for all  $x, y \in A$  the segment connecting  $x$  to  $y$  is contained in  $A$ , namely,

$$[x, y] := \{(1-t)x + ty : t \in [0, 1]\} \subseteq A.$$

Therefore we can define

$$\alpha : [0, 1] \rightarrow A, \quad \alpha(t) := (1-t)x + ty.$$

Clearly  $\alpha$  is continuous, and  $\alpha(0) = x, \alpha(1) = y$ .

### Example 2.60: Spaces of matrices

Let  $\mathbb{R}^{2 \times 2}$  denote the space of real  $2 \times 2$  matrices. Assume  $\mathbb{R}^{2 \times 2}$  has the euclidean topology obtained by identifying it with  $\mathbb{R}^4$ .

1. Consider the set of orthogonal matrices

$$\text{O}(2) = \{A \in \mathbb{R}^{2 \times 2} : A^T A = I\}.$$

Prove that  $\text{O}(2)$  is disconnected.

2. Consider the set of rotations

$$\text{SO}(2) = \{A \in \mathbb{R}^{2 \times 2} : A^T A = I, \det(A) = 1\}.$$

Prove that  $\text{SO}(2)$  is path-connected, and hence connected.

**Solution.** Let  $A \in \text{O}(2)$ , and denote its entries by  $a, b, c, d$ . By direct calculation, the condition  $A^T A = I$  is equivalent to

$$a^2 + b^2 = 1, \quad b^2 + c^2 = 1, \quad ac + bd = 0.$$

From the first condition, we get that  $a = \cos(t)$  and  $b = \sin(t)$ , for a suitable  $t \in [0, 2\pi)$ . From the second and third conditions, we get  $c = \pm \sin(t)$  and  $d = \mp \cos(t)$ . We decompose  $\text{O}(2)$  as

$$\text{O}(2) = A \cup B,$$

$$A = \text{SO}(2) = \left\{ \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, t \in [0, 2\pi) \right\}$$

$$B = \left\{ \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}, t \in [0, 2\pi) \right\}.$$

1. The determinant function  $\det : \text{O}(2) \rightarrow \mathbb{R}$  is continuous. If  $M \in A$ , we have  $\det(M) = 1$ . If instead  $M \in B$ , we have  $\det(M) = -1$ . Moreover,

$$\det^{-1}(\{1\}) = A, \quad \det^{-1}(\{-1\}) = B.$$

As  $\det$  is continuous, and  $\{0\}, \{1\}$  closed, we conclude that  $A$  and  $B$  are closed. Therefore,  $A$  and  $B$  are closed, proper and disjoint. Since  $\text{O}(2) = A \cup B$ , we conclude that  $\text{O}(2)$  is disconnected.

2. Define the function  $\psi : [0, 2\pi) \rightarrow \text{SO}(2)$  by

$$\psi(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Clearly,  $\psi$  is continuous. Let  $R, Q \in \text{SO}(2)$ . Then  $R$  is determined by an angle  $t_1$ , while  $Q$  by an angle  $t_2$ . Up to swapping  $R$  and  $Q$ , we can assume  $t_1 < t_2$ . Define the function  $f : [0, 1] \rightarrow \text{SO}(2)$  by

$$f(\lambda) = \psi(t_1(1-\lambda) + t_2\lambda).$$

Then,  $f$  is continuous and

$$f(0) = \psi(t_1) = R, \quad f(1) = \psi(t_2) = Q.$$

Thus  $\text{SO}(2)$  is path-connected.

### Warning

In general connectedness does not imply path-connectedness, as seen in Proposition 2.92.

# 3 Surfaces

## Definition 3.1: Topology of $\mathbb{R}^n$

The Euclidean norm on  $\mathbb{R}^n$  is denoted by

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^n x_i^2}, \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Define the Euclidean distance  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ .

1. The pair  $(\mathbb{R}^n, d)$  is a metric space.
2. The topology induced by the metric  $d$  is called the Euclidean topology, denoted by  $\mathcal{T}$ .
3. A set  $U \subseteq \mathbb{R}^n$  is **open** if for all  $\mathbf{x} \in U$  there exists  $\varepsilon > 0$  such that  $B_\varepsilon(\mathbf{x}) \subseteq U$ , where

$$B_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}$$

is the open ball of radius  $\varepsilon > 0$  centered at  $\mathbf{x}$ . We write  $U \in \mathcal{T}$ , with  $\mathcal{T}$  the Euclidean topology in  $\mathbb{R}^n$ .

4. A set  $V \subseteq \mathbb{R}^n$  is **closed** if  $V^c := \mathbb{R}^n \setminus V$  is open.

## Definition 3.2: Subspace Topology

Let  $A \subseteq \mathbb{R}^n$ . The **subspace topology** on  $A$  is the family

$$\mathcal{T}_A := \{U \subseteq A : \exists W \in \mathcal{T} \text{ s.t. } U = A \cap W\}.$$

If  $U \in \mathcal{T}_A$ , we say that  $U$  is open in  $A$ .

## Definition 3.3: Continuous Function

Let  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $U$  open. We say that  $f$  is **continuous** at  $\mathbf{x} \in U$  if  $\forall \varepsilon > 0, \exists \delta > 0$  such that

$$\|\mathbf{x} - \mathbf{y}\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon.$$

$f$  is continuous in  $U$  if it is continuous for all  $\mathbf{x} \in U$ .

## Theorem 3.4: Continuity: Topological definition

Let  $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ , with  $U, V$  open. We have that  $f$  is continuous if and only if  $f^{-1}(A)$  is open in  $U$ , for all  $A$  open in  $V$ .

## Definition 3.5: Homeomorphism

Let  $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$  with  $U, V$  open. We say that  $f$  is a **homeomorphism** if:

1.  $f$  is continuous;
2.  $f$  admits continuous inverse  $f^{-1} : V \rightarrow U$ .

## Definition 3.6: Differentiable Function

Let  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $U$  open. We say that  $f$  is **differentiable** at  $\mathbf{x} \in U$  if there exists a linear map  $d_{\mathbf{x}}f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$d_{\mathbf{x}}f(\mathbf{h}) = \lim_{\varepsilon \rightarrow 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{h}) - f(\mathbf{x})}{\varepsilon},$$

for all  $\mathbf{h} \in \mathbb{R}^n$ , where the limit is taken in  $\mathbb{R}^m$ . The linear map  $d_{\mathbf{x}}f$  is called the **differential** of  $f$  at  $\mathbf{x}$ .

## Definition 3.7: Partial Derivative

Let  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $U$  open,  $f$  differentiable. The **partial derivative** of  $f$  at  $\mathbf{x} \in U$  in direction  $\mathbf{e}_i$  is

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) := d_{\mathbf{x}}f(\mathbf{e}_i) = \lim_{\varepsilon \rightarrow 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{e}_i) - f(\mathbf{x})}{\varepsilon}.$$

## Definition 3.8: Jacobian Matrix

Let  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable. The **Jacobian** of  $f$  at  $\mathbf{x}$  is the  $m \times n$  matrix of partial derivatives:

$$Jf(\mathbf{x}) := \left( \frac{\partial f_i}{\partial x_j}(\mathbf{x}) \right)_{i,j} \in \mathbb{R}^{m \times n}.$$

If  $m = n$  then  $Jf \in \mathbb{R}^{n \times n}$  is a square matrix and we can compute its determinant, denoted by  $\det(Jf)$ .

## Proposition 3.9: Matrix representation of $d_{\mathbf{x}}f$

Let  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable. The matrix of the linear map  $d_{\mathbf{x}}f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with respect to the standard basis is given by the Jacobian matrix  $Jf(\mathbf{x})$ .

## Definition 3.10: Diffeomorphism

Let  $f : U \rightarrow V$ , with  $U, V \subseteq \mathbb{R}^n$  open. We say that  $f$  is a **diffeomorphism** between  $U$  and  $V$  if:

1.  $f$  is smooth,
2.  $f$  admits smooth inverse  $f^{-1} : V \rightarrow U$ .

## Definition 3.11: Local diffeomorphism

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a **local diffeomorphism** at  $\mathbf{x}_0 \in \mathbb{R}^n$  if:

1. There exists an open set  $U \subseteq \mathbb{R}^n$  such that  $\mathbf{x}_0 \in U$ ,
2. There exists an open set  $V \subseteq \mathbb{R}^n$  such that  $f(\mathbf{x}_0) \in V$ ,
3.  $f : U \rightarrow V$  is a diffeomorphism.

**Proposition 3.12**

Diffeomorphisms are local diffeomorphisms.

**Proposition 3.13:** Necessary condition for being diffeomorphism

Let  $f : U \rightarrow \mathbb{R}^n$  with  $U \subseteq \mathbb{R}^n$  open. Suppose  $f$  is a local diffeomorphism at  $\mathbf{x}_0 \in U$ . Then  $\det Jf(\mathbf{x}_0) \neq 0$ .

**Theorem 3.14:** Inverse Function Theorem

Let  $f : U \rightarrow \mathbb{R}^n$  with  $U \subseteq \mathbb{R}^n$  open,  $f$  smooth. Assume

$$\det Jf(\mathbf{x}_0) \neq 0,$$

for some  $\mathbf{x}_0 \in U$ . Then:

1. There exists an open set  $U_0 \subseteq U$  such that  $\mathbf{x}_0 \in U_0$ ,
2. There exists an open set  $V$  such that  $f(\mathbf{x}_0) \in V$ ,
3.  $f : U_0 \rightarrow V$  is a diffeomorphism.

**Example 3.15:** A local diffeomorphism which is not global

**Question.** Define the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x, y) = (e^x \cos(y), e^x \sin(y)).$$

Prove  $f$  is a local diffeomorphism but not a diffeomorphism.

**Solution.**  $f$  is a local diffeomorphism at each point  $(x, y) \in \mathbb{R}^2$  by the Inverse Function Theorem, since

$$Jf(x, y) = e^x \begin{pmatrix} \cos(y) & \sin(y) \\ -\sin(y) & \cos(y) \end{pmatrix}$$

$$\det Jf(x, y) = e^{2x} \neq 0.$$

However,  $f$  is not invertible because it is not injective, since

$$f(x, y) = f(x, y + 2n\pi), \quad \forall (x, y) \in \mathbb{R}^2, n \in \mathbb{N}.$$

Hence,  $f$  cannot be a diffeomorphism of  $\mathbb{R}^2$  into  $\mathbb{R}^2$ .

## 3.1 Regular Surfaces

**Definition 3.16:** Surface

Let  $\mathcal{S} \subseteq \mathbb{R}^3$  be a connected set. We say that  $\mathcal{S}$  is a **surface** if for every point  $\mathbf{p} \in \mathcal{S}$  there exist an open set  $U \subseteq \mathbb{R}^2$ , and a smooth map  $\sigma : U \rightarrow \sigma(U) \subseteq \mathcal{S}$  such that

1.  $\mathbf{p} \in \sigma(U)$ ,
2.  $\sigma(U)$  is open in  $\mathcal{S}$ ,
3.  $\sigma$  is a homeomorphism between  $U$  and  $\sigma(U)$ .

$\sigma$  is called a **surface chart** at  $\mathbf{p}$ .

**Definition 3.17:** Atlas of a surface

Let  $\mathcal{S}$  be a surface. Assume given a collection of charts

$$\mathcal{A} = \{\sigma_i\}_{i \in I}, \quad \sigma_i : U_i \rightarrow \sigma(U_i) \subseteq \mathcal{S}.$$

The family  $\mathcal{A}$  is an **atlas** of  $\mathcal{S}$  if

$$\mathcal{S} = \bigcup_{i \in I} \sigma_i(U_i).$$

**Definition 3.18:** Regular Chart

Let  $U \subseteq \mathbb{R}^2$  be open. A map  $\sigma = \sigma(u, v) : U \rightarrow \mathbb{R}^3$  is a **regular chart** if the partial derivatives

$$\sigma_u(u, v) = \frac{d\sigma}{du}(u, v), \quad \sigma_v(u, v) = \frac{d\sigma}{dv}(u, v)$$

are linearly independent vectors of  $\mathbb{R}^3$  for all  $(u, v) \in U$ .

**Definition 3.19:** Regular surface

Let  $\mathcal{S}$  be a surface. We say that:

- $\mathcal{A}$  is a **regular atlas** if any  $\sigma$  in  $\mathcal{A}$  is regular.
- $\mathcal{S}$  is a **regular surface** if it admits a regular atlas.

**Theorem 3.20:** Characterization of regular charts

Let  $\sigma : U \rightarrow \mathbb{R}^3$  with  $U \subseteq \mathbb{R}^2$  open. They are equivalent:

1.  $\sigma$  is a regular chart.
2.  $d_{\mathbf{x}}\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is injective for all  $\mathbf{x} \in U$ .
3. The Jacobian matrix  $J\sigma$  has rank 2 for all  $(u, v) \in U$ .
4.  $\sigma_u \times \sigma_v \neq 0$  for all  $(u, v) \in U$ .

**Example 3.21:** Unit cylinder

**Question.** Consider the infinite unit cylinder

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}.$$

$\mathcal{S}$  is a surface with atlas  $\mathcal{A} = \{\sigma_1, \sigma_2\}$ , with

$$\sigma(u, v) = (\cos(u), \sin(u), v), \quad \sigma_1 = \sigma|_{U_1}, \quad \sigma_2 = \sigma|_{U_2},$$

$$U_1 = \left(0, \frac{3\pi}{2}\right) \times \mathbb{R}, \quad U_2 = \left(\pi, \frac{5\pi}{2}\right) \times \mathbb{R}.$$

Prove that  $\mathcal{S}$  is a regular surface.

**Solution.** The map  $\sigma$  is regular because

$$\sigma_u = (-\sin(u), \cos(u), 0), \quad \sigma_v = (0, 0, 1),$$

are linearly independent, since the last components of  $\sigma_u$  and  $\sigma_v$  are 0 and 1. Therefore, also  $\sigma_1$  and  $\sigma_2$  are regular charts, being restrictions of  $\sigma$ . Thus,  $\mathcal{A}$  is a regular atlas and  $\mathcal{S}$  a regular surface.

**Example 3.22:** Graph of a function

**Question.** Let  $f : U \rightarrow \mathbb{R}$  be smooth,  $U \subseteq \mathbb{R}^2$  open. Define

$$\Gamma_f = \{(u, v, f(u, v)) : (u, v) \in U\},$$

the graph of  $f$ . Then  $\Gamma_f$  is surface with atlas  $\mathcal{A} = \{\sigma\}$ , where

$$\sigma : U \rightarrow \Gamma_f, \quad \sigma(u, v) := (u, v, f(u, v)).$$

Prove that  $\Gamma_f$  is a regular surface.

