Differential Geometry

Revision Guide

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6 Dec 2024

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Revision Guide

Revision Guide for the Exam of the module **Differential Geometry 661955** 2024/25 at the University of Hull. If you have any question or find any typo, please email me at

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Full lenght Lecture Notes of the module available at

silviofanzon.com/2024-Differential-Geometry-Notes

Recommended revision strategy

Make sure you are very comfortable with:

1. The Definitions, Theorems, Proofs, and Examples contained in this Revision Guide
2. The Homework questions
3. The 2022/23 and 2023/24 Exam Papers questions.
4. The Checklist below

Checklist

You should be comfortable with the following topics/taks:

Curves

- Regularity of curves
- Computing the length of a curve
- Computing arc-length function and arc-length reparametrization
- Calculating the curvature and torsion of unit-speed curves from the definitions
- Calculating the curvature and torsion of (possibly not unit-speed) curves from the formulae
- Calculating the Frenet frame of a unit-speed curve from the definitions
- Calculating the Frenet frame of a (possibly not unit-speed) unit-speed curve from the formulas
- Applying the Fundamental Theorem of Space Curves to determine if two curves coincide, up to a ridig motion
- Proving that a curve is contained in a plane, and computing the equation of such plane
- Proving that a curve is part of a circle

Topology:

- Proving that a given collection of sets is a topology
- Proving that a given set is open / closed
- Proving that a given topology is discrete
- Comparing two topology, determining which one is finer
- Studying convergent sequences in topological space
- Proving that a given set with a distance function is a metric space
- Studying the topology induced by the metric
- Studying convergent sequences in metric space
- Studying the interior, closure, boundary and limit points of a set
- Proving that a topological space is Hausdorff
- Proving that a given function between topological spaces is continuous / sequentially continuous
- Studying the subspace topology of a given subset of a topological space
- Identifying the basis of a given topology
- Showing that a given topological space is connected / path-connected
- Proving that two given topological spaces are not homeomorphic, by making use of connectedness arguments

Surfaces:

- Regularity of surface charts
- Computing reparametrizations of surface charts
- Calculating the standard unit normal of a surface chart
- Given a surface chart, compute a basis and the equation of the tangent plane
- Calculating the differential of a smooth function between surfaces
- Proving that a given level surface is regular, and computing its tangent plane
- Proving that a given surface is ruled
- Calculating the first fundamental form of a surface chart
- Proving that a given map is a local isometry / conformal
- Prove that a given parametrization is conformal
- Calculating length and angles of curves on surfaces
- Calculating the second fundamental form of a surface chart
- Calculating the matrix of the Weingarten map, the principal curvatures and vectors of a surface chart
- Calculating Gaussian and mean curvature of a surface chart
- Calculating normal and geodesic curvature of a unit-speed curve on a surface
- Calculating the normal and geodesic curvature of a (possibly not unit-speed) curve on a surface from the formulae
- Classifying surface points as elliptic, parabolic, hyperbolic, planar, umbilical

1 Curves

Definition 1.1: Length of a curve

The **length** of the curve $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ is

$$L(\boldsymbol{\gamma}) = \int_a^b \|\dot{\boldsymbol{\gamma}}(u)\| \, du \, .$$

Example 1.2: Length of the Helix

Question. Compute the length of the Helix

$$\boldsymbol{\gamma}(t) = (R\cos(t), R\sin(t), Ht) \, , \qquad t \in (0, 2\pi) \, .$$

Solution. We compute

$$\dot{\boldsymbol{\gamma}}(t) = (-R\sin(t), R\cos(t), H) \qquad \|\dot{\boldsymbol{\gamma}}(t)\| = \sqrt{R^2 + H^2}$$

$$L(\boldsymbol{\gamma}) = \int_0^{2\pi} \|\dot{\boldsymbol{\gamma}}(u)\| \, du = 2\pi\sqrt{R^2 + H^2}$$

Definition 1.3: Arc-Length of a curve

The **arc-length** along $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ from t_0 to t is

$$s : (a,b) \to \mathbb{R} \, , \qquad s(t) = \int_{t_0}^t \|\dot{\boldsymbol{\gamma}}(u)\| du \, .$$

Example 1.4: Arc-length of Logarithmic Spiral

Question. Compute the arc-length of

$$\boldsymbol{\gamma}(t) = (e^{kt}\cos(t), e^{kt}\sin(t), 0) \, .$$

Solution. The arc-length starting from t_0 is

$$\dot{\boldsymbol{\gamma}}(t) = e^{kt}(k\cos(t) - \sin(t), k\sin(t) + \cos(t), 0)$$

$$\|\dot{\boldsymbol{\gamma}}(t)\|^2 = (k^2 + 1)e^{2kt}$$

$$s(t) = \int_{t_0}^t \|\dot{\boldsymbol{\gamma}}(\tau)\| \, d\tau = \frac{\sqrt{k^2+1}}{k}(e^{kt} - e^{kt_0}) \, .$$

Definition 1.5: Unit-speed curve

A curve $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ is **unit-speed** if

$$\|\dot{\boldsymbol{\gamma}}(t)\| = 1 \, , \quad \forall \, t \in (a,b) \, .$$

Proposition 1.6

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be unit-speed. Then

$$\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}} = 0 \, , \quad \forall \, t \in (a,b) \, .$$

Proof

Since $\boldsymbol{\gamma}$ is unit-speed, we have $\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} = 1$. Differentiating both sides, we get the thesis:

$$0 = \frac{d}{dt}(\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}) = \ddot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}} = 2\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}} \, .$$

Definition 1.7: Reparametrization

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$. A **reparametrization** of $\boldsymbol{\gamma}$ is a curve $\tilde{\boldsymbol{\gamma}} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^3$ such that

$$\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\phi(t)) \, , \quad \forall \, t \in (\tilde{a}, \tilde{b}) \, ,$$

for $\phi : (\tilde{a}, \tilde{b}) \to (a,b)$ diffeomorphism. We call both ϕ and ϕ^{-1} **reparametrization maps**.

Definition 1.8: Unit-speed reparametrization

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$. A **unit-speed reparametrization** of $\boldsymbol{\gamma}$ is a reparametrization $\tilde{\boldsymbol{\gamma}} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^3$ which is unit-speed, that is,

$$\|\dot{\tilde{\boldsymbol{\gamma}}}(t)\| = 1 \, , \quad \forall \, t \in (\tilde{a}, \tilde{b}) \, .$$

Definition 1.9: Regular curve

A curve $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ is **regular** if

$$\|\dot{\boldsymbol{\gamma}}(t)\| \neq 0 \, , \quad \forall \, t \in (a,b)$$

Theorem 1.10: Existence of unit-speed reparametrization

Let $\boldsymbol{\gamma}$ be a curve. They are equivalent:

1. $\boldsymbol{\gamma}$ is regular,
2. $\boldsymbol{\gamma}$ admits unit-speed reparametrization.

Theorem 1.11: Characterization of unit-speed reparametrizations

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be a regular curve. Let $\tilde{\boldsymbol{\gamma}} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^3$ be a reparametrization of $\boldsymbol{\gamma}$, that is,

$$\boldsymbol{\gamma}(t) = \tilde{\boldsymbol{\gamma}}(\phi(t)), \quad \forall \, t \in (a,b)$$

for some diffeomorphism $\phi : (a,b) \to (\tilde{a}, \tilde{b})$. We have

1. If $\tilde{\boldsymbol{\gamma}}$ is unit-speed, there exists $c \in \mathbb{R}$ such that

$$\phi(t) = \pm s(t) + c \, , \quad \forall \, t \in (a,b) \, . \tag{1.1}$$

2. If ϕ is given by (1.1), then $\tilde{\boldsymbol{\gamma}}$ is unit-speed.

Definition 1.12: Arc-length reparametrization

Let $\boldsymbol{\gamma}$ be regular. The **arc-length reparametrization** of $\boldsymbol{\gamma}$ is

$$\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ s^{-1},$$

with s^{-1} inverse of the arc-length function of $\boldsymbol{\gamma}$.

Example 1.13: Reparametrization by arc-length

Question. Consider the curve

$$\boldsymbol{\gamma}(t) = (5\cos(t), 5\sin(t), 12t).$$

Prove that $\boldsymbol{\gamma}$ is regular, and reparametrize it by arc-length.
Solution. $\boldsymbol{\gamma}$ is regular because

$$\dot{\boldsymbol{\gamma}}(t) = (-5\sin(t), 5\cos(t), 12), \qquad \|\dot{\boldsymbol{\gamma}}(t)\| = 13 \neq 0$$

The arc-length of $\boldsymbol{\gamma}$ starting from $t_0 = 0$, and its inverse, are

$$s(t) = \int_0^t \|\dot{\boldsymbol{\gamma}}(u)\|\, du = 13t, \qquad t(s) = \frac{s}{13}.$$

The arc-length reparametrization of $\boldsymbol{\gamma}$ is

$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) = \left(5\cos\left(\frac{s}{13}\right), 5\sin\left(\frac{s}{13}\right), \frac{12}{13}s\right).$$

1.1 Curvature

Definition 1.14: Curvature of unit-speed curve

The **curvature** of a unit-speed curve $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ is

$$\kappa(t) = \|\ddot{\boldsymbol{\gamma}}(t)\|.$$

Example 1.15: Curvature of the Circle

Question. Compute the curvature of the circle of radius $R > 0$

$$\boldsymbol{\gamma}(t) = \left(x_0 + R\cos\left(\frac{t}{R}\right), y_0 + \sin\left(\frac{t}{R}\right), 0\right).$$

Solution. First, check that $\boldsymbol{\gamma}$ is unit-speed:

$$\dot{\boldsymbol{\gamma}}(t) = \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right), 0\right), \qquad \|\dot{\boldsymbol{\gamma}}(t)\| = 1$$

Now, compute second derivative and curvature

$$\ddot{\boldsymbol{\gamma}}(t) = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right), -\frac{1}{R}\sin\left(\frac{t}{R}\right), 0\right),$$
$$\kappa(t) = \|\ddot{\boldsymbol{\gamma}}(t)\| = \frac{1}{R}.$$

Definition 1.16: Curvature of regular curve

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be a regular curve and $\tilde{\boldsymbol{\gamma}}$ be a unit-speed reparametrization of $\boldsymbol{\gamma}$, with $\boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}} \circ \phi$ and $\phi : (a,b) \to (\tilde{a}, \tilde{b})$. Let $\tilde{\kappa} : (\tilde{a}, \tilde{b}) \to \mathbb{R}$ be the curvature of $\tilde{\boldsymbol{\gamma}}$. The **curvature** of $\boldsymbol{\gamma}$ is

$$\kappa(t) = \tilde{\kappa}(\phi(t)).$$

Remark 1.17: Computing curvature of regular $\boldsymbol{\gamma}$

1. Compute the arc-length $s(t)$ of $\boldsymbol{\gamma}$ and its inverse $t(s)$.
2. Compute the arc-length reparametrization
$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)).$$
3. Compute the curvature of $\tilde{\boldsymbol{\gamma}}$
$$\tilde{\kappa}(s) = \|\ddot{\tilde{\boldsymbol{\gamma}}}(s)\|.$$
4. The curvature of $\boldsymbol{\gamma}$ is
$$\kappa(t) = \tilde{\kappa}(s(t)).$$

Definition 1.18: Hyperbolic functions

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}$$
$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} \qquad \coth(t) = \frac{\cosh(t)}{\sinh(t)}$$
$$\operatorname{sech}(t) = \frac{1}{\cosh(t)} \qquad \operatorname{csch}(t) = \frac{1}{\sinh(t)}$$

Theorem 1.19: Properties of Hyperbolic Functions

$$\cosh^2(t) - \sinh^2(t) = 1 \qquad \operatorname{sech}^2(t) + \tanh^2(t) = 1$$
$$\sinh(t)' = \cosh(t) \qquad \cosh(t)' = \sinh(t)$$
$$\tanh(t)' = \operatorname{sech}^2(t) \qquad \operatorname{sech}(t)' = -\operatorname{sech}(t)\tanh(t)$$

Example 1.20: Curvature of the Catenary

Question. Consider the Catenary curve

$$\boldsymbol{\gamma}(t) = (t, \cosh(t)), \quad t \in \mathbb{R}.$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the arc-length reparametrization of $\boldsymbol{\gamma}$.
3. Compute the curvature of $\tilde{\boldsymbol{\gamma}}$.
4. Compute the curvature of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular because
$$\dot{\boldsymbol{\gamma}}(t) = (1, \sinh(t))$$
$$\|\dot{\boldsymbol{\gamma}}\| = \sqrt{1 + \sinh^2(t)} = \cosh(t) \geq 1$$
2. The arc-length of $\boldsymbol{\gamma}$ starting at $t_0 = 0$ is
$$s(t) = \int_0^t \|\dot{\boldsymbol{\gamma}}(u)\|\, du = \int_0^t \cosh(u)\, du = \sinh(t)$$
where we used that $\sinh(0) = 0$. Moreover,
$$s = \sinh(t) \iff s = \frac{e^t - e^{-t}}{2}$$
$$\iff e^{2t} - 2se^t - 1 = 0$$

Substitute $y = e^t$ to obtain

$$
\begin{aligned}
e^{2t} - 2se^t - 1 = 0 \quad &\iff \quad y^2 - 2sy - 1 = 0 \\
&\iff \quad y_\pm = s \pm \sqrt{1+s^2}\,.
\end{aligned}
$$

Notice that

$$y_+ = s + \sqrt{1+s^2} \geq s + \sqrt{s^2} = s + |s| \geq 0$$

by definition of absolute value. Therefore,

$$e^t = y_+ = s + \sqrt{1+s^2} \quad \Longrightarrow \quad t(s) = \log\left(s + \sqrt{1+s^2}\right)$$

The arc-length reparametrization of $\boldsymbol{\gamma}$ is

$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) = \left(\log\left(s + \sqrt{1+s^2}\right), \sqrt{1+s^2}\right)$$

3. Compute the curvature of $\tilde{\boldsymbol{\gamma}}$

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{\gamma}}}(s) &= \left(\frac{1}{\sqrt{1+s^2}}, \frac{s}{\sqrt{1+s^2}}\right) \\
\ddot{\tilde{\boldsymbol{\gamma}}}(s) &= \left(-\frac{s}{(1+s^2)^{3/2}}, \frac{1}{(1+s^2)^{3/2}}\right) \\
\tilde{\kappa}(s) &= \|\ddot{\tilde{\boldsymbol{\gamma}}}(s)\| = \frac{1}{1+s^2}
\end{aligned}
$$

4. Recalling that $s(t) = \sinh(t)$, the curvature of $\boldsymbol{\gamma}$ is

$$\kappa(t) = \tilde{\kappa}(s(t)) = \frac{1}{1+\sinh^2(t)} = \frac{1}{\cosh^2(t)}\,.$$

Definition 1.21: Vector product

The **vector product** of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_2 \\ v_1 & v_2 & v_3 \end{vmatrix}\,.$$

Theorem 1.22: Geometric Properties of vector product

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be linearly independent. Then

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to the plane spanned by $\mathbf{u}, \mathbf{v}$
- $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram with sides $\mathbf{u}, \mathbf{v}$
- The triple $(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})$ is a positive basis of $\mathbb{R}^3$

Theorem 1.23

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ it holds:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

Theorem 1.24

Let $\boldsymbol{\gamma}, \boldsymbol{\eta} : (a,b) \to \mathbb{R}^3$. Then, the curve $\boldsymbol{\gamma} \times \boldsymbol{\eta}$ is smooth, and

$$\frac{d}{dt}(\boldsymbol{\gamma} \times \boldsymbol{\eta}) = \dot{\boldsymbol{\gamma}} \times \boldsymbol{\eta} + \boldsymbol{\gamma} \times \dot{\boldsymbol{\eta}}\,.$$

Theorem 1.25: Curvature formula

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be regular. The curvature of $\boldsymbol{\gamma}$ is

$$\kappa(t) = \frac{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|}{\|\dot{\boldsymbol{\gamma}}(t)\|^3}\,.$$

Example 1.26: Curvature of the Helix

Question. Consider the Helix of radius $R > 0$ and rise H,

$$\boldsymbol{\gamma}(t) = (R\cos(t), R\sin(t), Ht)\,.$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the curvature of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular because

$$
\begin{aligned}
\dot{\boldsymbol{\gamma}}(t) &= (-R\sin(t), R\cos(t), H) \\
\|\dot{\boldsymbol{\gamma}}(t)\| &= \sqrt{R^2 + H^2} \geq R > 0
\end{aligned}
$$

2. Compute the curvature using the formula:

$$
\begin{aligned}
\ddot{\boldsymbol{\gamma}}(t) &= (-R\cos(t), -R\sin(t), 0) \\
\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} &= \left(RH\sin(t), -RH\cos(t), R^2\right) \\
\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| &= R\sqrt{R^2 + H^2} \\
\kappa(t) &= \frac{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|}{\|\dot{\boldsymbol{\gamma}}(t)\|^3} = \frac{R}{R^2 + H^2}
\end{aligned}
$$

Example 1.27: Calculation of curvature

Question. Define the curve

$$\boldsymbol{\gamma}(t) = \left(\frac{8}{5}\cos(t), 1 - 2\sin(t), \frac{6}{5}\cos(t)\right)\,.$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the curvature of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular because

$$\dot{\boldsymbol{\gamma}} = \left(-\frac{8}{5}\sin(t), -2\cos(t), -\frac{6}{5}\sin(t)\right), \qquad \|\dot{\boldsymbol{\gamma}}\| = 2 \neq 0\,.$$

2. Compute the curvature using the formula:

$$
\begin{aligned}
\ddot{\boldsymbol{\gamma}} &= \left(-\frac{8}{5}\cos(t), 2\sin(t), -\frac{6}{5}\cos(t)\right) & \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| &= 4 \\
\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} &= \left(-\frac{12}{5}, 0, \frac{16}{5}\right) & \kappa &= \frac{1}{2}\,.
\end{aligned}
$$

Example 1.28: Different curves, same curvature

Question Let $\boldsymbol{\gamma}$ be a circle

$$\boldsymbol{\gamma}(t) = (2\cos(t), 2\sin(t), 0)\,,$$

and $\boldsymbol{\eta}$ be a helix of radius $S > 0$ and rise $H > 0$

$$\boldsymbol{\eta}(t) = (S\cos(t), S\sin(t), Ht)\,.$$

Find S and H such that $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ have the same curvature.
Solution. Curvatures of $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ were already computed:

$$\kappa^{\boldsymbol{\gamma}} = \frac{1}{2}\,, \qquad \kappa^{\boldsymbol{\eta}} = \frac{S}{S^2+H^2}\,.$$

Imposing that $\kappa^{\boldsymbol{\gamma}} = \kappa^{\boldsymbol{\eta}}$, we get

$$\frac{1}{2} = \frac{S}{S^2+H^2} \qquad \Longrightarrow \qquad H^2 = 2S - S^2\,.$$

Choosing $S = 1$ and $H = 1$ yields $\kappa^{\boldsymbol{\gamma}} = \kappa^{\boldsymbol{\eta}}$.

1.2 Frenet frame and torsion

Definition 1.29: Frenet frame of unit-speed curve

Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be unit-speed, with $\kappa \neq 0$.

1. The **tangent vector** to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t)$ is
$$\mathbf{t}(t) = \dot{\boldsymbol{\gamma}}(t)\,.$$
2. The **principal normal vector** to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t)$ is
$$\mathbf{n}(t) = \frac{\ddot{\boldsymbol{\gamma}}(t)}{\kappa(t)}\,.$$
3. The **binormal vector** to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t)$ is
$$\mathbf{b}(t) = \dot{\boldsymbol{\gamma}}(t) \times \mathbf{n}(t)\,.$$
4. The **Frenet frame** of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t)$ is the triple
$$\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}\,.$$

Theorem 1.30: Frenet frame is orthonormal basis

Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be unit-speed, with $\kappa \neq 0$. The Frenet frame

$$\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$$

is a positive orthonomal basis of $\mathbb{R}^3$ for each $t \in (a,b)$.

Definition 1.31: Torsion of unit-speed curve with $\kappa \neq 0$

Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be unit-speed, with $\kappa \neq 0$. The **torsion** of $\boldsymbol{\gamma}$ is the unique scalar $\tau(t)$ such that

$$\dot{\mathbf{b}}(t) = -\tau(t)\mathbf{n}(t)\,.$$

In particular,

$$\tau(t) = -\dot{\mathbf{b}}(t) \cdot \mathbf{n}(t)\,.$$

Definition 1.32: Torsion of regular curve with $\kappa \neq 0$

Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be a regular curve with $\kappa \neq 0$. Let $\tilde{\boldsymbol{\gamma}}$ be a unit-speed reparametrization of $\boldsymbol{\gamma}$ with $\boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}} \circ \phi$ and $\phi\colon (a,b) \to (\tilde{a}, \tilde{b})$. Let $\tilde{\tau}\colon (\tilde{a}, \tilde{b}) \to \mathbb{R}$ be the torsion of $\tilde{\boldsymbol{\gamma}}$. The **torsion** of $\boldsymbol{\gamma}$ is

$$\tau(t) = \tilde{\tau}(\phi(t))\,.$$

Example 1.33: Curvature and torsion of Helix with Frenet frame

Question. Consider the Helix of radius $R > 0$ and rise H

$$\boldsymbol{\gamma}(t) = (R\cos(t), R\sin(t), tH)\,, \qquad t \in \mathbb{R}\,.$$

1. Compute the arc-length reparametrization $\tilde{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$.
2. Compute Frenet frame, curvature and torsion of $\tilde{\boldsymbol{\gamma}}$.
3. Compute curvature and torsion $\boldsymbol{\gamma}$.

Solution.

1. The arc-length of $\boldsymbol{\gamma}$ starting at $t_0 = 0$, and its inverse, are
$$\dot{\boldsymbol{\gamma}}(t) = (-R\sin(t), R\cos(t), H)$$
$$\|\dot{\boldsymbol{\gamma}}\| = \rho, \qquad \rho := \sqrt{R^2+H^2}$$
$$s(t) = \int_0^t \|\dot{\boldsymbol{\gamma}}(u)\|\, du = \rho t\,, \qquad t(s) = \frac{s}{\rho}\,.$$
The arc-length reparametrization $\tilde{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ is
$$\tilde{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)) = \left(R\cos\left(\frac{s}{\rho}\right), R\sin\left(\frac{s}{\rho}\right), \frac{Hs}{\rho} \right)\,.$$
2. Compute the tangent vector to $\tilde{\boldsymbol{\gamma}}$ and its derivative
$$\tilde{\mathbf{t}}(s) = \dot{\tilde{\boldsymbol{\gamma}}} = \frac{1}{\rho}\left(-R\sin\left(\frac{s}{\rho}\right), R\cos\left(\frac{s}{\rho}\right), H \right)$$
$$\dot{\tilde{\mathbf{t}}}(s) = \frac{R}{\rho^2}\left(-\cos\left(\frac{s}{\rho}\right), -\sin\left(\frac{s}{\rho}\right), 0 \right)$$
The curvature of $\tilde{\boldsymbol{\gamma}}$ is
$$\tilde{\kappa}(s) = \|\ddot{\tilde{\boldsymbol{\gamma}}}(s)\| = \|\dot{\tilde{\mathbf{t}}}(s)\| = \frac{R}{R^2+H^2}\,.$$
The principal normal vector and binormal are
$$\tilde{\mathbf{n}}(s) = \frac{\dot{\tilde{\mathbf{t}}}}{\tilde{\kappa}} = \left(-\cos\left(\frac{s}{\rho}\right), -\sin\left(\frac{s}{\rho}\right), 0 \right)$$
$$\tilde{\mathbf{b}}(s) = \tilde{\mathbf{t}} \times \tilde{\mathbf{n}} = \frac{1}{\rho}\left(H\sin\left(\frac{s}{\rho}\right), -H\cos\left(\frac{s}{\rho}\right), R \right)\,.$$
We are left to compute the torsion of $\tilde{\boldsymbol{\gamma}}$:
$$\dot{\tilde{\mathbf{b}}}(s) = \frac{H}{\rho^2}\left(\cos\left(\frac{s}{\rho}\right), \sin\left(\frac{s}{\rho}\right), 0 \right)$$
$$\dot{\tilde{\mathbf{b}}}(s) \cdot \tilde{\mathbf{n}}(s) = -\frac{H}{\rho^2}$$
$$\tilde{\tau}(s) = -\dot{\tilde{\mathbf{b}}}(s) \cdot \tilde{\mathbf{n}}(s) = \frac{H}{\rho^2} = \frac{H}{R^2+H^2}$$
3. The curvature and torsion of $\boldsymbol{\gamma}$ are
$$\kappa(t) = \tilde{\kappa}(s(t)) = \frac{R}{R^2+H^2}$$
$$\tau(t) = \tilde{\tau}(s(t)) = \frac{H}{R^2+H^2}$$

Theorem 1.34: Torsion formula

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be regular, with $\kappa \neq 0$. The torsion of $\boldsymbol{\gamma}$ is

$$\tau(t) = \frac{(\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)) \cdot \dddot{\boldsymbol{\gamma}}(t)}{\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|^2} .$$

Example 1.35: Torsion of the Helix with formula

Question. Consider the Helix of radius $R > 0$ and rise $H > 0$

$$\boldsymbol{\gamma}(t) = (R\cos(t), R\sin(t), Ht) , \quad t \in \mathbb{R} .$$

1. Prove that $\boldsymbol{\gamma}$ is regular with non-vanishing curvature.
2. Compute the torsion of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular with non-vanishing curvature, since

$$\|\dot{\boldsymbol{\gamma}}(t)\| = \sqrt{R^2 + H^2} \geq R > 0 , \qquad \kappa = \frac{R}{R^2 + H^2} > 0 .$$

2. We compute the torsion using the formula:

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= (-R\sin(t), R\cos(t), H) \\ \ddot{\boldsymbol{\gamma}}(t) &= (-R\cos(t), -R\sin(t), 0) \\ \dddot{\boldsymbol{\gamma}}(t) &= (R\sin(t), -R\cos(t), 0) \\ \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} &= \left(RH\sin(t), -RH\cos(t), R^2\right) \\ \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| &= R\sqrt{R^2 + H^2} \\ (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}} &= R^2 H \\ \tau(t) &= \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2} = \frac{H}{R^2 + H^2} \end{aligned}$$

Example 1.36: Calculation of torsion

Question. Compute the torsion of the curve

$$\boldsymbol{\gamma}(t) = \left(\frac{8}{5}\cos(t), 1 - 2\sin(t), \frac{6}{5}\cos(t)\right) .$$

Solution. Resuming calculations from Example 1.27,

$$\begin{aligned} \dddot{\boldsymbol{\gamma}} &= \left(\frac{8}{5}\sin(t), 2\cos(t), \frac{6}{5}\sin(t)\right) \\ (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}} &= \frac{96}{25}\sin(t) - \frac{96}{25}\sin(t) = 0 \\ \tau(t) &= \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2} = 0 \end{aligned}$$

Theorem 1.37: General Frenet frame formulas

The Frenet frame of a regular curve $\boldsymbol{\gamma}$ is

$$\mathbf{t} = \frac{\dot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\|} , \quad \mathbf{b} = \frac{\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|} , \quad \mathbf{n} = \mathbf{b} \times \mathbf{t} = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \times \dot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| \, \|\dot{\boldsymbol{\gamma}}\|} .$$

Example 1.38: Twisted cubic

Question. Let $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^3$ be the *twisted cubic*

$$\boldsymbol{\gamma}(t) = (t, t^2, t^3) .$$

1. Is $\boldsymbol{\gamma}$ regular/unit-speed? Justify your answer.
2. Compute the curvature and torsion of $\boldsymbol{\gamma}$.
3. Compute the Frenet frame of $\boldsymbol{\gamma}$.

