

Sparse recovery in Inverse Problems

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18 December 2025

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Sparse recovery in Inverse Problems

based on joint works with

Kristian Bredies, Marcello Carioni, Francisco Romero, Daniel Walter

Outline

- ① Introduction: Inverse Problems & Sparsity
- ② Algorithm for sparse solutions recovery
- ③ Dynamic Inverse Problems
- ④ Application to Dynamic MRI



Supported by Austrian Science Fund (FWF) and
Christian Doppler Research Society (CDG) Project PIR27
“Mathematical methods for motion-aware medical imaging”



- 1 **Introduction to Inverse Problems & Sparsity**
- 2 Algorithm for sparse solution recovery
- 3 Dynamic Inverse Problems
- 4 Application to Dynamic MRI



What is an Inverse Problem?

Inverse Problems: Link between model parameters and data

Inverse Problem: Given data $f \in Y$, find parameters $u \in X$ such that

$$Ku = f$$

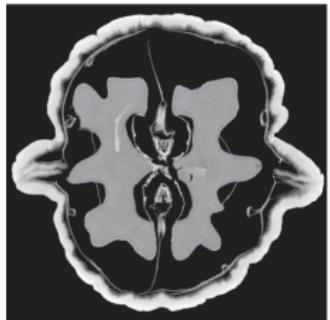
- ▶ f is data $\quad Y$ data space – Banach space or \mathbb{R}^n
- ▶ u are parameters $\quad X$ the parameters space – Same as above
- ▶ $K: X \rightarrow Y$ is Forward Operator
- ▶ K models the process to obtain the data from the parameters

Bredies, Lorenz. *Mathematical Image Processing*. Springer (2018)

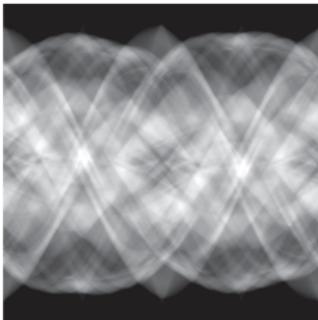
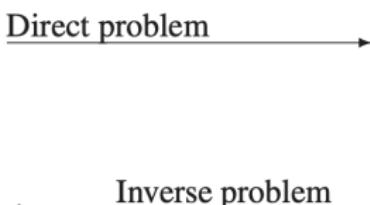
Mueller, Siltanen. *Linear and Nonlinear Inverse Problems with Practical Appl.* SIAM, 2012

Example: X-ray Imaging

Walnut



X-ray data (sinogram form)



Direct Problem: X-rays pass through walnut, detectors measure attenuation

Inverse problem: Given many X-ray measurements from different angles, reconstruct the walnut

Operator K = Radon transform

Example: Image Deblurring

Original Image



Blurred image



Direct problem →

← Inverse problem

Direct problem: Sharp image becomes blurred due to camera motion or focus issues

Inverse problem: Given the blurred image, recover the original sharp image

Operator $K = \text{convolution}$

Famous example: Hubble Space Telescope



- ▶ Hubble Space Telescope launched in 1990
- ▶ However images were blurred due to flawed lenses (Left)
- ▶ This issue was corrected through image processing (Right)

Link to [article](#) on NASA's website



III-posed Inverse Problems

Consider the inverse problem

$$Ku = f \quad (\text{P})$$

Problem (P) is **well-posed** if all three conditions hold:

- ① **Existence:** There exists at least one solution
- ② **Uniqueness:** There exists at most one solution
- ③ **Stability:** The solution depends continuously on the data, i.e., there exists a constant $C > 0$ such that

$$\|u - u'\|_X \leq C \|f - f'\|_Y \quad \text{where} \quad Ku = f, \quad Ku' = f'$$

Problem (P) is **ill-posed** if it is not well-posed



Measurements are noisy

Consider the inverse problem

$$Ku = f \quad (\text{P})$$

- ▶ **Ideal world:** Measurement comes from operator $\leadsto f = Ku$
- ▶ **Reality:** We can only observe noisy measurements

$$f^\varepsilon = Ku + \varepsilon, \quad \varepsilon \text{ random (unknown) noise}$$

Goal: To recover u from noisy measurement f^ε

Main difficulty: K^{-1} does not exist or is not continuous \leadsto **ill-posedness**



Variational Regularization

$$Ku = f \tag{P}$$

- (P) might not have solution. Find **approximate** solution by **least-squares**

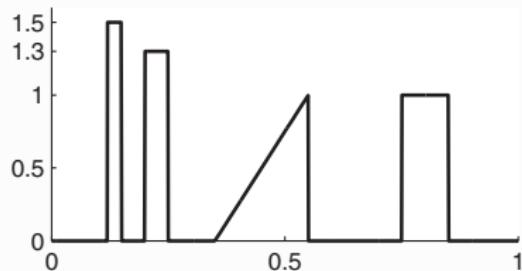
$$\min_{u \in X} \|Ku - f\|_Y^2 \tag{P'}$$

- **Problem:** Might still have **non-existence, non-uniqueness** and / or **instability**
(K is determined by the problem – Cannot make general assumptions on K)
- **Solution:** Replace (P) with the **regularized** least-squares problem

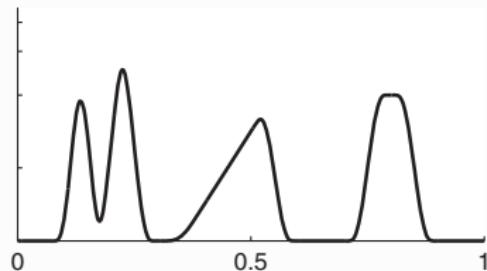
$$\min_{u \in X} \|Ku - f\|_X^2 + \alpha R(u), \quad R: X \rightarrow [0, +\infty], \quad \alpha > 0$$

- ➊ R makes the problem well-posed and stable – if chosen properly
- ➋ R favors certain solutions – the ones for which $R(u)$ is small

Example: 1D deconvolution



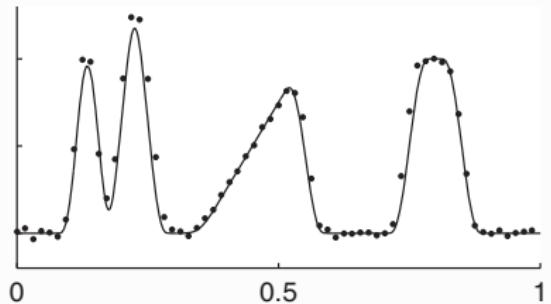
Original signal $\tilde{u}: [0, 1] \rightarrow \mathbb{R}$