**Solution.** The Jacobian matrix of  $\sigma$  is

$$J\sigma(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}.$$

$J\sigma$  has rank 2, because the first minor is the  $2 \times 2$  identity matrix. Therefore,  $\sigma$  is regular. This implies  $\mathcal{A}$  is a regular atlas, and  $\mathcal{S}$  is a regular surface.

### Definition 3.23: Spherical coordinates

The **spherical coordinates** of  $\mathbf{p} = (x, y, z) \neq \mathbf{0}$  are

$$\begin{aligned} \mathbf{p} &= (\rho \cos(\theta) \cos(\varphi), \rho \sin(\theta) \cos(\varphi), \rho \sin(\varphi)), \\ \rho &= \sqrt{x^2 + y^2 + z^2}, \quad \theta \in [-\pi, \pi], \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{aligned}$$

### Example 3.24: Unit sphere in spherical coordinates

**Question.** Consider the unit sphere in  $\mathbb{R}^3$

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Prove that  $\sigma : U \rightarrow \mathbb{R}^3$  is regular, where

$$\begin{aligned} \sigma(\theta, \varphi) &= (\cos(\theta) \cos(\varphi), \sin(\theta) \cos(\varphi), \sin(\varphi)), \\ U &= \left\{(\theta, \varphi) \in \mathbb{R}^2 : \theta \in (-\pi, \pi), \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}. \end{aligned}$$

**Solution.** The chart  $\sigma$  is regular because

$$\begin{aligned} \sigma_\theta &= (-\sin(\theta) \cos(\varphi), \cos(\theta) \cos(\varphi), 0) \\ \sigma_\varphi &= (-\cos(\theta) \sin(\varphi), -\sin(\theta) \sin(\varphi), \cos(\varphi)) \\ \sigma_\theta \times \sigma_\varphi &= (\cos(\theta) \cos^2(\varphi), \sin(\theta) \cos^2(\varphi), \cos(\varphi) \sin(\varphi)) \\ \|\sigma_\theta \times \sigma_\varphi\| &= |\cos(\varphi)| = \cos(\varphi) \neq 0, \end{aligned}$$

where we used that  $\cos(\varphi) > 0$ , since  $\varphi \in (-\pi/2, \pi/2)$ .

### Example 3.25: A non-regular chart

**Question.** Prove that the following chart is not regular

$$\sigma(u, v) = (u, v^2, v^3).$$

**Solution.** We have

$$\sigma_v = (0, 2v, 3v^2), \quad \sigma_v(u, 0) = (0, 0, 0).$$

$\sigma$  is not regular because  $\sigma_u$  and  $\sigma_v$  are linearly dependent along the line  $L = \{(u, 0) : u \in \mathbb{R}\}$ .

### Definition 3.26: Reparametrization

Suppose that  $U, \tilde{U} \subseteq \mathbb{R}^2$  are open sets and

$$\sigma : U \rightarrow \mathbb{R}^3, \quad \tilde{\sigma} : \tilde{U} \rightarrow \mathbb{R}^3,$$

are surface charts. We say that  $\tilde{\sigma}$  is a **reparametrization** of  $\sigma$  if

there exists a diffeomorphism  $\Phi : \tilde{U} \rightarrow U$  such that

$$\tilde{\sigma} = \sigma \circ \Phi.$$

### Theorem 3.27: Reparametrizations of regular charts are regular

Let  $U, \tilde{U} \subseteq \mathbb{R}^2$  be open and  $\sigma : U \rightarrow \mathbb{R}^3$  be regular. Suppose given a diffeomorphism  $\Phi : \tilde{U} \rightarrow U$ . The reparametrization

$$\tilde{\sigma} : \tilde{U} \rightarrow \mathbb{R}^3, \quad \tilde{\sigma} = \sigma \circ \Phi$$

is a regular chart, and it holds

$$\tilde{\sigma}_u \times \tilde{\sigma}_v = \det J\Phi (\sigma_u \times \sigma_v).$$

### Definition 3.28: Transition map

Let  $\mathcal{S}$  be a regular surface,  $\sigma : U \rightarrow \mathcal{S}$ ,  $\tilde{\sigma} : \tilde{U} \rightarrow \mathcal{S}$  regular charts. Suppose the images of  $\sigma$  and  $\tilde{\sigma}$  overlap

$$I := \sigma(U) \cap \tilde{\sigma}(\tilde{U}) \neq \emptyset.$$

$I$  is open in  $\mathcal{S}$ , being intersection of open sets. Define

$$V := \sigma^{-1}(I) \subseteq U, \quad \tilde{V} := \tilde{\sigma}^{-1}(I) \subseteq \tilde{U}.$$

$V$  and  $\tilde{V}$  are open, by continuity of  $\sigma$  and  $\tilde{\sigma}$ . Moreover, as  $\sigma$  and  $\tilde{\sigma}$  are homeomorphisms, we have  $\sigma(V) = \tilde{\sigma}(\tilde{V}) = I$ . Therefore, they are well defined the restriction homeomorphisms

$$\sigma|_V : V \rightarrow I, \quad \tilde{\sigma}|_{\tilde{V}} : \tilde{V} \rightarrow I.$$

The **transition map** from  $\sigma$  to  $\tilde{\sigma}$  is the homeomorphism

$$\Phi : \tilde{V} \rightarrow V, \quad \Phi := \sigma^{-1} \circ \tilde{\sigma}.$$

### Theorem 3.29

Transition maps between regular charts are diffeomorphisms.

### Theorem 3.30: Transition maps are reparametrizations

Let  $\mathcal{S}$  be a regular surface,  $\sigma : U \rightarrow \mathcal{S}$ ,  $\tilde{\sigma} : \tilde{U} \rightarrow \mathcal{S}$  regular charts, with  $I := \sigma(U) \cap \tilde{\sigma}(\tilde{U}) \neq \emptyset$ . Define the transition map

$$\Phi = \sigma^{-1} \circ \tilde{\sigma} : \tilde{V} \rightarrow V, \quad V = \sigma^{-1}(I), \quad \tilde{V} = \tilde{\sigma}^{-1}(I).$$

Then  $\sigma$  and  $\tilde{\sigma}$  are reparametrization of each other, with

$$\tilde{\sigma} = \sigma \circ \Phi, \quad \sigma = \tilde{\sigma} \circ \Phi^{-1}.$$

## 3.2 Smooth maps and tangent plane

### Definition 3.31: Smooth functions between surfaces

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be regular surfaces and  $f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  a map.

1.  $f$  is *smooth at*  $\mathbf{p} \in \mathcal{S}_1$ , if there exist charts

$$\sigma_i : U_i \rightarrow \mathcal{S}_i \text{ such that } \mathbf{p} \in \sigma_1(U_1), \quad f(\mathbf{p}) \in \sigma_2(U_2),$$

and that the following map is smooth

$$\Psi : U_1 \rightarrow U_2, \quad \Psi = \sigma_2^{-1} \circ f \circ \sigma_1.$$

2.  $f$  is *smooth*, if it is smooth for each  $\mathbf{p} \in \mathcal{S}_1$ .

**Proposition 3.32:** Inverse of a regular chart is smooth

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be regular. Then  $\sigma^{-1} : \sigma(U) \rightarrow U$  is smooth.

**Definition 3.33:** Diffeomorphism of surfaces

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be regular surfaces.

1.  $f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  is a **diffeomorphism**, if  $f$  is smooth and admits smooth inverse.
2.  $\mathcal{S}_1, \mathcal{S}_2$  are **diffeomorphic** if there exists  $f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  diffeomorphism.

**Proposition 3.34:** Image of charts under diffeomorphisms

Let  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  be regular surfaces,  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  diffeomorphism. If  $\sigma : U \rightarrow \mathcal{S}$  is a regular chart for  $\mathcal{S}$  at  $\mathbf{p}$ , then

$$\tilde{\sigma} : U \rightarrow \tilde{\mathcal{S}}, \quad \tilde{\sigma} := f \circ \sigma,$$

is a regular chart for  $\tilde{\mathcal{S}}$  at  $f(\mathbf{p})$ .

**Definition 3.35:** Local diffeomorphism

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be regular surfaces, and  $f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  smooth.

1.  $f$  is a **local diffeomorphism** at  $\mathbf{p} \in \mathcal{S}_1$  if:
  - There exists An open set  $V \subseteq \mathcal{S}_1$  with  $\mathbf{p} \in V$ ;
  - $f(V) \subseteq \mathcal{S}_2$  is open;
  - $f : V \rightarrow f(V)$  is smooth between surfaces.
2.  $f$  is a **local diffeomorphism** in  $\mathcal{S}_1$ , if it is a local diffeomorphism at each  $\mathbf{p} \in \mathcal{S}_1$ .
3.  $\mathcal{S}_1$  is **locally diffeomorphic** to  $\mathcal{S}_2$ , if for all  $\mathbf{p} \in \mathcal{S}_1$  there exists  $f$  local diffeomorphism at  $\mathbf{p}$ .

**Definition 3.36:** Tangent vectors and tangent plane

Let  $\mathcal{S}$  be a surface and  $\mathbf{p} \in \mathcal{S}$ .

1.  $\mathbf{v} \in \mathbb{R}^3$  is a **tangent vector** to  $\mathcal{S}$  at  $\mathbf{p}$ , if there exists a smooth curve  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$  such that

$$\gamma(-\varepsilon, \varepsilon) \subseteq \mathcal{S}, \quad \gamma(0) = \mathbf{p}, \quad \mathbf{v} = \dot{\gamma}(0).$$

2. The **tangent plane** of  $\mathcal{S}$  at  $\mathbf{p}$  is the set

$$T_{\mathbf{p}}\mathcal{S} := \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} \text{ tangent vector of } \mathcal{S} \text{ at } \mathbf{p}\}.$$

**Lemma 3.37:** Curves with values on surfaces

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be a regular chart and  $\mathcal{S} := \sigma(U)$ . Let  $\mathbf{p} \in \mathcal{S}$  and  $(u_0, v_0) = \sigma^{-1}(\mathbf{p})$ . Assume  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$  is a smooth curve such that

$$\gamma(-\varepsilon, \varepsilon) \subseteq \mathcal{S}, \quad \gamma(0) = \mathbf{p}.$$

There exist smooth functions  $u, v : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  such that

$$\gamma(t) = \sigma(u(t), v(t)), \quad \forall t \in (-\varepsilon, \varepsilon), \quad u(0) = u_0, \quad v(0) = v_0.$$

**Theorem 3.38:** Characterization of Tangent Plane

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be a regular chart and  $\mathcal{S} := \sigma(U)$ . Let  $\mathbf{p} \in \mathcal{S}$ . Then

$$T_{\mathbf{p}}\mathcal{S} = \text{span}\{\sigma_u, \sigma_v\} := \{\lambda\sigma_u + \mu\sigma_v : \lambda, \mu \in \mathbb{R}\},$$

where  $\sigma_u$  and  $\sigma_v$  are evaluated at  $(u, v) = \sigma^{-1}(\mathbf{p})$ .

**Theorem 3.39:** Equation of tangent plane

Let  $\sigma : U \rightarrow \mathcal{S}$  be regular,  $\mathcal{S} = \sigma(U)$ . Let  $\mathbf{p} \in \mathcal{S}$  and

$$\mathbf{n} := \sigma_u(u, v) \times \sigma_v(u, v), \quad (u, v) := \sigma^{-1}(\mathbf{p}).$$

The equation of the tangent plane  $T_{\mathbf{p}}\mathcal{S}$  is given by

$$\mathbf{x} \cdot \mathbf{n} = 0, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

**Example 3.40:** Calculation of tangent plane

**Question.** For  $u \in (0, 2\pi)$ ,  $v < 1$ , let  $\mathcal{S}$  charted by

$$\sigma(u, v) = (\sqrt{1-v}\cos(u), \sqrt{1-v}\sin(u), v).$$

1. Prove that  $\sigma$  charts the paraboloid  $x^2 + y^2 - z = 1$ .
2. Prove that  $\sigma$  is regular and compute  $\mathbf{n} = \sigma_u \times \sigma_v$ .
3. Give a basis for  $T_{\mathbf{p}}\mathcal{S}$  at  $\mathbf{p} = (\sqrt{2}/2, \sqrt{2}/2, 0)$ .
4. Compute the cartesian equation of  $T_{\mathbf{p}}\mathcal{S}$ .

**Solution.**

1. Denote  $\sigma(u, v) = (x, y, z)$ . We have

$$\begin{aligned} x^2 + y^2 &= (\sqrt{1-v}\cos(u))^2 + (\sqrt{1-v}\sin(u))^2 \\ &= 1-v = 1-z. \end{aligned}$$

2. We compute  $\mathbf{n} = \sigma_u \times \sigma_v$  and show that  $\sigma$  is regular:

$$\begin{aligned} \sigma_u &= (-\sqrt{1-v}\sin(u), \sqrt{1-v}\cos(u), 0) \\ \sigma_v &= \left(-\frac{1}{2}(1-v)^{-1/2}\cos(u), -\frac{1}{2}(1-v)^{-1/2}\sin(u), 1\right) \\ \mathbf{n} = \sigma_u \times \sigma_v &= \left(\sqrt{1-v}\cos(u), \sqrt{1-v}\sin(u), \frac{1}{2}\right) \neq \mathbf{0} \end{aligned}$$

3. Notice that  $\sigma(\pi/4, 0) = \mathbf{p}$ . A basis for  $T_{\mathbf{p}}\mathcal{S}$  is

$$\begin{aligned} \sigma_u\left(\frac{\pi}{4}, 0\right) &= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \\ \sigma_v\left(\frac{\pi}{4}, 0\right) &= \left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, 1\right). \end{aligned}$$



4. Using the calculation for  $\mathbf{n}$  in Point 2, we find

$$\mathbf{n}\left(\frac{\pi}{4}, 0\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2}\right).$$

The equation for  $T_{\mathbf{p}}\mathcal{S}$  is  $\mathbf{x} \cdot \mathbf{n} = 0$ , which reads

$$\sqrt{2}x + \sqrt{2}y - z = 0.$$

#### Definition 3.41: Standard unit normal of a chart

Let  $\mathcal{S}$  be a regular surface and  $\sigma : U \rightarrow \mathbb{R}^3$  a regular chart. The **standard unit normal** of  $\sigma$  is the smooth function

$$\mathbf{N}_\sigma : U \rightarrow \mathbb{R}^3, \quad \mathbf{N}_\sigma = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$

#### Example 3.42: Calculation of $\mathbf{N}$

**Question.** Compute the standard unit normal to

$$\sigma(u, v) = (e^u, u + v, v), \quad u, v \in \mathbb{R}.$$

**Solution.** The standard unit normal to  $\sigma$  is

$$\begin{aligned} \sigma_u &= (e^u, 1, 0), \quad \sigma_v = (0, 1, 1), & \|\sigma_u \times \sigma_v\| &= \sqrt{1 + 2e^{2u}} \\ \sigma_u \times \sigma_v &= (1, -e^u, e^u) & \mathbf{N}_\sigma &= \frac{(1, -e^u, e^u)}{\sqrt{1 + 2e^{2u}}} \end{aligned}$$

#### Definition 3.43: Unit normal of a surface

Let  $\mathcal{S}$  be a regular surface. A **unit normal** to  $\mathcal{S}$  is a smooth function  $\mathbf{N} : \mathcal{S} \rightarrow \mathbb{R}^3$  such that

$$\mathbf{N}(\mathbf{p}) \perp T_{\mathbf{p}}\mathcal{S}, \quad \|\mathbf{N}(\mathbf{p})\| = 1, \quad \forall \mathbf{p} \in \mathcal{S}.$$

#### Definition 3.44: Orientable surface

A regular surface  $\mathcal{S}$  is **orientable** if there exists a unit normal  $\mathbf{N} : \mathcal{S} \rightarrow \mathbb{R}^3$  and an atlas  $\mathcal{A}$  such that

$$\mathbf{N} \circ \sigma = \mathbf{N}_\sigma, \quad \forall \sigma \in \mathcal{A}.$$

#### Definition 3.45: Differential of smooth function

Let  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  be regular surfaces and  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  a smooth map. The differential  $d_{\mathbf{p}}f$  of  $f$  at  $\mathbf{p}$  is defined as the map

$$d_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \rightarrow T_{f(\mathbf{p})}\tilde{\mathcal{S}}, \quad d_{\mathbf{p}}f(\mathbf{v}) := (f \circ \gamma)'(0),$$

with  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{S}$  smooth curve,  $\gamma(0) = \mathbf{p}$ ,  $\dot{\gamma}(0) = \mathbf{v}$ .