Solution.

1. $\boldsymbol{\gamma}$ is regular, but not-unit speed, because

$$\begin{aligned} &\dot{\boldsymbol{\gamma}}(t) = (1, 2t, 3t^2) \\ &\|\dot{\boldsymbol{\gamma}}(t)\| = \sqrt{1 + 4t^2 + 9t^4} \geq 1 \qquad \|\dot{\boldsymbol{\gamma}}(1)\| = \sqrt{14} \neq 1 \end{aligned}$$

2. Compute the following quantities

$$\begin{aligned} &\ddot{\boldsymbol{\gamma}} = (0, 2, 6t) && \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| = 2\sqrt{1 + 9t^2 + 9t^4} \\ &\dddot{\boldsymbol{\gamma}} = (0, 0, 6) && (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}} = 12 \\ &\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} = (6t^2, -6t, 2) \end{aligned}$$

Compute curvature and torsion using the formulas:

$$\begin{aligned} \kappa(t) &= \frac{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|}{\|\dot{\boldsymbol{\gamma}}\|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}} \\ \tau(t) &= \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2} = \frac{3}{1 + 9t^2 + 9t^4} . \end{aligned}$$

3. By the Frenet frame formulas and the above calculations,

$$\begin{aligned} \mathbf{t} &= \frac{\dot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\|} = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} (1, 2t, 3t^2) \\ \mathbf{b} &= \frac{\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|} = \frac{1}{\sqrt{1 + 9t^2 + 9t^4}} (3t^2, -3t, 1) \\ \mathbf{n} = \mathbf{b} \times \mathbf{t} &= \frac{(-9t^3 - 2t, 1 - 9t^4, 6t^3 + 3t)}{\sqrt{1 + 9t^2 + 9t^4}\sqrt{1 + 4t^2 + 9t^4}} \end{aligned}$$

1.3 Frenet-Serret equations

Theorem 1.39: Frenet-Serret equations

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be unit-speed with $\kappa \neq 0$. The Frenet frame of $\boldsymbol{\gamma}$ solves the **Frenet-Serret** equations

$$\dot{\mathbf{t}} = \kappa\mathbf{n} , \qquad \dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b} , \qquad \dot{\mathbf{b}} = -\tau\mathbf{n} .$$

Definition 1.40: Rigid motion

A **rigid motion** of $\mathbb{R}^3$ is a map $M : \mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$M(\mathbf{v}) = R\mathbf{v} + \mathbf{p} , \qquad \mathbf{v} \in \mathbb{R}^3 ,$$

where $\mathbf{p} \in \mathbb{R}^3$, and $R \in SO(3)$ **rotation matrix**,

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \ : \ R^T R = I , \ \det(R) = 1\} .$$

Theorem 1.41: Fundamental Theorem of Space Curves

Let $\kappa, \tau : (a,b) \to \mathbb{R}$ be smooth, with $\kappa > 0$. Then:

1. There exists a unit-speed curve $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ with curvature $\kappa(t)$ and torsion $\tau(t)$.
2. Suppose that $\tilde{\boldsymbol{\gamma}} : (a,b) \to \mathbb{R}^3$ is a unit-speed curve whose curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ satisfy
$$\tilde{\kappa}(t) = \kappa(t), \quad \tilde{\tau}(t) = \tau(t), \quad \forall\, t \in (a,b).$$
There exists a rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ such that
$$\tilde{\boldsymbol{\gamma}}(t) = M(\boldsymbol{\gamma}(t)), \qquad \forall\, t \in (a,b).$$

Example 1.42: Application of FTSC

Question. Consider the curve
$$\boldsymbol{\gamma}(t) = \left(\sqrt{3}\,t - \sin(t), \sqrt{3}\sin(t) + t, 2\cos(t)\right).$$

1. Calculate the curvature and torsion of $\boldsymbol{\gamma}$.
2. The helix of radius R and rise H is parametrized by
$$\boldsymbol{\eta}(t) = (R\cos(t), R\sin(t), Ht).$$
Recall that $\boldsymbol{\eta}$ has curvature and torsion
$$\kappa^{\boldsymbol{\eta}} = \frac{R}{R^2 + H^2}, \qquad \tau^{\boldsymbol{\eta}} = \frac{H}{R^2 + H^2}.$$
Prove that there exist a rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ such that
$$\boldsymbol{\gamma}(t) = M(\boldsymbol{\eta}(t)), \quad \forall\, t \in \mathbb{R}. \tag{1.2}$$

Solution.

1. Compute curvature and torsion with the formulas
$$\dot{\boldsymbol{\gamma}}(t) = \left(\sqrt{3} - \cos(t), \sqrt{3}\cos(t) + 1, -2\sin(t)\right)$$
$$\ddot{\boldsymbol{\gamma}}(t) = \left(\sin(t), -\sqrt{3}\sin(t), -2\cos(t)\right)$$
$$\dddot{\boldsymbol{\gamma}}(t) = \left(\cos(t), -\sqrt{3}\cos(t), 2\sin(t)\right)$$
$$\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t) = \left(-2\left(\sqrt{3} + \cos(t)\right), 2\left(\sqrt{3}\cos(t) - 1\right), -4\sin(t)\right)$$
$$\|\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)\|^2 = 32$$
$$\|\dot{\boldsymbol{\gamma}}(t)\|^2 = 8$$
$$(\dot{\boldsymbol{\gamma}}(t) \times \ddot{\boldsymbol{\gamma}}(t)) \cdot \dddot{\boldsymbol{\gamma}}(t) = -8$$
$$\kappa(t) = \frac{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|}{\|\dot{\boldsymbol{\gamma}}\|^3} = \frac{\sqrt{32}}{8^{\frac{3}{2}}} = \frac{1}{4}$$
$$\tau(t) = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|^2} = \frac{-8}{32} = -\frac{1}{4}.$$
2. Equating $\kappa = \kappa^{\boldsymbol{\eta}}$ and $\tau = \tau^{\boldsymbol{\eta}}$, we obtain
$$\frac{R}{R^2 + H^2} = \frac{1}{4}, \qquad \frac{H}{R^2 + H^2} = -\frac{1}{4}$$
Rearranging both equalities we get
$$R^2 + H^2 = 4R, \qquad R^2 + H^2 = -4H,$$
from which we find the relation $R = -H$. Substituting into $R^2 + H^2 = -4H$, we get
$$H = -2, \quad R = -H = 2.$$
For these values of R and H we have $\kappa = \kappa^{\boldsymbol{\eta}}$ and $\tau = \tau^{\boldsymbol{\eta}}$. By the FTSC, there exists a rigid motion $M : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying (1.2).

Theorem 1.43: Curves contained in a plane - Part I

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be regular with $\kappa \neq 0$. They are equivalent:

1. The torsion of $\boldsymbol{\gamma}$ satisfies
$$\tau(t) = 0, \quad \forall\, t \in (a,b).$$
2. $\boldsymbol{\gamma}$ is contained in a plane: There exists a vector $\mathbf{P} \in \mathbb{R}^3$ and a scalar $d \in \mathbb{R}$ such that
$$\boldsymbol{\gamma}(t) \cdot \mathbf{P} = d, \quad \forall\, t \in (a,b).$$

Theorem 1.44: Curves contained in a plane - Part II

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be regular, with $\kappa \neq 0$ and $\tau = 0$. Then, the binormal $\mathbf{b}$ is a constant vector, and $\boldsymbol{\gamma}$ is contained in the plane of equation
$$(\mathbf{x} - \boldsymbol{\gamma}(t_0)) \cdot \mathbf{b} = 0.$$

Example 1.45: A planar curve

Question. Consider the curve
$$\boldsymbol{\gamma}(t) = (t, 2t, t^4), \quad t > 0.$$

1. Prove that $\boldsymbol{\gamma}$ is regular.
2. Compute the curvature and torsion of $\boldsymbol{\gamma}$.
3. Prove that $\boldsymbol{\gamma}$ is contained in a plane. Compute the equation of such plane.

Solution.

1. $\boldsymbol{\gamma}$ is regualar because $\dot{\boldsymbol{\gamma}}(t) = (1, 2, 4t^3) \neq \mathbf{0}$.
2. Compute the following quantities
$$\|\dot{\boldsymbol{\gamma}}\| = \sqrt{5 + 16t^4} \qquad \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} = 12\,(2t^2, -t^2, 0)$$
$$\ddot{\boldsymbol{\gamma}} = 12\,(0, 0, t^2) \qquad \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| = 12\sqrt{5}\,t^2$$
$$\dddot{\boldsymbol{\gamma}} = 24\,(0, 0, t) \qquad (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}} = 0$$
Compute curvature and torsion with the formulas
$$\kappa(t) = \frac{\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}}\|^3} = \frac{12\sqrt{5}\,t^2}{\sqrt{5 + 16t^4}}$$
$$\tau(t) = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|} = 0.$$
3. $\boldsymbol{\gamma}$ lies in a plane because $\tau = 0$. The binormal is
$$\mathbf{b} = \frac{\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}}{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|} = \frac{1}{\sqrt{5}}\,(2, -1, 0).$$
At $t_0 = 0$ we have $\boldsymbol{\gamma}\,(0) = \mathbf{0}$. The equation of the plane containing $\boldsymbol{\gamma}$ is then $\mathbf{x} \cdot \mathbf{b} = 0$, which reads
$$\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y = 0 \quad \Longrightarrow \quad 2x - y = 0.$$

Theorem 1.46: Curves contained in a circle

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be unit-speed. They are equivalent:

1. $\boldsymbol{\gamma}$ is contained in a circle of radius $R > 0$.
2. There exists $R > 0$ such that
$$\kappa(t) = \frac{1}{R}, \quad \tau(t) = 0, \quad \forall\, t \in (a,b).$$

Example 1.47: A curve contained in a circle

Question. Consider the curve

$$\boldsymbol{\gamma}(t) = \left(\frac{4}{5}\cos(t), 1 - \sin(t), -\frac{3}{5}\cos(t)\right).$$

1. Prove that $\boldsymbol{\gamma}$ is unit-speed.
2. Compute Frenet frame, curvature and torsion of $\boldsymbol{\gamma}$.
3. Prove that $\boldsymbol{\gamma}$ is part of a circle.

Solution.

1. $\boldsymbol{\gamma}$ is unit-speed because
$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= \left(-\frac{4}{5}\sin(t), -\cos(t), \frac{3}{5}\sin(t)\right) \\ \|\dot{\boldsymbol{\gamma}}(t)\|^2 &= \frac{16}{25}\sin^2(t) + \cos^2(t) + \frac{9}{25}\sin^2(t) = 1 \end{aligned}$$

2. As $\boldsymbol{\gamma}$ is unit-speed, the tangent vector is $\mathbf{t}(t) = \dot{\boldsymbol{\gamma}}(t)$. The curvature, normal, binormal and torsion are
$$\begin{aligned} \dot{\mathbf{t}}(t) &= \left(-\frac{4}{5}\cos(t), \sin(t), \frac{3}{5}\cos(t)\right) \\ \kappa(t) &= \|\dot{\mathbf{t}}(t)\| = \frac{16}{25}\cos^2(t) + \sin^2(t) + \frac{9}{25}\cos^2(t) = 1 \\ \mathbf{n}(t) &= \frac{1}{\kappa(t)}\ddot{\boldsymbol{\gamma}}(t) = \left(-\frac{4}{5}\cos(t), \sin(t), \frac{3}{5}\cos(t)\right) \\ \mathbf{b}(t) &= \dot{\boldsymbol{\gamma}}(t) \times \mathbf{n}(t) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right) \\ \dot{\mathbf{b}} &= \mathbf{0} \\ \tau &= -\dot{\mathbf{b}} \cdot \mathbf{n} = 0 \end{aligned}$$

3. The curvature of $\boldsymbol{\gamma}$ is constant and the torsion is zero. Therefore $\boldsymbol{\gamma}$ is contained in a circle of radius
$$R = \frac{1}{\kappa} = 1.$$

2 Topology

Definition 2.1: Topological space

Let X be a set and $\mathcal{T}$ a collection of subsets of X. We say that $\mathcal{T}$ is a **topology** on X if the following 3 properties hold:

- (A1) The sets $\emptyset, X$ belong to $\mathcal{T}$,
- (A2) If $\{A_i\}_{i\in I}$ is an arbitrary family of elements of $\mathcal{T}$, then

$$\bigcup_{i\in I} A_i \in \mathcal{T} .$$

- (A3) If $A, B \in \mathcal{T}$ then $A \cap B \in \mathcal{T}$.

Further, we say:

- The pair $(X, \mathcal{T})$ is a **topological space**.
- The elements of X are called **points**.
- The sets in the topology $\mathcal{T}$ are called **open sets**.

Definition 2.2: Trivial topology

Let X be a set. The **trivial topology** on X is the collection of sets

$$\mathcal{T}_{\text{trivial}} := \{\emptyset, X\} .$$

Definition 2.3: Discrete topology

Let X be a set. The **discrete topology** on X is the collection of all subsets of X

$$\mathcal{T}_{\text{discrete}} := \{A : \ A \subseteq X\} .$$

Definition 2.4: Open set of $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$. We say that the set A is **open** if it holds:

$$\forall\, \mathbf{x} \in A \,, \ \exists\, r > 0 \ \text{ s.t. } \ B_r(\mathbf{x}) \subseteq A \,, \qquad (2.1)$$

where $B_r(\mathbf{x})$ is the ball of radius $r > 0$ centered at $\mathbf{x}$

$$B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \ \|\mathbf{y} - \mathbf{x}\| < r\} ,$$

and the **Euclidean norm** of $\mathbf{x} \in \mathbb{R}^n$ is defined by

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^n x_i^2} .$$

Definition 2.5: Euclidean topology of $\mathbb{R}^n$

The **Euclidean topology** on $\mathbb{R}^n$ is the collection of sets

$$\mathcal{T}_{\text{euclid}} := \{A : \ A \subseteq \mathbb{R}^n \,, \ A \ \text{ is open}\} .$$

Proof: $\mathcal{T}_{\text{euclid}}$ is a topology on $\mathbb{R}^n$

To prove $\mathcal{T}_{\text{euclid}}$ is a topology on $\mathbb{R}^n$, we need to check the axioms:

- (A1) We have $\emptyset, \mathbb{R}^n \in \mathcal{T}_{\text{euclid}}$: Indeed $\emptyset$ is open because there is no point $\mathbf{x}$ for which (2.1) needs to be checked. Moreover, $\mathbb{R}^n$ is open because (2.1) holds with any radius $r > 0$.
- (A2) Let $A_i \in \mathcal{T}_{\text{euclid}}$ for all $i \in I$. Define the union $A = \bigcup_i A_i$. We need to check that A is open. Let $\mathbf{x} \in A$. By definition of union, there exists an index $i_0 \in I$ such that $\mathbf{x} \in A_{i_0}$. Since A_{i_0} is open, by (2.1) there exists $r > 0$ such that $B_r(\mathbf{x}) \subseteq A_{i_0}$. As $A_{i_0} \subseteq A$, we conclude that $B_r(\mathbf{x}) \subseteq A$, so that $A \in \mathcal{T}_{\text{euclid}}$.
- (A3) Let $A, B \in \mathcal{T}_{\text{euclid}}$. We need to check that $A \cap B$ is open. Let $\mathbf{x} \in A \cap B$. Therefore $\mathbf{x} \in A$ and $\mathbf{x} \in B$. Since A and B are open, by (2.1) there exist $r_1, r_2 > 0$ such that $B_{r_1}(\mathbf{x}) \subseteq A$ and $B_{r_2}(\mathbf{x}) \subseteq B$. Set $r := \min\{r_1, r_2\}$. Then

$$B_r(\mathbf{x}) \subseteq B_{r_1}(\mathbf{x}) \subseteq A \,, \quad B_r(\mathbf{x}) \subseteq B_{r_2}(\mathbf{x}) \subseteq B \,,$$

Hence $B_r(\mathbf{x}) \subseteq A \cap B$, showing that $A \cap B \in \mathcal{T}_{\text{euclid}}$.

This proves that $\mathcal{T}_{\text{euclid}}$ is a topology on $\mathbb{R}^n$.

Proposition 2.6: $B_r(\mathbf{x})$ is an open set of $\mathcal{T}_{\text{euclid}}$

Let $\mathbb{R}^n$ be equipped with the Euclidean topology $\mathcal{T}_{\text{euclid}}$. Let $r > 0$ and $\mathbf{x} \in \mathbb{R}^n$. Then $B_r(\mathbf{x}) \in \mathcal{T}_{\text{euclid}}$.

Proof

To prove $B_r(\mathbf{x}) \in \mathcal{T}_{\text{euclid}}$, we need to show that $B_r(\mathbf{x})$ satisfies (2.1). Therefore, let $\mathbf{y} \in B_r(\mathbf{x})$. In particular

$$\|\mathbf{x} - \mathbf{y}\| < r \,. \qquad (2.2)$$

Define $\varepsilon := r - \|\mathbf{x} - \mathbf{y}\|$. Note that $\varepsilon > 0$ by (2.2). We claim that

$$B_\varepsilon(\mathbf{y}) \subseteq B_r(\mathbf{x}) \,. \qquad (2.3)$$

Indeed, let $\mathbf{z} \in B_\varepsilon(\mathbf{y})$. By triangle inequality we have

$$\|\mathbf{z} - \mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| < \|\mathbf{x} - \mathbf{y}\| + \varepsilon = r \,,$$

where we used that $\|\mathbf{y} - \mathbf{z}\| < \varepsilon$ and the definition of ε. Hence $\mathbf{z} \in B_r(\mathbf{x})$, proving (2.3). Thus, $B_r(\mathbf{x})$ satisfies (2.1), ending the proof.

Definition 2.7: Closed set

Let $(X, \mathcal{T})$ be a topological space. A set $C \subseteq X$ is **closed** if

$$C^c \in \mathcal{T} \,,$$

where $C^c := X \setminus C$ is the complement of C in X.

Proposition 2.8

Let $(X, \mathcal{T})$ be a topological space. Properties (A1)-(A2)-(A3) of $\mathcal{T}$ are equivalent to (C1)-(C2)-(C3), where

- (C1) $\emptyset, X$ are closed.
- (C2) If C_i is closed for all $i \in I$, then $\bigcap_{i \in I} C_i$ is closed.
- (C3) If C_1, C_2 are closed then $C_1 \cup C_2$ is closed.

Definition 2.9: Comparing topologies

Let X be a set and let $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on X.

1. $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.
2. $\mathcal{T}_1$ is **strictly finer** than $\mathcal{T}_2$ if $\mathcal{T}_2 \subsetneq \mathcal{T}_1$.
3. $\mathcal{T}_1$ and $\mathcal{T}_2$ are the **same** topology if $\mathcal{T}_1 = \mathcal{T}_2$.

Example 2.10: Comparing $\mathcal{T}_{\text{trivial}}$ and $\mathcal{T}_{\text{discrete}}$

Let X be a set. Then $\mathcal{T}_{\text{trivial}} \subseteq \mathcal{T}_{\text{discrete}}$.

Example 2.11: Cofinite topology on $\mathbb{R}$

Question. The **cofinite topology** on $\mathbb{R}$ is the collection of sets

$$\mathcal{T}_{\text{cofinite}} := \{U \subseteq \mathbb{R} \, : \, U^c \text{ is finite, or } U^c = \mathbb{R}\}.$$

1. Prove that $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$ is a topological space.
2. Prove that $\mathcal{T}_{\text{cofinite}} \subseteq \mathcal{T}_{\text{euclid}}$.
3. Prove that $\mathcal{T}_{\text{cofinite}} \neq \mathcal{T}_{\text{euclid}}$.

Solution. Part 1. Show that the topology properties are satisfied:
(A1) We have $\emptyset \in \mathcal{T}_{\text{cofinite}}$, since $\emptyset^c = \mathbb{R}$. We have $\mathbb{R} \in \mathcal{T}_{\text{cofinite}}$ because $\mathbb{R}^c = \emptyset$ is finite.
(A2) Let $U_i \in \mathcal{T}_{\text{cofinite}}$ for all $i \in I$, and define $U := \bigcup_{i \in I} U_i$. By the De Morgan's laws we have

$$U^c = \left(\cup_{i \in I} U_i\right)^c = \cap_{i \in I} U_i^c .$$

We have two cases:

1. There exists $i_0 \in I$ such that $U_{i_0}^c$ is finite. Then

   $$U^c = \cap_{i \in I} U_i^c \subset U_{i_0}^c ,$$

   and therefore U^c is finite, showing that $U \in \mathcal{T}_{\text{cofinite}}$.

2. None of the sets U_i^c is finite. Therefore $U_i^c = \mathbb{R}$ for all $i \in I$, from which we deduce

   $$U^c = \cap_{i \in I} U_i^c = \mathbb{R} \quad \Longrightarrow \quad U \in \mathcal{T}_{\text{cofinite}} .$$

In both cases, we have $U \in \mathcal{T}_{\text{cofinite}}$, so that (A2) holds.
(A3) Let $U, V \in \mathcal{T}_{\text{cofinite}}$. Set $A = U \cap V$. Then

$$A^c = U^c \cup V^c .$$

We have 2 possibilities:

1. U^c, V^c finite: Then A^c is finite, and $A \in \mathcal{T}_{\text{cofinite}}$.
2. $U^c = \mathbb{R}$ or $V^c = \mathbb{R}$: Then $A^c = \mathbb{R}$, and $A \in \mathcal{T}_{\text{cofinite}}$.

In all cases, we have shown that $A \in \mathcal{T}_{\text{cofinite}}$, so that (A3) holds.
Part 2. Let $U \in \mathcal{T}_{\text{cofinite}}$. We have two cases:

- U^c is finite. Then $U^c = \{x_1, \dots, x_n\}$ for some points $x_i \in \mathbb{R}$. Up to relabeling the points, we can assume that $x_i < x_j$ when $i < j$. Therefore,

  $$U = \{x_1, \dots, x_n\}^c = \bigcup_{i=0}^{n} (x_i, x_{i+1}) , \quad x_0 := -\infty, \quad x_{n+1} := \infty .$$

  The sets (x_i, x_{i+1}) are open in $\mathcal{T}_{\text{euclid}}$, and therefore $U \in \mathcal{T}_{\text{euclid}}$.

- $U^c = \mathbb{R}$. Then $U = \emptyset$, which belongs to $\mathcal{T}_{\text{euclid}}$ by (A1).

In both cases, $U \in \mathcal{T}_{\text{euclid}}$. Therefore $\mathcal{T}_{\text{cofinite}} \subseteq \mathcal{T}_{\text{euclid}}$.
Part 3. consider the interval $U = (0, 1)$. Then $U \in \mathcal{T}_{\text{euclid}}$. However U^c is neither $\mathbb{R}$, nor finite. Thus $U \notin \mathcal{T}_{\text{cofinite}}$.

Definition 2.12: Convergent sequence

Let $(X, \mathcal{T})$ be a topological space. Consider a sequence $\{x_n\} \subseteq X$ and a point $x \in X$. We say that x_n converges to x_0 in the topology $\mathcal{T}$, if the following property holds:

$$\begin{aligned} &\forall\, U \in \mathcal{T} \text{ s.t. } x_0 \in U ,\ \exists\, N = N(U) \in \mathbb{N} \text{ s.t.} \\ &x_n \in U ,\ \forall\, n \geq N . \end{aligned} \tag{2.4}$$

The convergence of x_n to x_0 is denoted by $x_n \to x_0$.

Proposition 2.13: Convergent sequences in $\mathcal{T}_{\text{trivial}}$

Let X be equipped with $\mathcal{T}_{\text{trivial}}$. Let $\{x_n\} \subseteq X$, $x_0 \in X$. Then $x_n \to x_0$.

Proof

To show that $x_n \to x_0$ we need to check that (2.4) holds. Let $U \in \mathcal{T}_{\text{trivial}}$ with $x_0 \in U$. We have two cases:

- $U = \emptyset$: There is nothing to prove, since x_0 cannot be in U.
- $U = X$: Take $N = 1$. Since $U = X$, we have $x_n \in U$ for all $n \geq 1$.

Thus (2.4) holds for all the sets $U \in \mathcal{T}_{\text{trivial}}$, showing that $x_n \to x_0$.

Warning

Proposition 2.13 shows the topological limit may **not be unique**!

Proposition 2.14: Convergent sequences in $\mathcal{T}_{\text{discrete}}$

Let X be equipped with $\mathcal{T}_{\text{discrete}}$. Let $\{x_n\} \subseteq X$, $x_0 \in X$. They are equivalent:

1. $x_n \to x_0$ in the topology $\mathcal{T}_{\text{discrete}}$.
2. $\{x_n\}$ is eventually constant: $\exists\, N \in \mathbb{N}$ s.t. $x_n = x_0$, $\forall\, n \geq N$

Proof

Part 1. Assume that $x_n \to x_0$. Let $U = \{x_0\}$. Then $U \in \mathcal{T}_{\text{discrete}}$. Since $x_n \to x_0$, by (2.4) there exists $N \in \mathbb{N}$ such that

$$x_n \in U , \quad \forall\, n \geq N .$$

As $U = \{x_0\}$, we infer $x_n = x_0$ for all $n \geq N$. Hence x_n is eventually constant.

Part 2. Assume that x_n is eventually equal to x_0, that is, there exists $N \in \mathbb{N}$ such that

$$x_n = x_0 \,, \quad \forall\, n \geq N \,. \tag{2.5}$$

Let $U \in \mathcal{T}$ be an open set such that $x_0 \in U$. By (2.5) we have that

$$x_n \in U \,, \quad \forall\, n \geq N \,.$$

Since U was arbitrary, we conclude that $x_n \to x_0$.

Definition 2.15: Classical convergence in $\mathbb{R}^n$

Let $\{\mathbf{x}_n\} \subseteq \mathbb{R}^n$ and $\mathbf{x}_0 \in \mathbb{R}^n$. We say that $\mathbf{x}_n$ converges $\mathbf{x}_0$ in the classical sense if $\|\mathbf{x}_n - \mathbf{x}_0\| \to 0$, that is,

$$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N}, \quad \text{s.t.} \quad \|\mathbf{x}_n - \mathbf{x}_0\| < \varepsilon \,, \forall\, n \geq N \,.$$

Proposition 2.16: Convergent sequences in $\mathcal{T}_{\text{euclid}}$

Let $\mathbb{R}^n$ be equipped with $\mathcal{T}_{\text{euclid}}$. Let $\{x_n\} \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$. They are equivalent:

1. $\mathbf{x}_n \to \mathbf{x}_0$ in the topology $\mathcal{T}_{\text{euclid}}$.
2. $\mathbf{x}_n \to \mathbf{x}_0$ in the classical sense.

Proof

Part 1. Assume $\mathbf{x}_n \to \mathbf{x}_0$ with respect to $\mathcal{T}_{\text{euclid}}$. Fix $\varepsilon > 0$ and define $U := B_\varepsilon(\mathbf{x}_0)$. By Proposition 2.6, we have $U \in \mathcal{T}$. Moreover $\mathbf{x}_0 \in U$. As $\mathbf{x}_n \to \mathbf{x}_0$ with respect to $\mathcal{T}_{\text{euclid}}$, there exists $N \in \mathbb{N}$ such that

$$\mathbf{x}_n \in U \,, \quad \forall\, n \geq N \,.$$

As $U = B_\varepsilon(\mathbf{x}_0)$, the above reads

$$\|\mathbf{x}_n - \mathbf{x}_0\| < \varepsilon \,, \quad \forall\, n \geq N \,,$$

showing that $\mathbf{x}_n \to \mathbf{x}_0$ in the classical sense.

