Differential Geometry

Revision Guide

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Revision Guide

Revision Guide document for the module **Differential Geometry 661955** 2024/25 at the University of Hull. If you have any question or find any typo, please email me at

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Full lenght Lecture Notes of the module available at

silviofanzon.com/2024-Differential-Geometry-Notes

Recommended revision strategy

Make sure you are very comfortable with:

1. The Definitions, Theorems, Proofs, and Examples contained in this Revision Guide
2. The Homework questions
3. The 2022/23 and 2023/24 Exam Papers questions.
4. The Checklist below

Checklist

You should be comfortable with the following topics/taks:

You should be comfortable with the following topics/tasks:

Curves

- Regularity of curves
- Length, arc-length, and arc-length reparametrization
- Calculating the curvature and torsion of unit speed curves from the definitions
- Calculating the curvature and torsion of (possibly not unit speed) curves from the formulae
- Calculating the Frenet frame of a unit-speed curve
- Applying the Fundamental Theorem of Space Curves to determine if two curves coincide, up to a ridig motion
- Proving that a curve is contained in a plane, and computing the equation of such plane
- Proving that a curve is part of a circle

Topology: To be completed

Surfaces:

- Regularity of surface charts
- Computing reparametrizations
- Computing a basis and the equation of the tangent plane
- Calculating the standard unit normal of a surface chart
- Calculating the differential of a smooth function between surfaces
- Proving that a given level surface is regular, and computing its tangent plane
- Proving that a given surface is ruled
- Calculating the first fundamental form of a surface chart
- Proving that a given map is a local isometry / conformal
- Prove that a given parametrization is conformal
- Calculating length and angles of curves on surfaces
- Calculating the second fundamental form of a surface chart
- Calculating the matrix of the Weingarten map, the principal curvatures, and principal directions of a surface chart
- Calculating Gaussian and mean curvature of a surface chart
- Calculating normal and geodesic curvature of a curve on a surface
- Classifying points of a surface as elliptic, parabolic, hyperbolic, planar

1 Curves

Definition 1.1: Length

The **length** of the curve $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ is

$$L(\boldsymbol{\gamma}) = \int_a^b \|\dot{\boldsymbol{\gamma}}(u)\| \, du \,.$$

Example 1.2: Length of Circle

Question. Compute the length of the circle of radius R

$$\boldsymbol{\gamma}(t) = (x_0 + R\cos(t), y_0 + R\sin(t), 0)\,.$$

Solution. We compute

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= (-R\sin(t), R\cos(t), 0) \\ \|\dot{\boldsymbol{\gamma}}(t)\| &= \sqrt{\dot{\gamma}_1^2(t) + \dot{\gamma}_2^2(t)} = R \\ L(\boldsymbol{\gamma}) &= \int_0^{2\pi} \|\dot{\boldsymbol{\gamma}}(t)\| \, dt = \int_0^{2\pi} R \, dt = 2\pi R\,. \end{aligned}$$

Example 1.3: Length of Helix

Question. Compute the length of the Helix

$$\boldsymbol{\gamma}(t) = (R\cos(t), R\sin(t), Ht)\,, \quad t \in (0, 2\pi)\,.$$

Solution. We compute

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= (-R\sin(t), R\cos(t), H) \\ \|\dot{\boldsymbol{\gamma}}(t)\| &= \sqrt{R^2 + H^2} \\ L(\boldsymbol{\gamma}) &= \int_0^{2\pi} \|\dot{\boldsymbol{\gamma}}(u)\| \, du = 2\pi\sqrt{R^2 + H^2} \end{aligned}$$

Definition 1.4: Arc-Length

The **arc-length** along $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ from t_0 to t is

$$s : (a, b) \to \mathbb{R}\,, \quad s(t) = \int_{t_0}^t \|\dot{\boldsymbol{\gamma}}(u)\| du\,.$$

Example 1.5: Arc-length of Logarithmic Spiral

Question. Compute the arc-length of

$$\boldsymbol{\gamma}(t) = (e^{kt}\cos(t), e^{kt}\sin(t), 0)\,, \quad t \in (0, 2\pi)\,.$$

Solution. The arc-length starting from t_0 is

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= e^{kt}(k\cos(t) - \sin(t), k\sin(t) + \cos(t), 0) \\ \|\dot{\boldsymbol{\gamma}}(t)\|^2 &= (k^2 + 1)e^{2kt} \\ s(t) &= \int_{t_0}^t \|\dot{\boldsymbol{\gamma}}(\tau)\| \, d\tau = \frac{\sqrt{k^2+1}}{k}(e^{kt} - e^{kt_0})\,. \end{aligned}$$

Definition 1.6: Unit-speed curve

A curve $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ is **unit-speed** if

$$\|\dot{\boldsymbol{\gamma}}(t)\| = 1\,, \quad \forall\, t \in (a, b)\,.$$

Theorem 1.7

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^n$ be a unit-speed curve. Then

$$\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}} = 0\,, \quad \forall\, t \in (a, b)\,.$$

Proof

Since $\boldsymbol{\gamma}$ is unit-speed, we have $\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} = 1$. Differentiating both sides, we get the thesis:

$$0 = \frac{d}{dt}(\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}) = \ddot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}} = 2\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}}\,.$$

Definition 1.8: Reparametrization

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$. A **reparametrization** of $\boldsymbol{\gamma}$ is a curve $\tilde{\boldsymbol{\gamma}} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^3$ such that

$$\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\phi(t))\,, \quad \forall\, t \in (\tilde{a}, \tilde{b})\,,$$

for $\phi : (\tilde{a}, \tilde{b}) \to (a, b)$ diffeomorphism.

Definition 1.9: Unit-speed reparametrization

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$. A **unit-speed reparametrization** of $\boldsymbol{\gamma}$ is a reparametrization $\tilde{\boldsymbol{\gamma}} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^3$ which is unit-speed, that is,

$$\|\dot{\tilde{\boldsymbol{\gamma}}}(t)\| = 1\,, \quad \forall\, t \in (\tilde{a}, \tilde{b})\,.$$

Definition 1.10: Regular curve

A curve $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ is **regular** if

$$\|\dot{\boldsymbol{\gamma}}(t)\| \neq 0\,, \quad \forall\, t \in (a, b)$$

Theorem 1.11: Existence of unit-speed reparametrization

Let $\boldsymbol{\gamma}$ be a curve. They are equivalent:

1. $\boldsymbol{\gamma}$ is regular,
2. $\boldsymbol{\gamma}$ admits unit-speed reparametrization.

Theorem 1.12: Arc-length and unit-speed reparametrization

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a regular curve. Let $\tilde{\boldsymbol{\gamma}} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^3$ be a reparametrization of $\boldsymbol{\gamma}$, that is,

$$\boldsymbol{\gamma}(t) = \tilde{\boldsymbol{\gamma}}(\phi(t)), \quad \forall\, t \in (a, b)$$

for some diffeomorphism $\phi : (a, b) \to (\tilde{a}, \tilde{b})$. We have

1. If $\tilde{\boldsymbol{\gamma}}$ is unit-speed, there exists $c \in \mathbb{R}$ such that
$$\phi(t) = \pm s(t) + c\,, \quad \forall\, t \in (a, b)\,. \tag{1.1}$$
2. If ϕ is given by (1.1), then $\tilde{\boldsymbol{\gamma}}$ is unit-speed.

Definition 1.13: Arc-length reparametrization

Let $\boldsymbol{\gamma}$ be regular. The **arc-length reparametrization** of $\boldsymbol{\gamma}$ is the curve

$$\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ s^{-1}$$

with s^{-1} inverse of the arc-length function of $\boldsymbol{\gamma}$.

Example 1.14: Arc-length reparametrization of Circle

Question. The circle of radius $R > 0$ is

$$\boldsymbol{\gamma}(t) = (x_0 + R\cos(t), y_0 + \sin(t), 0)\,.$$

Reparametrize $\boldsymbol{\gamma}$ by arc-length.
Solution. The arc-length of $\boldsymbol{\gamma}$ starting from $t_0 = 0$ is

$$s(t) = \int_0^t \|\dot{\boldsymbol{\gamma}}(u)\|\, du = Rt$$

The inverse is $t(s) = s/R$. The arc-length reparametrization of $\boldsymbol{\gamma}$ is

$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) = \left(x_0 + R\cos\left(\frac{s}{R}\right), y_0 + \sin\left(\frac{s}{R}\right), 0\right)\,.$$

Example 1.15

Question. Consider the curve

$$\boldsymbol{\gamma}(t) = (5\cos(t), 5\sin(t), 12t)\,.$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Reparametrize $\boldsymbol{\gamma}$ by arc-length.

Solution.

1. $\boldsymbol{\gamma}$ is regular because
$$\dot{\boldsymbol{\gamma}}(t) = (-5\sin(t), 5\cos(t), 12)$$
$$\|\dot{\boldsymbol{\gamma}}(t)\| = 13 \neq 0$$
2. The arc-length of $\boldsymbol{\gamma}$ starting from $t_0 = 0$ is
$$s(t) = \int_0^t \|\dot{\boldsymbol{\gamma}}(u)\|\, du = 13t\,.$$
The inverse is $t(s) = s/13$. The arc-length reparametrization of $\boldsymbol{\gamma}$ is
$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) = \left(5\cos\left(\frac{s}{13}\right), 5\sin\left(\frac{s}{13}\right), \frac{12}{13}s\right)\,.$$

1.1 Curvature

Definition 1.16: Curvature of unit-speed curve

The **curvature** of a unit-speed curve $\boldsymbol{\gamma}$ is

$$\kappa(t) = \|\ddot{\boldsymbol{\gamma}}(t)\|\,.$$

Example 1.17: Curvature of the Circle

Question. Compute the curvature of the circle of radius $R > 0$

$$\boldsymbol{\gamma}(t) = \left(x_0 + R\cos\left(\frac{t}{R}\right), y_0 + \sin\left(\frac{t}{R}\right), 0\right)\,.$$

Solution. First, check that $\boldsymbol{\gamma}$ is unit-speed:

$$\dot{\boldsymbol{\gamma}}(t) = \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right), 0\right)$$
$$\|\dot{\boldsymbol{\gamma}}(t)\| = 1$$

Now, compute second derivative and curvature

$$\ddot{\boldsymbol{\gamma}}(t) = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right), 0\right)$$
$$\kappa(t) = \|\ddot{\boldsymbol{\gamma}}(t)\| = \frac{1}{R}$$

Definition 1.18: Curvature of regular curve

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be a regular curve and $\tilde{\boldsymbol{\gamma}}$ be a unit-speed reparametrization of $\boldsymbol{\gamma}$, with

$$\boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}} \circ \phi \,, \quad \phi : (a,b) \to (\tilde{a}, \tilde{b}) \,.$$

Let $\tilde{\kappa} : (\tilde{a}, \tilde{b}) \to \mathbb{R}$ be the curvature of $\tilde{\boldsymbol{\gamma}}$. The **curvature** of $\boldsymbol{\gamma}$ is

$$\kappa(t) = \tilde{\kappa}(\phi(t)) \,.$$

Remark 1.19: Computing curvature of regular $\boldsymbol{\gamma}$

1. Compute the arc-length $s(t)$ of $\boldsymbol{\gamma}$ and its inverse $t(s)$.
2. Compute the arc-length reparametrization

$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) \,.$$

3. Compute the curvature of $\tilde{\boldsymbol{\gamma}}$

$$\tilde{\kappa}(s) = \|\ddot{\tilde{\boldsymbol{\gamma}}}(s)\| \,.$$

4. The curvature of $\boldsymbol{\gamma}$ is

$$\kappa(t) = \tilde{\kappa}(s(t)) \,.$$

Definition 1.20: Hyperbolic functions

The **hyperbolic functions** are defined by:

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \,, \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$
$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} \,, \quad \coth(t) = \frac{\cosh(t)}{\sinh(t)}$$
$$\operatorname{sech}(t) = \frac{1}{\cosh(t)} \,, \quad \operatorname{csch}(t) = \frac{1}{\sinh(t)}$$

Key identities involving hyperbolic functions:

$$\cosh^2(t) - \sinh^2(t) = 1 \,, \quad \operatorname{sech}^2(t) - \tanh^2(t) = 1$$
$$\frac{d}{dt}[\sinh(t)] = \cosh(t) \,, \quad \frac{d}{dt}[\cosh(t)] = \sinh(t)$$
$$\frac{d}{dt}[\tanh(t)] = 1 - \tanh^2(t) = -\operatorname{csch}^2(t)$$

Example 1.21: Curvature of the Catenary

Question. Consider the Catenary curve

$$\boldsymbol{\gamma}(t) = (t, \cosh(t)) \,, \qquad t \in \mathbb{R} \,.$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the arc-length reparametrization of $\boldsymbol{\gamma}$.
3. Compute the curvature of $\tilde{\boldsymbol{\gamma}}$.
4. Compute the curvature of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular because

$$\dot{\boldsymbol{\gamma}}(t) = (1, \sinh(t))$$
$$\|\dot{\boldsymbol{\gamma}}\| = \sqrt{1 + \sinh^2(t)} = \cosh(t) \geq 1$$

2. The arc-length of $\boldsymbol{\gamma}$ starting at $t_0 = 0$ is

$$s(t) = \int_0^t \|\dot{\boldsymbol{\gamma}}(u)\| \, du = \int_0^t \cosh(u) \, du = \sinh(t)$$

where we used that $\sinh(0) = 0$. Moreover,

$$s = \sinh(t) \iff s = \frac{e^t - e^{-t}}{2}$$
$$\iff e^{2t} - 2se^t - 1 = 0$$

Substitute $y = e^t$ to obtain

$$e^{2t} - 2se^t - 1 = 0 \iff y^2 - 2sy - 1 = 0$$
$$\iff y_{\pm} = s \pm \sqrt{1 + s^2} \,.$$

Notice that

$$y_+ = s + \sqrt{1 + s^2} \geq s + \sqrt{s^2} = s + |s| \geq 0$$

by definition of absolute value. As we were looking for y in the form $y = e^t$, we only consider the positive solution y_+. Then,

$$e^t = y_+ = s + \sqrt{1 + s^2}$$
$$t(s) = \log\left(s + \sqrt{1 + s^2}\right)$$

The arc-length reparametrization of $\boldsymbol{\gamma}$ is

$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) = \left(\log\left(s + \sqrt{1 + s^2}\right), \sqrt{1 + s^2}\right)$$

3. Compute the curvature of $\tilde{\boldsymbol{\gamma}}$

$$\dot{\tilde{\boldsymbol{\gamma}}}(s) = \left(\frac{1}{\sqrt{1 + s^2}}, \frac{s}{\sqrt{1 + s^2}}\right)$$
$$\ddot{\tilde{\boldsymbol{\gamma}}}(s) = \left(-\frac{s}{(1 + s^2)^{3/2}}, \frac{1}{(1 + s^2)^{3/2}}\right)$$
$$\|\ddot{\tilde{\boldsymbol{\gamma}}}(s)\|^2 = \frac{1}{(1 + s^2)^2}$$
$$\tilde{\kappa}(s) = \|\ddot{\tilde{\boldsymbol{\gamma}}}(s)\| = \frac{1}{1 + s^2}$$

4. Recalling that $s(t) = \sinh(t)$, the curvature of $\boldsymbol{\gamma}$ is

$$\kappa(t) = \tilde{\kappa}(s(t)) = \frac{1}{1 + \sinh^2(t)} = \frac{1}{\cosh^2(t)} \,.$$

Definition 1.22: Vector product

The **vector product** of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{e}_3 \tag{1.2}$$

with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ vectors of the standard basis of R^3. Formula (1.2) is usually denoted by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_2 \\ v_1 & v_2 & v_3 \end{vmatrix} .$$

Theorem 1.23: Geometric Properties of vector product

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be linearly independent. Then

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to the plane spanned by $\mathbf{u}, \mathbf{v}$
- $\|\mathbf{u} \times \mathbf{v}\|$ equals the area of the parallelogram with sides $\mathbf{u}, \mathbf{v}$
- The following triple is a positive (right-handed) basis of $\mathbb{R}^3$
$$(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}) .$$

Theorem 1.24

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ it holds:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

Theorem 1.25

Suppose $\boldsymbol{\gamma}, \boldsymbol{\eta} : (a, b) \to \mathbb{R}^3$ are parametrized curves. Then, the curve $\boldsymbol{\gamma} \times \boldsymbol{\eta}$ is smooth, and

$$\frac{d}{dt}(\boldsymbol{\gamma} \times \boldsymbol{\eta}) = \dot{\boldsymbol{\gamma}} \times \boldsymbol{\eta} + \boldsymbol{\gamma} \times \dot{\boldsymbol{\eta}} .$$

