

Numbers Sequences and Series

Revision Guide

Dr. Silvio Fanzon

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Revision Guide

Revision Guide document for the module **Numbers Sequences and Series 400297** 2024/25 at the University of Hull.
If you have any question or find any typo, please email me at

S.Fanzon@hull.ac.uk

Full length Lecture Notes of the module available at

silviofanzon.com/2024-NSS-Notes

Recommended revision strategy

Make sure you are very comfortable with:

1. The Definitions, Theorems, Proofs, and Examples contained in this Revision Guide
2. The Tutorial and Homework questions
3. The 2023/24 Exam Paper questions.
4. The Checklist below

Checklist

You should be comfortable with the following topics/taks:

Preliminaries

- Prove that $\sqrt{p} \notin \mathbb{Q}$ for p a prime number
-

Complex Numbers

- Sum, multiplication and division of complex numbers
- Computing the complex conjugate
- Computing the inverse of a complex number
- Find modulus and argument of a complex number
- Compute Cartesian, Trigonometric and Exponential form of a complex number
- Complex exponential and its properties
- Computing powers of complex numbers
- Solving degree 2 polynomial equations in \mathbb{C}
- Long division of polynomials
- Solving higher degree polynomial equations in \mathbb{C}
- Finding the roots of unity
- Finding the n-th roots of a complex number

1 Preliminaries

Theorem 1.1

The number $\sqrt{2}$ does not belong to \mathbb{Q} .

Proof

Assume by contradiction that

$$\sqrt{2} \in \mathbb{Q}. \quad (1.1)$$

1. Therefore, there exists $m, n \in \mathbb{N}, n \neq 0$, such that

$$\frac{m}{n} = \sqrt{2}.$$

2. **Without loss of generality**, we can **assume** that m and n have no common factors.

3. Square the equation to get

$$\frac{m^2}{n^2} = 2. \quad (1.2)$$

4. Equation (1.2) implies

$$m^2 = 2n^2. \quad (1.3)$$

Therefore the integer m^2 is an even number.

5. Since m^2 is an even number, it follows that also m is an even number. Then there exists $p \in \mathbb{N}$ such that

$$m = 2p. \quad (1.4)$$

6. Substitute (1.4) in (1.3) to get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2. \quad (1.5)$$

7. We now make a series of observations:

- Equation (1.5) says that n^2 is even.
- The same argument in Step 5 guarantees that also n is even.
- We have already seen that m is even.
- Therefore n and m are both even.
- Hence n and m have 2 as common factor.

- But Step 2 says that n and m have no common factors.

• Contradiction

8. Our reasoning has run into a **contradiction**, stemming from assumption (1.1). Therefore (1.1) is **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

1.1 Set Theory

Definition 1.2

For two sets A and B we define their **union** as the set

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

The **intersection** of A and B is defined by

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

We denote the **empty set** by the symbol \emptyset . Two sets are **disjoint** if

$$A \cap B = \emptyset.$$

Given two sets A and B , we say that A is **contained** in B , in symbols

$$A \subseteq B,$$

if all the elements of A are also contained in B . Two sets A and B are **equal**, in symbols

$$A = B,$$

if they contain the same elements.

Proposition 1.3

Let A and B be sets. Then

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

Definition 1.4

Let Ω be a set, and $A_n \subseteq \Omega$ a family of subsets, where $n \in \mathbb{N}$.

1. The **infinte union** of the A_n is the set

$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for at least one } n \in \mathbb{N}\}.$$

2. The **infinte intersection** of the A_n is the set

$$\bigcap_{n \in \mathbb{N}} A_n := \{x \in \Omega : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

Example 1.5

Question. Define $\Omega := \mathbb{N}$ and a family A_n by

$$A_n = \{n, n+1, n+2, n+3, \dots\}, \quad n \in \mathbb{N}.$$

1. Prove that

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}. \quad (1.6)$$

2. Prove that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset. \quad (1.7)$$

Solution.

1. Assume that $m \in \bigcup_{n \in \mathbb{N}} A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subseteq \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows

$$\bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{N}.$$

Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index n , $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \bigcup_{n \in \mathbb{N}} A_n$, showing that

$$\mathbb{N} \subseteq \bigcup_{n \in \mathbb{N}} A_n.$$

This proves (1.6).

2. Suppose that (1.7) is false, i.e.,

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset.$$

This means there exists some $m \in \mathbb{N}$ such that $m \in \bigcap_{n \in \mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (1.7) holds.

Definition 1.6

Let $A, B \subseteq \Omega$. The **complement** of A with respect to B is the set of elements of B which do not belong to A , that is

$$B \setminus A := \{x \in \Omega : x \in B \text{ and } x \notin A\}.$$

In particular, the complement of A with respect to Ω is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : x \notin A\}.$$

Example 1.7

Question. Suppose $A, B \subseteq \Omega$. Prove that

$$A \subseteq B \iff B^c \subseteq A^c.$$

Solution. Let us prove the above claim:

- First implication \implies :

Suppose that $A \subseteq B$. We need to show that $B^c \subseteq A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subseteq B$, hence $x \in B$. But we had assumed $x \in B^c$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subseteq A^c$.

- Second implication \impliedby : Note that, for any set,

$$(A^c)^c = A.$$

Hence, by the first implication,

$$B^c \subseteq A^c \implies (A^c)^c \subseteq (B^c)^c \implies A \subseteq B.$$

Proposition 1.8: De Morgan's Laws

Suppose $A, B \subseteq \Omega$. Then

$$(A \cap B)^c = A^c \cup B^c, \quad (A \cup B)^c = A^c \cap B^c.$$

Definition 1.9

Let Ω be a set. The **power set** of Ω is

$$\mathcal{P}(\Omega) := \{A : A \subseteq \Omega\}.$$

Example 1.10

Question. Compute the power set of

$$\Omega = \{x, y, z\}.$$

Solution. $\mathcal{P}(\Omega)$ has $2^3 = 8$, and

$$\mathcal{P}(\Omega) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\} \quad (1.8)$$

$$\{x, z\}, \{y, z\}, \{x, y, z\}\}. \quad (1.9)$$

Definition 1.11

Let A, B be sets. The **product** of A and B is the set of pairs

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

1.2 Relations

Definition 1.12

Suppose A is a set. A **binary relation** R on A is a subset

$$R \subseteq A \times A.$$

Definition 1.13: Equivalence relation

A binary relation R is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive:** For each $x \in A$ one has

$$(x, x) \in R,$$

2. **Symmetric:** We have

$$(x, y) \in R \implies (y, x) \in R$$

3. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

If $(x, y) \in R$ we write

$$x \sim y$$

and we say that x and y are **equivalent**.

Definition 1.14: Equivalence classes

Suppose R is an **equivalence relation** on A . The **equivalence class** of an element $x \in A$ is the set

$$[x] := \{y \in A : y \sim x\}.$$

The set of equivalence classes of elements of A with re-

spect to the equivalence relation R is denoted by

$$A/R := A/\sim := \{[x] : x \in A\}.$$

Proposition 1.15

Let \sim be an equivalence relation on A . Then

1. For each $x \in A$ we have

$$[x] \neq \emptyset$$

2. For all $x, y \in A$ it holds

$$x \sim y \iff [x] = [y].$$

Example 1.16: Equality is an equivalence relation

Question. The equality defines a **binary relation** on $\mathbb{Q} \times \mathbb{Q}$, via

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x = y\}.$$

1. Prove that R is an **equivalence relation**.
2. Prove that $[x] = \{x\}$ and compute \mathbb{Q}/R .

Solution.

1. We need to check that R satisfies the 3 properties of an equivalence relation:

- Reflexive: It holds, since $x = x$ for all $x \in \mathbb{Q}$,
- Symmetric: Again $x = y$ if and only if $y = x$,
- Transitive: If $x = y$ and $y = z$ then $x = z$.

Therefore, R is an equivalence relation.

2. The class of equivalence of $x \in \mathbb{Q}$ is given by

$$[x] = \{x\},$$

that is, this relation is quite trivial, given that each element of \mathbb{Q} can only be related to itself. The quotient space is then

$$\mathbb{Q}/R = \{[x] : x \in \mathbb{Q}\} = \{\{x\} : x \in \mathbb{Q}\}.$$

Example 1.17

Question. Let R be the binary relation on the set \mathbb{Q} of rational numbers defined by

$$x \sim y \iff x - y \in \mathbb{Z}.$$

1. Prove that R is an equivalence relation on \mathbb{Q} .
2. Compute $[x]$ for each $x \in \mathbb{Q}$.
3. Compute \mathbb{Q}/R .

Solution.

1. We have:

- Reflexive: Let $x \in \mathbb{Q}$. Then $x - x = 0$ and $0 \in \mathbb{Z}$. Thus $x \sim x$.
- Symmetric: If $x \sim y$ then $x - y \in \mathbb{Z}$. But then also

$$-(x - y) = y - x \in \mathbb{Z}$$

and so $y \sim x$.

- Transitive: Suppose $x \sim y$ and $y \sim z$. Then

$$x - y \in \mathbb{Z} \text{ and } y - z \in \mathbb{Z}.$$

Thus, we have

$$x - z = (x - y) + (y - z) \in \mathbb{Z}$$

showing that $x \sim z$.

Thus, we have shown that R is an equivalence relation on \mathbb{Q} .

2. Note that

$$x \sim y \iff \exists n \in \mathbb{Z} \text{ s.t. } y = x + n.$$

Therefore the equivalence classes with respect to \sim are

$$[x] = \{x + n : n \in \mathbb{Z}\}.$$

Each equivalence class has exactly one element in $[0, 1) \cap \mathbb{Q}$, meaning that:

$$\forall x \in \mathbb{Q}, \exists! q \in \mathbb{Q} \text{ s.t. } 0 \leq q < 1 \text{ and } q \in [x]. \quad (1.10)$$

Indeed: take $x \in \mathbb{Q}$ arbitrary. Then $x \in [n, n + 1)$ for some $n \in \mathbb{Z}$. Setting $q := x - n$ we obtain that

$$x = q + n, \quad q \in [0, 1),$$

proving (1.10). In particular (1.10) implies that for each $x \in \mathbb{Q}$ there exists $q \in [0, 1) \cap \mathbb{Q}$ such that

$$[x] = [q].$$

3. From Point 2 we conclude that

$$\mathbb{Q}/R = \{[x] : x \in \mathbb{Q}\} = \{q \in \mathbb{Q} : 0 \leq q < 1\}.$$

Definition 1.18: Partial order

A binary relation R on A is called a **partial order** if it satisfies the following properties:

1. **Reflexive:** For each $x \in A$ one has

$$(x, x) \in R,$$

2. **Antisymmetric:** We have

$$(x, y) \in R \text{ and } (y, x) \in R \implies x = y$$

3. **Transitive:** We have

$$(x, y) \in R, (y, z) \in R \implies (x, z) \in R$$

Definition 1.19: Total order

A binary relation R on A is called a **total order relation** if it satisfies the following properties:

1. **Partial order:** R is a partial order on A .
2. **Total:** For each $x, y \in A$ we have

$$(x, y) \in R \text{ or } (y, x) \in R.$$

Example 1.20: Set inclusion is a partial order but not total order

Question. Let Ω be a non-empty set and consider its **power set**

$$\mathcal{P}(\Omega) = \{A : A \subseteq \Omega\}.$$

The inclusion defines **binary relation** on $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$, via

$$R := \{(A, B) \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) : A \subseteq B\}.$$

1. Prove that R is an **order relation**.
2. Prove that R is **not a total order**.

Solution.

1. Check that R is a partial order relation on $\mathcal{P}(\Omega)$:
 - Reflexive: It holds, since $A \subseteq A$ for all $A \in \mathcal{P}(\Omega)$.
 - Antisymmetric: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
 - Transitive: If $A \subseteq B$ and $B \subseteq C$, then, by definition of inclusion, $A \subseteq C$.
2. In general, R is **not** a total order. For example consider

$$\Omega = \{x, y\}.$$

Thus

$$\mathcal{P}(\Omega) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

If we pick $A = \{x\}$ and $B = \{y\}$ then $A \cap B = \emptyset$, meaning that

$$A \not\subseteq B, \quad B \not\subseteq A.$$

Example 1.21: Inequality is a total order

Question. Consider the binary relation

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x \leq y\}.$$

Prove that R is a **total order relation**.

Solution. We need to check that:

1. Reflexive: It holds, since $x \leq x$ for all $x \in \mathbb{Q}$,
2. Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$.
3. Transitive: If $x \leq y$ and $y \leq z$ then $x \leq z$.