Part 2. Assume $\mathbf{x}_n \to \mathbf{x}_0$ in the classical sense. Let $U \in \mathcal{T}_{\text{euclid}}$ be an arbitrary set such that $\mathbf{x}_0 \in U$. By definition of Euclidean topology, this means that there exists $r > 0$ such that $B_r(\mathbf{x}_0) \subseteq U$. As $\mathbf{x}_n \to \mathbf{x}_0$ in the classical sense, there exists $N \in \mathbb{N}$ such that

$$\|\mathbf{x}_n - \mathbf{x}_0\| < r \,, \quad \forall\, n \geq N \,.$$

By definition of $B_r(\mathbf{x}_0)$, and since $B_r(\mathbf{x}_0) \subseteq U$, the above is equivalent to

$$\mathbf{x}_n \in B_r(\mathbf{x}_0) \subseteq U \,, \quad \forall\, n \geq N \,.$$

As U is arbitrary, we infer $\mathbf{x}_n \to \mathbf{x}_0$ with respect to $\mathcal{T}_{\text{euclid}}$.

Proposition 2.17

Let X be a set and $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. Suppose that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Let $\{x_n\} \subset X$ and $x_0 \in X$. We have

$$x_n \to x_0 \;\text{ in }\; \mathcal{T}_1 \quad \Longrightarrow \quad x_n \to x_0 \;\text{ in }\; \mathcal{T}_2 \,.$$

2.1 Metric spaces

Definition 2.18: Distance and Metric space

Let X be a set. A **distance** on X is a function $d \colon X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$ they hold:

- (M1) Positivity: $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$
- (M2) Symmetry: $d(x,y) = d(y,x)$
- (M3) Triangle Inequality: $d(x,z) \leq d(x,y) + d(y,z)$

The pair (X,d) is called a **metric space**.

Definition 2.19: Euclidean distance on $\mathbb{R}^n$

The **Euclidean distance** over $\mathbb{R}^n$ is defined by

$$d(\mathbf{x},\mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \,, \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \,.$$

Proposition 2.20

Let d be the Euclidean distance on $\mathbb{R}^n$. Then $(\mathbb{R}^n, d)$ is a metric space.

Definition 2.21: Topology induced by the metric

Let (X,d) be a metric space. The set $A \subseteq X$ is **open** if it holds

$$\forall\, x \in U \,, \exists\, r \in \mathbb{R},\, r > 0 \quad \text{s.t.} \quad B_r(x) \subseteq U \,,$$

where $B_r(x)$ is the ball centered at x of radius r, defined by

$$B_r(x) = \{ y \in X \,:\, d(x,y) < r \} \,.$$

The topology **induced by the metric** d is the collection of sets

$$\mathcal{T}_d = \{ U \,:\, U \subseteq X,\, U \text{ open} \} \,.$$

Remark 2.22: Topology induced by Euclidean distance

Consider the metric space $(\mathbb{R}^n, d)$ with d the Euclidean distance. Then

$$\mathcal{T}_d = \mathcal{T}_{\text{euclid}} \,,$$

where $\mathcal{T}_{\text{euclid}}$ is the Euclidean topology on $\mathbb{R}^n$.

Example 2.23: Discrete distance

Question. Let X be a set. The **discrete distance** is the function $d \colon X \times X \to \mathbb{R}$ defined by

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

1. Prove that (X,d) is a metric space.
2. Prove that $\mathcal{T}_d = \mathcal{T}_{\text{discrete}}$.

Solution. See Question 3 in Homework 3.

Proposition 2.24

Let (X, d) be a metric space, $\mathcal{T}_d$ the topology induced by d. Then:

- For all $x \in X, r > 0$ we have $B_r(x) \in \mathcal{T}_d$.
- $U \in \mathcal{T}_d$ if and only if $\exists I$ family of indices s.t.

$$U = \bigcup_{i \in I} B_{r_i}(x_i) \,, \quad x_i \in X, \; r_i > 0 \,.$$

Proposition 2.25: Convergence in metric space

Suppose (X, d) is a metric space and $\mathcal{T}_d$ the topology induced by d. Let $\{x_n\} \subseteq X$ and $x_0 \in X$. They are equivalent:

1. $x_n \to x_0$ with respect to the topology $\mathcal{T}_d$.
2. $d(x_n, x_0) \to 0$ in $\mathbb{R}$.
3. For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$x_n \in B_r(x_0) \,, \; \forall\, n \geq \mathbb{N} \,.$$

2.2 Interior, closure and boundary

Definition 2.26: Interior of a set

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. The **interior** of A is

$$\operatorname{Int} A := \bigcup_{\substack{U \subseteq A \\ U \in \mathcal{T}}} U \,.$$

Proposition 2.27: $\operatorname{Int} A$ is the largest open set contained in A

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. Then $\operatorname{Int} A$ is the largest open set contained in A, that is:

1. $\operatorname{Int} A$ is open.
2. $\operatorname{Int} A \subseteq A$.
3. If $V \in \mathcal{T}$ and $V \subseteq A$, then $V \subseteq \operatorname{Int} A$.
4. A is open if and only if $A = \operatorname{Int} A$.

Definition 2.28: Closure of a set

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. The **closure** of A is

$$\overline{A} := \bigcap_{\substack{A \subseteq C \\ C \text{ closed}}} C \,.$$

Proposition 2.29: $\overline{A}$ is the smallest closed set containing A

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. Then $\overline{A}$ is the smallest closed set containing A, that is:

1. $\overline{A}$ is closed.
2. $A \subseteq \overline{A}$.
3. If V is closed $A \subseteq V$, then $\overline{A} \subseteq V$.
4. A is closed if and only if $A = \overline{A}$.

Lemma 2.30

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. They are equivalent:

1. $x_0 \in \overline{A}$.
2. For every $U \in \mathcal{T}$ such that $x_0 \in U$, it holds

$$U \cap A \neq \emptyset \,.$$

Definition 2.31: Boundary of a set

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. The **boundary** of A is

$$\partial A := \overline{A} \setminus \operatorname{Int} A \,.$$

Proposition 2.32

Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$. Then ∂A is closed.

Definition 2.33: Set of limit points

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. The set of **limit points** of A is defined as

$$L(A) := \{ x \in X \,:\, \exists \{x_n\} \subseteq A \text{ s.t. } x_n \to x \} \,.$$

Proposition 2.34

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$ a set. Let $\{x_n\} \subseteq A$ and $x_0 \in X$ be such that $x_n \to x_0$. Then $x_0 \in \overline{A}$. In particular,

$$L(A) \subseteq \overline{A} \,.$$

Proof

Suppose by contradiction $x_0 \notin \overline{A}$, so that

$$x_0 \in (\overline{A})^c \,.$$

Since $(\overline{A})^c$ is open and $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that

$$x_n \in (\overline{A})^c \,, \quad \forall\, n \geq N \,.$$

This is a contradiction, since we were assuming that $\{x_n\} \subseteq A$. This shows $x_0 \in \overline{A}$ and therefore $L(A) \subseteq \overline{A}$.

Warning

1. The converse of Proposition 2.34 is false in general, that is,

$$\overline{A} \not\subseteq L(A) \,.$$

   We show a counterexample in Example 2.35.

2. The relation

$$\overline{A} = L(A) \,.$$

   holds in the so-called first countable topological spaces, such as metric spaces, see Proposition 2.36 below.

Example 2.35: Co-countable topology on $\mathbb{R}$

Question. The **co-countable** topology on $\mathbb{R}$ is the collection of sets

$$\mathcal{T}_{\text{cc}} := \{A \subseteq \mathbb{R} \, : \, A^c = \mathbb{R} \ \text{ or } \ A^c \ \text{ countable} \, \}.$$

1. Prove that $\mathcal{T}_{\text{cc}}$ is a topology on $\mathbb{R}$.
2. Prove that a sequence $\{x_n\}$ is convergent in $\mathcal{T}_{\text{cc}}$ if and only if it is eventually constant.
3. Define the set $A = (-\infty, 0]$. Prove that $\overline{A} = \mathbb{R}$.
4. Conclude that $\overline{A} \not\subset L(A)$.

Solution.

1. See Question 2 in Homework 3.
2. See Question 2 in Homework 3.
3. Assume C is a closed set such that $A \subseteq C$. Since C is closed, it follows that $C^c \in \mathcal{T}_{\text{cc}}$. Therefore $(C^c)^c = C$ is either countable, or equal to $\mathbb{R}$. As $A \subseteq C$, we have that C is uncountable. Therefore, $C = \mathbb{R}$. As C is an arbitrary closed set containing A, we conclude that
$$\overline{A} = \bigcap_{\substack{A \subseteq C \\ C \text{ closed}}} C = \bigcap_{\substack{A \subseteq C \\ C \text{ closed}}} \mathbb{R} = \mathbb{R} \, .$$
4. By Point 2, convergent sequences are eventually constant. Therefore, if $\{x_n\} \subseteq A$ converges to x_0, we conclude that $x_0 \in A$. This shows
$$L(A) = A = [-\infty, 0) \, .$$
By Point 3, we have $\overline{A} = \mathbb{R}$. We conclude that $\overline{A} \not\subset L(A)$.

Proposition 2.36: Characterization of $\operatorname{Int} A$ and $\overline{A}$ in metric space

Let (X, d) be a metric space. Denote by $\mathcal{T}_d$ the topology induced by d. For any $A \subseteq X$, we have

1. $\operatorname{Int} A = \{x \in A \, : \, \exists \, r > 0 \text{ s.t. } B_r(x) \subseteq A\}\,,$
2. $\overline{A} = L(A) = \{x \in X \text{ s.t. } \exists \, \{x_n\} \subseteq A \text{ s.t. } x_n \to x\}.$

Proof

1. See Question 4 in Homework 3.
2. The inclusion $L(A) \subseteq \overline{A}$ holds by Proposition 2.34. We are left to show that
$$\overline{A} \subseteq L(A) \, .$$
To this end, let $x_0 \in \overline{A}$. For $n \in \mathbb{N}$, consider the ball $B_{1/n}(x_0)$. Since $B_{1/n}(x_0) \in \mathcal{T}_d$ and $x_0 \in B_\varepsilon(x_0)$, we can apply Lemma 2.30 and deduce that
$$B_{1/n}(x_0) \cap A \neq \emptyset \, .$$
Let $x_n \in B_{1/n}(x_0) \cap A$. Since n was arbitrary, we have constructed a sequence $\{x_n\} \subseteq A$ such that
$$x_n \in B_{1/n}(x_0) \, , \quad \forall \, n \in \mathbb{N} \, .$$
In particular, we have that
$$0 \leq d(x_n, x_0) < \frac{1}{n} \to 0$$
as $n \to \infty$. Thus $x_n \to x_0$, showing that $x_0 \in L(A)$.

Example 2.37

Question. Consider $\mathbb{R}$ equipped with the Euclidean topology. Let $A = [0, 1)$. Prove that:

$$\operatorname{Int} A = (0,1) \, , \quad \overline{A} = [0,1] \, , \quad \partial A = \{0, 1\} \, .$$

Note: In particular, this shows

$$\operatorname{Int} A \neq A \, , \quad \overline{A} \neq A \, ,$$

so that A is neither open, nor closed.
Solution. See Question 5 in Homework 3.

Definition 2.38: Density

Let $(X, \mathcal{T})$ be a topological space. We say that a subset $A \subseteq X$ is **dense** in X, if

$$A \cap U \neq \emptyset \, , \quad \forall \, U \in \mathcal{T} \, , \, U \neq \emptyset \, .$$

Proposition 2.39: Characterization of density

Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$. They are equivalent:

1. A is dense in X.
2. It holds $\overline{A} = X$.

Example 2.40: $\mathbb{Z}$ is not dense in $(\mathbb{R}, \mathcal{T}_{\text{euclid}})$

Question. Consider $\mathbb{R}$ equipped with the Euclidean topology $\mathcal{T}_{\text{euclid}}$. Prove that the set of integers $\mathbb{Z}$ is not dense in $\mathbb{R}$, with

$$\overline{\mathbb{Z}} = \mathbb{Z} \, .$$

Solution. The set of integers $\mathbb{Z}$ satisfies

$$\mathbb{Z}^c = \bigcup_{z \in \mathbb{Z}} (z, z+1) \, .$$

Since $(z, z+1)$ is open in $\mathbb{R}$, by (A2) we conclude that $\mathbb{Z}^c$ is open, so that $\mathbb{Z}$ is closed. Therefore

$$\overline{\mathbb{Z}} = \mathbb{Z} \, .$$

As $\overline{\mathbb{Z}} \neq \mathbb{R}$, by Proposition 2.39, we have that $\mathbb{Z}$ is not dense in $\mathbb{R}$.

Example 2.41: $\mathbb{Z}$ is dense in $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$

Question. Consider $\mathbb{R}$ equipped with the cofinite topology

$$\mathcal{T}_{\text{cofinite}} = \{U \subset \mathbb{R} \, : \, U^c \ \text{ is finite, } \ \text{or } U^c = \mathbb{R}\} \, .$$

Prove that $\mathbb{Z}$ is dense in $\mathbb{R}$. In particular,

$$\overline{\mathbb{Z}} = \mathbb{R} \, ,$$

Solution. Suppose C is a closed set such that $\mathbb{Z} \subseteq C$. By definition of $\mathcal{T}_{\text{cofinite}}$, we have that $(C^c)^c = C$ is either finite, or it coincides with $\mathbb{R}$. Since $\mathbb{Z} \subseteq C$, and $\mathbb{Z}$ is not finite, we conclude $C = \mathbb{R}$. As C is an arbitrary closed set containing $\mathbb{Z}$, we conclude that

$$\overline{\mathbb{Z}} = \bigcap_{\substack{\mathbb{Z}\subseteq C \\ C \text{ closed}}} C = \bigcap_{\substack{\mathbb{Z}\subseteq C \\ C \text{ closed}}} \mathbb{R} = \mathbb{R}.$$

In particular, by Proposition 2.39, $\mathbb{Z}$ is dense in $\mathbb{R}$.

2.3 Hausdorff spaces

Definition 2.42: Hausdorff space

We say that a topological space $(X, \mathcal{T})$ is **Hausdorff** if for every $x, y \in X$ with $x \neq y$, there exist $U, V \in \mathcal{T}$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Proposition 2.43

Let (X, d) be a metric space, $\mathcal{T}_d$ the topology induced by d. Then $(X, \mathcal{T}_d)$ is a Hausdorff space.

Proof

Let $x, y \in X$ with $x \neq y$. Define

$$U := B_\varepsilon(x), \quad V := B_\varepsilon(y), \quad \varepsilon := \frac{1}{2} d(x,y).$$

By Proposition 2.24, we know that $U, V \in \mathcal{T}_d$. Moreover $x \in U$, $y \in V$. We are left to show that $U \cap V = \emptyset$. Suppose by contradiction that $U \cap V \neq \emptyset$ and let $z \in U \cap V$. Therefore

$$d(x,z) < \varepsilon, \quad d(y,z) < \varepsilon.$$

By triangle inequality we have

$$d(x,y) \leq d(x,z) + d(y,z) < \varepsilon + \varepsilon = d(x,y),$$

where in the last inequality we used the definition of ε. This is a contradiction. Therefore $U \cap V = \emptyset$ and $(X, \mathcal{T}_d)$ is Hausdorff.

Definition 2.44: Metrizable space

Let $(X, \mathcal{T})$ be a topological space. We say that the topology $\mathcal{T}$ is **metrizable** if there exists a metric d on X such that

$$\mathcal{T} = \mathcal{T}_d,$$

with $\mathcal{T}_d$ the topology induced by d.

Corollary 2.45

Let $(X, \mathcal{T})$ be a metrizable space. Then X is Hausforff.

Example 2.46: $(X, \mathcal{T}_{\text{trivial}})$ is not Hausdorff

Question. Let X be equipped with the trivial topology $\mathcal{T}_{\text{trivial}}$. Then X is not Hausdorff.
Solution. Assume by contradiction $(X, \mathcal{T}_{\text{trivial}})$ is Hausdorff and let $x, y \in X$ with $x \neq y$. Then, there exist $U, V \in \mathcal{T}_{\text{trivial}}$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

In particular $U \neq \emptyset$ and $V \neq \emptyset$. Since $\mathcal{T} = \{\emptyset, X\}$, we conclude that

$$U = V = X \quad \Longrightarrow \quad U \cap V = X \neq \emptyset.$$

This is a contradiction, and thus $(X, \mathcal{T}_{\text{trivial}})$ is not Hausdorff.

Example 2.47: $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$ is not Hausdorff

Question. Consider the cofinite topology on $\mathbb{R}$

$$\mathcal{T}_{\text{cofinite}} = \{U \subseteq \mathbb{R} : U^c \text{ is finite, or } U^c = \mathbb{R}\}.$$

Prove that $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$ is not Hausdorff.
Solution. Assume by contradiction $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$ is Hausdorff and let $x, y \in \mathbb{R}$ with $x \neq y$. Then, there exist $U, V \in \mathcal{T}_{\text{cofinite}}$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Taking the complement of $U \cap V = \emptyset$, we infer

$$\mathbb{R} = (U \cap V)^c = U^c \cup V^c. \tag{2.6}$$

There are two possibilities:

1. U^c and V^c are finite. Then $U^c \cup V^c$ is finite, so that (2.6) is a contradiction.

2. Either $U^c = \mathbb{R}$ or $U^c = \mathbb{R}$. If $U^c = \mathbb{R}$, then $U = \emptyset$. This is a contradiction, since $x \in U$. If $V^c = \mathbb{R}$, then $V = \emptyset$. This is a contradiction, since $y \in V$.

Hence $(\mathbb{R}, \mathcal{T}_{\text{cofinite}})$ is not Hausdorff.

Example 2.48: Lower-limit topology on $\mathbb{R}$ is not Hausdorff

Question. The **lower-limit topology** on $\mathbb{R}$ is the collection of sets

$$\mathcal{T}_{\text{LL}} = \{\emptyset, \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\}.$$

1. Prove that $(\mathbb{R}, \mathcal{T}_{\text{LL}})$ is a topological space.
2. Prove that $(\mathbb{R}, \mathcal{T}_{\text{LL}})$ is not Hausdorff.

Solution. Part 1. We show that $(\mathbb{R}, \mathcal{T}_{\text{LL}})$ is a topological space by verifying the axioms:
(A1) By definition $\emptyset, \mathbb{R} \in \mathcal{T}_{\text{LL}}$.
(A2) Let $A_i \in \mathcal{T}_{\text{LL}}$ for all $i \in I$. We have 2 cases:

- If $A_i = \emptyset$ for all i, then $\cup_i A_i = \emptyset \in \mathcal{T}_{\text{LL}}$.

- At least one of the sets A_i is non-empty. As empty-sets do not contribute to the union, we can discard them. Therefore, $A_i = (-\infty, a_i)$ with $a_i \in \mathbb{R} \cup \{\infty\}$. Define:

$$a := \sup_{i \in I} a_i, \quad A := (-\infty, a).$$

Then $A \in \mathcal{T}$ and:

$$A = \cup_{i \in I} A_i.$$

To prove this, let $x \in A$. Then $x < a$, so there exists $i_0 \in I$ such that $x < a_{i_0}$. Thus, $x \in A_{i_0}$, showing $A \subseteq \cup_{i \in I} A_i$. Conversely, if $x \in \cup_{i \in I} A_i$, then $x \in A_{i_0}$ for some $i_0 \in I$, implying $x < a_{i_0} \leq a$. Thus, $x \in A$, proving $\cup_{i \in I} A_i \subseteq A$.

(A3) Let $A, B \in \mathcal{T}_{\mathrm{LL}}$. We have 3 cases:

- $A = \emptyset$ or $B = \emptyset$. Then $A \cap B = \emptyset \in \mathcal{T}_{\mathrm{LL}}$.
- $A \neq \emptyset$ and $B \neq \emptyset$. Therefore, $A = (-\infty, a)$ and $B = (-\infty, b)$ with $a, b \in \mathbb{R} \cup \{\infty\}$. Define
$$U := A \cap B, \quad z := \min\{a, b\}.$$
Then $U = (-\infty, z) \in \mathcal{T}_{\mathrm{LL}}$.

Thus, $(\mathbb{R}, \mathcal{T}_{\mathrm{LL}})$ is a topological space.
Part 2. To show $(\mathbb{R}, \mathcal{T}_{\mathrm{LL}})$ is not Hausdorff, assume otherwise. Let $x, y \in \mathbb{R}$ with $x \neq y$. Then there exist $U, V \in \mathcal{T}_{\mathrm{LL}}$ such that:

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

As U, V are non-empty, by definition of $\mathcal{T}_{\mathrm{LL}}$, there exist $a, b \in \mathbb{R} \cup \{\infty\}$ such that $U = (-\infty, a)$ and $V = (-\infty, b)$. Define:

$$z := \min\{a, b\}, \quad Z := U \cap V = (-\infty, z).$$

Hence $Z \neq \emptyset$, contradicting $U \cap V = \emptyset$. Thus, $(\mathbb{R}, \mathcal{T}_{\mathrm{LL}})$ is not Hausdorff.

Proposition 2.49: Uniqueness of limit in Hausdorff spaces

Let $(X, \mathcal{T})$ be a Hausdorff space. If a sequence $\{x_n\} \subseteq X$ converges, then the limit is unique.

Proof

Let $\{x_n\} \subseteq X$ be convergent, and suppose by contradiction that

$$x_n \to x_0, \quad x_n \to y_0, \quad x_0 \neq y_0.$$

Since X is Hausdorff, there exist $U, V \in \mathcal{T}$ such that

$$x_0 \in U, \quad y_0 \in V, \quad U \cap V = \emptyset.$$

As $x_n \to x_0$ and $U \in \mathcal{T}$ with $x_0 \in U$, there exists $N_1 \in \mathbb{N}$ such that

$$x_n \in U, \quad \forall n \geq N_1.$$

Similarly, since $x_n \to y_0$ and $V \in \mathcal{T}$ with $y_0 \in U$, there exists $N_2 \in \mathbb{N}$ such that

$$x_n \in V, \quad \forall n \geq N_2.$$

Take $N := \max\{N_1, N_2\}$. Then

$$x_n \in U \cap V, \quad \forall n \geq N.$$

Since $U \cap V = \emptyset$, the above is a contradiction. Therefore the limit is unique.

2.4 Continuity

Definition 2.50: Images and Pre-images

Let X, Y be sets and $f : X \to Y$ be a function.

1. Let $U \subseteq X$. The image of U under f is the subset of Y defined by
$$f(U) := \{y \in Y : \exists x \in X \text{ s.t. } y = f(x)\} = \{f(x) : x \in X\}.$$
2. Let $V \subseteq Y$. The pre-image of V under f is the subset of X defined by
$$f^{-1}(V) := \{x \in X : f(x) \in V\}.$$

Warning

The notation $f^{-1}(V)$ does not mean that we are inverting f. In fact, the pre-image is defined for all functions.

Definition 2.51: Continuous function

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a function.

1. Let $x_0 \in X$. We say that f is continuous at x_0 if it holds:
$$\forall V \in \mathcal{T}_Y \text{ s.t. } f(x_0) \in V, \ \exists U \in \mathcal{T}_X \text{ s.t. } x_0 \in U, \ f(U) \subseteq V.$$
2. We say that f is continuous from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$ if f is continuous at each point $x_0 \in X$.

Proposition 2.52

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a function. They are equivalent:

1. f is continuous from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$.
2. It holds: $f^{-1}(V) \in \mathcal{T}_X$ for all $V \in \mathcal{T}_Y$.

Example 2.53

Question. Let X be a set and $\mathcal{T}_1$, $\mathcal{T}_2$ be topologies on X. Define the identity map

$$\mathrm{Id}_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2), \quad \mathrm{Id}_X(x) := x.$$

Prove that they are equivalent:

1. Id_X is continuous from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$.
2. $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, that is, $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Solution. Id_X is continuous if and only if

$$\mathrm{Id}_X^{-1}(V) \in \mathcal{T}_1, \quad \forall V \in \mathcal{T}_2.$$

But $\mathrm{Id}_X^{-1}(V) = V$, so that the above reads

$$V \in \mathcal{T}_1, \quad \forall V \in \mathcal{T}_2,$$

which is equivalent to $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Definition 2.54: Continuity in the classical sense

Let $f : \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We say that f is continuous at $\mathbf{x}_0$ if it holds:

$$\forall \varepsilon > 0 \,,\, \exists \delta > 0 \text{ s.t. } \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon \text{ if } \|\mathbf{x} - \mathbf{x}_0\| < \delta \,.$$

Proposition 2.55

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and suppose $\mathbb{R}^n, \mathbb{R}^m$ are equipped with the Euclidean topology. Let $\mathbf{x}_0 \in \mathbb{R}^n$. They are equivalent:

1. f is continuous at $\mathbf{x}_0$ in the topological sense.
2. f is continuous at $\mathbf{x}_0$ in the classical sense.

Proposition 2.56

Let (X, d_X) and (Y, d_Y) be metric spaces. Denote by $\mathcal{T}_X$ and $\mathcal{T}_Y$ the topologies induced by the metrics. Let $f : X \to Y$ and $x_0 \in X$. They are equivalent:

1. f is continuous at x_0 in the topological sense.
2. It holds:

$$\forall \varepsilon > 0 \,,\, \exists \delta > 0 \text{ s.t.}$$
$$d_Y(f(x), f(x_0)) < \varepsilon \text{ if } d_X(x, x_0) < \delta \,.$$

Example 2.57

Question. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be a topological space. Suppose that $\mathcal{T}_Y$ is the trivial topology, that is,

$$\mathcal{T}_Y = \{\emptyset, Y\} \,.$$

Prove that every function $f : X \to Y$ is continuous.
Solution. f is continuous if $f^{-1}(V) \in \mathcal{T}_X$ for all $V \in \mathcal{T}_Y$. We have two cases:

- $V = \emptyset$: Then $f^{-1}(V) = f^{-1}(\emptyset) = \emptyset \in \mathcal{T}_X$.
- $V = Y$: Then $f^{-1}(V) = f^{-1}(Y) = X \in \mathcal{T}_X$.

Therefore f is continuous.

Example 2.58

Question. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Suppose that $\mathcal{T}_Y$ is the discrete topology, that is,

$$\mathcal{T}_Y = \{V \text{ s.t. } V \subseteq Y\} \,.$$

Let $f : X \to Y$. Prove that they are equivalent:

1. f is continuous from X to Y.
2. $f^{-1}(\{y\}) \in \mathcal{T}_X$ for all $y \in Y$.

Solution. Suppose that f is continuous. Then

$$f^{-1}(V) \in \mathcal{T}_X \,, \quad \forall\, V \in \mathcal{T}_Y \,.$$

As $V = \{y\} \in \mathcal{T}_Y$, we conclude that $f^{-1}(\{y\}) \in \mathcal{T}_X$.
Conversely, assume that $f^{-1}(\{y\}) \in \mathcal{T}_X$ for all $y \in Y$. Let $V \in \mathcal{T}_Y$. Trivially, we have $V = \cup_{y \in V} \{y\}$. Therefore

$$f^{-1}(V) = f^{-1}\left(\bigcup_{y \in V} \{y\} \right) = \bigcup_{y \in V} f^{-1}(\{y\}) \,.$$

As $f^{-1}(\{y\}) \in \mathcal{T}_X$ for all $y \in Y$, by property (A2) we conclude that $f^{-1}(V) \in \mathcal{T}_X$. Therefore f is continuous.