Theorem 1.26: Curvature formula

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a regular curve. The curvature of $\boldsymbol{\gamma}$ is

$$\kappa(t) = \frac{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|}{\|\dot{\boldsymbol{\gamma}}(t)\|^3} .$$

Example 1.27: Curvature of the Helix

Question. Consider the Helix of radius $R > 0$ and rise H,

$$\boldsymbol{\gamma}(t) = (R \cos(t), R \sin(t), Ht) .$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the curvature of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular because
$$\dot{\boldsymbol{\gamma}}(t) = (-R \sin(t), R \cos(t), H)$$
$$\|\dot{\boldsymbol{\gamma}}(t)\| = \sqrt{R^2 + H^2} \geq R > 0$$
2. Compute the curvature using the formula:
$$\begin{aligned} \ddot{\boldsymbol{\gamma}}(t) &= (-R \cos(t), -R \sin(t), 0) \\ \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} &= \left(RH \sin(t), -RH \cos(t), R^2\right) \\ \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| &= R\sqrt{R^2 + H^2} \\ \kappa(t) &= \frac{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|}{\|\dot{\boldsymbol{\gamma}}(t)\|^3} = \frac{R}{R^2 + H^2} \end{aligned}$$

Example 1.28

Question. Define the curve

$$\boldsymbol{\gamma}(t) = \left(\frac{8}{5} \cos(t), 1 - 2 \sin(t), \frac{6}{5} \cos(t)\right) .$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the curvature of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular because
$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= \left(-\frac{8}{5} \sin(t), -2 \cos(t), -\frac{6}{5} \sin(t)\right) \\ \|\dot{\boldsymbol{\gamma}}(t)\| &= 2 \neq 0 \end{aligned}$$
2. Compute the curvature using the formula:
$$\begin{aligned} \ddot{\boldsymbol{\gamma}}(t) &= \left(-\frac{8}{5} \cos(t), 2 \sin(t), -\frac{6}{5} \cos(t)\right) \\ \dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t) &= \left(-\frac{12}{5}, 0, \frac{16}{5}\right) \\ \|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\| &= 4 \\ \kappa(t) &= \frac{1}{2} . \end{aligned}$$

Example 1.29: Different curves, same curvature

Question Let $\boldsymbol{\gamma}$ be a circle

$$\boldsymbol{\gamma}(t) = (2 \cos(t), 2 \sin(t), 0) ,$$

and $\boldsymbol{\eta}$ be a helix of radius $S > 0$ and rise $H > 0$

$$\boldsymbol{\eta}(t) = (S \cos(t), S \sin(t), Ht) .$$

Find S and H such that $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ have the same curvature.
Solution. Curvatures of $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ were already computed:

$$\kappa^{\boldsymbol{\gamma}} = \frac{1}{2}, \quad \kappa^{\boldsymbol{\eta}} = \frac{S}{S^2 + H^2}.$$

Imposing that $\kappa^{\boldsymbol{\gamma}} = \kappa^{\boldsymbol{\eta}}$, we get

$$\frac{1}{2} = \frac{S}{S^2 + H^2} \quad \Longrightarrow \quad H^2 = 2S - S^2.$$

Choosing $S = 1$ and $H = 1$ yields $\kappa^{\boldsymbol{\gamma}} = \kappa^{\boldsymbol{\eta}}$.

1.2 Frenet frame and torsion

Definition 1.30: Frenet frame of unit-speed curve

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a unit-speed curve with $\kappa \neq 0$.

1. The **tangent vector** to $\boldsymbol{\gamma}$ is
$$\mathbf{t}(t) = \dot{\boldsymbol{\gamma}}(t).$$
2. The **principal normal vector** to $\boldsymbol{\gamma}$ is
$$\mathbf{n}(t) = \frac{1}{\kappa(t)} \ddot{\boldsymbol{\gamma}}(t).$$
3. The **binormal vector** to $\boldsymbol{\gamma}$ is
$$\mathbf{b}(t) = \dot{\boldsymbol{\gamma}}(t) \times \mathbf{n}(t).$$
4. The **Frenet frame** of $\boldsymbol{\gamma}$ is the triple
$$\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}.$$

Theorem 1.31

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a unit-speed curve with $\kappa \neq 0$. The Frenet frame

$$\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$$

is a positive orthonormal basis of $\mathbb{R}^3$ for each $t \in (a, b)$.

Definition 1.32: Torsion of unit-speed curve with $\kappa \neq 0$

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a unit-speed curve with $\kappa \neq 0$. The **torsion** of $\boldsymbol{\gamma}$ is the unique scalar $\tau(t)$ such that

$$\dot{\mathbf{b}}(t) = -\tau(t)\mathbf{n}(t).$$

In particular,

$$\tau(t) = -\dot{\mathbf{b}}(t) \cdot \mathbf{n}(t).$$

Definition 1.33: Torsion of regular curve with $\kappa \neq 0$

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a regular curve with $\kappa \neq 0$. Let $\tilde{\boldsymbol{\gamma}}$ be a unit-speed reparametrization of $\boldsymbol{\gamma}$ with

$$\boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}} \circ \phi, \quad \phi : (a, b) \to (\tilde{a}, \tilde{b}).$$

Let $\tilde{\tau} : (\tilde{a}, \tilde{b}) \to \mathbb{R}$ be the torsion of $\tilde{\boldsymbol{\gamma}}$. The **torsion** of $\boldsymbol{\gamma}$ is

$$\tau(t) = \tilde{\tau}(\phi(t)).$$

Example 1.34: Frenet frame of Helix

Question. Consider the Helix of radius $R > 0$ and rise H

$$\boldsymbol{\gamma}(t) = (R\cos(t), R\sin(t), tH), \quad t \in \mathbb{R}.$$

1. Compute the arc-length reparametrization $\tilde{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$.
2. Compute Frenet frame, curvature and torsion of $\tilde{\boldsymbol{\gamma}}$.
3. Compute curvature and torsion $\boldsymbol{\gamma}$.

Solution.

1. The arc-length of $\boldsymbol{\gamma}$ starting at $t_0 = 0$ is
$$\dot{\boldsymbol{\gamma}}(t) = (-R\sin(t), R\cos(t), H)$$
$$\|\dot{\boldsymbol{\gamma}}\| = \rho, \qquad \rho := \sqrt{R^2 + H^2}$$
$$s(t) = \int_0^t \|\dot{\boldsymbol{\gamma}}(u)\| \, du = \rho t,$$
which is invertible, with inverse
$$t(s) = \frac{s}{\rho}.$$
The arc-length reparametrization $\tilde{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ is
$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) = \left(R\cos\left(\frac{s}{\rho}\right), R\sin\left(\frac{s}{\rho}\right), \frac{Hs}{\rho}\right).$$
2. Compute the tangent vector to $\tilde{\boldsymbol{\gamma}}$ and its derivative
$$\tilde{\mathbf{t}}(s) = \dot{\tilde{\boldsymbol{\gamma}}} = \frac{1}{\rho}\left(-R\sin\left(\frac{s}{\rho}\right), R\cos\left(\frac{s}{\rho}\right), H\right)$$
$$\dot{\tilde{\mathbf{t}}}(s) = \frac{R}{\rho^2}\left(-\cos\left(\frac{s}{\rho}\right), -\sin\left(\frac{s}{\rho}\right), 0\right)$$
The curvature of $\tilde{\boldsymbol{\gamma}}$ is
$$\tilde{\kappa}(s) = \|\ddot{\tilde{\boldsymbol{\gamma}}}(s)\| = \|\dot{\tilde{\mathbf{t}}}(s)\| = \frac{R}{R^2 + H^2}.$$
The principal normal vector and binormal are
$$\tilde{\mathbf{n}}(s) = \frac{\dot{\tilde{\mathbf{t}}}}{\tilde{\kappa}} = \left(-\cos\left(\frac{s}{\rho}\right), -\sin\left(\frac{s}{\rho}\right), 0\right)$$
$$\tilde{\mathbf{b}}(s) = \tilde{\mathbf{t}} \times \tilde{\mathbf{n}} = \frac{1}{\rho}\left(H\sin\left(\frac{s}{\rho}\right), -H\cos\left(\frac{s}{\rho}\right), R\right).$$

We are left to compute the torsion of $\tilde{\boldsymbol{\gamma}}$:

$$\dot{\tilde{\mathbf{b}}}(s) = \frac{H}{\rho^2}\left(\cos\left(\frac{s}{\rho}\right), \sin\left(\frac{s}{\rho}\right), 0\right)$$
$$\dot{\tilde{\mathbf{b}}}(s)\cdot\tilde{\mathbf{n}}(s) = -\frac{H}{\rho^2}$$
$$\tilde{\tau}(s) = -\dot{\tilde{\mathbf{b}}}(s)\cdot\tilde{\mathbf{n}}(s) = \frac{H}{\rho^2} = \frac{H}{R^2+H^2}$$

3. The curvature and torsion of $\boldsymbol{\gamma}$ are

$$\kappa(t) = \tilde{\kappa}(s(t)) = \frac{R}{R^2+H^2}$$
$$\tau(t) = \tilde{\tau}(s(t)) = \frac{H}{R^2+H^2}$$

Theorem 1.35: Torsion formula

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be a regular curve with $\kappa \neq 0$. The torsion of $\boldsymbol{\gamma}$ is

$$\tau(t) = \frac{(\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t))\cdot\dddot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t)\|^2}\,.$$

Example 1.36: Torsion of Helix with formula

Question. Consider the Helix of radius $R > 0$ and rise $H > 0$

$$\boldsymbol{\gamma}(t) = (R\cos(t), R\sin(t), Ht)\,, \quad t\in\mathbb{R}\,.$$

1. Prove that $\boldsymbol{\gamma}$ is regular with non-vanishing curvature.
2. Compute the torsion of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular with non-vanishing curvature, since

$$\|\dot{\boldsymbol{\gamma}}(t)\| = \sqrt{R^2+H^2} \geq R > 0\,,$$
$$\kappa = \frac{R}{R^2+H^2} > 0\,.$$

2. We compute the torsion using the formula:

$$\dot{\boldsymbol{\gamma}}(t) = (-R\sin(t), R\cos(t), H)$$
$$\ddot{\boldsymbol{\gamma}}(t) = (-R\cos(t), -R\sin(t), 0)$$
$$\dddot{\boldsymbol{\gamma}}(t) = (R\sin(t), -R\cos(t), 0)$$
$$\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}} = \left(RH\sin(t), -RH\cos(t), R^2\right)$$
$$\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\| = R\sqrt{R^2+H^2}$$
$$(\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}} = R^2H$$
$$\tau(t) = \frac{(\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|^2} = \frac{H}{R^2+H^2}$$

Example 1.37

Question. Compute the torsion of the curve

$$\boldsymbol{\gamma}(t) = \left(\frac{8}{5}\cos(t), 1-2\sin(t), \frac{6}{5}\cos(t)\right)\,.$$

Solution. Resuming calculations from Example 1.28,

$$\dddot{\boldsymbol{\gamma}}(t) = \left(\frac{8}{5}\sin(t), 2\cos(t), \frac{6}{5}\sin(t)\right)$$
$$(\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t))\cdot\dddot{\boldsymbol{\gamma}}(t) = \frac{96}{25}\sin(t) - \frac{96}{25}\sin(t) = 0$$
$$\tau(t) = \frac{(\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|^2} = 0$$

Theorem 1.38: General Frenet frame formulas

The Frenet frame of a regular curve $\boldsymbol{\gamma}$ is

$$\mathbf{t} = \frac{\dot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\|}\,, \qquad \mathbf{b} = \frac{\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|}\,,$$
$$\mathbf{n} = \mathbf{b}\times\mathbf{t} = \frac{(\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}})\times\dot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|\,\|\dot{\boldsymbol{\gamma}}\|}\,.$$

Example 1.39: Twisted cubic

Question. Let $\boldsymbol{\gamma} : \mathbb{R}\to\mathbb{R}^3$ be the *twisted cubic*

$$\boldsymbol{\gamma}(t) = (t, t^2, t^3)\,.$$

1. Is $\boldsymbol{\gamma}$ regular/unit-speed? Justify your answer.
2. Compute the curvature and torsion of $\boldsymbol{\gamma}$.
3. Compute the Frenet frame of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular, but not-unit speed, because

$$\dot{\boldsymbol{\gamma}}(t) = (1, 2t, 3t^2)$$
$$\|\dot{\boldsymbol{\gamma}}(t)\| = \sqrt{1+4t^2+9t^4} \geq 1$$
$$\|\dot{\boldsymbol{\gamma}}(1)\| = \sqrt{14} \neq 1$$

2. Compute curvature and torsion using the formulas:

$$\ddot{\boldsymbol{\gamma}}(t) = (0, 2, 6t)$$
$$\dddot{\boldsymbol{\gamma}}(t) = (0, 0, 6)$$
$$\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t) = (6t^2, -6t, 2)$$
$$\|\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t)\| = 2\sqrt{1+9t^2+9t^4}$$
$$(\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}} = 12$$
$$\kappa(t) = \frac{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|}{\|\dot{\boldsymbol{\gamma}}\|^3} = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}$$
$$\tau(t) = \frac{(\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|^2} = \frac{3}{1+9t^2+9t^4}\,.$$

3. By the Frenet frame formulas and the above calculations,

$$\mathbf{t} = \frac{\dot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\|} = \frac{1}{\sqrt{1+4t^2+9t^4}}\,(1,2t,3t^2)$$

$$\mathbf{b} = \frac{\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|} = \frac{1}{\sqrt{1+9t^2+9t^4}}(3t^2,-3t,1)$$

$$\mathbf{n} = \mathbf{b}\times\mathbf{t} = \frac{(-9t^3-2t,\,1-9t^4,\,6t^3+3t)}{\sqrt{1+9t^2+9t^4}\,\sqrt{1+4t^2+9t^4}}$$

1.3 Frenet-Serret equations

Theorem 1.40: Frenet frame is right-handed

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be a unit-speed curve with $\kappa \neq 0$. Then

$$\mathbf{b} = \mathbf{t}\times\mathbf{n}\,, \qquad \mathbf{n} = \mathbf{b}\times\mathbf{t}\,, \qquad \mathbf{t} = \mathbf{n}\times\mathbf{b}\,.$$

Theorem 1.41: Frenet-Serret equations

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be unit-speed with $\kappa \neq 0$. The Frenet frame of $\boldsymbol{\gamma}$ solves the **Frenet-Serret** equations

$$\dot{\mathbf{t}} = \kappa\mathbf{n}\,, \qquad \dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b}\,, \qquad \dot{\mathbf{b}} = -\tau\mathbf{n}\,.$$

Definition 1.42: Rigid motion

A **rigid motion** of $\mathbb{R}^3$ is a map $M : \mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$M(\mathbf{v}) = R\mathbf{v} + \mathbf{p}\,, \qquad \mathbf{v} \in \mathbb{R}^3\,,$$

where $\mathbf{p} \in \mathbb{R}^3$, and $R \in \mathbb{R}^{3\times 3}$ **rotation matrix**,

$$R \in \mathrm{SO}(3) := \{R \in \mathbb{R}^{3\times 3} \ : \ R^T R = I\,,\ \det(R) = 1\}\,.$$

Theorem 1.43: Fundamental Theorem of Space Curves

Let $\kappa, \tau \ : (a,b) \to \mathbb{R}$ be smooth, with $\kappa > 0$. Then:

1. There exists a unit-speed curve $\boldsymbol{\gamma} \ : (a,b) \to \mathbb{R}^3$ with curvature $\kappa(t)$ and torsion $\tau(t)$.
2. Suppose that $\tilde{\boldsymbol{\gamma}} \ : (a,b) \to \mathbb{R}^3$ is a unit-speed curve whose curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ satisfy

$$\tilde{\kappa}(t) = \kappa(t)\,, \quad \tilde{\tau}(t) = \tau(t)\,, \quad \forall\, t \in (a,b)\,.$$

There exists a rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\tilde{\boldsymbol{\gamma}}(t) = M(\boldsymbol{\gamma}(t))\,, \qquad \forall\, t \in (a,b)\,.$$

Example 1.44: Application of FTSC

Question. Consider the curve

$$\boldsymbol{\gamma}(t) = \left(\sqrt{3}\,t - \sin(t), \sqrt{3}\sin(t) + t, 2\cos(t)\right)\,.$$

1. Calculate the curvature and torsion of $\boldsymbol{\gamma}$.
2. The helix of radius R and rise H is parametrized by

$$\boldsymbol{\eta}(t) = (R\cos(t), R\sin(t), Ht)\,.$$

Recall that $\boldsymbol{\eta}$ has curvature and torsion

$$\kappa^{\boldsymbol{\eta}} = \frac{R}{R^2+H^2}\,, \qquad \tau^{\boldsymbol{\eta}} = \frac{H}{R^2+H^2}\,.$$

Prove that there exist a rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\boldsymbol{\gamma}(t) = M(\boldsymbol{\eta}(t))\,, \qquad \forall\, t \in \mathbb{R}\,. \tag{1.3}$$

Solution.