Finally, we also have that R is a **total order** on \mathbb{Q} , since for all $x, y \in \mathbb{Q}$ we have

$$x \leq y \text{ or } y \leq x.$$

Lemma 1.24

Let $x, y \in \mathbb{R}$. Then

$$|x| < v \iff -v < x < v.$$

Corollary 1.25

Let $x, y \in \mathbb{R}$. Then

$$|x| < v \iff -v < x < v.$$

Theorem 1.26: Triangle inequality

For every $x, y \in \mathbb{R}$ we have

$$||x| - |y|| \leq |x + y| \leq |x| + |y|. \quad (1.11)$$

Proposition 1.27

For any $x, y \in \mathbb{R}$ it holds

$$||x| - |y|| \leq |x - y| \leq |x| + |y|. \quad (1.12)$$

Moreover for any $x, y, z \in \mathbb{R}$ it holds

$$|x - y| \leq |x - z| + |z - y|.$$

1.3 Absolute value

Definition 1.22: Absolute value

For $x \in \mathbb{R}$ we define its **absolute value** as the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Proposition 1.23

For all $x \in \mathbb{R}$ they hold:

1. $|x| \geq 0$.
2. $|x| = 0$ if and only if $x = 0$.
3. $|x| = |-x|$.

1.4 Induction

Definition 1.28: Principle of Induction

Let $\alpha(n)$ be a statement which depends on $n \in \mathbb{N}$. Suppose that

1. $\alpha(1)$ is true, and
2. Whenever $\alpha(n)$ is true, then $\alpha(n + 1)$ is true.

Then $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Example 1.29: Formula for summing first n natural numbers

Question. Prove by induction that the following formula holds for all $n \in \mathbb{N}$:

$$1 + 2 + 3 + \dots + (n - 1) + n = \frac{n(n + 1)}{2}. \quad (1.13)$$

Solution. Define

$$S(n) = 1 + 2 + \dots + n.$$

This way the formula at (1.13) is equivalent to

$$S(n) = \frac{n(n+1)}{2}, \quad \forall n \in \mathbb{N}.$$

1. It is immediate to check that (1.13) holds for $n = 1$.
2. Suppose (1.13) holds for $n = k$. Then

$$S(k+1) = 1 + \dots + k + (k+1) \quad (1.14)$$

$$= S(k) + (k+1) \quad (1.15)$$

$$= \frac{k(k+1)}{2} + (k+1) \quad (1.16)$$

$$= \frac{k(k+1) + 2(k+1)}{2} \quad (1.17)$$

$$= \frac{(k+1)(k+2)}{2} \quad (1.18)$$

where in the first equality we used that (1.13) holds for $n = k$. We have proven that

$$S(k+1) = \frac{(k+1)(k+2)}{2}.$$

The RHS in the above expression is exactly the RHS of (1.13) computed at $n = k+1$. Therefore, we have shown that formula (1.13) holds for $n = k+1$.

By the Principle of Induction, we conclude that (1.13) holds for all $n \in \mathbb{N}$.

where we used that $kx^2 \geq 0$. Then (1.19) holds for $n = k+1$.

By induction we conclude (1.19).

Example 1.30: Bernoulli's inequality

Question. Let $x \in \mathbb{R}$ with $x > -1$. Bernoulli's inequality states that

$$(1+x)^n \geq 1+nx, \quad \forall n \in \mathbb{N}. \quad (1.19)$$

Prove Bernoulli's inequality by induction.

Solution. Let $x \in \mathbb{R}, x > -1$. We prove the statement by induction:

- Base case: (1.19) holds with equality when $n = 1$.
- Induction hypothesis: Let $k \in \mathbb{N}$ and suppose that (1.19) holds for $n = k$, i.e.,

$$(1+x)^k \geq 1+kx.$$

Then

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \\ &\geq (1+kx)(1+x) \\ &= 1+kx+x+kx^2 \\ &\geq 1+(k+1)x, \end{aligned}$$

2 Real Numbers

2.1 Fields

Definition 2.1: Binary operation

A binary operation on a set K is a function

$$\circ : K \times K \rightarrow K$$

which maps the ordered pair (x, y) into $x \circ y$.

Definition 2.2

Let K be a set and $\circ : K \times K \rightarrow K$ be a binary operation on K . We say that:

1. \circ is **commutative** if

$$x \circ y = y \circ x, \quad \forall x, y \in K$$

2. \circ is **associative** if

$$(x \circ y) \circ z = x \circ (y \circ z), \quad \forall x, y, z \in K$$

3. An element $e \in K$ is called **neutral element** of \circ if

$$x \circ e = e \circ x = x, \quad \forall x \in K$$

4. Let e be a neutral element of \circ and let $x \in K$. An element $y \in K$ is called an **inverse** of x with respect to \circ if

$$x \circ y = y \circ x = e.$$

Example 2.3

Question. Let $K = \{0, 1\}$ be a set with binary operation \circ defined by the table

\circ	0	1
0	1	1
1	0	0

1. Is \circ commutative? Justify your answer.
2. Is \circ associative? Justify your answer.

Solution.

1. We have

$$0 \circ 1 = 1, \quad 1 \circ 0 = 0$$

and therefore

$$0 \circ 1 \neq 1 \circ 0.$$

showing that \circ is not commutative.

2. We have

$$(0 \circ 1) \circ 1 = 1 \circ 1 = 0,$$

while

$$0 \circ (1 \circ 1) = 0 \circ 0 = 1,$$

so that

$$(0 \circ 1) \circ 1 \neq 0 \circ (1 \circ 1).$$

Thus, \circ is not associative.

Definition 2.4: Field

Let K be a set with binary operations of **addition**

$$+ : K \times K \rightarrow K, \quad (x, y) \mapsto x + y$$

and **multiplication**

$$\cdot : K \times K \rightarrow K, \quad (x, y) \mapsto x \cdot y = xy.$$

We call the triple $(K, +, \cdot)$ a **field** if:

1. The addition $+$ satisfies: $\forall x, y, z \in K$

- (A1) **Commutativity and Associativity:**

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

- (A2) **Additive Identity:** There exists a **neutral element** in K for $+$, which we call 0. It holds:

$$x + 0 = 0 + x = x$$

- (A3) **Additive Inverse:** There exists an **inverse** of x with respect to $+$. We call this element the **additive inverse** of x and denote it by $-x$. It holds

$$x + (-x) = (-x) + x = 0$$

2. The multiplication \cdot satisfies: $\forall x, y, z \in K$

- (M1) **Commutativity and Associativity:**

$$x \cdot y = y \cdot x$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

- (M₂) **Multiplicative Identity:** There exists a **neutral element** in K for \cdot , which we call 1. It holds:

$$x \cdot 1 = 1 \cdot x = x$$

- (M₃) **Multiplicative Inverse:** If $x \neq 0$ there exists an **inverse** of x with respect to \cdot . We call this element the **multiplicative inverse** of x and denote it by x^{-1} . It holds

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

3. The operations $+$ and \cdot are related by

- (AM) **Distributive Property:** $\forall x, y, z \in K$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

Theorem 2.5

Consider the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ with the usual operations $+$ and \cdot . We have:

- $(\mathbb{N}, +, \cdot)$ is **not a field**.
- $(\mathbb{Z}, +, \cdot)$ is **not a field**.
- $(\mathbb{Q}, +, \cdot)$ is **a field**.

Theorem 2.6

Let K with $+$ and \cdot defined by

$+$	0	1	\cdot	0	1
0	0	1	0	0	0
1	1	0	1	0	1

Then $(K, +, \cdot)$ is a field.

Proposition 2.7: Uniqueness of neutral elements and inverses

Let $(K, +, \cdot)$ be a field. Then

1. There is a unique element in K with the property of 0.
2. There is a unique element in K with the property of 1.
3. For all $x \in K$ there is a unique additive inverse $-x$.
4. For all $x \in K, x \neq 0$, there is a unique multiplicative inverse x^{-1} .

Proof

1. Suppose that $0 \in K$ and $\tilde{0} \in K$ are both neutral element of $+$, that is, they both satisfy (A₂). Then

$$0 + \tilde{0} = 0$$

since $\tilde{0}$ is a neutral element for $+$. Moreover

$$\tilde{0} + 0 = \tilde{0}$$

since 0 is a neutral element for $+$. By commutativity of $+$, see property (A₁), we have

$$0 = 0 + \tilde{0} = \tilde{0} + 0 = \tilde{0},$$

showing that $0 = \tilde{0}$. Hence the neutral element for $+$ is unique.

2. Exercise.
3. Let $x \in K$ and suppose that $y, \tilde{y} \in K$ are both additive inverses of x , that is, they both satisfy (A₃). Therefore

$$x + y = 0$$

since y is an additive inverse of x and

$$x + \tilde{y} = 0$$

since \tilde{y} is an additive inverse of x . Therefore we can use commutativity and associativity of $+$, see property (A₁), and the fact that 0 is the neutral element of $+$, to infer

$$\begin{aligned} y &= y + 0 = y + (x + \tilde{y}) \\ &= (y + x) + \tilde{y} = (x + y) + \tilde{y} \\ &= 0 + \tilde{y} = \tilde{y}, \end{aligned}$$

concluding that $y = \tilde{y}$. Thus there is a unique additive inverse of x , and

$$y = \tilde{y} = -x,$$

with $-x$ the element from property (A₃).

4. Exercise.

Definition 2.8

Let K be a set with binary operations $+$ and \cdot , and with an order relation \leq . We call $(K, +, \cdot, \leq)$ an **ordered field** if:

1. $(K, +, \cdot)$ is a field
2. There \leq is of **total order** on K : $\forall x, y, z \in K$
 - (O₁) **Reflexivity:**

$$x \leq x$$

- (O₂) **Antisymmetry**:

$$x \leq y \text{ and } y \leq x \implies x = y$$

- (O₃) **Transitivity**:

$$x \leq y \text{ and } y \leq z \implies x \leq z$$

- (O₄) **Total order**:

$$x \leq y \text{ or } y \leq x$$

3. The operations $+$ and \cdot , and the total order \leq , are related by the following properties: $\forall x, y, z \in K$

- (AM) **Distributive**: Relates addition and multiplication via

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

- (AO) Relates addition and order with the requirement:

$$x \leq y \implies x + z \leq y + z$$

- (MO) Relates multiplication and order with the requirement:

$$x \geq 0, y \geq 0 \implies x \cdot y \geq 0$$

Theorem 2.9

$(\mathbb{Q}, +, \cdot, \leq)$ is an **ordered field**.

2.2 Supremum and infimum

Definition 2.10: Upper bound, bounded above, supremum, maximum

Let $A \subseteq K$:

1. We say that $b \in K$ is an **upper bound** for A if

$$a \leq b, \quad \forall a \in A.$$

2. We say that A is **bounded above** if there exists an upper bound $b \in K$ for A .

3. We say that $s \in K$ is the **least upper bound** or **supremum** of A if:

- s is an upper bound for A ,

- s is the smallest upper bound of A , that is,

If $b \in K$ is upper bound for A then $s \leq b$.

If it exists, the supremum is denoted by

$$s = \sup A.$$

4. Let $A \subseteq K$. We say that $M \in K$ is the **maximum** of A if:

$$M \in A \text{ and } a \leq M, \forall a \in A.$$

If it exists, we denote the maximum by

$$M = \max A.$$

Remark 2.11

Note that if a set $A \subseteq K$ in **NOT** bounded above, then the supremum does not exist, as there are no upper bounds of A .

Proposition 2.12: Relationship between Max and Sup

Let $A \subseteq K$. If the maximum of A exists, then also the supremum exists, and

$$\sup A = \max A.$$

Definition 2.13: Upper bound, bounded below, infimum, minimum

Let $A \subseteq K$:

1. We say that $l \in K$ is a **lower bound** for A if

$$l \leq a, \quad \forall a \in A.$$

2. We say that A is **bounded below** if there exists a lower bound $l \in K$ for A .

3. We say that $i \in K$ is the **greatest lower bound** or **infimum** of A if:

- i is a lower bound for A ,
- i is the largest lower bound of A , that is,

If $l \in K$ is a lower bound for A then $l \leq i$.

If it exists, the infimum is denoted by

$$i = \inf A.$$

4. We say that $m \in K$ is the **minimum** of A if:

$$m \in A \text{ and } m \leq a, \forall a \in A.$$

If it exists, we denote the minimum by

$$m = \min A.$$

Proposition 2.14

Let $A \subseteq K$. If the minimum of A exists, then also the infimum exists, and

$$\inf A = \min A.$$

Proposition 2.15

Let $A \subseteq K$. If $\inf A$ and $\sup A$ exist, then

$$\inf A \leq a \leq \sup A, \quad \forall a \in A.$$

Proposition 2.16: Relationship between sup and inf

Let $A \subseteq K$. Define

$$-A := \{-a : a \in A\}.$$

They hold

1. If $\sup A$ exists, then $\inf A$ exists and

$$\inf(-A) = -\sup A.$$

2. If $\inf A$ exists, then $\sup A$ exists and

$$\sup(-A) = -\inf A.$$

- A is bounded above,
- $\sup A$ does not exist in \mathbb{Q} .