Definition 2.59: Sequential continuity

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$. We say that f is **sequentially continuous** if the following condition holds:

$$\{x_n\} \subset X \,,\; x_n \to x_0 \text{ in } X \implies f(x_n) \to f(x_0) \text{ in } Y \,.$$

Proposition 2.60

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be continuous. Then f is sequentially continuous.

Warning

1. The reverse implication of Proposition 2.60 is false:

$$\text{sequential continuity} \quad \not\Longrightarrow \quad \text{continuity}$$

A counterexample is given in Example 2.62 below.

2. Continuity is equivalent to sequential continuity if the topologies on X and Y are first countable. This is the case for metrizable topologies, see Proposition 2.61 below.

Proposition 2.61

Let (X, d_X) and (Y, d_Y) be metric spaces. They are equivalent:

1. f is continuous.
2. f is sequentially continuous.

Example 2.62: Sequential continuity does not imply continuity

Question. Consider the co-countable and discrete topologies on $\mathbb{R}$

$$\mathcal{T}_{\text{cc}} = \{A \subseteq \mathbb{R} : A^c = \mathbb{R} \text{ or } A^c \text{ countable}\}$$
$$\mathcal{T}_{\text{discrete}} = \{A \subseteq \mathbb{R}\}$$

Consider the identity function

$$f : (\mathbb{R}, \mathcal{T}_{\text{cc}}) \to (\mathbb{R}, \mathcal{T}_{\text{discrete}}) \,, \quad f(x) := x \,.$$

Prove that

1. f is not continuous.
2. f is sequentially continuous.

Hint: You can use the following fact: Sequences in $(\mathbb{R}, \mathcal{T}_{cc})$ and $(\mathbb{R}, \mathcal{T}_{discrete})$ converge if and only if they are eventually constant.
Solution.

1. We have $\{x\} \in \mathcal{T}_{\text{discrete}}$. However,

$$f^{-1}(\{x\}) = \{x\} \notin \mathcal{T}_{\text{cc}} \,,$$

since $\{x\}^c = \mathbb{R} \setminus \{x\}$ is neither equal to $\mathbb{R}$, nor countable. Therefore f is not continuous.

2. Assume that $\{x_n\}$ is convergent in $\mathcal{T}_{cc}$. By the Hint, we have that $\{x_n\}$ is eventually constant. Again by the Hint, we infer that $\{x_n\}$ is convergent in $\mathcal{T}_{\text{discrete}}$. Since $f(x_n) = x_n$, we conclude that f is sequentially continuous.

Proposition 2.63: Continuity of compositions

Let $(X,\mathcal{T}_X), (Y,\mathcal{T}_Y), (Z,\mathcal{T}_Z)$ be topological spaces. Assume $f : X \to Y$ and $g : Y \to Z$ are continuous. Then $(g \circ f) : X \to Z$ is continuous.

Definition 2.64: Homeomoprhim

Let $(X,\mathcal{T}_X)$, $(Y,\mathcal{T}_Y)$ be topological space. A function $f : X \to Y$ is called an **homeomorphism** if they hold:

1. f is continuous.
2. f admits continuous inverse $f^{-1} : Y \to X$.

2.5 Subspace, basis and product

Definition 2.65: Subspace topology

Let $(X,\mathcal{T})$ be a topological space and $Y \subseteq X$ a subset. Define the family of sets

$$\begin{aligned}\mathcal{S} &:= \{A \subseteq Y \,:\, \exists\, U \in \mathcal{T} \text{ s.t. } A = U \cap Y\} \\ &= \{U \cap Y,\ U \in \mathcal{T}\}.\end{aligned}$$

The family $\mathcal{S}$ is the **subspace topology** on Y induced by the inclusion $Y \subseteq X$.

Proposition 2.66

Let $(X,\mathcal{T})$ be a topological space and $Y \in \mathcal{T}$. Let $A \subseteq Y$. Then

$$A \in \mathcal{S} \quad \Longleftrightarrow \quad A \in \mathcal{T}.$$

Warning

Let $(X,\mathcal{T})$ be a topological space, $A \subseteq Y \subseteq X$. In general we could have

$$A \in \mathcal{S} \quad \text{and} \quad A \notin \mathcal{T}.$$

Example. Let $X = \mathbb{R}$ with $\mathcal{T}_{\text{euclid}}$. Consider the subset $Y = [0,2)$, and equip Y with the subspace topology $\mathcal{S}$. Let $A = [0,1)$. Then $A \notin \mathcal{T}_{\text{euclid}}$ but $A \in \mathcal{S}$, since

$$A = (-1,1) \cap Y, \qquad (-1,1) \in \mathcal{T}_{\text{euclid}}.$$

Example 2.67

Question. Let $X = \mathbb{R}$ be equipped with $\mathcal{T}_{\text{euclid}}$. Let $\mathcal{S}$ be the subspace topology on $\mathbb{Z}$. Prove that

$$\mathcal{S} = \mathcal{T}_{\text{discrete}}.$$

Solution. To prove that $\mathcal{S} = \mathcal{T}_{\text{discrete}}$, we need to show that all the subsets of $\mathbb{Z}$ are open in $\mathcal{S}$.

1. Let $z \in \mathbb{Z}$ be arbitrary. Notice that

$$\{z\} = (z-1, z+1) \cap \mathbb{Z}$$

and $(z-1,z+1) \in \mathcal{T}_{\text{euclid}}$. Thus $\{z\} \in \mathcal{S}$.

2. Let now $A \subseteq \mathbb{Z}$ be an arbitrary subset. Trivially,

$$A = \cup_{z \in A} \{z\}.$$

As $\{z\} \in \mathcal{S}$, we infer that $A \in \mathcal{S}$ by (A2).

Definition 2.68: Topological basis

Let $(X,\mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{T}$. We say that $\mathcal{B}$ is a **topological basis** for the topology $\mathcal{T}$, if for all $U \in \mathcal{T}$ there exist open sets $\{B_i\}_{i \in I} \subseteq \mathcal{B}$, with I family of indices, such that

$$U = \bigcup_{i \in I} B_i. \tag{2.7}$$

Example 2.69

Question. Prove the following statements.

1. Let $(X,\mathcal{T})$ be a topological space. Then $\mathcal{B} := \mathcal{T}$ is a basis for $\mathcal{T}$.

2. Let (X,d) be a metric space with topology $\mathcal{T}_d$ induced by the metric. Then

$$\mathcal{B} := \{B_r(x) \,:\, x \in X,\ r > 0\}$$

is a basis for $\mathcal{T}_d$.

3. Let X be equipped with $\mathcal{T}_{\text{discrete}}$. Then

$$\mathcal{B} := \{\{x\} \,:\, x \in X\}$$

is a basis for $\mathcal{T}_{\text{discrete}}$.

Solution.

1. This is true because one can just take $B = U$ in (2.7).

2. This is true by Propostion 2.24.

3. This is true because for any $U \in \mathcal{T}$ we have

$$U = \bigcup_{x \in U} \{x\}.$$

Example 2.70

Question. Consider $\mathbb{R}$ equipped with the Euclidean topology $\mathcal{T}_{\text{euclid}}$. Which of the following collection of sets are basis for $\mathcal{T}_{\text{euclid}}$? Motivate your answer.

1. $\mathcal{B}_1 = \{(a,b) \,:\, a,b \in \mathbb{R}\}$.
2. $\mathcal{B}_2 = \{(a,b) \,:\, a,b \in \mathbb{Q}\}$.
3. $\mathcal{B}_3 = \{(a,b) \,:\, a,b \in \mathbb{Z}\}$.

Solution.

1. $\mathcal{B}_1$ is a basis for $\mathcal{T}_{\text{euclid}}$, for the following reason. Let d be the

Euclidean distance on $\mathbb{R}$, and $\mathcal{T}_d$ the topology induced by d. We know that $\mathcal{T}_d = \mathcal{T}_{\text{euclid}}$, therefore

$$\mathcal{B} := \{B_r(x) \,:\, x \in \mathbb{R}\,,\ r > 0\}$$

is a basis for $\mathcal{T}_{\text{euclid}}$ by Propostion 2.24. Note that balls in $\mathbb{R}$ are just open intervals

$$B_r(x) = (x - r, x + r)\,.$$

Hence $\mathcal{B}_1 = \mathcal{B}$, so that $\mathcal{B}_1$ is a basis for $\mathcal{T}_{\text{euclid}}$.

2. $\mathcal{B}_2$ is a basis for $\mathcal{T}_{\text{euclid}}$. This is because any open interval (a, b) with $a, b \in \mathbb{R}$ can be written as

$$\bigcup_{q,r \in \mathbb{Q},\, a<q,\, s<b} (q, s) = (a, b)\,.$$

Therefore, since $\mathcal{B}_1$ is a basis for $\mathcal{T}_{\text{euclid}}$, we conclude that also $\mathcal{B}_2$ is a basis for $\mathcal{T}_{\text{euclid}}$.

3. $\mathcal{B}_3$ is not a basis for $\mathcal{T}_{\text{euclid}}$. Indeed, consider $U = (0, 1/2)$, which is open in $\mathcal{T}_{\text{euclid}}$. It is clear that U cannot be obtained as the union of intervals (q, s) with $q, s \in \mathbb{Z}$.

Proposition 2.71

Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B}$ a basis for $\mathcal{T}$. They hold:

- (B1) We have
$$\bigcup_{B \in \mathcal{B}} B = X\,.$$
- (B2) If $U_1, U_2 \in \mathcal{B}$ then there exist $\{B_i\} \subseteq \mathcal{B}$ such that
$$U_1 \cap U_2 = \bigcup_{i \in I} B_i\,.$$

Proposition 2.72

Let X be a set, and $\mathcal{B}$ a collection of subsets of X satisfying (B1)-(B2). Define

$$\mathcal{T} := \left\{ U \subseteq X \,:\, U = \bigcup_{i \in I} B_i\,,\ B_i \in \mathcal{B} \right\}.$$

Then $\mathcal{T}$ is a topology on X, with basis given by $\mathcal{B}$.

Proposition 2.73

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Define the family $\mathcal{B}$ of subsets of $X \times Y$ as

$$\mathcal{B} := \{U \times V \,:\, U \in \mathcal{T}_X\,,\ V \in \mathcal{T}_Y\} \subseteq \mathcal{T}_X \times \mathcal{T}_Y\,.$$

Then $\mathcal{B}$ satisfies properties (B1) and (B2) from Proposition 2.71. In particular,

$$\mathcal{T}_{X \times Y} := \left\{ Z \,:\, Z = \bigcup_{i \in I} U_i \times V_i,\ U_i \times V_i \in \mathcal{B} \right\} \tag{2.8}$$

is a topology on $X \times Y$.

Definition 2.74: Product topology

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The **product topology** on $X \times Y$ is the collection of sets $\mathcal{T}_{X \times Y}$ at (2.8).

Example 2.75

Let $\mathbb{R}$ be equipped with the (one dimensional) Euclidean topology. The product topology on $\mathbb{R} \times \mathbb{R}$ coincides with the topology on $\mathbb{R}^2$ equipped with the (two dimensional) Euclidean topology.

2.6 Connectedness

Definition 2.76: Connected space

Let $(X, \mathcal{T})$ be a topological space. We say that:

1. X is **connected** if the only subsets of X which are both open and closed are $\emptyset$ and X.
2. X is **disconnected** if it is not connected.

Definition 2.77: Proper subset

Let X be a set. A subset $A \subseteq X$ is **proper** if $A \neq \emptyset$ and $A \neq X$.

Proposition 2.78: Equivalent definition for connectedness

Let $(X, \mathcal{T})$ be a topological space. They are equivalent:

1. X is disconnected.
2. X is the disjoint union of two proper open subsets.
3. X is the disjoint union of two proper closed subsets.

Example 2.79

Question. Consider the set $X = \{0, 1\}$ with the subspace topology induced by the inclusion $X \subseteq \mathbb{R}$, where $\mathbb{R}$ is equipped with the Euclidean topology $\mathcal{T}_{\text{euclid}}$. Prove that X is disconnected.
Solution. Note that

$$X = \{0\} \cup \{1\}\,, \quad \{0\} \cap \{1\} = \emptyset\,.$$

The set $\{0\}$ is open for the subspace topology, since

$$\{0\} = X \cap (-1, 1)\,, \quad (-1, 1) \in \mathcal{T}_{\text{euclid}}\,.$$

Similarly, also $\{1\}$ is open for the subspace topology, since

$$\{1\} = X \cap (0, 2)\,, \quad (0, 2) \in \mathcal{T}_{\text{euclid}}\,.$$

Since $\{0\}$ and $\{1\}$ are proper subsets of X, we conclude that X is disconnected.

Example 2.80

Question. Let $\mathbb{R}$ be equipped with $\mathcal{T}_{\text{euclid}}$, and let $p \in \mathbb{R}$. Prove that the set $X = \mathbb{R} \smallsetminus \{p\}$ is disconnected.
Solution. Define the sets

$$A = (-\infty, p)\,, \quad B = (p, \infty)\,.$$

A and B are proper subsets of X. Moreover

$$X = A \cup B, \quad A \cap B = \emptyset.$$

Finally, A, B are open for the subspace topology on X, since they are open in $(\mathbb{R}, \mathcal{T}_{\text{euclid}})$. Therefore X is disconnected.

Theorem 2.81

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces. Suppose that $f : X \to Y$ is continuous and let $f(X) \subseteq Y$ be equipped with the subspace topology. If X is connected, then $f(X)$ is connected.

Theorem 2.82: Connectedness is topological invariant

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be homeomorhic topological spaces. Then

$$X \text{ is connected} \quad \iff \quad Y \text{ is connected}$$

Example 2.83

Question. Define the one dimensional unit circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 = 1\}.$$

Prove that $\mathbb{S}^1$ and $[0, 1]$ are not homeomorphic.
Solution. Suppose by contradiction that there exists a homeomorphism

$$f : [0, 1] \to \mathbb{S}^1.$$

The restriction of f to $[0, 1] \setminus \{\frac{1}{2}\}$ defines a homeomorphism

$$g : \left([0, 1] \setminus \left\{\frac{1}{2}\right\}\right) \to (\mathbb{S}^1 \setminus \{\mathbf{p}\}), \quad \mathbf{p} := f\left(\frac{1}{2}\right).$$

The set $[0, 1] \setminus \left\{\frac{1}{2}\right\}$ is disconnected, since

$$[0, 1] \setminus \{1/2\} = [0, 1/2) \cup (1/2, 1]$$

with $[0, 1/2)$ and $(1/2, 1]$ open for the subset topology, non-empty and disjoint. Therefore, using that g is a homeomorphism, we conclude that also $\mathbb{S}^1 \setminus \{\mathbf{p}\}$ is disconnected. Let $\theta_0 \in [0, 2\pi)$ be the unique angle such that

$$\mathbf{p} = (\cos(\theta_0), \sin(\theta_0)).$$

Thus $\mathbb{S}^1 \setminus \{\mathbf{p}\}$ is parametrized by

$$\boldsymbol{\gamma}(t) := (\cos(t), \sin(t)), \quad t \in (\theta_0, \theta_0 + 2\pi).$$

Since $\boldsymbol{\gamma}$ is continuous and $(\theta_0, \theta_0 + 2\pi)$ is connected, by Theorem 2.81, we conclude that $\mathbb{S}^1 \setminus \{\mathbf{p}\}$ is connected. Contradiction.

Definition 2.84: Interval

A subset $I \subset \mathbb{R}$ is an interval if it holds:

$$\forall\, a, b \in I, \, x \in \mathbb{R} \;\text{ s.t. }\; a < x < b \quad \implies \quad x \in I.$$

Theorem 2.85: Intervals are connected

Let $\mathbb{R}$ be equipped with the Euclidean topology and let $I \subseteq \mathbb{R}$. They are equivalent:

1. I is connected.
2. I is an interval.

Theorem 2.86: Intermediate Value Theorem

Let $(X, \mathcal{T})$ be a connected topological space. Suppose that $f : X \to \mathbb{R}$ is continuous. Suppose that $a, b \in X$ are such that $f(a) < f(b)$. It holds:

$$\forall\, c \in \mathbb{R} \;\text{ s.t. }\; f(a) < c < f(b), \; \exists\, \xi \in X \;\text{ s.t. }\; f(\xi) = c.$$

Example 2.87: Intervals are connected - Alternative proof

Question. Prove the following statements.

1. Let $(X, \mathcal{T})$ be a disconnected topological space. Prove that there exists a function $f : X \to \{0, 1\}$ which is continuous and surjective.

2. Consider $\mathbb{R}$ equipped with the Euclidean topology. Let $I \subseteq \mathbb{R}$ be an interval. Use point (1), and the Intermediate Value Theorem in $\mathbb{R}$ (see statement below), to show that I is connected.

Intermediate Value Theorem in $\mathbb{R}$: Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) < f(b)$. Let $c \in \mathbb{R}$ be such that $f(a) \leq c \leq f(b)$. Then, there exists $\xi \in [a, b]$ such that $f(\xi) = c$.
Solution. Part 1. Since X is disconnected, there exist $A, B \in \mathcal{T}$ proper and such that

$$X = A \cup B, \quad A \cap B = \emptyset.$$

Define $f : X \to \{0, 1\}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

Since A and B are non-empty, it follows that f is surjective. Moreover f is continuous: Indeed suppose $U \subseteq \mathbb{R}$ is open. We have 4 cases:

- $0, 1 \notin U$. Then $f^{-1}(U) = \emptyset \in \mathcal{T}$.
- $0 \in U$, $1 \notin U$. Then $f^{-1}(U) = A \in \mathcal{T}$.
- $0 \notin U$, $1 \in U$. Then $f^{-1}(U) = B \in \mathcal{T}$.
- $0, 1 \in U$. Then $f^{-1}(U) = X \in \mathcal{T}$.

Then $f^{-1}(U) \in \mathcal{T}$ for all $U \subseteq \mathbb{R}$ open, showing that f is continuous.
Part 2. Let $I \subseteq \mathbb{R}$ be an interval. Suppose by contradiction I is disconnected. By Point (1), there exists a map $f : I \to \{0, 1\}$ which is continuous and surjective. As f is surjective, there exist $a, b \in I$ such that

$$f(a) = 0, \quad f(b) = 1.$$

Since f is continuous, and $f(a) = 0 < 1 = f(b)$, by the *Intermediate Value Theorem in* $\mathbb{R}$, there exists $\xi \in [a, b]$ such that $f(\xi) = 1/2$. As I is an interval, $a, b \in I$, and $a \leq \xi \leq b$, it follows that $\xi \in I$. This is a contradiction, since f maps I into $\{0, 1\}$, and $f(\xi) = 1/2 \notin \{0, 1\}$. Therefore I is connected.

Definition 2.88: Path-connectedness

Let $(X, \mathcal{T})$ be a topological space. We say that X is **path-connected** if for every $x, y \in X$ there exist $a, b \in \mathbb{R}$ with $a < b$, and a continuous function

$$\alpha : [a, b] \to X \quad \text{s.t.} \quad \alpha(a) = x \,, \quad \alpha(b) = y \,.$$

Theorem 2.89: Path-connectedness implies connectedness

Let $(X, \mathcal{T})$ be a path-connected topological space. Then X is connected.

Example 2.90

Question. Let $A \subseteq \mathbb{R}^n$ be convex. Show that A is path-connected, and hence connected.
Solution. A is convex if for all $x, y \in A$ the segment connecting x to y is contained in A, namely,

$$[x, y] := \{(1-t)x + ty \; : \; t \in [0, 1]\} \subseteq A \,.$$

Therefore we can define

$$\alpha : [0, 1] \to A \,, \quad \alpha(t) := (1-t)x + ty \,.$$

Clearly α is continuous, and $\alpha(0) = x, \alpha(1) = y$.

Example 2.91: Spaces of matrices

Let $\mathbb{R}^{2\times 2}$ denote the space of real 2×2 matrices. Assume $\mathbb{R}^{2\times 2}$ has the euclidean topology obtained by identifying it with $\mathbb{R}^4$.

1. Consider the set of orthogonal matrices

$$O(2) = \{A \in \mathbb{R}^{2\times 2} \; : \; A^T A = I\} \,.$$

Prove that $O(2)$ is disconnected.

2. Consider the set of rotations

$$SO(2) = \{A \in \mathbb{R}^{2\times 2} \; : \; A^T A = I, \; \det(A) = 1\} \,.$$

Prove that $SO(2)$ is path-connected, and hence connected.

Solution. Let $A \in O(2)$, and denote its entries by a, b, c, d. By direct calculation, the condition $A^T A = I$ is equivalent to

$$a^2 + b^2 = 1 \,, \qquad b^2 + c^2 = 1 \,, \qquad ac + bd = 0 \,.$$

From the first condition, we get that $a = \cos(t)$ and $b = \sin(t)$, for a suitable $t \in [0, 2\pi)$. From the second and third conditions, we get $c = \pm \sin(t)$ and $d = \mp \cos(t)$. We decompose $O(2)$ as

$$\begin{aligned} O(2) &= A \cup B \,, \\ A = SO(2) &= \left\{ \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \, t \in [0, 2\pi) \right\} \\ B &= \left\{ \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}, \, t \in [0, 2\pi) \right\} . \end{aligned}$$

1. The determinant function $\det : O(2) \to \mathbb{R}$ is continuous. If $M \in A$, we have $\det(M) = 1$. If instead $M \in B$, we have $\det(M) = -1$. Moreover,

$$\det^{-1}(\{1\}) = A \,, \qquad \det^{-1}(\{0\}) = B \,.$$

As det is continuous, and $\{0\}, \{1\}$ closed, we conclude that A and B are closed. Therefore, A and B are closed, proper and disjoint. Since $O(2) = A \cup B$, we conclude that $O(2)$ is disconnected.

2. Define the function $\psi : [0, 2\pi) \to SO(2)$ by

$$\psi(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} .$$

Clearly, ψ is continuous. Let $R, Q \in SO(2)$. Then R is determined by an angle t_1, while Q by an angle t_2. Up to swapping R and Q, we can assume $t_1 < t_2$. Define the function $f : [0, 1] \to SO(2)$ by

$$f(\lambda) = \psi(t_1(1-\lambda) + t_2\lambda) \,.$$

Then, f is continuous and

$$f(0) = \psi(t_1) = R, \quad f(1) = \psi(t_2) = Q \,.$$

Thus $SO(2)$ is path-connected.

Warning

In general connectedness does not imply path-connectedness, as seen in Proposition 2.92.

Proposition 2.92: Topologist curve

Consider $\mathbb{R}^2$ with the Euclidean topology, and define the sets

$$\begin{aligned} A &:= \left\{ \left(t, \sin\left(\frac{1}{t}\right)\right) \; : \; t > 0 \right\} \\ B &:= \{(0, t) \; : \; t \in [-1, 1]\} \,, \qquad X := A \cup B \,. \end{aligned}$$

Then X is connected, but not path-connected.

Theorem 2.93

Let $A \subseteq \mathbb{R}^n$ be **open** for the Euclidean topology. Then A is connected if and only if it is path-connected.

3 Surfaces

Definition 3.1: Topology of $\mathbb{R}^n$

The Euclidean norm on $\mathbb{R}^n$ is denoted by

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^{n} x_i^2}\,, \qquad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n\,.$$

Define the Euclidean distance $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

1. The pair $(\mathbb{R}^n, d)$ is a metric space.
2. The topology induced by the metric d is called the Euclidean topology, denoted by $\mathcal{T}$.
3. A set $U \subseteq \mathbb{R}^n$ is **open** if for all $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subseteq U$, where

$$B_\varepsilon(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n \;:\; \|\mathbf{x} - \mathbf{y}\| < \varepsilon\}$$

   is the open ball of radius $\varepsilon > 0$ centered at $\mathbf{x}$. We write $U \in \mathcal{T}$, with $\mathcal{T}$ the Euclidean topology in $\mathbb{R}^n$.
4. A set $V \subseteq \mathbb{R}^n$ is **closed** if $V^c := \mathbb{R}^n \setminus U$ is open.

Definition 3.2: Subspace Topology

Let $A \subseteq \mathbb{R}^n$. The **subspace topology** on A is the family

$$\mathcal{T}_A := \{U \subseteq A \;:\; \exists\, W \in \mathcal{T} \text{ s.t. } U = A \cap W\}\,.$$

If $U \in \mathcal{T}_A$, we say that U is open in A.

Definition 3.3: Continuous Function

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with U open. We say that f is **continuous** at $\mathbf{x} \in U$ if $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that

$$\|\mathbf{x} - \mathbf{y}\| < \delta \quad \Longrightarrow \quad \|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon\,.$$

f is continuous in U if it is continuous for all $\mathbf{x} \in U$.

Theorem 3.4: Continuity: Topological definition

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$, with U, V open. We have that f is continuous if and only if $f^{-1}(A)$ is open in U, for all A open in V.

Definition 3.5: Homeomorphism

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ with U, V open. We say that f is a **homeomorphism** if:

1. f is continuous;
2. f admits continuous inverse $f^{-1} : V \to U$.

Definition 3.6: Differentiable Function

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with U open. We say that f is **differentiable** at $\mathbf{x} \in U$ if there exists a linear map $d_{\mathbf{x}} f : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$d_{\mathbf{x}} f(\mathbf{h}) = \lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{h}) - f(\mathbf{x})}{\varepsilon}\,,$$

for all $\mathbf{h} \in \mathbb{R}^n$, where the limit is taken in $\mathbb{R}^m$. The linear map $d_{\mathbf{x}} f$ is called the **differential** of f at $\mathbf{x}$.

Definition 3.7: Partial Derivative

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m, U$ open, f differentiable. The **partial derivative** of f at $\mathbf{x} \in U$ in direction $\mathbf{e}_i$ is

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) := d_{\mathbf{x}} f(\mathbf{e}_i) = \lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{e}_i) - f(\mathbf{x})}{\varepsilon}\,.$$

Definition 3.8: Jacobian Matrix

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. The **Jacobian** of f at $\mathbf{x}$ is the $m \times n$ matrix of partial derivatives:

$$Jf(\mathbf{x}) := \left(\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right)_{i,j} \in \mathbb{R}^{m \times n}\,.$$

If $m = n$ then $Jf \in \mathbb{R}^{n \times n}$ is a square matrix and we can compute its determinant, denoted by $\det(Jf)$.

Proposition 3.9: Matrix representation of $d_{\mathbf{x}} f$

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. The matrix of the linear map $d_{\mathbf{x}} f : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the standard basis is given by the Jacobian matrix $Jf(\mathbf{x})$.

Definition 3.10: Diffeomorphism

Let $f : U \to V$, with $U, V \subseteq \mathbb{R}^n$ open. We say that f is a **diffeomorphism** between U and V if:

1. f is smooth,
2. f admits smooth inverse $f^{-1} : V \to U$.

Definition 3.11: Local diffeomorphism

$f : \mathbb{R}^n \to \mathbb{R}^n$ is a **local diffeomorphism** at $\mathbf{x}_0 \in \mathbb{R}^n$ if:

1. There exists an open set $U \subseteq \mathbb{R}^n$ such that $\mathbf{x}_0 \in U$,
2. There exists an open set $V \subseteq \mathbb{R}^n$ such that $f(\mathbf{x}_0) \in V$,
3. $f : U \to V$ is a diffeomorphism.