#### Example 3.46: Computing $d_{\mathbf{p}}f$ using the definition

**Question.** Consider the plane  $\mathcal{S} = \{z = 0\}$ , the unit cylinder  $\tilde{\mathcal{S}} = \{x^2 + y^2 = 1\}$ , and the map

$$f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}, \quad f(x, y, 0) = (\cos x, \sin x, y).$$

1. Compute  $T_{\mathbf{p}}\mathcal{S}$ .

2. Compute  $d_{\mathbf{p}}f$  at  $\mathbf{p} = (u_0, v_0, 0)$  and  $\mathbf{v} = (\lambda, \mu, 0)$ .

**Solution.**

1. A chart for  $\mathcal{S}$  is given by  $\sigma(u, v) = (u, v, 0)$ . Hence,

$$\sigma_u = (1, 0, 0), \quad \sigma_v = (0, 1, 0),$$

and the tangent space is

$$T_{\mathbf{p}}\mathcal{S} = \text{span}\{\sigma_u, \sigma_v\} = \{(\lambda, \mu, 0) : \lambda, \mu \in \mathbb{R}\}.$$

2. Define the curve  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{S}$  by setting

$$\gamma(t) := \sigma(u_0 + t\lambda, v_0 + t\mu) = (u_0 + t\lambda, v_0 + t\mu, 0).$$

Note that  $\gamma(0) = \mathbf{p}$  and  $\dot{\gamma}(0) = \mathbf{v} = (\lambda, \mu, 0)$ . Therefore, the differential is given by

$$\begin{aligned} (f \circ \gamma)(t) &= (\cos(u_0 + t\lambda), \sin(u_0 + t\lambda), v_0 + t\mu), \\ (f \circ \gamma)'(t) &= (-\lambda \sin(u_0 + t\lambda), \lambda \cos(u_0 + t\lambda), \mu), \\ d_{\mathbf{p}}f(\mathbf{v}) &= (f \circ \gamma)'(0) = (-\lambda \sin(u_0), \lambda \cos(u_0), \mu). \end{aligned}$$

#### Theorem 3.47: Matrix of $d_{\mathbf{p}}f$

Let  $\mathcal{S}, \tilde{\mathcal{S}}$  be regular surfaces, and  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  smooth.

- $d_{\mathbf{p}}f(\mathbf{v})$  depends only on  $f, \mathbf{p}, \mathbf{v}$  (and not on  $\gamma$ ).
- $d_{\mathbf{p}}f$  is linear, that is, for all  $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$  and  $\lambda, \mu \in \mathbb{R}$

$$d_{\mathbf{p}}f(\lambda\mathbf{v} + \mu\mathbf{w}) = \lambda d_{\mathbf{p}}f(\mathbf{v}) + \mu d_{\mathbf{p}}f(\mathbf{w}).$$

- Let  $\sigma : U \rightarrow \mathcal{S}, \tilde{\sigma} : \tilde{U} \rightarrow \tilde{\mathcal{S}}$  be regular charts at  $\mathbf{p}, f(\mathbf{p})$ . Let  $\alpha$  and  $\beta$  be the components of  $\Psi = \tilde{\sigma}^{-1} \circ f \circ \sigma$ , so that

$$\tilde{\sigma}(\alpha(u, v), \beta(u, v)) = f(\sigma(u, v)), \quad \forall (u, v) \in U.$$

The matrix of  $d_{\mathbf{p}}f$  with respect to the basis

$$\{\sigma_u, \sigma_v\} \text{ on } T_{\mathbf{p}}\mathcal{S}, \quad \{\tilde{\sigma}_u, \tilde{\sigma}_v\} \text{ on } T_{f(\mathbf{p})}\tilde{\mathcal{S}},$$

is given by the Jacobian of the map  $\Psi$ , that is,

$$J\Psi = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}.$$

#### Example 3.48: Computing the matrix of $d_{\mathbf{p}}f$

**Question.** Let  $\mathcal{S}$  be the cylinder, and  $\tilde{\mathcal{S}}$  the plane, charted by

$$\sigma(u, v) = (\cos u, \sin u, v), \quad \tilde{\sigma}(u, v) = (u, v, 0),$$

defined on  $U = (0, 2\pi) \times \mathbb{R}$  and  $\tilde{U} = \mathbb{R}^2$ . Define the map

$$f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}, \quad f(x, y, z) = (y, xz, 0).$$

Compute the matrix of  $d_{\mathbf{p}}f$  with respect to  $\{\sigma_u, \sigma_v\}$  and  $\{\tilde{\sigma}_u, \tilde{\sigma}_v\}$ .

**Solution.** Note that  $\tilde{\sigma}^{-1}(u, v, 0) = (u, v)$ . Hence,

$$\begin{aligned}\Psi(u, v) &= \tilde{\sigma}^{-1}(f(\sigma(u, v))) = \tilde{\sigma}^{-1}(f(\cos u, \sin u, v)) \\ &= \tilde{\sigma}^{-1}(\sin(u), \cos(u)v, 0) = (\sin(u), \cos(u)v).\end{aligned}$$

The components of  $\Psi$  are

$$\alpha(u, v) = \sin(u), \quad \beta(u, v) = \cos(u)v.$$

The matrix of  $d_{\mathbf{p}}f$  is hence

$$J\Psi = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix} = \begin{pmatrix} \cos(u) & 0 \\ -\sin(u)v & \cos(u) \end{pmatrix}.$$

### 3.3 Examples of Surfaces

#### Definition 3.49: Level surface

Let  $f : V \rightarrow \mathbb{R}$  be smooth,  $V \subseteq \mathbb{R}^3$  open. The **level surface** associated to  $f$  is the set

$$\mathcal{S}_f = f^{-1}(\{0\}) = \{(x, y, z) \in V : f(x, y, z) = 0\}.$$

#### Theorem 3.50: Regularity of level surfaces

Let  $f : V \rightarrow \mathbb{R}$  be smooth, with  $V \subseteq \mathbb{R}^3$  open. Assume

$$\nabla f(x, y, z) \neq \mathbf{0}, \quad \forall (x, y, z) \in V.$$

Then  $\mathcal{S}_f$  is a regular surface.

#### Example 3.51: Circular cone

**Question.** Prove the circular cone is a regular surface

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\}.$$

**Solution.** Define the open set  $V \subset \mathbb{R}^3$  and  $f : V \rightarrow \mathbb{R}$  by

$$V = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}, \quad f(x, y, z) = x^2 + y^2 - z^2.$$

$\mathcal{S}$  is a regular surface, since  $\mathcal{S} = \mathcal{S}_f$  and

$$\nabla f(x, y, z) = (2x, 2y, -2z) \neq \mathbf{0}, \quad \forall (x, y, z) \in V.$$

#### Theorem 3.52: Tangent plane of level surfaces

Let  $f : V \rightarrow \mathbb{R}$  be smooth, with  $V \subseteq \mathbb{R}^3$  open. Assume

$$\nabla f(x, y, z) \neq \mathbf{0}, \quad \forall (x, y, z) \in V.$$

Let  $\mathbf{p} \in \mathcal{S}_f$ . Then  $\nabla f(\mathbf{p}) \perp T_{\mathbf{p}}\mathcal{S}_f$  and  $T_{\mathbf{p}}\mathcal{S}_f$  has equation

$$\nabla f(\mathbf{p}) \cdot \mathbf{x} = 0, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

#### Example 3.53: Unit cylinder

**Question.** Consider the unit cylinder  $\mathcal{S} = \{x^2 + y^2 = 1\}$ .

1. Prove that  $\mathcal{S}$  is a regular surface.
2. Find the equation of  $T_{\mathbf{p}}\mathcal{S}$  at  $\mathbf{p} = (\sqrt{2}/2, \sqrt{2}/2, 5)$ .

**Solution.**

1. Define the open set  $V \subseteq \mathbb{R}^3$  and  $f : V \rightarrow \mathbb{R}$  by

$$V = \mathbb{R}^3 \setminus \{(0, 0, z) : z \in \mathbb{R}\}, \quad f(x, y, z) := x^2 + y^2 - 1.$$

$\mathcal{S}$  is a regular surface, since  $\mathcal{S} = \mathcal{S}_f$  and

$$\nabla f(x, y, z) = (2x, 2y, 0) \neq \mathbf{0}, \quad \forall (x, y, z) \in V.$$

2. Using the expression for  $\nabla f$  in Point 1, we get

$$\nabla f(\mathbf{p}) = \nabla f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 5\right) = (\sqrt{2}, \sqrt{2}, 0).$$

The equation for  $T_{\mathbf{p}}\mathcal{S}$  is

$$\nabla f(\mathbf{p}) \cdot \mathbf{x} = 0 \iff x + y = 0.$$

#### Definition 3.54: Ruled surface

A **ruled surface** is a surface with chart

$$\sigma(u, v) = \gamma(u) + v\mathbf{a}(u),$$

where  $\gamma, \mathbf{a} : (a, b) \rightarrow \mathbb{R}^3$  are smooth curves, such that

$\dot{\gamma}(t)$  and  $\mathbf{a}(t)$  are linearly independent for all  $t \in (a, b)$ .

$\gamma$  is the **base curve** and the lines  $v \mapsto v\mathbf{a}(u)$  the **ruledings**.

#### Theorem 3.55: Regularity of ruled surfaces

A ruled surface  $\mathcal{S}$  is regular if  $v$  is sufficiently small.

#### Example 3.56: Unit Cylinder is ruled surface

**Question.** Prove that the unit cylinder is a ruled surface.

**Solution.** The unit cylinder  $\mathcal{S}$  is charted by

$$\begin{aligned}\sigma(u, v) &= (\cos(u), \sin(u), v) = \gamma(u) + v\mathbf{a}(u) \\ \gamma(u) &= (\cos(u), \sin(u), 0), \quad \mathbf{a} = (0, 0, 1)\end{aligned}$$

$\mathcal{S}$  is a ruled surface, since the vectors

$$\dot{\gamma} = (-\sin(u), \cos(u), 0), \quad \mathbf{a} = (0, 0, 1)$$

are orthogonal, and hence linearly independent.

#### Example 3.57: A ruled surface

**Question.** Show that the following surface is ruled

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 10xy + 16x^2 - z = 0\}.$$

**Solution.** We can rearrange

$$x^2 + 10xy + 16x^2 - z = 0 \iff (x + 8y)(x + 2y) = z.$$



Let  $u = x + 8y$  and  $v = x + 2y$ . Therefore  $uv = z$  and

$$u - v = 6y \implies y = \frac{u - v}{6} \implies x = u - 8y = \frac{4v - u}{3}.$$

It follows that if  $(x, y, z) \in S$  then

$$\begin{aligned} (x, y, z) &= \left( \frac{4v - u}{3}, \frac{u - v}{6}, uv \right) \\ &= \left( -\frac{u}{3}, \frac{u}{6}, 0 \right) + v \left( \frac{4}{3}, -\frac{1}{6}, u \right) = \boldsymbol{\gamma}(u) + v\mathbf{a}(u). \end{aligned}$$

When  $u \neq 0$ , the vectors

$$\mathbf{a}(u) = \left( \frac{4}{3}, -\frac{1}{6}, u \right), \quad \dot{\boldsymbol{\gamma}}(u) = \left( -\frac{1}{3}, \frac{1}{6}, 0 \right),$$

are linearly independent, as the last component of  $\dot{\boldsymbol{\gamma}}(u)$  is 0. Also  $\mathbf{a}(0)$  and  $\dot{\boldsymbol{\gamma}}(0)$  are linearly independent. Thus,  $\mathcal{S}$  is a ruled surface.

### Definition 3.58: Surface of revolution

Let  $\boldsymbol{\gamma} : (a, b) \rightarrow \mathbb{R}^3$  be a smooth curve in the  $(x, z)$ -plane,

$$\boldsymbol{\gamma}(v) = (f(v), 0, g(v)), \quad f > 0.$$

The surface  $\mathcal{S}$  formed by rotating  $\boldsymbol{\gamma}$  about the  $z$ -axis, called a **surface of revolution**, is charted by  $\boldsymbol{\sigma} : U \rightarrow \mathbb{R}^3$

$$\boldsymbol{\sigma}(u, v) = (\cos(u)f(v), \sin(u)f(v), g(v)), \quad U = (0, 2\pi) \times (a, b).$$

### Theorem 3.59: Regularity of surfaces of revolution

A surface of revolution is regular if and only if  $\boldsymbol{\gamma}$  is regular.

### Example 3.60: Catenoid is surface of revolution

**Question.** The Catenoid  $\mathcal{S}$  is the surface of revolution formed by rotating the catenary  $\boldsymbol{\gamma}(v) = (\cosh(v), 0, v)$  about the  $z$ -axis. A chart for  $\mathcal{S}$  is given by

$$\boldsymbol{\sigma}(u, v) = (\cos(u) \cosh(v), \sin(u) \cosh(v), v),$$

with  $u \in (0, 2\pi), v \in \mathbb{R}$ . Prove that  $\mathcal{S}$  is a regular surface.