Proposition 2.19

Let $(K, +, \cdot, <)$ be a complete ordered field. Suppose that

Definition 2.20: System of Real Numbers \mathbb{R}

A system of Real Numbers is a set \mathbb{R} with two operations $+$ and \cdot , and a total order relation \leq , such that

- $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field
- \mathbb{R} satisfies the Axiom of Completeness

2.3.1 Inductive sets

2.3 Axioms of Real Numbers

Definition 2.17: Completeness

Let $(K, +, \cdot, \leq)$ be an ordered field. We say that K is **complete** if it holds the property:

- (AC) For every $A \subseteq K$ non-empty and bounded above

$$\sup A \in K.$$

Theorem 2.18

\mathbb{Q} is not complete. In particular, there exists a set $A \subseteq \mathbb{Q}$ such that

- A is non-empty,

Definition 2.21: Inductive set

Let $S \subseteq \mathbb{R}$. We say that S is an inductive set if they are satisfied:

Example 2.22

Question. Prove the following:

1. \mathbb{R} is an inductive set.
2. The set $A = \{0, 1\}$ is not an inductive set.

Solution.

1. We have that $1 \in \mathbb{R}$ by axiom (M2). Moreover $(x + 1) \in \mathbb{R}$ for every $x \in \mathbb{R}$, by definition of sum $+$.

2. We have $1 \in A$ but $(1 + 1) \notin A$, since $1 + 1 \neq 0$.

Proposition 2.23

Let \mathcal{M} be a collection of inductive subsets of \mathbb{R} . Then

$$S := \bigcap_{M \in \mathcal{M}} M$$

is an inductive subset of \mathbb{R} .

Definition 2.24: Set of Natural Numbers

Let \mathcal{M} be the collection of **all** inductive subsets of \mathbb{R} . We define the set of natural numbers in \mathbb{R} as

$$\mathbb{N} := \bigcap_{M \in \mathcal{M}} M.$$

Proposition 2.25: $\mathbb{N}_{\mathbb{R}}$ is the smallest inductive subset of \mathbb{R}

Let $C \subseteq \mathbb{R}$ be an inductive subset. Then

$$\mathbb{N} \subseteq C.$$

In other words, \mathbb{N} is the smallest inductive set in \mathbb{R} .

Theorem 2.26

Let $x \in \mathbb{N}$. Then

$$x \geq 1.$$

3 Properties of \mathbb{R}

Theorem 3.1: Archimedean Property

Let $x \in \mathbb{R}$ be given. Then:

1. There exists $n \in \mathbb{N}$ such that

$$n > x.$$

2. Suppose in addition that $x > 0$. There exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < x.$$

Theorem 3.2: Archimedean Property (Alternative formulation)

Let $x, y \in \mathbb{R}$, with $0 < x < y$. There exists $n \in \mathbb{N}$ such that

$$nx > y.$$

Theorem 3.3: Nested Interval Property

For each $n \in \mathbb{N}$ assume given a closed interval

$$I_n := [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Suppose that the intervals are nested, that is,

$$I_n \supset I_{n+1}, \quad \forall n \in \mathbb{N}.$$

Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset. \quad (3.1)$$

Example 3.4

Question. Consider the **open** intervals

$$I_n := \left(0, \frac{1}{n}\right).$$

These are clearly nested

$$I_n \supset I_{n+1}, \quad \forall n \in \mathbb{N}.$$

Prove that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset. \quad (3.2)$$

Solution. Suppose by contradiction that the intersection is non-empty. Then there exists $x \in \mathbb{N}$ such that

$$x \in I_n, \quad \forall n \in \mathbb{N}.$$

By definition of I_n the above reads

$$0 < x < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \quad (3.3)$$

Since $x > 0$, by the Archimedean Property in Theorem 3.1 Point 2, there exists $n_0 \in \mathbb{N}$ such that

$$0 < \frac{1}{n_0} < x.$$

The above contradicts (3.3). Therefore (3.2) holds.

3.1 Revisiting Sup and Inf

Proposition 3.5: Characterization of Supremum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $s \in \mathbb{R}$ is an upper bound for A . They are equivalent:

1. $s = \sup A$
2. For every $\varepsilon > 0$ there exists $x \in A$ such that

$$s - \varepsilon < x.$$

Proposition 3.6: Characterization of Infimum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $i \in \mathbb{R}$ is a lower bound for A . They are equivalent:

1. $i = \inf A$
2. For every $\varepsilon \in \mathbb{R}$, with $\varepsilon > 0$, there exists $x \in A$ such that

$$x < i + \varepsilon.$$

Proposition 3.7

Let $a, b \in \mathbb{R}$ with $a < b$. Let

$$A := (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Then

$$\inf A = a, \quad \sup A = b.$$

Corollary 3.8

Let $a, b \in \mathbb{R}$ with $a < b$. Let

$$A := (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Then $\min A$ and $\max A$ do not exist.

Corollary 3.9

Let $a, b \in \mathbb{R}$ with $a < b$. Let

$$A := [a, b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

Then

$$\min A = \inf A = a, \quad \sup A = b,$$

$\max A$ does not exist.

Proposition 3.10

Define the set

$$A := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then

$$\inf A = 0, \quad \sup A = \max A = 1.$$

Proof

Part 1. We have

$$\frac{1}{n} \leq 1, \quad \forall n \in \mathbb{N}.$$

Therefore 1 is an upper bound for A . Since $1 \in A$, by definition of maximum we conclude that

$$\max A = 1.$$

Since the maximum exists, we conclude that also the supremum exists, and

$$\sup A = \max A = 1.$$

Part 2. We have

$$\frac{1}{n} > 0, \quad \forall n \in \mathbb{N},$$

showing that 0 is a lower bound for A . Suppose by contradiction that 0 is not the infimum. Therefore 0 is not the largest lower bound. Then there exists $\varepsilon \in \mathbb{R}$ such that:

- ε is a lower bound for A , that is,

$$\varepsilon \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}, \quad (3.4)$$

- ε is larger than 0:

$$0 < \varepsilon.$$

As $\varepsilon > 0$, by the Archimedean Property there exists $n_0 \in \mathbb{N}$ such that

$$0 < \frac{1}{n_0} < \varepsilon.$$

This contradicts (3.4). Thus 0 is the largest lower bound of A , that is, $0 = \inf A$.

Part 3. We have that $\min A$ does not exist. Indeed suppose by contradiction that $\min A$ exists. Then

$$\min A = \inf A.$$

As $\inf A = 0$ by Part 2, we conclude $\min A = 0$. As $\min A \in A$, we obtain $0 \in A$, which is a contradiction.

3.2 Cardinality

Definition 3.11: Cardinality, Finite, Countable, Uncountable

Let X be a set. The **cardinality** of X is the number of elements in X . We denote the cardinality of X by

$$|X| := \# \text{ of elements in } X.$$

Further, we say that:

1. X is **finite** if there exists a natural number $n \in \mathbb{N}$ and a bijection

$$f : \{1, 2, \dots, n\} \rightarrow X.$$

In particular

$$|X| = n.$$

2. X is **countable** if there exists a bijection

$$f : \mathbb{N} \rightarrow X.$$

In this case we denote the cardinality of X by

$$|X| = |\mathbb{N}|.$$

3. X is **uncountable** if X is neither finite, nor countable.

Proposition 3.12

Let X be a countable set and $A \subseteq X$. Then either A is finite or countable.

Example 3.13

Question. Prove that $X = \{a, b, c\}$ is finite.

Solution. Set $Y = \{1, 2, 3\}$. The function $f : X \rightarrow Y$ defined by

$$f(1) = a, \quad f(2) = b, \quad f(3) = c,$$

is bijective. Therefore X is finite, with $|X| = 3$.

Example 3.14

Question. Prove that the set of natural numbers \mathbb{N} is countable.

Solution. The function $f : X \rightarrow \mathbb{N}$ defined by

$$f(n) := n,$$

is bijective. Therefore $X = \mathbb{N}$ is countable.

Example 3.15

Question. Let X be the set of even numbers

$$X = \{2n : n \in \mathbb{N}\}.$$

Prove that X is countable.

Solution. Define the map $f : \mathbb{N} \rightarrow X$ by

$$f(n) := 2n.$$

We have that:

1. f is injective, because

$$f(m) = f(k) \implies 2m = 2k \quad m = k.$$

2. f is surjective: Suppose that $m \in X$. By definition of X , there exists $n \in \mathbb{N}$ such that $m = 2n$. Therefore, $f(n) = m$.

We have shown that f is bijective. Thus, X is countable.

Example 3.16

Question. Prove that the set of integers \mathbb{Z} is countable.

Solution. Define $f : \mathbb{N} \rightarrow \mathbb{Z}$ by

$$f(n) := \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ -\frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

For example

$$\begin{aligned} f(0) &= 0, & f(1) &= -1, & f(2) &= 1, & f(3) &= -2, \\ f(4) &= 2, & f(5) &= -3, & f(6) &= 3, & f(7) &= -4. \end{aligned}$$

We have:

1. f is injective: Indeed, suppose that $m \neq n$. If n and m are both even or both odd we have, respectively

$$\begin{aligned} f(m) &= \frac{m}{2} \neq \frac{n}{2} = f(n) \\ f(m) &= -\frac{m+1}{2} \neq -\frac{n+1}{2} = f(n). \end{aligned}$$

If instead m is even and n is odd, we get

$$f(m) = \frac{m}{2} \neq -\frac{n+1}{2} = f(n),$$

since the LHS is positive and the RHS is negative. The case when m is odd and n even is similar.

2. f is surjective: Let $z \in \mathbb{Z}$. If $z \geq 0$, then $m := 2z$ belongs to \mathbb{N} , is even, and

$$f(m) = f(2z) = z.$$

If instead $z < 0$, then $m := -2z - 1$ belongs to \mathbb{N} , is odd, and

$$f(m) = f(-2z - 1) = z.$$

Therefore f is bijective, showing that \mathbb{Z} is countable.

Proposition 3.17

Let the set A_n be countable for all $n \in \mathbb{N}$. Define

$$A = \bigcup_{n \in \mathbb{N}} A_n.$$

Then A is countable.

Theorem 3.18: \mathbb{Q} is countable

The set of rational numbers \mathbb{Q} is countable.

Theorem 3.19: \mathbb{R} is uncountable

The set of Real Numbers \mathbb{R} is **uncountable**.

Theorem 3.20

The set of irrational numbers

$$\mathcal{I} := \mathbb{R} \setminus \mathbb{Q}$$

is uncountable.

Proof

We know that \mathbb{R} is uncountable and \mathbb{Q} is countable. Suppose by contradiction that \mathcal{I} is countable. Then

$$\mathbb{Q} \cup \mathcal{I}$$

is countable by Proposition 3.17, being union of countable sets. Since by definition

$$\mathbb{R} = \mathbb{Q} \cup \mathcal{I},$$

we conclude that \mathbb{R} is countable. Contradiction.

4 Complex Numbers

Definition 4.1: Complex Numbers

The set of complex numbers \mathbb{C} is defined as

$$\mathbb{C} := \mathbb{R} + i\mathbb{R} := \{x + iy : x, y \in \mathbb{R}\}.$$

For a complex number

$$z = x + iy \in \mathbb{C}$$

we say that

- x is the **real part** of z , and denote it by

$$x = \operatorname{Re}(z)$$

- y is the **imaginary part** of z , and denote it by

$$y = \operatorname{Im}(z)$$

We say that

- If $\operatorname{Re} z = 0$ then z is a **purely imaginary** number.
- If $\operatorname{Im} z = 0$ then z is a **real** number.

Definition 4.2: Addition and multiplication in \mathbb{C}

Let $z_1, z_2 \in \mathbb{C}$, so that

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2,$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$:

1. The sum of z_1 and z_2 is

$$z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2).$$

2. The multiplication of z_1 and z_2 is

$$z_1 \cdot z_2 := (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1),$$

Example 4.3

Question. Compute zw , where

$$z = -2 + 3i, \quad w = 1 - i.$$

Solution. Using the definition we compute

$$\begin{aligned} z \cdot w &= (-2 + 3i) \cdot (1 - i) \\ &= (-2 - (-3)) + (2 + 3)i \\ &= 1 + 5i. \end{aligned}$$

Alternatively, we can proceed formally: We just need to recall that i^2 has to be replaced with -1 :

$$\begin{aligned} z \cdot w &= (-2 + 3i) \cdot (1 - i) \\ &= -2 + 2i + 3i - 3i^2 \\ &= (-2 + 3) + (2 + 3)i \\ &= 1 + 5i. \end{aligned}$$

Proposition 4.4: Additive inverse in \mathbb{C}

The neutral element of addition in \mathbb{C} is the number

$$0 := 0 + 0i.$$

For any $z = x + iy \in \mathbb{C}$, the unique additive inverse is given by

$$-z := -x - iy.$$

Proposition 4.5: Multiplicative inverse in \mathbb{C}

The neutral element of multiplication in \mathbb{C} is the number

$$1 := 1 + 0i.$$

For any $z = x + iy \in \mathbb{C}$, the unique multiplicative inverse is given by

$$z^{-1} := \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}.$$

Proof

It is immediate to check that 1 is the neutral element of multiplication in \mathbb{C} . For the remaining part of the statement, set

$$w := \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}.$$

We need to check that $z \cdot w = 1$

$$\begin{aligned} z \cdot w &= (x + iy) \cdot \left(\frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} \right) \\ &= \left(\frac{x^2}{x^2 + y^2} - \frac{y \cdot (-y)}{x^2 + y^2} \right) + i \left(\frac{x \cdot (-y)}{x^2 + y^2} + \frac{xy}{x^2 + y^2} \right) \\ &= 1, \end{aligned}$$

so indeed $z^{-1} = w$.