Blurred signal $f = \psi \star \tilde{u}$

Goal: Recover \tilde{u} from noisy data f^ε

$$\psi \star u = f^\varepsilon$$



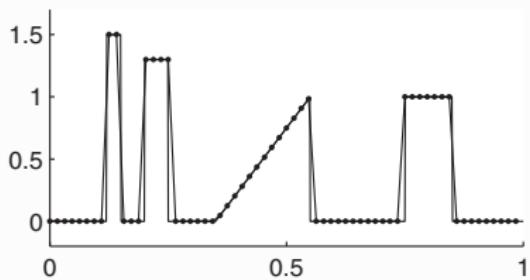
Blur + Noise $f^\varepsilon = f + \varepsilon$

Naive deconvolution

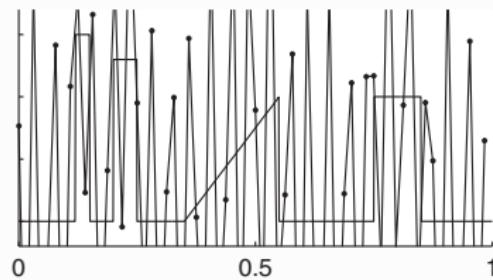
Solve the discrete 1D-deconvolution problem by least-squares:

$$u^\varepsilon \in \arg \min_{u \in L^2(0,1)} \|\psi * u - f^\varepsilon\|_{L^2(0,1)}^2$$

- ▶ Solution behaves well when noise $\varepsilon = 0$ but is terrible when $\varepsilon \neq 0$
 - ▶ Instability amplifies noise in the reconstruction
- ▶ Below the solid line represents the ground truth \tilde{u}
- ▶ We need regularizer which penalizes oscillations



u^0



u^ε

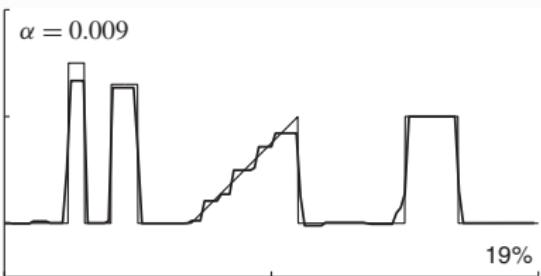
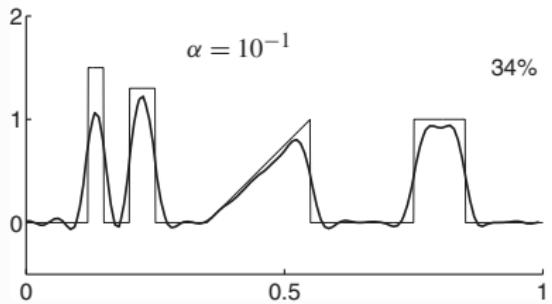
Two different regularizers

Regularize with L^2 norm

$$\min_{u \in L^2(0,1)} \|\psi * u - f^\varepsilon\|_{L^2(0,1)}^2 + \alpha \|u\|_{L^2(0,1)}^2$$

Reg. with Total Variation (BV semi-norm)

$$\min_{u \in L^1(0,1)} \|\psi * u - f^\varepsilon\|_{L^2(0,1)}^2 + \alpha \text{TV}(u)$$



- ▶ Notice the smoothing effect of L^2 regularization
- ▶ Smoothness not always desirable (e.g. if u is image with sharp edges – like here)
- ▶ Notice the **sparsifying effect** of TV (staircase effect)
- ▶ **Extremal points of regularizer describe features of sparse solutions**



Summary

Setting: X, Y Banach spaces, $K: X \rightarrow Y$ linear continuous operator

Inverse Problem: Given $f \in Y$, find $u \in X$ such that

$$Ku = f$$

Main difficulty: K^{-1} does not exist or is not continuous

Variational regularization: Given $f \in Y$, find $u \in X$ which solves

$$\min_{u \in X} \|Ku - f\|_Y^2 + \alpha R(u) \quad (\text{P})$$

Goals of the Talk:

- ▶ Algorithm to recover sparse solutions to (P)
- ▶ Framework for regularizing dynamic inverse problems



Outline

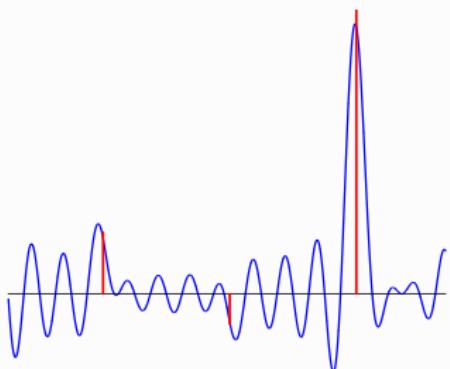
1 Introduction to Inverse Problems & Sparsity

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Motivation: Sparse peak recovery



- ▶ $\Omega \subset \mathbb{R}^d$ finite set, $\mathcal{M}(\Omega)$ Radon measures
- ▶ $\mathfrak{F}: \mathcal{M}(\Omega) \rightarrow \mathbb{C}^n$ undersampled Fourier transform
- ▶ **Sparsity assumption:** $\bar{u} = \sum_{i=1}^N \lambda_i \delta_{x_i}$
- ▶ Data $f = \mathfrak{F}\bar{u}$

Well-studied problem: Superresolution

$$\text{Solve} \quad \mathfrak{F}u = f \quad \text{on } \Omega$$

Radon-norm regularization: \mathfrak{F} not injective \rightsquigarrow need to regularize

$$\min_{u \in \mathcal{M}(\Omega)} \|\mathfrak{F}u - f\|_{L^2(\Omega)}^2 + \alpha \|u\|_{\mathcal{M}(\Omega)}$$

Goal: Recover sparse sol. $\bar{u} = \sum_{i=1}^N \lambda_i \delta_{x_i}$ \rightsquigarrow **(Fast) algorithms for general setting**