**Solution.** Note that  $f > 0$ .  $\mathcal{S}$  is regular because  $\boldsymbol{\gamma}$  is regular, as

$$\dot{\boldsymbol{\gamma}} = (\sinh(v), 0, 1), \quad \|\dot{\boldsymbol{\gamma}}\|^2 = 1 + \sinh(v)^2 \geq 1.$$

## 3.4 First fundamental form

### Definition 3.61: First fundamental form (FFF)

Let  $\mathcal{S}$  be a regular surface and  $\mathbf{p} \in \mathcal{S}$ . The **first fundamental form (FFF)** of  $\mathcal{S}$  at  $\mathbf{p}$  is the bilinear symmetric map

$$I_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{S} \times T_{\mathbf{p}}\mathcal{S} \rightarrow \mathbb{R}, \quad I_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) := \mathbf{v} \cdot \mathbf{w}.$$

### Definition 3.62: Coordinate functions on tangent plane

Let  $\boldsymbol{\sigma} : U \rightarrow \mathbb{R}^3$  be regular,  $\mathcal{S} = \boldsymbol{\sigma}(U)$ . The **coordinate functions** on  $T_{\mathbf{p}}\mathcal{S}$  are the linear maps

$$du, dv : T_{\mathbf{p}}\mathcal{S} \rightarrow \mathbb{R}, \quad du(\mathbf{v}) := \lambda, \quad dv(\mathbf{v}) := \mu,$$

where  $\mathbf{v} = \lambda\boldsymbol{\sigma}_u + \mu\boldsymbol{\sigma}_v$ , since  $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$  is a basis for  $T_{\mathbf{p}}\mathcal{S}$ .

### Definition 3.63: FFF of a chart

Let  $\boldsymbol{\sigma} : U \rightarrow \mathbb{R}^3$  be regular,  $\mathcal{S} = \boldsymbol{\sigma}(U)$ . Define  $E, F, G : U \rightarrow \mathbb{R}$

$$E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \quad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \quad G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v.$$

The **FFF** of  $\boldsymbol{\sigma}$  is the quadratic form  $\mathcal{F}_1 : T_{\mathbf{p}}\mathcal{S} \rightarrow \mathbb{R}$

$$\mathcal{F}_1(\mathbf{v}) = E du^2(\mathbf{v}) + 2F du(\mathbf{v}) dv(\mathbf{v}) + G dv^2(\mathbf{v}), \quad \forall \mathbf{v} \in T_{\mathbf{p}}\mathcal{S},$$

for all  $\mathbf{p} \in \boldsymbol{\sigma}(U)$ , with  $E, F, G$  evaluated at  $(u, v) = \boldsymbol{\sigma}^{-1}(\mathbf{p})$ .

### Theorem 3.64: Matrix of FFF

Let  $\boldsymbol{\sigma} : U \rightarrow \mathbb{R}^3$  be regular,  $\mathcal{S} = \boldsymbol{\sigma}(U)$ , and  $\mathbf{p} \in \boldsymbol{\sigma}(U)$ . Then

$$I_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = (du(\mathbf{v}), dv(\mathbf{v})) \begin{pmatrix} E & F \\ F & G \end{pmatrix} (du(\mathbf{w}), dv(\mathbf{w}))^T,$$

for all  $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$ . In particular, it holds

$$\mathcal{F}_1(\mathbf{v}) = I_{\mathbf{p}}(\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in T_{\mathbf{p}}\mathcal{S}.$$

### Example 3.65: FFF of Unit cylinder

**Question.** Consider the unit cylinder with chart

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v), \quad (u, v) \in (0, 2\pi) \times \mathbb{R}.$$

Prove that the FFF of  $\boldsymbol{\sigma}$  is

$$\mathcal{F}_1 = du^2 + dv^2.$$

**Solution.** We have

$$\begin{aligned} \boldsymbol{\sigma}_u &= (-\sin(u), \cos(u), 0) & F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0 \\ \boldsymbol{\sigma}_v &= (0, 0, 1) & G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 1 \\ E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 1 & \mathcal{F}_1 &= du^2 + dv^2 \end{aligned}$$

### Proposition 3.66: FFF and reparametrizations

Let  $\boldsymbol{\sigma} : U \rightarrow \mathbb{R}^3$  be regular, and  $\tilde{\boldsymbol{\sigma}} : \tilde{U} \rightarrow \mathbb{R}^3$  a reparametrization, with  $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi$  and  $\Phi : \tilde{U} \rightarrow U$  diffeomorphism. The matrices  $\mathcal{F}_1$  and  $\tilde{\mathcal{F}}_1$  of the FFF of  $\boldsymbol{\sigma}$  and  $\tilde{\boldsymbol{\sigma}}$  are related by

$$\tilde{\mathcal{F}}_1 = (J\Phi)^T \mathcal{F}_1 J\Phi, \quad \mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \tilde{\mathcal{F}}_1 = \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix}.$$

### Example 3.67: FFF of Plane

**Question.** Let  $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$ , with  $\mathbf{p}, \mathbf{q}$  orthonormal. The plane in

cartesian and polar coordinates is charted by, respectively,

$$\begin{aligned}\sigma(u, v) &= \mathbf{a} + u\mathbf{p} + v\mathbf{q}, \quad (u, v) \in \mathbb{R}^2, \\ \tilde{\sigma}(\rho, \theta) &= \mathbf{a} + \rho \cos(\theta)\mathbf{p} + \rho \sin(\theta)\mathbf{q}, \quad \rho > 0, \theta \in (0, 2\pi).\end{aligned}$$

1. Show that the FFF of  $\sigma$  and  $\tilde{\sigma}$  are

$$\mathcal{F}_1 = du^2 + dv^2, \quad \tilde{\mathcal{F}}_1 = d\rho^2 + \rho^2 d\theta^2.$$

2. Let  $\Phi$  be the change of variables from polar to cartesian coordinates. Show that

$$\tilde{\mathcal{F}}_1 = (J\Phi)^T \mathcal{F}_1 J\Phi.$$

**Solution.**

1. Using that  $\mathbf{p}$  and  $\mathbf{q}$  are orthonormal,

$$\begin{aligned}\sigma_u &= \mathbf{p}, & \tilde{\sigma}_\rho &= \cos(\theta)\mathbf{p} + \sin(\theta)\mathbf{q} \\ \sigma_v &= \mathbf{q}, & \tilde{\sigma}_\theta &= -\rho \sin(\theta)\mathbf{p} + \rho \cos(\theta)\mathbf{q} \\ E &= \sigma_u \cdot \sigma_u = 1, & \tilde{E} &= \tilde{\sigma}_\rho \cdot \tilde{\sigma}_\rho = 1 \\ F &= \sigma_u \cdot \sigma_v = 0, & \tilde{F} &= \tilde{\sigma}_\rho \cdot \tilde{\sigma}_\theta = 0 \\ G &= \sigma_v \cdot \sigma_v = 1, & \tilde{G} &= \tilde{\sigma}_\theta \cdot \tilde{\sigma}_\theta = \rho^2 \\ \mathcal{F}_1 &= du^2 + dv^2, & \tilde{\mathcal{F}}_1 &= d\rho^2 + \rho^2 d\theta^2\end{aligned}$$

2. We have  $\Phi(\rho, \theta) = (\rho \cos(\theta), \rho \sin(\theta))$ . Then

$$\begin{aligned}(J\Phi)^T \mathcal{F}_1 J\Phi &= (J\Phi)^T J\Phi \\ &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\rho \sin(\theta) & \rho \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\rho \sin(\theta) \\ \sin(\theta) & \rho \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix} = \tilde{\mathcal{F}}_1.\end{aligned}$$

## 3.5 Length of curves

**Proposition 3.68:** Length of curves and FFF

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be regular,  $\mathcal{S} = \sigma(U)$ . Let  $\gamma : (a, b) \rightarrow \mathcal{S}$  be a smooth curve. Then

$$\gamma(t) = \sigma(u(t), v(t)),$$

for some smooth functions  $u, v : (a, b) \rightarrow \mathbb{R}$ , and

$$\int_a^b \|\dot{\gamma}(t)\| dt = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} dt,$$

where  $\dot{u}, \dot{v}$  are computed at  $t$ , and  $E, F, G$  at  $(u(t), v(t))$ .

**Example 3.69:** Curves on the Cone

**Question.** Consider the cone with chart

$$\sigma(u, v) = (\cos(u)v, \sin(u)v, v), \quad u \in (0, 2\pi), v > 0.$$

1. Compute the first fundamental form of  $\sigma$ .
2. Compute the length of  $\gamma(t) = \sigma(t, t)$  for  $t \in (\pi/2, \pi)$ .

**Solution.**

1. The first fundamental form of  $\sigma$  is

$$\begin{aligned}\sigma_u &= (-\sin(u)v, \cos(u)v, 0) & F &= \sigma_u \cdot \sigma_v = 0 \\ \sigma_v &= (\cos(u), \sin(u), 1) & G &= \sigma_v \cdot \sigma_v = 2 \\ E &= \sigma_u \cdot \sigma_u = v^2 & \mathcal{F}_1 &= v^2 du^2 + 2 dv^2\end{aligned}$$

2.  $\gamma(t) = \sigma(u(t), v(t))$  with  $u(t) = t$  and  $v(t) = t$ . Then

$$\begin{aligned}\dot{u} &= 1, \quad \dot{v} = 1 & F(u(t), v(t)) &= F(t, t) = 0 \\ E(u(t), v(t)) &= E(t, t) = t^2 & G(u(t), v(t)) &= G(t, t) = 2\end{aligned}$$

The length of  $\gamma$  between  $\pi/2$  and  $\pi$  is

$$\int_{\pi/2}^{\pi} \|\dot{\gamma}(t)\| dt = \int_{\pi/2}^{\pi} \sqrt{t^2 + 2} dt.$$

## 3.6 Local isometries

**Definition 3.70:** Local isometry

Let  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  be regular and  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  smooth. We say that  $f$  is a **local isometry**, if for all  $\mathbf{p} \in \mathcal{S}$

$$\mathbf{v} \cdot \mathbf{w} = d_{\mathbf{p}}f(\mathbf{v}) \cdot d_{\mathbf{p}}f(\mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}.$$

In this case,  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  are said to be **locally isometric**.

**Proposition 3.71**

Local isometries are local diffeomorphisms.

**Theorem 3.72:** Local isometries preserve lengths

Let  $\mathcal{S}, \tilde{\mathcal{S}}$  be regular surfaces,  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  smooth. Equivalently:

1.  $f$  is a local isometry.
2. Let  $\gamma$  be a curve on  $\mathcal{S}$  and define the curve  $\tilde{\gamma} = f \circ \gamma$  on  $\tilde{\mathcal{S}}$ . Then  $\gamma$  and  $\tilde{\gamma}$  have the same length.

**Theorem 3.73:** Local isometries preserve FFF

Let  $\mathcal{S}, \tilde{\mathcal{S}}$  be regular surfaces,  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  smooth. Equivalently:

1.  $f$  is a local isometry.
2. Let  $\sigma : U \rightarrow \mathcal{S}$  be regular chart of  $\mathcal{S}$ , and define a chart of  $\tilde{\mathcal{S}}$  as  $\tilde{\sigma} : U \rightarrow \tilde{\mathcal{S}}$ , with  $\tilde{\sigma} = f \circ \sigma$ . Then  $\sigma$  and  $\tilde{\sigma}$  have the same FFF

$$E = \tilde{E}, \quad F = \tilde{F}, \quad G = \tilde{G}.$$

**Theorem 3.74:** Sufficient condition for local isometry

Let  $\mathcal{S}, \tilde{\mathcal{S}}$  be regular surfaces, with charts  $\sigma : U \rightarrow \mathcal{S}$  and  $\tilde{\sigma} : U \rightarrow \tilde{\mathcal{S}}$ . Assume that  $\sigma$  and  $\tilde{\sigma}$  have the same FFF. We have

1. The surfaces  $\sigma(U)$  and  $\tilde{\sigma}(U)$  are locally isometric.

2. A local isometry is given by

$$f : \sigma(U) \rightarrow \tilde{\mathcal{S}}, \quad f = \tilde{\sigma} \circ \sigma^{-1}.$$

**Example 3.75:** Plane and Cylinder are locally isometric

**Question.** Consider the plane  $\mathcal{S} = \{x = 0\}$  and the unit cylinder  $\tilde{\mathcal{S}} = \{x^2 + y^2 = 1\}$ . Define the function

$$f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}, \quad f(0, y, z) = (\cos(y), \sin(y), z).$$

Prove that  $f$  is a local isometry (you may assume  $f$  smooth).

**Solution.** The plane  $\mathcal{S}$  is charted by

$$\sigma(u, v) = (0, u, v), \quad u, v \in \mathbb{R}.$$

We already know that  $\sigma$  is regular, with FFF coefficients

$$E = 1, \quad F = 0, \quad G = 1 \quad \implies \quad \mathcal{F}_1 = du^2 + dv^2.$$

Define  $\tilde{\sigma} = f \circ \sigma$ . Therefore,

$$\tilde{\sigma}(u, v) = f(0, u, v) = (\cos(u), \sin(u), v).$$

The FFF of  $\tilde{\sigma}$  is

$$\begin{aligned} \tilde{\sigma}_u &= (-\sin(u), \cos(u), 0) & \tilde{F} &= \tilde{\sigma}_u \cdot \tilde{\sigma}_v = 0 \\ \tilde{\sigma}_v &= (0, 0, 1) & \tilde{G} &= \tilde{\sigma}_v \cdot \tilde{\sigma}_v = 1 \\ \tilde{E} &= \tilde{\sigma}_u \cdot \tilde{\sigma}_u = 1 & \tilde{\mathcal{F}}_1 &= du^2 + dv^2 \end{aligned}$$

Thus,  $\sigma$  and  $\tilde{\sigma}$  have the same FFF. Since  $\mathcal{A} = \{\sigma\}$  is an atlas for  $\mathcal{S}$ , by Theorem 1.74 we conclude that  $f$  is a local isometry of  $\mathcal{S}$  into  $\tilde{\mathcal{S}}$ .

**Example 3.76:** Plane and Cone are locally isometric

**Question.** Consider the cone without tip

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, \quad z > 0\},$$

and the plane  $\tilde{\mathcal{S}} = \{z = 0\}$ .

