Sparse optimization Algorithms in Medical Imaging

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Faculty member

University of Graz

- Inverse Problems
- Numerical Analysis
- Medical Imaging
- Optimal Transport
- Machine Learning

PhD Mathematics

University of Sussex

- PDEs
- Calculus of Variations
- Materials Science

Sparse optimization Algorithms in Medical Imaging

based on joint works with

Kristian Bredies, Marcello Carioni, Francisco Romero, Daniel Walter

Outline

1. Minimization Problem / Sparsity
2. Sparse optimization Algorithm
3. Application: Particle Tracking

Minimization Problem

X Banach space. Solve

$$\min_{u \in X} \; L(u) + R(u)$$

- **L** $\rightsquigarrow$ **Loss function:** **Smooth** $+$ **Convex**

  (Close to data)

  $$L : X \to [0, \infty)$$

- **R** $\rightsquigarrow$ **Regularizer:** **Convex** $+$ **1-homogeneous**

  (Promotes Sparsity)

  $$R \colon X \to [0, \infty]$$

Note: Compactness assumptions $\Longrightarrow$ Minimizer exists

[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2023)

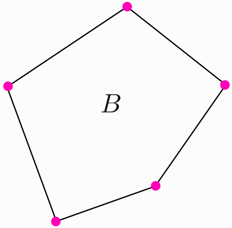
Sparsity

Unit Ball of regularizer R

$$B := \{u \in X : \ R(u) \leq 1\}$$

Extremal Points: $u \in B$ s.t.

$$\begin{cases} u = \alpha u_1 + (1-\alpha) u_2 \\ \alpha \in (0,1)\,, \ u_1, u_2 \in B \end{cases} \implies u = u_1 = u_2$$



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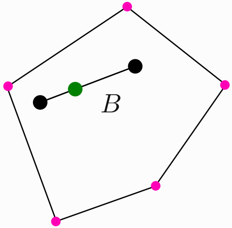

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2023)

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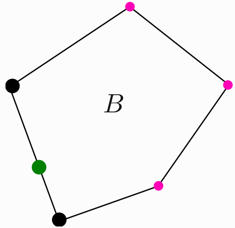

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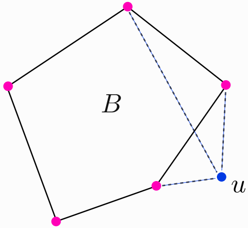


Definition: $u \in X$ **sparse**

Conic combination

$$u = \sum_{i=1}^{N} \lambda_i \, u_i \,, \quad \lambda_i \geq 0\,, \quad u_i \in \mathrm{Ext}(B)$$

[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2023)

Main Task

Numerical **Algorithm** to compute $\bar{u}$ solution of

$$\min_{u \in X} \; L(u) + R(u)$$

which is **sparse**

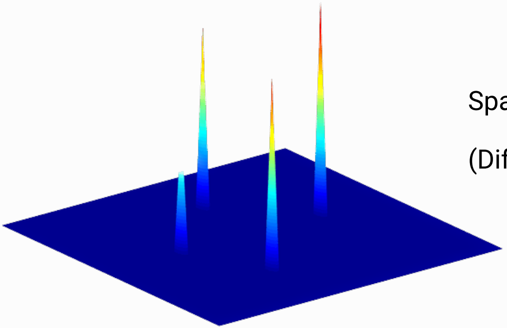
$$\bar{u} = \sum_{i=1}^{N} \lambda_i \, u_i \, , \quad \lambda_i \geq 0 \, , \quad u_i \in \operatorname{Ext}(B)$$

Examples:

- Portfolio optimization $\rightsquigarrow$ $\mathbb{R}^d$
- Training 2-Layer Neural Networks $\rightsquigarrow$ $\mathcal{M}(\mathbb{R}^d)$ Radon Measures
- Microstructures in Materials $\rightsquigarrow$ $\mathrm{BV}(\mathbb{R}^d)$ Bounded Variation

Example: Radon Measures

Banach space: $X = \mathcal{M}(\mathbb{R}^d)$ Radon measures



Sparse Source Identification

(Diffusion-Advection Equation)

$$\mu = \sum_i \lambda_i \, \delta_{x_i}$$

Regularizer: $R(\mu) := \|\mu\|_{\mathcal{M}}$, $\mathrm{Ext}(B) = \{\pm\delta_x : \; x \in \mathbb{R}^d\}$

Figure from: Monge, Zuazua. **Systems** & **Control Letters** (2020)

