Sparse recovery in Inverse Problems

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Sparse recovery in Inverse Problems

based on joint works with

Kristian Bredies, Marcello Carioni, Francisco Romero, Daniel Walter

Outline

1. Introduction: Inverse Problems & Sparsity
2. Algorithm for sparse solutions recovery
3. Dynamic Inverse Problems
4. Application to Dynamic MRI

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Christian Doppler Research Society (CDG) Project PIR27
"Mathematical methods for motion-aware medical imaging"

Outline

What is an Inverse Problem?

Mathematical models and data play a big role in modern science

- Inverse Problems: Link between model parameters and data

Direct Problem: Model Parameters $\rightsquigarrow$ Data (easy)

Inverse Problem: Data $\rightsquigarrow$ Model Parameters (hard)

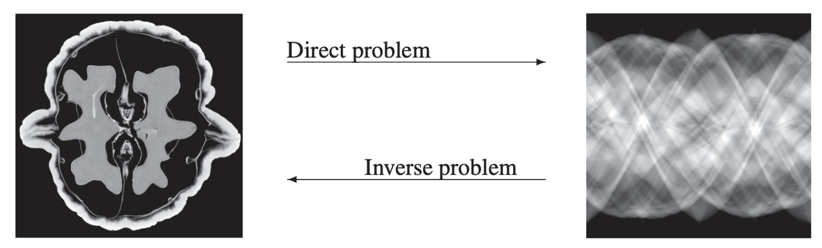
Bredies, Lorenz. *Mathematical Image Processing*. Springer (2018)

Mueller, Siltanen. *Linear and Nonlinear Inverse Problems with Practical Appl.* SIAM, 2012

van Leeuwen, Brune. *10 Lectures on Inverse Problems and Imaging.* Online link

Example: X-ray Imaging

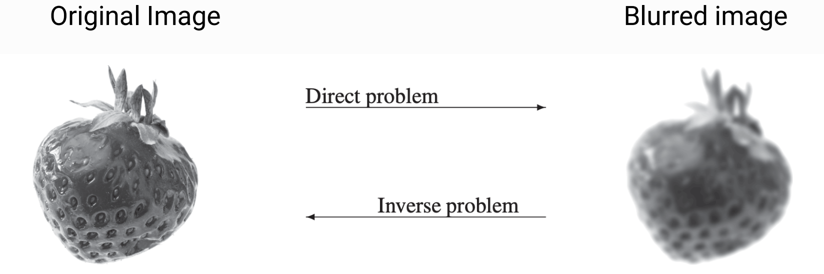
Walnut | X-ray data (sinogram form)



Direct Problem: X-rays pass through walnut, detectors measure attenuation

Inverse problem: Given many X-ray measurements from different angles, reconstruct the walnut

Example: Image Deblurring



Direct problem: Sharp image becomes blurred due to camera motion or focus issues

Inverse problem: Given the blurred image, recover the original sharp image

Famous example: Hubble Space Telescope

- Hubble Space Telescope launched in 1990
- However images were blurred due to flawed lenses (Left)
- This issue was corrected through image processing (Right)

Link to article on NASA's website

Mathematical formulation

Inverse Problem: Given data $f \in Y$, find parameters $u \in X$ such that

$$Ku = f$$

- f is data — Y data space – Banach space or $\mathbb{R}^n$
- u are parameters — X the parameters space – Same as above
- $K : X \to Y$ is Forward Operator
- K models the process to obtain the data from the parameters

Ill-posed Inverse Problems

Consider the inverse problem

$$Ku = f \tag{P}$$

Problem (P) is **well-posed** if all three conditions hold:

1. **Existence**: There exists at least one solution
2. **Uniqueness**: There exists at most one solution
3. **Stability:** The solution depends continuously on the data, i.e., there exists a constant $C > 0$ such that

$$\|u - u'\|_X \leq C \, \|f - f'\|_Y \quad \text{where} \quad Ku = f \, , \; Ku' = f'$$

Problem (P) is **ill-posed** if it is not well-posed

Measurements are noisy

Consider the inverse problem

$$Ku = f \tag{P}$$

- **Ideal world:** Measurement comes from operator $\rightsquigarrow \; f = Ku$
- **Reality:** We can only observe noisy measurements

$$f^{\varepsilon} = Ku + \varepsilon \,, \qquad \varepsilon \text{ random (unknown) noise}$$

Goal: To recover u from noisy measurement f^{ε}

Main difficulty: K^{-1} does not exist or is not continuous $\rightsquigarrow$ **ill-posedness**

Variational Regularization

$$Ku = f \tag{P}$$

- (P) might not have solution. Find **approximate** solution by **least-squares**

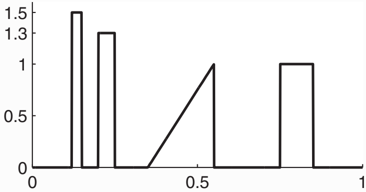
$$\min_{u \in X} \|Ku - f\|_Y^2 \tag{P'}$$

- **Problem:** Might still have **non-existence**, **non-uniqueness** and / or **instability**
- **Solution:** Replace (P) with the **regularized** least-squares problem

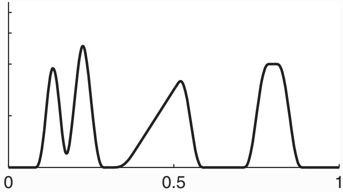
$$\min_{u \in X} \|Ku - f\|_X^2 + \alpha R(u)\,, \qquad R\colon X \to [0, +\infty]\,,\ \alpha > 0$$

1. R makes the problem well-posed and stable – if chosen properly
2. R favors certain solutions – the ones for which $R(u)$ is small

Example: 1D deconvolution



Original signal $\tilde{u} \colon [0,1] \to \mathbb{R}$



Blurred signal $f = \psi \star \tilde{u}$

- Goal: Recover $\tilde{u}$ from noisy data $f^\varepsilon = f + \varepsilon$
- This means solving the 1D-deconvolution problem:

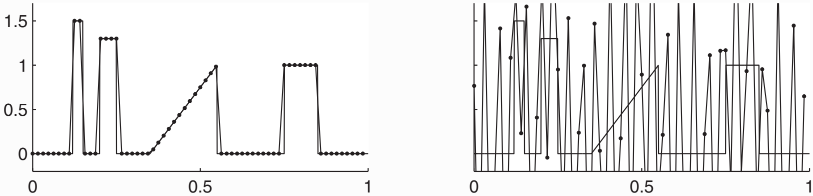
$$\text{Find } u \text{ such that } \psi \star u = f^\varepsilon$$

Naive deconvolution

Solve the discrete 1D-deconvolution problem by least-squares:

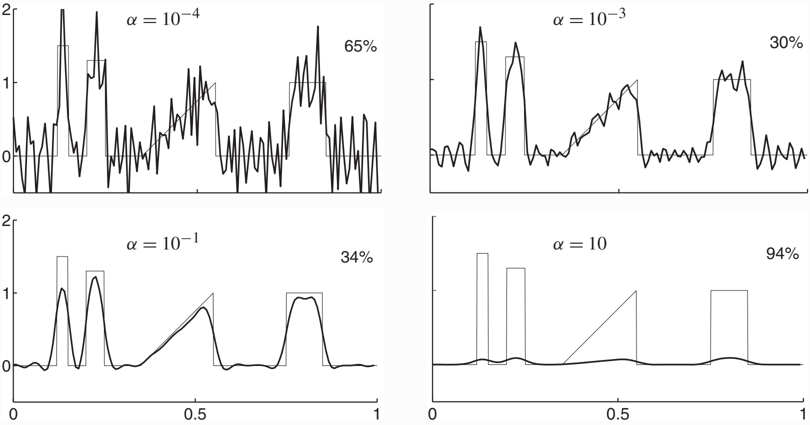
$$u^{\varepsilon} \in \underset{u \in L^2(0,1)}{\arg\min} \; \|\psi \star u - f^{\varepsilon}\|^2_{L^2(0,1)}$$

- Solution behaves well when noise $\varepsilon = 0$ but is terrible when $\varepsilon \neq 0$
  - Instability amplifies noise in the reconstruction
- Below the solid line represents the ground truth $\tilde{u}$
- We need regularizer which penalizes oscillations



u^0 $\qquad$ u^{ε}

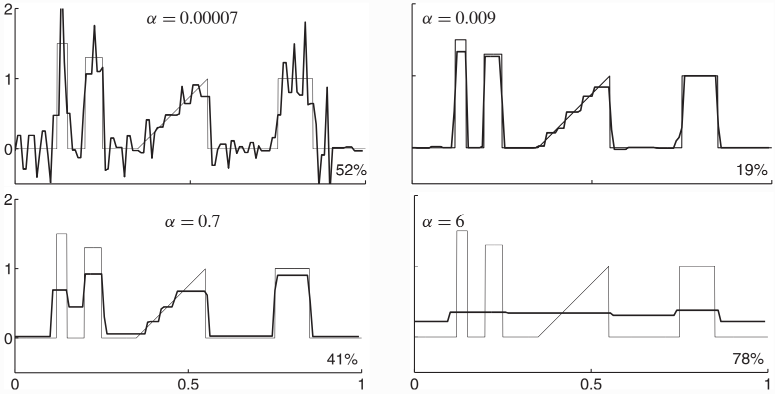
Regularize with L^2 norm: $$\min_{u \in L^2(0,1)} \|\psi \star u - f^\varepsilon\|^2_{L^2(0,1)} + \alpha \|u\|^2_{L^2(0,1)}$$



- Best is $\alpha = 10^{-1}$. Notice the smoothing effect of L^2 regularization
- Smoothness not always desirable (e.g. if u is image with sharp edges – like here)

Reg. with Total Variation (BV semi-norm): $\mathrm{TV}(u) = \sup_{\text{partitions of } [0,1]} \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})|$

$$\min_{u \in L^1(0,1)} \|\psi \star u - f^\varepsilon\|^2_{L^2(0,1)} + \alpha\, \mathrm{TV}(u)$$



- Best is $\alpha = 0.009$. Notice the **sparsifying effect** of TV (the jumps)
- Sparsity assumes different forms, depending on the setting
- **Extremal points of regularizer describe features of sparse solutions**

Summary

Setting: X, Y Banach spaces, $K \colon X \to Y$ linear continuous operator

Inverse Problem: Given $f \in Y$, find $u \in X$ such that

$$Ku = f$$

Main difficulty: K^{-1} does not exist or is not continuous

Variational regularization: Given $f \in Y$, find $u \in X$ which solves

$$\min_{u \in X} \|Ku - f\|_Y^2 + \alpha R(u) \tag{P}$$

Goals of the Talk:

- Algorithm to recover sparse solutions to (P)
- Framework for regularizing dynamic inverse problems

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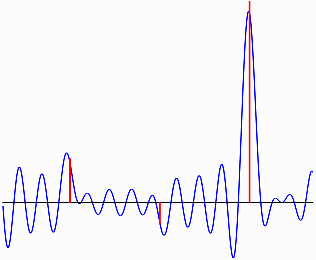
Motivation: Sparse peak recovery



Well-studied problem: Superresolution

- Solve $\qquad \mathfrak{F}u = f \qquad$ on $\quad \Omega$
- $\mathfrak{F}$ Fourier transform, $\Omega \subset \mathbb{R}^d$ finite set
- **Sparsity assumption:** $\overline{u} = \sum_{i=1}^{N} \lambda_i \delta_{x_i}$

Radon-norm regularization:

- Solve variational problem in space of Radon measures (noisy data)

$$\min_{u \in \mathcal{M}(\Omega)} \left\| \mathfrak{F}u - f \right\|_{L^2(\Omega)}^2 + \alpha \left\| u \right\|_{\mathcal{M}(\Omega)}$$

- Want to recover sparse solution $\overline{u} = \sum_{i=1}^{N} \lambda_i \delta_{x_i}$
- [Candès, Fernandez-Granda. **CPAM** (2013)] and many more

$\rightsquigarrow$ **(Fast) algorithms for a general setting**

Minimization Problem in General Setting

$$\min_{u \in X} \; F(Ku) + R(u)$$

- **Parameters:** X separable Banach space with predual X_*
- **Data:** Y Hilbert space
- **Forward operator:** $K : X \to Y$ linear and weak*-to-strong continuous
- **F** $\rightsquigarrow$ **Loss function:** **Smooth** $+$ **Strictly Convex**

$$F : Y \to [0, \infty) \qquad \left(F(y) = \|y - f\|_Y^2 \right)$$

- **R** $\rightsquigarrow$ **Regulariser:** **Convex** $+$ **1-homogeneous** $+$ **Coercive**

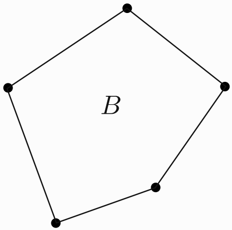
$$R : X \to [0, \infty] \qquad \text{(Promotes Sparsity)}$$