1. Compute the FFF of the chart of the Cone

$$\begin{aligned} \sigma : U \rightarrow \mathcal{S}, \quad \sigma(\rho, \theta) &= (\rho \cos(\theta), \rho \sin(\theta), \rho), \\ U &= \{(\rho, \theta) \in \mathbb{R}^2 : \rho > 0, \theta \in (0, 2\pi)\}. \end{aligned}$$

2. Compute the FFF of the chart of the plane

$$\tilde{\sigma} : U \rightarrow \tilde{\mathcal{S}}, \quad \tilde{\sigma}(\rho, \theta) = (a\rho \cos(b\theta), a\rho \sin(b\theta), 0),$$

where  $a > 0$  and  $b \in (0, 1]$  are constants.

3. Prove that  $f = \tilde{\sigma} \circ \sigma^{-1}$  is a local isometry between  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$ , for suitable coefficients  $a, b$ .

**Solution.**

1. As seen in Example 1.71, the coefficients of the FFF of  $\sigma$  are

$$E = 2, \quad F = 0, \quad G = \rho^2.$$

2. Note that  $\tilde{\sigma}$  is well defined for all  $(\rho, \theta) \in U$ , as

$$\theta \in (0, 2\pi), \quad b \in (0, 1] \quad \implies \quad b\theta \in (0, 2\pi).$$

The coefficients of the FFF of  $\tilde{\sigma}$  are

$$\begin{aligned} \tilde{\sigma}_\rho &= a(\cos(b\theta), \sin(b\theta), 0) & \tilde{F} &= \tilde{\sigma}_\rho \cdot \tilde{\sigma}_\theta = 0 \\ \tilde{\sigma}_\theta &= ab\rho(-\sin(b\theta), \cos(b\theta), 0) & \tilde{G} &= \tilde{\sigma}_\theta \cdot \tilde{\sigma}_\theta = a^2 b^2 \rho^2 \\ \tilde{E} &= \tilde{\sigma}_\rho \cdot \tilde{\sigma}_\rho = a^2 \end{aligned}$$

3. Imposing that  $\tilde{E} = E$ ,  $\tilde{F} = F$  and  $\tilde{G} = G$ , we obtain

$$a^2 = 2, \quad a^2 b^2 = 1 \quad \implies \quad a = \sqrt{2}, \quad b = \frac{1}{\sqrt{2}}.$$

Note that  $a > 0$  and  $0 < b < 1$ , showing that  $a, b$  are admissible. Hence, for  $a = \sqrt{2}$  and  $b = 1/\sqrt{2}$ , the charts  $\sigma$  and  $\tilde{\sigma}$  have the same FFF. By Theorem 1.73, we conclude that  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  are locally isometric, with local isometry given by  $f = \tilde{\sigma} \circ \sigma^{-1}$ .

### 3.7 Angle between curves

**Definition 3.77:** Angle between curves

Let  $\mathcal{S}$  be a regular surface, and  $\gamma, \tilde{\gamma}$  curves on  $\mathcal{S}$  intersecting at

$$\gamma(t_0) = \mathbf{p} = \tilde{\gamma}(t_0).$$

The angle  $\theta$  between  $\gamma$  and  $\tilde{\gamma}$  is

$$\cos(\theta) = \frac{\dot{\gamma}(t_0) \cdot \dot{\tilde{\gamma}}(t_0)}{\|\dot{\gamma}(t_0)\| \|\dot{\tilde{\gamma}}(t_0)\|}.$$

**Theorem 3.78:** Angle between curves and FFF

Let  $\mathcal{S}$  be a regular surface,  $\sigma$  regular chart at  $\mathbf{p}$ , and  $\gamma, \tilde{\gamma}$  curves on  $\mathcal{S}$  intersecting at  $\gamma(t_0) = \mathbf{p} = \tilde{\gamma}(t_0)$ . There exist smooth functions  $u, v, \tilde{u}, \tilde{v}$  such that

$$\gamma(t) = \sigma(u(t), v(t)), \quad \tilde{\gamma}(t) = \sigma(\tilde{u}(t), \tilde{v}(t)).$$

The angle between  $\gamma$  and  $\tilde{\gamma}$  is

$$\cos(\theta) = \frac{E\dot{u}\dot{u} + F(\dot{u}\dot{v} + \dot{\tilde{u}}\dot{\tilde{v}}) + G\dot{v}\dot{v}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}},$$

with  $E, F, G, \tilde{E}, \tilde{F}, \tilde{G}$  evaluated at  $(u(t_0), v(t_0))$ , and  $\dot{u}, \dot{v}, \dot{\tilde{u}}, \dot{\tilde{v}}$  at  $t_0$ .

**Example 3.79:** Calculation of angle between curves

**Question.** Let  $S$  be a surface charted by

$$\sigma(u, v) = (u, v, e^{uv}).$$

1. Calculate the FFF of  $\sigma$ .

2. Calculate  $\cos(\theta)$ , where  $\theta$  is the angle between the two curves

$$\begin{aligned} \gamma(t) &= \sigma(u(t), v(t)), & u(t) &= t, \quad v(t) = t, \\ \tilde{\gamma}(t) &= \sigma(\tilde{u}(t), \tilde{v}(t)), & \tilde{u}(t) &= 1, \quad \tilde{v}(t) = t. \end{aligned}$$

**Solution.**

1. The coefficients of the FFF are

$$\begin{aligned}\sigma_u &= (1, 0, e^{uv}v) & F(u, v) &= e^{2uv}uv \\ \sigma_v &= (0, 1, e^{uv}u) & G(u, v) &= 1 + e^{2uv}u^2 \\ E(u, v) &= 1 + e^{2uv}v^2\end{aligned}$$

2.  $\gamma$  and  $\tilde{\gamma}$  intersect at  $\gamma(1) = \tilde{\gamma}(1) = \sigma(1, 1)$ . We compute

$$\begin{aligned}\dot{u}(1) &= 1 & E(1, 1) &= 1 + e^2 \\ \dot{v}(1) &= 1 & F(1, 1) &= e^2 \\ \ddot{u}(1) &= 0 & G(1, 1) &= 1 + e^2 \\ \ddot{v}(1) &= 1\end{aligned}$$

Therefore, the angle  $\theta$  satisfies

$$\cos(\theta) = \frac{1 + 2e^2}{\sqrt{2 + 4e^2}\sqrt{1 + e^2}} = \sqrt{\frac{1 + 2e^2}{2 + 2e^2}}.$$

### 3.8 Conformal maps

#### Definition 3.80: Conformal map

Let  $\mathcal{S}, \tilde{\mathcal{S}}$  be regular surfaces,  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  local diffeomorphism. We say that  $f$  is a **conformal map**, if for all  $\mathbf{p} \in \mathcal{S}$

$$\theta = \tilde{\theta}, \quad \forall \mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S},$$

- $\theta$  is the angle between  $\mathbf{v}$  and  $\mathbf{w}$ ,
- $\tilde{\theta}$  is the angle between  $d_{\mathbf{p}}f(\mathbf{v})$  and  $d_{\mathbf{p}}f(\mathbf{w})$ .

In this case, we say that  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  are **conformal**.

#### Proposition 3.81

Local isometries are conformal maps.

#### Theorem 3.82: Conformal maps and FFF

Let  $\mathcal{S}, \tilde{\mathcal{S}}$  be regular surfaces,  $f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$  a local diffeomorphism. Equivalently:

1.  $f$  is a conformal map.
2. Let  $\sigma : U \rightarrow \mathcal{S}$  be regular chart of  $\mathcal{S}$ , and define a chart of  $\tilde{\mathcal{S}}$  as  $\tilde{\sigma} : U \rightarrow \tilde{\mathcal{S}}$ , with  $\tilde{\sigma} = f \circ \sigma$ . Then the FFF of  $\sigma$  and  $\tilde{\sigma}$  satisfy

$$\tilde{\mathcal{F}}_1 = \lambda(u, v)\mathcal{F}_1, \quad \forall (u, v) \in U,$$

for some smooth map  $\lambda : U \rightarrow \mathbb{R}$ .

#### Theorem 3.83: Sufficient condition for conformality

Let  $\mathcal{S}, \tilde{\mathcal{S}}$  be regular surfaces, with charts  $\sigma : U \rightarrow \mathcal{S}$  and  $\tilde{\sigma} : U \rightarrow \tilde{\mathcal{S}}$ . Assume that  $\tilde{\mathcal{F}}_1 = \lambda\mathcal{F}_1$  for some  $\lambda : U \rightarrow \mathbb{R}$ . We have

1. The surfaces  $\sigma(U)$  and  $\tilde{\mathcal{S}}$  are conformal.
2. A conformal map is given by  $f : \sigma(U) \rightarrow \tilde{\mathcal{S}}$  with  $f = \tilde{\sigma} \circ \sigma^{-1}$ .

#### Example 3.84: Stereographic Projection

**Question.** Consider the unit sphere  $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$  and define the surface  $\mathcal{S} = \mathbb{S}^2 \setminus \{N\}$ , where  $N = (0, 0, 1)$ . Consider the plane  $\tilde{\mathcal{S}} = \{z = 0\}$ . The *Stereographic Projection* is

$$f : \mathcal{S} \rightarrow \tilde{\mathcal{S}}, \quad f(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z}, 0 \right).$$

Prove that  $f$  is a conformal map.

**Solution.** It is easy to prove that  $f^{-1} = \sigma$ , with

$$\sigma(u, v) = \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, 1 - \frac{2}{u^2 + v^2 + 1} \right).$$

It is straightforward to compute that the FFF of  $\sigma$  is

$$\mathcal{F}_1 = \lambda(u, v)(du^2 + dv^2), \quad \lambda(u, v) = \frac{4}{(u^2 + v^2 + 1)^2}.$$

Let  $\tilde{\sigma} = f \circ \sigma$ . Since  $\sigma = f^{-1}$ , we have that  $\tilde{\sigma}(u, v) = (u, v, 0)$ . As already computed, the FFF of  $\tilde{\sigma}$  is  $\tilde{\mathcal{F}}_1 = du^2 + dv^2$ . Therefore,

$$\tilde{\mathcal{F}}_1 = \frac{1}{\lambda}\mathcal{F}_1.$$

Since  $\mathcal{A} = \{\sigma\}$  is an atlas for  $\mathcal{S}$ , by Theorem 3.82 we conclude that  $f$  is a conformal map.

#### Definition 3.85: Conformal parametrization

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be regular. We say that  $\sigma$  is a **conformal parametrization** if the FFF of  $\sigma$  satisfies

$$\mathcal{F}_1 = \lambda(u, v)(du^2 + dv^2),$$

for some smooth function  $\lambda : U \rightarrow \mathbb{R}$ .

#### Example 3.86: Mercator projection

**Question.** Prove that the parametrization of  $\mathbb{S}^2$  is conformal

$$\sigma(u, v) := (\cos(u) \operatorname{sech}(v), \sin(u) \operatorname{sech}(v), \tanh(v)).$$

**Solution.** Recall the identities  $\operatorname{sech}^2(v) + \tanh^2(v) = 1$  and

$$\operatorname{sech}(v)' = -\operatorname{sech}(v)\tanh(v), \quad \tanh(v)' = \operatorname{sech}^2(v).$$

The chart  $\sigma$  is a conformal parametrization because the FFF is

$$\begin{aligned}\tilde{\sigma}_u &= \operatorname{sech}(v)(-\sin(u), \cos(u), 0) \\ \tilde{\sigma}_v &= \operatorname{sech}(v)(-\cos(u)\tanh(v), -\sin(u)\tanh(v), \operatorname{sech}(v)) \\ \tilde{E} &= \tilde{\sigma}_u \cdot \tilde{\sigma}_u = \operatorname{sech}^2(v)(\cos^2(u) + \sin^2(u)) = \operatorname{sech}^2(v) \\ \tilde{F} &= \tilde{\sigma}_u \cdot \tilde{\sigma}_v = 0 \\ \tilde{G} &= \tilde{\sigma}_v \cdot \tilde{\sigma}_v = \operatorname{sech}^2(v)(\tanh^2(v) + \operatorname{sech}^2(v)) = \operatorname{sech}^2(v) \\ \tilde{\mathcal{F}}_1 &= \operatorname{sech}^2(v)(du^2 + dv^2).\end{aligned}$$

#### Theorem 3.87: Conformal parametrizations preserve angles

Let  $\sigma$  be a conformal parametrization, and  $\gamma_1(t), \gamma_2(t)$  be curves in  $\mathbb{R}^2$  such that  $\dot{\gamma}_1(t_0), \dot{\gamma}_2(t_0)$  make angle  $\theta$ . Let  $\tilde{\gamma}_1 = \sigma \circ \gamma_1$  and  $\tilde{\gamma}_2 = \sigma \circ \gamma_2$ . Then  $\dot{\tilde{\gamma}}_1(t_0), \dot{\tilde{\gamma}}_2(t_0)$  also make angle  $\theta$ .

### 3.9 Second fundamental form

**Definition 3.88:** Second fundamental form of a chart

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be regular,  $\mathcal{S} = \sigma(U)$ . Define  $L, M, N : U \rightarrow \mathbb{R}$

$$L := \sigma_{uu} \cdot \mathbf{N}, \quad M := \sigma_{uv} \cdot \mathbf{N}, \quad N := \sigma_{vv} \cdot \mathbf{N},$$

where  $\mathbf{N}$  is the standard unit normal to  $\sigma$ . The **second fundamental form (SFF)** of  $\sigma$  is the quadratic form  $\mathcal{F}_2 : T_{\mathbf{p}}\mathcal{S} \rightarrow \mathbb{R}$

$$\mathcal{F}_2(\mathbf{v}) = L du^2(\mathbf{v}) + 2M du(\mathbf{v}) dv(\mathbf{v}) + N dv^2(\mathbf{v}), \quad \forall \mathbf{v} \in T_{\mathbf{p}}\mathcal{S},$$

for all  $\mathbf{p} \in \sigma(U)$ , with  $L, M, N$  evaluated at  $(u, v) = \sigma^{-1}(\mathbf{v})$ , and  $du, dv$  the coordinate functions in Definition 1.62.

**Example 3.89:** SFF of Plane

**Question.** Let  $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$ , with  $\mathbf{p}, \mathbf{q}$  orthonormal. The plane is charted by

$$\sigma(u, v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}, \quad (u, v) \in \mathbb{R}^2.$$

Prove that the SFF of  $\sigma$  is  $\mathcal{F}_2 = 0$ .