Example 4.6

Question. Let $z = 3 + 2i$. Compute z^{-1} .

Solution. By the formula in Proposition 4.5 we immediately get

$$z^{-1} = \frac{3}{3^2 + 2^2} + \frac{-2}{3^2 + 2^2}i = \frac{3}{13} - \frac{2}{13}i.$$

Alternatively, we can proceed formally:

$$\begin{aligned} (3 + 2i)^{-1} &= \frac{1}{3 + 2i} \\ &= \frac{1}{3 + 2i} \frac{3 - 2i}{3 - 2i} \\ &= \frac{3 - 2i}{3^2 + 2^2} \\ &= \frac{3}{13} - \frac{2}{13}i, \end{aligned}$$

and obtain the same result.

Theorem 4.7

$(\mathbb{C}, +, \cdot)$ is a field.

Example 4.8

Question. Let $w = 1 + i$ and $z = 3 - i$. Compute $\frac{w}{z}$.

Solution. We compute w/z using the two options we have:

- Using the formula for the inverse from Proposition 4.5 we compute

$$\begin{aligned} z^{-1} &= \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} \\ &= \frac{3}{3^2 + 1^2} - i \frac{-1}{3^2 + 1^2} \\ &= \frac{3}{10} + \frac{1}{10}i \end{aligned}$$

and therefore

$$\begin{aligned} \frac{w}{z} &= w \cdot z^{-1} \\ &= (1 + i) \left(\frac{3}{10} + \frac{1}{10}i \right) \\ &= \left(\frac{3}{10} - \frac{1}{10} \right) + \left(\frac{1}{10} + \frac{3}{10} \right)i \\ &= \frac{2}{10} + \frac{4}{10}i \\ &= \frac{1}{5} + \frac{2}{5}i \end{aligned}$$

- We proceed formally, using the multiplication by 1 trick. We have

$$\begin{aligned} \frac{w}{z} &= \frac{1 + i}{3 - i} \\ &= \frac{1 + i}{3 - i} \frac{3 + i}{3 + i} \\ &= \frac{3 - 1 + (3 + 1)i}{3^2 + 1^2} \\ &= \frac{2}{10} + \frac{4}{10}i \\ &= \frac{1}{5} + \frac{2}{5}i \end{aligned}$$

Definition 4.9: Complex conjugate

Let $z = x + iy$. We call the **complex conjugate** of z , denoted by \bar{z} , the complex number

$$\bar{z} = x - iy.$$

Theorem 4.10

For all $z_1, z_2 \in \mathbb{C}$ it holds:

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

4.1 The complex plane

Definition 4.11: Modulus

The **modulus** of a complex number $z = x + iy$ is defined by

$$|z| := \sqrt{x^2 + y^2}.$$

Definition 4.12: Distance in \mathbb{C}

Given $z_1, z_2 \in \mathbb{C}$, we define the **distance** between z_1 and z_2 as the quantity

$$|z_1 - z_2|.$$

Theorem 4.13

Given $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 4.14

Question. Compute the distance between

$$z = 2 - 4i, \quad w = -5 + i.$$

Solution. The distance is

$$\begin{aligned} |z - w| &= |(2 - 4i) - (-5 + i)| \\ &= |7 - 5i| \\ &= \sqrt{7^2 + (-5)^2} \\ &= \sqrt{74} \end{aligned}$$

Theorem 4.15

Let $z, z_1, z_2 \in \mathbb{C}$. Then

1. $|z_1 \cdot z_2| = |z_1| |z_2|$
2. $|z^n| = |z|^n$ for all $n \in \mathbb{N}$
3. $z \cdot \bar{z} = |z|^2$

Theorem 4.16: Triangle inequality in \mathbb{C}

For all $x, y, z \in \mathbb{C}$,

1. $|x + y| \leq |x| + |y|$
2. $|x - z| \leq |x - y| + |y - z|$

Definition 4.17: Argument

Let $z \in \mathbb{C}$. The angle θ between the line connecting the origin and z and the positive real axis is called the **argument** of z , and is denoted by

$$\theta := \arg(z).$$

Example 4.18

We have the following arguments:

$$\begin{aligned} \arg(1) &= 0 & \arg(i) &= \frac{\pi}{2} \\ \arg(-1) &= \pi & \arg(-i) &= -\frac{\pi}{2} \\ \arg(1+i) &= \frac{1}{4}\pi & \arg(-1-i) &= -\frac{3}{4}\pi \end{aligned}$$

Theorem 4.19: Polar coordinates

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$x = \rho \cos(\theta), \quad y = \rho \sin(\theta),$$

where

$$\rho := |z| = \sqrt{x^2 + y^2}, \quad \theta := \arg(z).$$

Definition 4.20: Trigonometric form

Let $z \in \mathbb{C}$. The trigonometric form of z is

$$z = |z| [\cos(\theta) + i \sin(\theta)],$$

where $\theta = \arg(z)$.

Example 4.21

Question. Suppose that $z \in \mathbb{C}$ has polar coordinates

$$\rho = \sqrt{8}, \quad \theta = \frac{3}{4}\pi.$$

Therefore, the trigonometric form of z is

$$z = \sqrt{8} \left[\cos\left(\frac{3}{4}\pi\right) + i \sin\left(\frac{3}{4}\pi\right) \right].$$

Write z in cartesian form.

Solution. We have

$$\begin{aligned} x &= \rho \cos(\theta) = \sqrt{8} \cos\left(\frac{3}{4}\pi\right) = -\sqrt{8} \cdot \frac{\sqrt{2}}{2} = -2 \\ y &= \rho \sin(\theta) = \sqrt{8} \sin\left(\frac{3}{4}\pi\right) = \sqrt{8} \cdot \frac{\sqrt{2}}{2} = 2. \end{aligned}$$

Therefore, the cartesian form of z is

$$z = x + iy = -2 + 2i.$$

Corollary 4.22: Computing $\arg(z)$

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$\arg(z) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$

where \arctan is the inverse of \tan .

Example 4.23

Question. Compute the arguments of the complex numbers

$$z = 3 + 4i, \quad \bar{z} = 3 - 4i, \quad -\bar{z} = -3 + 4i, \quad -z = -3 - 4i.$$

Solution. Using the formula for \arg in Corollary 4.22 we have

$$\begin{aligned} \arg(3 + 4i) &= \arctan\left(\frac{4}{3}\right) \\ \arg(3 - 4i) &= \arctan\left(-\frac{4}{3}\right) = -\arctan\left(\frac{4}{3}\right) \\ \arg(-3 + 4i) &= \arctan\left(-\frac{4}{3}\right) + \pi = -\arctan\left(\frac{4}{3}\right) + \pi \\ \arg(-3 - 4i) &= \arctan\left(\frac{4}{3}\right) - \pi \end{aligned}$$

Theorem 4.24: Euler's identity

For all $\theta \in \mathbb{R}$ it holds

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Theorem 4.25

For all $\theta \in \mathbb{R}$ it holds

$$|e^{i\theta}| = 1.$$

Theorem 4.26

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$z = \rho e^{i\theta},$$

where

$$\rho := |z| = \sqrt{x^2 + y^2}, \quad \theta := \arg(z).$$

Definition 4.27: Exponential form

The **exponential form** of a complex number $z \in \mathbb{C}$ is

$$z = \rho e^{i\theta} = |z| e^{i \arg(z)}.$$

Example 4.28

Question. Write the number

$$z = -2 + 2i$$

in exponential form.

Solution. From Example 4.21 we know that $z = -2 + 2i$ can be written in trigonometric form as

$$z = \sqrt{8} \left[\cos\left(\frac{3}{4}\pi\right) + i \sin\left(\frac{3}{4}\pi\right) \right].$$

By Euler's identity we hence obtain the exponential form

$$z = \sqrt{8} e^{i\frac{3}{4}\pi}.$$

Remark 4.29: Periodicity of exponential

For all $k \in \mathbb{Z}$ we have

$$e^{i\theta} = e^{i(\theta+2\pi k)}, \quad (4.1)$$

meaning that the complex exponential is 2π -periodic.

Proposition 4.30

Let $z, z_1, z_2 \in \mathbb{C}$ and suppose that

$$z = \rho e^{i\theta}, \quad z_1 = \rho_1 e^{i\theta_1}, \quad z_2 = \rho_2 e^{i\theta_2}.$$

We have

$$z_1 \cdot z_2 = \rho_1 \rho_2 e^{i(\theta_1+\theta_2)}, \quad z^n = \rho^n e^{in\theta},$$

for all $n \in \mathbb{N}$.

Example 4.31

Question. Compute $(-2 + 2i)^4$.

Solution. We have two possibilities:

4.2 Fundamental Theorem of Algebra

1. Use the binomial theorem:

$$\begin{aligned} (-2 + 2i)^4 &= (-2)^4 + \binom{4}{1}(-2)^3 \cdot 2i + \binom{4}{2}(-2)^2 \cdot (2i)^2 \\ &\quad + \binom{4}{3}(-2) \cdot (2i)^3 + (2i)^4 \\ &= 16 - 4 \cdot 8 \cdot 2i - 6 \cdot 4 \cdot 4 + 4 \cdot 2 \cdot 8i + 16 \\ &= 16 - 64i - 96 + 64i + 16 = -64. \end{aligned}$$

2. A much simpler calculation is possible by using the exponential form: We know that

$$-2 + 2i = \sqrt{8}e^{i\frac{3}{4}\pi}$$

by Example 4.28. Hence

$$(-2 + 2i)^4 = \left(\sqrt{8}e^{i\frac{3}{4}\pi}\right)^4 = 8^2 e^{i3\pi} = -64,$$

where we used that

$$e^{i3\pi} = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

by 2π periodicity of $e^{i\theta}$ and Euler's identity.

Definition 4.32: Complex exponential

The complex exponential of $z \in \mathbb{C}$ is defined as

$$e^z = |z|e^{i\theta}, \quad \theta = \arg(z).$$

Theorem 4.33

Let $z, w \in \mathbb{C}$. Then

$$e^{z+w} = e^z e^w, \quad (e^z)^w = e^{zw}. \quad (4.2)$$

Example 4.34

Question. Compute i^i .

Solution. We know that

$$|i| = 1, \quad \arg(i) = \frac{\pi}{2}.$$

Hence we can write i in exponential form

$$i = |i|e^{i\arg(i)} = e^{i\frac{\pi}{2}}.$$

Therefore

$$i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}.$$

Theorem 4.35: Fundamental theorem of algebra

Let $p_n(z)$ be a polynomial of degree n with complex coefficients, i.e.,

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

for some coefficients $a_n, \dots, a_0 \in \mathbb{C}$ with $a_n \neq 0$. There exist

$$z_1, \dots, z_n \in \mathbb{C}$$

solutions to the polynomial equation

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0. \quad (4.3)$$

In particular, p_n factorizes as

$$p_n(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_n). \quad (4.4)$$

Example 4.36

Question. Find all the complex solutions to

$$z^2 = -1 \quad (4.5)$$

Solution. The equation $z^2 = -1$ is equivalent to

$$p(z) = 0, \quad p(z) := z^2 + 1.$$

Since p has degree $n = 2$, the Fundamental Theorem of Algebra tells us that there are two solutions to (4.5). We have already seen that these two solutions are $z = i$ and $z = -i$. Then p factorizes as

$$p(z) = z^2 + 1 = (z - i)(z + i).$$

Example 4.37

Question. Find all the complex solutions to

$$z^4 - 1 = 0. \quad (4.6)$$

Solution The associated polynomial equation is

$$p(z) = 0, \quad p(z) := z^4 - 1.$$

Since p has degree $n = 4$, the Fundamental Theorem of Algebra tells us that there are 4 solutions to (4.6). Let us find such solutions. We use the well known identity

$$a^2 - b^2 = (a + b)(a - b), \quad \forall a, b \in \mathbb{R},$$

to factorize p . We get:

$$p(z) = (z^4 - 1) = (z^2 + 1)(z^2 - 1).$$

4.3 Solving polynomial equations

We know that

$$z^2 + 1 = 0$$

has solutions $z = \pm i$. Instead

$$z^2 - 1 = 0$$

has solutions $x = \pm 1$. Hence, the four solutions of (4.6) are given by

$$z = 1, -1, i, -i,$$

and p factorizes as

$$p(z) = z^4 - 1 = (z - 1)(z + 1)(z - i)(z + i).$$

Definition 4.38

Suppose that the polynomial p_n factorizes as

$$p_n(z) = a_n(z - z_1)^{k_1}(z - z_2)^{k_2} \dots (z - z_m)^{k_m}$$

with $a_n \neq 0$, $z_1, \dots, z_m \in \mathbb{C}$ and $k_1, \dots, k_m \in \mathbb{N}$, $k_i \geq 1$. In this case p_n has degree

$$n = k_1 + \dots + k_m = \sum_{i=1}^m k_i.$$

Note that z_i solves the equation

$$p_n(z) = 0$$

exactly k_i times. We call k_i the **multiplicity** of the solution z_i .