Candès, Fernandez-Granda. **CPAM** (2013) and many more



Minimization Problem in General Setting

$$\min_{u \in X} F(Ku) + R(u)$$

- **Parameters:** X separable Banach space with predual X_*
- **Data:** Y Hilbert space
- **Forward operator:** $K: X \rightarrow Y$ linear and weak*-to-strong continuous
- **F ~ Loss function:** Smooth + Strictly Convex

$$F: Y \rightarrow [0, \infty) \quad \left(F(y) = \|y - f\|_Y^2 \right)$$

- **R ~ Regulariser:** Convex + 1-homogeneous + Coercive

$$R: X \rightarrow [0, \infty] \quad (\text{Promotes Sparsity})$$

Theorem [1]: Direct method \implies Minimizer exists

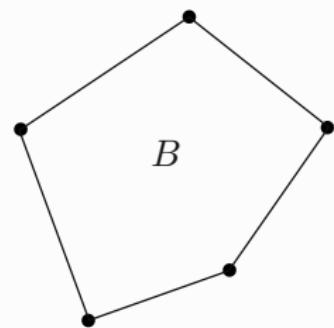
[1] Bredies, Carioni, **Fanzon**, Walter. Mathematical Programming (2024)



Sparsity

Unit Ball of regularizer R

$$B := \{u \in X : R(u) \leq 1\}$$



[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)



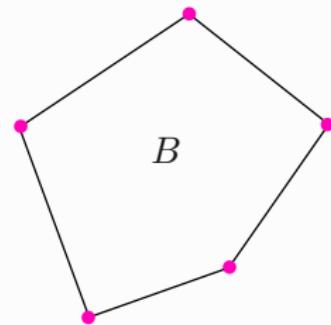
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$$B := \{u \in X : R(u) \leq 1\}$$

Extremal Points: $u \in B$ s.t.

$$\begin{cases} u = \alpha u_1 + (1 - \alpha) u_2 \\ \alpha \in (0, 1), \ u_1, u_2 \in B \end{cases} \implies u = u_1 = u_2$$



[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

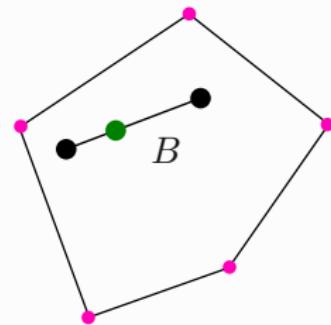
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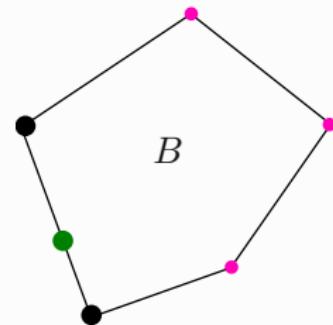
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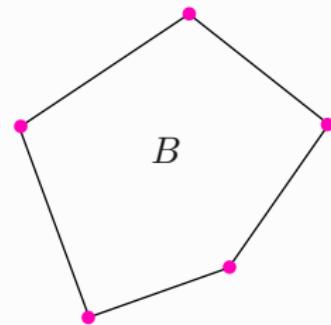
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[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

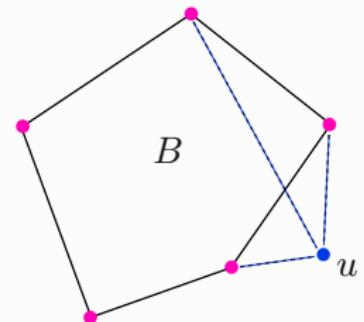
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$$\begin{cases} u = \alpha u_1 + (1 - \alpha) u_2 \\ \alpha \in (0, 1), \quad u_1, u_2 \in B \end{cases} \implies u = u_1 = u_2$$



Definition: $u \in X$ **sparse**

$$u = \sum_{i=1}^N \lambda_i u_i, \quad \lambda_i \geq 0, \quad u_i \in \text{Ext}(B)$$

Conic combination

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)



Main Task

Numerical **Algorithm** to compute

$$\bar{u} \in \arg \min_{u \in X} F(Ku) + R(u)$$

which is **sparse**

$$\bar{u} = \sum_{i=1}^N \lambda_i u_i, \quad \lambda_i \geq 0, \quad u_i \in \text{Ext}(B)$$

Existence of sparse solutions: Proven for $K: X \rightarrow \mathbb{R}^n$ [1,2]

Very general setting \rightsquigarrow **Important Examples:**

- ▶ Training of Machine Learning models $\rightsquigarrow X = \mathbb{R}^d$
- ▶ Microstructures in Materials $\rightsquigarrow X = \text{BV}(\mathbb{R}^d)$ Bounded Variation
- ▶ Recovery of sparse sources $\rightsquigarrow X = \mathcal{M}(\mathbb{R}^d)$ Radon Measures

[1] Bredies, Carioni. **Calc. Var. PDE** (2020)

[2] Boyer, Chambolle, De Castro, Duval, De Gournay, Weiss. **SIAM Optimization** (2019)

Example: Training of Machine Learning models

Parameters: vector $\Theta = (\theta_1, \dots, \theta_d) \in \mathbb{R}^d$

ML Model: Fit model to given data

$$\min_{\Theta \in \mathbb{R}^d} F(\theta) + \|\Theta\|_1$$

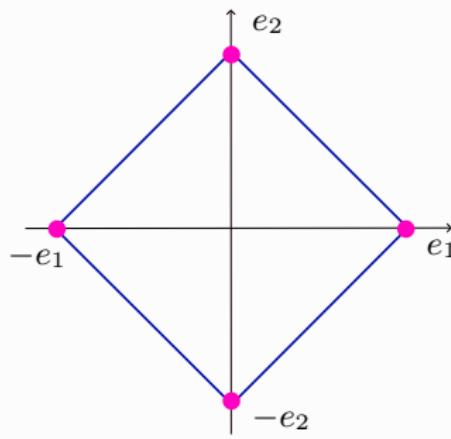
- ▶ Fidelity term F promotes data fit
- ▶ 1-norm promotes sparsity – e.g. solutions will have lots of zeros