Theorem [1]: Direct method $\Longrightarrow$ Minimizer exists

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

Unit Ball of regularizer R

$$B := \{u \in X :\ R(u) \leq 1\}$$



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Sparsity

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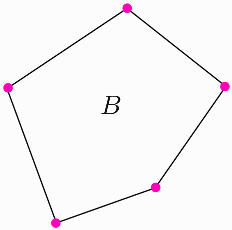
Extremal Points: $u \in B$ s.t.

$$\begin{cases} u = \alpha u_1 + (1-\alpha) u_2 \\ \alpha \in (0,1)\,, \; u_1, u_2 \in B \end{cases} \implies u = u_1 = u_2$$



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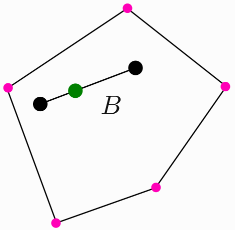

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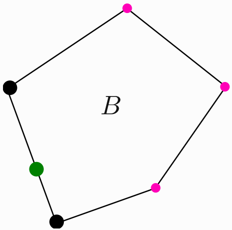

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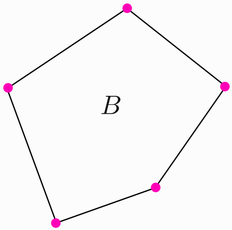

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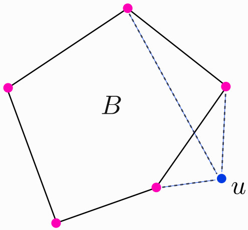

Definition: $u \in X$ **sparse**

Conic combination

$$u = \sum_{i=1}^{N} \lambda_i \, u_i \,, \quad \lambda_i \geq 0 \,, \quad u_i \in \mathrm{Ext}(B)$$

[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2024)

Main Task

Numerical **Algorithm** to compute

$$\bar{u} \in \operatorname*{arg\,min}_{u \in X} \; F(Ku) + R(u)$$

which is **sparse**

$$\bar{u} = \sum_{i=1}^{N} \lambda_i \, u_i \,, \quad \lambda_i \geq 0 \,, \quad u_i \in \operatorname{Ext}(B)$$

Existence of sparse solutions: Proven for $K \colon X \to \mathbb{R}^n$ [1,2]

Very general setting $\rightsquigarrow$ **Important Examples:**

- Training of Machine Learning models $\rightsquigarrow$ $X = \mathbb{R}^d$
- Microstructures in Materials $\rightsquigarrow$ $X = \mathrm{BV}(\mathbb{R}^d)$ Bounded Variation
- Recovery of sparse sources $\rightsquigarrow$ $X = \mathcal{M}(\mathbb{R}^d)$ Radon Measures

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[1] Bredies, Carioni. **Calc. Var. PDE** (2020)
[2] Boyer, Chambolle, De Castro, Duval, De Gournay, Weiss. **SIAM Optimization** (2019)

Example: Training of Machine Learning models

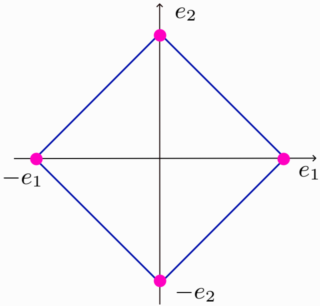
Parameters: vector $\Theta = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$

ML Model: Fit model to given data

$$\min_{\Theta \in \mathbb{R}^d} F(\theta) + \|\Theta\|_1$$

- Fidelity term F promotes data fit
- 1-norm promotes sparsity – e.g. solutions will have lots of zeros

$$\hat{\Theta} = (0, 0, \theta_{\mathbf{i}}, 0, 0, \dots, 0, 0, \theta_{\mathbf{d}})$$

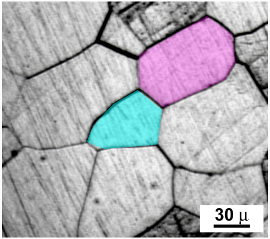


Banach space: $X = \mathbb{R}^d$

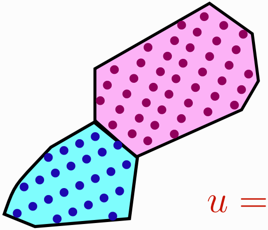
Regularizer: $\|x\|_1 := \sum_{i=1}^{d} |x_i|$

$$\mathrm{Ext}(B) = \{\pm e_i\}_{i=1}^d$$

Example: Microstructures in Materials



Polycrystalline Metal



$E_i \subset \mathbb{R}^2$

$A_i \in \mathbb{R}^{2\times 2}$

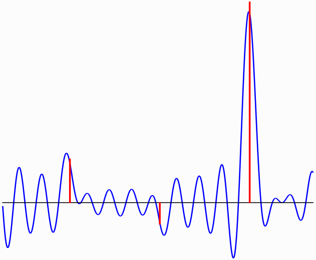
$$u = \sum_{i=1}^{N} A_i \, \chi_{E_i}$$

Banach space: $X = \mathrm{BV}(\mathbb{R}^2)$ functions of bounded variation

Regularizer: $R(u) := \|Du\|_{\mathcal{M}}$, $\mathrm{Ext}(B) = \{\chi_E : E \subset \mathbb{R}^2$ simply connected$\}$

[2] **Fanzon**, Palombaro, Ponsiglione. **SIAM Journal on Mathematical Analysis** (2019)

Example: Recovery of sparse sources



Well-studied problem: Superresolution

- Solve $\mathfrak{F}u = f$ on Ω
- $\mathfrak{F}$ Fourier transform, $\Omega \subset \mathbb{R}^d$ finite set
- **Sparsity assumption:** $\overline{u} = \sum_{i=1}^{N} \lambda_i \delta_{x_i}$

Banach space: $X = \mathcal{M}(\mathbb{R}^d)$ Radon measures

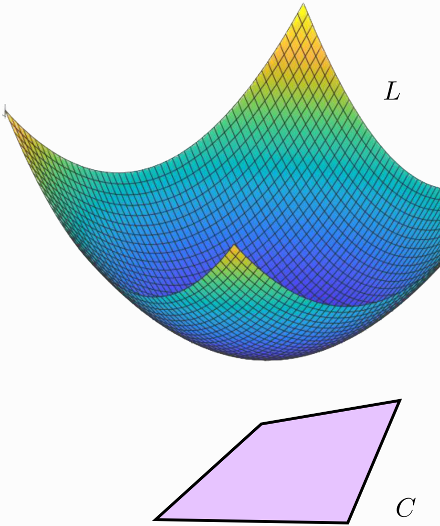
Regularizer: $R(u) := \|u\|_{\mathcal{M}}$ $\qquad \mathrm{Ext}(B) = \{\pm\delta_x : \; x \in \mathbb{R}^d\}$

Starting Point: Classic Frank-Wolfe

Problem: Constrained minimization

$$\min_{x \in C} \; L(x)$$

- $L \colon \mathbb{R}^N \to \mathbb{R}$ regular convex
- $C \subset \mathbb{R}^N$ convex compact set



M. Jaggi. **Proceedings of Machine Learning Research** (2013)

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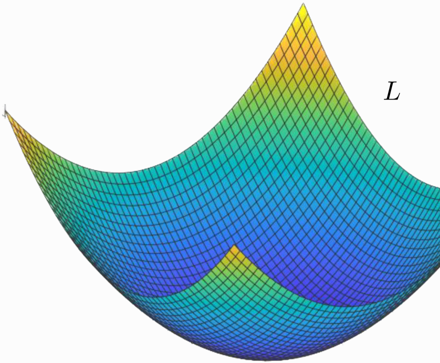


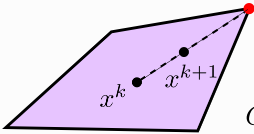
Frank-Wolfe Algorithm: Given $x^k \in C$

1. **Insertion:** Solve linearized problem

$$\min_{x \in C} \langle \nabla L(x^k), x \rangle \quad \mapsto \quad \widehat{x}$$

2. **Convex update:** Set

$$x^{k+1} := x^k + \alpha(\widehat{x} - x^k)\,, \quad \alpha := \frac{2}{k+2}$$

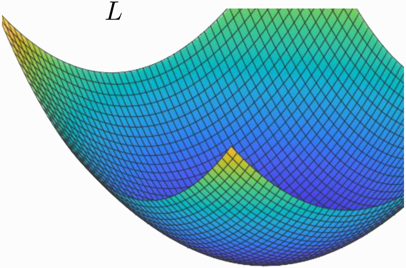


M. Jaggi. **Proceedings of Machine Learning Research** (2013)

Proposed Algorithm: Generalized Frank-Wolfe

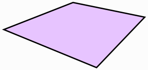
$$\min_{u \in X} L(u) + R(u)\,, \qquad L(u) = F(Ku)$$

L



Idea: Set $B = \{R \leq 1\}$. Consider

$$\min_{u \in X} L(u) + \chi_B(u) \iff \min_{u \in B} L(u)$$



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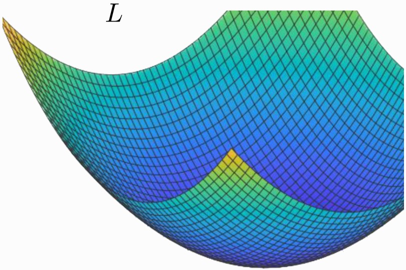
[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)



$$\min_{u\in X} L(u)+R(u)\,, \quad L(u) = F(Ku)$$

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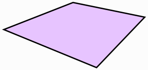
Descent Direction: Solve

$$\min_{v\in B} \langle \nabla L(u), v\rangle \quad \mapsto \quad \hat{v}$$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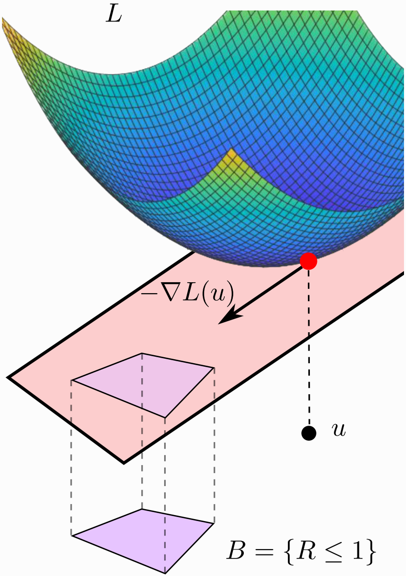

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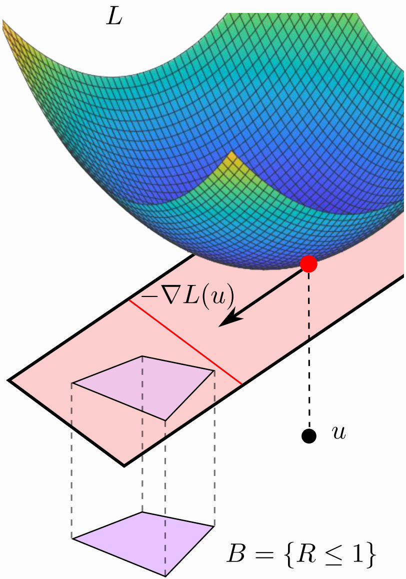

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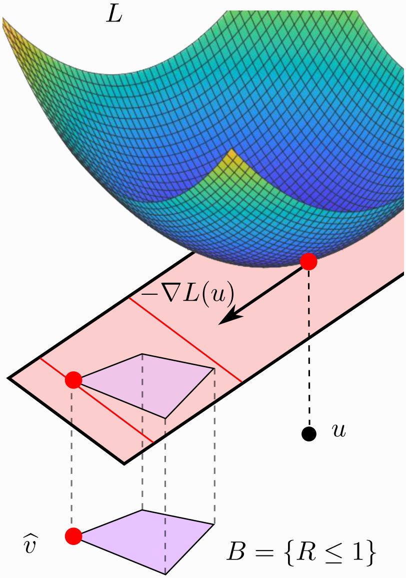

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Descent Direction: Solve

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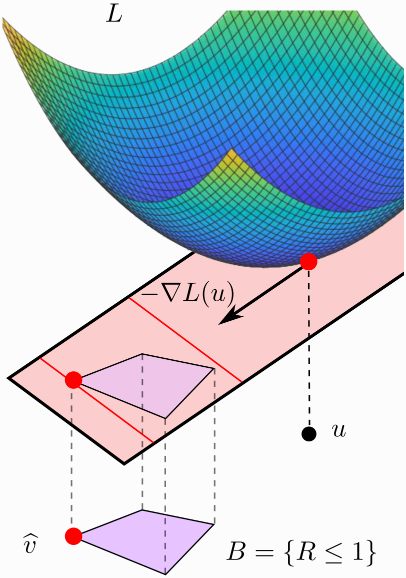
Lemma [1]. $\hat{v} \in \mathrm{Ext}(B)$