Example 4.39

The equation

$$(z - 1)(z - 2)^2(z + i)^3 = 0$$

has 6 solutions:

- $z = 1$ with multiplicity 1
- $z = 2$ with multiplicity 2
- $z = -i$ with multiplicity 3

Proposition 4.40: Quadratic formula

Let $a, b, c \in \mathbb{R}$, $a \neq 0$ and consider the equation

$$ax^2 + bx + c = 0. \quad (4.7)$$

Define

$$\Delta := b^2 - 4ac \in \mathbb{R}.$$

The following hold:

1. If $\Delta > 0$ then (4.7) has two distinct real solutions $z_1, z_2 \in \mathbb{R}$ given by

$$z_1 = \frac{-b - \sqrt{\Delta}}{2a}, \quad z_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

2. If $\Delta = 0$ then (4.7) has one real solution $z \in \mathbb{R}$ with multiplicity 2. Such solution is given by

$$z = z_1 = z_2 = \frac{-b}{2a}.$$

3. If $\Delta < 0$ then (4.7) has two distinct complex solutions $z_1, z_2 \in \mathbb{C}$ given by

$$z_1 = \frac{-b - i\sqrt{-\Delta}}{2a}, \quad z_2 = \frac{-b + i\sqrt{-\Delta}}{2a},$$

where $\sqrt{-\Delta} \in \mathbb{R}$, since $-\Delta > 0$.

In all cases, the polynomial at (4.7) factorizes as

$$az^2 + bz + c = a(z - z_1)(z - z_2).$$

Example 4.41

Question. Solve the following equations:

1. $3z^2 - 6z + 2 = 0$
2. $4z^2 - 8z + 4 = 0$
3. $z^2 + 2z + 3 = 0$

Solution.

1. We have that

$$\Delta = (-6)^2 - 4 \cdot 3 \cdot 2 = 12 > 0$$

Therefore the equation has two distinct real solutions, given by

$$z = \frac{-(-6) \pm \sqrt{12}}{2 \cdot 3} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$

In particular we have the factorization

$$3z^2 - 6z + 2 = 3 \left(z - 1 - \frac{\sqrt{3}}{3} \right) \left(z - 1 + \frac{\sqrt{3}}{3} \right).$$

2. We have that

$$\Delta = (-8)^2 - 4 \cdot 4 \cdot 4 = 0.$$

Therefore there exists one solution with multiplicity

2. This is given by

$$z = \frac{-(-8)}{2 \cdot 4} = 1.$$

In particular we have the factorization

$$4z^2 - 8z + 4 = 4(z - 1)^2.$$

3. We have

$$\Delta = 2^2 - 4 \cdot 1 \cdot 3 = -8 < 0.$$

Therefore there are two complex solutions given by

$$z = \frac{-2 \pm i\sqrt{8}}{2 \cdot 1} = -1 \pm i\sqrt{2}.$$

In particular we have the factorization

$$z^2 + 2z + 3 = (z + 1 - i\sqrt{2})(z + 1 + i\sqrt{2}).$$

Proposition 4.42: Quadratic formula with complex coefficients

Let $a, b, c \in \mathbb{C}, a \neq 0$. The two solutions to

$$az^2 + bz + c = 0$$

are given by

$$z_1 = \frac{-b + S_1}{2a}, \quad z_2 = \frac{-b + S_2}{2a},$$

where S_1 and S_2 are the two solutions to

$$z^2 = \Delta, \quad \Delta := b^2 - 4ac.$$

Example 4.43

Question Find all the solutions to

$$\frac{1}{2}z^2 - (3+i)z + (4-i) = 0. \quad (4.8)$$

Solution. We have

$$\begin{aligned} \Delta &= (-(3+i))^2 - 4 \cdot \frac{1}{2} \cdot (4-i) \\ &= 8 + 6i - 8 + 2i \\ &= 8i. \end{aligned}$$

Therefore $\Delta \in \mathbb{C}$. We have to find solutions S_1 and S_2 to the equation

$$z^2 = \Delta = 8i. \quad (4.9)$$

We look for solutions of the form $z = a + ib$. Then we must have that

$$z^2 = (a + ib)^2 = a^2 - b^2 + 2abi = 8i.$$

Thus

$$a^2 - b^2 = 0, \quad 2ab = 8.$$

From the first equation we conclude that $|a| = |b|$. From the second equation we have that $ab = 4$, and therefore a and b must have the same sign. Hence $a = b$, and

$$2ab = 8 \implies a = b = \pm 2.$$

From this we conclude that the solutions to (4.9) are

$$S_1 = 2 + 2i, \quad S_2 = -2 - 2i.$$

Hence the solutions to (4.8) are

$$\begin{aligned} z_1 &= \frac{3 + i + S_1}{2 \cdot \frac{1}{2}} = 3 + i + S_1 \\ &= 3 + i + 2 + 2i = 5 + 3i, \end{aligned}$$

and

$$\begin{aligned} z_2 &= \frac{3 + i + S_2}{2 \cdot \frac{1}{2}} = 3 + i + S_2 \\ &= 3 + i - 2 - 2i = 1 - i. \end{aligned}$$

In particular, the given polynomial factorizes as

$$\begin{aligned} \frac{1}{2}z^2 - (3+i)z + (4-i) &= \frac{1}{2}(z - z_1)(z - z_2) \\ &= \frac{1}{2}(z - 5 - 3i)(z - 1 + i). \end{aligned}$$

Example 4.44

Question. Consider the equation

$$z^3 - 7z^2 + 6z = 0.$$

1. Check whether $z = 0, 1, -1$ are solutions.
2. Using your answer from Point 1, and polynomial division, find all the solutions.

Solution.

1. By direct inspection we see that $z = 0$ and $z = 1$ are solutions.

2. Since $z = 0$ is a solution, we can factorize

$$z^3 - 7z^2 + 6z = z(z^2 - 7z + 6).$$

We could now use the quadratic formula on the term $z^2 - 7z + 6$ to find the remaining two roots. However, we have already observed that $z = 1$ is a solution. Therefore $z - 1$ divides $z^2 - 7z + 6$. Using polynomial long division, we find that

$$\frac{z^2 - 7z + 6}{z - 1} = z - 6.$$

Therefore the last solution is $z = 6$, and

Example 4.45

Question. Find all the complex solutions to

$$z^3 - 7z + 6 = 0.$$

Solution. It is easy to see $z = 1$ is a solution. This means that $z - 1$ divides $z^3 - 7z + 6$. By using polynomial long division, we compute that

$$\frac{z^3 - 7z + 6}{z - 1} = z^2 + z - 6.$$

We are now left to solve

$$z^2 + z - 6 = 0.$$

Using the quadratic formula, we see that the above is solved by $z = 2$ and $z = -3$. Therefore the given polynomial factorizes as

$$z^3 - 7z + 6 = (z - 1)(z - 2)(z + 3).$$

4.4 Roots

Theorem 4.46

Let $n \in \mathbb{N}$ and consider the equation

$$z^n = 1. \quad (4.10)$$

All the n solutions to (4.10) are given by

$$z_k = \exp\left(i\frac{2\pi k}{n}\right), \quad k = 0, \dots, n-1,$$

where $\exp(x)$ denotes e^x .

Definition 4.47

The n solutions to

$$z^n = 1$$

are called the **roots of unity**.

Example 4.48

Question. Find all the solutions to

$$z^4 = 1.$$

Solution. The 4 solutions are given by

$$z_k = \exp\left(i\frac{2\pi k}{4}\right) = \exp\left(i\frac{\pi k}{2}\right),$$

for $k = 0, 1, 2, 3$. We compute:

$$\begin{aligned} z_0 &= e^{i0} = 1, & z_1 &= e^{i\frac{\pi}{2}} = i, \\ z_2 &= e^{i\pi} = -1, & z_3 &= e^{i\frac{3\pi}{2}} = -i. \end{aligned}$$

Note that for $k = 4$ we would again get the solution $z = e^{i2\pi} = 1$.

Example 4.49

Question. Find all the solutions to

$$z^3 = 1.$$

Solution. The 3 solutions are given by

$$z_k = \exp\left(i\frac{2\pi k}{3}\right),$$

for $k = 0, 1, 2$. We compute:

$$z_0 = e^{i0} = 1, \quad z_1 = e^{i\frac{2\pi}{3}}, \quad z_2 = e^{i\frac{4\pi}{3}}.$$

We can write z_1 and z_2 in cartesian form:

$$z_1 = e^{i\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

and

$$z_2 = e^{i\frac{4\pi}{3}} = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Theorem 4.50

Let $n \in \mathbb{N}$, $c \in \mathbb{C}$ and consider the equation

$$z^n = c. \quad (4.11)$$

All the n solutions to (4.11) are given by

$$z_k = \sqrt[n]{|c|} \exp\left(i \frac{\theta + 2\pi k}{n}\right), \quad k = 0, \dots, n-1,$$

where $\sqrt[n]{|c|}$ is the n -th root of the real number $|c|$, and $\theta = \arg(c)$.

The 4 solutions are given by

$$\begin{aligned} z_k &= \sqrt[4]{9} \exp\left(i \frac{\pi/3 + 2\pi k}{4}\right) \\ &= \sqrt{3} \exp\left(i\pi \frac{1+6k}{12}\right) \end{aligned}$$

for $k = 0, 1, 2, 3$. We compute

$$\begin{aligned} z_0 &= \sqrt{3} e^{i\pi \frac{1}{12}} & z_1 &= \sqrt{3} e^{i\pi \frac{7}{12}} \\ z_2 &= \sqrt{3} e^{i\pi \frac{13}{12}} & z_3 &= \sqrt{3} e^{i\pi \frac{19}{12}} \end{aligned}$$

Example 4.51

Question. Find all the $z \in \mathbb{C}$ such that

$$z^5 = -32.$$

Solution. Let $c = -32$. We have

$$|c| = |-32| = 32 = 2^5, \quad \theta = \arg(-32) = \pi.$$

The 5 solutions are given by

$$z_k = (2^5)^{\frac{1}{5}} \exp\left(i\pi \frac{1+2k}{5}\right), \quad k \in \mathbb{Z},$$

for $k = 0, 1, 2, 3, 4$. We get

$$\begin{aligned} z_0 &= 2e^{i\frac{\pi}{5}} & z_1 &= 2e^{i\frac{3\pi}{5}} \\ z_2 &= 2e^{i\pi} = -2 & z_3 &= 2e^{i\frac{7\pi}{5}} \\ z_4 &= 2e^{i\frac{9\pi}{5}} \end{aligned}$$

Example 4.52

Question. Find all the $z \in \mathbb{C}$ such that

$$z^4 = 9 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right).$$

Solution. Set

$$c := 9 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right).$$

The complex number c is already in the trigonometric form, so that we can immediately obtain

$$|c| = 9, \quad \theta = \arg(c) = \frac{\pi}{3}.$$

5 Sequences in \mathbb{R}

Definition 5.1: Convergent sequence

The real sequence (a_n) **converges** to a , or equivalently has limit a , denoted by

$$\lim_{n \rightarrow \infty} a_n = a,$$

if for all $\varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq N$ it holds that

$$|a_n - a| < \varepsilon.$$

Using quantifiers, we can write this as

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |a_n - a| < \varepsilon.$$

The sequence $(a_n)_{n \in \mathbb{N}}$ is **convergent** if it admits limit.

Theorem 5.2

Let $p > 0$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0.$$

Proof

Let $p > 0$. We have to show that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \left| \frac{1}{n^p} - 0 \right| < \varepsilon.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon^{1/p}}. \quad (5.1)$$

Let $n \geq N$. Since $p > 0$, we have $n^p \geq N^p$, which implies

$$\frac{1}{n^p} \leq \frac{1}{N^p}.$$

By (5.1) we deduce

$$\frac{1}{N^p} < \varepsilon.$$

Then

$$\left| \frac{1}{n^p} - 0 \right| = \frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon.$$

Example 5.3

Question. Using the definition of convergence, prove that

$$\lim_{n \rightarrow \infty} \frac{n}{2n+3} = \frac{1}{2}.$$

Solution.