1. Compute curvature and torsion with the formulas

$$\begin{aligned}
\dot{\boldsymbol{\gamma}}(t) &= \left(\sqrt{3} - \cos(t), \sqrt{3}\cos(t) + 1, -2\sin(t)\right)\\
\ddot{\boldsymbol{\gamma}}(t) &= \left(\sin(t), -\sqrt{3}\sin(t), -2\cos(t)\right)\\
\dddot{\boldsymbol{\gamma}}(t) &= \left(\cos(t), -\sqrt{3}\cos(t), 2\sin(t)\right)\\
\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t) &= \left(-2\left(\sqrt{3}+\cos(t)\right), 2\left(\sqrt{3}\cos(t) - 1\right), -4\sin(t)\right)\\
\|\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t)\|^2 &= 32\\
\|\dot{\boldsymbol{\gamma}}(t)\|^2 &= 8\\
(\dot{\boldsymbol{\gamma}}(t)\times\ddot{\boldsymbol{\gamma}}(t))\cdot\dddot{\boldsymbol{\gamma}}(t) &= -8\\
\kappa(t) &= \frac{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|}{\|\dot{\boldsymbol{\gamma}}\|^3} = \frac{\sqrt{32}}{8^{\frac{3}{2}}} = \frac{1}{4}\\
\tau(t) &= \frac{(\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}})\cdot\dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|^2} = \frac{-8}{32} = -\frac{1}{4}\,.
\end{aligned}$$

2. Equating $\kappa = \kappa^{\boldsymbol{\eta}}$ and $\tau = \tau^{\boldsymbol{\eta}}$, we obtain

$$\frac{R}{R^2+H^2} = \frac{1}{4}\,, \qquad \frac{H}{R^2+H^2} = -\frac{1}{4}$$

Rearranging both equalities we get

$$R^2 + H^2 = 4R\,, \qquad R^2 + H^2 = -4H\,,$$

from which we find the relation $R = -H$. Substituting into $R^2 + H^2 = -4H$, we get

$$H = -2\,, \quad R = -H = 2\,.$$

For these values of R and H we have $\kappa = \kappa^{\boldsymbol{\eta}}$ and $\tau = \tau^{\boldsymbol{\eta}}$. By the FTSC, there exists a rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying (1.3).

Theorem 1.45: Curves contained in a plane

For $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ regular with $\kappa \neq 0$,they are equivalent

1. The torsion of $\boldsymbol{\gamma}$ satisfies
$$\tau(t) = 0\,, \quad \forall\, t \in (a, b)\,.$$
2. The image of $\boldsymbol{\gamma}$ is contained in a plane: There exists a vector $\mathbf{P} \in \mathbb{R}^3$ and a scalar $d \in \mathbb{R}$ such that
$$\boldsymbol{\gamma}(t) \cdot \mathbf{P} = d\,, \quad \forall\, t \in (a, b)\,.$$

Theorem 1.46: Curves contained in a plane

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be regular, with $\kappa \neq 0$ and $\tau = 0$. Then, the binormal $\mathbf{b}$ is a constant vector, and $\boldsymbol{\gamma}$ is contained in the plane of equation
$$(\mathbf{x} - \boldsymbol{\gamma}(t_0)) \cdot \mathbf{b} = 0\,.$$

Example 1.47

Question. Consider the curve
$$\boldsymbol{\gamma}(t) = (t, 2t, t^4)\,, \quad t > 0\,.$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the curvature and torsion of $\boldsymbol{\gamma}$.
3. Prove that $\boldsymbol{\gamma}$ is contained in a plane. Compute the equation of such plane.

Solution.

1. $\boldsymbol{\gamma}$ is regualar because $\dot{\boldsymbol{\gamma}}(t) = (1, 2, 4t^3) \neq \mathbf{0}$.
2. Compute curvature and torsion with the formulas
$$\begin{aligned}
\|\dot{\boldsymbol{\gamma}}(t)\| &= \sqrt{5 + 16t^4} \\
\ddot{\boldsymbol{\gamma}}(t) &= 12\,(0, 0, t^2) \\
\dddot{\boldsymbol{\gamma}}(t) &= 24\,(0, 0, t) \\
\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t) &= 12\,(2t^2, -t^2, 0) \\
\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\| &= 12\sqrt{5}\,t^2 \\
(\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)) \cdot \dddot{\boldsymbol{\gamma}}(t) &= 0 \\
\kappa(t) &= \frac{\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\|^3} = \frac{12\sqrt{5}\,t^2}{\sqrt{5 + 16t^4}} \\
\tau(t) &= \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|} = 0\,.
\end{aligned}$$
3. $\boldsymbol{\gamma}$ lies in a plane because $\tau = 0$. The binormal is
$$\mathbf{b} = \frac{\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|} = \frac{1}{\sqrt{5}}\,(2, -1, 0)\,.$$
At $t_0 = 0$ we have $\boldsymbol{\gamma}(0) = \mathbf{0}$. The equation of the plane containing $\boldsymbol{\gamma}$ is then
$$\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y = 0 \quad \Longrightarrow \quad 2x - y = 0\,.$$

Theorem 1.48: Curves contained in a circle

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a unit-speed curve. They are equivalent:

1. $\boldsymbol{\gamma}$ is contained in a circle of radius $R > 0$.
2. There exists $R > 0$ such that
$$\kappa(t) = \frac{1}{R}\,, \quad \tau(t) = 0\,, \quad \forall\, t \in (a, b)\,.$$

Example 1.49

Question. Consider the curve
$$\boldsymbol{\gamma}(t) = \left(\frac{4}{5}\cos(t), 1 - \sin(t), -\frac{3}{5}\cos(t)\right)\,.$$

1. Prove that $\boldsymbol{\gamma}$ is unit-speed.
2. Compute Frenet frame, curvature and torsion of $\boldsymbol{\gamma}$.
3. Prove that $\boldsymbol{\gamma}$ is part of a circle.

Solution.

1. $\boldsymbol{\gamma}$ is unit-speed because
$$\begin{aligned}
\dot{\boldsymbol{\gamma}}(t) &= \left(-\frac{4}{5}\sin(t), -\cos(t), \frac{3}{5}\sin(t)\right) \\
\|\dot{\boldsymbol{\gamma}}(t)\|^2 &= \frac{16}{25}\sin^2(t) + \cos^2(t) + \frac{9}{25}\sin^2(t) = 1
\end{aligned}$$
2. As $\boldsymbol{\gamma}$ is unit-speed, the tangent vector is $\mathbf{t}(t) = \dot{\boldsymbol{\gamma}}(t)$. The curvature, normal, binormal and torsion are
$$\begin{aligned}
\dot{\mathbf{t}}(t) &= \left(-\frac{4}{5}\cos(t), \sin(t), \frac{3}{5}\cos(t)\right) \\
\kappa(t) &= \|\dot{\mathbf{t}}(t)\| = \frac{16}{25}\cos^2(t) + \sin^2(t) + \frac{9}{25}\cos^2(t) = 1 \\
\mathbf{n}(t) &= \frac{1}{\kappa(t)}\ddot{\boldsymbol{\gamma}}(t) = \left(-\frac{4}{5}\cos(t), \sin(t), \frac{3}{5}\cos(t)\right) \\
\mathbf{b}(t) &= \dot{\boldsymbol{\gamma}}(t) \times \mathbf{n}(t) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right) \\
\dot{\mathbf{b}} &= \mathbf{0} \\
\tau &= -\dot{\mathbf{b}} \cdot \mathbf{n} = 0
\end{aligned}$$
3. The curvature of $\boldsymbol{\gamma}$ is constant and the torsion is zero. Therefore $\boldsymbol{\gamma}$ is contained in a circle of radius
$$R = \frac{1}{\kappa} = 1\,.$$

2 Topology

3 Surfaces

3.1 Preliminaries

Definition 3.1: Topology of $\mathbb{R}^n$

The Euclidean norm on $\mathbb{R}^n$ is denoted by

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^{n} x_i^2}\,, \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n\,.$$

The Euclidean norm induces the distance

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\,.$$

In particular, we have:

1. The pair $(\mathbb{R}^n, d)$ is a metric space.
2. The topology induced by the metric d is called the Euclidean topology, denoted by $\mathcal{T}$.
3. A set $U \subseteq \mathbb{R}^n$ is **open** if for all $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subseteq U$, where

$$B_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n \,:\, \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}$$

is the open ball of radius $\varepsilon > 0$ and centered at $\mathbf{x}$. In this case we write $U \in \mathcal{T}$, with $\mathcal{T}$ the Euclidean topology in $\mathbb{R}^n$.

4. A set $V \subseteq \mathbb{R}^n$ is **closed** if $V^c := \mathbb{R}^n \setminus U$ is open.

Definition 3.2: Subspace Topology

Given a subset $A \subseteq \mathbb{R}^n$ the **subspace topology** on A is the family of sets

$$\mathcal{T}_A := \{U \subseteq A \,:\, \exists\, W \in \mathcal{T} \text{ s.t. } U = A \cap W\}\,.$$

If $U \in \mathcal{T}_A$, we say that U is open in A.

Definition 3.3: Continuous Function

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with U open. We say that f is **continuous** at $\mathbf{x} \in U$ if $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that

$$\|\mathbf{x} - \mathbf{y}\| < \delta \quad \Longrightarrow \quad \|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon\,.$$

f is continuous in U if it is continuous for all $\mathbf{x} \in U$.

Theorem 3.4: Continuity: Topological definition

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$, with U, V open. We have that f is continuous if and only if $f^{-1}(A)$ is open in U, for all A open in V.

Definition 3.5: Homeomorphism

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ with U, V open. We say that f is a **homeomorphism** if:

1. f is continuous;
2. There exists continuous inverse $f^{-1} : V \to U$.

Definition 3.6: Differentiable Function

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with U open. We say that f is **differentiable** at $\mathbf{x} \in U$ if there exists a linear map $d_{\mathbf{x}} f : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$d_{\mathbf{x}} f(\mathbf{h}) = \lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{h}) - f(\mathbf{x})}{\varepsilon}\,,$$

for all $\mathbf{h} \in \mathbb{R}^n$, where the limit is taken in $\mathbb{R}^m$. The linear map $d_{\mathbf{x}} f$ is called the **differential** of f at $\mathbf{x}$.

Definition 3.7: Partial Derivative

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, U open, f differentiable. The **partial derivative** of f at $\mathbf{x} \in U$ in direction $\mathbf{e}_i$ is given by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) := d_{\mathbf{x}} f(\mathbf{e}_i) = \lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{e}_i) - f(\mathbf{x})}{\varepsilon}\,.$$

Definition 3.8: Jacobian Matrix

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. The **Jacobian** of f at $\mathbf{x}$ is the $m \times n$ matrix of partial derivatives:

$$Jf(\mathbf{x}) := \left(\frac{\partial f_i}{\partial x_j}(\mathbf{x}) \right)_{i,j} \in \mathbb{R}^{m \times n}\,.$$

If $m = n$ then $Jf \in \mathbb{R}^{n \times n}$ is a square matrix and we can compute its determinant, denoted by

$$\det(Jf)\,.$$

Proposition 3.9: Matrix representation of $d_{\mathbf{x}} f$

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. The matrix of the linear map $d_{\mathbf{x}} f : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the standard basis is given by the Jacobian matrix $Jf(\mathbf{x})$.

Definition 3.10: Smooth Function

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with U open. We say that f is smooth if the derivatives

$$\frac{\partial^{|\alpha|} f}{d\mathbf{x}^{\alpha}} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$$

exist for each multi-index $\alpha \in \mathbb{N}^n$. Note that in this case all the derivatives of f are automatically continuous.

Notation 3.11: Gradient and partial derivatives

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be smooth. We denote the partial derivatives by

$$\partial_{x_i} f = f_{x_i} = \frac{\partial f}{\partial x_i}$$

$$\partial_{x_i x_j} f = f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$\partial_{x_i x_j x_k} f = f_{x_i x_j x_k} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}$$

For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ smooth we denote the **gradient** by

$$\nabla f(\mathbf{x}) = \left(f_{x_1}(\mathbf{x}), \dots, f_{x_n}(\mathbf{x}) \right) .$$

Note that $\nabla f(\mathbf{x})$ coincides with $Jf(\mathbf{x})$.

Definition 3.12: Diffeomorphism

Let $f : U \to V$, with $U, V \subseteq \mathbb{R}^n$ open. We say that f is a **diffeomorphism** between U and V if:

1. f is smooth,
2. There exists the inverse $f^{-1} : V \to U$,
3. f^{-1} is smooth.

Definition 3.13: Local diffeomorphism

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a **local diffeomorphism** at $\mathbf{x}_0 \in \mathbb{R}^n$ if:

1. There exists an open set $U \subseteq \mathbb{R}^n$ such that $\mathbf{x}_0 \in U$,
2. There exists an open set $V \subseteq \mathbb{R}^n$ such that $f(\mathbf{x}_0) \in V$,
3. $f : U \to V$ is a diffeomorphism.

Theorem 3.14

If $f : U \to V$ is a diffeomorphism, then f is a local diffeomorphism at each $\mathbf{x}_0 \in U$.

Theorem 3.15

Let $f : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^n$ open. Suppose f is a local diffeomorhism at $\mathbf{x}_0 \in U$. Then

$$\det Jf(\mathbf{x}_0) \neq 0 .$$

Theorem 3.16: Inverse Function Theorem

Let $f : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^n$ open, f smooth. Assume

$$\det Jf(\mathbf{x}_0) \neq 0 ,$$

for some $\mathbf{x}_0 \in U$. Then:

1. There exists an open set $U_0 \subseteq U$ such that $\mathbf{x}_0 \in U_0$,
2. There exists an open set V such that $f(\mathbf{x}_0) \in V$,
3. $f : U_0 \to V$ is a diffeomorphism.

Example 3.17

Question. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x, y) = (e^x \cos(y), e^x \sin(y)) .$$

1. Prove that f is a local diffeomorphism at each point.
2. Prove that f is not a diffeomorphism.

Solution.

1. We compute

$$Jf(x, y) = e^x \begin{pmatrix} \cos(y) & \sin(y) \\ -\sin(y) & \cos(y) \end{pmatrix}$$

$$\det Jf(x, y) = e^{2x} \neq 0$$

By the Inverse Function Theorem, f is a local diffeomorphism at each point $(x, y) \in \mathbb{R}^2$.

2. f is not injective, since

$$f(x, y) = f(x, y + 2n\pi) , \quad \forall (x, y) \in \mathbb{R}^2, \, n \in \mathbb{N} .$$

Therefore f is not invertible from $\mathbb{R}^2$ into $\mathbb{R}^2$. This implies f cannot be a diffeomorphism of $\mathbb{R}^2$ into $\mathbb{R}^2$.

3.2 Regular surfaces

Definition 3.18: Surface

Let $\mathscr{S} \subseteq \mathbb{R}^3$ be a connected set. We say that $\mathscr{S}$ is a **surface** if for every point $\mathbf{p} \in \mathscr{S}$ there exist

1. An open set $U \subseteq \mathbb{R}^2$,
2. A smooth map $\boldsymbol{\sigma} : U \to \boldsymbol{\sigma}(U) \subseteq \mathscr{S}$ such that
   - $\mathbf{p} \in \boldsymbol{\sigma}(U)$,
   - $\boldsymbol{\sigma}(U)$ is open in $\mathscr{S}$
   - $\boldsymbol{\sigma}$ is a homeomorphism between U and $\boldsymbol{\sigma}(U)$

The homeomorphism $\boldsymbol{\sigma}$ is called a **surface chart** at $\mathbf{p}$.