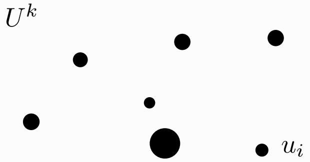


[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2024)

Sparse k-th iterate

$$U^k = \sum_{i=1}^{n} \lambda_i \, u_i$$

$$\lambda_i \geq 0, \quad u_i \in \mathrm{Ext}(B)$$



[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

Proposed Algorithm: Generalized Frank-Wolfe

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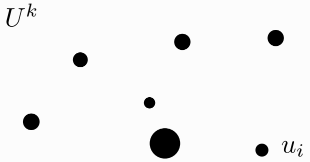

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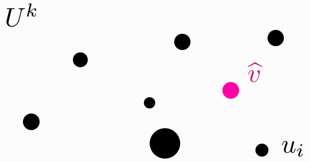

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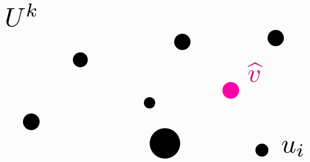

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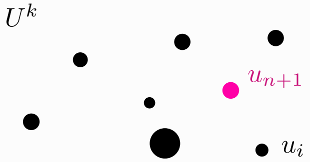

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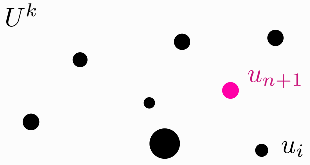

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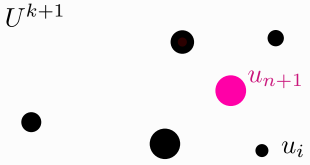

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1. Non-linear problem (usually)
   - Non-linearity due to $\mathrm{Ext}(B)$
   - Expensive and / or hard to solve
2. Quadratic program – Easy to solve

1. **Insertion Step:** Solve

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Convergence Analysis

Theorem [1]

U^k sparse iterate from the Generalized Frank-Wolfe Algorithm. Then

$$U^k \overset{*}{\rightharpoonup} \bar{u} \,, \qquad \bar{u} \in \arg\min \, G \,, \quad G := L + R$$

General convergence rate is **sublinear** (expected for gradient methods)

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Highlight: $\bar{u}$ **sparse** $+$ **"Source Condition"** $+$ **"Quadratic Growth"**

$\implies$ **linear** convergence: $G(U^k) - \min G \lesssim \dfrac{1}{2^k}$

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Linear Convergence Assumptions

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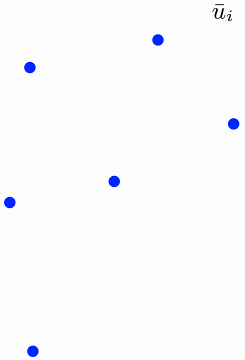

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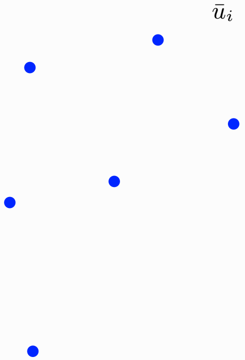

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$$\bar{p} := \nabla L(\bar{u}) = K_* \nabla F(K\bar{u})$$

is maximized exactly at $\bar{u}_i$

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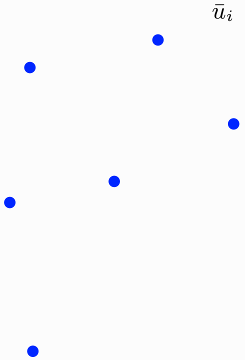

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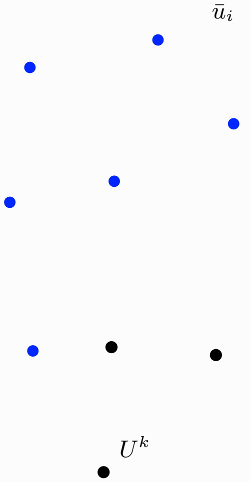

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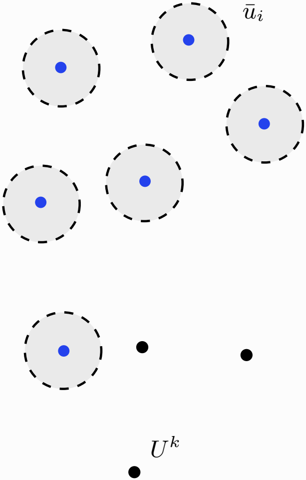

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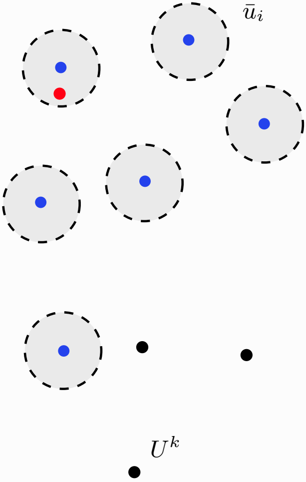

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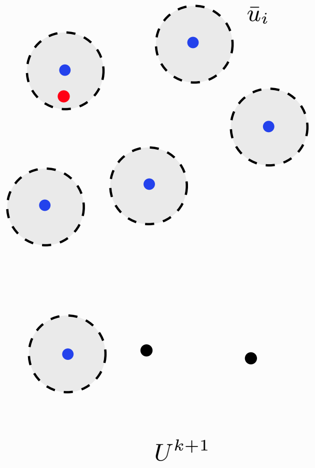

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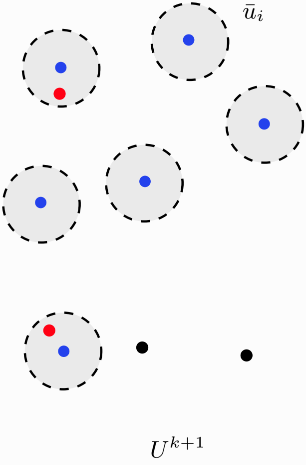

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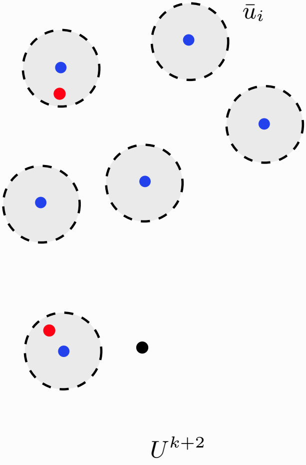

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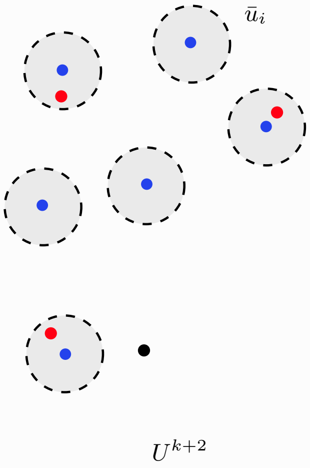

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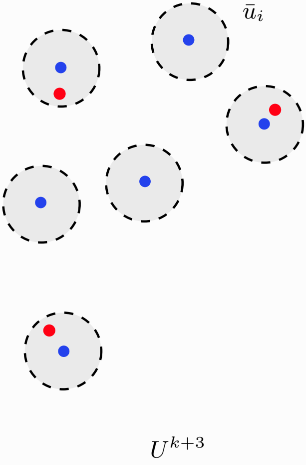

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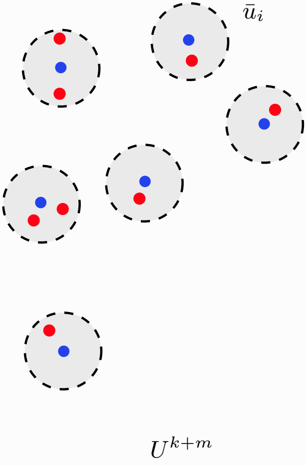

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Comments on Linear Convergence Assumptions

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- (S) + (SC) widely accepted
  - only proven in few cases [2]
  - can be verified numerically
- (QG) is novelty
- In applications we need to:
  - Characterize $\mathrm{Ext}(B)$
  - Define suitable distance g
  - Show (QG) under reasonable assumptions
- **Applications:** [1] Prove fast convergence of Gen. Frank-Wolfe
  - Minimum effort prob.
  - Trace-norm regularized prob.
  - Sparse source identification (heat eqn)

[1] Bredies, Carioni, F., Walter. **Math. Prog.** (’24) [2] Candès, Fernandez-Granda. **CPAM** (’13)

Application: Sparse peak recovery [1, 2]

$$\min_{u\in\mathcal{M}(\Omega)} \frac{1}{2}\|Ku - f^\varepsilon\|^2_{L^2(\Omega)} + \|u\|_{\mathcal{M}(\Omega)}$$

- **Given:** $\Omega \subset \mathbb{R}^d$ and $f^\varepsilon \in L^2(\Omega)$ noisy data
- **Forw. operator:** $K\colon \mathcal{M}(\Omega) \to L^2(\Omega)$

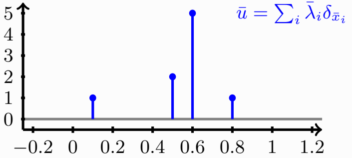
$$Ku = \psi \star u\,, \quad \psi = \text{Gauss. Kern.}$$

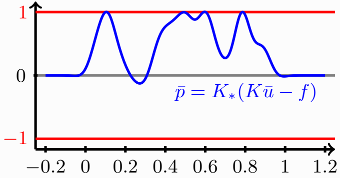
- **Extr. points:** $B = \{\|u\|_{\mathcal{M}(\Omega)} \leq 1\}$

$$\mathrm{Ext}(B) = \{\pm\delta_x : x \in \Omega\}$$

- **(S) $\exists$ sparse solution:**

$$\bar{u} = \sum_{i=1}^{M} \bar{\lambda}_i \delta_{\bar{x}_i}\,, \quad \bar{\lambda}_i > 0\,,\ \bar{x}_i \in \Omega$$





- **(SC)** $\bar{p} = K_*(K\bar{u} - f^\varepsilon) \in C(\Omega)$

$$\arg\max_{v\in\mathrm{Ext}(B)} \langle \bar{p}, v\rangle = \{\delta_{\bar{x}_1}, \dots, \delta_{\bar{x}_M}\}$$

$$\max_{v\in\mathrm{Ext}(B)} \langle \bar{p}, v\rangle = \max_{x\in\Omega} \bar{p}(x) = 1$$

[1] Bredies, Carioni, F., Walter. **Math. Prog.** (2024) [2] Pieper, Walter. **ESAIM: COCV** (2021)

Application: Sparse peak recovery [1, 2]

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2} \|Ku - f^\varepsilon\|^2_{L^2(\Omega)} + \|u\|_{\mathcal{M}(\Omega)}$$

- **(HP)** $\bar{p}$ strictly concave at x_i

$$\text{sign}(\bar{p}(x_i)) \, \langle \xi, \nabla^2 \bar{p}(x_i)\xi \rangle \gtrsim |\xi|^2$$

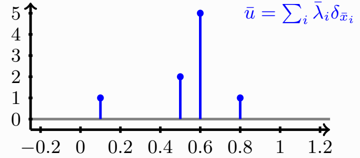
- **Metric:** $g\colon \ \mathrm{Ext}(B) \times \mathrm{Ext}(B) \to [0, \infty)$

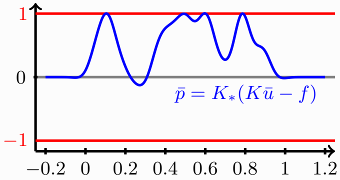
$$g(s_1\delta_{x_1}, s_2\delta_{x_2}) := |s_1 - s_2| + |x_1 - x_2|$$

- **(QG) Quadratic growth** of $\bar{p}$ around $\bar{u}_i$

$$1 - \langle \bar{p}, u \rangle \gtrsim g(u, u_i)^2 , \quad u \sim u_i$$

Theorem [1,2]: (HP) $\Longrightarrow$ (QG)
Gen. Frank-Wolfe converges **linearly**





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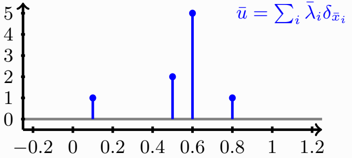
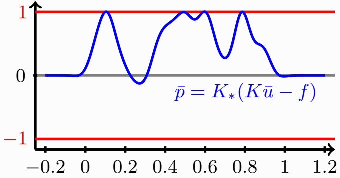


Gen. Frank-Wolfe: $\mathrm{Ext}(B) = \{\pm\delta_x : x \in \Omega\}$

Initialize: $u^0 = 0$

Iterate: Given $u^k = \sum_{i=1}^n \lambda_i \delta_{x_i}$



1. **Insertion Step:** $p^k = K_*(Ku^k - f^\varepsilon)$

$$\max_{v \in \mathrm{Ext}(B)} \langle p^k, v\rangle = \max_{x \in \Omega} p^k(x) \quad \rightsquigarrow \quad \hat{x}$$