$$\hat{\Theta} = (0, 0, \theta_i, 0, 0, \dots, 0, 0, \theta_d)$$

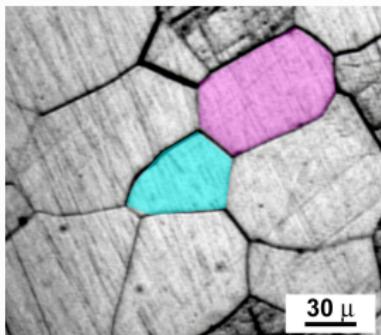
Banach space: $X = \mathbb{R}^d$

Regularizer: $\|x\|_1 := \sum_{i=1}^d |x_i|$

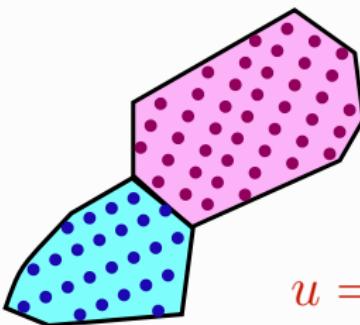
$$\text{Ext}(B) = \{\pm e_i\}_{i=1}^d$$



Example: Microstructures in Materials



Polycrystalline Metal



$$E_i \subset \mathbb{R}^2$$

$$A_i \in \mathbb{R}^{2 \times 2}$$

$$u = \sum_{i=1}^N A_i \chi_{E_i}$$

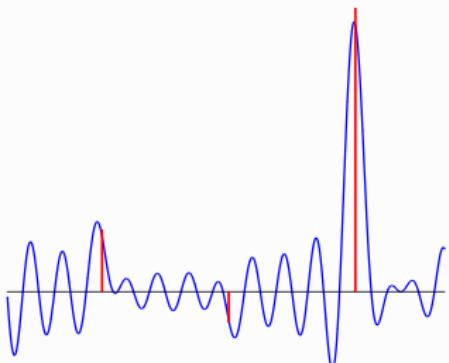
Banach space: $X = \text{BV}(\mathbb{R}^2)$ functions of bounded variation

Regularizer: $R(u) := \|Du\|_{\mathcal{M}}$, $\text{Ext}(B) = \{\chi_E : E \subset \mathbb{R}^2 \text{ simply connected}\}$

[2] Fanzon, Palombaro, Ponsiglione. **SIAM Journal on Mathematical Analysis** (2019)

Example: Recovery of sparse sources

Well-studied problem: Superresolution



- ▶ $\mathfrak{F}: \mathcal{M}(\Omega) \rightarrow \mathbb{C}^n$ undersampled Fourier transform
- ▶ **Sparsity assumption:** $\bar{u} = \sum_{i=1}^N \lambda_i \delta_{x_i}$
- ▶ Data $f = \mathfrak{F}\bar{u}$

$$\min_{u \in \mathcal{M}(\Omega)} \|\mathfrak{F}u - f\|_{L^2(\Omega)}^2 + \alpha \|u\|_{\mathcal{M}(\Omega)}$$

Banach space: $X = \mathcal{M}(\Omega)$ Radon measures

Regularizer: $R(u) := \|u\|_{\mathcal{M}}$ $\text{Ext}(B) = \{\pm \delta_x : x \in \Omega\}$

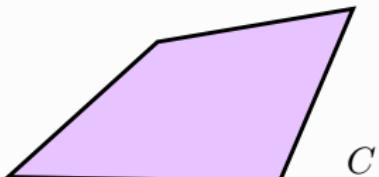
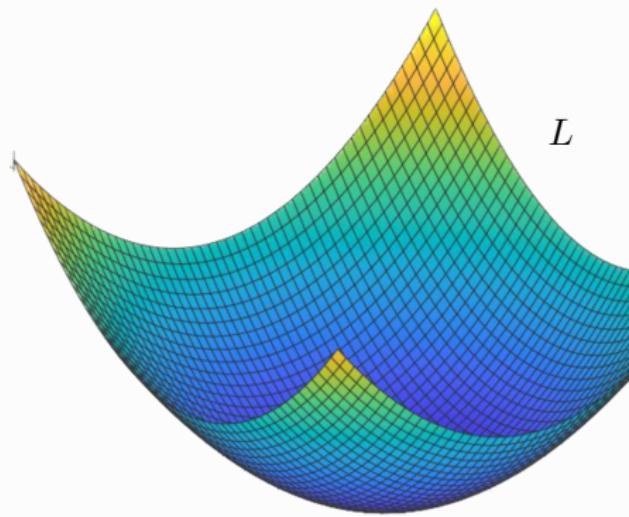
Starting Point: Classic Frank-Wolfe



Problem: Constrained minimization

$$\min_{x \in C} L(x)$$

- $L: \mathbb{R}^N \rightarrow \mathbb{R}$ regular convex
- $C \subset \mathbb{R}^N$ convex compact set



M. Jaggi. Proceedings of Machine Learning Research (2013)

Starting Point: Classic Frank-Wolfe



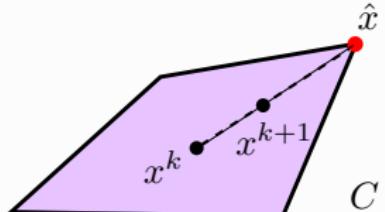
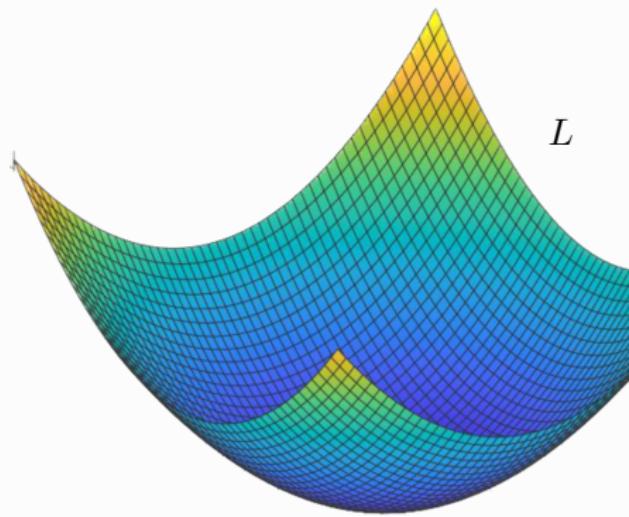
Frank-Wolfe Algorithm: Given $x^k \in C$