Proposition 3.12

Diffeomorphisms are local diffeomorphisms.

Proposition 3.13: Necessary condition for being diffeomorphism

Let $f : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^n$ open. Suppose f is a local diffeomorhism at $\mathbf{x}_0 \in U$. Then $\det Jf(\mathbf{x}_0) \neq 0$.

Theorem 3.14: Inverse Function Theorem

Let $f : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^n$ open, f smooth. Assume

$$\det Jf(\mathbf{x}_0) \neq 0,$$

for some $\mathbf{x}_0 \in U$. Then:

1. There exists an open set $U_0 \subseteq U$ such that $\mathbf{x}_0 \in U_0$,
2. There exists an open set V such that $f(\mathbf{x}_0) \in V$,
3. $f : U_0 \to V$ is a diffeomorphism.

Example 3.15: A local diffeomorphism which is not global

Question. Define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x, y) = (e^x \cos(y), e^x \sin(y)).$$

Prove f is a local diffeomorphism but not a diffeomorphism.
Solution. f is a local diffeomorphism at each point $(x, y) \in \mathbb{R}^2$ by the Inverse Function Theorem, since

$$Jf(x, y) = e^x \begin{pmatrix} \cos(y) & \sin(y) \\ -\sin(y) & \cos(y) \end{pmatrix}$$
$$\det Jf(x, y) = e^{2x} \neq 0.$$

However, f is not invertible because it is not injective, since

$$f(x, y) = f(x, y + 2n\pi), \quad \forall (x, y) \in \mathbb{R}^2, n \in \mathbb{N}.$$

Hence, f cannot be a diffeomorphism of $\mathbb{R}^2$ into $\mathbb{R}^2$.

3.1 Regular Surfaces

Definition 3.16: Surface

Let $\mathcal{S} \subseteq \mathbb{R}^3$ be a connected set. We say that $\mathcal{S}$ is a **surface** if for every point $\mathbf{p} \in \mathcal{S}$ there exist an open set $U \subseteq \mathbb{R}^2$, and a smooth map $\boldsymbol{\sigma} : U \to \boldsymbol{\sigma}(U) \subseteq \mathcal{S}$ such that

1. $\mathbf{p} \in \boldsymbol{\sigma}(U)$,
2. $\boldsymbol{\sigma}(U)$ is open in $\mathcal{S}$,
3. $\boldsymbol{\sigma}$ is a homeomorphism between U and $\boldsymbol{\sigma}(U)$.

$\boldsymbol{\sigma}$ is called a **surface chart** at $\mathbf{p}$.

Definition 3.17: Atlas of a surface

Let $\mathcal{S}$ be a surface. Assume given a collection of charts

$$\mathcal{A} = \{\boldsymbol{\sigma}_i\}_{i \in I}, \qquad \boldsymbol{\sigma}_i : U_i \to \boldsymbol{\sigma}(U_i) \subseteq \mathcal{S}.$$

The family $\mathcal{A}$ is an **atlas** of $\mathcal{S}$ if

$$\mathcal{S} = \bigcup_{i \in I} \boldsymbol{\sigma}_i(U_i).$$

Definition 3.18: Regular Chart

Let $U \subseteq \mathbb{R}^2$ be open. A map $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u, v) : U \to \mathbb{R}^3$ is a **regular chart** if the partial derivatives

$$\boldsymbol{\sigma}_u(u, v) = \frac{d\boldsymbol{\sigma}}{du}(u, v), \quad \boldsymbol{\sigma}_v(u, v) = \frac{d\boldsymbol{\sigma}}{dv}(u, v)$$

are linearly independent vectors of $\mathbb{R}^3$ for all $(u, v) \in U$.

Definition 3.19: Regular surface

Let $\mathcal{S}$ be a surface. We say that:

- $\mathcal{A}$ is a **regular atlas** if any $\boldsymbol{\sigma}$ in $\mathcal{A}$ is regular.
- $\mathcal{S}$ is a **regular surface** if it admits a regular atlas.

Theorem 3.20: Characterization of regular charts

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ with $U \subseteq \mathbb{R}^2$ open. They are equivalent:

1. $\boldsymbol{\sigma}$ is a regular chart.
2. $d_{\mathbf{x}}\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $\mathbf{x} \in U$.
3. The Jacobian matrix $J\boldsymbol{\sigma}$ has rank 2 for all $(u, v) \in U$.
4. $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq 0$ for all $(u, v) \in U$.

Example 3.21: Unit cylinder

Question. Consider the infinite unit cylinder

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}.$$

$\mathcal{S}$ is a surface with atlas $\mathcal{A} = \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\}$, with

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v), \qquad \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}|_{U_1}, \quad \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}|_{U_2},$$
$$U_1 = \left(0, \frac{3\pi}{2}\right) \times \mathbb{R}, \qquad U_2 = \left(\pi, \frac{5\pi}{2}\right) \times \mathbb{R}.$$

Prove that $\mathcal{S}$ is a regular surface.
Solution. The map $\boldsymbol{\sigma}$ is regular because

$$\boldsymbol{\sigma}_u = (-\sin(u), \cos(u), 0), \quad \boldsymbol{\sigma}_v = (0, 0, 1),$$

are linearly independent, since the last components of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are 0 and 1. Therefore, also $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are regular charts, being restrictions of $\boldsymbol{\sigma}$. Thus, $\mathcal{A}$ is a regular atlas and $\mathcal{S}$ a regular surface.

Example 3.22: Graph of a function

Question. Let $f : U \to \mathbb{R}$ be smooth, $U \subseteq \mathbb{R}^2$ open. Define

$$\Gamma_f = \{(u, v, f(u, v)) : (u, v) \in U\},$$

the graph of f. Then Γ_f is surface with atlas $\mathcal{A} = \{\boldsymbol{\sigma}\}$, where

$$\boldsymbol{\sigma} : U \to \Gamma_f, \quad \boldsymbol{\sigma}(u, v) := (u, v, f(u, v)).$$

Prove that Γ_f is a regular surface.
Solution. The Jacobian matrix of $\boldsymbol{\sigma}$ is

$$J\boldsymbol{\sigma}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}.$$

$J\boldsymbol{\sigma}$ has rank 2, because the first minor is the 2×2 identity matrix. Therefore, $\boldsymbol{\sigma}$ is regular. This implies $\mathscr{A}$ is a regular atlas, and $\mathscr{S}$ is a regular surface.

Definition 3.23: Spherical coordinates

The **spherical coordinates** of $\mathbf{p} = (x,y,z) \neq \mathbf{0}$ are

$$\mathbf{p} = (\rho\cos(\theta)\cos(\varphi), \rho\sin(\theta)\cos(\varphi), \rho\sin(\varphi)),$$
$$\rho = \sqrt{x^2+y^2+z^2}, \quad \theta \in [-\pi,\pi], \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Example 3.24: Unit sphere in spherical coordinates

Question. Consider the unit sphere in $\mathbb{R}^3$

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \, : \, x^2+y^2+z^2 = 1\}.$$

Prove that $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ is regular, where

$$\boldsymbol{\sigma}(\theta,\varphi) = (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), \sin(\varphi)),$$
$$U = \left\{(\theta,\varphi) \in \mathbb{R}^2 \, : \, \theta \in (-\pi,\pi), \, \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}.$$

Solution. The chart $\boldsymbol{\sigma}$ is regular because

$$\begin{aligned}
\boldsymbol{\sigma}_\theta &= (-\sin(\theta)\cos(\varphi), \cos(\theta)\cos(\varphi), 0) \\
\boldsymbol{\sigma}_\varphi &= (-\cos(\theta)\sin(\varphi), -\sin(\theta)\sin(\varphi), \cos(\varphi)) \\
\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_\varphi &= (\cos(\theta)\cos^2(\varphi), \sin(\theta)\cos^2(\varphi), \cos(\varphi)\sin(\varphi)) \\
\|\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_\varphi\| &= |\cos(\varphi)| = \cos(\varphi) \neq 0,
\end{aligned}$$

where we used that $\cos(\phi) > 0$, since $\varphi \in (-\pi/2, \pi/2)$.

Example 3.25: A non-regular chart

Question. Prove that the following chart is not regular

$$\boldsymbol{\sigma}(u,v) = (u, v^2, v^3).$$

Solution. We have

$$\boldsymbol{\sigma}_v = (0, 2v, 3v^2), \qquad \boldsymbol{\sigma}_v(u,0) = (0,0,0).$$

$\boldsymbol{\sigma}$ is not regular because $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly dependent along the line $L = \{(u,0) \, : \, u \in \mathbb{R}\}$.

Definition 3.26: Reparametrization

Suppose that $U, \widetilde{U} \subseteq \mathbb{R}^2$ are open sets and

$$\boldsymbol{\sigma} : U \to \mathbb{R}^3, \quad \tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathbb{R}^3,$$

are surface charts. We say that $\tilde{\boldsymbol{\sigma}}$ is a **reparametrization** of $\boldsymbol{\sigma}$ if there exists a diffeomorphism $\Phi : \widetilde{U} \to U$ such that

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi.$$

Theorem 3.27: Reparametrizations of regular charts are regular

Let $U, \widetilde{U} \subseteq \mathbb{R}^2$ be open and $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular. Suppose given a diffeomorphism $\Phi : \widetilde{U} \to U$. The reparametrization

$$\tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathbb{R}^3, \qquad \tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi$$

is a regular chart, and it holds

$$\tilde{\boldsymbol{\sigma}}_{\tilde{u}} \times \tilde{\boldsymbol{\sigma}}_{\tilde{v}} = \det J\Phi \, (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v).$$

Definition 3.28: Transition map

Let $\mathscr{S}$ be a regular surface, $\boldsymbol{\sigma} : U \to \mathscr{S}$, $\tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathscr{S}$ regular charts. Suppose the images of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ overlap

$$I := \boldsymbol{\sigma}(U) \cap \tilde{\boldsymbol{\sigma}}(\widetilde{U}) \neq \emptyset.$$

I is open in $\mathscr{S}$, being intersection of open sets. Define

$$V := \boldsymbol{\sigma}^{-1}(I) \subseteq U, \quad \widetilde{V} := \tilde{\boldsymbol{\sigma}}^{-1}(I) \subseteq \widetilde{U}.$$

V and $\widetilde{V}$ are open, by continuity of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$. Moreover, as $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are homeomorphisms, we have $\boldsymbol{\sigma}(V) = \tilde{\boldsymbol{\sigma}}(\widetilde{V}) = I$. Therefore, they are well defined the restriction homeomorphisms

$$\boldsymbol{\sigma}|_V : V \to I, \quad \tilde{\boldsymbol{\sigma}}|_{\widetilde{V}} : \widetilde{V} \to I.$$

The **transition map** from $\boldsymbol{\sigma}$ to $\tilde{\boldsymbol{\sigma}}$ is the homeomorphism

$$\Phi : \widetilde{V} \to V, \quad \Phi := \boldsymbol{\sigma}^{-1} \circ \tilde{\boldsymbol{\sigma}}.$$

Theorem 3.29

Transition maps between regular charts are diffeomorphisms.

Theorem 3.30: Transition maps are reparametrizations

Let $\mathscr{S}$ be a regular surface, $\boldsymbol{\sigma} : U \to \mathscr{S}$, $\tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathscr{S}$ regular charts, with $I := \boldsymbol{\sigma}(U) \cap \tilde{\boldsymbol{\sigma}}(\widetilde{U}) \neq \emptyset$. Define the transition map

$$\Phi = \boldsymbol{\sigma}^{-1} \circ \tilde{\boldsymbol{\sigma}} : \widetilde{V} \to V, \quad V = \boldsymbol{\sigma}^{-1}(I), \quad \widetilde{V} = \tilde{\boldsymbol{\sigma}}^{-1}(I).$$

Then $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are reparametrization of each other, with

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi, \qquad \boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} \circ \Phi^{-1}.$$

3.2 Smooth maps and tangent plane

Definition 3.31: Smooth functions between surfaces

Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be regular surfaces and $f : \mathscr{S}_1 \to \mathscr{S}_2$ a map.

1. f is *smooth at* $\mathbf{p} \in \mathscr{S}_1$, if there exist charts

$$\boldsymbol{\sigma}_i : U_i \to \mathscr{S}_i \text{ such that } \mathbf{p} \in \boldsymbol{\sigma}_1(U_1), \; f(\mathbf{p}) \in \boldsymbol{\sigma}_2(U_2),$$

and that the following map is smooth

$$\Psi : U_1 \to U_2 \,, \quad \Psi = \boldsymbol{\sigma}_2^{-1} \circ f \circ \boldsymbol{\sigma}_1 \,.$$

2. f is *smooth*, if it is smooth for each $\mathbf{p} \in \mathcal{S}_1$.

Proposition 3.32: Inverse of a regular chart is smooth

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular. Then $\boldsymbol{\sigma}^{-1} : \boldsymbol{\sigma}(U) \to U$ is smooth.

Definition 3.33: Diffeomorphism of surfaces

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be regular surfaces.

1. $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a **diffeomorphism**, if f is smooth and admits smooth inverse.
2. $\mathcal{S}_1$, $\mathcal{S}_2$ are **diffeomorphic** if there exists $f : \mathcal{S}_1 \to \mathcal{S}_2$ diffeomorphism.

Proposition 3.34: Image of charts under diffeomorphisms

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces, $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ diffeomorphism. If $\boldsymbol{\sigma} : U \to \mathcal{S}$ is a regular chart for $\mathcal{S}$ at $\mathbf{p}$, then

$$\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}} \,, \qquad \tilde{\boldsymbol{\sigma}} := f \circ \boldsymbol{\sigma} \,,$$

is a regular chart for $\widetilde{\mathcal{S}}$ at $f(\mathbf{p})$.

Definition 3.35: Local diffeomorphism

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be regular surfaces, and $f : \mathcal{S}_1 \to \mathcal{S}_2$ smooth.

1. f is a **local diffeomorphism** at $\mathbf{p} \in \mathcal{S}_1$ if:
   - There exists An open set $V \subseteq \mathcal{S}_1$ with $\mathbf{p} \in V$;
   - $f(V) \subseteq \mathcal{S}_2$ is open;
   - $f : V \to f(V)$ is smooth between surfaces.
2. f is a **local diffeomorphism** in $\mathcal{S}_1$, if it is a local diffeomorphism at each $\mathbf{p} \in \mathcal{S}_1$.
3. $\mathcal{S}_1$ is **locally diffeomorphic** to $\mathcal{S}_2$, if for all $\mathbf{p} \in \mathcal{S}_1$ there exists f local diffeomorphism at $\mathbf{p}$.

Definition 3.36: Tangent vectors and tangent plane

Let $\mathcal{S}$ be a surface and $\mathbf{p} \in \mathcal{S}$.

1. $\mathbf{v} \in \mathbb{R}^3$ is a **tangent vector** to $\mathcal{S}$ at $\mathbf{p}$, if there exists a smooth curve $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that

$$\boldsymbol{\gamma}(-\varepsilon, \varepsilon) \subseteq \mathcal{S} \,, \quad \boldsymbol{\gamma}(0) = \mathbf{p} \,, \quad \mathbf{v} = \dot{\boldsymbol{\gamma}}(0) \,.$$

2. The **tangent plane** of $\mathcal{S}$ at $\mathbf{p}$ is the set

$$T_{\mathbf{p}}\mathcal{S} := \{ \mathbf{v} \in \mathbb{R}^3 \; : \; \mathbf{v} \text{ tangent vector of } \mathcal{S} \text{ at } \mathbf{p} \} \,.$$

Lemma 3.37: Curves with values on surfaces

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart and $\mathcal{S} := \boldsymbol{\sigma}(U)$. Let $\mathbf{p} \in \mathcal{S}$ and $(u_0, v_0) = \boldsymbol{\sigma}^{-1}(\mathbf{p})$. Assume $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ is a smooth curve such that

$$\boldsymbol{\gamma}(-\varepsilon, \varepsilon) \subseteq \mathcal{S} \,, \quad \boldsymbol{\gamma}(0) = \mathbf{p} \,.$$

There exist smooth functions $u, v : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t)) \,, \;\; \forall\, t \in (-\varepsilon, \varepsilon) \,, \quad u(0) = u_0 \,, \;\; v(0) = v_0 \,.$$

Theorem 3.38: Characterization of Tangent Plane

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart and $\mathcal{S} := \boldsymbol{\sigma}(U)$. Let $\mathbf{p} \in \mathcal{S}$. Then

$$T_{\mathbf{p}}\mathcal{S} = \operatorname{span}\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} := \{ \lambda \boldsymbol{\sigma}_u + \mu \boldsymbol{\sigma}_v \; : \; \lambda, \mu \in \mathbb{R} \} \,,$$

where $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are evaluated at $(u, v) = \boldsymbol{\sigma}^{-1}(\mathbf{p})$.

Theorem 3.39: Equation of tangent plane

Let $\boldsymbol{\sigma} : U \to \mathcal{S}$ be regular, $\mathcal{S} = \boldsymbol{\sigma}(U)$. Let $\mathbf{p} \in \mathcal{S}$ and

$$\mathbf{n} := \boldsymbol{\sigma}_u(u, v) \times \boldsymbol{\sigma}_v(u, v) \,, \quad (u, v) := \boldsymbol{\sigma}^{-1}(\mathbf{p}) \,.$$

The equation of the tangent plane $T_{\mathbf{p}}\mathcal{S}$ is given by

$$\mathbf{x} \cdot \mathbf{n} = 0 \,, \quad \forall\, \mathbf{x} \in \mathbb{R}^3 \,.$$

Example 3.40: Calculation of tangent plane

Question. For $u \in (0, 2\pi)$, $v < 1$, let $\mathcal{S}$ charted by

$$\boldsymbol{\sigma}(u, v) = \left(\sqrt{1-v}\cos(u), \sqrt{1-v}\sin(u), v \right) \,.$$

1. Prove that $\boldsymbol{\sigma}$ charts the paraboloid $x^2 + y^2 - z = 1$.
2. Prove that $\boldsymbol{\sigma}$ is regular and compute $\mathbf{n} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$.
3. Give a basis for $T_{\mathbf{p}}\mathcal{S}$ at $\mathbf{p} = (\sqrt{2}/2, \sqrt{2}/2, 0)$.
4. Compute the cartesian equation of $T_{\mathbf{p}}\mathcal{S}$.

Solution.

1. Denote $\boldsymbol{\sigma}(u, v) = (x, y, z)$. We have

$$\begin{aligned} x^2 + y^2 &= \left(\sqrt{1-v}\cos(u) \right)^2 + \left(\sqrt{1-v}\sin(u) \right)^2 \\ &= 1 - v = 1 - z \,. \end{aligned}$$

2. We compute $\mathbf{n} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$ and show that $\boldsymbol{\sigma}$ is regular:

$$\begin{aligned} \boldsymbol{\sigma}_u &= \left(-\sqrt{1-v}\sin(u), \sqrt{1-v}\cos(u), 0 \right) \\ \boldsymbol{\sigma}_v &= \left(-\frac{1}{2}(1-v)^{-1/2}\cos(u), -\frac{1}{2}(1-v)^{-1/2}\sin(u), 1 \right) \\ \mathbf{n} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= \left(\sqrt{1-v}\cos(u), \sqrt{1-v}\sin(u), \frac{1}{2} \right) \neq \mathbf{0} \end{aligned}$$

3. Notice that $\boldsymbol{\sigma}(\pi/4, 0) = \mathbf{p}$. A basis for $T_{\mathbf{p}}\mathcal{S}$ is

$$\begin{aligned} \boldsymbol{\sigma}_u\left(\frac{\pi}{4}, 0 \right) &= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right) \,, \\ \boldsymbol{\sigma}_v\left(\frac{\pi}{4}, 0 \right) &= \left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, 1 \right) \,. \end{aligned}$$

4. Using the calculation for $\mathbf{n}$ in Point 2, we find

$$\mathbf{n}\left(\frac{\pi}{4}, 0\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2}\right).$$

The equation for $T_{\mathbf{p}}\mathcal{S}$ is $\mathbf{x} \cdot \mathbf{n} = 0$, which reads

$$\sqrt{2}\,x + \sqrt{2}\,y - z = 0.$$

Definition 3.41: Standard unit normal of a chart

Let $\mathcal{S}$ be a regular surface and $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ a regular chart. The **standard unit normal** of $\boldsymbol{\sigma}$ is the smooth function

$$\mathbf{N}_{\boldsymbol{\sigma}} : U \to \mathbb{R}^3, \quad \mathbf{N}_{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}.$$

Example 3.42: Calculation of $\mathbf{N}$

Question. Compute the standard unit normal to

$$\boldsymbol{\sigma}(u,v) = (e^u, u+v, v), \quad u, v \in \mathbb{R}.$$

Solution. The standard unit normal to $\boldsymbol{\sigma}$ is

$$\boldsymbol{\sigma}_u = (e^u, 1, 0), \quad \boldsymbol{\sigma}_v = (0,1,1), \qquad \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| = \sqrt{1+2e^{2u}}$$

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (1, -e^u, e^u) \qquad \mathbf{N}_{\boldsymbol{\sigma}} = \frac{(1, -e^u, e^u)}{\sqrt{1+2e^{2u}}}$$

Definition 3.43: Unit normal of a surface

Let $\mathcal{S}$ be a regular surface. A **unit normal** to $\mathcal{S}$ is a smooth function $\mathbf{N} : \mathcal{S} \to \mathbb{R}^3$ such that

$$\mathbf{N}(\mathbf{p}) \perp T_{\mathbf{p}}\mathcal{S}, \quad \|\mathbf{N}(\mathbf{p})\| = 1, \quad \forall\, \mathbf{p} \in \mathcal{S}.$$

Definition 3.44: Orientable surface

A regular surface $\mathcal{S}$ is **orientable** if there exists a unit normal $\mathbf{N} : \mathcal{S} \to \mathbb{R}^3$ and an atlas $\mathscr{A}$ such that

$$\mathbf{N} \circ \boldsymbol{\sigma} = \mathbf{N}_{\boldsymbol{\sigma}}, \qquad \forall\, \boldsymbol{\sigma} \in \mathscr{A}.$$

Definition 3.45: Differential of smooth function

Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be regular surfaces and $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ a smooth map. The differential $d_{\mathbf{p}}f$ of f at $\mathbf{p}$ is defined as the map

$$d_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\widetilde{\mathcal{S}}, \quad d_{\mathbf{p}}f(\mathbf{v}) := (f \circ \boldsymbol{\gamma})'(0),$$

with $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to \mathcal{S}$ smooth curve, $\boldsymbol{\gamma}(0) = \mathbf{p}$, $\dot{\boldsymbol{\gamma}}(0) = \mathbf{v}$.

Example 3.46: Computing $d_{\mathbf{p}}f$ using the definition

Question. Consider the plane $\mathcal{S} = \{z = 0\}$, the unit cylinder $\widetilde{\mathcal{S}} = \{x^2 + y^2 = 1\}$, and the map

$$f : S \to \widetilde{\mathcal{S}}, \qquad f(x,y,0) = (\cos x, \sin x, y).$$

1. Compute $T_{\mathbf{p}}\mathcal{S}$.
2. Compute $d_{\mathbf{p}}f$ at $\mathbf{p} = (u_0, v_0, 0)$ and $\mathbf{v} = (\lambda, \mu, 0)$.

Solution.

1. A chart for $\mathcal{S}$ is given by $\boldsymbol{\sigma}(u,v) = (u,v,0)$. Hence,

$$\boldsymbol{\sigma}_u = (1,0,0), \quad \boldsymbol{\sigma}_v = (0,1,0),$$

and the tangent space is

$$T_{\mathbf{p}}\mathcal{S} = \operatorname{span}\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} = \{(\lambda, \mu, 0) \,:\, \lambda, \mu \in \mathbb{R}\}.$$

2. Define the curve $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to \mathcal{S}$ by setting

$$\boldsymbol{\gamma}(t) := \boldsymbol{\sigma}(u_0 + t\lambda, v_0 + t\mu) = (u_0 + t\lambda, v_0 + t\mu, 0).$$

Note that $\boldsymbol{\gamma}(0) = \mathbf{p}$ and $\dot{\boldsymbol{\gamma}}(0) = \mathbf{v} = (\lambda, \mu, 0)$. Therefore, the differential is given by

$$\begin{aligned}
(f \circ \boldsymbol{\gamma})(t) &= (\cos(u_0 + t\lambda), \sin(u_0 + t\lambda), v_0 + t\mu),\\
(f \circ \boldsymbol{\gamma})'(t) &= (-\lambda \sin(u_0 + t\lambda), \lambda\cos(u_0 + t\lambda), \mu),\\
d_{\mathbf{p}}f(\mathbf{v}) &= (f \circ \boldsymbol{\gamma})'(0) = (-\lambda\sin(u_0), \lambda\cos(u_0), \mu).
\end{aligned}$$

Theorem 3.47: Matrix of $d_{\mathbf{p}}f$

Let $\mathcal{S}, \widetilde{\mathcal{S}}$ be regular surfaces, and $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ smooth.

1. $d_{\mathbf{p}}f(\mathbf{v})$ depends only on $f, \mathbf{p}, \mathbf{v}$ (and not on $\boldsymbol{\gamma}$).
2. $d_{\mathbf{p}}f$ is linear, that is, for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$ and $\lambda, \mu \in \mathbb{R}$

$$d_{\mathbf{p}}f(\lambda\mathbf{v} + \mu\mathbf{w}) = \lambda d_{\mathbf{p}}f(\mathbf{v}) + \mu d_{\mathbf{p}}f(\mathbf{w}).$$

3. Let $\boldsymbol{\sigma} : U \to \mathcal{S}$, $\tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \widetilde{\mathcal{S}}$ be regular charts at $\mathbf{p}$, $f(\mathbf{p})$. Let α and β be the components of $\Psi = \tilde{\boldsymbol{\sigma}}^{-1} \circ f \circ \boldsymbol{\sigma}$, so that

$$\tilde{\boldsymbol{\sigma}}(\alpha(u,v), \beta(u,v)) = f(\boldsymbol{\sigma}(u,v)), \quad \forall\,(u,v) \in U.$$

The matrix of $d_{\mathbf{p}}f$ with respect to the basis

$$\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} \text{ on } T_{\mathbf{p}}\mathcal{S}, \quad \{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\} \text{ on } T_{f(\mathbf{p})}\widetilde{\mathcal{S}},$$

is given by the Jacobian of the map Ψ, that is,

$$J\Psi = \begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}.$$

Example 3.48: Computing the matrix of $d_{\mathbf{p}}f$

Question. Let $\mathcal{S}$ be the cylinder, and $\widetilde{\mathcal{S}}$ the plane, charted by

$$\boldsymbol{\sigma}(u,v) = (\cos u, \sin u, v), \quad \tilde{\boldsymbol{\sigma}}(u,v) = (u,v,0),$$

defined on $U = (0, 2\pi) \times \mathbb{R}$ and $\widetilde{U} = \mathbb{R}^2$. Define the map

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}, \quad f(x,y,z) = (y, xz, 0).$$

Compute the matrix of $d_{\mathbf{p}}f$ with respect to $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ and $\{\tilde{\boldsymbol{\sigma}}_u, \tilde{\boldsymbol{\sigma}}_v\}$.