1. *Rough Work:* Let $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ such that

$$\left| \frac{n}{2n+3} - \frac{1}{2} \right| < \varepsilon, \quad \forall n \geq N.$$

To this end, we compute:

$$\begin{aligned} \left| \frac{n}{2n+3} - \frac{1}{2} \right| &= \left| \frac{2n - (2n+3)}{2(2n+3)} \right| \\ &= \left| \frac{-3}{4n+6} \right| \\ &= \frac{3}{4n+6}. \end{aligned}$$

Therefore

$$\begin{aligned} \left| \frac{n}{2n+3} - \frac{1}{2} \right| < \varepsilon &\iff \frac{3}{4n+6} < \varepsilon \\ &\iff \frac{4n+6}{3} > \frac{1}{\varepsilon} \\ &\iff 4n+6 > \frac{3}{\varepsilon} \\ &\iff 4n > \frac{3}{\varepsilon} - 6 \\ &\iff n > \frac{3}{4\varepsilon} - \frac{6}{4}. \end{aligned}$$

Looking at the above equivalences, it is clear that $N \in \mathbb{N}$ has to be chosen so that

$$N > \frac{3}{4\varepsilon} - \frac{6}{4}.$$

2. *Formal Proof:* We have to show that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \left| \frac{n}{2n+3} - \frac{1}{2} \right| < \varepsilon.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{3}{4\varepsilon} - \frac{6}{4}. \quad (5.2)$$

By the rough work shown above, inequality (5.2) is equivalent to

$$\frac{3}{4N+6} < \varepsilon.$$

Let $n \geq N$. Then

$$\begin{aligned} \left| \frac{n}{2n+3} - \frac{1}{2} \right| &= \frac{3}{4n+6} \\ &\leq \frac{3}{4N+6} \\ &< \varepsilon, \end{aligned}$$

where in the third line we used that $n \geq N$.

Definition 5.4: Divergent sequence

We say that a sequence $(a_n)_{n \in \mathbb{N}}$ in \mathbb{R} is **divergent** if it is not convergent.

Theorem 5.5

Let (a_n) be the sequence defined by

$$a_n = (-1)^n.$$

Then (a_n) does not converge.

Proof

Suppose by contradiction that $a_n \rightarrow a$ for some $a \in \mathbb{R}$. Let

$$\varepsilon := \frac{1}{2}.$$

Since $a_n \rightarrow a$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon = \frac{1}{2} \quad \forall n \geq N.$$

If we take $n = 2N$, then $n \geq N$ and

$$|a_{2N} - a| = |1 - a| < \frac{1}{2}.$$

If we take $n = 2N + 1$, then $n \geq N$ and

$$|a_{2N+1} - a| = |-1 - a| < \frac{1}{2}.$$

Therefore

$$\begin{aligned} 2 &= |(1 - a) - (-1 - a)| \\ &\leq |1 - a| + |-1 - a| \\ &< \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

which is a contradiction. Hence (a_n) is divergent.

Theorem 5.6: Uniqueness of limit

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence. Suppose that

$$\lim_{n \rightarrow \infty} a_n = a, \quad \lim_{n \rightarrow \infty} a_n = b.$$

Then $a = b$.

Definition 5.7: Bounded sequence

A sequence $(a_n)_{n \in \mathbb{N}}$ is called **bounded** if there exists a constant $M \in \mathbb{R}$, with $M > 0$, such that

$$|a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

Theorem 5.8

Every convergent sequence is bounded.

Example 5.9

The sequence

$$a_n = (-1)^n$$

is bounded but not convergent.

Corollary 5.10

If a sequence is not bounded, then the sequence does not converge.

Remark 5.11

For a sequence (a_n) to be unbounded, it means that

$$\forall M > 0, \exists n \in \mathbb{N} \text{ s.t. } |a_n| > M.$$

Theorem 5.12

For all $p > 0$, the sequence

$$a_n = n^p$$

does not converge.

Theorem 5.13

The sequence

$$a_n = \log n$$

does not converge.

Theorem 5.14: Algebra of limits

Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be sequences in \mathbb{R} . Suppose that

$$\lim_{n \rightarrow \infty} a_n = a, \quad \lim_{n \rightarrow \infty} b_n = b,$$

for some $a, b \in \mathbb{R}$. Then,

1. Limit of sum is the sum of limits:

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = a \pm b$$

2. Limit of product is the product of limits:

$$\lim_{n \rightarrow \infty} (a_n b_n) = ab$$

3. If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then

$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{a}{b}$$

Example 5.15

Question. Prove that

$$\lim_{n \rightarrow \infty} \frac{3n}{7n+4} = \frac{3}{7}.$$

Solution. We can rewrite

$$\frac{3n}{7n+4} = \frac{3}{7 + \frac{4}{n}}$$

From Theorem 5.2, we know that

$$\frac{1}{n} \rightarrow 0.$$

Hence, it follows from Theorem 5.14 Point 2 that

$$\frac{4}{n} = 4 \cdot \frac{1}{n} \rightarrow 4 \cdot 0 = 0.$$

By Theorem 5.14 Point 1 we have

$$7 + \frac{4}{n} \rightarrow 7 + 0 = 7.$$

Finally we can use Theorem 5.14 Point 3 to infer

$$\frac{3n}{7n+4} = \frac{3}{7 + \frac{4}{n}} \rightarrow \frac{3}{7}.$$

Example 5.16

Question. Prove that

$$\lim_{n \rightarrow \infty} \frac{n^2 - 1}{2n^2 - 3} = \frac{1}{2}.$$

Solution. Factor n^2 to obtain

$$\frac{n^2 - 1}{2n^2 - 3} = \frac{1 - \frac{1}{n^2}}{2 - \frac{3}{n^2}}.$$

By Theorem 5.2 we have

$$\frac{1}{n^2} \rightarrow 0.$$

We can then use the Algebra of Limits Theorem 5.14 Point 2 to infer

$$\frac{3}{n^2} \rightarrow 3 \cdot 0 = 0$$

and Theorem 5.14 Point 1 to get

$$1 - \frac{1}{n^2} \rightarrow 1 - 0 = 1, \quad 2 - \frac{3}{n^2} \rightarrow 2 - 0 = 2.$$

Finally we use Theorem 5.14 Point 3 and conclude

$$\frac{1 - \frac{1}{n^2}}{2 - \frac{3}{n^2}} \rightarrow \frac{1}{2}.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{n^2 - 1}{2n^2 - 3} = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{n^2}}{2 - \frac{3}{n^2}} = \frac{1}{2}.$$

Example 5.17

Question. Prove that the sequence

$$a_n = \frac{4n^3 + 8n + 1}{7n^2 + 2n + 1}$$

does not converge.

Solution. To show that the sequence (a_n) does not converge, we divide by the largest power in the denominator,

which in this case is n^2

$$\begin{aligned} a_n &= \frac{4n^3 + 8n + 1}{7n^2 + 2n + 1} \\ &= \frac{4n + \frac{8}{n} + \frac{1}{n^2}}{7 + \frac{2}{n} + \frac{1}{n^2}} \\ &= \frac{b_n}{c_n} \end{aligned}$$

where we set

$$b_n := 4n + \frac{8}{n} + \frac{1}{n^2}, \quad c_n := 7 + \frac{2}{n} + \frac{1}{n^2}.$$

Using the Algebra of Limits Theorem 5.14 we see that

$$c_n = 7 + \frac{2}{n} + \frac{1}{n^2} \rightarrow 7.$$

Suppose by contradiction that

$$a_n \rightarrow a$$

for some $a \in \mathbb{R}$. Then, by the Algebra of Limits, we would infer

$$b_n = c_n \cdot a_n \rightarrow 7a,$$

concluding that b_n is convergent to $7a$. We have that

$$b_n = 4n + d_n, \quad d_n := \frac{8}{n} + \frac{1}{n^2}.$$

Again by the Algebra of Limits Theorem 5.14 we get that

$$d_n = \frac{8}{n} + \frac{1}{n^2} \rightarrow 0,$$

and hence

$$4n = b_n - d_n \rightarrow 7a - 0 = 7a.$$

This is a contradiction, since the sequence $(4n)$ is unbounded, and hence cannot be convergent. Hence (a_n) is not convergent.

Example 5.18

Question. Define the sequence

$$a_n := \frac{2n^3 + 7n + 1}{5n + 9} \cdot \frac{8n + 9}{6n^3 + 8n^2 + 3}.$$

Prove that

$$\lim_{n \rightarrow \infty} a_n = \frac{8}{15}.$$

Solution. The first fraction in (a_n) does not converge, as it is unbounded. Therefore we cannot use Point 2 in

Theorem 5.14 directly. However, we note that

$$\begin{aligned} a_n &= \frac{2n^3 + 7n + 1}{5n + 9} \cdot \frac{8n + 9}{6n^3 + 8n^2 + 3} \\ &= \frac{8n + 9}{5n + 9} \cdot \frac{2n^3 + 7n + 1}{6n^3 + 8n^2 + 3}. \end{aligned}$$

Factoring out n and n^3 , respectively, and using the Algebra of Limits, we see that

$$\frac{8n + 9}{5n + 9} = \frac{8 + 9/n}{5 + 9/n} \rightarrow \frac{8 + 0}{5 + 0} = \frac{8}{5}$$

and

$$\frac{2 + 7/n^2 + 1/n^3}{6 + 8/n + 3/n^3} \rightarrow \frac{2 + 0 + 0}{6 + 0 + 0} = \frac{1}{3}$$

Therefore Theorem 5.14 Point 2 ensures that

$$a_n \rightarrow \frac{8}{5} \cdot \frac{1}{3} = \frac{8}{15}.$$

Example 5.19

Question. Prove that

$$a_n = \frac{n^{7/3} + 2\sqrt{n} + 7}{4n^{3/2} + 5n}$$

does not converge.

Solution. The largest power of n in the denominator is $n^{3/2}$. Hence we factor out $n^{3/2}$

$$\begin{aligned} a_n &= \frac{n^{7/3} + 2\sqrt{n} + 7}{4n^{3/2} + 5n} \\ &= \frac{n^{7/3-3/2} + 2n^{1/2-3/2} + 7n^{-3/2}}{4 + 5n^{-3/2}} \\ &= \frac{n^{5/6} + 2n^{-1} + 7n^{-3/2}}{4 + 5n^{-3/2}} \\ &= \frac{b_n}{c_n} \end{aligned}$$

where we set

$$b_n := n^{5/6} + 2n^{-1} + 7n^{-3/2}, \quad c_n := 4 + 5n^{-3/2}.$$

We see that b_n is unbounded while $c_n \rightarrow 4$. By the Algebra of Limits (and usual contradiction argument) we conclude that (a_n) is divergent.

Theorem 5.20

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} such that

$$\lim_{n \rightarrow \infty} a_n = a,$$

for some $a \in \mathbb{R}$. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a \geq 0$, then

$$\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}.$$

Example 5.21

Question. Define the sequence

$$a_n = \sqrt{9n^2 + 3n + 1} - 3n.$$

Prove that

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2}.$$

Solution. We first rewrite

$$\begin{aligned} a_n &= \sqrt{9n^2 + 3n + 1} - 3n \\ &= \frac{(\sqrt{9n^2 + 3n + 1} - 3n)(\sqrt{9n^2 + 3n + 1} + 3n)}{\sqrt{9n^2 + 3n + 1} + 3n} \\ &= \frac{9n^2 + 3n + 1 - (3n)^2}{\sqrt{9n^2 + 3n + 1} + 3n} \\ &= \frac{3n + 1}{\sqrt{9n^2 + 3n + 1} + 3n}. \end{aligned}$$

The biggest power of n in the denominator is n . Therefore we factor out n :

$$\begin{aligned} a_n &= \frac{\sqrt{9n^2 + 3n + 1} - 3n}{3n + 1} \\ &= \frac{3n + 1}{\sqrt{9n^2 + 3n + 1} + 3n} \\ &= \frac{3 + \frac{1}{n}}{\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 3}. \end{aligned}$$

By the Algebra of Limits we have

$$9 + \frac{3}{n} + \frac{1}{n^2} \rightarrow 9 + 0 + 0 = 9.$$

Therefore we can use Theorem 5.20 to infer

$$\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} \rightarrow \sqrt{9}.$$

By the Algebra of Limits we conclude:

$$a_n = \frac{3 + \frac{1}{n}}{\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 3} \rightarrow \frac{3 + 0}{\sqrt{9} + 3} = \frac{1}{2}.$$

Example 5.22

Question. Prove that the sequence

$$a_n = \sqrt{9n^2 + 3n + 1} - 2n$$

does not converge.