- ① **Insertion:** Solve linearized problem

$$\min_{x \in C} \langle \nabla L(x^k), x \rangle \quad \mapsto \quad \hat{x}$$

- ② **Convex update:** Set

$$x^{k+1} := x^k + \alpha(\hat{x} - x^k), \quad \alpha := \frac{2}{k+2}$$

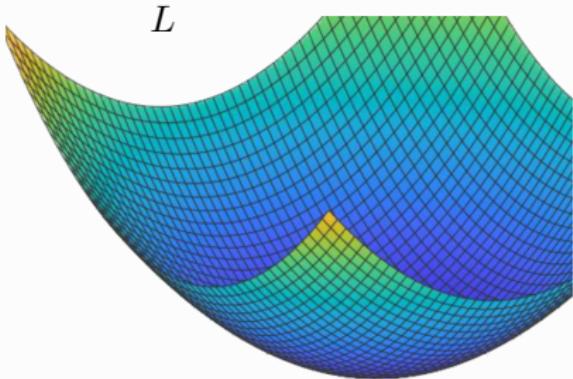


M. Jaggi. Proceedings of Machine Learning Research (2013)

Proposed Algorithm: Generalized Frank-Wolfe

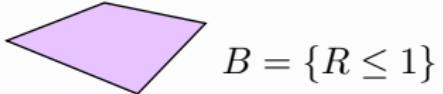


$$\min_{u \in X} L(u) + R(u), \quad L(u) = F(Ku)$$



Idea: Set $B = \{R \leq 1\}$. Consider

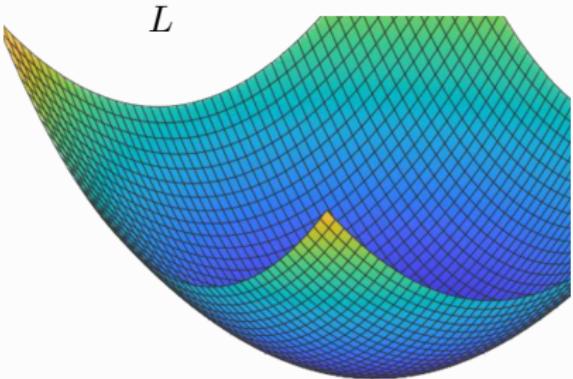
$$\min_{u \in X} L(u) + \chi_B(u) \iff \min_{u \in B} L(u)$$



[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

Proposed Algorithm: Generalized Frank-Wolfe

$$\min_{u \in X} L(u) + R(u), \quad L(u) = F(Ku)$$

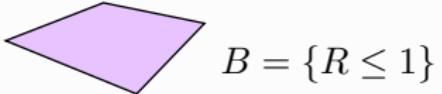


Idea: Set $B = \{R \leq 1\}$. Consider

$$\min_{u \in X} L(u) + \chi_B(u) \iff \min_{u \in B} L(u)$$

Descent Direction: Solve

$$\min_{v \in B} \langle \nabla L(u), v \rangle \mapsto \hat{v}$$



[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

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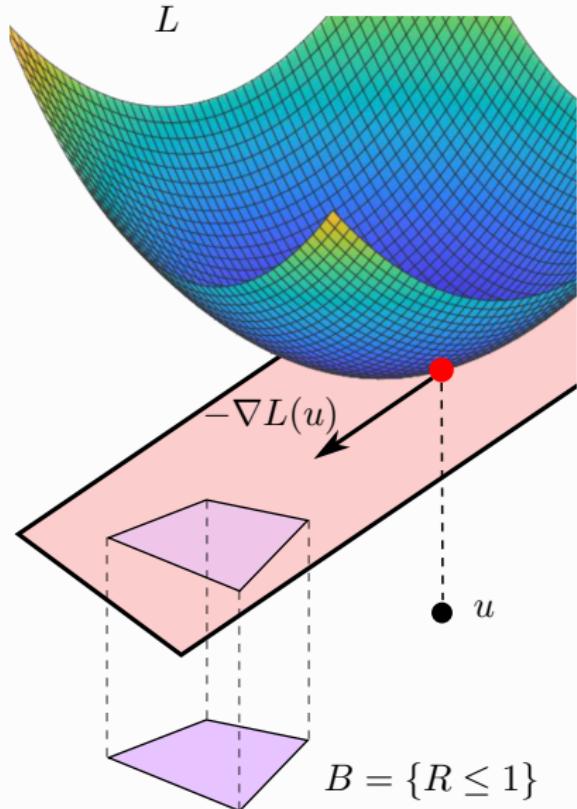
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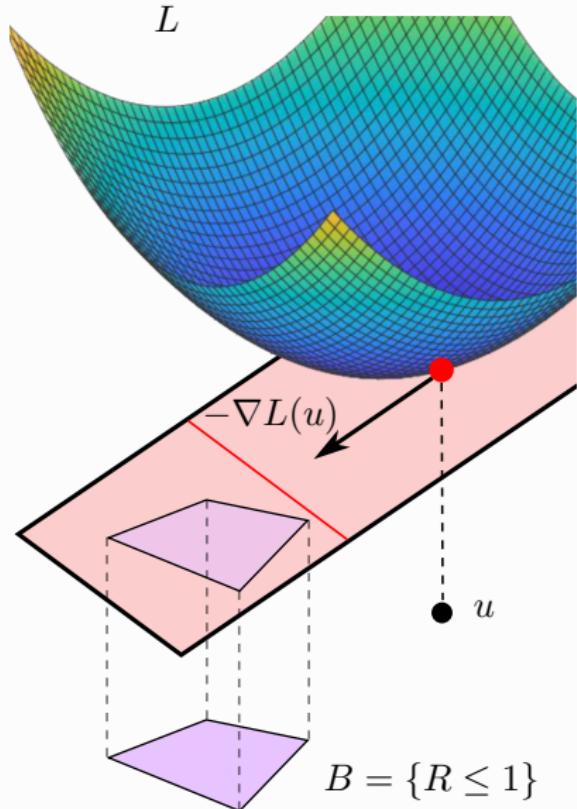
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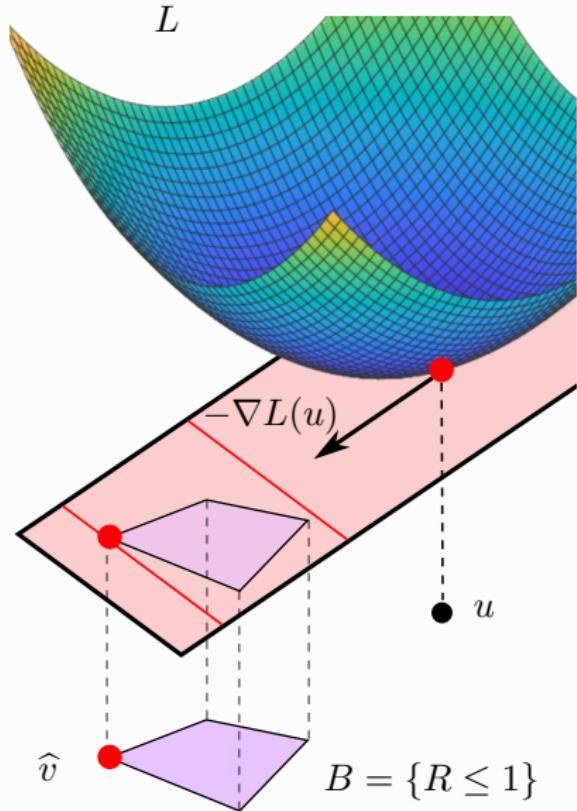
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Proposed Algorithm: Generalized Frank-Wolfe