Definition 3.19: Atlas of a surface

Let $\mathscr{S}$ be a surface. Assume given a collection of charts

$$\mathscr{A} = \{\boldsymbol{\sigma}_i\}_{i\in I}\,, \qquad \boldsymbol{\sigma}_i : U_i \to \boldsymbol{\sigma}(U_i) \subseteq \mathscr{S}\,.$$

The family $\mathscr{A}$ is an **atlas** of $\mathscr{S}$ if

$$\mathscr{S} = \bigcup_{i\in I} \boldsymbol{\sigma}_i(U_i)\,.$$

Definition 3.20: Regular Chart

Let $U \subseteq \mathbb{R}^2$ be open. A map

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(u,v) : U \to \mathbb{R}^3$$

is called a **regular chart** if the partial derivatives

$$\boldsymbol{\sigma}_u(u,v) = \frac{d\boldsymbol{\sigma}}{du}(u,v)\,, \quad \boldsymbol{\sigma}_v(u,v) = \frac{d\boldsymbol{\sigma}}{dv}(u,v)$$

are linearly independent vectors of $\mathbb{R}^3$ for all $(u,v) \in U$.

Definition 3.21: Regular surface

Let $\mathscr{S}$ be a surface. We say that:

- $\mathscr{A}$ is a **regular atlas** if any $\boldsymbol{\sigma}$ in $\mathscr{A}$ is regular.
- $\mathscr{S}$ is a **regular surface** if it admits a regular atlas.

Theorem 3.22

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ with $U \subseteq \mathbb{R}^2$ open. They are equivalent

1. $\boldsymbol{\sigma}$ is a regular chart.
2. $d_{\mathbf{x}}\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $\mathbf{x} \in U$.
3. The Jacobian matrix $J\boldsymbol{\sigma}$ has rank 2 for all $(u,v) \in U$.
4. $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq 0$ for all $(u,v) \in U$.

Example 3.23: 2D Plane in $\mathbb{R}^3$

Question. Let $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, with $\mathbf{p}$ and $\mathbf{q}$ orthonormal. The plane

$$\boldsymbol{\pi} = \{\mathbf{a} + u\mathbf{p} + v\mathbf{q} \;:\; u,v \in \mathbb{R}\}$$

is a surface with atlas $\mathscr{A} = \{\boldsymbol{\sigma}\}$, where

$$\boldsymbol{\sigma} : \mathbb{R}^2 \to \boldsymbol{\pi}\,, \quad \boldsymbol{\sigma}(u,v) := \mathbf{a} + u\mathbf{p} + v\mathbf{q}\,.$$

Prove that $\boldsymbol{\pi}$ is a regular surface.
Solution. We have $\boldsymbol{\sigma}_u = \mathbf{p}, \boldsymbol{\sigma}_v = \mathbf{q}$. Since $\mathbf{p}$ and $\mathbf{q}$ are orthonormal, we conclude that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent and $\boldsymbol{\sigma}$ is regular. $\boldsymbol{\pi}$ is a regular surface because $\boldsymbol{\sigma}$ is a regular chart.

Example 3.24: Unit cylinder

Question. Consider the infinite unit cylinder

$$\mathscr{S} = \{(x,y,z) \in \mathbb{R}^3 \;:\; x^2 + y^2 = 1\}\,.$$

$\mathscr{S}$ is a surface with atlas $\mathscr{A} = \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$, where

$$\begin{aligned}
\boldsymbol{\sigma} &: \mathbb{R}^2 \to \mathbb{R}^3\,, & \boldsymbol{\sigma}(u,v) &= (\cos(u), \sin(u), v)\,,\\
\boldsymbol{\sigma}_1 &:= \boldsymbol{\sigma}|_{U_1}\,, & \boldsymbol{\sigma}_2 &:= \boldsymbol{\sigma}|_{U_2}\,,\\
U_1 &:= \left(0, \frac{3\pi}{2}\right) \times \mathbb{R}\,, & U_2 &:= \left(\pi, \frac{5\pi}{2}\right) \times \mathbb{R}\,.
\end{aligned}$$

Prove that $\mathscr{S}$ is a regular surface.
Solution. The map $\boldsymbol{\sigma}$ is regular because

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (-\sin(u), \cos(u), 0)\\
\boldsymbol{\sigma}_v &= (0,0,1)\\
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (\cos(u), \sin(u), 0)\\
\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| &= 1 \neq 0\,.
\end{aligned}$$

Therefore $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are regular charts, being restrictions of $\boldsymbol{\sigma}$. Thus, $\mathscr{A}$ is a regular atlas, making $\mathscr{S}$ a regular surface.

Example 3.25: Graph of a function

Question. Let $U \subseteq \mathbb{R}^2$ be open and $f : U \to \mathbb{R}$ be smooth. The graph of f is the set

$$\Gamma_f := \{(u,v,f(u,v)) \;:\; (u,v) \in U\}\,.$$

Γ_f is surface with atlas given by $\mathscr{A} = \{\boldsymbol{\sigma}\}$, where

$$\boldsymbol{\sigma} : U \to \Gamma_f\,, \quad \boldsymbol{\sigma}(u,v) := (u,v,f(u,v))\,.$$

Prove that Γ_f is a regular surface.

Solution. The Jacobian matrix of $\boldsymbol{\sigma}$ is

$$J\boldsymbol{\sigma}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}.$$

The first minor of $J\boldsymbol{\sigma}$ is the identity matrix, which has determinant 1, and is hence invertible. Therefore $J\boldsymbol{\sigma}$ has rank 2, showing that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent. Hence $\boldsymbol{\sigma}$ is regular. This implies $\mathscr{A}$ is a regular atlas, and $\mathscr{S}$ is a regular surface.

Definition 3.26: Spherical coordinates

A point $\mathbf{p} = (x,y,z) \neq \mathbf{0}$ is represented in **spherical coordinates** by

$$\begin{aligned} x &= \rho\cos(\theta)\cos(\varphi) \\ y &= \rho\sin(\theta)\cos(\varphi) \\ z &= \rho\sin(\varphi) \end{aligned}$$

where

$$\rho := \sqrt{x^2+y^2+z^2}, \quad \theta \in [-\pi,\pi], \quad \varphi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

Example 3.27: Unit sphere in spherical coordinates

Question. Consider the unit sphere in $\mathbb{R}^3$

$$\mathbb{S}^2 = \{(x,y,z)\in\mathbb{R}^3 \ : \ x^2+y^2+z^2 = 1\}.$$

Define the set

$$U = \left\{(\theta,\varphi)\in\mathbb{R}^2 \ : \ \theta\in(-\pi,\pi),\ \varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right\},$$

and the chart $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ by

$$\boldsymbol{\sigma}(\theta,\varphi) := (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), \sin(\varphi)).$$

Prove that $\boldsymbol{\sigma}$ is regular.
Solution. We compute

$$\begin{aligned} \boldsymbol{\sigma}_\theta &= (-\sin(\theta)\cos(\varphi), \cos(\theta)\cos(\varphi), 0) \\ \boldsymbol{\sigma}_\varphi &= (-\cos(\theta)\sin(\varphi), -\sin(\theta)\sin(\varphi), \cos(\varphi)). \end{aligned}$$

Since $(\theta,\varphi)\in U$, we have $\varphi\in(-\pi/2,\pi/2)$. Therefore, the last component $\cos(\varphi)$ of $\boldsymbol{\sigma}_\varphi$ is non-zero. As the last component of $\boldsymbol{\sigma}_\theta$ is 0, we conclude that $\boldsymbol{\sigma}_\theta$ and $\boldsymbol{\sigma}_\varphi$ are linearly independent for all $(\theta,\varphi)\in U$. Therefore $\boldsymbol{\sigma}$ is regular. Alternatively, compute

$$\begin{aligned} \boldsymbol{\sigma}_\theta\times\boldsymbol{\sigma}_\varphi &= (\cos(\theta)\cos^2(\varphi), \sin(\theta)\cos^2(\varphi), \cos(\varphi)\sin(\varphi)) \\ \left\|\boldsymbol{\sigma}_\theta\times\boldsymbol{\sigma}_\varphi\right\| &= |\cos(\varphi)| = \cos(\varphi)\neq 0, \end{aligned}$$

since $\varphi\in(-\pi/2,\pi/2)$, showing that $\boldsymbol{\sigma}$ is regular.

Example 3.28

Question. Prove that the following chart is not regular

$$\boldsymbol{\sigma}(u,v) = (u,v^2,v^3).$$

Solution. We have

$$\begin{aligned} \boldsymbol{\sigma}_u &= (1,0,0) \\ \boldsymbol{\sigma}_v &= (0,2v,3v^2) \\ \boldsymbol{\sigma}_v(u,0) &= (0,0,0), \end{aligned}$$

showing that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly dependent along the line

$$L = \{(u,0) \ : \ u\in\mathbb{R}\}.$$

Hence $\boldsymbol{\sigma}$ is not a regular chart.

3.3 Reparametrizations

Definition 3.29: Reparametrization

Suppose that $U,\widetilde{U}\subseteq\mathbb{R}^2$ are open sets and

$$\boldsymbol{\sigma} : U\to\mathbb{R}^3, \quad \tilde{\boldsymbol{\sigma}} : \widetilde{U}\to\mathbb{R}^3,$$

are surface charts. We say that $\tilde{\boldsymbol{\sigma}}$ is a **reparametrization** of $\boldsymbol{\sigma}$ if there exists a diffeomorphism $\Phi : \widetilde{U}\to U$ such that

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\circ\Phi.$$

We call Φ a **reparametrization map**.

Theorem 3.30

Let $U,\widetilde{U}\subseteq\mathbb{R}^2$ be open and $\boldsymbol{\sigma} : U\to\mathbb{R}^3$ be regular. Suppose given a diffeomorphism $\Phi : \widetilde{U}\to U$. The reparametrization

$$\tilde{\boldsymbol{\sigma}} : \widetilde{U}\to\mathbb{R}^3, \qquad \tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\circ\Phi$$

is a regular chart, and it holds

$$\tilde{\boldsymbol{\sigma}}_{\tilde{u}}\times\tilde{\boldsymbol{\sigma}}_{\tilde{v}} = \det J\Phi\,(\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v).$$

Definition 3.31: Transition map

Let $\mathscr{S}$ be a regular surface and

$$\boldsymbol{\sigma} : U\to\mathscr{S}, \quad \tilde{\boldsymbol{\sigma}} : \widetilde{U}\to\mathscr{S}$$

be regular charts. Suppose the images of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ overlap, that is,

$$I := \boldsymbol{\sigma}(U)\cap\tilde{\boldsymbol{\sigma}}(\widetilde{U})\neq\emptyset.$$

The set I is open in $\mathcal{S}$, being intersection of open sets. Define the sets

$$V := \boldsymbol{\sigma}^{-1}(I) \subseteq U, \quad \widetilde{V} := \tilde{\boldsymbol{\sigma}}^{-1}(I) \subseteq \widetilde{U}.$$

Note that V and $\widetilde{V}$ are open, by continuity of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$. Since $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are homeomorphisms, by construction

$$\boldsymbol{\sigma}(V) = \tilde{\boldsymbol{\sigma}}(\widetilde{V}) = I.$$

Therefore, they are well defined the restrictions

$$\boldsymbol{\sigma}|_V : V \to I, \quad \tilde{\boldsymbol{\sigma}}|_{\widetilde{V}} : \widetilde{V} \to I.$$

The maps $\boldsymbol{\sigma}|_V$ and $\tilde{\boldsymbol{\sigma}}|_{\widetilde{V}}$ are homeomorphisms, being restrictions of homeomorphisms. The composition homeomorphism

$$\Phi : \widetilde{V} \to V, \quad \Phi := \boldsymbol{\sigma}^{-1} \circ \tilde{\boldsymbol{\sigma}}$$

is called a **transition map** from $\boldsymbol{\sigma}$ to $\tilde{\boldsymbol{\sigma}}$.

Theorem 3.32

Let $\mathcal{S}$ be a regular surface. The transition maps between regular charts are diffeomorphisms.

Theorem 3.33

Let $\mathcal{S}$ be a regular surface and

$$\boldsymbol{\sigma} : U \to \mathcal{S}, \quad \tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathcal{S}$$

be regular charts. Suppose the images of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ overlap, that is,

$$I := \boldsymbol{\sigma}(U) \cap \tilde{\boldsymbol{\sigma}}(\widetilde{U}) \neq \emptyset.$$

Define the open sets

$$V := \boldsymbol{\sigma}^{-1}(I) \subseteq U, \quad \widetilde{V} := \tilde{\boldsymbol{\sigma}}^{-1}(I) \subseteq \widetilde{U},$$

and the transition map

$$\Phi : \widetilde{V} \to V, \quad \Phi := \boldsymbol{\sigma}^{-1} \circ \tilde{\boldsymbol{\sigma}}.$$

Then $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are reparametrization of each other, with reparametrization map given by Φ, that is,

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi, \qquad \boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} \circ \Phi^{-1}.$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be regular surfaces and $f : \mathcal{S}_1 \to \mathcal{S}_2$ a map.

1. f is *smooth at* $\mathbf{p} \in \mathcal{S}_1$, if $\exists$ charts $\boldsymbol{\sigma}_i : U_i \to \mathcal{S}_i$ for $i = 1, 2$ such that
$$\mathbf{p} \in \boldsymbol{\sigma}_1(U_1), \quad f(\mathbf{p}) \in \boldsymbol{\sigma}_2(U_2)$$
and
$$(\boldsymbol{\sigma}_2^{-1} \circ f \circ \boldsymbol{\sigma}_1) : U_1 \to U_2$$
is smooth.
2. f is *smooth*, if it is smooth for each $\mathbf{p} \in \mathcal{S}_1$.

Theorem 3.35

If $f : \mathcal{S}_1 \to \mathcal{S}_2$ and $g : \mathcal{S}_2 \to \mathcal{S}_3$ are smooth maps between surfaces, then the composition

$$(g \circ f) : \mathcal{S}_1 \to \mathcal{S}_3$$

is smooth.

Theorem 3.36

Let $\mathcal{S}$ be a regular surface, and $\boldsymbol{\sigma} : U \to \mathcal{S}$ a regular chart. The inverse

$$\boldsymbol{\sigma}^{-1} : \boldsymbol{\sigma}(U) \to U \subseteq \mathbb{R}^2,$$

is a differentiable function.

Theorem 3.37

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be regular surfaces. Make the following assumptions:

1. $V \subseteq \mathbb{R}^3$ is an open set such that
$$\mathcal{S}_1 \subseteq V,$$
2. $f : V \to \mathbb{R}^3$ is a differentiable function such that
$$f(\mathcal{S}_1) \subseteq \mathcal{S}_2.$$

Then the restriction

$$f|_{\mathcal{S}_1} : \mathcal{S}_1 \to \mathcal{S}_2$$

is a differentiable map between surfaces.

Definition 3.38: Diffeomorphism of surfaces

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be regular surfaces.

1. A map $f \colon \mathcal{S}_1 \to \mathcal{S}_2$ is a **diffeomorphism**, if f is smooth and admits smooth inverse.
2. We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are **diffeomorphic**, if there exists $f \colon \mathcal{S}_1 \to \mathcal{S}_2$ diffeomorphism.

Theorem 3.39

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces, $f \colon \mathcal{S} \to \widetilde{\mathcal{S}}$ diffeomorphism. If $\boldsymbol{\sigma} : U \to \mathcal{S}$ is a regular chart for $\mathcal{S}$ at $\mathbf{p}$, then

$$\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}}\,, \qquad \tilde{\boldsymbol{\sigma}} := f \circ \boldsymbol{\sigma}\,,$$

is a regular chart for $\widetilde{\mathcal{S}}$ at $f(\mathbf{p})$.

Definition 3.40: Local diffeomorphism

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be regular surfaces, and

$$f \colon \mathcal{S}_1 \to \mathcal{S}_2\,,$$

be a differentiable map.

1. f is a **local diffeomorphism** at $\mathbf{p} \in \mathcal{S}_1$ if:
   - There exists An open set $V \subseteq \mathcal{S}_1$ with $\mathbf{p} \in V$;
   - $f(V) \subseteq \mathcal{S}_2$ is open;
   - $f \colon V \to f(V)$ is a diffeomorphism between surfaces.
2. f is a **local diffeomorphism** in $\mathcal{S}_1$, if it is a local diffeomorphism at each $\mathbf{p} \in \mathcal{S}_1$.
3. $\mathcal{S}_1$ is **locally diffeomorphic** to $\mathcal{S}_2$, if for each $\mathbf{p} \in \mathcal{S}_1$ there exists f local diffeomorphism at $\mathbf{p}$.