Solution. We rewrite a_n as

$$\begin{aligned} a_n &= \sqrt{9n^2 + 3n + 1} - 2n \\ &= \frac{(\sqrt{9n^2 + 3n + 1} - 2n)(\sqrt{9n^2 + 3n + 1} + 2n)}{\sqrt{9n^2 + 3n + 1} + 2n} \\ &= \frac{9n^2 + 3n + 1 - (2n)^2}{\sqrt{9n^2 + 3n + 1} + 2n} \\ &= \frac{5n^2 + 3n + 1}{\sqrt{9n^2 + 3n + 1} + 2n} \\ &= \frac{5n + 3 + \frac{1}{n}}{\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2} \\ &= \frac{b_n}{c_n}, \end{aligned}$$

where we factored n , being it the largest power of n in the denominator, and defined

$$b_n := 5n + 3 + \frac{1}{n}, \quad c_n := \sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2.$$

Note that

$$9 + \frac{3}{n} + \frac{1}{n^2} \rightarrow 9$$

by the Algebra of Limits. Therefore

$$\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} \rightarrow \sqrt{9} = 3$$

by Theorem 5.20. Hence

$$c_n = \sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2 \rightarrow 3 + 2 = 5.$$

The numerator

$$b_n = 5n + 3 + \frac{1}{n}$$

is instead unbounded. Therefore (a_n) is not convergent, by the Algebra of Limits and the usual contradiction argument.

5.1 Limit Tests

Theorem 5.23: Squeeze theorem

Let (a_n) , (b_n) and (c_n) be sequences in \mathbb{R} . Suppose that

$$b_n \leq a_n \leq c_n, \quad \forall n \in \mathbb{N},$$

and that

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = L.$$

Then

$$\lim_{n \rightarrow \infty} a_n = L.$$

Example 5.24

Question. Prove that

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0.$$

Solution. For all $n \in \mathbb{N}$ we can estimate

$$-1 \leq (-1)^n \leq 1.$$

Therefore

$$\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Moreover

$$\lim_{n \rightarrow \infty} \frac{-1}{n} = -1 \cdot 0 = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

By the Squeeze Theorem 5.23 we conclude

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0.$$

Example 5.25

Question. Prove that

$$\lim_{n \rightarrow \infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15 \sin(17n)} = \frac{9}{11}.$$

Solution. We know that

$$-1 \leq \cos(x) \leq 1, \quad -1 \leq \sin(x) \leq 1, \quad \forall x \in \mathbb{R}.$$

Therefore, for all $n \in \mathbb{N}$

$$-1 \leq \cos(3n) \leq 1, \quad -1 \leq \sin(17n) \leq 1.$$

We can use the above to estimate the numerator in the given sequence:

$$-1 + 9n^2 \leq \cos(3n) + 9n^2 \leq 1 + 9n^2. \quad (5.3)$$

Concerning the denominator, we have

$$11n^2 - 15 \leq 11n^2 + 15 \sin(17n) \leq 11n^2 + 15$$

and therefore

$$\frac{1}{11n^2 + 15} \leq \frac{1}{11n^2 + 15 \sin(17n)} \leq \frac{1}{11n^2 - 15}. \quad (5.4)$$

Putting together (5.3)-(5.4) we obtain

$$\frac{-1 + 9n^2}{11n^2 + 15} \leq \frac{\cos(3n) + 9n^2}{11n^2 + 15 \sin(17n)} \leq \frac{1 + 9n^2}{11n^2 - 15}.$$

By the Algebra of Limits we infer

$$\frac{-1 + 9n^2}{11n^2 + 15} = \frac{-\frac{1}{n^2} + 9}{11 + \frac{15}{n^2}} \rightarrow \frac{0 + 9}{11 + 0} = \frac{9}{11}$$

and

$$\frac{1 + 9n^2}{11n^2 - 15} = \frac{\frac{1}{n^2} + 9}{11 - \frac{15}{n^2}} \rightarrow \frac{0 + 9}{11 + 0} = \frac{9}{11}.$$

Applying the Squeeze Theorem 5.23 we conclude

$$\lim_{n \rightarrow \infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15 \sin(17n)} = \frac{9}{11}.$$

Theorem 5.26: Geometric Sequence Test

Let $x \in \mathbb{R}$ and let (a_n) be the geometric sequence defined by

$$a_n := x^n.$$

We have:

1. If $|x| < 1$, then

$$\lim_{n \rightarrow \infty} a_n = 0.$$

2. If $|x| > 1$, then sequence (a_n) is unbounded, and hence divergent.

Example 5.27

We can apply Theorem 5.26 to prove convergence or divergence for the following sequences.

1. We have

$$\left(\frac{1}{2}\right)^n \rightarrow 0$$

since

$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1.$$

2. We have

$$\left(\frac{-1}{2}\right)^n \rightarrow 0$$

since

$$\left|\frac{-1}{2}\right| = \frac{1}{2} < 1.$$

3. The sequence

$$a_n = \left(\frac{-3}{2}\right)^n$$

does not converge, since

$$\left|\frac{-3}{2}\right| = \frac{3}{2} > 1.$$

4. As $n \rightarrow \infty$,

$$\frac{3^n}{(-5)^n} = \left(-\frac{3}{5}\right)^n \rightarrow 0$$

since

$$\left|-\frac{3}{5}\right| = \frac{3}{5} < 1.$$

5. The sequence

$$a_n = \frac{(-7)^n}{2^{2n}}$$

does not converge, since

$$\frac{(-7)^n}{2^{2n}} = \frac{(-7)^n}{(2^2)^n} = \left(-\frac{7}{4}\right)^n$$

and

$$\left|-\frac{7}{4}\right| = \frac{7}{4} > 1.$$

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \geq L, \quad \forall n \geq N.$$

Then the sequence (a_n) is unbounded, and hence does not converge.

Example 5.29

Question. Let

$$a_n = \frac{3^n}{n!},$$

where we recall that $n!$ (pronounced n factorial) is defined by

$$n! := n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1.$$

Prove that

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Solution. We have

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{\left(\frac{3^{n+1}}{(n+1)!}\right)}{\left(\frac{3^n}{n!}\right)} \\ &= \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \\ &= \frac{3 \cdot 3^n}{3^n} \frac{n!}{(n+1)n!} \\ &= \frac{3}{n+1} \rightarrow L = 0. \end{aligned}$$

Hence, $L = 0 < 1$ so $a_n \rightarrow 0$ by the Ratio Test in Theorem 5.28.

Theorem 5.28: Ratio Test

Let (a_n) be a sequence in \mathbb{R} such that

$$a_n \neq 0, \quad \forall n \in \mathbb{N}.$$

1. Suppose that the following limit exists:

$$L := \lim_{n \rightarrow \infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then,

- If $L < 1$ we have

$$\lim_{n \rightarrow \infty} a_n = 0.$$

- If $L > 1$, the sequence (a_n) is unbounded, and hence does not converge.

Example 5.30

Question. Consider the sequence

$$a_n = \frac{n! \cdot 3^n}{\sqrt{(2n)!}}.$$

Prove that (a_n) is divergent.

Solution. We have

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)! \cdot 3^{n+1} \sqrt{(2n)!}}{\sqrt{(2(n+1))!} \cdot n! \cdot 3^n} \\ &= \frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}} \end{aligned}$$

For the first two fractions we have

$$\frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} = 3(n+1),$$

while for the third fraction

$$\begin{aligned}\frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}} &= \sqrt{\frac{(2n)!}{(2n+2)!}} \\ &= \sqrt{\frac{(2n)!}{(2n+2) \cdot (2n+1) \cdot (2n)!}} \\ &= \frac{1}{\sqrt{(2n+1)(2n+2)}}.\end{aligned}$$

Therefore, using the Algebra of Limits,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{3(n+1)}{\sqrt{(2n+1)(2n+2)}} \\ &= \frac{3n\left(1+\frac{1}{n}\right)}{\sqrt{n^2\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}} \\ &= \frac{3\left(1+\frac{1}{n}\right)}{\sqrt{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}} \rightarrow \frac{3}{\sqrt{4}} = \frac{3}{2} > 1.\end{aligned}$$

By the Ratio Test we conclude that (a_n) is divergent.

Example 5.31

Question. Prove that the following sequence is divergent

$$a_n = \frac{n!}{100^n}.$$

Solution. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{100^n}{100^{n+1}} \frac{(n+1)!}{n!} = \frac{n+1}{100}.$$

Choose $N = 101$. Then for all $n \geq N$,

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{n+1}{100} \\ &\geq \frac{N+1}{100} \\ &= \frac{101}{100} > 1.\end{aligned}$$

Hence a_n is divergent by the Ratio Test.

Let (a_n) be a real sequence. We say that:

1. (a_n) is **increasing** if

$$a_n \leq a_{n+1}, \quad \forall n \geq N.$$

2. (a_n) is **decreasing** if

$$a_n \geq a_{n+1}, \quad \forall n \geq N.$$

3. (a_n) is **monotone** if it is either increasing or decreasing.

Example 5.33

Question. Prove that the following sequence is increasing

$$a_n = \frac{n-1}{n}.$$

Solution. We have

$$a_{n+1} = \frac{n}{n+1} > \frac{n-1}{n} = a_n,$$

where the inequality holds because

$$\begin{aligned}\frac{n}{n+1} > \frac{n-1}{n} &\iff n^2 > (n-1)(n+1) \\ &\iff n^2 > n^2 - 1 \\ &\iff 0 > -1\end{aligned}$$

Example 5.34

Question. Prove that the following sequence is decreasing

$$a_n = \frac{1}{n}.$$

Solution. We have

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1},$$

concluding.

Theorem 5.35: Monotone Convergence Theorem

Let (a_n) be a sequence in \mathbb{R} . Suppose that (a_n) is bounded and monotone. Then (a_n) converges.

Proof

Assume (a_n) is bounded and monotone. Since (a_n) is bounded, the set

$$A := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$$

is bounded below and above. By the Axiom of Completeness of \mathbb{R} there exist $i, s \in \mathbb{R}$ such that

$$i = \inf A, \quad s = \sup A.$$

We have two cases:

1. (a_n) is increasing: We are going to prove that

$$\lim_{n \rightarrow \infty} a_n = s.$$

Equivalently, we need to prove that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |a_n - s| < \varepsilon. \quad (5.5)$$

Let $\varepsilon > 0$. Since s is the smallest upper bound for A , this means

$$s - \varepsilon$$

is not an upper bound. Therefore there exists $N \in \mathbb{N}$ such that

$$s - \varepsilon < a_N. \quad (5.6)$$

Let $n \geq N$. Since a_n is increasing, we have

$$a_N \leq a_n, \quad \forall n \geq N. \quad (5.7)$$

Moreover s is the supremum of A , so that

$$a_n \leq s < s + \varepsilon, \quad \forall n \in \mathbb{N}. \quad (5.8)$$

Putting together estimates (5.6)-(5.7)-(5.8) we get

$$s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon, \quad \forall n \geq N.$$

The above implies

$$s - \varepsilon < a_n < s + \varepsilon, \quad \forall n \geq N,$$

which is equivalent to (5.5).

2. (a_n) is decreasing: With a similar proof, one can show that

$$\lim_{n \rightarrow \infty} a_n = i.$$

This is left as an exercise.

5.3 Example: Euler's Number

As an application of the Monotone Convergence Theorem we can give a formal definition for the Euler's Number

$$e = 2.71828182845904523536 \dots$$

Theorem 5.36

Consider the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

We have that:

1. (a_n) is monotone increasing,
2. (a_n) is bounded.

In particular (a_n) is convergent.

Proof

Part 1. We prove that (a_n) is increasing

$$a_n \geq a_{n-1}, \quad \forall n \in \mathbb{N},$$

which by definition is equivalent to

$$\left(1 + \frac{1}{n}\right)^n \geq \left(1 + \frac{1}{n-1}\right)^{n-1}, \quad \forall n \in \mathbb{N}.$$

Summing the fractions we get

$$\left(\frac{n+1}{n}\right)^n \geq \left(\frac{n}{n-1}\right)^{n-1}.$$

Multiplying by $((n-1)/n)^n$ we obtain

$$\left(\frac{n-1}{n}\right)^n \left(\frac{n+1}{n}\right)^n \geq \frac{n-1}{n},$$

which simplifies to

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 - \frac{1}{n}, \quad \forall n \in \mathbb{N}. \quad (5.9)$$

Therefore (a_n) is increasing if and only if (5.9) holds. Recall Bernoulli's inequality from Lemma ?? : For $x \in \mathbb{R}$, $x > -1$, it holds

$$(1+x)^n \geq 1+nx, \quad \forall n \in \mathbb{N}.$$

Applying Bernoulli's inequality with

$$x = -\frac{1}{n^2}$$

yields

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 + n\left(-\frac{1}{n^2}\right) = 1 - \frac{1}{n},$$

which is exactly (5.9). Then (a_n) is increasing.
Part 2. We have to prove that (a_n) is bounded, that is, that there exists $M > 0$ such that

$$|a_n| \leq M, \quad \forall n \in \mathbb{N}.$$

To this end, introduce the sequence (b_n) by setting

$$b_n := \left(1 + \frac{1}{n}\right)^{n+1}.$$

The sequence (b_n) is decreasing.