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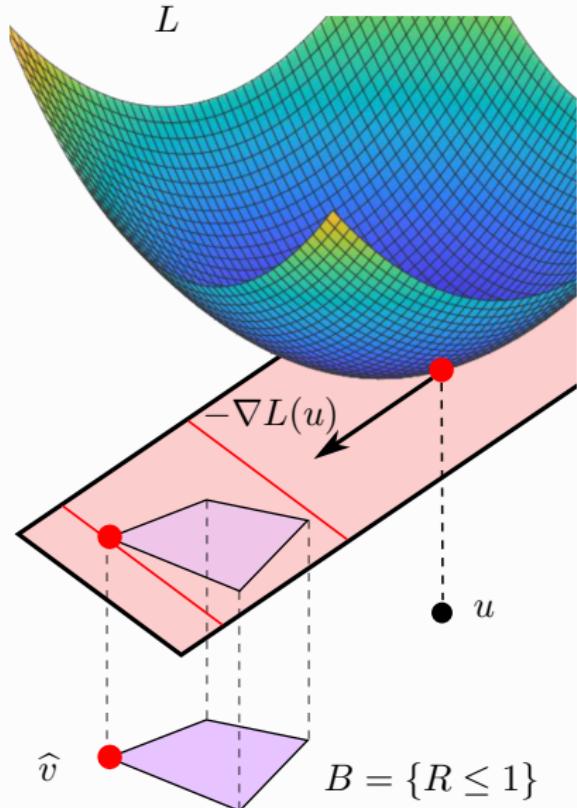
Idea: Set $B = \{R \leq 1\}$. Consider

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Descent Direction: Solve

$$\min_{v \in B} \langle \nabla L(u), v \rangle \mapsto \hat{v}$$

Lemma [1]. $\hat{v} \in \text{Ext}(B)$
(Krein-Milman)



[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

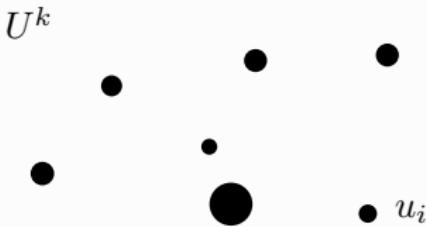


Proposed Algorithm: Generalized Frank-Wolfe

Sparse k -th iterate

$$U^k = \sum_{i=1}^n \lambda_i u_i$$

$$\lambda_i \geq 0, \quad u_i \in \text{Ext}(B)$$



[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)

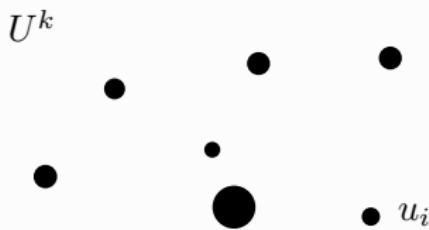


Proposed Algorithm: Generalized Frank-Wolfe

Sparse k -th iterate

$$U^k = \sum_{i=1}^n \lambda_i u_i$$

$$\lambda_i \geq 0, \quad u_i \in \text{Ext}(B)$$



① **Insertion Step:** Solve

$$\hat{v} \in \arg \max_{v \in \text{Ext}(B)} \langle \nabla L(U^k), v \rangle$$

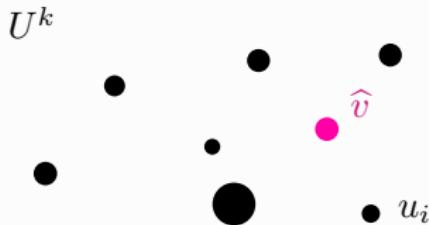
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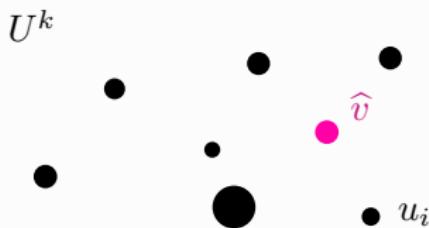
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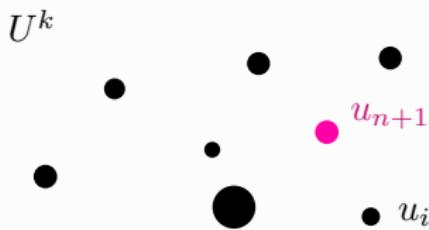
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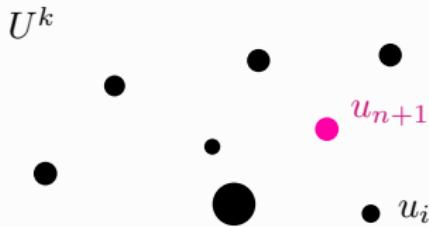


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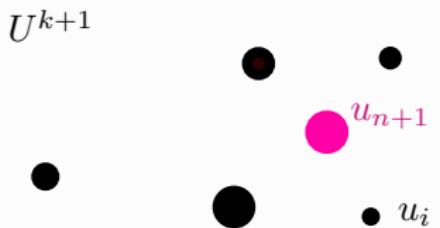
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- ① Non-linear problem (usually)
 - ▶ Non-linearity due to $\text{Ext}(B)$
 - ▶ Expensive and / or hard to solve
- ② Quadratic program – Easy to solve