3.5 Tangent plane

Definition 3.41: Tangent vectors and tangent plane

Let $\mathcal{S}$ be a surface and $\mathbf{p} \in \mathcal{S}$.

1. We say that $\mathbf{v} \in \mathbb{R}^3$ is a **tangent vector** to $\mathcal{S}$ at $\mathbf{p}$, if there exists a smooth curve

   $$\boldsymbol{\gamma} \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3\,,$$

   such that

   $$\boldsymbol{\gamma}(-\varepsilon, \varepsilon) \subseteq \mathcal{S}\,, \quad \boldsymbol{\gamma}(0) = \mathbf{p}\,, \quad \mathbf{v} = \dot{\boldsymbol{\gamma}}(0)\,,$$

   where $\varepsilon > 0$.

2. The **tangent plane** of $\mathcal{S}$ at $\mathbf{p}$ is the set

   $$T_{\mathbf{p}}\mathcal{S} := \{\mathbf{v} \in \mathbb{R}^3 \,:\, \mathbf{v} \text{ tangent vector of } \mathcal{S} \text{ at } \mathbf{p}\}\,.$$

Theorem 3.42: Curves with values on surfaces

Let $\mathcal{S}$ be a regular surface, $\mathbf{p} \in \mathcal{S}$, and $\boldsymbol{\sigma} : U \to \mathcal{S}$ be a chart at $\mathbf{p}$. Denote

$$(u_0, v_0) = \boldsymbol{\sigma}^{-1}(\mathbf{p})\,.$$

Suppose $\boldsymbol{\gamma} \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ is a smooth curve such that

$$\boldsymbol{\gamma}(t) \in \mathcal{S}\,, \quad \forall\, t \in (-\varepsilon, \varepsilon)\,.$$

Then, there exist smooth functions

$$u, v \colon (\varepsilon_0, \varepsilon_0) \to \mathbb{R}\,, \quad u(0) = u_0\,, \quad v(0) = v_0\,,$$

such that

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))\,, \quad \forall\, t \in (-\varepsilon, \varepsilon)\,.$$

Theorem 3.43: Characterization of Tangent Plane

Let $\mathcal{S}$ be a regular surface and $\mathbf{p} \in \mathcal{S}$. Let $\boldsymbol{\sigma} : U \to \mathcal{S}$ be a chart at $\mathbf{p}$. Denote

$$(u_0, v_0) = \boldsymbol{\sigma}^{-1}(\mathbf{p})\,.$$

The tangent plane satisfies

$$T_{\mathbf{p}}\mathcal{S} = \operatorname{span}\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} := \{\lambda\boldsymbol{\sigma}_u + \mu\boldsymbol{\sigma}_v \,:\, \lambda, \mu \in \mathbb{R}\}\,,$$

where $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are evaluated at (u_0, v_0).

Theorem 3.44: Equation of tangent plane

Let $\mathcal{S}$ be a regular surface and $\mathbf{p} \in \mathcal{S}$. Let $\boldsymbol{\sigma} : U \to \mathcal{S}$ be a regular chart at $\mathbf{p}$. Set

$$(u_0, v_0) := \boldsymbol{\sigma}^{-1}(\mathbf{p})\,, \quad \mathbf{n} := \boldsymbol{\sigma}_u(u_0, v_0) \times \boldsymbol{\sigma}_v(u_0, v_0)\,.$$

They hold:

1. The equation of the tangent plane $T_{\mathbf{p}}\mathcal{S}$ is given by

   $$\mathbf{x} \cdot \mathbf{n} = 0\,, \quad \forall\, \mathbf{x} \in \mathbb{R}^3\,.$$

2. The equation of the affine tangent plane $\mathbf{p} + T_{\mathbf{p}}\mathcal{S}$ is given by

   $$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0\,, \quad \forall\, \mathbf{x} \in \mathbb{R}^3\,.$$

Example 3.45

Question. Consider the surface $\mathcal{S}$ defined by the chart

$$\boldsymbol{\sigma}(u,v) := \left(\sqrt{1-v}\cos(u), \sqrt{1-v}\sin(u), v\right),$$

for $u \in (0,2\pi)$, $v < 1$.

1. Prove that $\boldsymbol{\sigma}$ parametrizes a paraboloid.
2. Prove that $\boldsymbol{\sigma}$ is regular.
3. Compute the vector $\mathbf{n} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$
4. Consider the point

$$\mathbf{p} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right).$$

   Give a basis for $T_{\mathbf{p}}\mathcal{S}$ at the point

5. Compute the cartesian equation of $T_{\mathbf{p}}\mathcal{S}$. Give your answer in the form

$$ax + by + cz = d\,,$$

   for suitable $a,b,c,d \in \mathbb{R}$.

Solution.

1. Denote the coordinates of $\boldsymbol{\sigma}$ by

$$\boldsymbol{\sigma}(u,v) = (x,y,z)\,.$$

   We have

$$\begin{aligned} x^2 + y^2 &= \left(\sqrt{1-v}\cos(u)\right)^2 + \left(\sqrt{1-v}\cos(u)\right)^2 \\ &= 1 - v \\ &= 1 - z\,, \end{aligned}$$

   showing that $\boldsymbol{\sigma}$ parametrizes the paraboloid

$$z = -x^2 - y^2 + 1\,.$$

2. Proof that $\boldsymbol{\sigma}$ is regular:

$$\boldsymbol{\sigma}_u = \left(-\sqrt{1-v}\sin(u), \sqrt{1-v}\cos(u), 0\right)$$
$$\boldsymbol{\sigma}_v = \left(-\frac{1}{2}(1-v)^{-1/2}\cos(u), -\frac{1}{2}(1-v)^{-1/2}\sin(u), 1\right)$$

   The last component of $\boldsymbol{\sigma}_u$ is 0, and the last component of $\boldsymbol{\sigma}_v$ is 1, thus $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent. Hence $\boldsymbol{\sigma}$ is regular.

3. We compute:

$$\begin{aligned} \mathbf{n} &= \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{1-v}\sin(u) & \sqrt{1-v}\cos(u) & 0 \\ -\frac{1}{2}(1-v)^{-1/2}\cos(u) & -\frac{1}{2}(1-v)^{-1/2}\sin(u) & 1 \end{vmatrix} \\ &= \left(\sqrt{1-v}\cos(u), \sqrt{1-v}\sin(u), \frac{1}{2}\right) \end{aligned}$$

4. Notice that

$$\boldsymbol{\sigma}\left(\frac{\pi}{4}, 0\right) = \mathbf{p}\,.$$

   A basis for $T_{\mathbf{p}}\mathcal{S}$ is given by the vectors

$$\begin{aligned} \boldsymbol{\sigma}_u\left(\frac{\pi}{4}, 0\right) &= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \\ \boldsymbol{\sigma}_v\left(\frac{\pi}{4}, 0\right) &= \left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, 1\right). \end{aligned}$$

5. Using the calculation for $\mathbf{n}$ in Point 3, we find

$$\mathbf{n}\left(\frac{\pi}{4}, 0\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2}\right).$$

   The equation for $T_{\mathbf{p}}\mathcal{S}$ is therefore

$$\mathbf{x} \cdot \mathbf{n} = 0\,, \quad \forall\, \mathbf{x} \in \mathbb{R}^3\,.$$

   The above reads

$$\frac{\sqrt{2}}{2}\,x + \frac{\sqrt{2}}{2}\,y - \frac{1}{2}\,z = 0\,,$$

   which implies

$$\sqrt{2}\,x + \sqrt{2}\,y - z = 0\,,$$

3.6 Unit normal and orientability

Definition 3.46: Standard unit normal of a chart

Let $\mathcal{S}$ be a regular surface and $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ a regular chart. The **standard unit normal** of $\boldsymbol{\sigma}$ is the smooth function

$$\mathbf{N}_{\boldsymbol{\sigma}} : U \to \mathbb{R}^3\,, \quad \mathbf{N}_{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}\,.$$

Example 3.47: Standard unit normal of the plane

Let $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, with $\mathbf{p}$ and $\mathbf{q}$ linearly independent. Consider the plane charted by

$$\boldsymbol{\sigma}(u,v) := \mathbf{a} + \mathbf{p}u + \mathbf{q}v\,, \quad \forall\,(u,v) \in \mathbb{R}^2\,.$$

Compute the standard unit normal to $\boldsymbol{\sigma}$.
Solution. We have

$$\boldsymbol{\sigma}_u = \mathbf{p}\,, \quad \boldsymbol{\sigma}_v = \mathbf{q}\,,$$

and therefore

$$\mathbf{N}_{\boldsymbol{\sigma}} = \frac{\mathbf{p} \times \mathbf{q}}{\|\mathbf{p} \times \mathbf{q}\|}\,.$$

Definition 3.48: Unit normal of a surface

Let $\mathcal{S}$ be a regular surface. A **standard unit normal** to $\mathcal{S}$ is a smooth function $\mathbf{N}: \mathcal{S} \to \mathbb{R}^3$ such that

$$\mathbf{N}(\mathbf{p}) \perp T_{\mathbf{p}}\mathcal{S}\,, \quad \|\mathbf{N}(\mathbf{p})\| = 1\,, \quad \forall\, \mathbf{p} \in \mathcal{S}\,.$$

Definition 3.49: Orientable surface

Let $\mathcal{S}$ be a regular surface. We say that $\mathcal{S}$ is **orientable** if there exists a standard unit normal $\mathbf{N}: \mathcal{S} \to \mathbb{R}^3$ and an atlas $\mathscr{A}$ such that

$$\mathbf{N} \circ \boldsymbol{\sigma} = \mathbf{N}_{\boldsymbol{\sigma}}\,, \qquad \forall\, \boldsymbol{\sigma} \in \mathscr{A}\,.$$

Example 3.50

Question. Let $\mathcal{S}$ be the surface described by the chart

$$\boldsymbol{\sigma}(u,v) = (e^u, u+v, v)\,, \quad u,v \in \mathbb{R}\,.$$

1. Prove that $\boldsymbol{\sigma}$ is regular.
2. Compute the standard unit normal to $\boldsymbol{\sigma}$.

Solution.

1. Compute the following quantities:

$$\begin{aligned} \boldsymbol{\sigma}_u &= (e^u, 1, 0) \\ \boldsymbol{\sigma}_v &= (0,1,1) \\ \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^u & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\ &= (1, -e^u, e^u) \end{aligned}$$

Since

$$\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| = \sqrt{1 + 2e^{2u}} \geq 1\,,$$

we see that $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq \mathbf{0}$, showing that $\boldsymbol{\sigma}$ is regular.

2. The standard unit normal to $\boldsymbol{\sigma}$ is

$$\mathbf{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = \frac{1}{\sqrt{1+2e^{2u}}}\,(1, -e^u, e^u)\,.$$

3.7 Differential of smooth functions

Definition 3.51: Differential of smooth function

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and $f: \mathcal{S} \to \widetilde{\mathcal{S}}$ a smooth map. The differential $d_{\mathbf{p}}f$ of f at $\mathbf{p}$ is defined as the map

$$d_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\widetilde{\mathcal{S}}\,, \quad d_{\mathbf{p}}f(\mathbf{v}) := (f \circ \boldsymbol{\gamma})'(0)\,,$$

where $\boldsymbol{\gamma}: (-\varepsilon, \varepsilon) \to \mathcal{S}$ is any smooth curve such that

$$\boldsymbol{\gamma}(0) = \mathbf{p}\,, \qquad \dot{\boldsymbol{\gamma}}(0) = \mathbf{v}\,.$$

Example 3.52: Computing $d_{\mathbf{p}}f$ using the definition

Question. Consider the portion of the plane $\{z=0\}$,

$$\mathcal{S} = \left\{(x,y,0) \in \mathbb{R}^3 \,:\, x \in (0, 2\pi)\,,\ y \in \mathbb{R}\right\}\,,$$

and the unit cylinder

$$\widetilde{\mathcal{S}} = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2 + y^2 = 1\}\,.$$

Define the map

$$f : S \to \widetilde{\mathcal{S}}\,, \quad f(x,y,0) := (\cos x, \sin x, y)\,.$$

1. Compute $T_{\mathbf{p}}\mathcal{S}$.
2. Prove that the differential $d_{\mathbf{p}}f$ at

$$\mathbf{p} = (u_0, v_0, 0)\,, \quad \mathbf{v} = (\lambda, \mu, 0)\,,$$

is given by

$$d_{\mathbf{p}}f(\mathbf{v}) = (-\lambda\sin(u_0), \lambda\cos(u_0), \mu)\,.$$

Solution.

1. A chart for $\mathcal{S}$ is given by

$$\boldsymbol{\sigma}(u,v) = (u,v,0)\,.$$

Therefore

$$\boldsymbol{\sigma}_u = (1,0,0)\,, \qquad \boldsymbol{\sigma}_v = (0,1,0)\,,$$

from which

$$\begin{aligned} T_{\mathbf{p}}\mathcal{S} &= \operatorname{span}\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} \\ &= \operatorname{span}\{(1,0,0),(0,1,0)\} \\ &= \{(\lambda,\mu,0) \,:\, \lambda,\mu \in \mathbb{R}\}\,. \end{aligned}$$

2. From the answer to Point 1, we see that

$$\mathbf{v} = (\lambda, \mu, 0) \in T_{\mathbf{p}}\mathcal{S}\,.$$

Define the curve $\boldsymbol{\gamma}: (-\varepsilon,\varepsilon) \to \mathcal{S}$ by setting

$$\boldsymbol{\gamma}(t) := \boldsymbol{\sigma}(u_0 + t\lambda, v_0 + t\mu) = (u_0 + t\lambda, v_0 + t\mu, 0)\,.$$

Clearly,

$$\boldsymbol{\gamma}(0) = \mathbf{p}\,, \quad \dot{\boldsymbol{\gamma}}(0) = \mathbf{v} = (\lambda, \mu, 0)\,.$$

We have

$$\begin{aligned}(f \circ \boldsymbol{\gamma})(t) &= f(u_0 + t\lambda, v_0 + t\mu, 0)\\ &= (\cos(u_0 + t\lambda), \sin(u_0 + t\lambda), v_0 + t\mu)\,,\\ (f \circ \boldsymbol{\gamma})'(t) &= (-\lambda \sin(u_0 + t\lambda), \lambda \cos(u_0 + t\lambda), \mu)\,.\end{aligned}$$

Therefore, the differential is given by

$$d_{\mathbf{p}} f(\mathbf{v}) = (f \circ \boldsymbol{\gamma})'(0) = (-\lambda \sin(u_0), \lambda \cos(u_0), \mu)\,.$$

Theorem 3.53

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and $f \colon \mathcal{S} \to \widetilde{\mathcal{S}}$ a smooth map. Denote the differential of f by

$$d_{\mathbf{p}} f \colon T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\widetilde{\mathcal{S}}\,.$$

We have:

1. $d_{\mathbf{p}} f(\mathbf{v})$ depends only on $f, \mathbf{p}, \mathbf{v}$ (and not on $\boldsymbol{\gamma}$).
2. $d_{\mathbf{p}} f$ is linear, that is,

   $$d_{\mathbf{p}} f(\lambda \mathbf{v} + \mu \mathbf{w}) = \lambda d_{\mathbf{p}} f(\mathbf{v}) + \mu d_{\mathbf{p}} f(\mathbf{w})\,,$$

   for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$ and $\lambda, \mu \in \mathbb{R}$.
3. Let

   $$\boldsymbol{\sigma} : U \to \mathcal{S}\,, \quad \tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \widetilde{\mathcal{S}}\,,$$

   be regular charts at $\mathbf{p}$ and $f(\mathbf{p})$, respectively. Denote by

   $$(u, v) \mapsto (\alpha(u, v), \beta(u, v))$$

   the components of the smooth map

   $$\Psi := \tilde{\boldsymbol{\sigma}}^{-1} \circ f \circ \boldsymbol{\sigma} : U \to \widetilde{U}\,.$$

   In particular it holds

   $$\tilde{\boldsymbol{\sigma}}(\alpha(u, v), \beta(u, v)) = f(\boldsymbol{\sigma}(u, v))\,, \quad \forall\, (u, v) \in U\,.$$

   The matrix of the linear map $d_{\mathbf{p}} f$ with respect to the basis

   $$\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} \text{ on } T_{\mathbf{p}}\mathcal{S}\,, \quad \{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\} \text{ on } T_{f(\mathbf{p})}\widetilde{\mathcal{S}}\,,$$

   is given by the Jacobian of the map Ψ, that is,

   $$d_{\mathbf{p}} f = J\Psi = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}.$$