**Solution.** We have that  $\mathcal{F}_2 = 0$ , since

$$\begin{aligned} \sigma_u &= \mathbf{p}, & \sigma_v &= \mathbf{q}, & \sigma_{uu} &= \sigma_{uv} = \sigma_{vv} = \mathbf{0}, \\ L &= \sigma_{uu} \cdot \mathbf{N} = 0, & M &= \sigma_{uv} \cdot \mathbf{N} = 0, & N &= \sigma_{vv} \cdot \mathbf{N} = 0. \end{aligned}$$

**Example 3.90:** SFF of Unit cylinder

**Question.** Consider the unit cylinder with chart

$$\sigma(u, v) = (\cos(u), \sin(u), v), \quad (u, v) \in (0, 2\pi) \times \mathbb{R}.$$

Prove that the SFF of  $\sigma$  is

$$\mathcal{F}_2 = -du^2.$$

**Solution.** We have

$$\begin{aligned} \sigma_u &= (-\sin(u), \cos(u), 0) & \mathbf{N} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \\ \sigma_v &= (0, 0, 1) & &= (\cos(u), \sin(u), 0) \\ \sigma_{uu} &= (-\cos(u), -\sin(u), 0) & L &= \sigma_{uu} \cdot \mathbf{N} = -1 \\ \sigma_{uv} &= \sigma_{vv} = \mathbf{0} & M &= \sigma_{uv} \cdot \mathbf{N} = 0 \\ \sigma_u \times \sigma_v &= (\cos(u), \sin(u), 0) & N &= \sigma_{vv} \cdot \mathbf{N} = 0 \\ \|\sigma_u \times \sigma_v\| &= 1 & \mathcal{F}_2 &= -du^2 \end{aligned}$$

**Remark 3.91:** SFF and reparametrizations

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be regular, and  $\tilde{\sigma} : \tilde{U} \rightarrow \mathbb{R}^3$  a reparametrization, with  $\tilde{\sigma} = \sigma \circ \Phi$  and  $\Phi : \tilde{U} \rightarrow U$  diffeomorphism. The matrices  $\mathcal{F}_2$  and  $\tilde{\mathcal{F}}_2$  of the SFF of  $\sigma$  and  $\tilde{\sigma}$  are related by

$$\tilde{\mathcal{F}}_2 = \pm (J\Phi)^T \mathcal{F}_2 J\Phi, \quad \mathcal{F}_2 = \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix}, \quad \tilde{\mathcal{F}}_2 = \begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix},$$

where the formula holds with the plus sign if  $\det J\Phi > 0$ , and with the minus sign if  $\det J\Phi < 0$ .

### 3.10 Gauss and Weingarten maps

**Definition 3.92:** Gauss map

Let  $\mathcal{S}$  be an oriented surface with standard unit normal  $\mathbf{N}$ . The **Gauss map** of  $\mathcal{S}$  is

$$\mathcal{G}_{\mathcal{S}} : \mathcal{S} \rightarrow \mathbb{S}^2, \quad \mathcal{G}_{\mathcal{S}}(\mathbf{p}) := \mathbf{N}(\mathbf{p}).$$

**Definition 3.93:** Weingarten map

Let  $\mathcal{S}$  be an orientable surface with Gauss map  $\mathcal{G} : \mathcal{S} \rightarrow \mathbb{S}^2$ . The **Weingarten map**  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$  of  $\mathcal{S}$  at  $\mathbf{p}$  is

$$\mathcal{W}_{\mathbf{p}, \mathcal{S}} : T_{\mathbf{p}}\mathcal{S} \rightarrow T_{\mathbf{p}}\mathcal{S}, \quad \mathcal{W}_{\mathbf{p}, \mathcal{S}}(\mathbf{v}) = -d_{\mathbf{p}}\mathcal{G}(\mathbf{v}).$$

**Lemma 3.94**

Let  $\mathcal{S}$  be an orientable surface with Weingarten map  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$ , and  $\sigma$  a regular chart at  $\mathbf{p}$ . Then

$$\mathcal{W}_{\mathbf{p}, \mathcal{S}}(\sigma_u) = -\mathbf{N}_u, \quad \mathcal{W}_{\mathbf{p}, \mathcal{S}}(\sigma_v) = -\mathbf{N}_v,$$

where  $\sigma_u, \sigma_v, \mathbf{N}_u, \mathbf{N}_v$  are evaluated at  $(u, v) = \sigma^{-1}(\mathbf{p})$ .

**Definition 3.95:** SFF of a surface

Let  $\mathcal{S}$  be an orientable surface with Weingarten map  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$ . The **SFF** of  $\mathcal{S}$  at  $\mathbf{p}$  is the bilinear map

$$II_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{S} \times T_{\mathbf{p}}\mathcal{S} \rightarrow \mathbb{R}, \quad II_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) := \mathcal{W}_{\mathbf{p}, \mathcal{S}}(\mathbf{v}) \cdot \mathbf{w}.$$

**Theorem 3.96:** Matrix of the SFF

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be regular,  $\mathcal{S} = \sigma(U)$ , and  $\mathbf{p} \in \sigma(U)$ . Then

$$II_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = (du(\mathbf{v}), dv(\mathbf{v})) \begin{pmatrix} L & M \\ M & N \end{pmatrix} (du(\mathbf{w}), dv(\mathbf{w}))^T,$$

for all  $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$ . In particular, it holds

$$\mathcal{F}_2(\mathbf{v}) = II_{\mathbf{p}}(\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in T_{\mathbf{p}}\mathcal{S}.$$

**Theorem 3.97:** Matrix of Weingarten map

Let  $\mathcal{S}$  be an orientable surface with Weingarten map  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$ . Let  $\sigma$  be a regular chart at  $\mathbf{p}$ . The matrix of the Weingarten map with respect to the basis  $\{\sigma_u, \sigma_v\}$  of  $T_{\mathbf{p}}\mathcal{S}$  is

$$\mathcal{W} = \mathcal{F}_1^{-1} \mathcal{F}_2,$$

where the FFF and SFF are evaluated at  $(u, v) = \sigma^{-1}(\mathbf{p})$ .

**Remark 3.98:** Matrix inverse

A matrix  $A \in \mathbb{R}^{2 \times 2}$  is invertible if and only if  $\det(A) \neq 0$ . In such case the inverse  $A^{-1}$  is computed via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det(A) = ad - bc.$$

If the matrix is diagonal, then

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^{-1} = \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1/\mu \end{pmatrix}.$$

### Example 3.99: Weingarten map of Helicoid

**Question.** The Helicoid is charted by

$$\sigma(u, v) = (u \cos(v), u \sin(v), \lambda v), \quad u \in \mathbb{R}, v \in (0, 2\pi),$$

with  $\lambda > 0$  constant. Compute the matrix of the Weingarten map.

**Solution.** We compute all the derivatives of  $\sigma$

$$\begin{aligned} \sigma_u &= (\cos(v), \sin(v), 0) & \sigma_{uv} &= (-\sin(v), \cos(v), 0) \\ \sigma_v &= (-u \sin(v), u \cos(v), \lambda) & \sigma_{vv} &= -u(\cos(v), \sin(v), 0) \\ \sigma_{uu} &= (0, 0, 0) \end{aligned}$$

The FFF and its inverse are

$$\begin{aligned} E &= \sigma_u \cdot \sigma_u = 1 & F &= \sigma_u \cdot \sigma_v = 0 \\ G &= \sigma_v \cdot \sigma_v = u^2 + \lambda^2 \\ \mathcal{F}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & u^2 + \lambda^2 \end{pmatrix} & \mathcal{F}_1^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u^2 + \lambda^2} \end{pmatrix}. \end{aligned}$$

The standard unit normal to  $\sigma$  is

$$\begin{aligned} \sigma_u \times \sigma_v &= (\lambda \sin(v), -\lambda \cos(v), u) \\ \|\sigma_u \times \sigma_v\| &= \sqrt{u^2 + \lambda^2} \\ \mathbf{N} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{1}{\sqrt{u^2 + \lambda^2}} (\lambda \sin(v), -\lambda \cos(v), u). \end{aligned}$$

The SFF of  $\sigma$  is

$$\begin{aligned} L &= \sigma_{uu} \cdot \mathbf{N} = 0 & M &= \sigma_{uv} \cdot \mathbf{N} = -\frac{\lambda}{\sqrt{u^2 + \lambda^2}} \\ N &= \sigma_{vv} \cdot \mathbf{N} = 0 \\ \mathcal{F}_2 &= \begin{pmatrix} 0 & -\frac{\lambda}{\sqrt{u^2 + \lambda^2}} \\ -\frac{\lambda}{\sqrt{u^2 + \lambda^2}} & 0 \end{pmatrix}. \end{aligned}$$

Finally, the matrix of the Weingarten map is

$$\mathcal{W} = \mathcal{F}_1^{-1} \mathcal{F}_2 = \begin{pmatrix} 0 & -\frac{\lambda}{(u^2 + \lambda^2)^{1/2}} \\ -\frac{\lambda}{(u^2 + \lambda^2)^{3/2}} & 0 \end{pmatrix}.$$

## 3.11 Curvatures

### Definition 3.100: Gaussian and mean curvature

Let  $\mathcal{S}$  be an orientable surface. Let  $\mathcal{W}$  be the matrix of the Weingarten map  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$  of  $\mathcal{S}$  at  $\mathbf{p}$ . We define:

1. The **Gaussian curvature** of  $\mathcal{S}$  at  $\mathbf{p}$  is

$$K := \det(\mathcal{W}),$$

2. The **mean curvature** of  $\mathcal{S}$  at  $\mathbf{p}$  is

$$H := \frac{1}{2} \operatorname{Tr}(\mathcal{W}),$$

### Notation 3.101: Trace of a matrix

The **trace** of a  $2 \times 2$  matrix is the sum of the diagonal entries.

### Proposition 3.102: Formulas for $K$ and $H$

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be a regular chart, and  $\mathcal{S} = \sigma(U)$ . Then

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{LG - 2MF - NE}{2(EG - F^2)}.$$

### Example 3.103: Curvatures of the Plane

**Question.** Let  $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$ , with  $\mathbf{p}, \mathbf{q}$  orthonormal. Consider the plane charted by

$$\sigma(u, v) = \mathbf{a} + \mathbf{p}u + \mathbf{q}v.$$

1. Compute the matrix of the Weingarten map of  $\sigma$ .
2. Compute the Gaussian and mean curvatures of the plane.

**Solution.**

1. From Examples 1.68, 1.89, the FFF and SFF of  $\sigma$  are

$$\mathcal{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore the matrix of the Weingarten map is

$$\mathcal{W} = \mathcal{F}_1^{-1} \mathcal{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

2. The Gaussian and mean curvatures are

$$K = \det(\mathcal{W}) = 0, \quad H = \frac{1}{2} \operatorname{Tr}(\mathcal{W}) = 0.$$

### Example 3.104: Curvatures of the Unit cylinder

**Question.** Consider the unit cylinder  $\mathcal{S}$  charted by

$$\sigma(u, v) = (\cos(u), \sin(u), v).$$

1. Compute the matrix of the Weingarten map of  $\sigma$ .
2. Compute the Gaussian and mean curvatures of  $\mathcal{S}$ .

**Solution.**

1. From Examples 1.65, 3.90, the FFF and SFF of  $\sigma$  are

$$\mathcal{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$



Therefore the matrix of the Weingarten map is

$$\mathcal{W} = \mathcal{F}_1^{-1} \mathcal{F}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

2. The Gaussian and mean curvatures are

$$K = \det(\mathcal{W}) = 0, \quad H = \frac{1}{2} \operatorname{Tr}(\mathcal{W}) = -\frac{1}{2}.$$

### Theorem 3.105: Eigenvalues of Weingarten map

Let  $\mathcal{S}$  be an orientable surface and  $\sigma$  a regular chart at  $\mathbf{p}$ . Let  $\mathcal{W}$  be the matrix of the Weingarten map  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$  with respect to the basis  $\{\sigma_u, \sigma_v\}$  of  $T_{\mathbf{p}}\mathcal{S}$ . Then

1. There exist scalars  $\kappa_1, \kappa_2 \in \mathbb{R}$  and an orthonormal basis  $\{\mathbf{t}_1, \mathbf{t}_2\}$  of  $T_{\mathbf{p}}\mathcal{S}$  such that

$$\mathcal{W}_{\mathbf{p}, \mathcal{S}}(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1, \quad \mathcal{W}_{\mathbf{p}, \mathcal{S}}(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2.$$

2. Let  $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$  be such that

$$\mathbf{t}_1 = \lambda_1 \sigma_u + \mu_1 \sigma_v, \quad \mathbf{t}_2 = \lambda_2 \sigma_u + \mu_2 \sigma_v.$$

Denote  $\mathbf{x}_1 = (\lambda_1, \mu_1)$  and  $\mathbf{x}_2 = (\lambda_2, \mu_2)$ . Then  $\kappa_1, \kappa_2$  are eigenvalues of  $\mathcal{W}$  of eigenvectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$

$$\mathcal{W} \mathbf{x}_1 = \kappa_1 \mathbf{x}_1, \quad \mathcal{W} \mathbf{x}_2 = \kappa_2 \mathbf{x}_2.$$

In particular, the matrix  $\mathcal{W}$  is diagonalizable, with

$$\mathcal{W} = P^{-1}DP, \quad D = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix}.$$

### Definition 3.106: Principal curvatures and vectors

Let  $\mathcal{S}$  be an orientable surface. Let  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$  the Weingarten map of  $\mathcal{S}$  at  $\mathbf{p}$ . We define:

1. The **principal curvatures** of  $\mathcal{S}$  at  $\mathbf{p}$  are the eigenvalues  $\kappa_1, \kappa_2$  of  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$ .
2. The **principal vectors** corresponding to  $\kappa_1$  and  $\kappa_2$  are the eigenvectors  $\mathbf{t}_1, \mathbf{t}_2$  of  $\mathcal{W}_{\mathbf{p}, \mathcal{S}}$ .

### Remark 3.107: Computing principal curvatures and vectors

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be a regular chart and  $\mathcal{S} = \sigma(U)$ .

1. Compute the FFF and SFF of  $\sigma$ , and the matrix of the Weingarten map

$$\mathcal{W} = \mathcal{F}_1^{-1} \mathcal{F}_2.$$

2. Compute the eigenvalues of  $\mathcal{W}$ , by solving for  $\lambda$  the equation

$$\det(\mathcal{W} - \lambda I) = 0.$$

The two solutions are the principal curvatures  $\kappa_1$  and  $\kappa_2$ .

3. Find scalars  $\lambda, \mu$  which solve the linear system

$$(\mathcal{W} - \kappa_i I) \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0.$$

The solution(s) gives the eigenvector(s) of  $\mathcal{W}$

$$\mathbf{x}_i = (\lambda, \mu)$$

corresponding to the eigenvalue  $\kappa_i$ .