① **Insertion Step:** Solve

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Convergence Analysis

Theorem [1]

U^k sparse iterate from the Generalized Frank-Wolfe Algorithm. Then

$$U^k \xrightarrow{*} \bar{u}, \quad \bar{u} \in \arg \min G, \quad G := L + R$$

General convergence rate is **sublinear**

(expected for gradient methods)

$$G(U^k) - \min G \lesssim \frac{1}{k}$$

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Highlight: \bar{u} **sparse** + **“Source Condition”** + **“Quadratic Growth”**

$$\implies \text{linear convergence:} \quad G(U^k) - \min G \lesssim \frac{1}{2^k}$$

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2024)



Key ingredient: Novel lifting argument

Classical Theorem (Choquet)

- X locally convex space, $K \subset X$ non-empty, convex, metrizable, compact
- For each $v \in K$, there is $\mu \in \mathcal{P}(X)$ concentrated on $\text{Ext}(K)$ with

$$T(v) = \int_X T d\mu \quad \forall T \in X^*$$

Theorem [1]. Let $\mathcal{B} = \overline{\text{Ext}(R \leq 1)}^*$. There exists $\mathcal{K}: \mathcal{M}(\mathcal{B}) \rightarrow Y$ linear bounded s.t. the two problems are equivalent

$$\min_{u \in X} F(Ku) + R(u) \quad \min_{\mu \in \mathcal{M}(\mathcal{B})} F(\mathcal{K}\mu) + \|\mu\|_{\mathcal{M}(\mathcal{B})}$$

Linear convergence can be obtained by carefully extending arguments in [2]

$$\min_{\mu \in \mathcal{M}(\mathbb{R}^d)} F(\tilde{K}\mu) + \|\mu\|_{\mathcal{M}(\mathbb{R}^d)}$$

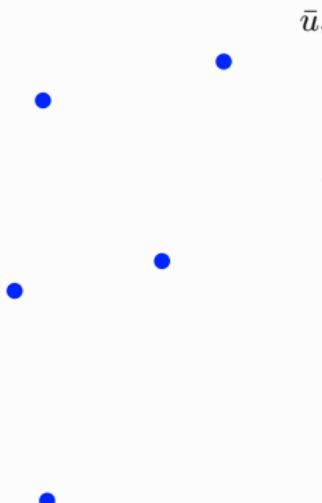
[1] Bredies, Carioni, F., Walter. **Math. Prog.** ('24)

[2] Pieper, Walter. **ESAIM: COCV** (2021)

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① (S) \exists sparse minimizer of $G := L + R$

$$\bar{u} = \sum_{i=1}^M \bar{\lambda}_i \bar{u}_i , \quad \bar{u}_i \in \text{Ext}(B)$$



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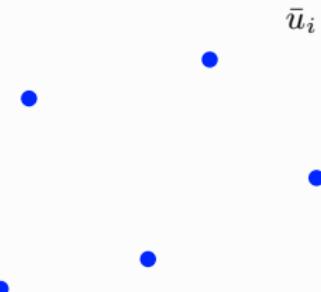
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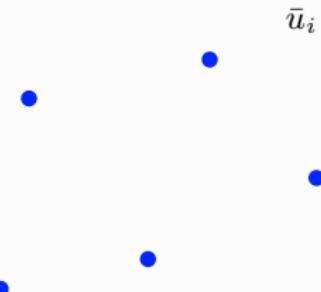
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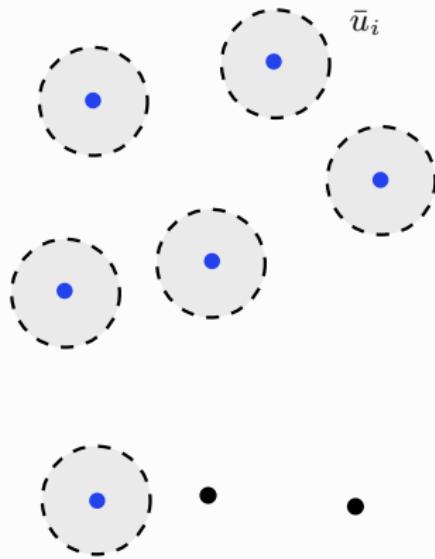
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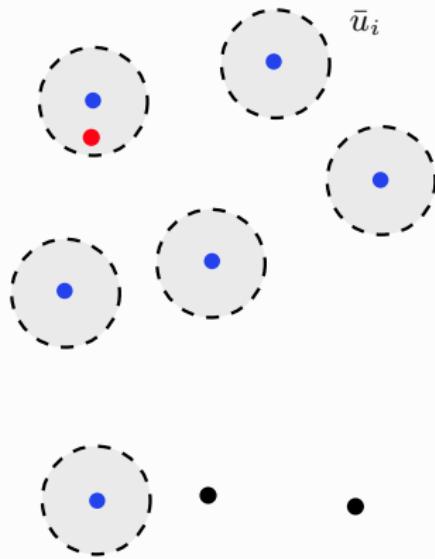
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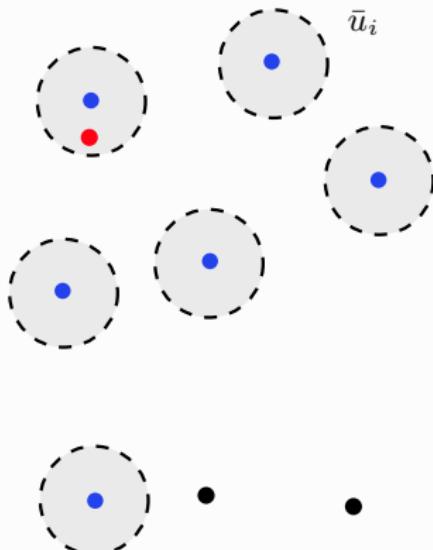
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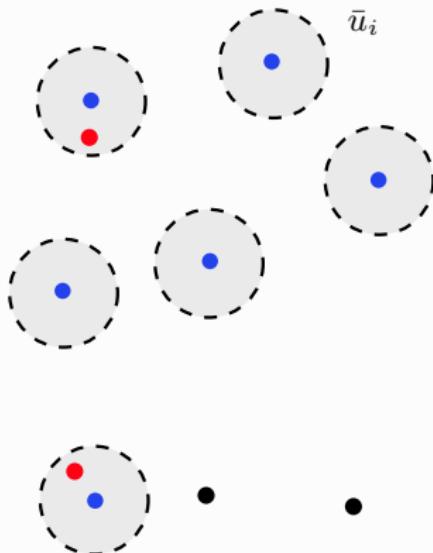
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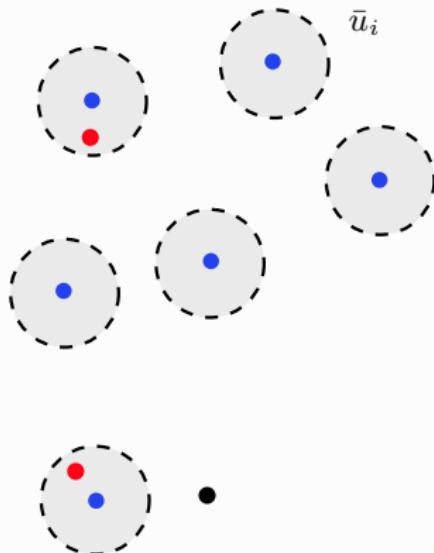
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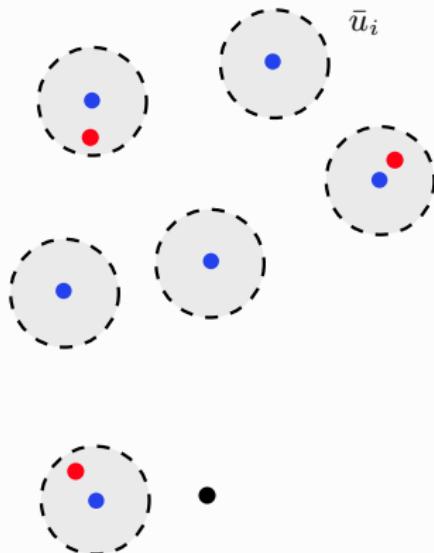
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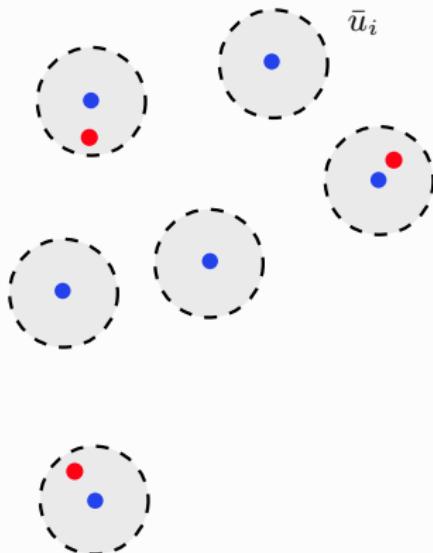
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U^{k+3}