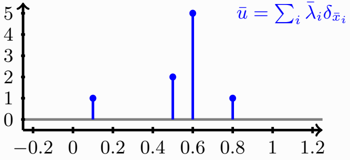
2. **Fully-corrective Step:** Solve

$$(\lambda_1^*, \dots, \lambda_{n+1}^*) \in \arg\min_{\lambda_i \geq 0} G\left(u^k + \lambda_{n+1}\delta_{\hat{x}}\right) \rightsquigarrow u^{k+1} := \left(\sum_{i=1}^n \lambda_i^* \delta_{x_i}\right) + \lambda_{n+1}^* \delta_{\hat{x}}$$

Stop: Based on Primal-Dual gap

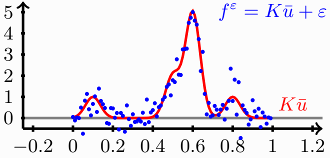
[1] Bredies, Carioni, F., Walter. **Math. Prog.** (2024) [2] Pieper, Walter. **ESAIM: COCV** (2021)

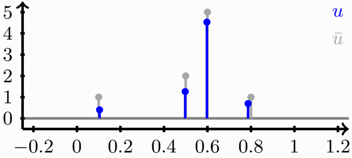
$$\min_{u\in\mathcal{M}(\Omega)} \frac{1}{2}\|Ku - f^\varepsilon\|^2_{L^2(\Omega)} + \|u\|_{\mathcal{M}(\Omega)}$$



Numerical experiment:

- Ground truth $\bar{u}$ with 4 peaks
- Noiseless data $f = K\bar{u}$
- Noisy data $f^\varepsilon = K\bar{u} + \varepsilon$
- Run Gen. Frank-Wolfe $\rightsquigarrow u$
  - u is minimizer (by Thm)
  - u correctly has 4 peaks
  - Weights of peaks are shrunk (effect of regularization)
  - Empirical **linear convergence**



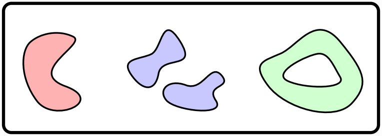


[1] Bredies, Carioni, F., Walter. **Math. Prog.** (2024) [2] Pieper, Walter. **ESAIM: COCV** (2021)

An open problem

Total variation: $X = \mathrm{BV}(\Omega)$, $\Omega \subset \mathbb{R}^d$

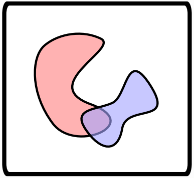
$$G(u) := F(Ku) + \|\nabla u\|_{\mathcal{M}} \,, \qquad \mathrm{Ext}(B) = \left\{ \frac{\chi_E}{\mathrm{Per}(E)} : E \subset \Omega \text{ simple} \right\}$$



Assume: sparse solution $\hat{u} = \sum_{i=1}^{M} \lambda_i \, \chi_{E_i}$

Fast convergence: Which “metric”???

$$g(E_i, E_j) := |E_i \triangle E_j|^{-1} \ \ ???$$

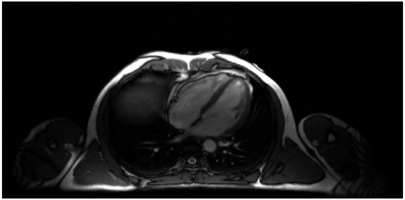


- **Connected problems:** Exact recovery, Noise Robustness

Outline

1. Introduction to Inverse Problems & Sparsity
2. Algorithm for sparse solution recovery
3. **Dynamic Inverse Problems**
4. Application to Dynamic MRI

MRI Scanner



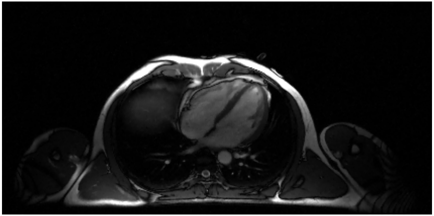
Human Heart

MRI: Medical imaging device, producing gray-scale images

$$u: \ \Omega \subset \mathbb{R}^2 \to \mathbb{R}$$

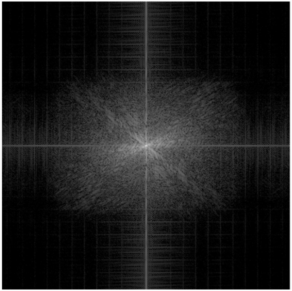
Mathematical model for MRI

Image $u \colon \Omega \to \mathbb{R}$



$\xrightarrow{\mathfrak{F}}$

$\mathfrak{F}u \colon \mathbb{R}^2 \to \mathbb{C}$



$$(\mathfrak{F}u)\,[\xi] = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x)\, e^{i\xi \cdot x}\, dx\,, \qquad \xi \in \mathbb{R}^2$$

MRI machine measures Fourier coefficients

Inverse Problem:

- Given MRI data y
- Find image $u \colon \Omega \to \mathbb{R}$ s.t.

$$\mathfrak{F} u = y$$

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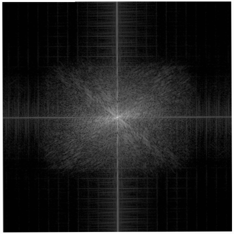
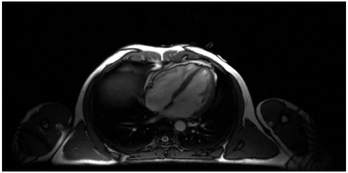
Ideal World: Fourier transform is invertible. Unique solution is $u = \mathfrak{F}^{-1}y$



Data y

$$\xrightarrow{\mathfrak{F}^{-1}}$$



Reconstruction u

MRI Inverse Problem

Reality: Things are not straightforward

- Machine is slow in acquiring data $\Longrightarrow$ can only sample **limited data**
- Measurement process is inherently **noisy**

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- Machine is slow in acquiring data $\implies$ can only sample **limited data**
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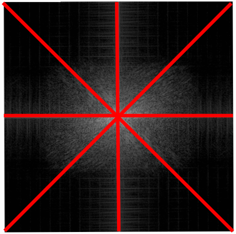
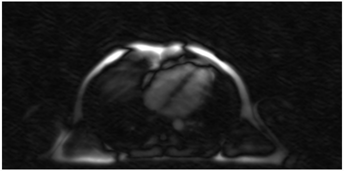
Issue: Plain inversion $\rightsquigarrow$ poor reconstructions



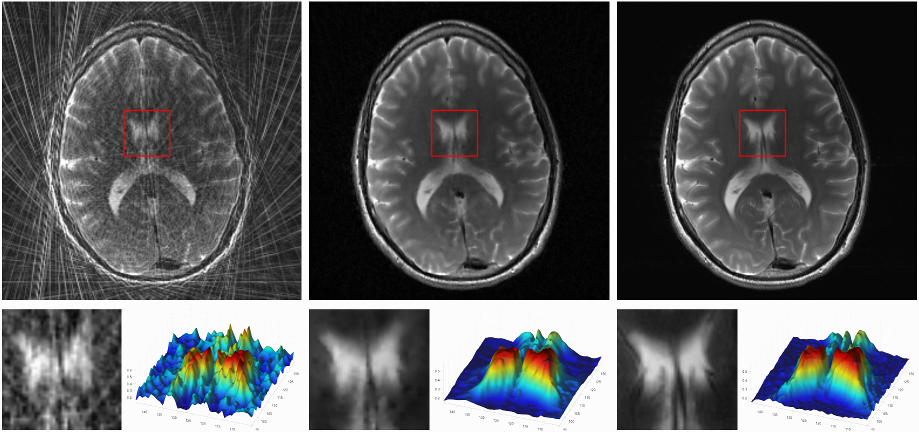
Undersampled noisy data y

$$\xrightarrow{\mathfrak{F}^{-1}}$$



Reconstruction u

Benchmark Regularizer: TGV



Unders. Noisy Data
Least Squares

Unders. Noisy Data
Regularized (TGV)

Full Data
Least Squares

Bredies, Kunisch, Pock. Total Generalized Variation. **SIAM Imaging** (2010)

Motivation: Undersampled Dynamic MRI

Goal: Dynamic MRI $\rightsquigarrow$ **Motion** is big challenge to accurate reconstructions

- High resolution imaging
- Imaging moving organs

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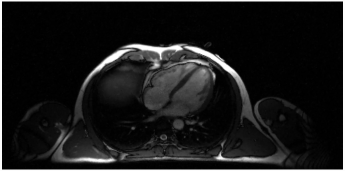
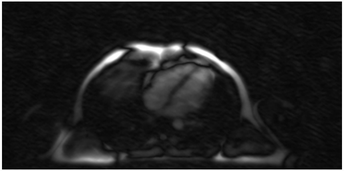
Dynamic IP: Reconstruct movie u_t from undersampled data series y_t

$$\mathfrak{F}(u_t) = y_t \quad \text{for all} \quad t \in [0,1]$$



Fully sampled data



Undersampled data

Solution: We need regularization for **dynamic inverse problems**

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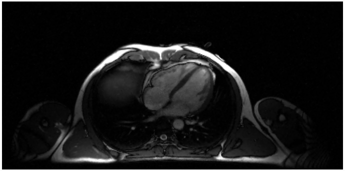

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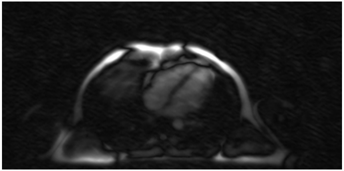

Undersampled data

Solution: We need regularization for **dynamic inverse problems**

Dynamic Inverse Problem

- **Images:** Radon Measures $\mu \in \mathcal{M}(\Omega)$ ($\Omega \subset \mathbb{R}^N$ bounded closed domain)
- **Data spaces:** Hilbert spaces H_t for $t \in [0,1]$
- **Measurement Operators:** linear continuous maps

$$K_t \colon \mathcal{M}(\Omega) \to H_t$$

- **Data points:** Curve $t \mapsto y_t$ with $y_t \in H_t$

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

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$$K_t \mu_t = y_t \quad \text{for all} \quad t \in [0,1] \tag{P}$$

Assumptions: weak time-regularity for $\{H_t\}_t$ and K_t^* (wk*-measurability)

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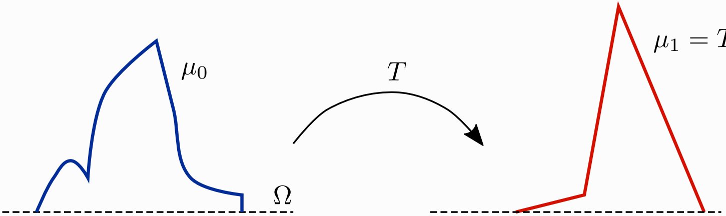
Proposal: Regularize (P) with an **Optimal Transport Energy**

[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Optimal Transport - Static Formulation

$\Omega \subset \mathbb{R}^d$ bounded domain, $\mu_0, \mu_1 \in \mathcal{P}(\Omega)$, $T\colon \Omega \to \Omega$ measurable displacement



Goal: move μ_0 to μ_1 in the cheapest way, with cost of moving mass from x to y

$$c(x,y) := |x-y|^2$$

Optimal Transport: a transport plan $\hat{T}$ solving

$$\hat{T} \in \arg\min \left\{ \int_\Omega |T(x)-x|^2 \, d\mu_0(x) \, : \, T\colon \Omega \to \Omega, \, T_\# \mu_0 = \mu_1 \right\}$$

Optimal Transport - Dynamic Formulation

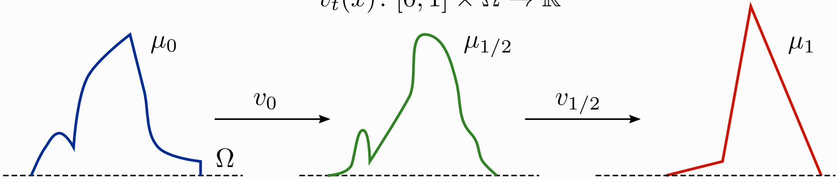
Idea: introduce a time variable $t \in [0,1]$ and consider the density evolution

- time dependent probability measures

$$t \mapsto \mu_t \in \mathcal{P}(\Omega) \quad \text{for} \quad t \in [0,1]$$

- μ_t is advected by the velocity field

$$v_t(x) \colon [0,1] \times \Omega \to \mathbb{R}^d$$



Dynamic model: (μ_t, v_t) solves the continuity equation with initial conditions

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\mu_t v_t) = 0 \\ \text{Initial data } \mu_0, \text{ final data } \mu_1 \end{cases} \tag{CE-IC}$$