Solution. Note that $\tilde{\boldsymbol{\sigma}}^{-1}(u,v,0)=(u,v)$. Hence,

$$\begin{aligned}\Psi(u,v) &= \tilde{\boldsymbol{\sigma}}^{-1}\left(f(\boldsymbol{\sigma}(u,v))\right)=\tilde{\boldsymbol{\sigma}}^{-1}\left(f(\cos u,\sin u,v)\right)\\ &= \tilde{\boldsymbol{\sigma}}^{-1}\left(\sin(u),\cos(u)v,0\right)=\left(\sin(u),\cos(u)v\right).\end{aligned}$$

The components of Ψ are

$$\alpha(u,v)=\sin(u)\,,\quad \beta(u,v)=\cos(u)v\,.$$

The matrix of $d_{\mathbf{p}}f$ is hence

$$J\Psi=\begin{pmatrix}\alpha_u & \alpha_v\\ \beta_u & \beta_v\end{pmatrix}=\begin{pmatrix}\cos(u) & 0\\ -\sin(u)v & \cos(u)\end{pmatrix}.$$

3.3 Examples of Surfaces

Definition 3.49: Level surface

Let $f : V \to \mathbb{R}$ be smooth, $V \subseteq \mathbb{R}^3$ open. The **level surface** associated to f is the set

$$\mathcal{S}_f = f^{-1}(\{0\}) = \{(x,y,z)\in V \,:\, f(x,y,z)=0\}\,.$$

Theorem 3.50: Regularity of level surfaces

Let $f : V \to \mathbb{R}$ be smooth, with $V \subseteq \mathbb{R}^3$ open. Assume

$$\nabla f(x,y,z)\neq \mathbf{0}\,,\quad \forall\,(x,y,z)\in V\,.$$

Then $\mathcal{S}_f$ is a regular surface.

Example 3.51: Circular cone

Question. Prove the circular cone is a regular surface

$$\mathcal{S}=\{(x,y,z)\in\mathbb{R}^3 \,:\, x^2+y^2=z^2\,,\, z>0\}\,.$$

Solution. Define the open set $V\subset\mathbb{R}^3$ and $f : V\to\mathbb{R}$ by

$$V=\{(x,y,z)\in\mathbb{R}^3 \,:\, z>0\}\,,\quad f(x,y,z)=x^2+y^2-z^2\,.$$

$\mathcal{S}$ is a regular surface, since $\mathcal{S}=\mathcal{S}_f$ and

$$\nabla f(x,y,z)=(2x,2y,-2z)\neq\mathbf{0}\,,\quad \forall\,(x,y,z)\in V\,.$$

Theorem 3.52: Tangent plane of level surfaces

Let $f : V \to \mathbb{R}$ be smooth, with $V \subseteq \mathbb{R}^3$ open. Assume

$$\nabla f(x,y,z)\neq \mathbf{0}\,,\quad \forall\,(x,y,z)\in V\,.$$

Let $\mathbf{p}\in\mathcal{S}_f$. Then $\nabla f(\mathbf{p})\perp T_{\mathbf{p}}\mathcal{S}_f$ and $T_{\mathbf{p}}\mathcal{S}_f$ has equation

$$\nabla f(\mathbf{p})\cdot\mathbf{x}=0\,,\quad \forall\,\mathbf{x}\in\mathbb{R}^3\,.$$

Example 3.53: Unit cylinder

Question. Consider the unit cylinder $\mathcal{S}=\{x^2+y^2=1\}$.

1. Prove that $\mathcal{S}$ is a regular surface.
2. Find the equation of $T_{\mathbf{p}}\mathcal{S}$ at $\mathbf{p}=\left(\sqrt{2}/2,\sqrt{2}/2,5\right)$.

Solution.

1. Define the open set $V\subseteq\mathbb{R}^3$ and $f : V\to\mathbb{R}$ by

$$V=\mathbb{R}^3\setminus\{(0,0,z) \,:\, z\in\mathbb{R}\}\,,\quad f(x,y,z):=x^2+y^2-1\,.$$

$\mathcal{S}$ is a regular surface, since $\mathcal{S}=\mathcal{S}_f$ and

$$\nabla f(x,y,z)=(2x,2y,0)\neq\mathbf{0}\,,\quad \forall\,(x,y,z)\in V\,.$$

2. Using the expression for ∇f in Point 1, we get

$$\nabla f(\mathbf{p})=\nabla f\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},5\right)=(\sqrt{2},\sqrt{2},0)\,.$$

The equation for $T_{\mathbf{p}}\mathcal{S}$ is

$$\nabla f(\mathbf{p})\cdot\mathbf{x}=0 \quad\iff\quad x+y=0\,.$$

Definition 3.54: Ruled surface

A **ruled surface** is a surface with chart

$$\boldsymbol{\sigma}(u,v)=\boldsymbol{\gamma}(u)+v\mathbf{a}(u)\,,$$

where $\boldsymbol{\gamma},\mathbf{a} : (a,b)\to\mathbb{R}^3$ are smooth curves, such that

$$\dot{\boldsymbol{\gamma}}(t)\ \text{ and }\ \mathbf{a}(t)\ \text{ are linearly independent for all }\ t\in(a,b).$$

$\boldsymbol{\gamma}$ is the **base curve** and the lines $v\mapsto v\mathbf{a}(u)$ the **rulings**.

Theorem 3.55: Regularity of ruled surfaces

A ruled surface $\mathcal{S}$ is regular if v is sufficiently small.

Example 3.56: Unit Cylinder is ruled surface

Question. Prove that the unit cylinder is a ruled surface.
Solution. The unit cylinder $\mathcal{S}$ is charted by

$$\begin{gathered}\boldsymbol{\sigma}(u,v)=(\cos(u),\sin(u),v)=\boldsymbol{\gamma}(u)+v\mathbf{a}(u)\\ \boldsymbol{\gamma}(u)=(\cos(u),\sin(u),0)\,,\quad \mathbf{a}=(0,0,1)\end{gathered}$$

$\mathcal{S}$ is a ruled surface, since the vectors

$$\dot{\boldsymbol{\gamma}}=(-\sin(u),\cos(u),0)\,,\quad \mathbf{a}=(0,0,1)$$

are orthogonal, and hence linearly independent.

Example 3.57: A ruled surface

Question. Show that the following surface is ruled

$$S=\left\{(x,y,z)\in\mathbb{R}^3 \,:\, x^2+10xy+16x^2-z=0\right\}.$$

Solution. We can rearrange

$$x^2+10xy+16x^2-z=0 \quad\iff\quad (x+8y)(x+2y)=z\,.$$

Let $u = x + 8y$ and $v = x + 2y$. Therefore $uv = z$ and

$$u - v = 6y \implies y = \frac{u-v}{6} \implies x = u - 8y = \frac{4v-u}{3}.$$

It follows that if $(x,y,z) \in S$ then

$$\begin{aligned}(x,y,z) &= \left(\frac{4v-u}{3}, \frac{u-v}{6}, uv\right) \\ &= \left(-\frac{u}{3}, \frac{u}{6}, 0\right) + v\left(\frac{4}{3}, -\frac{1}{6}, u\right) = \boldsymbol{\gamma}(u) + v\mathbf{a}(u).\end{aligned}$$

When $u \neq 0$, the vectors

$$\mathbf{a}(u) = \left(\frac{4}{3}, -\frac{1}{6}, u\right), \quad \dot{\boldsymbol{\gamma}}(u) = \left(-\frac{1}{3}, \frac{1}{6}, 0\right),$$

are linearly independent, as the last component of $\dot{\boldsymbol{\gamma}}(u)$ is 0. Also $\mathbf{a}(0)$ and $\dot{\boldsymbol{\gamma}}(0)$ are linearly independent. Thus, $\mathcal{S}$ is a ruled surface.

Definition 3.58: Surface of revolution

Let $\boldsymbol{\gamma} : (a,b) \to \mathbb{R}^3$ be a smooth curve in the (x,z)-plane,

$$\boldsymbol{\gamma}(v) = (f(v), 0, g(v)), \qquad f > 0.$$

The surface $\mathcal{S}$ formed by rotating $\boldsymbol{\gamma}$ about the z-axis, called a **surface of revolution**, is charted by $\boldsymbol{\sigma} : U \to \mathbb{R}^3$

$$\boldsymbol{\sigma}(u,v) = (\cos(u)f(v), \sin(u)f(v), g(v)), \ U = (0, 2\pi) \times (a,b).$$

Theorem 3.59: Regularity of surfaces of revolution

A surface of revolution is regular if and only if $\boldsymbol{\gamma}$ is regular.

Example 3.60: Catenoid is surface of revolution

Question. The Catenoid $\mathcal{S}$ is the surface of revolution formed by rotating the catenary $\boldsymbol{\gamma}(v) = (\cosh(v), 0, v)$ about the z-axis. A chart for $\mathcal{S}$ is given by

$$\boldsymbol{\sigma}(u,v) = (\cos(u)\cosh(v), \sin(u)\cosh(v), v),$$

with $u \in (0, 2\pi), v \in \mathbb{R}$. Prove that $\mathcal{S}$ is a regular surface.
Solution. Note that $f > 0$. $\mathcal{S}$ is regular because $\boldsymbol{\gamma}$ is regular, as

$$\dot{\boldsymbol{\gamma}} = (\sinh(v), 0, 1), \quad \|\dot{\boldsymbol{\gamma}}\|^2 = 1 + \sinh(v)^2 \geq 1.$$

3.4 First fundamental form

Definition 3.61: First fundamental form (FFF)

Let $\mathcal{S}$ be a regular surface and $\mathbf{p} \in \mathcal{S}$. The **first fundamental form (FFF)** of $\mathcal{S}$ at $\mathbf{p}$ is the bilinear symmetric map

$$I_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{S} \times T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}, \quad I_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) := \mathbf{v} \cdot \mathbf{w}.$$

Definition 3.62: Coordinate functions on tangent plane

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, $\mathcal{S} = \boldsymbol{\sigma}(U)$. The **coordinate functions** on $T_{\mathbf{p}}\mathcal{S}$ are the linear maps

$$du, dv : T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}, \quad du(\mathbf{v}) := \lambda, \quad dv(\mathbf{v}) := \mu,$$

where $\mathbf{v} = \lambda\boldsymbol{\sigma}_u + \mu\boldsymbol{\sigma}_v$, since $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ is a basis for $T_{\mathbf{p}}\mathcal{S}$.

Definition 3.63: FFF of a chart

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, $\mathcal{S} = \boldsymbol{\sigma}(U)$. Define $E, F, G : U \to \mathbb{R}$

$$E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u, \quad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \quad G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v.$$

The **FFF** of $\boldsymbol{\sigma}$ is the quadratic form $\mathcal{F}_1 : T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}$

$$\mathcal{F}_1(\mathbf{v}) = E\,du^2(\mathbf{v}) + 2F\,du(\mathbf{v})\,dv(\mathbf{v}) + G\,dv^2(\mathbf{v}), \quad \forall\, v \in T_{\mathbf{p}}\mathcal{S},$$

for all $\mathbf{p} \in \boldsymbol{\sigma}(U)$, with E, F, G evaluated at $(u,v) = \boldsymbol{\sigma}^{-1}(\mathbf{p})$.

Theorem 3.64: Matrix of FFF

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, $\mathcal{S} = \boldsymbol{\sigma}(U)$, and $\mathbf{p} \in \boldsymbol{\sigma}(U)$. Then

$$I_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = (du(\mathbf{v}), dv(\mathbf{v})) \begin{pmatrix} E & F \\ F & G \end{pmatrix} (du(\mathbf{w}), dv(\mathbf{w}))^T,$$

for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$. In particular, it holds

$$\mathcal{F}_1(\mathbf{v}) = I_{\mathbf{p}}(\mathbf{v}, \mathbf{v}), \quad \forall\, \mathbf{v} \in T_{\mathbf{p}}\mathcal{S}.$$

Example 3.65: FFF of Unit cylinder

Question. Consider the unit cylinder with chart

$$\boldsymbol{\sigma}(u,v) = (\cos(u), \sin(u), v), \quad (u,v) \in (0, 2\pi) \times \mathbb{R}.$$

Prove that the FFF of $\boldsymbol{\sigma}$ is

$$\mathcal{F}_1 = du^2 + dv^2.$$

Solution. We have

$$\begin{aligned} \boldsymbol{\sigma}_u &= (-\sin(u), \cos(u), 0) & F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0 \\ \boldsymbol{\sigma}_v &= (0,0,1) & G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 1 \\ E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 1 & \mathcal{F}_1 &= du^2 + dv^2 \end{aligned}$$

Proposition 3.66: FFF and reparametrizations

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, and $\tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathbb{R}^3$ a reparametrization, with $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi$ and $\Phi : \widetilde{U} \to U$ diffeomorphism. The matrices $\mathcal{F}_1$ and $\widetilde{\mathcal{F}}_1$ of the FFF of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are related by

$$\widetilde{\mathcal{F}}_1 = (J\Phi)^T \mathcal{F}_1\, J\Phi, \quad \mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \widetilde{\mathcal{F}}_1 = \begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix}.$$

Example 3.67: FFF of Plane

Question. Let $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, with $\mathbf{p}$, $\mathbf{q}$ orthonormal. The plane in cartesian and polar coordinates is charted by, respectively,

$$\boldsymbol{\sigma}(u,v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}\,, \quad (u,v) \in \mathbb{R}^2\,,$$
$$\tilde{\boldsymbol{\sigma}}(\rho,\theta) = \mathbf{a} + \rho\cos(\theta)\mathbf{p} + \rho\sin(\theta)\mathbf{q}\,, \quad \rho > 0,\ \theta \in (0,2\pi)\,.$$

1. Show that the FFF of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are

$$\mathscr{F}_1 = du^2 + dv^2\,, \qquad \widetilde{\mathscr{F}}_1 = d\rho^2 + \rho^2 d\theta^2\,.$$

2. Let Φ be the change of variables from polar to cartesian coordinates. Show that

$$\widetilde{\mathscr{F}}_1 = (J\Phi)^T\, \mathscr{F}_1\, J\Phi\,.$$

Solution.

1. Using that $\mathbf{p}$ and $\mathbf{q}$ are orthonormal,

$$\begin{aligned}
&\boldsymbol{\sigma}_u = \mathbf{p}\,, && \tilde{\boldsymbol{\sigma}}_\rho = \cos(\theta)\mathbf{p} + \sin(\theta)\mathbf{q}\\
&\boldsymbol{\sigma}_v = \mathbf{q} && \tilde{\boldsymbol{\sigma}}_\theta = -\rho\sin(\theta)\mathbf{p} + \rho\cos(\theta)\mathbf{q}\\
&E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 1 && \widetilde{E} = \tilde{\boldsymbol{\sigma}}_\rho \cdot \tilde{\boldsymbol{\sigma}}_\rho = 1\\
&F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0 && \widetilde{F} = \tilde{\boldsymbol{\sigma}}_\rho \cdot \tilde{\boldsymbol{\sigma}}_\theta = 0\\
&G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 1 && \widetilde{G} = \tilde{\boldsymbol{\sigma}}_\theta \cdot \tilde{\boldsymbol{\sigma}}_\theta = r^2\\
&\mathscr{F}_1 = du^2 + dv^2 && \widetilde{\mathscr{F}}_1 = d\rho^2 + \rho^2 d\theta^2
\end{aligned}$$

2. We have $\Phi(\rho,\theta) = (\rho\cos(\theta), \rho\sin(\theta))$. Then

$$\begin{aligned}
(J\Phi)^T\, \mathscr{F}_1 J\Phi &= (J\Phi)^T\, J\Phi\\
&= \begin{pmatrix} \cos(\theta) & \sin(\theta)\\ -\rho\sin(\theta) & \rho\cos(\theta)\end{pmatrix}\begin{pmatrix} \cos(\theta) & -\rho\sin(\theta)\\ \sin(\theta) & \rho\cos(\theta)\end{pmatrix}\\
&= \begin{pmatrix} 1 & 0\\ 0 & \rho^2\end{pmatrix} = \widetilde{\mathscr{F}}_1\,.
\end{aligned}$$

Proposition 3.68: Length of curves and FFF

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, $\mathscr{S} = \boldsymbol{\sigma}(U)$. Let $\boldsymbol{\gamma} : (a,b) \to \mathscr{S}$ be a smooth curve. Then

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))\,,$$

for some smooth functions $u, v : (a,b) \to \mathbb{R}$, and

$$\int_a^b \|\dot{\boldsymbol{\gamma}}(t)\|\, dt = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt\,,$$

where $\dot{u}, \dot{v}$ are computed at t, and E, F, G at $(u(t), v(t))$.

Example 3.69: Curves on the Cone

Question. Consider the cone with chart

$$\boldsymbol{\sigma}(u,v) = (\cos(u)v, \sin(u)v, v)\,, \quad u \in (0,2\pi),\ v > 0\,.$$

1. Compute the first fundamental form of $\boldsymbol{\sigma}$.

2. Compute the length of $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(t,t)$ for $t \in (\pi/2, \pi)$.

Solution.

1. The first fundamental form of $\boldsymbol{\sigma}$ is

$$\begin{aligned}
&\boldsymbol{\sigma}_u = (-\sin(u)v, \cos(u)v, 0) && F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0\\
&\boldsymbol{\sigma}_v = (\cos(u), \sin(u), 1) && G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 2\\
&E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = v^2 && \mathscr{F}_1 = v^2\, du^2 + 2\, dv^2
\end{aligned}$$

2. $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ with $u(t) = t$ and $v(t) = t$. Then

$$\begin{aligned}
&\dot{u} = 1\,,\ \dot{v} = 1 && F(u(t),v(t)) = F(t,t) = 0\\
&E(u(t),v(t)) = E(t,t) = t^2 && G(u(t),v(t)) = G(t,t) = 2
\end{aligned}$$

The length of $\boldsymbol{\gamma}$ between $\pi/2$ and π is

$$\int_{\pi/2}^{\pi} \|\dot{\boldsymbol{\gamma}}(t)\|\, dt = \int_{\pi/2}^{\pi} \sqrt{t^2 + 2}\, dt\,.$$

Definition 3.70: Local isometry

Let $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ be regular and $f : \mathscr{S} \to \widetilde{\mathscr{S}}$ smooth. We say that f is a **local isometry**, if for all $\mathbf{p} \in \mathscr{S}$

$$\mathbf{v} \cdot \mathbf{w} = d_{\mathbf{p}} f(\mathbf{v}) \cdot d_{\mathbf{p}} f(\mathbf{w})\,, \quad \forall\, \mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathscr{S}\,.$$

In this case, $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ are said to be **locally isometric**.

Proposition 3.71

Local isometries are local diffeomorphims.

Theorem 3.72: Local isometries preserve lengths

Let $\mathscr{S}, \widetilde{\mathscr{S}}$ be regular surfaces, $f : \mathscr{S} \to \widetilde{\mathscr{S}}$ smooth. Equivalently:

1. f is a local isometry.
2. Let $\boldsymbol{\gamma}$ be a curve on $\mathscr{S}$ and define the curve $\tilde{\boldsymbol{\gamma}} = f \circ \boldsymbol{\gamma}$ on $\widetilde{\mathscr{S}}$. Then $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same length.

Theorem 3.73: Local isometries preserve FFF

Let $\mathscr{S}, \widetilde{\mathscr{S}}$ be regular surfaces, $f : \mathscr{S} \to \widetilde{\mathscr{S}}$ smooth. Equivalently:

1. f is a local isometry.
2. Let $\boldsymbol{\sigma} : U \to \mathscr{S}$ be regular chart of $\mathscr{S}$, and define a chart of $\widetilde{\mathscr{S}}$ as $\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathscr{S}}$, with $\tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}$. Then $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ have the same FFF

$$E = \widetilde{E}\,, \quad F = \widetilde{F}\,, \quad G = \widetilde{G}\,.$$

Theorem 3.74: Sufficient condition for local isometry

Let $\mathscr{S}, \widetilde{\mathscr{S}}$ be regular surfaces, with charts $\boldsymbol{\sigma} : U \to \mathscr{S}$ and $\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathscr{S}}$. Assume that $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ have the same FFF. We have

1. The surfaces $\boldsymbol{\sigma}(U)$ and $\widetilde{\mathscr{S}}$ are locally isometric.
2. A local isometry is given by

$$f : \boldsymbol{\sigma}(U) \to \widetilde{\mathscr{S}}\,, \qquad f = \tilde{\boldsymbol{\sigma}} \circ \boldsymbol{\sigma}^{-1}\,.$$

Example 3.75: Plane and Cylinder are locally isometric

Question. Consider the plane $\mathcal{S} = \{x = 0\}$ and the unit cylinder $\widetilde{\mathcal{S}} = \{x^2 + y^2 = 1\}$. Define the function

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}\,, \qquad f(0, y, z) = (\cos(y), \sin(y), z)\,.$$

Prove that f is a local isometry (you may assume f smooth).
Solution. The plane $\mathcal{S}$ is charted by

$$\boldsymbol{\sigma}(u, v) = (0, u, v)\,, \quad u, v \in \mathbb{R}\,.$$

We already know that $\boldsymbol{\sigma}$ is regular, with FFF coefficients

$$E = 1\,, \; F = 0\,, \; G = 1 \quad \Longrightarrow \quad \mathcal{F}_1 = du^2 + dv^2\,.$$

Define $\tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}$. Therefore,

$$\tilde{\boldsymbol{\sigma}}(u, v) = f(0, u, v) = (\cos(u), \sin(u), v)\,.$$

The FFF of $\tilde{\boldsymbol{\sigma}}$ is

$$\begin{aligned} &\tilde{\boldsymbol{\sigma}}_u = (-\sin(u), \cos(u), 0) && \widetilde{F} = \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_v = 0 \\ &\tilde{\boldsymbol{\sigma}}_v = (0, 0, 1) && \widetilde{G} = \tilde{\boldsymbol{\sigma}}_v \cdot \tilde{\boldsymbol{\sigma}}_v = 1 \\ &\widetilde{E} = \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_u = 1 && \widetilde{\mathcal{F}}_1 = du^2 + dv^2 \end{aligned}$$

Thus, $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ have the same FFF. Since $\mathcal{A} = \{\boldsymbol{\sigma}\}$ is an atlas for $\mathcal{S}$, by Theorem 1.74 we conclude that f is a local isometry of $\mathcal{S}$ into $\widetilde{\mathcal{S}}$.

Example 3.76: Plane and Cone are locally isometric

Question. Consider the cone without tip

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \; : \; x^2 + y^2 = z^2\,, \; z > 0\}\,,$$

and the plane $\widetilde{\mathcal{S}} = \{z = 0\}$.

1. Compute the FFF of the chart of the Cone

$$\begin{aligned} &\boldsymbol{\sigma} : U \to \mathcal{S}\,, \qquad \boldsymbol{\sigma}(\rho, \theta) = (\rho\cos(\theta), \rho\sin(\theta), \rho)\,, \\ &U = \left\{(\rho, \theta) \in \mathbb{R}^2 \; : \; \rho > 0, \, \theta \in (0, 2\pi)\right\}\,. \end{aligned}$$

2. Compute the FFF of the chart of the plane

$$\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}}\,, \qquad \tilde{\boldsymbol{\sigma}}(\rho, \theta) = (a\rho\cos(b\theta), a\rho\sin(b\theta), 0)\,,$$

where $a > 0$ and $b \in (0, 1]$ are constants.

3. Prove that $f = \tilde{\boldsymbol{\sigma}} \circ \boldsymbol{\sigma}^{-1}$ is a local isometry between $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, for suitable coefficients a, b.

Solution.

1. As seen in Example 1.71, the coefficients of the FFF of $\boldsymbol{\sigma}$ are

$$E = 2\,, \qquad F = 0\,, \qquad G = \rho^2\,.$$

2. Note that $\tilde{\boldsymbol{\sigma}}$ is well defined for all $(\rho, \theta) \in U$, as

$$\theta \in (0, 2\pi), \quad b \in (0, 1] \quad \Longrightarrow \quad b\theta \in (0, 2\pi)\,.$$

The coefficients of the FFF of $\tilde{\boldsymbol{\sigma}}$ are

$$\begin{aligned} &\tilde{\boldsymbol{\sigma}}_\rho = a\,(\cos(b\theta), \sin(b\theta), 0) && \widetilde{F} = \tilde{\boldsymbol{\sigma}}_\rho \cdot \tilde{\boldsymbol{\sigma}}_\theta = 0 \\ &\tilde{\boldsymbol{\sigma}}_\theta = ab\rho\,(-\sin(b\theta), \cos(b\theta), 0) && \widetilde{G} = \tilde{\boldsymbol{\sigma}}_\theta \cdot \tilde{\boldsymbol{\sigma}}_\theta = a^2 b^2 \rho^2 \\ &\widetilde{E} = \tilde{\boldsymbol{\sigma}}_\rho \cdot \tilde{\boldsymbol{\sigma}}_\rho = a^2 && \end{aligned}$$

3. Imposing that $\widetilde{E} = E$, $\widetilde{F} = F$ and $\widetilde{G} = G$, we obtain

$$a^2 = 2, \; a^2 b^2 = 1 \quad \Longrightarrow \quad a = \sqrt{2}\,, \; b = \frac{1}{\sqrt{2}}\,.$$

Note that $a > 0$ and $0 < b < 1$, showing that a, b are admissible. Hence, for $a = \sqrt{2}$ and $b = 1/\sqrt{2}$, the charts $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ have the same FFF. By Theorem 1.73, we conclude that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are locally isometric, with local isometry given by $f = \tilde{\boldsymbol{\sigma}} \circ \boldsymbol{\sigma}^{-1}$.

Definition 3.77: Angle between curves

Let $\mathcal{S}$ be a regular surface, and $\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}$ curves on $\mathcal{S}$ intersecting at

$$\boldsymbol{\gamma}(t_0) = \mathbf{p} = \tilde{\boldsymbol{\gamma}}(t_0)\,.$$

The angle θ between $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ is

$$\cos(\theta) = \frac{\dot{\boldsymbol{\gamma}}(t_0) \cdot \dot{\tilde{\boldsymbol{\gamma}}}(t_0)}{\|\dot{\boldsymbol{\gamma}}(t_0)\| \, \|\dot{\tilde{\boldsymbol{\gamma}}}(t_0)\|}\,.$$

Theorem 3.78: Angle between curves and FFF

Let $\mathcal{S}$ be a regular surface, $\boldsymbol{\sigma}$ regular chart at $\mathbf{p}$, and $\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}$ curves on $\mathcal{S}$ intersecting at $\boldsymbol{\gamma}(t_0) = \mathbf{p} = \tilde{\boldsymbol{\gamma}}(t_0)$. There exist smooth functions $u, v, \tilde{u}, \tilde{v}$ such that

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))\,, \quad \tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\sigma}(\tilde{u}(t), \tilde{v}(t))\,.$$

The angle between $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ is

$$\cos(\theta) = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}) + G\dot{v}\dot{\tilde{v}}}{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}}\,,$$

with $E, F, G, \widetilde{E}, \widetilde{F}, \widetilde{G}$ evaluated at $(u(t_0), v(t_0))$, and $\dot{u}, \dot{v}, \dot{\tilde{u}}, \dot{\tilde{v}}$ at t_0.

Example 3.79: Calculation of angle between curves

Question. Let S be a surface charted by

$$\boldsymbol{\sigma}(u, v) = (u, v, e^{uv})\,.$$

1. Calculate the FFF of $\boldsymbol{\sigma}$.
2. Calculate $\cos(\theta)$, where θ is the angle between the two curves

$$\begin{aligned} &\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t)), \quad u(t) = t, \; v(t) = t\,, \\ &\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\sigma}(\tilde{u}(t), \tilde{v}(t))\,, \quad \tilde{u}(t) = 1, \; \tilde{v}(t) = t\,. \end{aligned}$$

Solution.