4. The principal vector(s) associated to  $\kappa_i$  is

$$\mathbf{t}_i = \lambda \sigma_u + \mu \sigma_v$$

### Remark 3.108: The case of $\mathcal{W}$ diagonal

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be a regular chart and  $\mathcal{S} = \sigma(U)$ . Assume the matrix of the Weingarten map is diagonal

$$\mathcal{W} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Then, the eigenvalues of  $\mathcal{W}$  are  $\kappa_1$  and  $\kappa_2$ , with eigenvectors

$$\mathbf{x}_1 = (1, 0), \quad \mathbf{x}_2 = (0, 1).$$

Therefore  $\kappa_1, \kappa_2$  are the principal curvatures of  $\mathcal{S}$ , with principal vectors given by

$$\mathbf{t}_1 = \sigma_u, \quad \mathbf{t}_2 = \sigma_v.$$

### Proposition 3.109: Relationships between curvatures

Let  $\mathcal{S}$  be an orientable surface. Then

$$K = \kappa_1 \kappa_2, \quad H = \frac{\kappa_1 + \kappa_2}{2},$$

$$k_i = H \pm \sqrt{H^2 - K}.$$

### Example 3.110: Principal curvatures of Unit Cylinder

**Question.** Consider the unit cylinder charted by

$$\sigma(u, v) = (\cos(u), \sin(u), v).$$

Compute the principal curvature and principal vectors.

**Solution.** By Example 3.104, the matrix of the Weingarten map is

$$\mathcal{W} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since  $\mathcal{W}$  is diagonal, the eigenvalues are the diagonal entries of  $\mathcal{W}$  and the eigenvectors are

$$\mathbf{x}_1 = (1, 0), \quad \mathbf{x}_2 = (0, 1).$$

Therefore, the principal curvatures and principal vectors are

$$\kappa_1 = -1, \quad \kappa_2 = 0,$$

$$\mathbf{t}_1 = \sigma_u = (-\sin(u), \cos(u), 0),$$

$$\mathbf{t}_2 = \sigma_v = (0, 0, 1).$$

**Example 3.111: Curvatures of Sphere**

**Question.** Consider the chart for the sphere

$$\sigma(u, v) = (\cos(u) \cos(v), \sin(u) \cos(v), \sin(v)),$$

where  $u \in (0, 2\pi)$ ,  $v \in (-\pi/2, \pi/2)$ . Prove that

$$\mathcal{F}_1 = \mathcal{F}_2 = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$K = H = \kappa_1 = \kappa_2 = 1, \quad \mathbf{t}_1 = \sigma_u, \quad \mathbf{t}_2 = \sigma_v.$$

**Solution.** Compute the FFF of  $\sigma$

$$\begin{aligned} \sigma_u &= (-\sin(u) \cos(v), \cos(u) \cos(v), 0) \\ \sigma_v &= (-\cos(u) \sin(v), -\sin(u) \sin(v), \cos(v)) \\ E &= \sigma_u \cdot \sigma_u = \cos^2(v) \\ F &= \sigma_u \cdot \sigma_v = 0 \\ G &= \sigma_v \cdot \sigma_v = 1 \\ \mathcal{F}_1 &= \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Moreover

$$\begin{aligned} \sigma_u \times \sigma_v &= (\cos(u) \cos^2(v), \sin(u) \cos^2(v), \cos(v) \sin(v)) \\ \|\sigma_u \times \sigma_v\| &= |\cos(v)| = \cos(v), \end{aligned}$$

where we used that  $\cos(v) > 0$  since  $v \in (-\pi/2, \pi/2)$ . Therefore,

$$\begin{aligned} \mathbf{N} &= (\cos(u) \cos(v), \sin(u) \cos(v), \sin(v)) \\ \sigma_{uu} &= (-\cos(u) \cos(v), -\sin(u) \cos(v), 0) \\ \sigma_{uv} &= (\sin(u) \sin(v), -\cos(u) \sin(v), 0) \\ \sigma_{vv} &= (-\cos(u) \cos(v), -\sin(u) \cos(v), -\sin(v)) \\ L &= \sigma_{uu} \cdot \mathbf{N} = \cos^2(v) \\ M &= \sigma_{uv} \cdot \mathbf{N} = 0 \\ N &= \sigma_{vv} \cdot \mathbf{N} = 1 \end{aligned}$$

Hence, the SFF and matrix of the Weingarten map are

$$\mathcal{F}_2 = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{W} = \mathcal{F}_1^{-1} \mathcal{F}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since  $\mathcal{W}$  is diagonal, the principal curvatures and vectors are

$$\kappa_1 = \kappa_2 = 1, \quad \mathbf{t}_1 = \sigma_u, \quad \mathbf{t}_2 = \sigma_v.$$

Finally, the mean and Gaussian curvatures are

$$H = \frac{\kappa_1 + \kappa_2}{2} = 1, \quad K = \kappa_1 \kappa_2 = 1.$$

## 3.12 Normal and Geodesic curvatures

### Definition 3.112: Darboux frame

Let  $\mathcal{S}$  be a regular surface,  $\gamma : (a, b) \rightarrow \mathcal{S}$  a unit-speed curve. The **Darboux frame** of  $\gamma$  at  $t$  is the triple

$$\{\dot{\gamma}, \mathbf{N}, \mathbf{N} \times \dot{\gamma}\},$$

where  $\gamma$  is evaluated at  $t$ , and  $\mathbf{N}$  is the standard unit normal to  $\mathcal{S}$ , evaluated at  $\mathbf{p} = \gamma(t)$ .

### Proposition 3.113: Darboux frame is orthonormal basis

Let  $\mathcal{S}$  be a regular surface,  $\gamma : (a, b) \rightarrow \mathcal{S}$  a unit-speed curve. The Darboux frame is an orthonormal basis of  $\mathbb{R}^3$  for all  $t \in (a, b)$ .

### Proposition 3.114: Coefficients of $\ddot{\gamma}$ in the Darboux frame

Let  $\mathcal{S}$  be a regular surface,  $\gamma : (a, b) \rightarrow \mathcal{S}$  a unit-speed curve. Then

$$\ddot{\gamma} = \kappa_n \mathbf{N} + \kappa_g (\mathbf{N} \times \dot{\gamma}),$$

where  $\mathbf{N}$  is evaluated at  $\mathbf{p} := \gamma(t)$  and  $\kappa_n, \kappa_g$  are scalars dependent on  $\mathbf{p}$ . Moreover

$$\kappa_n = \ddot{\gamma} \cdot \mathbf{N}, \quad \kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma}),$$

$$\kappa^2 = \kappa_n^2 + \kappa_g^2,$$

$$\kappa_n = \kappa \cos(\phi), \quad \kappa_g = \pm \kappa \sin(\phi),$$

where  $\kappa$  is the curvature of  $\gamma$ , and  $\phi$  is the angle between  $\mathbf{N}$  and  $\mathbf{n}$ , the principal unit normal of  $\gamma$ .

### Definition 3.115: Normal and geodesic curvatures

Let  $\mathcal{S}$  be regular and  $\gamma : (a, b) \rightarrow \mathcal{S}$  a unit-speed curve. Let  $\mathbf{N}$  be the standard unit normal to  $\mathcal{S}$ .

1. The **normal curvature** of  $\gamma$  is

$$\kappa_n = \ddot{\gamma} \cdot \mathbf{N},$$

2. The **geodesic curvature** of  $\gamma$  is

$$\kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma}).$$

### Theorem 3.116: Computing $\kappa_n$ with SFF

Let  $\mathcal{S}$  be a regular surface and  $\gamma : (a, b) \rightarrow \mathcal{S}$  a unit-speed curve. Denote  $\mathbf{p} := \gamma(t)$ . We have:

1. The normal curvature  $\kappa_n$  satisfies

$$\kappa_n = II_{\mathbf{p}}(\dot{\gamma}, \dot{\gamma}).$$

2. Let  $\sigma$  be a chart for  $\mathcal{S}$  at  $\mathbf{p} = \gamma(t)$ . Then

$$\gamma(t) = \sigma(u(t), v(t))$$

for some smooth functions  $u, v : (a, b) \rightarrow \mathbb{R}$ , and

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,$$

where  $L, M, N$  are evaluated at  $(u(t), v(t))$ , and  $\dot{u}, \dot{v}$  at  $t$ .



**Example 3.117:** Curves on the sphere**Question.** Consider the unit sphere  $\mathbb{S}^2$  with chart

$$\sigma(u, v) = (\cos(u) \cos(v), \sin(u) \cos(v), \sin(v)).$$

Show that, for all unit-speed curves on  $\mathbb{S}^2$ ,

$$\kappa_n(t) = 1.$$

**Solution.** Let  $\gamma(t) = \sigma(u(t), v(t))$  be a unit-speed curve on  $\mathbb{S}^2$ . Differentiating, we get

$$\begin{aligned}\dot{\gamma}(t) &= \frac{d}{dt}(\cos(u(t)) \cos(v(t)), \sin(u(t)) \cos(v(t)), \sin(v(t))) \\ &= (-\dot{u} \sin(u) \cos(v) - \dot{v} \cos(u) \sin(v), \\ &\quad \dot{u} \cos(u) \cos(v) - \dot{v} \sin(u) \sin(v), \\ &\quad \dot{v} \cos(v))\end{aligned}$$

$$\|\dot{\gamma}(t)\|^2 = \cos^2(v) \dot{u}^2 + \dot{v}^2.$$

Since  $\gamma$  is unit-speed, we have  $\|\dot{\gamma}\| = 1$ . Therefore,

$$\cos^2(v) \dot{u}^2 + \dot{v}^2 = 1.$$

By Example 3.111, the coefficients of the SFF of  $\sigma$  are

$$L = \cos^2(v), \quad M = 0, \quad N = 1.$$

By Theorem 3.116, the normal curvature of  $\gamma$  is

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = \cos^2(v) \dot{u}^2 + \dot{v}^2 = 1.$$

**Theorem 3.118:** Euler's TheoremLet  $\mathcal{S}$  be a regular surface with principal curvatures  $\kappa_1, \kappa_2$  and principal vectors  $\mathbf{t}_1, \mathbf{t}_2$ . Let  $\gamma$  be a unit-speed curve on  $\mathcal{S}$ . The normal curvature of  $\gamma$  is given by

$$\kappa_n = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta),$$

where  $\theta$  is the angle between  $\dot{\gamma}$  and  $\mathbf{t}_1$ .**Example 3.119:** Curves on the sphere (again)**Question.** Same question as in Example 3.117.**Solution.** By Example 3.111, the principal curvatures of the unit sphere are  $\kappa_1 = \kappa_2 = 1$ . By Euler's Theorem, for any unit-speed curve  $\gamma$  on the sphere we have

$$\kappa_n = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta) = \cos^2(\theta) + \sin^2(\theta) = 1.$$

**Definition 3.120:**  $\kappa_n$  and  $\kappa_g$  for regular  $\gamma$ Let  $\mathcal{S}$  be regular, and  $\gamma : (a, b) \rightarrow \mathcal{S}$  a regular curve. Let  $\tilde{\gamma}$  be a unit-speed reparametrization of  $\gamma$ , with

$$\gamma = \tilde{\gamma} \circ \phi, \quad \phi : (a, b) \rightarrow (\tilde{a}, \tilde{b}).$$

Let  $\tilde{\kappa}_n$  and  $\tilde{\kappa}_g$  be the normal and geodesic curvatures of  $\tilde{\gamma}$ . The normal and geodesic curvatures of  $\gamma$  are

$$\kappa_n(t) = \tilde{\kappa}_n(\phi(t)), \quad \kappa_g(t) = \tilde{\kappa}_g(\phi(t)).$$

**Theorem 3.121:** Formulas for  $\kappa_n$  and  $\kappa_g$ Let  $\mathcal{S}$  be regular, and  $\gamma : (a, b) \rightarrow \mathcal{S}$  a regular curve.

1. The normal and geodesic curvatures of  $\gamma$  are given by

$$\kappa_n = \frac{\ddot{\gamma} \cdot \mathbf{N}}{\|\dot{\gamma}\|^2}, \quad \kappa_g = \frac{\ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma})}{\|\dot{\gamma}\|^3}.$$

2. Denote by  $\kappa$  the curvature of  $\gamma$ . It holds

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

3. Let  $\sigma$  be a chart for  $\mathcal{S}$  at  $\mathbf{p} = \gamma(t)$ . Then

$$\gamma(t) = \sigma(u(t), v(t))$$

for some smooth functions  $u, v : (a, b) \rightarrow \mathbb{R}$ , and

$$\kappa_n = \frac{II_{\mathbf{p}}(\dot{\gamma}, \dot{\gamma})}{I_{\mathbf{p}}(\dot{\gamma}, \dot{\gamma})} = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2},$$

with  $E, F, G, L, M, N$  evaluated at  $(u(t), v(t))$ , and  $\dot{u}, \dot{v}$  at  $t$ .**Example 3.122:** Calculation of normal and geodesic curvatures**Question.** For  $v \neq 0$  and  $t \neq 0$ , consider the surface chart and curve

$$\sigma(u, v) = \left(u, v, \frac{u}{v}\right), \quad \gamma(t) = \sigma(t^2, t).$$

1. Prove that  $\sigma$  is regular.
2. Compute the principal unit normal to  $\sigma$ .
3. Prove that  $\gamma$  is regular.
4. Compute the normal and geodesic curvatures of  $\gamma$ .
5. Compute  $\kappa$ , the curvature of  $\gamma$ . Verify that

$$\kappa^2 = \kappa_n^2 + \kappa_g^2.$$

**Solution.**

1. The chart  $\sigma$  is regular because

$$\sigma_u = \left(1, 0, \frac{1}{v}\right), \quad \sigma_v = \left(0, 1, -\frac{u}{v^2}\right)$$

$$\sigma_u \times \sigma_v = \left(-\frac{1}{v}, \frac{u}{v^2}, 1\right) \neq \mathbf{0}$$

2. The principal unit normal is

$$\begin{aligned}\|\sigma_u \times \sigma_v\| &= \frac{(u^2 + v^2 + v^4)^{1/2}}{v^2} \\ \mathbf{N} &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-v, u, v^2)}{(u^2 + v^2 + v^4)^{1/2}}.\end{aligned}$$

3. The curve  $\gamma$  is regular because

$$\begin{aligned}\gamma(t) &= \sigma(t^2, t) = (t^2, t, t) \\ \dot{\gamma}(t) &= (2t, 1, 1) \neq \mathbf{0}\end{aligned}$$

4. Compute the following quantities

$$\begin{aligned}\|\dot{\gamma}(t)\| &= 2^{1/2} (2t^2 + 1)^{1/2} & \ddot{\gamma} \cdot \mathbf{N} &= -\frac{2}{(2t^2 + 1)^{1/2}} \\ \dot{\gamma}(t) &= (2, 0, 0) & \mathbf{N} \times \dot{\gamma} &= (1 + 2t^2)^{1/2} (0, 1, -1) \\ \mathbf{N}(t^2, t) &= \frac{(-1, t, t)}{(2t^2 + 1)^{1/2}} & \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma}) &= 0\end{aligned}$$

The normal and geodesic curvatures are

$$\begin{aligned}\kappa_n &= \frac{\ddot{\gamma} \cdot \mathbf{N}}{\|\dot{\gamma}\|^2} = -\frac{1}{(2t^2 + 1)^{3/2}}, \\ \kappa_g &= \frac{\ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma})}{\|\dot{\gamma}\|^3} = 0.\end{aligned}$$

5. The curvature of  $\gamma$  is

$$\begin{aligned}\dot{\gamma} \times \ddot{\gamma} &= (0, 2, -2), \quad \|\dot{\gamma} \times \ddot{\gamma}\| = 2^{3/2} \\ \kappa &= \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} = \frac{1}{(2t^2 + 1)^{3/2}}\end{aligned}$$

Thus  $\kappa = -\kappa_n$ . Since  $\kappa_g = 0$ , we conclude that  $\kappa^2 = \kappa_n^2 + \kappa_g^2$ .