Benamou-Brenier Formula

Theorem: Benamou-Brenier [1]

$$\min_{\substack{(\mu_t, v_t) \\ \text{solving (CE-IC)}}} \int_0^1 \int_\Omega |v_t(x)|^2 \, \mu_t(x) dx \, dt = \min_{\substack{T \colon \Omega \to \Omega \\ T_\# \mu_0 = \mu_1}} \int_\Omega |T(x) - x|^2 \, \mu_0(x) \, dx$$

Advantages of Dynamic Formulation:

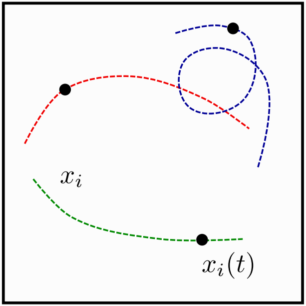
1. By introducing the momentum $m_t := \rho_t v_t$ we have
$$\int_0^1 \int_\Omega |v_t(x)|^2 \, \mu_t(x) \, dx \, dt = \int_0^1 \int_\Omega \frac{|m_t(x)|^2}{\mu_t(x)} \, dx \, dt$$
which is **convex** in (μ_t, m_t)
2. The continuity equation becomes **linear**
$$\partial_t \mu_t + \operatorname{div} m_t = 0$$
3. We know the full trajectory μ_t and can recover the velocity field v_t from m_t

[1] Benamou, Brenier. **Numerische Mathematik** (2000)

Optimal Transport Regularization

Trajectories: Curve of measures

$$t \mapsto \mu_t \in \mathcal{M}(\Omega)\,, \qquad t \in [0,1]$$



$$\mu_t = \sum_i \delta_{x_i(t)}$$

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

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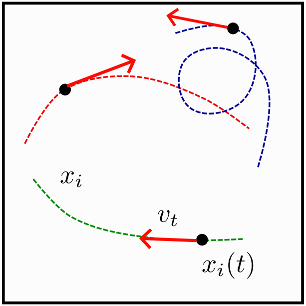
Assumptions:

- μ_t satisfies **Continuity Equation**

$$\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0$$

for some velocity field (to find)

$$v_t : \mathbb{R}^2 \to \mathbb{R}^2$$



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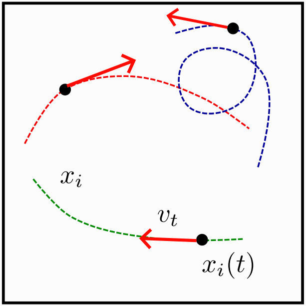
- Finite **Kinetic Energy**

  $$\int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 \, d\mu_t(x) \, dt < \infty$$



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[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Optimal Transport Regularization

Minimization Problem: Given data $t \mapsto y_t \in H_t$

$$K_t \mu_t = y_t \quad \rightsquigarrow \quad \min_{\mu, v} \; L(\mu) + R(\mu, v)$$

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

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- $L \rightsquigarrow$ **Loss Function:** Fits $t \mapsto \mu_t$ to data $t \mapsto y_t$ (Generalized Bochner spaces [2])

$$L(\mu) := \int_0^1 \|K_t \mu_t - y_t\|_{H_t}^2 \, dt$$

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- $R \rightsquigarrow$ **Regularizer:** Connected to **Optimal Transport** (Benamou-Brenier formula)

$$R(\mu, v) := \underbrace{\int_0^1 \int_\Omega |v_t(x)|^2 \; d\mu_t(x) \, dt}_{\text{Kinetic Energy}} + \underbrace{\int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} \; dt}_{\text{Radon Norm}}$$

$$\text{s.t.} \quad \underbrace{\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0}_{\text{Continuity Equation}}$$

- **Theorem [2]:** R provides stable regularization for the dynamic inverse problem

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Extremal Points

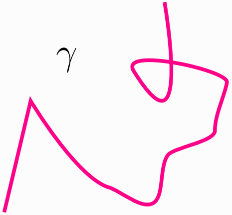
$$R(\mu, v) := \int_0^1 \int_\Omega |v_t(x)|^2 \, d\mu_t(x) \, dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} \, dt$$

$$\text{s.t.} \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \qquad \text{(CE)}$$

Theorem [1]

Let $B = \{R \leq 1\}$. Then $\operatorname{Ext}(B)$ are measures $t \mapsto \mu_t$ supported on **Sobolev Curves**

$$t \mapsto \mu_t = \delta_{\gamma(t)} \, , \qquad \gamma \in H^1([0,1]; \Omega)$$



Superposition Principle [2]: μ_t solves (CE) $\iff \exists \, \sigma \in P(H^1)$ concentrated on sol. to

$$\dot{\gamma}(t) = v(t, \gamma(t)) \ \text{ and s.t. } \int_\Omega \varphi \, d\mu_t = \int_{H^1} \varphi(\gamma(t)) \, d\sigma(\gamma) \, , \ \forall \, \varphi \in C(\Omega)$$

Proof Idea: μ_t extr. for $R \overset{\text{SP}}{\iff} \sigma$ extr. for $\|\cdot\|_{\mathcal{M}(H^1)} \overset{\text{Known}}{\iff} \sigma = \delta_\gamma \overset{\text{SP}}{\iff} \mu_t = \delta_{\gamma(t)}$

[1] Bredies, Carioni, Fanzon, Romero. **Bulletin London Mathematical Society** (2021)
[2] Ambrosio. **Inv. Math.** (2004)

Non-homogeneous case

- Homogeneous continuity equation implies mass preservation

$$\mu_t(\Omega) \quad \text{is constant for all } t$$

- Restrictive in applications – e.g. contrast agent in MRI
- Modify the regularizer to allow change of mass
- Based on **Unbalanced OT** – a.k.a. Hellinger-Kantorovich distance [2,3]

$$R(\mu, v) := \int_0^1 \int_\Omega |v_t(x)|^2 + |g_t(x)|^2 \, d\mu_t(x) \, dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} \, dt$$

$$\text{s.t.} \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = g_t \mu_t \qquad \text{(CE)}$$

Theorem [1]: R is stable regularizer for the dynamic inverse problem

[1] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)
[2] Chizat, Peyré, Schmitzer, Vialard. **Found. of Comp. Math** (2018)
[3] Liero, Mielke, Savaré. **Inv. Math.** (2018)

Extremal Points – Non-homogeneous case

$$R(\mu, v) := \int_0^1 \int_\Omega |v_t(x)|^2 + |g_t(x)|^2 \, d\mu_t(x) \, dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} \, dt$$

$$\text{s.t.} \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = g_t \mu_t \qquad \text{(CE)}$$

Theorem [1]

Let $B = \{R \leq 1\}$. Then $\operatorname{Ext}(B)$ are curves of measures of the form

$$t \mapsto \mu_t = h(t)\delta_{\gamma(t)}$$

- $h, \sqrt{h} \in \mathrm{AC}^2(0,1)$, $\gamma \in C(\{h > 0\}; \Omega)$, $\sqrt{h}\gamma \in \mathrm{AC}^2([0,1]; \mathbb{R}^d)$
- $\{h > 0\}$ is connected

Proof Idea: Novel Probabilistic Superposition Principle to (CE)

[1] Bredies, Carioni, Fanzon. **Communications in PDEs** (2022)

Numerical optimization

Dynamic IP: Given $t \mapsto y_t \in H_t$ find $t \mapsto \mu_t \in \mathcal{M}(\Omega)$ s.t.

$$K_t \mu_t = y_t \quad \text{for all} \quad t \in [0,1]$$

Optimal Transport Regularization: $\min_{\mu,v} \; L(\mu) + R(\mu, v)$

$$L = \int_0^1 \|K_t\mu_t - y_t\|_{H_t}^2 \,, \qquad R = \int_0^1 \int_\Omega |v_t(x)|^2 \, dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} \, dt$$

$$\text{s.t. } \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

- **Given:** $\Omega \subset \mathbb{R}^d$ and $t \mapsto y_t \in H_t$ data
- **Forw. operator:** $K \colon \mathcal{M}(\Omega) \to H_t$
- **Extr. points:** $B = \{R \leq 1\}$

$$\operatorname{Ext}(B) = \left\{ t \mapsto \delta_{\gamma(t)} : \gamma \in H^1([0,1]; \Omega) \right\}$$

Generalized Frank-Wolfe [1]

$$\min_{\mu,v} \int_0^1 \|K_t\mu_t - y_t\|_{H_t}^2 + \int_0^1 \int_\Omega |v_t(x)|^2 \, dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} \, dt$$

s.t. $\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$

Algorithm: **Initialize:** $\mu^0 = 0$ **Iterate:** Given $\mu^k = \sum_{i=1}^n \lambda_i \delta_{\gamma_i}$

❶ **Insertion Step:** $p_t^k = K_*(K\mu_t^k - y_t)$ $\quad p_t^k \in L^\infty([0,1]; C(\Omega))$

$$\max_{w \in \mathrm{Ext}(B)} \langle p^k, w\rangle = \max_{\gamma \in H^1([0,1];\Omega)} \left(\int_0^1 |\dot\gamma(t)|^2 \, dt + 1 \right)^{-1} \int_0^1 p_t^k(\gamma(t)) \, dt \quad \rightsquigarrow \quad \hat\gamma$$

❷ **Fully-corrective Step:** Solve

$$\lambda_i^* \in \operatorname*{arg\,min}_{\lambda_i \geq 0} \; G\left(\mu^k + \lambda_{n+1}\delta_{\hat\gamma}\right) \quad \rightsquigarrow \quad \mu^{k+1} := \left(\sum_{i=1}^n \lambda_i^* \delta_{\gamma_i}\right) + \lambda_{n+1}^* \delta_{\hat\gamma}$$

[1] Bredies, Carioni, Fanzon, Romero. **Found. of Computational Mathematics** (2023)

Convergence Analysis

Theorem [1]

μ^k sparse iterate from the Generalized Frank-Wolfe Algorithm. Then

$$\mu^k \overset{*}{\rightharpoonup} \bar{\mu}\,, \qquad \bar{\mu} \in \arg\min\, G\,, \quad G := L + R$$

General convergence rate is **sublinear** (expected for gradient methods)

$$G(\mu^k) - \min G \lesssim \frac{1}{k}$$

Work in Progress: $\bar{\mu}$ **sparse** $+$ **"Source Condition"** $+$ **"Quadratic Growth"**

$\Longrightarrow$ **linear** convergence: $$G(\mu^k) - \min G \lesssim \frac{1}{2^k}$$

[1] Bredies, Carioni, **Fanzon**, Romero. **Found. of Computational Mathematics** (2023)

Details and additional tweaks

- Solve the curve insertion problem

$$\hat{\gamma} \in \operatorname*{arg\,max}_{\gamma \in H^1([0,1];\Omega)} \left(\int_0^1 |\dot{\gamma}(t)|^2 \, dt + 1 \right)^{-1} \int_0^1 p_t^k(\gamma(t)) \, dt$$

  via gradient descent with suitable stepsize rule

Theorem [1]

Under suitable regularity assumptions, the gradient descent procedure converges subsequentially to stationary points and strongly in $\mathrm{AC}^2([0,1];\Omega)$.