Sparse Algorithm

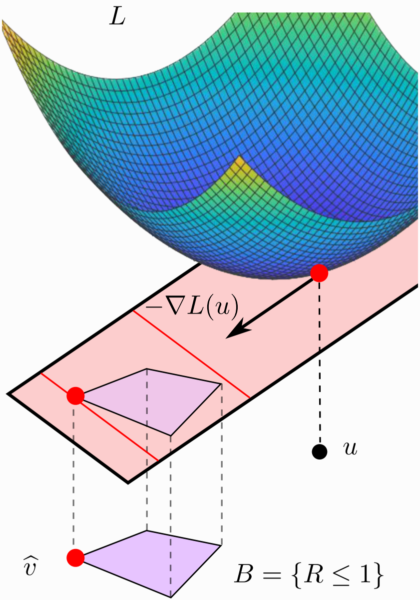
$$\min_{u \in X} \; G(u)\,, \quad G := L + R$$

Idea: Gradient descent (Frank-Wolfe)

Descent Direction: $\widehat{v} \in B$

$$\widehat{v} \in \operatorname*{arg\,min}_{v \in B} \langle \nabla L(u), v \rangle$$

Lemma [1]. $\widehat{v} \in \operatorname{Ext}(B)$



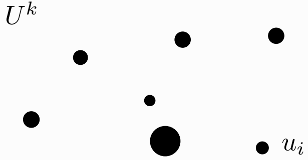
[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2023)

Algorithm: Generalized Frank-Wolfe

Sparse k-th iterate

$$U^k = \sum_{i=1}^{n} \lambda_i \, u_i$$

$$\lambda_i \geq 0, \quad u_i \in \mathrm{Ext}(B)$$



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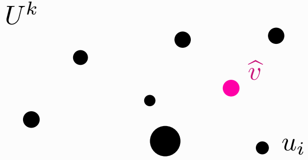

❶ **Insertion Step:** Solve

$$\widehat{v} \in \underset{v \in \mathrm{Ext}(B)}{\arg\min} \; \langle \nabla L(U^k), v \rangle$$

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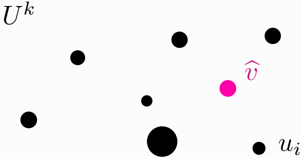

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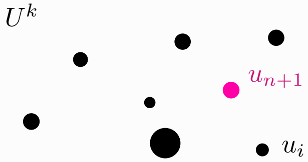

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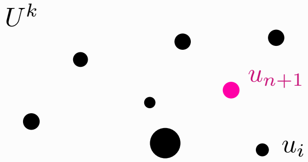

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❷ **Correction Step:** Set $u_{n+1} := \widehat{v}$. Optimize coefficients

$$(\lambda_1^*, \ldots, \lambda_{n+1}^*) \in \operatorname*{arg\,min}_{\lambda_i \geq 0} G\left(\sum_{i=1}^{n+1} \lambda_i \, u_i \right) \quad \rightsquigarrow \quad U^{k+1} := \sum_{i=1}^{n+1} \lambda_i^* \, u_i$$

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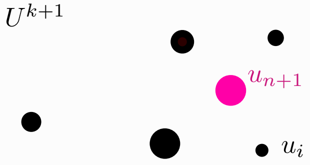

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Convergence Analysis

Theorem [1]

U^k sparse iterate generated by Algorithm. Then

$$U^k \overset{*}{\rightharpoonup} \bar{u}\,, \qquad \bar{u} \in \arg\min\, G\,, \quad G := L + R$$

General convergence is **sublinear**

$$G(U^k) - \min G \lesssim \frac{1}{k}$$

Highlight: $\bar{u}$ **sparse** $+$ **"source condition"** $\Longrightarrow$ **linear** convergence

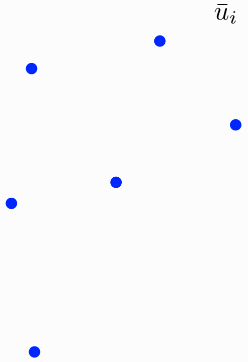
$$G(U^k) - \min G \lesssim \frac{1}{2^k}$$

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2023)

Fast Convergence Assumptions

❶ Minimizer is **sparse**

$$\bar{u} = \sum_{i=1}^{M} \bar{\lambda}_i \, \bar{u}_i \ , \qquad \bar{u}_i \in \operatorname{Ext}(B)$$