1. The coefficients of the FFF are

$$\begin{aligned} &\boldsymbol{\sigma}_u = (1, 0, e^{uv} v) && F(u, v) = e^{2uv} uv \\ &\boldsymbol{\sigma}_v = (0, 1, e^{uv} u) && G(u, v) = 1 + e^{2uv} u^2 \\ &E(u, v) = 1 + e^{2uv} v^2 && \end{aligned}$$

2. $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ intersect at $\boldsymbol{\gamma}(1) = \tilde{\boldsymbol{\gamma}}(1) = \boldsymbol{\sigma}(1, 1)$. We compute

$$\begin{aligned} &\dot{u}(1) = 1 && E(1, 1) = 1 + e^2 \\ &\dot{v}(1) = 1 && F(1, 1) = e^2 \\ &\dot{\tilde{u}}(1) = 0 && G(1, 1) = 1 + e^2 \\ &\dot{\tilde{v}}(1) = 1 && \end{aligned}$$

Therefore, the angle θ satisfies

$$\cos(\theta) = \frac{1+2e^2}{\sqrt{2+4e^2}\sqrt{1+e^2}} = \sqrt{\frac{1+2e^2}{2+2e^2}}\,.$$

Definition 3.80: Conformal map

Let $\mathcal{S}, \widetilde{\mathcal{S}}$ be regular surfaces, $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ local diffeomorphism. We say that f is a **conformal map**, if for all $\mathbf{p} \in \mathcal{S}$

$$\theta = \tilde{\theta}\,, \quad \forall\, \mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}\,,$$

- θ is the angle between $\mathbf{v}$ and $\mathbf{w}$,
- $\tilde{\theta}$ is the angle between $d_{\mathbf{p}}f(\mathbf{v})$ and $d_{\mathbf{p}}f(\mathbf{w})$.

In this case, we say that $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are **conformal**.

Proposition 3.81

Local isometries are conformal maps.

Theorem 3.82: Conformal maps and FFF

Let $\mathcal{S}$, $\widetilde{\mathcal{S}}$ be regular surfaces, $f : \mathcal{S} \to \widetilde{\mathcal{S}}$ a local diffeomorphism. Equivalently:

1. f is a conformal map.
2. Let $\boldsymbol{\sigma} : U \to \mathcal{S}$ be regular chart of $\mathcal{S}$, and define a chart of $\widetilde{\mathcal{S}}$ as $\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}}$, with $\tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}$. Then the FFF of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ satisfy

$$\widetilde{\mathcal{F}}_1 = \lambda(u,v)\mathcal{F}_1\,, \quad \forall\,(u,v) \in U\,,$$

   for some smooth map $\lambda : U \to \mathbb{R}$.

Theorem 3.83: Sufficient condition for conformality

Let $\mathcal{S}, \widetilde{\mathcal{S}}$ be regular surfaces, with charts $\boldsymbol{\sigma} : U \to \mathcal{S}$ and $\tilde{\boldsymbol{\sigma}} : U \to \widetilde{\mathcal{S}}$. Assume that $\widetilde{\mathcal{F}}_1 = \lambda\mathcal{F}_1$ for some $\lambda : U \to \mathbb{R}$. We have

1. The surfaces $\boldsymbol{\sigma}(U)$ and $\widetilde{\mathcal{S}}$ are conformal.
2. A conformal map is given by $f : \boldsymbol{\sigma}(U) \to \widetilde{\mathcal{S}}$ with $f = \tilde{\boldsymbol{\sigma}} \circ \boldsymbol{\sigma}^{-1}$.

Example 3.84: Stereographic Projection

Question. Consider the unit sphere $\mathbb{S}^2 = \{x^2+y^2+z^2 = 1\}$ and define the surface $\mathcal{S} = \mathbb{S}^2 \smallsetminus \{N\}$, where $N = (0,0,1)$. Consider the plane $\widetilde{\mathcal{S}} = \{z = 0\}$. The *Stereographic Projection* is

$$f : \mathcal{S} \to \widetilde{\mathcal{S}}\,, \quad f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)\,.$$

Prove that f is a conformal map.
Solution. It is easy to prove that $f^{-1} = \boldsymbol{\sigma}$, with

$$\boldsymbol{\sigma}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, 1 - \frac{2}{u^2+v^2+1}\right)\,.$$

It is straightforward to compute that the FFF of $\boldsymbol{\sigma}$ is

$$\mathcal{F}_1 = \lambda(u,v)(du^2+dv^2)\,, \quad \lambda(u,v) = \frac{4}{(u^2+v^2+1)^2}\,.$$

Let $\tilde{\boldsymbol{\sigma}} = f \circ \boldsymbol{\sigma}$. Since $\boldsymbol{\sigma} = f^{-1}$, we have that $\tilde{\boldsymbol{\sigma}}(u,v) = (u,v,0)$. As already computed, the FFF of $\tilde{\boldsymbol{\sigma}}$ is $\widetilde{\mathcal{F}}_1 = du^2 + dv^2$. Therefore,

$$\widetilde{\mathcal{F}}_1 = \frac{1}{\lambda}\mathcal{F}_1\,.$$

Since $\mathcal{A} = \{\boldsymbol{\sigma}\}$ is an atlas for $\mathcal{S}$, by Theorem 3.82 we conclude that f is a conformal map.

Definition 3.85: Conformal parametrization

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular. We say that $\boldsymbol{\sigma}$ is a **conformal parametrization** if the FFF of $\boldsymbol{\sigma}$ satisfies

$$\mathcal{F}_1 = \lambda(u,v)(du^2+dv^2)\,,$$

for some smooth function $\lambda : U \to \mathbb{R}$.

Example 3.86: Mercator projection

Question. Prove that the parametrization of $\mathbb{S}^2$ is conformal

$$\boldsymbol{\sigma}(u,v) := (\cos(u)\operatorname{sech}(v), \sin(u)\operatorname{sech}(v), \tanh(v))\,.$$

Solution. Recall the identities $\operatorname{sech}^2(v) + \tanh^2(v) = 1$ and

$$\operatorname{sech}(v)' = -\operatorname{sech}(v)\tanh(v)\,, \qquad \tanh(v)' = \operatorname{sech}^2(v)\,.$$

The chart $\boldsymbol{\sigma}$ is a conformal parametrization because the FFF is

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}}_u &= \operatorname{sech}(v)(-\sin(u), \cos(u), 0)\\
\tilde{\boldsymbol{\sigma}}_v &= \operatorname{sech}(v)(-\cos(v)\tanh(v), -\sin(u)\tanh(v), \operatorname{sech}(v))\\
\widetilde{E} &= \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_u = \operatorname{sech}^2(v)(\cos^2(u) + \sin^2(u)) = \operatorname{sech}^2(v)\\
\widetilde{F} &= \tilde{\boldsymbol{\sigma}}_u \cdot \tilde{\boldsymbol{\sigma}}_v = 0\\
\widetilde{G} &= \tilde{\boldsymbol{\sigma}}_v \cdot \tilde{\boldsymbol{\sigma}}_v = \operatorname{sech}^2(v)(\tanh^2(v) + \operatorname{sech}^2(v)) = \operatorname{sech}^2(v)\\
\mathcal{F}_1 &= \operatorname{sech}^2(v)\,\left(du^2+dv^2\right)\,.
\end{aligned}$$

Theorem 3.87: Conformal parametrizations preserve angles

Let $\boldsymbol{\sigma}$ be a conformal parametrization, and $\boldsymbol{\gamma}_1(t), \boldsymbol{\gamma}_2(t)$ be curves in $\mathbb{R}^2$ such that $\dot{\boldsymbol{\gamma}}_1(t_0), \dot{\boldsymbol{\gamma}}_2(t_0)$ make angle θ. Let $\tilde{\boldsymbol{\gamma}}_1 = \boldsymbol{\sigma} \circ \boldsymbol{\gamma}_1$ and $\tilde{\boldsymbol{\gamma}}_2 = \boldsymbol{\sigma} \circ \boldsymbol{\gamma}_2$. Then $\dot{\tilde{\boldsymbol{\gamma}}}_1(t_0), \dot{\tilde{\boldsymbol{\gamma}}}_2(t_0)$ also make angle θ.

3.5 Second fundamental form

Definition 3.88: Second fundamental form of a chart

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, $\mathcal{S} = \boldsymbol{\sigma}(U)$. Define $L, M, N : U \to \mathbb{R}$

$$L := \boldsymbol{\sigma}_{uu} \cdot \mathbf{N}\,, \quad M := \boldsymbol{\sigma}_{uv} \cdot \mathbf{N}\,, \quad N := \boldsymbol{\sigma}_{vv} \cdot \mathbf{N}\,,$$

where $\mathbf{N}$ is the standard unit normal to $\boldsymbol{\sigma}$. The **second fundamental form (SFF)** of $\boldsymbol{\sigma}$ is the quadratic form $\mathcal{F}_2 : T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}$

$$\mathcal{F}_2(\mathbf{v}) = L\,du^2(\mathbf{v}) + 2M\,du(\mathbf{v})\,dv(\mathbf{v}) + N\,dv^2(\mathbf{v})\,, \quad \forall\, \mathbf{v} \in T_{\mathbf{p}}\mathcal{S}\,,$$

for all $\mathbf{p} \in \boldsymbol{\sigma}(U)$, with L, M, N evaluated at $(u,v) = \boldsymbol{\sigma}^{-1}(\mathbf{v})$, and du, dv the coordinate functions in Definition 1.62.

Example 3.89: SFF of Plane

Question. Let $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, with $\mathbf{p}$, $\mathbf{q}$ orthonormal. The plane is charted by

$$\boldsymbol{\sigma}(u,v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}\,, \quad (u,v) \in \mathbb{R}^2\,.$$

Prove that the SFF of $\boldsymbol{\sigma}$ is $\mathcal{F}_2 = 0$.
Solution. We have that $\mathcal{F}_2 = 0$, since

$$\boldsymbol{\sigma}_u = \mathbf{p}\,, \quad \boldsymbol{\sigma}_v = \mathbf{q}\,, \quad \boldsymbol{\sigma}_{uu} = \boldsymbol{\sigma}_{uv} = \boldsymbol{\sigma}_{vv} = \mathbf{0}\,,$$
$$L = \boldsymbol{\sigma}_{uu} \cdot \mathbf{N} = 0\,, \quad M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N} = 0\,, \quad N = \boldsymbol{\sigma}_{vv} \cdot \mathbf{N} = 0\,.$$

Example 3.90: SFF of Unit cylinder

Question. Consider the unit cylinder with chart

$$\boldsymbol{\sigma}(u,v) = (\cos(u), \sin(u), v)\,, \quad (u,v) \in (0,2\pi) \times \mathbb{R}\,.$$

Prove that the SFF of $\boldsymbol{\sigma}$ is

$$\mathcal{F}_2 = -du^2\,.$$

Solution. We have

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (-\sin(u), \cos(u), 0) & \mathbf{N} &= \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} \\
\boldsymbol{\sigma}_v &= (0,0,1) & &= (\cos(u), \sin(u), 0) \\
\boldsymbol{\sigma}_{uu} &= (-\cos(u), -\sin(u), 0) & L &= \boldsymbol{\sigma}_{uu} \cdot \mathbf{N} = -1 \\
\boldsymbol{\sigma}_{uv} &= \boldsymbol{\sigma}_{vv} = \mathbf{0} & M &= \boldsymbol{\sigma}_{uv} \cdot \mathbf{N} = 0 \\
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (\cos(u), \sin(u), 0) & N &= \boldsymbol{\sigma}_{vv} \cdot \mathbf{N} = 0 \\
\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| &= 1 & \mathcal{F}_2 &= -du^2
\end{aligned}$$

Remark 3.91: SFF and reparametrizations

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, and $\tilde{\boldsymbol{\sigma}} : \widetilde{U} \to \mathbb{R}^3$ a reparametrization, with $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Phi$ and $\Phi : \widetilde{U} \to U$ diffeomorphism. The matrices $\mathcal{F}_2$ and $\widetilde{\mathcal{F}}_2$ of the SFF of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are related by

$$\widetilde{\mathcal{F}}_2 = \pm (J\Phi)^T \mathcal{F}_2 J\Phi\,, \quad \mathcal{F}_2 = \begin{pmatrix} \tilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix}, \quad \widetilde{\mathcal{F}}_2 \begin{pmatrix} \tilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix},$$

where the formula holds with the plus sign if $\det J\Phi > 0$, and with the minus sign if $\det J\Phi < 0$.

Definition 3.92: Gauss map

Let $\mathcal{S}$ be an oriented surface with standard unit normal $\mathbf{N}$. The **Gauss map** of $\mathcal{S}$ is

$$\mathcal{G}_{\mathcal{S}} : \mathcal{S} \to \mathbb{S}^2\,, \quad \mathcal{G}_{\mathcal{S}}(\mathbf{p}) := \mathbf{N}(\mathbf{p})\,.$$

Definition 3.93: Weingarten map

Let $\mathcal{S}$ be an orientable surface with Gauss map $\mathcal{G} : \mathcal{S} \to \mathbb{S}^2$. The **Weingarten map** $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ of $\mathcal{S}$ at $\mathbf{p}$ is

$$\mathcal{W}_{\mathbf{p},\mathcal{S}} : T_{\mathbf{p}}\mathcal{S} \to T_{\mathbf{p}}\mathcal{S}\,, \quad \mathcal{W}_{\mathbf{p},\mathcal{S}}(\mathbf{v}) = -d_{\mathbf{p}}\mathcal{G}(\mathbf{v})\,.$$

Lemma 3.94

Let $\mathcal{S}$ be an orientable surface with Weingarten map $\mathcal{W}_{\mathbf{p},\mathcal{S}}$, and $\boldsymbol{\sigma}$ a regular chart at $\mathbf{p}$. Then

$$\mathcal{W}_{\mathbf{p},\mathcal{S}}(\boldsymbol{\sigma}_u) = -\mathbf{N}_u\,, \quad \mathcal{W}_{\mathbf{p},\mathcal{S}}(\boldsymbol{\sigma}_v) = -\mathbf{N}_v\,,$$

where $\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v, \mathbf{N}_u, \mathbf{N}_v$ are evaluated at $(u,v) = \boldsymbol{\sigma}^{-1}(\mathbf{p})$.

Definition 3.95: SFF of a surface

Let $\mathcal{S}$ be an orientable surface with Weingarten map $\mathcal{W}_{\mathbf{p},\mathcal{S}}$. The **SFF** of $\mathcal{S}$ at $\mathbf{p}$ is the bilinear map

$$II_{\mathbf{p}} : T_{\mathbf{p}}\mathcal{S} \times T_{\mathbf{p}}\mathcal{S} \to \mathbb{R}\,, \quad II_{\mathbf{p}}(\mathbf{v},\mathbf{w}) := \mathcal{W}_{\mathbf{p},\mathcal{S}}(\mathbf{v}) \cdot \mathbf{w}\,.$$

Theorem 3.96: Matrix of the SFF

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be regular, $\mathcal{S} = \boldsymbol{\sigma}(U)$, and $\mathbf{p} \in \boldsymbol{\sigma}(U)$. Then

$$II_{\mathbf{p}}(\mathbf{v},\mathbf{w}) = (du(\mathbf{v}), dv(\mathbf{v})) \begin{pmatrix} L & M \\ M & N \end{pmatrix} (du(\mathbf{w}), dv(\mathbf{w}))^T\,,$$

for all $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$. In particular, it holds

$$\mathcal{F}_2(\mathbf{v}) = II_{\mathbf{p}}(\mathbf{v},\mathbf{v})\,, \quad \forall\, \mathbf{v} \in T_{\mathbf{p}}\mathcal{S}\,.$$

Theorem 3.97: Matrix of Weingarten map

Let $\mathcal{S}$ be an orientable surface with Weingarten map $\mathcal{W}_{\mathbf{p},\mathcal{S}}$. Let $\boldsymbol{\sigma}$ be a regular chart at $\mathbf{p}$. The matrix of the Weingarten map with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of $T_{\mathbf{p}}\mathcal{S}$ is

$$\mathcal{W} = \mathcal{F}_1^{-1}\mathcal{F}_2\,,$$

where the FFF and SFF are evaluated at $(u,v) = \boldsymbol{\sigma}^{-1}(\mathbf{p})$.

Remark 3.98: Matrix inverse

A matrix $A \in \mathbb{R}^{2\times 2}$ is invertible if and only if $\det(A) \neq 0$. In such case the inverse A^{-1} is computed via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det(A) = ad - bc\,.$$

If the matrix is diagonal, then

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^{-1} = \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1/\mu \end{pmatrix}.$$

Example 3.99: Weingarten map of Helicoid

Question. The Helicoid is charted by

$$\boldsymbol{\sigma}(u,v) = (u\cos(v), u\sin(v), \lambda v)\,, \quad u \in \mathbb{R}\,,\ v \in (0,2\pi)\,,$$

with $\lambda > 0$ constant. Compute the matrix of the Weingarten map.
Solution. We compute all the derivatives of $\boldsymbol{\sigma}$

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (\cos(v), \sin(v), 0) & \boldsymbol{\sigma}_{uv} &= (-\sin(v), \cos(v), 0) \\
\boldsymbol{\sigma}_v &= (-u\sin(v), u\cos(v), \lambda) & \boldsymbol{\sigma}_{vv} &= -u\,(\cos(v), \sin(v), 0) \\
\boldsymbol{\sigma}_{uu} &= (0,0,0)
\end{aligned}$$

The FFF and its inverse are

$$E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 1 \qquad F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0$$
$$G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = u^2 + \lambda^2$$
$$\mathcal{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & u^2+\lambda^2 \end{pmatrix} \qquad \mathcal{F}_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{u^2+\lambda^2} \end{pmatrix}.$$

The standard unit normal to $\boldsymbol{\sigma}$ is

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (\lambda \sin(v), -\lambda \cos(v), u)$$
$$\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| = \sqrt{u^2+\lambda^2}$$
$$\mathbf{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = \frac{1}{\sqrt{u^2+\lambda^2}} (\lambda \sin(v), -\lambda \cos(v), u).$$

The SFF of $\boldsymbol{\sigma}$ is

$$L = \boldsymbol{\sigma}_{uu} \cdot \mathbf{N} = 0 \qquad M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N} = -\frac{\lambda}{\sqrt{u^2+\lambda^2}}$$
$$N = \boldsymbol{\sigma}_{vv} \cdot \mathbf{N} = 0$$
$$\mathcal{F}_2 = \begin{pmatrix} 0 & -\dfrac{\lambda}{\sqrt{u^2+\lambda^2}} \\ -\dfrac{\lambda}{\sqrt{u^2+\lambda^2}} & 0 \end{pmatrix}.$$

Finally, the matrix of the Weingarten map is

$$\mathcal{W} = \mathcal{F}_1^{-1}\mathcal{F}_2 = \begin{pmatrix} 0 & -\dfrac{\lambda}{(u^2+\lambda^2)^{1/2}} \\ -\dfrac{\lambda}{(u^2+\lambda^2)^{3/2}} & 0 \end{pmatrix}.$$

3.6 Curvatures

Definition 3.100: Gaussian and mean curvature

Let $\mathcal{S}$ be an orientable surface. Let $\mathcal{W}$ be the matrix of the Weingarten map $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ of $\mathcal{S}$ at $\mathbf{p}$. We define:

1. The **Gaussian curvature** of $\mathcal{S}$ at $\mathbf{p}$ is
$$K := \det(\mathcal{W}),$$
2. The **mean curvature** of $\mathcal{S}$ at $\mathbf{p}$ is
$$H := \frac{1}{2}\operatorname{Tr}(\mathcal{W}),$$

Notation 3.101: Trace of a matrix

The **trace** of a 2×2 matrix is the sum of the diagonal entries.

Proposition 3.102: Formulas for K and H

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart, and $\mathcal{S} = \boldsymbol{\sigma}(U)$. Then

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{LG - 2MF - NE}{2(EG-F^2)}.$$

Example 3.103: Curvatures of the Plane

Question. Let $\mathbf{a}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, with $\mathbf{p}$, $\mathbf{q}$ orthonormal. Consider the plane charted by
$$\boldsymbol{\sigma}(u,v) = \mathbf{a} + \mathbf{p}u + \mathbf{q}v.$$

1. Compute the matrix of the Weingarten map of $\boldsymbol{\sigma}$.
2. Compute the Gaussian and mean curvatures of the plane.

Solution.

1. From Examples 1.68, 1.89, the FFF and SFF of $\boldsymbol{\sigma}$ are
$$\mathcal{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
Therefore the matrix of the Weingarten map is
$$\mathcal{W} = \mathcal{F}_1^{-1}\mathcal{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
2. The Gaussian and mean curvatures are
$$K = \det(\mathcal{W}) = 0, \qquad H = \frac{1}{2}\operatorname{Tr}(\mathcal{W}) = 0.$$

Example 3.104: Curvatures of the Unit cylinder

Question. Consider the unit cylinder $\mathcal{S}$ charted by
$$\boldsymbol{\sigma}(u,v) = (\cos(u), \sin(u), v).$$

1. Compute the matrix of the Weingarten map of $\boldsymbol{\sigma}$.
2. Compute the Gaussian and mean curvatures of $\mathcal{S}$.

Solution.

1. From Examples 1.65, 3.90, the FFF and SFF of $\boldsymbol{\sigma}$ are
$$\mathcal{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$
Therefore the matrix of the Weingarten map is
$$\mathcal{W} = \mathcal{F}_1^{-1}\mathcal{F}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$
2. The Gaussian and mean curvatures are
$$K = \det(\mathcal{W}) = 0, \qquad H = \frac{1}{2}\operatorname{Tr}(\mathcal{W}) = -\frac{1}{2}.$$

Theorem 3.105: Eigenvalues of Weingarten map

Let $\mathcal{S}$ be an orientable surface and $\boldsymbol{\sigma}$ a regular chart at $\mathbf{p}$. Let $\mathcal{W}$ be the matrix of the Weingarten map $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of $T_{\mathbf{p}}\mathcal{S}$. Then

1. There exist scalars $\kappa_1, \kappa_2 \in \mathbb{R}$ and an orthonormal basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ of $T_{\mathbf{p}}\mathcal{S}$ such that
$$\mathcal{W}_{\mathbf{p},\mathcal{S}}(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1, \quad \mathcal{W}_{\mathbf{p},\mathcal{S}}(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2.$$
2. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ be such that
$$\mathbf{t}_1 = \lambda_1 \boldsymbol{\sigma}_u + \mu_1 \boldsymbol{\sigma}_v, \quad \mathbf{t}_2 = \lambda_2 \boldsymbol{\sigma}_u + \mu_2 \boldsymbol{\sigma}_v.$$

Denote $\mathbf{x}_1 = (\lambda_1, \mu_1)$ and $\mathbf{x}_2 = (\lambda_2, \mu_2)$. Then κ_1, κ_2 are eigenvalues of $\mathscr{W}$ of eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\mathscr{W}\mathbf{x}_1 = \kappa_1 \mathbf{x}_1 , \quad \mathscr{W}\mathbf{x}_2 = \kappa_2 \mathbf{x}_2 .$$

In particular, the matrix $\mathscr{W}$ is diagonalizable, with

$$\mathscr{W} = P^{-1}DP, \quad D = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix}.$$

Definition 3.106: Principal curvatures and vectors

Let $\mathcal{S}$ be an orientable surface. Let $\mathscr{W}_{\mathbf{p},\mathcal{S}}$ the Weingarten map of $\mathcal{S}$ at $\mathbf{p}$. We define:

1. The **principal curvatures** of $\mathcal{S}$ at $\mathbf{p}$ are the eigenvalues κ_1, κ_2 of $\mathscr{W}_{\mathbf{p},\mathcal{S}}$.
2. The **principal vectors** corresponding to κ_1 and κ_2 are the eigenvectors $\mathbf{t}_1, \mathbf{t}_2$ of $\mathscr{W}_{\mathbf{p},\mathcal{S}}$.

Remark 3.107: Computing principal curvatures and vectors

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart and $\mathcal{S} = \boldsymbol{\sigma}(U)$.

1. Compute the FFF and SFF of $\boldsymbol{\sigma}$, and the matrix of the Weingarten map
$$\mathscr{W} = \mathcal{F}_1^{-1}\mathcal{F}_2 .$$
2. Compute the eigenvalues of $\mathscr{W}$, by solving for λ the equation
$$\det(\mathscr{W} - \lambda I) = 0 .$$
The two solutions are the principal curvatures κ_1 and κ_2.
3. Find scalars λ, μ which solve the linear system
$$(\mathscr{W} - \kappa_i I)\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0 .$$
The solution(s) gives the eigenvector(s) of $\mathscr{W}$
$$\mathbf{x}_i = (\lambda, \mu)$$
corresponding to the eigenvalue κ_i.
4. The principal vector(s) associated to κ_i is
$$\mathbf{t}_i = \lambda\boldsymbol{\sigma}_u + \mu\boldsymbol{\sigma}_v$$

Remark 3.108: The case of $\mathscr{W}$ diagonal

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart and $\mathcal{S} = \boldsymbol{\sigma}(U)$. Assume the matrix of the Weingarten map is diagonal

$$\mathscr{W} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Then, the eigenvalues of $\mathscr{W}$ are κ_1 and κ_2, with eigenvectors

$$\mathbf{x}_1 = (1, 0) , \quad \mathbf{x}_2 = (0, 1) .$$

Therefore κ_1, κ_2 are the principal curvatures of $\mathcal{S}$, with principal vectors given by

$$\mathbf{t}_1 = \boldsymbol{\sigma}_u , \quad \mathbf{t}_2 = \boldsymbol{\sigma}_v .$$

Proposition 3.109: Relationships between curvatures

Let $\mathcal{S}$ be an orientable surface. Then

$$K = \kappa_1\kappa_2 , \quad H = \frac{\kappa_1 + \kappa_2}{2} ,$$
$$k_i = H \pm \sqrt{H^2 - K} .$$

Example 3.110: Principal curvatures of Unit Cylinder

Question. Consider the unit cylinder charted by

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v) .$$

Compute the principal curvature and principal vectors.
Solution. By Example 3.104, the matrix of the Weingarten map is

$$\mathscr{W} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\mathscr{W}$ is diagonal, the eigenvalues are the diagonal entries of $\mathscr{W}$ and the eigenvectors are

$$\mathbf{x}_1 = (1, 0), \quad \mathbf{x}_2 = (0, 1) .$$

Therefore, the principal curvatures and principal vectors are

$$\begin{aligned} &\kappa_1 = -1 , \quad \kappa_2 = 0 , \\ &\mathbf{t}_1 = \boldsymbol{\sigma}_u = (-\sin(u), \cos(v), 0) , \\ &\mathbf{t}_2 = \boldsymbol{\sigma}_v = (0, 0, 1) . \end{aligned}$$

Example 3.111: Curvatures of Sphere

Question. Consider the chart for the sphere

$$\boldsymbol{\sigma}(u, v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)) ,$$

where $u \in (0, 2\pi)$, $v \in (-\pi/2, \pi/2)$. Prove that

$$\mathcal{F}_1 = \mathcal{F}_2 = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathscr{W} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$K = H = \kappa_1 = \kappa_2 = 1 , \quad \mathbf{t}_1 = \boldsymbol{\sigma}_u , \quad \mathbf{t}_2 = \boldsymbol{\sigma}_v .$$

Solution. Compute the FFF of $\boldsymbol{\sigma}$

$$\begin{aligned} \boldsymbol{\sigma}_u &= (-\sin(u)\cos(v), \cos(u)\cos(v), 0) \\ \boldsymbol{\sigma}_v &= (-\cos(u)\sin(v), -\sin(u)\sin(v), \cos(v)) \\ E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = \cos^2(v) \\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0 \\ G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 1 \\ \mathcal{F}_1 &= \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Moreover

$$\begin{aligned} \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (\cos(u)\cos^2(v), \sin(u)\cos^2(v), \cos(v)\sin(v)) \\ \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| &= |\cos(v)| = \cos(v) , \end{aligned}$$

where we used that $\cos(v) > 0$ since $v \in (-\pi/2, \pi/2)$. Therefore,

$$\begin{aligned} \mathbf{N} &= (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)) \\ \boldsymbol{\sigma}_{uu} &= (-\cos(u)\cos(v), -\sin(u)\cos(v), 0) \\ \boldsymbol{\sigma}_{uv} &= (\sin(u)\sin(v), -\cos(u)\sin(v), 0) \\ \boldsymbol{\sigma}_{vv} &= (-\cos(u)\cos(v), -\sin(u)\cos(v), -\sin(v)) \\ L &= \boldsymbol{\sigma}_{uu} \cdot \mathbf{N} = \cos^2(v) \\ M &= \boldsymbol{\sigma}_{uv} \cdot \mathbf{N} = 0 \\ N &= \boldsymbol{\sigma}_{vv} \cdot \mathbf{N} = 1 \end{aligned}$$

Hence, the SFF and matrix of the Weingarten map are

$$\mathcal{F}_2 = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{W} = \mathcal{F}_1^{-1}\mathcal{F}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathcal{W}$ is diagonal, the principal curvatures and vectors are

$$\kappa_1 = \kappa_2 = 1, \quad \mathbf{t}_1 = \boldsymbol{\sigma}_u, \quad \mathbf{t}_2 = \boldsymbol{\sigma}_v.$$

Finally, the mean and Gaussian curvatures are

$$H = \frac{\kappa_1 + \kappa_2}{2} = 1, \quad K = \kappa_1\kappa_2 = 1.$$

Definition 3.112: Darboux frame

Let $\mathcal{S}$ be a regular surface, $\boldsymbol{\gamma} : (a,b) \to \mathcal{S}$ a unit-speed curve. The **Darboux frame** of $\boldsymbol{\gamma}$ at t is the triple

$$\{\dot{\boldsymbol{\gamma}}, \mathbf{N}, \mathbf{N} \times \dot{\boldsymbol{\gamma}}\},$$

where $\boldsymbol{\gamma}$ is evaluated at t, and $\mathbf{N}$ is the standard unit normal to $\mathcal{S}$, evaluated at $\mathbf{p} = \boldsymbol{\gamma}(t)$.