- Multiple starts with suitable initial guess (crossovers, random curves, etc.) to increase chance to obtain global minimizer
- Multiple insertion $\rightsquigarrow$ insert all obtained stationary points

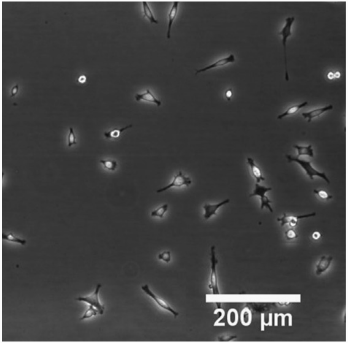
[1] Bredies, Carioni, Fanzon, Romero. **Found. of Computational Mathematics** (2023)

Outline

1. Introduction to Inverse Problems & Sparsity
2. Algorithm for sparse solution recovery
3. Dynamic Inverse Problems
4. **Application to Dynamic MRI**

Imaging Method $\Longrightarrow$

- Stars from ground-based telescope
- Microbubbles in blood vessels
- Cell migration



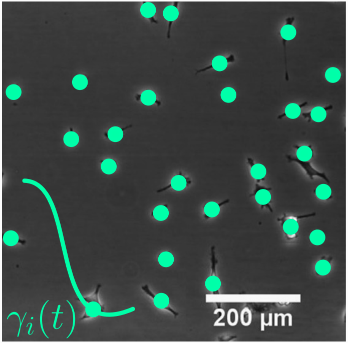
Microscopy image of cells

Image from: Yang, Venkataraman, Styles, et al. **Journal of Biomechanics** (2016)

Motivation: Particle Tracking

Imaging Method $\Longrightarrow$

- Stars from ground-based telescope
- Microbubbles in blood vessels
- Cell migration



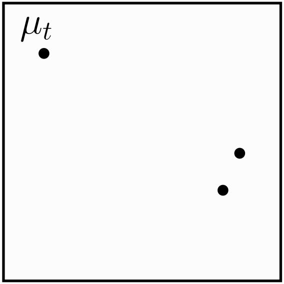
Microscopy image of cells

$$\mu_t = \sum_{i=1}^{M} \delta_{\gamma_i(t)}$$

Image from: Yang, Venkataraman, Styles, et al. **Journal of Biomechanics** (2016)

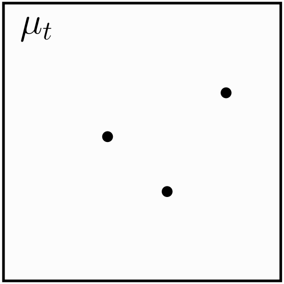
Application: Peak tracking for Dynamic MRI

y_t



$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

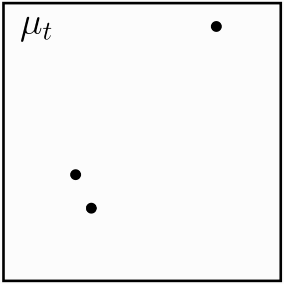


$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

Application: Peak tracking for Dynamic MRI

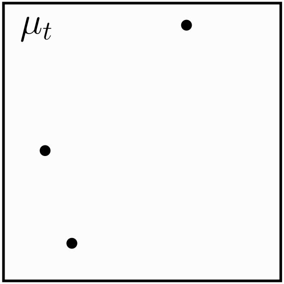
y_t



$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

y_t

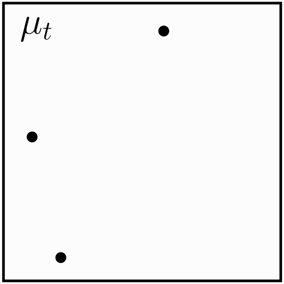


$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data y_t $\rightsquigarrow$ Image μ_t

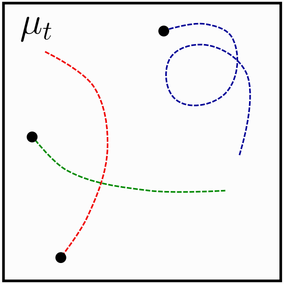
Application: Peak tracking for Dynamic MRI

y_t



$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data y_t $\rightsquigarrow$ Image μ_t

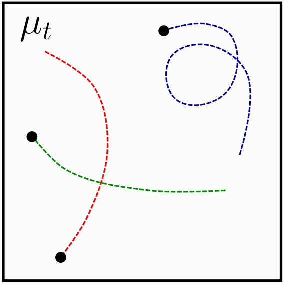


$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image $\mu_t \implies$ Interpolate Trajectories

Application: Peak tracking for Dynamic MRI

y_t



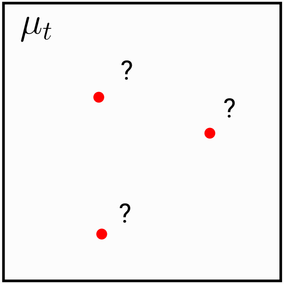
$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image $\mu_t \implies$ Interpolate Trajectories

Issue: Motion $\implies$ Low Scan Time $\implies$ **Low Data** $\mathbf{y}_t$

Application: Peak tracking for Dynamic MRI

y_t



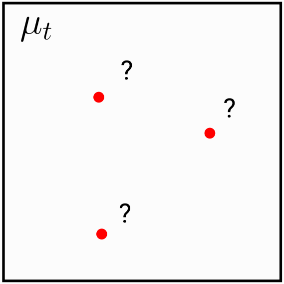
$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data y_t $\rightsquigarrow$ Image μ_t $\Longrightarrow$ Interpolate Trajectories

Issue: Motion $\Longrightarrow$ Low Scan Time $\Longrightarrow$ **Low Data** y_t $\rightsquigarrow$ **Particles?**

Application: Peak tracking for Dynamic MRI

y_t



$$\mu_t = \sum_{i=1}^{3} \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image $\mu_t \implies$ Interpolate Trajectories

Issue: Motion $\implies$ Low Scan Time $\implies$ **Low Data** $y_t \rightsquigarrow$ **Particles?**

Global-in-Time: Full Time-Series $t \mapsto y_t \rightsquigarrow$ Trajectories $t \mapsto \mu_t$

The Dynamic Undersampled MRI problem

Dynamic IP MRI: Given $t \mapsto y_t \in \mathbb{C}^{M_t}$ find $t \mapsto \mu_t \in \mathcal{M}(\Omega)$ s.t.

$$K_t \mu_t = y_t \quad \text{for all} \quad t \in [0,1]$$

Fourier Transform: For $\mu \in \mathcal{M}(\Omega)$

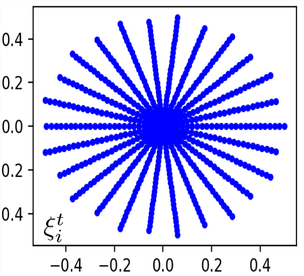
$$\hat{\mu} \colon \mathbb{C} \to \mathbb{C}\,, \quad \hat{\mu}\,[\xi] := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\xi \cdot x}\, d\mu(x)$$

Sampling Frequencies: M_t points

$$\xi_1^t\,, \ldots\,, \xi_{M_t}^t \in \mathbb{C}$$

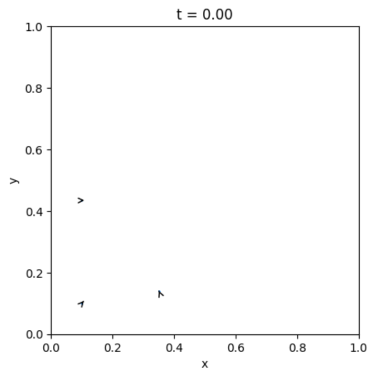
Forward operators: linear cont. $K_t \colon \mathcal{M}(\Omega) \to \mathbb{C}^{M_t}$

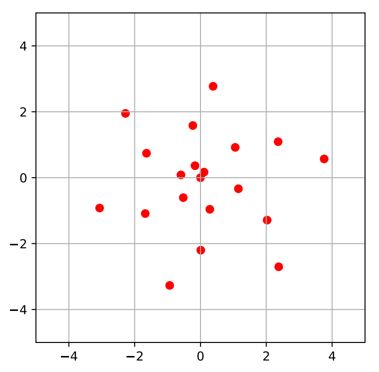
$$K_t \mu := \left(\hat{\mu}[\xi_1^t]\,, \ldots\,, \hat{\mu}[\xi_{M_t}^t]\right)$$



Example: Radial Sampling

Experiment 1: Spiral sampling (static)





Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

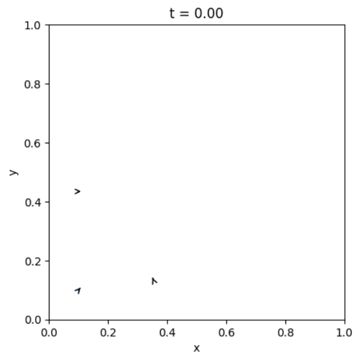
Sampling Freq: $\xi_1, \ldots, \xi_{20}$

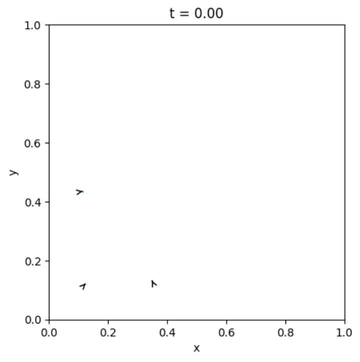
Data: Defined by

$$y_t := K\bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 1: Spiral sampling (static)

Algorithm: Generalized Frank-Wolfe $\rightsquigarrow \quad t \mapsto \mu_t^k = \sum_{i=1}^{M} \lambda_i \, \delta_{\gamma_i(t)}$





Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

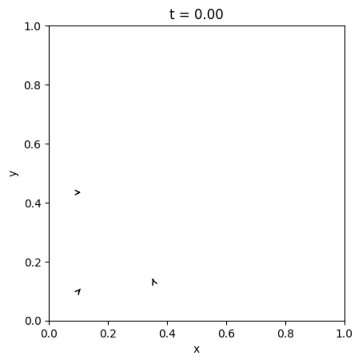
Reconstruction: from data

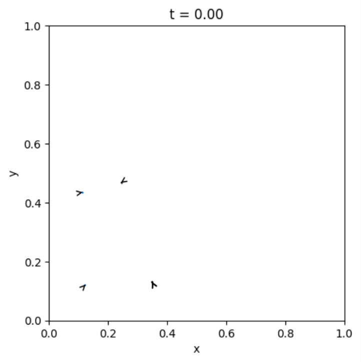
$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

(Thresholded at 0.05)

Experiment 1: Spiral sampling (static)

Algorithm: Generalized Frank-Wolfe $\rightsquigarrow \quad t \mapsto \mu_t^k = \sum_{i=1}^{M} \lambda_i \, \delta_{\gamma_i(t)}$





Ground truth: Curve of measures

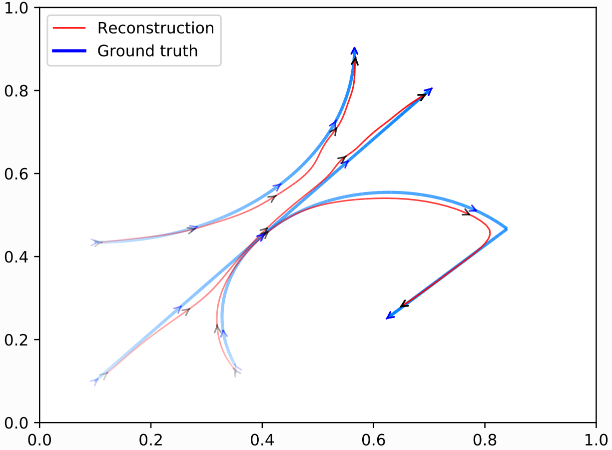
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Reconstruction: from data

$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

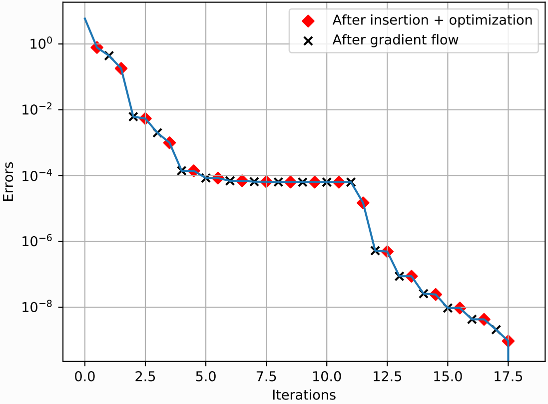
(No Thresholding)

Experiment 1: Spiral sampling (static)



Reconstructed trajectories

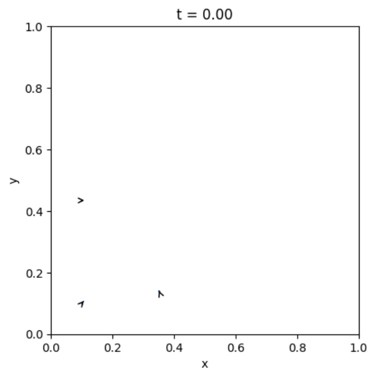
Experiment 1: Spiral sampling (static)

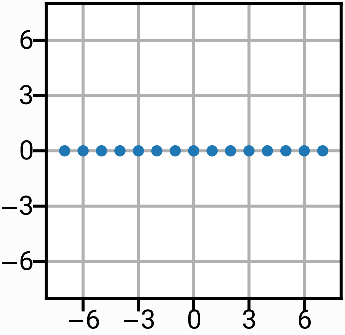


Convergence plot: exhibits linear rate

$$\text{Error} = G(\mu^k) - G(\mu^{k+1})$$

Experiment 2: Dynamic sampling on lines





$t = 0$

Ground truth: Curve of measures

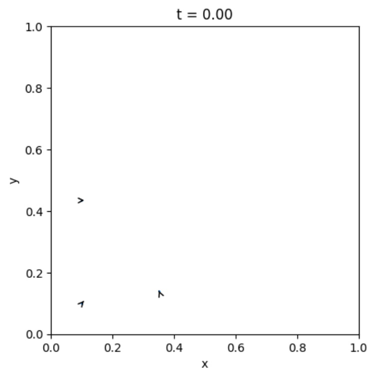
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

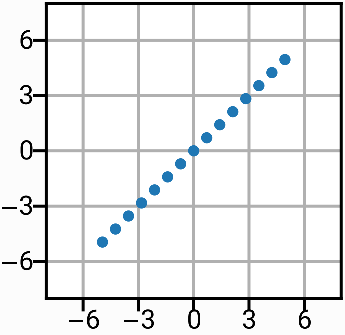
Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines





$t = 1/50$

Ground truth: Curve of measures

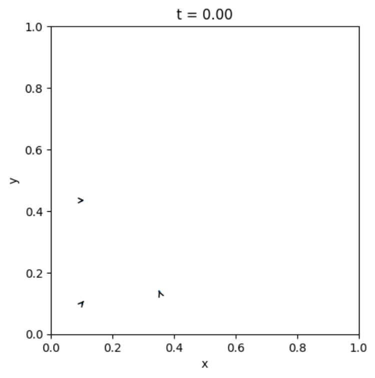
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

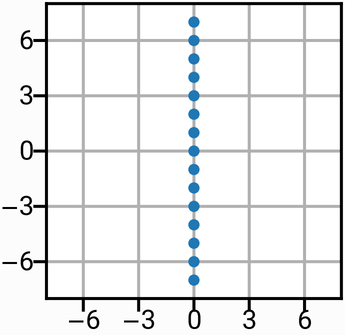
Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines





$t = 2/50$

Ground truth: Curve of measures

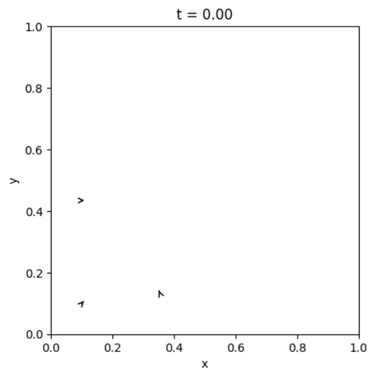
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

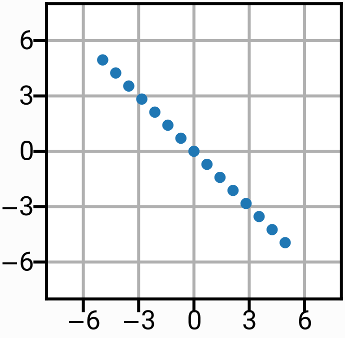
Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines





$t = 3/50$

Ground truth: Curve of measures

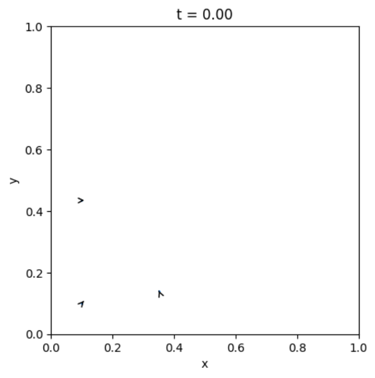
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

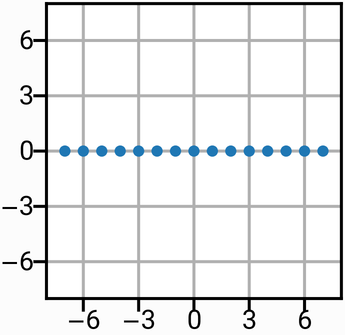
Sampling Freq: $\xi_1^t, \ldots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines





$t = 4/50$

Ground truth: Curve of measures

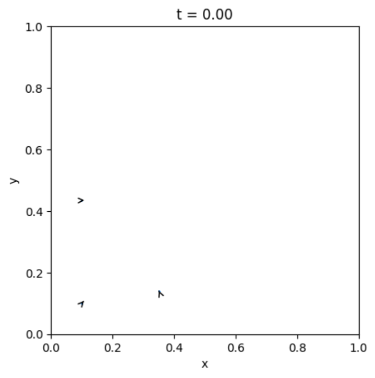
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

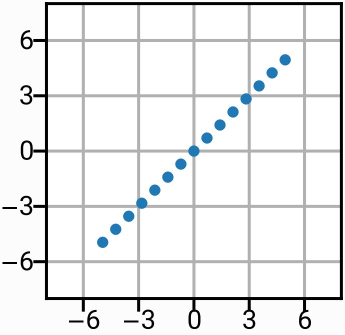
Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\%$$ Noise

Experiment 2: Dynamic sampling on lines





$t = 5/50$

Ground truth: Curve of measures

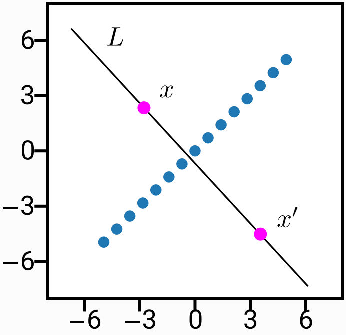
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines



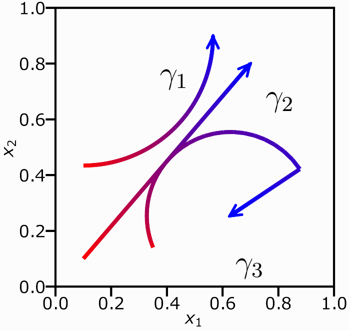
- L line orthogonal to the line of sampling frequences at time t

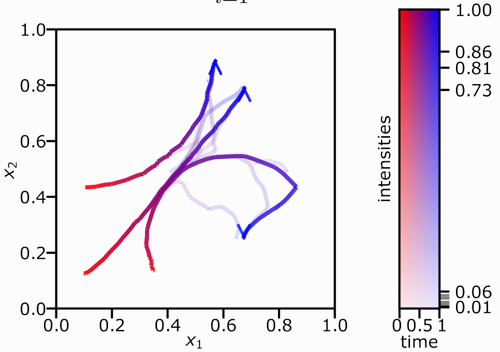
$$x, x' \in L \quad \Longrightarrow \quad K_t \delta_x = K_t \delta_{x'}$$

- Impossible to locate source along L at time t
- Only way to locate source is to enforce time regularity
- Example showcases need for time regularization

Experiment 2: Dynamic sampling on lines

Algorithm: Generalized Frank-Wolfe $\rightsquigarrow \quad t \mapsto \mu_t^k = \sum_{i=1}^{M} \lambda_i \, \delta_{\gamma_i(t)}$





Ground truth: Curve of measures

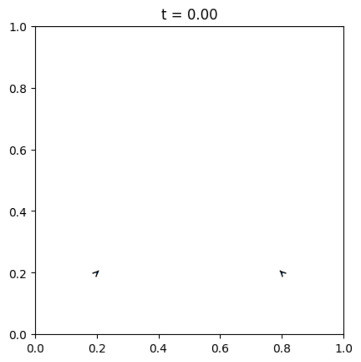
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

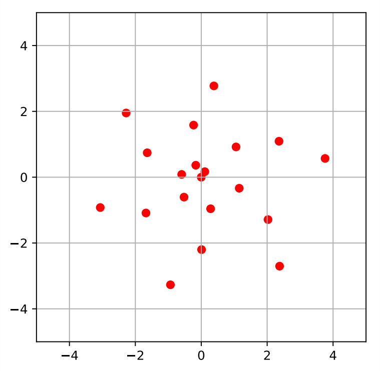
Reconstruction: from data

$$y_t = K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Remarkable reconstructions – considering unfavorable sampling pattern

Experiment 3: Crossing





Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$$

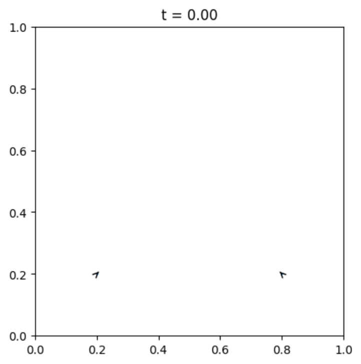
Sampling Freq: $\xi_1, \dots, \xi_{20}$

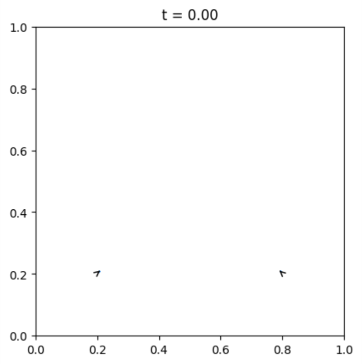
Data: Defined by

$$y_t := K\bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 3: Crossing

Algorithm: Generalized Frank-Wolfe $\rightsquigarrow \quad t \mapsto \mu_t^k = \sum_{i=1}^{M} \lambda_i \, \delta_{\gamma_i(t)}$





Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$$

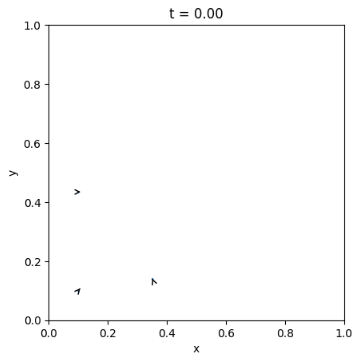
Reconstruction: from data

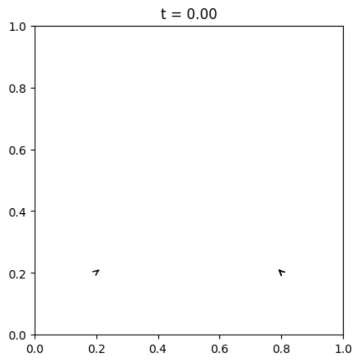
$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

(Thresholded at 0.01)

Experiment 3: Crossing

Algorithm: Generalized Frank-Wolfe $\rightsquigarrow \quad t \mapsto \mu_t^k = \sum_{i=1}^{M} \lambda_i \, \delta_{\gamma_i(t)}$





Ground truth: Curve of measures

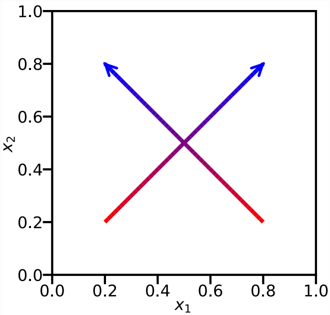
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$$

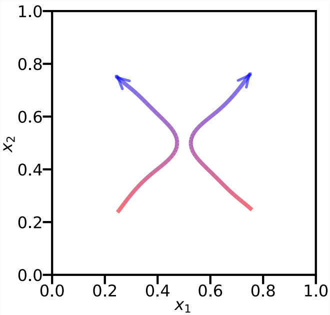
Reconstruction: from data

$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

(No Thresholding)

Experiment 3: Crossing





Ground truth: $\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$ **Reconstruction:** $\tilde{\mu}_t := \delta_{\tilde{\gamma}_1(t)} + \delta_{\tilde{\gamma}_2(t)}$

Question: Why do reconstructed trajectories differ from ground truth ones?

Answer: They don't! When regarded as measures they are basically the same

$$dt \otimes \bar{\mu}_t \approx dt \otimes \tilde{\mu}_t$$

Regularizer is Dynamic OT $\implies$ Particles chose shortest path

What to do? Maybe could include curvature penalization

Conclusion

1. Algorithm for computing **sparse** solutions to

$$\min_{u \in X} \; F(Ku) + R(u)$$

in **Banach** space

Conclusion

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Conclusion

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   in **Banach** space

2. **Linear** convergence if solution is **Sparse** + **"Source Condition"** + **"Quadratic Growth"**

3. General framework for **dynamic** inverse problems

Conclusion

❶ Algorithm for computing **sparse** solutions to

$$\min_{u \in X} \; F(Ku) + R(u)$$

in **Banach** space

❷ **Linear** convergence if solution is **Sparse** + **“Source Condition”** + **“Quadratic Growth”**

❸ General framework for **dynamic** inverse problems

❹ Application to **Dynamic MRI**

Thank You!