❷ **Source condition:** dual variable

$$\bar{p} := -\nabla L(\bar{u})$$

is maximized exactly at $\bar{u}_i$

$$\operatorname*{arg\,max}_{v \in \operatorname{Ext}(B)} \ \langle \bar{p}, v \rangle = \{\bar{u}_1, \dots, \bar{u}_M\}$$

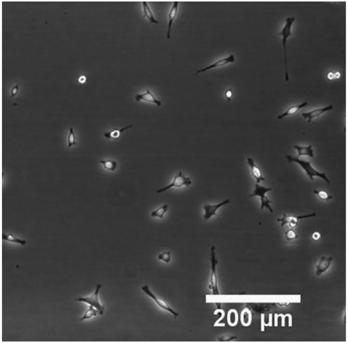
Note:

- Verifiable a posteriori
- Proven in some cases

Application: Particle Tracking

Imaging Method $\Longrightarrow$

- Stars from ground-based telescope
- Microbubbles in blood vessels
- Cell migration



Microscopy image of cells

Image from: Yang, Venkataraman, Styles, et al. **Journal of Biomechanics** (2016)

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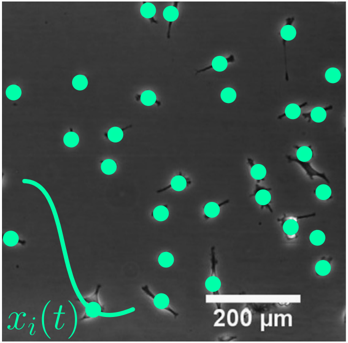

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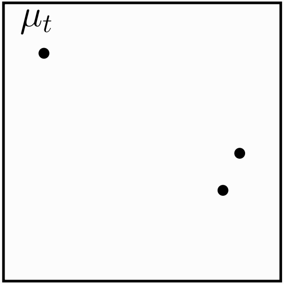
$$\mu_t = \sum_{i=1}^{M} \delta_{x_i(t)}$$

Image from: Yang, Venkataraman, Styles, et al. **Journal of Biomechanics** (2016)

Magnetic Resonance Imaging (MRI)

y_t



Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

Magnetic Resonance Imaging (MRI)

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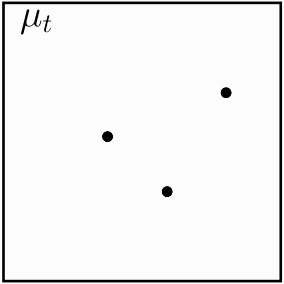

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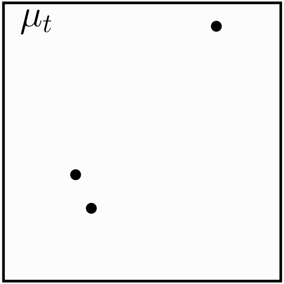

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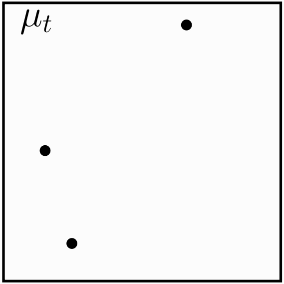

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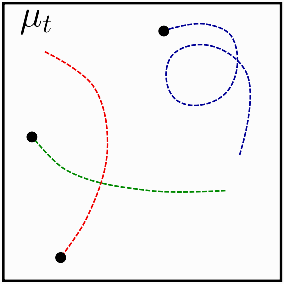

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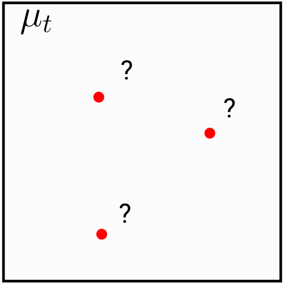

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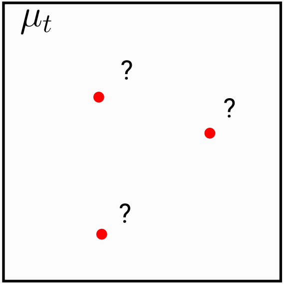
Motion $\implies$ Ultra-Low Scan Time $\implies$ **Ultra-Low Data $\mathbf{y}_t$** $\rightsquigarrow$ **Particles?**

Magnetic Resonance Imaging (MRI)

y_t



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Motion $\implies$ Ultra-Low Scan Time $\implies$ **Ultra-Low Data $\mathbf{y}_t$** $\rightsquigarrow$ **Particles?**

Global-in-Time: Full Time-Series $t \mapsto y_t \rightsquigarrow$ Trajectories $t \mapsto \mu_t$