Proposition 3.113: Darboux frame is orthonormal basis

Let $\mathcal{S}$ be a regular surface, $\boldsymbol{\gamma} : (a,b) \to \mathcal{S}$ a unit-speed curve. The Darboux frame is an orthornormal basis of $\mathbb{R}^3$ for all $t \in (a,b)$.

Proposition 3.114: Coefficients of $\ddot{\boldsymbol{\gamma}}$ in the Darboux frame

Let $\mathcal{S}$ be a regular surface, $\boldsymbol{\gamma} : (a,b) \to \mathcal{S}$ a unit-speed curve. Then

$$\ddot{\boldsymbol{\gamma}} = \kappa_n \mathbf{N} + \kappa_g \, (\mathbf{N} \times \dot{\boldsymbol{\gamma}}),$$

where $\mathbf{N}$ is evaluated at $\mathbf{p} := \boldsymbol{\gamma}(t)$ and κ_n, κ_g are scalars depedent on $\mathbf{p}$. Moreover

$$\kappa_n = \ddot{\boldsymbol{\gamma}} \cdot \mathbf{N}, \quad \kappa_g = \ddot{\boldsymbol{\gamma}} \cdot (\mathbf{N} \times \dot{\boldsymbol{\gamma}}),$$

$$\kappa^2 = \kappa_n^2 + \kappa_g^2,$$

$$\kappa_n = \kappa \cos(\phi), \quad \kappa_g = \pm\kappa \sin(\phi),$$

where κ is the curvature of $\boldsymbol{\gamma}$, and ϕ is the angle between $\mathbf{N}$ and $\mathbf{n}$, the principal unit normal of $\boldsymbol{\gamma}$.

Definition 3.115: Normal and geodesic curvatures

Let $\mathcal{S}$ be regular and $\boldsymbol{\gamma} : (a,b) \to \mathcal{S}$ a unit-speed curve. Let $\mathbf{N}$ bet the standard unit normal to $\mathcal{S}$.

1. The **normal curvature** of $\boldsymbol{\gamma}$ is

$$\kappa_n = \ddot{\boldsymbol{\gamma}} \cdot \mathbf{N},$$

2. The **geodesic curvature** of $\boldsymbol{\gamma}$ is

$$\kappa_g = \ddot{\boldsymbol{\gamma}} \cdot (\mathbf{N} \times \dot{\boldsymbol{\gamma}}).$$

Theorem 3.116: Computing κ_n with SFF

Let $\mathcal{S}$ be a regular surface and $\boldsymbol{\gamma} : (a,b) \to \mathcal{S}$ a unit-speed curve. Denote $\mathbf{p} := \boldsymbol{\gamma}(t)$. We have:

1. The normal curvature κ_n satisfies

$$\kappa_n = II_{\mathbf{p}}(\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}}).$$

2. Let $\boldsymbol{\sigma}$ be a chart for $\mathcal{S}$ at $\mathbf{p} = \boldsymbol{\gamma}(t)$. Then

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$$

for some smooth functions $u, v : (a,b) \to \mathbb{R}$, and

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,$$

where L, M, N are evaluated at $(u(t), v(t))$, and $\dot{u}, \dot{v}$ at t.

Example 3.117: Curves on the sphere

Question. Consider the unit sphere $\mathbb{S}^2$ with chart

$$\boldsymbol{\sigma}(u,v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)).$$

Show that, for all unit-speed curves on $\mathbb{S}^2$,

$$\kappa_n(t) = 1.$$

Solution. Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ be a unit-speed curve on $\mathbb{S}^2$. Differentiating, we get

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= \frac{d}{dt}(\cos(u(t))\cos(v(t)), \sin(u(t))\cos(v(t)), \sin(v(t))) \\ &= (-\dot{u}\sin(u)\cos(v) - \dot{v}\cos(u)\sin(v), \\ &\quad\ \dot{u}\cos(u)\cos(v) - \dot{v}\sin(u)\sin(v), \\ &\quad\ \dot{v}\cos(v)) \\ \|\dot{\boldsymbol{\gamma}}(t)\|^2 &= \cos^2(v)\dot{u}^2 + \dot{v}^2. \end{aligned}$$

Since $\boldsymbol{\gamma}$ is unit-speed, we have $\|\dot{\boldsymbol{\gamma}}\| = 1$. Therefore,

$$\cos^2(v)\dot{u}^2 + \dot{v}^2 = 1.$$

By Example 3.111, the coefficients of the SFF of $\boldsymbol{\sigma}$ are

$$L = \cos^2(v), \quad M = 0, \quad N = 1.$$

By Theorem 3.116, the normal curvature of $\boldsymbol{\gamma}$ is

$$\kappa_n = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = \cos^2(v)\dot{u}^2 + \dot{v}^2 = 1.$$

Theorem 3.118: Euler's Theorem

Let $\mathcal{S}$ be a regular surface with principal curvatures κ_1, κ_2 and principal vectors $\mathbf{t}_1, \mathbf{t}_2$. Let $\boldsymbol{\gamma}$ be a unit-speed curve on $\mathcal{S}$. The normal curvature of $\boldsymbol{\gamma}$ is given by

$$\kappa_n = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta)\,,$$

where θ is the angle between $\dot{\boldsymbol{\gamma}}$ and $\mathbf{t}_1$.

Example 3.119: Curves on the sphere (again)

Question. Same question as in Example 3.117.
Solution. By Example 3.111, the principal curvatures of the unit sphere are $\kappa_1 = \kappa_2 = 1$. By Euler's Theorem, for any unit-speed curve $\boldsymbol{\gamma}$ on the sphere we have

$$\kappa_n = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta) = \cos^2(\theta) + \sin^2(\theta) = 1\,.$$

Definition 3.120: κ_n and κ_g for regular $\boldsymbol{\gamma}$

Let $\mathcal{S}$ be regular, and $\boldsymbol{\gamma} : (a,b) \to \mathcal{S}$ a regular curve. Let $\tilde{\boldsymbol{\gamma}}$ be a unit-speed reparametrization of $\boldsymbol{\gamma}$, with

$$\boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}} \circ \phi\,, \quad \phi : (a,b) \to (\tilde{a}, \tilde{b})\,.$$

Let $\tilde{\kappa}_n$ and $\tilde{\kappa}_g$ be the normal and geodesic curvatures of $\tilde{\boldsymbol{\gamma}}$. The normal and geodesic curvatures of $\boldsymbol{\gamma}$ are

$$\kappa_n(t) = \tilde{\kappa}_n(\phi(t))\,, \qquad \kappa_g(t) = \tilde{\kappa}_g(\phi(t))\,.$$

Theorem 3.121: Formulas for κ_n and κ_g

Let $\mathcal{S}$ be regular, and $\boldsymbol{\gamma} : (a,b) \to \mathcal{S}$ a regular curve.

1. The normal and geodesic curvatures of $\boldsymbol{\gamma}$ are given by
$$\kappa_n = \frac{\ddot{\boldsymbol{\gamma}} \cdot \mathbf{N}}{\|\dot{\boldsymbol{\gamma}}\|^2}\,, \qquad \kappa_g = \frac{\ddot{\boldsymbol{\gamma}} \cdot (\mathbf{N} \times \dot{\boldsymbol{\gamma}})}{\|\dot{\boldsymbol{\gamma}}\|^3}\,.$$
2. Denote by κ the curvature of $\boldsymbol{\gamma}$. It holds
$$\kappa^2 = \kappa_n^2 + \kappa_g^2\,.$$
3. Let $\boldsymbol{\sigma}$ be a chart for $\mathcal{S}$ at $\mathbf{p} = \boldsymbol{\gamma}(t)$. Then
$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$$
for some smooth functions $u, v : (a,b) \to \mathbb{R}$, and
$$\kappa_n = \frac{II_{\mathbf{p}}(\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}})}{I_{\mathbf{p}}(\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}})} = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\,,$$
with E, F, G, L, M, N evaluated at $(u(t), v(t))$, and $\dot{u}, \dot{v}$ at t.

Example 3.122: Calculation of normal and geodesic curvatures

Question. For $v \neq 0$ and $t \neq 0$, consider the surface chart and curve

$$\boldsymbol{\sigma}(u,v) = \left(u, v, \frac{u}{v}\right), \quad \boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(t^2, t)\,.$$

1. Prove that $\boldsymbol{\sigma}$ is regular.
2. Compute the principal unit normal to $\boldsymbol{\sigma}$.
3. Prove that $\boldsymbol{\gamma}$ is regular.
4. Compute the normal and geodesic curvatures of $\boldsymbol{\gamma}$.
5. Compute κ, the curvature of $\boldsymbol{\gamma}$. Verify that
$$\kappa^2 = \kappa_n^2 + \kappa_g^2\,.$$

Solution.

1. The chart $\boldsymbol{\sigma}$ is regular because
$$\boldsymbol{\sigma}_u = \left(1, 0, \frac{1}{v}\right), \quad \boldsymbol{\sigma}_v = \left(0, 1, -\frac{u}{v^2}\right)$$
$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = \left(-\frac{1}{v}, \frac{u}{v^2}, 1\right) \neq \mathbf{0}$$
2. The principal unit normal is
$$\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| = \frac{\left(u^2 + v^2 + v^4\right)^{1/2}}{v^2}$$
$$\mathbf{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|} = \frac{\left(-v, u, v^2\right)}{\left(u^2 + v^2 + v^4\right)^{1/2}}.$$
3. The curve $\boldsymbol{\gamma}$ is regular because
$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(t^2, t) = (t^2, t, t)$$
$$\dot{\boldsymbol{\gamma}}(t) = (2t, 1, 1) \neq \mathbf{0}$$
4. Compute the following quantities
$$\|\dot{\boldsymbol{\gamma}}(t)\| = 2^{1/2}\,(2t^2+1)^{1/2} \qquad \ddot{\boldsymbol{\gamma}} \cdot \mathbf{N} = -\frac{2}{(2t^2+1)^{1/2}}$$
$$\ddot{\boldsymbol{\gamma}}(t) = (2, 0, 0) \qquad \mathbf{N} \times \dot{\boldsymbol{\gamma}} = \left(1 + 2t^2\right)^{1/2} (0, 1, -1)$$
$$\mathbf{N}(t^2, t) = \frac{(-1, t, t)}{\left(2t^2 + 1\right)^{1/2}} \qquad \ddot{\boldsymbol{\gamma}} \cdot (\mathbf{N} \times \dot{\boldsymbol{\gamma}}) = 0$$
The normal and geodesic curvatures are
$$\kappa_n = \frac{\ddot{\boldsymbol{\gamma}} \cdot \mathbf{N}}{\|\dot{\boldsymbol{\gamma}}\|^2} = -\frac{1}{(2t^2+1)^{3/2}}\,,$$
$$\kappa_g = \frac{\ddot{\boldsymbol{\gamma}} \cdot (\mathbf{N} \times \dot{\boldsymbol{\gamma}})}{\|\dot{\boldsymbol{\gamma}}\|^3} = 0\,.$$
5. The curvature of $\boldsymbol{\gamma}$ is
$$\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} = (0, 2, -2)\,, \quad \|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\| = 2^{3/2}$$
$$\kappa = \frac{\|\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}\|}{\|\dot{\boldsymbol{\gamma}}\|^3} = \frac{1}{(2t^2+1)^{3/2}}$$
Thus $\kappa = -\kappa_n$. Since $\kappa_g = 0$, we conclude that $\kappa^2 = \kappa_n^2 + \kappa_g^2$.

3.7 Local shape of a surface

Theorem 3.123: Local structure of surfaces

Let $\mathcal{S}$ be a regular surface and $\mathbf{p} \in \mathcal{S}$. In the vicinity of $\mathbf{p}$, the surface $\mathcal{S}$ is approximated by the quadric surface of equation

$$z = \frac{1}{2}\left(x^2\kappa_1(\mathbf{p}) + y^2\kappa_2(\mathbf{p})\right),$$

where $\kappa_1(\mathbf{p}), \kappa_2(\mathbf{p})$ are the principal curvatures of $\mathcal{S}$ at $\mathbf{p}$.

Definition 3.124: Local shape types

Let $\mathcal{S}$ be a regular surface, with $\kappa_1(\mathbf{p})$ and $\kappa_2(\mathbf{p})$ the principal curvatures at $\mathbf{p}$. The point $\mathbf{p}$ is

- **Elliptic** if
$$\kappa_1(\mathbf{p}) > 0\,,\ \kappa_2(\mathbf{p}) > 0 \quad \text{or} \quad \kappa_1(\mathbf{p}) < 0\,,\ \kappa_2(\mathbf{p}) < 0$$
- **Hyperbolic** if
$$\kappa_1(\mathbf{p}) < 0 < \kappa_2(\mathbf{p}) \quad \text{or} \quad \kappa_2(\mathbf{p}) < 0 < \kappa_1(\mathbf{p})$$
- **Parabolic** if
$$\kappa_1(\mathbf{p}) = 0\,,\ \kappa_2(\mathbf{p}) \neq 0 \quad \text{or} \quad \kappa_2(\mathbf{p}) \neq 0,\ \kappa_1(\mathbf{p}) = 0$$
- **Planar** if
$$\kappa_1(\mathbf{p}) = \kappa_2(\mathbf{p}) = 0$$

Proposition 3.125: Gaussian curvature and local shape

Let $\mathcal{S}$ be a regular surface, with $K(\mathbf{p})$ the Gaussian curvature at $\mathbf{p}$. The point $\mathbf{p}$ is

- **Elliptic** if $K(\mathbf{p}) > 0$,
- **Hyperbolic** if $K(\mathbf{p}) < 0$,
- **Parabolic** or **Planar** if $K(\mathbf{p}) = 0$.

Example 3.126: Analysis of local shape

Question. Consider the surface chart

$$\boldsymbol{\sigma}(u,v) = \left(u - v, u + v, u^2 + v^2\right).$$

1. Compute the first fundamental form of $\boldsymbol{\sigma}$.
2. Compute the second fundamental form of $\boldsymbol{\sigma}$.
3. Compute the matrix of the Weingarten map.
4. Show that $\mathbf{p} = \boldsymbol{\sigma}(1,0)$ is an elliptic point.
5. Can there be points which are not elliptic?

Solution.

1. The FFF of $\boldsymbol{\sigma}$ is
$$\begin{aligned} &\boldsymbol{\sigma}_u = (1,1,2u) && F = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 4uv \\ &\boldsymbol{\sigma}_v = (-1,1,2v) && G = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 2(1+2v^2) \\ &E = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 2(1+2u^2) && \mathcal{F}_1 = 2\begin{pmatrix} 1+2u^2 & 2uv \\ 2uv & 1+2v^2 \end{pmatrix} \end{aligned}$$

2. The standard unit normal is
$$\begin{aligned} \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= 2(v-u, -u-v, 1) \\ \|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| &= 2\left(1+2u^2+2v^2\right)^{\frac{1}{2}} \\ \mathbf{N} &= \frac{(v-u,-u-v,1)}{\left(1+2u^2+2v^2\right)^{\frac{1}{2}}} \end{aligned}$$
The SFF of $\boldsymbol{\sigma}$ is
$$\begin{aligned} &\boldsymbol{\sigma}_{uu} = (0,0,2) && L = \boldsymbol{\sigma}_{uu} \cdot \mathbf{N} = 2\left(1+2u^2+2v^2\right)^{-\frac{1}{2}} \\ &\boldsymbol{\sigma}_{uv} = (0,0,0) && M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N} = 0 \\ &\boldsymbol{\sigma}_{vv} = (0,0,2) && N = \boldsymbol{\sigma}_{vv} \cdot \mathbf{N} = 2\left(1+2u^2+2v^2\right)^{-\frac{1}{2}} \\ &\mathcal{F}_2 = \left(1+2u^2+2v^2\right)^{-\frac{1}{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

3. The inverse of $\mathcal{F}_1$ is
$$\begin{aligned} \mathcal{F}_1^{-1} &= \frac{1}{\det(\mathcal{F}_1)}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \\ &= \frac{1}{2(1+2u^2+2v^2)}\begin{pmatrix} 1+2v^2 & -2uv \\ -2uv & 1+2u^2 \end{pmatrix}. \end{aligned}$$
The matrix of the Weingarten map is
$$\begin{aligned} \mathcal{W} &= \mathcal{F}_1^{-1}\mathcal{F}_2 \\ &= \frac{1}{(1+2u^2+2v^2)^{\frac{3}{2}}}\begin{pmatrix} 1+2v^2 & -2uv \\ -2uv & 1+2u^2 \end{pmatrix}. \end{aligned}$$

4. For $u = 1$ and $v = 0$ we obtain
$$\mathcal{W} = \frac{1}{3^{\frac{3}{2}}}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3^{-\frac{3}{2}} & 0 \\ 0 & 3^{-\frac{1}{2}} \end{pmatrix}.$$
Therefore the principal curvatures at $\mathbf{p}$ are
$$\kappa_1(\mathbf{p}) = 3^{-\frac{3}{2}} > 0\,, \quad \kappa_2(\mathbf{p}) = 3^{-\frac{1}{2}} > 0\,.$$
Therefore $\mathbf{p}$ is an elliptic point.

5. No. This is because the Gaussian curvature is
$$K = \det(\mathcal{W}) = \frac{1}{(1+2u^2+2v^2)^2} > 0\,.$$
By Proposition 3.125 we conclude that every point is elliptic.

Definition 3.127: Umbilical point

Let $\mathcal{S}$ be a regular surface, with $\kappa_1(\mathbf{p})$ and $\kappa_2(\mathbf{p})$ the principal curvatures at $\mathbf{p}$. We say that $\mathbf{p}$ is an **umbilical point** if

$$\kappa_1(\mathbf{p}) = \kappa_2(\mathbf{p})\,.$$

Theorem 3.128: Structure theorem at umbilics

Let $\mathcal{S}$ be a regular surface such that every point $\mathbf{p} \in \mathcal{S}$ is umbilic. Then $\mathcal{S}$ is an open subset of plane or a sphere.

Proposition 3.129: Criterion for umbilics

Let $\mathcal{S}$ be a regular surface. The point $\mathbf{p}$ is umbilical if and only if

$$H^2(\mathbf{p}) = K(\mathbf{p}) .$$

In particular, $\mathbf{p}$ cannot be umbilical if

$$K(\mathbf{p}) < 0 .$$

Proposition 3.130: Chart criterion for umbilics

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular chart and $\mathcal{S} = \boldsymbol{\sigma}(U)$. A point $\mathbf{p}$ is umbilic if and only if there exists a scalar κ such that

$$\mathcal{F}_2 = \kappa \mathcal{F}_1 .$$

Example 3.131: Plane and Sphere

1. If the plane is charted as in Example 3.103, the FFF and SFF are

$$\mathcal{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $\mathcal{F}_2 = \kappa \mathcal{F}_1$ with $\kappa = 0$, and all points are umbilical.

2. If the sphere is charted as in Example 3.111, the FFF and SFF are

$$\mathcal{F}_1 = \mathcal{F}_2 = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathcal{F}_2 = \mathcal{F}_1$, all points on the sphere are umbilical.

Remark 3.132: How to find umbilics

Condition $\mathcal{F}_2 = \kappa \mathcal{F}_1$ is equivalent to

$$(E, F, G) \times (L, M, N) = \mathbf{0} .$$

In practice, umbilics can be found by solving the above equations. Common factors may be discarded, if convenient.

Example 3.133: Local shape of the Monkey Saddle

Question. Consider the *Monkey Saddle* surface $\mathcal{S}$ described by

$$z = x^3 - 3xy^2 .$$

1. Compute the Gaussian curvature of $\mathcal{S}$.
2. Does $\mathcal{S}$ contain any hyperbolic point?
3. Prove that the origin is the only umbilical point.

Solution. The Monkey Saddle is charted by

$$\boldsymbol{\sigma}(u, v) = (u, v, u^3 - 3uv^2) .$$

The FFF of $\boldsymbol{\sigma}$ is

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (1, 0, 3(u^2 - v^2)) & F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = -18uv(u^2 - v^2) \\
\boldsymbol{\sigma}_v &= (0, 1, -6uv) & G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 1 + 36u^2v^2 \\
E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 1 + 9(u^2 - v^2)^2
\end{aligned}$$

The SFF of $\boldsymbol{\sigma}$ is

$$\begin{aligned}
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (-3(u^2 - v^2), 6uv, 1) \\
\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\| &= 1 + 36u^2v^2 + 9(u^2 - v^2)^2 \\
&= 1 + 9u^4 + 9v^4 + 18u^2v^2 \\
&= 1 + 9(u^2 + v^2)^2 \\
\mathbf{N} &= \frac{(-3(u^2 - v^2), 6uv, 1)}{\sqrt{1 + 9(u^2 + v^2)^2}} \\
\boldsymbol{\sigma}_{uu} &= (0, 0, 6u) \\
\boldsymbol{\sigma}_{uv} &= (0, 0, -6v) \\
\boldsymbol{\sigma}_{vv} &= (0, 0, -6u) \\
L = \boldsymbol{\sigma}_{uu} \cdot \mathbf{N} &= \frac{6u}{\sqrt{1 + 9(u^2 + v^2)^2}} \\
M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N} &= \frac{-6v}{\sqrt{1 + 9(u^2 + v^2)^2}} \\
N = \boldsymbol{\sigma}_{vv} \cdot \mathbf{N} &= \frac{6u}{\sqrt{1 + 9(u^2 + v^2)^2}}
\end{aligned}$$

1. We have that

$$\begin{aligned}
EG - F^2 &= (1 + 9(u^2 - v^2)^2)(1 + 36u^2v^2) - (-18uv(u^2 - v^2))^2 \\
&= 1 + 36u^2v^2 + 9(u^2 - v^2)^2 \\
&= 1 + 9u^4 + 9v^4 + 18u^2v^2 \\
&= 1 + 9(u^2 + v^2)^2 \\
LN - M^2 &= -\frac{36(u^2 + v^2)}{1 + 9(u^2 + v^2)^2}
\end{aligned}$$

Therefore the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{36(u^2 + v^2)}{[1 + 9(u^2 + v^2)^2]^2} .$$

2. Note that

$$K < 0 , \quad \forall (u, v) \neq (0, 0) .$$

By Proposition 3.125, we conclude that all the points outside of the origin are hyperbolic.

3. Since $K < 0$ everywhere except at the origin, Proposition 3.129 implies that points outside the origin cannot be umbilic. At $(0, 0)$, we have

$$\mathcal{F}_1 = du^2 + dv^2 , \quad \mathcal{F}_2 = 0 .$$

Therefore $\mathcal{F}_2$ is a multiple of $\mathcal{F}_1$, and by Proposition 3.130 we conclude that $(0, 0)$ is an umbilical point. Note: the matrix of the Weingarten map is $\mathcal{W} = \mathcal{F}_1^{-1}\mathcal{F}_2 = 0$. Therefore the principal curvatures are $\kappa_1 = \kappa_2 = 0$, showing that $(0, 0)$ is a planar point.

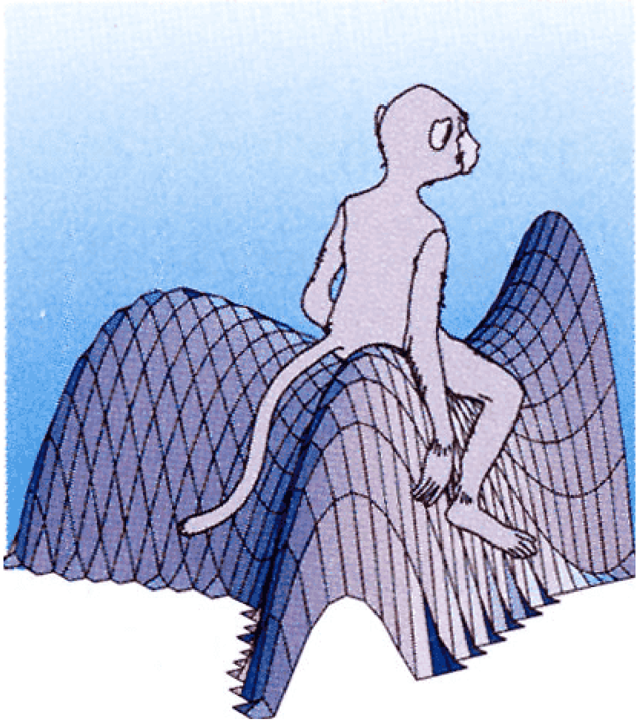

Figure 3.1: The Monkey Saddle surface $z = x^3 - 3xy^2$.

Good Luck with the Exam

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BibTex citation:

```
@electronic{Fanzon-Differential-Geometry-Revision-2024,
    author = {Fanzon, Silvio},
    title = {Revision Guide of Differential Geometry},
    url = {https://www.silviofanzon.com/2024-Differential-
     ↪  Geometry-Revision/},
    year = {2024}}
```