Example 3.54: Computing the matrix of $d_{\mathbf{p}} f$

Question. Consider the unit cylinder, charted by

$$\boldsymbol{\sigma}(u, v) = (\cos u, \sin u, v)\,, \quad (u, v) \in U := (0, 2\pi) \times \mathbb{R}\,.$$

Consider the plane $\widetilde{\mathcal{S}}$ charted by

$$\tilde{\boldsymbol{\sigma}}(u, v) = (u, v, 0)\,, \quad (u, v) \in \widetilde{U} = \mathbb{R}^2\,.$$

Define the map

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}\,, \quad f(x, y, z) = (y, xz, 0)\,.$$

Compute the matrix of $d_{\mathbf{p}} f$ with respect to these charts.
Solution. We need to compute the map

$$\Psi : U \to \widetilde{U}\,, \quad \Psi := \tilde{\boldsymbol{\sigma}}^{-1} \circ f \circ \boldsymbol{\sigma}\,.$$

Clearly, we have

$$\tilde{\boldsymbol{\sigma}}^{-1}(u, v, 0) = (u, v)\,.$$

Therefore

$$\begin{aligned}\Psi(u, v) &= \tilde{\boldsymbol{\sigma}}^{-1}\left(f(\boldsymbol{\sigma}(u, v))\right)\\ &= \tilde{\boldsymbol{\sigma}}^{-1}\left(f(\cos u, \sin u, v)\right)\\ &= \tilde{\boldsymbol{\sigma}}^{-1}\left(\sin(u), \cos(u)v, 0\right)\\ &= (\sin(u), \cos(u)v)\,.\end{aligned}$$

The components of Ψ are

$$\alpha(u, v) = \sin(u)\,, \qquad \beta(u, v) = \cos(u)v\,.$$

Therefore

$$\begin{aligned}\alpha_u &= \cos(u)\,, & \alpha_v &= 0\\ \beta_u &= -\sin(u)v\,, & \beta_v &= \cos(u)\end{aligned}$$

The matrix of $d_{\mathbf{p}} f$ is hence

$$d_{\mathbf{p}} f = J\Psi = \begin{pmatrix} \cos(u) & 0 \\ -\sin(u)v & \cos(u) \end{pmatrix}.$$

Theorem 3.55

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be regular surfaces. Suppose that

$$f : \mathcal{S}_1 \to \mathcal{S}_2$$

is smooth, and let $\mathbf{p} \in \mathcal{S}_1$. They are equivalent:

1. f is a local diffeomorphism at $\mathbf{p}$.
2. The differential $d_{\mathbf{p}} f : T_{\mathbf{p}}\mathcal{S}_1 \to T_{f(\mathbf{p})}\mathcal{S}_2$ is invertible at $\mathbf{p}$.

3.8 Level surfaces

Definition 3.56: Level surface

Let $V \subseteq \mathbb{R}^3$ be an open set and $f : V \to \mathbb{R}$ be smooth. The **level surface** associated with f is the set

$$\mathcal{S}_f := f^{-1}(0) = \{(x,y,z) \in V : f(x,y,z) = 0\}.$$

Theorem 3.57

Let $V \subseteq \mathbb{R}^3$ be an open set and $f : V \to \mathbb{R}$ be smooth. Consider the level surface

$$\mathcal{S}_f = \{(x,y,z) \in V : f(x,y,z) = 0\}.$$

Suppose that

$$\nabla f(x,y,z) \neq 0, \quad \forall (x,y,z) \in V.$$

Then $\mathcal{S}_f$ is a regular surface.

Example 3.58: Circular cone

Question. Consider the circular cone

$$\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, z > 0\}.$$

Prove that $\mathcal{S}$ is a regular surface.
Solution. Define the open set

$$V = \{(x,y,z) \in \mathbb{R}^3 : z > 0\},$$

and the function $f : V \to \mathbb{R}$ by

$$f(x,y,z) = x^2 + y^2 - z^2.$$

We have

$$\nabla f(x,y,z) = (2x, 2y, -2z) \neq 0, \quad \forall (x,y,z) \in V.$$

Since

$$\mathcal{S} = \mathcal{S}_f,$$

we conclude that $\mathcal{S}$ is a regular surface.

Theorem 3.59: Tangent plane of level surfaces

Let $V \subseteq \mathbb{R}^3$ be an open set and $f : V \to \mathbb{R}$ be smooth. Suppose that

$$\nabla f(x,y,z) \neq 0, \quad \forall (x,y,z) \in V.$$

Let $\mathbf{p} \in \mathcal{S}_f$. We have

$$\nabla f(\mathbf{p}) \perp T_{\mathbf{p}}\mathcal{S}_f.$$

In particular, they hold:

1. The cartesian equation of $T_{\mathbf{p}}\mathcal{S}_f$ is given by

$$\nabla f(\mathbf{p}) \cdot \mathbf{x} = 0, \quad \forall \mathbf{x} \in \mathbb{R}^3,$$

2. The cartesian equation for $\mathbf{p} + T_{\mathbf{p}}\mathcal{S}_f$ is given by

$$\nabla f(\mathbf{p}) \cdot (\mathbf{x} - \mathbf{p}) = 0, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

Example 3.60: Unit cylinder

Question. Consider the unit cylinder

$$\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}.$$

1. Prove that $\mathcal{S}$ is a regular surface.
2. Let

$$\mathbf{p} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 5 \right) \in \mathcal{S}$$

Show that $T_{\mathbf{p}}\mathcal{S}$ has cartesian equation

$$x + y = 0.$$

Solution.

1. Define the open set

$$V = \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}.$$

Note that V is obtained by removing the z-axis from $\mathbb{R}^3$. Also define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) := x^2 + y^2 - 1.$$

We have

$$\nabla f(x,y,z) = (2x, 2y, 0) \neq 0, \quad \forall (x,y,z) \in V.$$

Since

$$\mathcal{S} = \mathcal{S}_f,$$

we conclude that $\mathcal{S}$ is a regular surface.

2. Using the expression for ∇f found in Point 1, we get

$$\nabla f(\mathbf{p}) = \nabla f \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 5 \right) = (\sqrt{2}, \sqrt{2}, 0).$$

The equation for $T_{\mathbf{p}}\mathcal{S}$ is

$$\nabla f(\mathbf{p}) \cdot \mathbf{x} = 0, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

Therefore, we find

$$\begin{aligned} \nabla f(\mathbf{p}) \cdot \mathbf{x} = 0 \quad &\Longleftrightarrow \quad (\sqrt{2}, \sqrt{2}, 0) \cdot (x,y,z) = 0 \\ &\Longleftrightarrow \quad x + y = 0. \end{aligned}$$

where

1. $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ is a smooth curve,
2. $\mathbf{a} : (a, b) \to \mathbb{R}^3$ is a smooth curve,
3. $\dot{\boldsymbol{\gamma}}(t)$ and $\mathbf{a}(t)$ are linearly independent for all $t \in (a, b)$.

We say that:

1. $\boldsymbol{\gamma}$ is the **base curve**.
2. The lines $v \mapsto v\mathbf{a}(u)$ are the **rulings**.

Theorem 3.62

A ruled surface $\mathcal{S}$ is regular if v is sufficiently small.

Example 3.63: Unit Cylinder is ruled surface

Question. Prove that the unit cylinder

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = 1\} .$$

is a ruled surface.
Solution. We know that the unit cylinder is charted by

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \cos(u), v) , \quad (u, v) \in (0, 2\pi) \times \mathbb{R} .$$

We can rewrite $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\mathbf{a}(u) ,$$

with

$$\boldsymbol{\gamma}(u) := (\cos(u), \cos(u), 0) , \quad \mathbf{a} = (0, 0, 1) .$$

Note that the vectors

$$\dot{\boldsymbol{\gamma}} = (-\sin(u), \cos(u), 0) , \quad \mathbf{a} = (0, 0, 1)$$

are orthogonal, and hence linearly independent. Therefore $\mathcal{S}$ is a ruled surface.

Example 3.64: Möbius band

Question. The Möbius band is a ruled surface with chart

$$\boldsymbol{\sigma} = \boldsymbol{\gamma}(u) + v\mathbf{a}(u) , \quad u \in (0, 2\pi), \ v \in \left(-\frac{1}{2}, \frac{1}{2}\right) ,$$

where

$$\boldsymbol{\gamma}(u) = (\cos(u), \sin(u), 0)$$

is the unit circle, and

$$\mathbf{a} = \left(-\sin\left(\frac{u}{2}\right)\cos(u), -\sin\left(\frac{u}{2}\right)\sin(u), \cos\left(\frac{u}{2}\right)\right)$$

is a vector which does a half rotation while going around the unit circle $\boldsymbol{\gamma}$. In particular

$$\boldsymbol{\sigma}(u, v) = \left(\left[1 - v\sin\left(\frac{u}{2}\right)\right]\cos(u), \left[1 - v\sin\left(\frac{u}{2}\right)\right]\sin(u), v\cos\left(\frac{u}{2}\right)\right)$$

1. Compute the standard unit normal to $\boldsymbol{\sigma}$.
2. Prove that $\mathcal{S}$ is **non orientable**.

Solution.

1. From the formula for $\boldsymbol{\sigma}$, it is easy to compute that

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = \left(-\cos(u)\cos\left(\frac{u}{2}\right), -\sin(u)\cos\left(\frac{u}{2}\right), -\sin\left(\frac{u}{2}\right)\right) .$$

   It is also immediate to check that $\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| = 1$, and therefore the principal unit normal of $\boldsymbol{\sigma}$ is

$$\mathbf{N}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v .$$

2. Suppose by contradiction that $\mathcal{S}$ is orientable. This means there exists a globally defined principal unit normal vector

$$\mathbf{N} : \ \mathcal{S} \to \mathbb{R}^3 .$$

   By definition of principal normal, we have either

$$\mathbf{N} \circ \boldsymbol{\sigma} = \mathbf{N}_{\boldsymbol{\sigma}} .$$

   Consider the point $\mathbf{p} = (1, 0, 0)$ on $\mathcal{S}$. Notice that, by continuity, $\mathbf{p}$ can be obtained via $\boldsymbol{\sigma}$ through the limits

$$\mathbf{p} = \lim_{u \to 0^+} \boldsymbol{\sigma}(u, 0) = \lim_{u \to 2\pi^-} \boldsymbol{\sigma}(u, 0) .$$

   Since $\mathbf{N}$ is continuous, the above implies

$$\mathbf{N}(\mathbf{p}) = \lim_{u \to 0^+} \mathbf{N} \circ \boldsymbol{\sigma}(u, 0) = \lim_{u \to 2\pi^-} \mathbf{N} \circ \boldsymbol{\sigma}(u, 0) . \tag{3.1}$$

   However, by direct calculation:

$$\lim_{u \to 0^+} \mathbf{N} \circ \boldsymbol{\sigma}(u, 0) = \lim_{u \to 0^+} \mathbf{N}_{\boldsymbol{\sigma}}(u, 0) = (-1, 0, 0)$$
$$\lim_{u \to 2\pi^-} \mathbf{N} \circ \boldsymbol{\sigma}(u, 0) = \lim_{u \to 2\pi^-} \mathbf{N}_{\boldsymbol{\sigma}}(u, 0) = (1, 0, 0)$$

   This clearly contradicts (3.1). Therefore $\mathbf{N}$ cannot exist, and $\mathcal{S}$ is not orientable.

Example 3.65

Question. Show that the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + 10xy + 16x^2 - z = 0\}$$

is a ruled surface.
Solution. We shall make a change of variables. Notice that we can rearrange

$$\begin{aligned} x^2 + 10xy + 16x^2 - z &= 0 \\ (x + 8y)(x + 2y) &= z. \end{aligned}$$

Let

$$u = x + 8y, \qquad v = x + 2y.$$

Then $uv = z$ and

$$\begin{aligned} u - v = 6y \quad &\implies \quad y = \frac{u - v}{6} \\ x &= u - 8y \\ &= u - \frac{8(u - v)}{6} \\ &= \frac{4v - u}{3}. \end{aligned}$$

It follows that if $(x, y, z) \in S$ then

$$\begin{aligned} (x, y, z) &= \left(\frac{4v - u}{3}, \frac{u - v}{6}, uv\right) \\ &= \left(-\frac{u}{3}, \frac{u}{6}, 0\right) + v\left(\frac{4}{3}, -\frac{1}{6}, u\right) \\ &= \boldsymbol{\gamma}(u) + v\mathbf{a}(u), \end{aligned}$$

where we have set

$$\begin{aligned} \boldsymbol{\gamma}(u) &= \left(-\frac{u}{3}, \frac{u}{6}, 0\right) \\ \mathbf{a}(u) &= \left(\frac{4}{3}, -\frac{1}{6}, u\right). \end{aligned}$$

Notice that

$$\dot{\boldsymbol{\gamma}}(u) = \left(-\frac{1}{3}, \frac{1}{6}, 0\right).$$

For $u \neq 0$, we clearly have that $\mathbf{a}(u)$ and $\dot{\boldsymbol{\gamma}}(u)$ are linearly independent (the last component of $\dot{\boldsymbol{\gamma}}(u)$ is 0). For $u = 0$ we have

$$\dot{\boldsymbol{\gamma}}(0) = \left(-\frac{1}{3}, \frac{1}{6}, 0\right), \qquad \mathbf{a}(0) = \left(\frac{4}{3}, -\frac{1}{6}, 0\right),$$

which are clearly linearly independent. Therefore, S is a ruled surface.

3.10 Surfaces of Revolution

Definition 3.66: Surface of revolution

Let $\boldsymbol{\gamma} : (a, b) \to \mathbb{R}^3$ be a smooth curve in the (x, z)-plane, that is,

$$\boldsymbol{\gamma}(v) = (f(v), 0, g(v)).$$

Suppose that $f > 0$. The surface $\mathcal{S}$ obtained by rotating $\boldsymbol{\gamma}$ about the z-axis is called **surface of revolution**. A chart for $\mathcal{S}$ is given by

$$\boldsymbol{\sigma}(u, v) = (\cos(u)f(v), \sin(u)f(v), g(v)),$$

with $u \in (0, 2\pi)$ and $v \in (a, b)$.

Theorem 3.67

A surface of revolution is regular if and only if $\boldsymbol{\gamma}$ is regular.

Example 3.68: Catenoid

Question. The catenary function is defined by

$$f(v) = \cosh(v).$$

The Catenoid $\mathcal{S}$ is the surface of revolution obtained by rotating the catenary about the z-axis, that is, by rotating the curve

$$\boldsymbol{\gamma}(v) = (\cosh(v), 0, v).$$

A chart $\mathcal{S}$ is given by

$$\boldsymbol{\sigma}(u, v) = (\cos(u)\cosh(v), \sin(u)\cosh(v), v),$$

where $u \in [0, 2\pi)$ and $v \in \mathbb{R}$. Prove that $\mathcal{S}$ is a regular surface.
Solution. Note that $f > 0$ and

$$\dot{\boldsymbol{\gamma}} = (\sinh(v), 0, 1), \quad \|\dot{\boldsymbol{\gamma}}\|^2 = 1 + \sinh(v)^2 \geq 1.$$

Therefore $\boldsymbol{\gamma}$ is regular. As $\mathcal{S}$ is a surface of revolution, we conclude that $\mathcal{S}$ is regular.