### 3.13 Local shape of a surface

#### Theorem 3.123: Local structure of surfaces

Let  $\mathcal{S}$  be a regular surface and  $\mathbf{p} \in \mathcal{S}$ . In the vicinity of  $\mathbf{p}$ , the surface  $\mathcal{S}$  is approximated by the quadric surface of equation

$$z = \frac{1}{2} (x^2 \kappa_1(\mathbf{p}) + y^2 \kappa_2(\mathbf{p})),$$

where  $\kappa_1(\mathbf{p}), \kappa_2(\mathbf{p})$  are the principal curvatures of  $\mathcal{S}$  at  $\mathbf{p}$ .

#### Definition 3.124: Local shape types

Let  $\mathcal{S}$  be a regular surface, with  $\kappa_1(\mathbf{p})$  and  $\kappa_2(\mathbf{p})$  the principal curvatures at  $\mathbf{p}$ . The point  $\mathbf{p}$  is

• **Elliptic** if

$$\kappa_1(\mathbf{p}) > 0, \kappa_2(\mathbf{p}) > 0 \quad \text{or} \quad \kappa_1(\mathbf{p}) < 0, \kappa_2(\mathbf{p}) < 0$$

• **Hyperbolic** if

$$\kappa_1(\mathbf{p}) < 0 < \kappa_2(\mathbf{p}) \quad \text{or} \quad \kappa_2(\mathbf{p}) < 0 < \kappa_1(\mathbf{p})$$

• **Parabolic** if

$$\kappa_1(\mathbf{p}) = 0, \kappa_2(\mathbf{p}) \neq 0 \quad \text{or} \quad \kappa_2(\mathbf{p}) \neq 0, \kappa_1(\mathbf{p}) = 0$$

• **Planar** if

$$\kappa_1(\mathbf{p}) = \kappa_2(\mathbf{p}) = 0$$

#### Proposition 3.125: Gaussian curvature and local shape

Let  $\mathcal{S}$  be a regular surface, with  $K(\mathbf{p})$  the Gaussian curvature at  $\mathbf{p}$ . The point  $\mathbf{p}$  is

- **Elliptic** if  $K(\mathbf{p}) > 0$ ,
- **Hyperbolic** if  $K(\mathbf{p}) < 0$ ,
- **Parabolic or Planar** if  $K(\mathbf{p}) = 0$ .

#### Example 3.126: Analysis of local shape

**Question.** Consider the surface chart

$$\sigma(u, v) = (u - v, u + v, u^2 + v^2).$$

1. Compute the first fundamental form of  $\sigma$ .
2. Compute the second fundamental form of  $\sigma$ .
3. Compute the matrix of the Weingarten map.
4. Show that  $\mathbf{p} = \sigma(1, 0)$  is an elliptic point.
5. Can there be points which are not elliptic?

**Solution.**

1. The FFF of  $\sigma$  is

$$\begin{aligned}\sigma_u &= (1, 1, 2u) & F &= \sigma_u \cdot \sigma_v = 4uv \\ \sigma_v &= (-1, 1, 2v) & G &= \sigma_v \cdot \sigma_v = 2(1 + 2v^2) \\ E &= \sigma_u \cdot \sigma_u = 2(1 + 2u^2) & \mathcal{F}_1 &= 2 \begin{pmatrix} 1 + 2u^2 & 2uv \\ 2uv & 1 + 2v^2 \end{pmatrix}\end{aligned}$$

2. The standard unit normal is

$$\begin{aligned}\sigma_u \times \sigma_v &= 2(v - u, -u - v, 1) \\ \|\sigma_u \times \sigma_v\| &= 2(1 + 2u^2 + 2v^2)^{1/2} \\ \mathbf{N} &= \frac{(v - u, -u - v, 1)}{(1 + 2u^2 + 2v^2)^{1/2}}\end{aligned}$$

The SFF of  $\sigma$  is

$$\begin{aligned}\sigma_{uu} &= (0, 0, 2) & L &= \sigma_{uu} \cdot \mathbf{N} = 2(1 + 2u^2 + 2v^2)^{-1/2} \\ \sigma_{uv} &= (0, 0, 0) & M &= \sigma_{uv} \cdot \mathbf{N} = 0 \\ \sigma_{vv} &= (0, 0, 2) & N &= \sigma_{vv} \cdot \mathbf{N} = 2(1 + 2u^2 + 2v^2)^{-1/2} \\ \mathcal{F}_2 &= (1 + 2u^2 + 2v^2)^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

3. The inverse of  $\mathcal{F}_1$  is

$$\begin{aligned}\mathcal{F}_1^{-1} &= \frac{1}{\det(\mathcal{F}_1)} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{2(1 + 2u^2 + 2v^2)} \begin{pmatrix} 1 + 2v^2 & -2uv \\ -2uv & 1 + 2u^2 \end{pmatrix}.\end{aligned}$$

The matrix of the Weingarten map is

$$\begin{aligned}\mathcal{W} &= \mathcal{F}_1^{-1} \mathcal{F}_2 \\ &= \frac{1}{(1 + 2u^2 + 2v^2)^{3/2}} \begin{pmatrix} 1 + 2v^2 & -2uv \\ -2uv & 1 + 2u^2 \end{pmatrix}.\end{aligned}$$

4. For  $u = 1$  and  $v = 0$  we obtain

$$\mathcal{W} = \frac{1}{3^{\frac{3}{2}}} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3^{-\frac{3}{2}} & 0 \\ 0 & 3^{-\frac{1}{2}} \end{pmatrix}.$$

Therefore the principal curvatures at  $\mathbf{p}$  are

$$\kappa_1(\mathbf{p}) = 3^{-\frac{3}{2}} > 0, \quad \kappa_2(\mathbf{p}) = 3^{-\frac{1}{2}} > 0.$$

Therefore  $\mathbf{p}$  is an elliptic point.

5. No. This is because the Gaussian curvature is

$$K = \det(\mathcal{W}) = \frac{1}{(1 + 2u^2 + 2v^2)^2} > 0.$$

By Proposition 3.125 we conclude that every point is elliptic.

2. If the sphere is charted as in Example 3.111, the FFF and SFF are

$$\mathcal{F}_1 = \mathcal{F}_2 = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}.$$

Since  $\mathcal{F}_2 = \mathcal{F}_1$ , all points on the sphere are umbilical.

**Remark 3.132:** How to find umbilics

Condition  $\mathcal{F}_2 = \kappa \mathcal{F}_1$  is equivalent to

$$(E, F, G) \times (L, M, N) = \mathbf{0}.$$

In practice, umbilics can be found by solving the above equations. Common factors may be discarded, if convenient.

**Example 3.133:** Local shape of the Monkey Saddle

**Question.** Consider the *Monkey Saddle* surface  $\mathcal{S}$  described by

$$z = x^3 - 3xy^2.$$

1. Compute the Gaussian curvature of  $\mathcal{S}$ .
2. Does  $\mathcal{S}$  contain any hyperbolic point?
3. Prove that the origin is the only umbilical point.

**Solution.** The Monkey Saddle is charted by

$$\sigma(u, v) = (u, v, u^3 - 3uv^2).$$

The FFF of  $\sigma$  is

$$\begin{aligned} \sigma_u &= (1, 0, 3(u^2 - v^2)) & F &= \sigma_u \cdot \sigma_v = -18uv(u^2 - v^2) \\ \sigma_v &= (0, 1, -6uv) & G &= \sigma_v \cdot \sigma_v = 1 + 36u^2v^2 \\ E &= \sigma_u \cdot \sigma_u = 1 + 9(u^2 - v^2)^2 \end{aligned}$$

The SFF of  $\sigma$  is

$$\begin{aligned} \sigma_u \times \sigma_v &= (-3(u^2 - v^2), 6uv, 1) \\ \|\sigma_u \times \sigma_v\| &= 1 + 36u^2v^2 + 9(u^2 - v^2)^2 \\ &= 1 + 9u^4 + 9v^4 + 18u^2v^2 \\ &= 1 + 9(u^2 + v^2)^2 \\ \mathbf{N} &= \frac{(-3(u^2 - v^2), 6uv, 1)}{\sqrt{1 + 9(u^2 + v^2)^2}} \\ \sigma_{uu} &= (0, 0, 6u) \\ \sigma_{uv} &= (0, 0, -6v) \\ \sigma_{vv} &= (0, 0, -6u) \\ L &= \sigma_{uu} \cdot \mathbf{N} = \frac{6u}{\sqrt{1 + 9(u^2 + v^2)^2}} \\ M &= \sigma_{uv} \cdot \mathbf{N} = \frac{-6v}{\sqrt{1 + 9(u^2 + v^2)^2}} \\ N &= \sigma_{vv} \cdot \mathbf{N} = \frac{-6u}{\sqrt{1 + 9(u^2 + v^2)^2}} \end{aligned}$$

## 3.14 Umbilical points

**Definition 3.127:** Umbilical point

Let  $\mathcal{S}$  be a regular surface, with  $\kappa_1(\mathbf{p})$  and  $\kappa_2(\mathbf{p})$  the principal curvatures at  $\mathbf{p}$ . We say that  $\mathbf{p}$  is an **umbilical point** if

$$\kappa_1(\mathbf{p}) = \kappa_2(\mathbf{p}).$$

**Theorem 3.128:** Structure theorem at umbilics

Let  $\mathcal{S}$  be a regular surface such that every point  $\mathbf{p} \in \mathcal{S}$  is umbilic. Then  $\mathcal{S}$  is an open subset of plane or a sphere.

**Proposition 3.129:** Criterion for umbilics

Let  $\mathcal{S}$  be a regular surface. The point  $\mathbf{p}$  is umbilical if and only if

$$H^2(\mathbf{p}) = K(\mathbf{p}).$$

In particular,  $\mathbf{p}$  cannot be umbilical if

$$K(\mathbf{p}) < 0.$$

**Proposition 3.130:** Chart criterion for umbilics

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be a regular chart and  $\mathcal{S} = \sigma(U)$ . A point  $\mathbf{p}$  is umbilic if and only if there exists a scalar  $\kappa$  such that

$$\mathcal{F}_2 = \kappa \mathcal{F}_1.$$

**Example 3.131:** Plane and Sphere

1. If the plane is charted as in Example 3.103, the FFF and SFF are

$$\mathcal{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore  $\mathcal{F}_2 = \kappa \mathcal{F}_1$  with  $\kappa = 0$ , and all points are umbilical.

1. We have that

$$\begin{aligned}
 EG - F^2 &= (1 + 9(u^2 - v^2)^2)(1 + 36u^2v^2) - (-18uv(u^2 - v^2))^2 \\
 &= 1 + 36u^2v^2 + 9(u^2 - v^2)^2 \\
 &= 1 + 9u^4 + 9v^4 + 18u^2v^2 \\
 &= 1 + 9(u^2 + v^2)^2 \\
 LN - M^2 &= -\frac{36(u^2 + v^2)}{1 + 9(u^2 + v^2)^2}
 \end{aligned}$$

Therefore the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{36(u^2 + v^2)}{[1 + 9(u^2 + v^2)^2]^2}.$$

2. Note that

$$K < 0, \quad \forall (u, v) \neq (0, 0).$$

By Proposition 3.125, we conclude that all the points outside of the origin are hyperbolic.

3. Since  $K < 0$  everywhere except at the origin, Proposition 3.129 implies that points outside the origin cannot be umbilic. At  $(0, 0)$ , we have

$$\mathcal{F}_1 = du^2 + dv^2, \quad \mathcal{F}_2 = 0.$$

Therefore  $\mathcal{F}_2$  is a multiple of  $\mathcal{F}_1$ , and by Proposition 3.130 we conclude that  $(0, 0)$  is an umbilical point. Note: the matrix of the Weingarten map is  $\mathcal{W} = \mathcal{F}_1^{-1}\mathcal{F}_2 = 0$ . Therefore the principal curvatures are  $\kappa_1 = \kappa_2 = 0$ , showing that  $(0, 0)$  is a planar point.

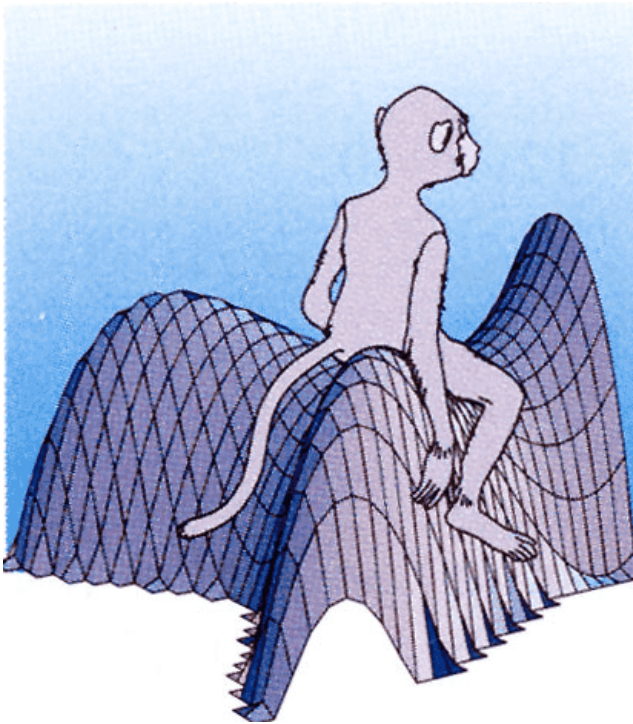


Figure 3.1: The Monkey Saddle surface  $z = x^3 - 3xy^2$ .

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