To prove (b_n) is decreasing, we need to show that

$$b_{n-1} \geq b_n, \quad \forall n \in \mathbb{N}.$$

By definition of b_n , the above reads

$$\left(1 + \frac{1}{n-1}\right)^n \geq \left(1 + \frac{1}{n}\right)^{n+1}, \quad \forall n \in \mathbb{N}.$$

Summing the terms inside the brackets, the above is equivalent to

$$\left(\frac{n}{n-1}\right)^n \geq \left(\frac{n+1}{n}\right)^n \left(\frac{n+1}{n}\right).$$

Multiplying by $(n/(n+1))^n$ we get

$$\left(\frac{n^2}{n^2-1}\right)^n \geq \left(\frac{n+1}{n}\right).$$

The above is equivalent to

$$\left(1 + \frac{1}{n^2-1}\right)^n \geq \left(1 + \frac{1}{n}\right). \quad (5.10)$$

Therefore (b_n) is decreasing if and only if (5.10) holds for all $n \in \mathbb{N}$. By choosing

$$x = \frac{1}{n^2-1}$$

in Bernoulli's inequality, we obtain

$$\begin{aligned} \left(1 + \frac{1}{n^2-1}\right)^n &\geq 1 + n \left(\frac{1}{n^2-1}\right) \\ &= 1 + \frac{n}{n^2-1} \\ &\geq 1 + \frac{1}{n}, \end{aligned}$$

where in the last inequality we used that

$$\frac{n}{n^2-1} > \frac{1}{n},$$

which holds, being equivalent to $n^2 > n^2 - 1$. We have therefore proven (5.10), and hence (b_n) is decreasing.

We now observe that For all $n \in \mathbb{N}$

$$\begin{aligned} b_n &= \left(1 + \frac{1}{n}\right)^{n+1} \\ &= \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) \\ &= a_n \left(1 + \frac{1}{n}\right) \\ &> a_n. \end{aligned}$$

Since (a_n) is increasing and (b_n) is decreasing, in particular

$$a_n \geq a_1, \quad b_n \leq b_1.$$

Therefore

$$a_1 \leq a_n < b_n \leq b_1, \quad \forall n \in \mathbb{N}.$$

We compute

$$a_1 = 2, \quad b_1 = 4,$$

from which we get

$$2 \leq a_n \leq 4, \quad \forall n \in \mathbb{N}.$$

Therefore

$$|a_n| \leq 4, \quad \forall n \in \mathbb{N},$$

showing that (a_n) is bounded.

Part 3. The sequence (a_n) is increasing and bounded above. Therefore (a_n) is convergent by the Monotone Convergence Theorem 5.35.

Thanks to Theorem 5.36 we can define the Euler's Number e .

Definition 5.37: Euler's Number

The Euler's number is defined as

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

Setting $n = 1000$ in the formula for (a_n) , we get an approximation of e :

$$e \approx a_{1000} = 2.7169.$$

5.4 Some important limits

In this section we investigate limits of some sequences to which the Limit Tests do not apply.

Theorem 5.38

Let $x \in \mathbb{R}$, with $x > 0$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{x} = 1.$$

Proof

Step 1. Assume $x \geq 1$. In this case

$$\sqrt[n]{x} \geq 1.$$

Define

$$b_n := \sqrt[n]{x} - 1,$$

so that $b_n \geq 0$. By Bernoulli's Inequality we have

$$x = (1 + b_n)^n \geq 1 + nb_n.$$

Therefore

$$0 \leq b_n \leq \frac{x-1}{n}.$$

Since

$$\frac{x-1}{n} \rightarrow 0,$$

by the Squeeze Theorem we infer $b_n \rightarrow 0$, and hence

$$\sqrt[n]{x} = 1 + b_n \rightarrow 1 + 0 = 1,$$

by the Algebra of Limits.

Step 2. Assume $0 < x < 1$. In this case

$$\frac{1}{x} > 1.$$

Therefore

$$\lim_{n \rightarrow \infty} \sqrt[n]{1/x} = 1.$$

by Step 1. Therefore

$$\sqrt[n]{x} = \frac{1}{\sqrt[n]{1/x}} \rightarrow \frac{1}{1} = 1,$$

by the Algebra of Limits.

Theorem 5.39

Let (a_n) be a sequence such that $a_n \rightarrow 0$. Then

$$\sin(a_n) \rightarrow 0, \quad \cos(a_n) \rightarrow 1.$$

Proof

Assume that $a_n \rightarrow 0$ and set

$$\varepsilon := \frac{\pi}{2}.$$

By the convergence $a_n \rightarrow 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n| < \varepsilon = \frac{\pi}{2} \quad \forall n \geq N. \quad (5.11)$$

Step 1. We prove that

$$\sin(a_n) \rightarrow 0.$$

By elementary trigonometry we have

$$0 \leq |\sin(x)| = \sin|x| \leq |x|, \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Therefore, since (5.11) holds, we can substitute $x = a_n$ in the above inequality to get

$$0 \leq |\sin(a_n)| \leq |a_n|, \quad \forall n \geq N.$$

Since $a_n \rightarrow 0$, we also have $|a_n| \rightarrow 0$. Therefore $|\sin(a_n)| \rightarrow 0$ by the Squeeze Theorem. This immediately implies $\sin(a_n) \rightarrow 0$.

Step 2. We prove that

$$\cos(a_n) \rightarrow 1.$$

Inverting the relation

$$\cos^2(x) + \sin^2(x) = 1,$$

we obtain

$$\cos(x) = \pm \sqrt{1 - \sin^2(x)}.$$

We have that $\cos(x) \geq 0$ for $-\pi/2 \leq x \leq \pi/2$. Thus

$$\cos(x) = \sqrt{1 - \sin^2(x)}, \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Since (5.11) holds, we can set $x = a_n$ in the above inequality and obtain

$$\cos(a_n) = \sqrt{1 - \sin^2(a_n)}, \quad \forall n \geq N.$$

By Step 1 we know that $\sin(a_n) \rightarrow 0$. Therefore, by the Algebra of Limits,

$$1 - \sin^2(a_n) \rightarrow 1 - 0 \cdot 0 = 1.$$

Using Theorem 5.20 we have

$$\cos(a_n) = \sqrt{1 - \sin^2(a_n)} \rightarrow \sqrt{1} = 1,$$

concluding the proof.

Theorem 5.40

Suppose (a_n) is such that $a_n \rightarrow 0$ and

$$a_n \neq 0, \quad \forall n \in \mathbb{N}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\sin(a_n)}{a_n} = 1.$$

Proof

The following elementary trigonometric inequality holds:

$$\sin(x) < x < \tan(x), \quad \forall x \in \left[0, \frac{\pi}{2}\right].$$

Note that $\sin x > 0$ for $0 < x < \pi/2$. Therefore we can divide the above inequality by $\sin(x)$ and take the reciprocals to get

$$\cos(x) < \frac{\sin(x)}{x} < 1, \quad \forall x \in \left(0, \frac{\pi}{2}\right).$$

If $-\pi/2 < x < 0$, we can apply the above inequality to $-x$ to obtain

$$\cos(-x) < \frac{\sin(-x)}{-x} < 1.$$

Recalling that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, we get

$$\cos(x) < \frac{\sin(x)}{x} < 1, \quad \forall x \in \left(-\frac{\pi}{2}, 0\right).$$

Thus

$$\cos(x) < \frac{\sin(x)}{x} < 1, \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}. \quad (5.12)$$

Let

$$\varepsilon := \frac{\pi}{2}.$$

Since $a_n \rightarrow 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n| < \varepsilon = \frac{\pi}{2}, \quad \forall n \geq N.$$

Since $a_n \neq 0$ by assumption, the above shows that

$$a_n \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}, \quad \forall n \geq N.$$

Therefore we can substitute $x = a_n$ into (5.12) to get

$$\cos(a_n) < \frac{\sin(a_n)}{a_n} < 1, \quad \forall n \geq N.$$

We have

$$\cos(a_n) \rightarrow 1$$

by Theorem 5.39. By the Squeeze Theorem we conclude that

$$\lim_{n \rightarrow \infty} \frac{\sin(a_n)}{a_n} = 1.$$

Warning

You might be tempted to apply L'Hôpital's rule (which we did not cover in these Lecture Notes) to compute

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x}.$$

This would yield the correct limit

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0} \frac{(\sin(x))'}{(x)'} = \lim_{x \rightarrow 0} \cos(x) = 1.$$

However this is a circular argument, since the derivative of $\sin(x)$ at $x = 0$ is defined as the limit

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x}.$$

Theorem 5.41

Suppose (a_n) is such that $a_n \rightarrow 0$ and

$$a_n \neq 0, \quad \forall n \in \mathbb{N}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{1 - \cos(a_n)}{(a_n)^2} = \frac{1}{2}, \quad \lim_{n \rightarrow \infty} \frac{1 - \cos(a_n)}{a_n} = 0.$$

Proof

Step 1. By Theorem 5.39 and Theorem 5.40, we have

$$\cos(a_n) \rightarrow 1, \quad \frac{\sin(a_n)}{a_n} \rightarrow 1.$$

Therefore

$$\begin{aligned} \frac{1 - \cos(a_n)}{(a_n)^2} &= \frac{1 - \cos(a_n)}{(a_n)^2} \frac{1 + \cos(a_n)}{1 + \cos(a_n)} \\ &= \frac{1 - \cos^2(a_n)}{(a_n)^2} \frac{1}{1 + \cos(a_n)} \\ &= \left(\frac{\sin(a_n)}{a_n}\right)^2 \frac{1}{1 + \cos(a_n)} \rightarrow 1 \cdot \frac{1}{1 + 1} = \frac{1}{2}, \end{aligned}$$

where in the last line we use the Algebra of Limits.

Step 2. We have

$$\frac{1 - \cos(a_n)}{a_n} = a_n \cdot \frac{1 - \cos(a_n)}{(a_n)^2} \rightarrow 0 \cdot \frac{1}{2} = 0,$$

using Step 1 and the Algebra of Limits.

Example 5.42

Question. Prove that

$$\lim_{n \rightarrow \infty} n \sin\left(\frac{1}{n}\right) = 1. \quad (5.13)$$

Solution. This is because

$$n \sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \rightarrow 1,$$

by Theorem 5.40 with $a_n = 1/n$.

Example 5.43

Question. Prove that

$$\lim_{n \rightarrow \infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}. \quad (5.14)$$

Solution. Indeed,

$$n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1 - \cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \rightarrow \frac{1}{2},$$

by applying Theorem 5.41 with $a_n = 1/n$.

Example 5.44

Question. Prove that

$$\lim_{n \rightarrow \infty} \frac{n \left(1 - \cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} = \frac{1}{2}.$$

Solution. Using (5.14)-(5.13) and the Algebra of Limits

$$\begin{aligned} \frac{n \left(1 - \cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} &= \frac{n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right)}{n \sin\left(\frac{1}{n}\right)} \\ &\rightarrow \frac{1/2}{1} = \frac{1}{2}. \end{aligned}$$

Example 5.45

Question. Prove that

$$\lim_{n \rightarrow \infty} n \cos\left(\frac{2}{n}\right) \sin\left(\frac{2}{n}\right) = 2.$$

Solution. We have

$$\cos\left(\frac{2}{n}\right) \rightarrow 1,$$

by Theorem 5.39 applied with $a_n = 2/n$. Moreover

$$\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} \rightarrow 1,$$

by Theorem 5.40 applied with $a_n = 2/n$. Therefore

$$\begin{aligned} n \cos\left(\frac{2}{n}\right) \sin\left(\frac{2}{n}\right) &= 2 \cdot \cos\left(\frac{2}{n}\right) \cdot \frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} \\ &\rightarrow 2 \cdot 1 \cdot 1 = 2, \end{aligned}$$

where we used the Algebra of Limits.

Example 5.46

Question. Prove that

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n + 1} \sin\left(\frac{1}{n}\right) = 1.$$

Solution. Note that

$$\begin{aligned} \frac{n^2 + 1}{n + 1} \sin\left(\frac{1}{n}\right) &= \left(\frac{1 + \frac{1}{n^2}}{1 + \frac{1}{n}} \right) \cdot \left(n \sin\left(\frac{1}{n}\right) \right) \\ &\rightarrow \frac{1 + 0}{1 + 0} \cdot 1 = 1, \end{aligned}$$

where we used (5.13) and the Algebra of Limits.

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