3.11 First fundamental form

Definition 3.69: First fundamental form

Let $\mathcal{S}$ be a regular surface and $\mathbf{p} \in \mathcal{S}$. The **first fundamental form** of $\mathcal{S}$ at $\mathbf{p}$ is the bilinear symmetric map

$$I_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{S} \times T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}, \quad I_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) := \mathbf{v} \cdot \mathbf{w}.$$

Definition 3.70: Coordinate functions on tangent plane

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart of $\mathcal{S}$. For each $\mathbf{p} \in \boldsymbol{\sigma}(U)$ we have

$$T_{\mathbf{p}}\mathcal{S} = \operatorname{span}\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\},$$

where $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are evaluated at the point $(u,v) \in U$ such that

$$\boldsymbol{\sigma}(u,v) = \mathbf{p}.$$

Therefore, for each $\mathbf{v} \in T_{\mathbf{p}}\mathcal{S}$, there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\mathbf{v} = \lambda \boldsymbol{\sigma}_u + \mu \boldsymbol{\sigma}_v.$$

The **coordinate functions** on $T_{\mathbf{p}}\mathcal{S}$ are the linear maps

$$du, dv : T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}, \quad du(\mathbf{v}) := \lambda, \quad dv(\mathbf{v}) := \mu.$$

Definition 3.71: First fundamental form of a chart

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart of $\mathcal{S}$. Define the functions

$$E, F, G : U \to \mathbb{R}$$

by setting

$$E := \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \quad F := \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \quad G := \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v.$$

The **first fundamental form** of $\boldsymbol{\sigma}$ is the quadratic form

$$\mathcal{F}_1 : T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}$$

defined by

$$\mathcal{F}_1(\mathbf{v}) := E\, du^2(\mathbf{v}) + 2F\, du(\mathbf{v})\, dv(\mathbf{v}) + G\, dv^2(\mathbf{v}),$$

for all $\mathbf{v} \in T_{\mathbf{p}}\mathcal{S}$, and $\mathbf{p} \in \boldsymbol{\sigma}(U)$, where E, F, G are evaluated at

$$(u,v) = \boldsymbol{\sigma}^{-1}(\mathbf{p}).$$

Theorem 3.72

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart of $\mathcal{S}$, and $\mathbf{p} \in \boldsymbol{\sigma}(U)$. Then

$$I_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = (du(\mathbf{v}), dv(\mathbf{v})) \begin{pmatrix} E & F \\ F & G \end{pmatrix} (du(\mathbf{w}), dv(\mathbf{w}))^T,$$

for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$. In particular, $\mathcal{F}_1$ is the quadratic form associated to the symmetric bilinear form $I_{\mathbf{p}}$, that is,

$$\mathcal{F}_1(\mathbf{v}) = I_{\mathbf{p}}(\mathbf{v}, \mathbf{v}), \quad \forall\, \mathbf{v} \in T_{\mathbf{p}}\mathcal{S}.$$

Definition 3.73

With a little abuse of notation, we also denote by $\mathcal{F}_1$ the 2×2 matrix

$$\mathcal{F}_1 := \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Proposition 3.74: First fundamental form and reparametrizations

Let $\mathcal{S}$ be a regular surface and $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ a regular chart. Suppose that $\tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathbb{R}^3$ is a reparametrization of $\boldsymbol{\sigma}$, that is,

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi,$$

where $\Phi : \widetilde{U} \to U$ is a diffeomorphism. Denote the first fundamental forms of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ by, respectively,

$$\mathcal{F}_1 = E\, du^2 + 2F\, du dv + G\, dv^2,$$
$$\widetilde{\mathcal{F}}_1 = \widetilde{E}\, d\tilde{u}^2 + 2\widetilde{F}\, d\tilde{u} d\tilde{v} + \widetilde{G}\, d\tilde{v}^2.$$

1. The matrices of $\mathcal{F}_1$ and $\widetilde{\mathcal{F}}_1$ are related by

$$\begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} = (J\Phi)^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J\Phi, \tag{3.2}$$

   where $J\Phi$ is the Jacobian of Φ

$$J\Phi = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}.$$

2. The linear maps du, dv and $d\tilde{u}, d\tilde{v}$ are related by

$$\begin{aligned} du &= \frac{\partial u}{\partial \tilde{u}}\, d\tilde{u} + \frac{\partial u}{\partial \tilde{v}}\, d\tilde{v} \\ dv &= \frac{\partial v}{\partial \tilde{u}}\, d\tilde{u} + \frac{\partial v}{\partial \tilde{v}}\, d\tilde{v} \end{aligned} \tag{3.3}$$

Example 3.75: FFF of Plane

Question. Let $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, with $\mathbf{p}$ and $\mathbf{q}$ orthonormal. Consider the plane with chart

$$\boldsymbol{\sigma}(u,v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}, \quad (u,v) \in \mathbb{R}^2.$$

Prove that the first fundamental form of $\boldsymbol{\sigma}$ is

$$\mathcal{F}_1 = du^2 + dv^2.$$

Solution. We have

$$\boldsymbol{\sigma}_u = \mathbf{p}, \quad \boldsymbol{\sigma}_v = \mathbf{q},$$

and therefore, using that $\mathbf{p}$ and $\mathbf{q}$ are orthonormal,

$$\begin{aligned} E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = \|\mathbf{p}\|^2 = 1 \\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = \mathbf{p} \cdot \mathbf{q} = 0 \\ G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = \|\mathbf{q}\|^2 = 1 \end{aligned}$$

The first fundamental form of $\boldsymbol{\sigma}$ is, therefore

$$\mathscr{F}_1 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = du^2 + dv^2 \,.$$

Example 3.76: FFF of Plane in polar coordinates

Question. Let $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, with $\mathbf{p}, \mathbf{q}$ orthonormal. In polar coordinates, the plane is charted by

$$\boldsymbol{\sigma}(\rho, \theta) = \mathbf{a} + \rho\cos(\theta)\mathbf{p} + \rho\sin(\theta)\mathbf{q} \,, \quad \rho > 0,\ \theta \in (0, 2\pi) \,.$$

1. By direct calculation, show that the first fundamental form of $\boldsymbol{\sigma}$ is
$$\mathscr{F}_1 = d\rho^2 + \rho^2 d\theta^2 \,.$$
2. The first fundamental form of the plane in cartesian coordinates is
$$\widetilde{\mathscr{F}}_1 = du^2 + dv^2 \,.$$
Verify that
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (J\Phi)^T \begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} J\Phi \,,$$
where Φ is the change of variables from polar to cartesian coordinates.

Solution.

1. Compute $\mathscr{F}_1$ directly:
$$\begin{aligned} \boldsymbol{\sigma}_\rho &= \cos(\theta)\mathbf{p} + \sin(\theta)\mathbf{q} \\ \boldsymbol{\sigma}_\theta &= -\rho\sin(\theta)\mathbf{p} + \rho\cos(\theta)\mathbf{q} \end{aligned}$$
and therefore
$$\begin{aligned} E &= \boldsymbol{\sigma}_\rho \cdot \boldsymbol{\sigma}_\rho \\ &= \cos^2(\theta)\,\|\mathbf{p}\|^2 + \sin^2(\theta)\,\|\mathbf{q}\|^2 + 2\cos(\theta)\sin(\theta)\mathbf{p}\cdot\mathbf{q} \\ &= 1 \\ F &= \boldsymbol{\sigma}_\rho \cdot \boldsymbol{\sigma}_\theta = 0 \\ G &= \boldsymbol{\sigma}_\theta \cdot \boldsymbol{\sigma}_\theta = r^2 \end{aligned}$$
Then the first fundamental form is
$$\mathscr{F}_1 = E\,d\rho^2 + 2F\,d\rho\,d\theta + G\,d\theta^2 = d\rho^2 + \rho^2 d\theta^2 \,.$$
2. The change of variables from polar to cartesian coordinates is
$$\Psi(\rho, \theta) = (\rho\cos(\theta), \rho\sin(\theta)) \,.$$
The Jacobian of Ψ is
$$J\Phi = \begin{pmatrix} \cos(\theta) & -\rho\sin(\theta) \\ \sin(\theta) & \rho\cos(\theta) \end{pmatrix} .$$
The matrix of $\widetilde{\mathscr{F}}_1$ is just the identity:
$$\begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$
Therefore, we have
$$\begin{aligned} (J\Phi)^T \begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} J\Phi &= (J\Phi)^T\, J\Phi \\ &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\rho\sin(\theta) & \rho\cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\rho\sin(\theta) \\ \sin(\theta) & \rho\cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} . \end{aligned}$$

Example 3.77: FFF of Unit cylinder

Question. Consider the unit cylinder with chart

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v) \,, \quad (u, v) \in (0, 2\pi) \times \mathbb{R} \,.$$

Prove that the first fundamental form of $\boldsymbol{\sigma}$ is

$$\mathscr{F}_1 = du^2 + dv^2 \,.$$

Solution. We have

$$\boldsymbol{\sigma}_u = (-\sin(u), \cos(u), 0) \,, \quad \boldsymbol{\sigma}_v = (0, 0, 1) \,,$$

and therefore

$$\begin{aligned} E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 1 \\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0 \\ G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 1 \end{aligned}$$

Then the first fundamental form is

$$\mathscr{F}_1 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = du^2 + dv^2 \,.$$

Example 3.78

Question. Find the first fundamental form of the surface chart

$$\sigma(u,v) = \left(u - v, u + v, u^2 + v^2\right) .$$

Solution. We compute

$$\begin{aligned} \sigma_u &= (1,1,2u) \\ \sigma_v &= (-1,1,2v) \\ E = \sigma_u \cdot \sigma_v &= 2\left(1 + 2u^2\right) \\ F = \sigma_u \cdot \sigma_v &= 4uv \\ G = \sigma_v \cdot \sigma_v &= \left(1 + 2v^2\right) \end{aligned}$$

so that

$$\mathscr{F}_1 = \begin{pmatrix} 2\left(1+2u^2\right) & 4uv \\ 4uv & 2\left(1+2v^2\right) \end{pmatrix} .$$

3.12 Length of curves

Proposition 3.79

Let $\mathscr{S}$ be a regular surface with chart $\sigma : U \to \mathbb{R}^3$. Suppose

$$\gamma : (a,b) \to \sigma(U) \subseteq \mathscr{S}$$

is a smooth curve. Then

$$\gamma(t) = \sigma(u(t), v(t)) ,$$

for some smooth functions $u, v : (a,b) \to \mathbb{R}$, and

$$\int_a^b \|\dot{\gamma}(t)\| \, dt = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} \, dt ,$$

where in the above formula:

- $\dot{u}, \dot{v}$ are computed at t,
- E, F, G are computed at $(u(t), v(t))$.

Example 3.80: Curves on the Cone

Question. Consider the cone with chart

$$\sigma(u,v) = (\cos(u)v, \sin(u)v, v) ,$$

where $u \in (0, 2\pi)$ and $v > 0$. Prove the following:

1. The first fundamental form of σ is

$$\mathscr{F}_1 = 2\, du^2 + u^2\, dv^2 .$$

2. Let $\gamma(t) := \sigma(t,t)$. The length of γ is

$$\int_{\pi/2}^{\pi} \|\dot{\gamma}(t)\| \, dt = \int_{\pi/2}^{\pi} \sqrt{t^2 + 2}\, dt .$$

Solution.

1. We have

$$\begin{aligned} \sigma_u &= (-\sin(u)v, \cos(u)v, 0) \\ \sigma_v &= (\cos(u), \sin(u), 1) \\ E = \sigma_u \cdot \sigma_u &= v^2 \\ F = \sigma_u \cdot \sigma_v &= 0 \\ G = \sigma_v \cdot \sigma_v &= 2 \end{aligned}$$

The first fundamental form of σ is

$$\mathscr{F}_1 = v^2\, du^2 + 2\, dv^2 .$$

2. By definition we have

$$\gamma(t) = \sigma(t,t) ,$$

so that

$$\gamma(t) = \sigma(u(t), v(t))$$

with

$$u(t) = t , \quad v(t) = t .$$

In particular

$$\dot{u} = 1 , \quad \dot{v} = 1$$

and

$$\begin{aligned} E(u(t), v(t)) &= E(t,t) = t^2 \\ F(u(t), v(t)) &= F(t,t) = 0 \\ G(u(t), v(t)) &= G(t,t) = 2 . \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\pi/2}^{\pi} \|\dot{\gamma}(t)\| \, dt &= \int_{\pi/2}^{\pi} \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt \\ &= \int_{\pi/2}^{\pi} \sqrt{t^2 + 2}\, dt . \end{aligned}$$

3.13 Isometries

Definition 3.81: Local Isometry and Isometry

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}$$

be a smooth map. Denote the differential of f by

$$d_{\mathbf{p}} f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\widetilde{\mathcal{S}} .$$

We say that:

1. f is a **local isometry**, if for all $\mathbf{p} \in \mathcal{S}$

$$\mathbf{v} \cdot \mathbf{w} = d_{\mathbf{p}} f(\mathbf{v}) \cdot d_{\mathbf{p}} f(\mathbf{w}) , \quad \forall\, \mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S} . \tag{3.4}$$

   In this case, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are said to be **locally isometric**.

2. f is an **isometry** if:
   - f is a local isometry;
   - f is a diffeomorphism of $\mathcal{S}$ into $\widetilde{\mathcal{S}}$.

   In this case, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are said to be **isometric**.

Theorem 3.82

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}$$

a local isometry. Then f is a local diffeomorphims.

Theorem 3.83

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ be a smooth map. They are equivalent:

1. f is a local isometry.
2. Let $\boldsymbol{\gamma}$ be a curve in $\mathcal{S}$ and consider the curve $\tilde{\boldsymbol{\gamma}} = f \circ \boldsymbol{\gamma}$ on $\widetilde{\mathcal{S}}$. Then $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same length.

Theorem 3.84

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ be a local diffeomorphism. They are equivalent:

1. f is a local isometry.
2. Let $\boldsymbol{\sigma} : U \to \mathcal{S}$ be a regular chart of $\mathcal{S}$ and consider the chart of $\widetilde{\mathcal{S}}$ given by

$$\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}} , \qquad \tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma} .$$

   Then $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ have the same first fundamental form, that is,

$$E = \widetilde{E} , \quad F = \widetilde{F} , \quad G = \widetilde{G} .$$

Theorem 3.85

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and consider charts

$$\boldsymbol{\sigma} : U \to \mathcal{S} , \qquad \tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}} .$$

Assume that $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ have the same first fundamental form:

$$E = \widetilde{E} , \quad F = \widetilde{F} , \quad G = \widetilde{G} .$$

We have:

1. The surfaces $\boldsymbol{\sigma}(U)$ and $\widetilde{\mathcal{S}}$ are locally isometric.
2. A local isometry is given by

$$f : \boldsymbol{\sigma}(U) \to \widetilde{\mathcal{S}} , \qquad f = \tilde{\boldsymbol{\sigma}} \circ \boldsymbol{\sigma}^{-1} .$$

Example 3.86: Plane and Cylinder are locally isometric

Question. Consider the plane

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \ : \ x = 0\} ,$$

and the unit cylinder

$$\widetilde{\mathcal{S}} = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = 1\} .$$

Define the function

$$f : \mathcal{S} \to \widetilde{\mathcal{S}} , \qquad f(0, y, z) = (\cos(y), \sin(y), z) .$$

1. Prove that f is smooth.
2. Prove that f is a local isometry.

Note: This shows that the Plane and the Cylinder are locally isometric.
Solution.

1. Note that $f \in \widetilde{\mathcal{S}}$ because

$$\cos(y)^2 + \sin(y)^2 = 1 ,$$

   therefore f is well-defined. Moreover, f is the restriction to $\mathcal{S}$ of the function

$$g : \mathbb{R}^3 \to \mathbb{R}^3 , \qquad g(x, y, z) = (\cos(y), \sin(y), z) .$$

   Since g is smooth, and $g(\mathcal{S}) = \widetilde{\mathcal{S}}$, by Theorem 3.37 we infer that $g|_{\mathcal{S}} = f$ is smooth between $\mathcal{S}$ and $\widetilde{\mathcal{S}}$.

2. Define the chart of $\mathcal{S}$:

$$\boldsymbol{\sigma}(u,v) = (0,u,v)\,, \quad u,v \in \mathbb{R}\,.$$

We already know that $\boldsymbol{\sigma}$ is regular, with first fundamental form coefficients given by

$$E = 1\,, \quad F = 1\,, \quad G = 1\,,$$

and corresponding first fundamental form

$$\mathcal{F}_1 = du^2 + dv^2\,.$$

Define $\tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}$. Therefore,

$$\tilde{\boldsymbol{\sigma}}(u,v) = f(0,u,v) = (\cos(u), \sin(u), v)\,.$$

We have that

$$\begin{aligned} \tilde{\boldsymbol{\sigma}}_u &= (-\sin(u), \cos(u), 0) \\ \tilde{\boldsymbol{\sigma}}_v &= (0,0,1) \\ \widetilde{E} &= \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_u = 1 \\ \widetilde{F} &= \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_v = 0 \\ \widetilde{G} &= \tilde{\boldsymbol{\sigma}}_v \cdot \tilde{\boldsymbol{\sigma}}_v = 1 \end{aligned}$$

Therefore, the first fundamental form of $\tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}$ is

$$\widetilde{\mathcal{F}}_1 = du^2 + dv^2\,.$$

In particular, we have shown that $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ have the same first fundamental form:

$$\mathcal{F}_1 = \widetilde{\mathcal{F}}_1\,.$$

Since $\mathcal{A} = \{\boldsymbol{\sigma}\}$ is an atlas for $\mathcal{S}$, by Theorem 3.85 we conclude that f is a local isometry of $\mathcal{S}$ into $\widetilde{\mathcal{S}}$.