References

Generalized Frank-Wolfe Algorithm

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

Particles Tracking + Dynamic Inverse Problems

[2] **Fanzon**, Bredies. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

[3] Bredies, Carioni, **Fanzon**, Romero. **Bulletin London Mathematical Society** (2021)

[4] Bredies, Carioni, **Fanzon**. **Communications in PDEs** (2022)

[5] Bredies, Carioni, **Fanzon**, Romero. **Found. of Computational Mathematics** (2023)

FWF

Christian Doppler Forschungsgesellschaft

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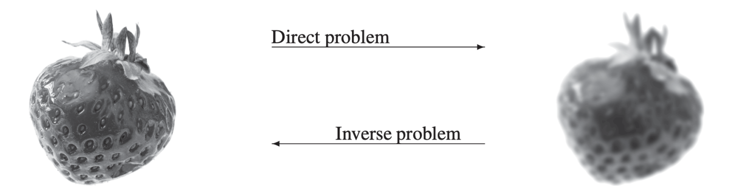
Christian Doppler Research Society (CDG) Project PIR27
“Mathematical methods for motion-aware medical imaging”

Infinite dimensional Example: Deblurring



Original Image $u\colon \Omega \to \mathbb{R}$ Blurred image $f\colon \Omega \to \mathbb{R}$

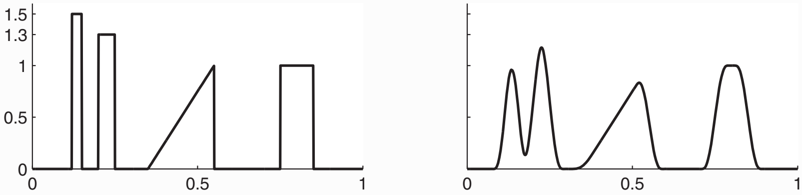
- Deblurring can be achieved by **deconvolution**

  Solve $\quad Ku = f\,, \qquad (Ku)(x) = \displaystyle\int_{\mathbb{R}^2} \psi(y)u(x-y)\,dy = (\psi \star u)(x)$

  with ψ suitable kernel (e.g. point-spread function)

- $K\colon L^2(\Omega) \to L^2(\Omega)$ compact operator $\implies$ K^{-1} unbounded (ill-posed)

Simpler case: 1D deconvolution

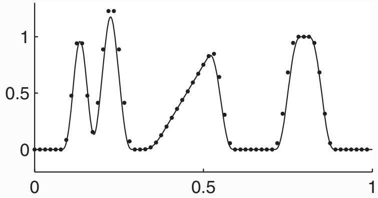


Original signal $\tilde{u} \colon [0,1] \to \mathbb{R}$

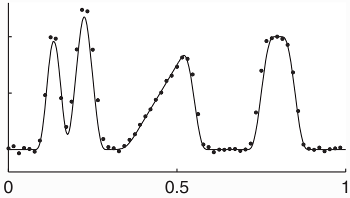
Blurred signal $f = \psi \star \tilde{u}$

- Goal: Recover $\tilde{u}$ from noisy data $f^{\varepsilon} = f + \varepsilon$
- This means solving the 1D-deconvolution problem: Find u such that

$$\psi \star u = f^{\varepsilon}$$



Discrete f with $n = 64$ grid points



Add 1% noise to obtain $f^\varepsilon \in \mathbb{R}^{64}$

- The convolution $\psi \star u$ can be discretized using Riemann sums
- The discrete inverse problem is therefore: Find $u \in \mathbb{R}^{64}$ such that

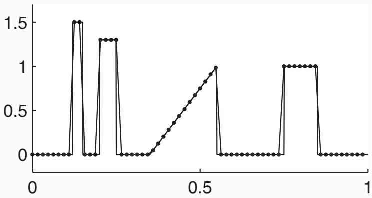
$$Ku = f^\varepsilon \,, \qquad K \in \mathbb{R}^{64 \times 64}$$

Naive deconvolution

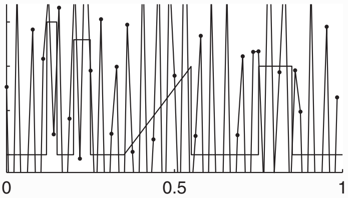
Solve the discrete 1D-deconvolution problem: Find $u \in \mathbb{R}^{64}$ such that

$$Ku = f^\varepsilon \,, \qquad K \in \mathbb{R}^{64\times 64}$$

- The naive solution is $u = K^{-1} f^\varepsilon$
- This behaves well when $\varepsilon = 0$ but is terrible when $\varepsilon \neq 0$
- Below the solid line represents the ground truth $\tilde{u}$
- We need regularizer which penalizes oscillations



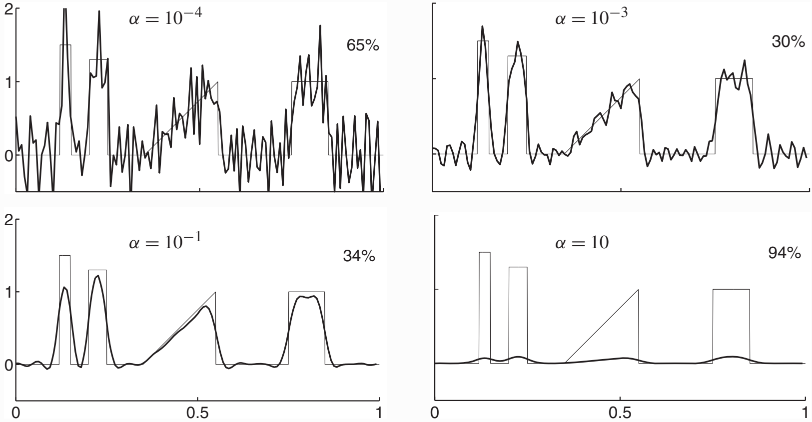
$$u = K^{-1} f$$



$$u = K^{-1} f^\varepsilon$$

Regularize the discrete inverse problem with the ℓ_2 norm:

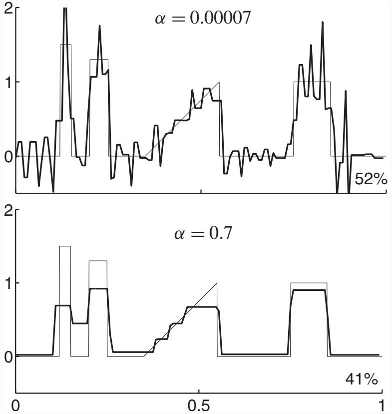
$$\min_{u \in L^2(0,1)} \|Ku - f^\varepsilon\|^2_{L^2(0,1)} + \alpha \|u'\|^2_{L^2(0,1)}$$



Best is $\alpha = 10^{-1}$. Notice the smoothing effect of ℓ_2 Smoothness is not always desirable (e.g. if u is image with sharp edges)

Regularize inverse problem with the Total Variation (BV) semi-norm:

$$\min_{u \in L^1(0,1)} \|Ku - f^\varepsilon\|^2_{L^2(0,1)} + \alpha \, \mathrm{TV}(u)$$



Best is $\alpha = 0.009$. Notice the sparsifying effect of TV (the jumps)

Elementary example: Matrix inversion

Given $f \in \mathbb{R}^m$ and a matrix $K \in \mathbb{R}^{m \times n}$ we want to find $u \in \mathbb{R}^n$ such that

$$Ku = f + \varepsilon \tag{P}$$

What could go wrong:

1. $m > n \implies$ Range$(K) \neq \mathbb{R}^m \implies$ No solution when $f + \varepsilon \notin$ Range(K)
2. $m < n \implies$ ker$(K) \neq \{0\} \implies$ There are several solutions
3. $m = n$ and K^{-1} exists: However condition number $\kappa = \lambda_1 / \lambda_n$ could be large. Then K is almost singular and

$$\left\| K^{-1} \varepsilon \right\| \approx \frac{\|\varepsilon\|}{\lambda_n} \quad \implies \quad \text{Naive reconstruction is dominated by noise}$$

$$\tilde{u} = u + K^{-1} \varepsilon \quad \implies \quad \text{instability}$$

Therefore (P) is in general **ill-posed**

Least-squares solution

Given $f \in \mathbb{R}^m$ and a matrix $K \in \mathbb{R}^{m \times n}$ we want to find $u \in \mathbb{R}^n$ such that

$$Ku = f \tag{P}$$

- (P) might not have solution. Find approximate solution by **least-squares**

$$\min_{u \in \mathbb{R}^n} \; \|Ku - f\|_2^2 \tag{P'}$$

  with $\|\cdot\|_2$ the Euclidean norm

- (P') always has the explicit solution (seen by differentiation)

$$\tilde{u} = (K^T K)^{-1} K^T f$$

- **Problem 1:** Solution to (P') **not unique** (if K is not injective)
- **Problem 2:** Solution might be **instable** (depends on eigenvalues of $K^T K$)

Variational Regularization

Given $f \in \mathbb{R}^m$ and a matrix $K \in \mathbb{R}^{m \times n}$ we want to find $u \in \mathbb{R}^n$ such that

$$Ku = f \tag{P}$$

- **Question:** Non uniqueness and / or instability. What to do?
- **Answer:** Replace (P) with the **regularized** least-squares problem

$$\min_{u \in \mathbb{R}^n} \|Ku - f\|_2^2 + \alpha R(u)$$

with $R \colon \mathbb{R}^n \to [0, +\infty]$ **regularizer** and $\alpha > 0$ to be chosen

1. R promotes stability (if chosen properly)
2. R selects only some solutions (the ones for which $R(u)$ is small)

First Example: ℓ_2 regularization

Regularize using the ℓ^2 norm:

$$\min_{u \in \mathbb{R}^n} \|Ku - f\|_2^2 + \alpha \|u\|_2^2 \tag{P}$$

- (P) is known as Ridge-regression in Statistics
- (P) always has the explicit solution (seen by differentiation)

$$\tilde{u} = (K^T K + \alpha I)^{-1} K^T f$$

- (P) more stable because eigenvalues of $K^T K + \alpha I$ are away from zero
- ℓ_2 norm **shrinks components** $\implies$ mitigates effects of noise

Second Example: ℓ_1 regularization

Regularize using the ℓ^1 norm

$$\min_{u \in \mathbb{R}^n} \|Ku - f\|_2^2 + \alpha \|u\|_1 \tag{P}$$

- (P) is known as LASSO-regression is statistics
- (P) always admits a solution (no explicit formula available)
- ℓ_1 norm automatically **sets some components to zero** $\rightsquigarrow$ **sparsity**

$$\tilde{u} = (0, 0, 0, *, 0, 0, \dots, 0, *, 0, 0, 0, 0)$$

- Desirable when n is large (many parameters), as it simplifies the model

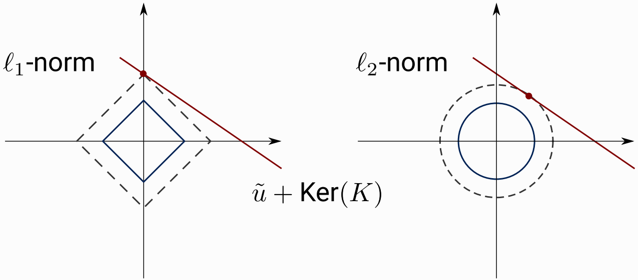
Models like GPT-5 have 10s of trillions of parameters

Why does ℓ_1 set components to zero?

$$\min_{u \in \mathbb{R}^n} \|Ku - f\|_2^2 + \alpha \|u\|_1 \qquad \min_{u \in \mathbb{R}^n} \|Ku - f\|_2^2 + \alpha \|u\|_2^2$$

$$\Updownarrow \qquad \Updownarrow$$

$$\min_{\|u\|_1 \leq s} \|Ku - f\|_2^2 \qquad \min_{\|u\|_2 \leq s} \|Ku - f\|_2^2$$



Extremal points are different $\Longrightarrow$ ℓ_1 and ℓ_2 select different solutions

Extremal points of regularizer describe features of sparse solutions

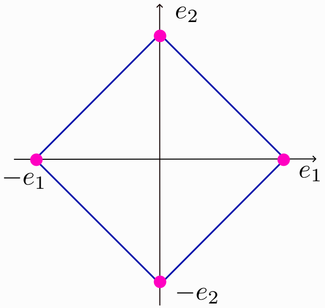
Example: Portfolio Optimization

Portfolio: Vector

$$P = (w_1, \dots, w_d) \qquad w_i = \text{capital to invest in asset } i$$

Sparsity: Invest in few assets

$$P = (0, 0, \mathbf{w_i}, 0, 0, \dots, 0, 0, \mathbf{w_d}) \implies \text{lower managing fees}$$



Banach space: $X = \mathbb{R}^d$

Regularizer: $\|x\|_1 := \sum_{i=1}^{d} |x_i|$

$$\mathrm{Ext}(B) = \{\pm e_i\}_{i=1}^d$$

Extremal Points

$$R(\mu, v) := \int_0^1 \int_\Omega |v_t(x)|^2 \, d\mu_t(x) \, dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} \, dt$$

$$\text{s.t.} \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \qquad \text{(CE)}$$

[3] Bredies, Carioni, **Fanzon**, Romero. **Bulletin London Mathematical Society** (2021)

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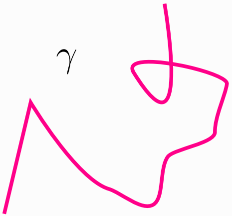
s.t. $\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$ (CE)

Theorem [3]

Let $B = \{R \leq 1\}$. Then $\operatorname{Ext}(B)$ are measures $t \mapsto \mu_t$ supported on **Sobolev Curves**

$$t \mapsto \mu_t = \delta_{\gamma(t)} \,, \qquad \gamma \in H^1([0,1];\mathbb{R}^2)$$



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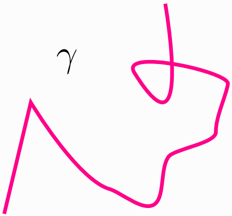

Proof Idea: Probabilistic Superposition Principle for measure solutions to

$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \qquad (= g_t \mu_t \;)$$

[3] Bredies, Carioni, **Fanzon**, Romero. **Bulletin London Mathematical Society** (2021)