Motion-Aware Regularization

Video: Curve of measures

$$t \mapsto \mu_t \in \mathcal{M}(\mathbb{R}^2)\,, \qquad t \in [0,1]$$

Assumptions:

- μ_t satisfies **Continuity Equation**

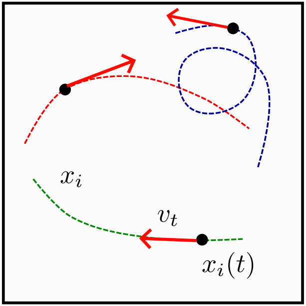
$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

  for some velocity field (to find)

$$v_t \colon \mathbb{R}^2 \to \mathbb{R}^2$$

- Finite **Kinetic Energy**

$$\int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 \, d\mu_t(x)\, dt < \infty$$



$$\mu_t = \sum_i \delta_{x_i(t)}$$

[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Motion-Aware Regularization

Minimization Problem: Given MRI data $t \mapsto y_t$

$$\min_{\mu, v} \; L(\mu) + R(\mu, v)$$

- $L \rightsquigarrow$ **Loss Function:** Fits $t \mapsto \mu_t$ to given MRI data $t \mapsto y_t$

- $R \rightsquigarrow$ **Regularizer:**

$$R(\mu, v) := \underbrace{\int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 \, d\mu_t(x) \, dt}_{\text{Kinetic Energy}} \quad \text{s.t.} \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

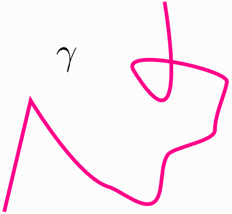
Extremal Points

$$R(\mu, v) := \int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 \, d\mu_t(x) \, dt \quad \text{s.t.} \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

Theorem [3]

Let $B = \{R \leq 1\}$. Then $\operatorname{Ext}(B)$ are measures $t \mapsto \mu_t$ supported on **Sobolev Curves**

$$t \mapsto \mu_t = \delta_{\gamma(t)} \,, \qquad \gamma \in H^1([0,1]; \mathbb{R}^2)$$



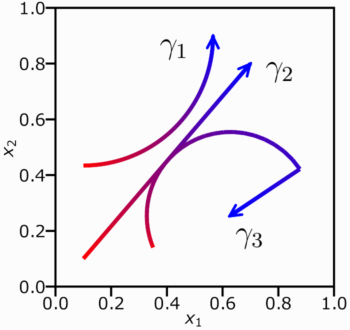
Proof Idea: Probabilistic Superposition Principle for measure solutions to

$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \qquad (= g_t \mu_t)$$

[3] Bredies, Carioni, **Fanzon**, Romero. **Bulletin London Mathematical Society** (2021)
[4] Bredies, Carioni, **Fanzon**. **Communications in PDEs** (2022)

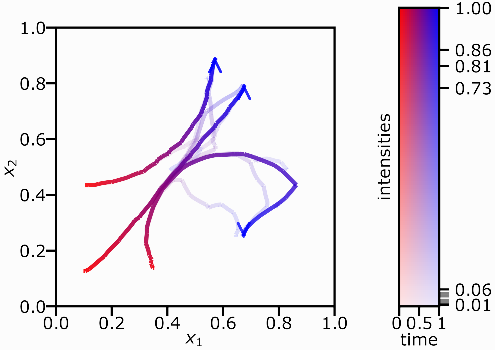
Reconstructions

Algorithm: **Generalized Frank-Wolfe** $\rightsquigarrow \quad t \mapsto \mu_t^k = \sum_{i=1}^{M} \lambda_i \, \delta_{\gamma_i(t)}$



Real Trajectories

$$\bar{\mu}_t = \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$



Reconstruction

MRI Data $\quad t \mapsto y_t$

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[5] Bredies, Carioni, Fanzon, Romero. **Found. of Computational Mathematics** (2022)

Conclusion

❶ Algorithm for computing **sparse** solutions to

$$\min_{u \in X} \; L(u) + R(u)$$

❷ **Fast** convergence when solution is **sparse**

❸ Application to **particles tracking**

Thank You!

References

Sparse optimization Algorithm

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2023)

Particles Tracking

[2] **Fanzon**, Bredies. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

[3] Bredies, Carioni, **Fanzon**, Romero. **Bulletin London Mathematical Society** (2021)

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[5] Bredies, Carioni, **Fanzon**, Romero. **Found. of Computational Mathematics** (2022)