Example 3.87: Plane and Cone are locally isometric

Question. Consider the cone without tip

$$\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \;:\; x^2 + y^2 = z^2\,,\; z > 0\}\,,$$

and the plane

$$\widetilde{\mathcal{S}} = \{(x,y,z) \in \mathbb{R}^3 \;:\; z = 0\}\,.$$

1. Let $\boldsymbol{\sigma} : U \to \mathcal{S}$ be the chart of the Cone

$$\boldsymbol{\sigma}(\rho,\theta) = (\rho\cos(\theta), \rho\sin(\theta), \rho)\,,$$

where we define

$$U := \left\{(\rho,\theta) \in \mathbb{R}^2 \;:\; \rho > 0,\; \theta \in (0,2\pi)\right\}\,.$$

Compute the first fundamental form $\mathcal{F}_1$ of $\boldsymbol{\sigma}$.

2. Let $\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}}$ be the chart of the Plane

$$\tilde{\boldsymbol{\sigma}}(\rho,\theta) = (a\rho\cos(b\theta), a\rho\sin(b\theta), 0)$$

where $a > 0$ and $b \in (0,1]$ are constants. Compute the first fundamental form $\widetilde{\mathcal{F}}_1$ of $\tilde{\boldsymbol{\sigma}}$.

3. Find coefficients a, b such that

$$\mathcal{F}_1 = \widetilde{\mathcal{F}}_1\,.$$

4. Conclude that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are locally isometric.

Solution.

1. We have already computed in Example 3.81, that the first fundamental form of $\boldsymbol{\sigma}$ is

$$\mathcal{F}_1 = 2d\rho^2 + \rho^2 d\theta^2\,.$$

2. First of all, note that

$$\theta \in (0,2\pi), \quad b \in (0,1] \quad \Longrightarrow \quad b\theta \in (0,2\pi)\,,$$

showing that $\tilde{\boldsymbol{\sigma}}$ is well defined for all $(\rho,\theta) \in U$. We compute

$$\begin{aligned} \tilde{\boldsymbol{\sigma}}_\rho &= (a\cos(b\theta), a\sin(b\theta), 0) \\ \tilde{\boldsymbol{\sigma}}_\theta &= (-ab\rho\sin(b\theta), ab\rho\cos(b\theta), 0) \\ \widetilde{E} &= \tilde{\boldsymbol{\sigma}}_\rho \cdot \tilde{\boldsymbol{\sigma}}_\rho = a^2 \\ \widetilde{F} &= \tilde{\boldsymbol{\sigma}}_\rho \cdot \tilde{\boldsymbol{\sigma}}_\theta = 0 \\ \widetilde{G} &= \tilde{\boldsymbol{\sigma}}_\theta \cdot \tilde{\boldsymbol{\sigma}}_\theta = a^2 b^2 \rho^2 \end{aligned}$$

from which we conclude that the first fundamental form of $\tilde{\boldsymbol{\sigma}}$ is

$$\widetilde{\mathcal{F}}_1 = a^2 d\rho^2 + a^2 b^2 \rho^2 d\rho^2\,.$$

3. Equating $\mathcal{F}_1$ and $\widetilde{\mathcal{F}}_1$ we obtain

$$a^2 = 2,\; a^2b^2 = 1 \quad \Longrightarrow \quad a = \sqrt{2}\,,\; b = \frac{1}{\sqrt{2}}\,.$$

Note that $a > 0$ and $0 < b < 1$, showing that a, b are admissible.

4. For $a = \sqrt{2}$ and $b = 1/\sqrt{2}$ we have that

$$\mathcal{F}_1 = \widetilde{\mathcal{F}}_1\,.$$

Since $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are regular charts for $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, respectively, from Theorem 3.84 we conclude that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are locally isometric. Furthermore, the local isometry is given by

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}\,, \qquad f = \tilde{\boldsymbol{\sigma}} \circ \boldsymbol{\sigma}^{-1}\,.$$

3.14 Angles between curves

Definition 3.88: Angle between curves

Let $\mathcal{S}$ be a regular surface. Let $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ be curves on $\mathcal{S}$ such that

$$\boldsymbol{\gamma}(t_0) = \mathbf{p} = \tilde{\boldsymbol{\gamma}}(t_0)\,.$$

The angle θ between $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ is

$$\cos(\theta) = \frac{\dot{\boldsymbol{\gamma}} \cdot \dot{\tilde{\boldsymbol{\gamma}}}}{\|\dot{\boldsymbol{\gamma}}\| \, \|\dot{\tilde{\boldsymbol{\gamma}}}\|}\,,$$

where $\dot{\boldsymbol{\gamma}}$ and $\dot{\tilde{\boldsymbol{\gamma}}}$ are evaluated at t_0.

Theorem 3.89

Let $\mathcal{S}$ be a regular surface, $\boldsymbol{\sigma}$ a regular chart at $\mathbf{p}$, and $\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}$ smooth curves on $\mathcal{S}$ such that

$$\boldsymbol{\gamma}(t_0) = \mathbf{p} = \tilde{\boldsymbol{\gamma}}(t_0)\,.$$

There exist smooth functions $u, v, \tilde{u}, \tilde{v}$ such that

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))\,, \quad \tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\sigma}(\tilde{u}(t), \tilde{v}(t))\,.$$

The angle between $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ is

$$\cos(\theta) = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G\dot{v}\dot{\tilde{v}}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}}\,,$$

where $E, F, G, \widetilde{E}, \widetilde{F}, \widetilde{G}$ are evaluated at $(u(t_0), v(t_0))$ and $\dot{u}, \dot{v}, \dot{\tilde{u}}, \dot{\tilde{v}}$ are evaluated at t_0.

Example 3.90

Question. Let S be a surface with surface chart

$$\boldsymbol{\sigma}(u, v) = (u, v, e^{uv})\,.$$

1. Calculate its first fundamental form.
2. Calculate $\cos(\theta)$ where θ is the angle between the two curves

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t)), \quad u(t) = t,\ v(t) = t\,,$$
$$\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\sigma}(\tilde{u}(t), \tilde{v}(t))\,, \quad \tilde{u}(t) = 1,\ \tilde{v}(t) = t\,.$$

Solution.

1. We calculate

$$\boldsymbol{\sigma}_u = (1, 0, e^{uv} v)\,, \quad \boldsymbol{\sigma}_v = (0, 1, e^{uv} u)\,.$$

Therefore, the coefficients of the first fundamental form are

$$E(u, v) = 1 + e^{2uv} v^2$$
$$F(u, v) = e^{2uv} uv$$
$$G(u, v) = 1 + e^{2uv} u^2$$

2. The curves $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ intersect at

$$\boldsymbol{\gamma}(1) = \tilde{\boldsymbol{\gamma}}(1) = \boldsymbol{\sigma}(1, 1)\,.$$

We calculate

$$\dot{u}(1) = 1, \quad \dot{v}(1) = 1, \quad \dot{\tilde{u}}(1) = 0, \quad \dot{\tilde{v}}(1) = 1\,,$$

and

$$E(1, 1) = 1 + e^2$$
$$F(1, 1) = e^2$$
$$G(1, 1) = 1 + e^2$$

Therefore

$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F\left(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}\right) + G\dot{v}\dot{\tilde{v}}}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\sqrt{E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2}}$$
$$= \frac{1 + 2e^2}{\sqrt{2 + 4e^2}\sqrt{1 + e^2}} = \sqrt{\frac{1 + 2e^2}{2 + 2e^2}}\,.$$

3.15 Conformal maps

Definition 3.91: Conformal map

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces, and

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}$$

a local diffeomorphism. We say that f is a **conformal map** if for all $\mathbf{p} \in \mathcal{S}$ and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$ is holds

$$\theta = \tilde{\theta}\,,$$

where:

- θ is the angle between $\mathbf{v}$ and $\mathbf{w}$,
- $\tilde{\theta}$ is the angle between $d_{\mathbf{p}}f(\mathbf{v})$ and $d_{\mathbf{p}}f(\mathbf{w})$.

In this case, we say that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are **conformal**.

Proposition 3.92

Let f be a local isometry. Then f is a conformal map.

Theorem 3.93

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ be a local diffeomorphism. They are equivalent:

1. f is a conformal map.
2. Let $\boldsymbol{\sigma} : U \to \mathcal{S}$ be a regular chart of $\mathcal{S}$ and consider the chart of $\widetilde{\mathcal{S}}$ given by

$$\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}}, \qquad \tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}.$$

Then, there exists $\lambda : U \to \mathbb{R}$ such that

$$\widetilde{\mathcal{F}_1} = \lambda(u,v)\mathcal{F}_1, \quad \forall (u,v) \in U,$$

where $\mathcal{F}_1$ and $\widetilde{\mathcal{F}_1}$ are the first fundamental forms of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$, respectively.

Theorem 3.94

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and consider charts

$$\boldsymbol{\sigma} : U \to \mathcal{S}, \qquad \tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}}.$$

Assume there exists $\lambda : U \to \mathbb{R}$ such that

$$\widetilde{\mathcal{F}_1} = \lambda(u,v)\mathcal{F}_1, \quad \forall (u,v) \in U,$$

where $\mathcal{F}_1$ and $\widetilde{\mathcal{F}_1}$ are the first fundamental forms of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$, respectively. We have:

1. The surfaces $\boldsymbol{\sigma}(U)$ and $\widetilde{\mathcal{S}}$ are conformal.
2. A conformal map is given by

$$f : \boldsymbol{\sigma}(U) \to \widetilde{\mathcal{S}}, \qquad f = \tilde{\boldsymbol{\sigma}} \circ \boldsymbol{\sigma}^{-1}.$$

Example 3.95: Stereographic Projection

Question. Denote the unit sphere by

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \, : \, x^2 + y^2 + z^2 = 1\},$$

and consider the surface

$$\mathcal{S} = \mathbb{S}^2 \smallsetminus \{N\},$$

where the point $N = (0,0,1)$ is the North Pole. Denote the plane $\{z = 0\}$ by

$$\widetilde{\mathcal{S}} = \{(x,y,z) \in \mathbb{R}^3 \, : \, z = 0\},$$

The plane $\{z = 0\}$ slices through the equator of the sphere. Let $P = (x,y,z)$ be any point on $\mathbb{S}^2$ except the north pole.

The line joining the north pole to P intersects the plane $\{z = 0\}$ at the point P'. The point P' defines the *Stereographic Projection* map, which is easily computed to be:

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}, \qquad f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right).$$

Prove that:

1. f is a conformal map.
2. f is not a local isometry.

Note: In particular, the Sphere and the Plane are conformal.

Solution. It is not difficult to prove that f is invertible, with inverse given by

$$\boldsymbol{\sigma}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, 1 - \frac{2}{u^2+v^2+1} \right).$$

We have that $\boldsymbol{\sigma}$ is a regular chart for $\mathcal{S}$, forming an altas of one chart. It is straightforward to compute that the coefficients of the first fundamental form of $\boldsymbol{\sigma}$ are

$$E = G = \lambda(u,v) := \frac{4}{(u^2+v^2+1)^2}, \qquad F = 0.$$

In particualar the fisrt fundamental form is

$$\mathcal{F}_1 = \lambda(u,v)(du^2 + dv^2).$$

Define the chart of $\widetilde{\mathcal{S}}$:

$$\tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}.$$

Since $\boldsymbol{\sigma}$ is the inverse of f, we have that

$$\tilde{\boldsymbol{\sigma}}(u,v) = (u,v,0).$$

As already computed, the first fundamental form of $\tilde{\boldsymbol{\sigma}}$ is

$$\widetilde{\mathcal{F}_1} = du^2 + dv^2.$$

We can now conclude:

1. We have that
$$\widetilde{\mathcal{F}_1} = \frac{1}{\lambda}\mathcal{F}_1.$$
Since $\mathcal{A} = \{\boldsymbol{\sigma}\}$ is an atlas for $\mathcal{S}$, by Theorem 3.93 we conclude that f is a conformal map.

2. Since λ is not always equal to 1, we have that
$$\widetilde{\mathcal{F}_1} \neq \mathcal{F}_1.$$
Therefore, by Theorem 3.84, we conclude that f cannot be a local isometry.

Example 3.96: Sphere and Plane are conformal

Question. Let $\mathcal{S}$ be the plane $\{z = 0\}$ with chart

$$\boldsymbol{\sigma}(u,v) := (u,v,0)\,, \quad u,v \in \mathbb{R}\,.$$

Let $\widetilde{\mathcal{S}}$ be the sphere $\mathbb{S}^2$ with parametrization

$$\tilde{\boldsymbol{\sigma}}(u,v) := (\cos(u)\operatorname{sech}(v), \sin(u)\operatorname{sech}(v), \tanh(v))\,.$$

1. Compute the first fundamental forms of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$.
2. Show that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are conformal.

Solution.

1. As already computed, the first fundamental form of $\boldsymbol{\sigma}$ is
$$\mathcal{F}_1 = du^2 + dv^2\,.$$
Using the identitities
$$\frac{d}{dv}(\operatorname{sech}(v)) = -\operatorname{sech}(v)\tanh(v)\,,$$
$$\frac{d}{dv}(\tanh(v)) = \operatorname{sech}^2(v)\,,$$
we obtain
$$\tilde{\boldsymbol{\sigma}}_u = (-\sin(u)\operatorname{sech}(v), \cos(u)\operatorname{sech}(v), 0)$$
$$\tilde{\boldsymbol{\sigma}}_v = (-\cos(v)\operatorname{sech}(v)\tanh(v), -\sin(u)\operatorname{sech}(v)\tanh(v), \operatorname{sech}^2(v))$$
By recalling that
$$\operatorname{sech}^2(v) + \tanh^2(v) = 1\,,$$
we compute
$$\widetilde{E} = \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_u = \operatorname{sech}^2(v)(\cos^2(u) + \sin^2(u)) = \operatorname{sech}^2(v)$$
$$\widetilde{F} = \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_v = 0$$
$$\widetilde{G} = \tilde{\boldsymbol{\sigma}}_v \cdot \tilde{\boldsymbol{\sigma}}_v = \operatorname{sech}^2(v)(\tanh^2(v) + \operatorname{sech}^2(v)) = \operatorname{sech}^2(v)$$
Hence the first fundamental form of $\widetilde{\mathcal{S}}$ is
$$\widetilde{\mathcal{F}_1} = \operatorname{sech}^2(v)\,\left(du^2 + dv^2\right)\,.$$
2. We have computed that
$$\widetilde{\mathcal{F}_1} = \operatorname{sech}^2(v)\,\left(du^2 + dv^2\right) = \operatorname{sech}^2(v)\,\mathcal{F}_1\,.$$
Since $A = \{\boldsymbol{\sigma}\}$ is an atlas for $\mathcal{S}$, by Theorem 3.94 we conclude that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are conformal.

3.16 Conformal parametrizations

Definition 3.97: Conformal parametrization

Let $\mathcal{S}$ be a regular surface and

$$\boldsymbol{\sigma} : U \to \mathcal{S}$$

be a regular chart of $\mathcal{S}$. We say that $\boldsymbol{\sigma}$ is a **conformal parametrization** if the first fundamental form of $\boldsymbol{\sigma}$ satisfies

$$\mathcal{F}_1 = \lambda(u,v)(du^2 + dv^2)$$

for some function $\lambda : U \to \mathbb{R}$.

Theorem 3.98

A conformal parametrization $\boldsymbol{\sigma}$ preserves angles between vectors. Specifically, let $\boldsymbol{\gamma}_1(t), \boldsymbol{\gamma}_2(t)$ be curves in $\mathbb{R}^2$ such that $\dot{\boldsymbol{\gamma}}_1(t_0), \dot{\boldsymbol{\gamma}}_2(t_0)$ make angle θ. If

$$\boldsymbol{\gamma}_3(t) = \boldsymbol{\sigma}\left(\boldsymbol{\gamma}_1(t)\right)\,, \qquad \boldsymbol{\gamma}_4(t) = \boldsymbol{\sigma}\left(\boldsymbol{\gamma}_2(t)\right)\,,$$

then $\dot{\boldsymbol{\gamma}}_3(t_0), \dot{\boldsymbol{\gamma}}_4(t_0)$ also make angle θ.

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