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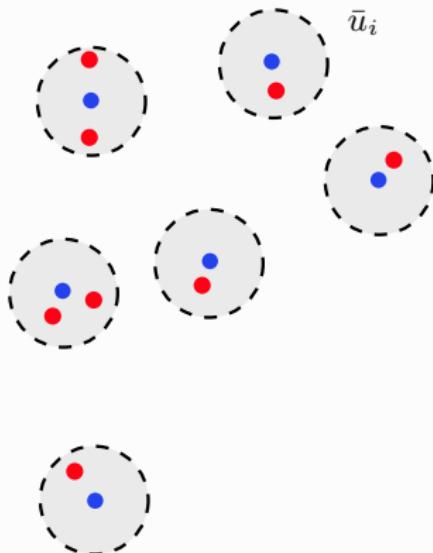
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Comments on Linear Convergence Assumptions

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- ▶ (S) + (SC) widely accepted

- ▶ only proven in few cases [2]
- ▶ can be verified numerically

- ▶ (QG) is novelty

- ▶ In applications we need to:

- ▶ Characterize $\text{Ext}(B)$
- ▶ Define suitable distance g
- ▶ Show (QG) under reasonable assumptions

- ▶ Applications: [1] Prove fast convergence of Gen. Frank-Wolfe

- ▶ Minimum effort prob.
- ▶ Trace-norm regularized prob.
- ▶ Sparse source identification (heat eqn)

[1] Bredies, Carioni, F., Walter. **Math. Prog.** ('24) [2] Candès, Fernandez-Granda. **CPAM** ('13)

Application: Sparse peak recovery [1, 2]

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2} \|Ku - f^\varepsilon\|_{L^2(\Omega)}^2 + \|u\|_{\mathcal{M}(\Omega)}$$

- Given: $\Omega \subset \mathbb{R}^d$ and $f^\varepsilon \in L^2(\Omega)$ noisy data
- Forw. operator: $K: \mathcal{M}(\Omega) \rightarrow L^2(\Omega)$

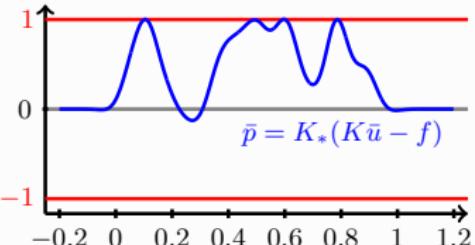
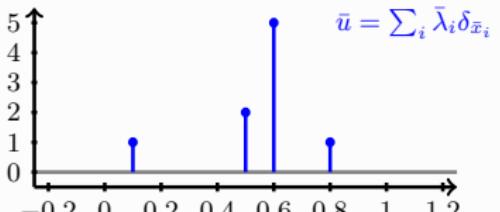
$$Ku = \psi \star u, \quad \psi = \text{Gauss. Kern.}$$

- Ext. points: $B = \{\|u\|_{\mathcal{M}(\Omega)} \leq 1\}$

$$\text{Ext}(B) = \{\pm \delta_x : x \in \Omega\}$$

- (S) \exists sparse solution:

$$\bar{u} = \sum_{i=1}^M \bar{\lambda}_i \delta_{\bar{x}_i}, \quad \bar{\lambda}_i > 0, \quad \bar{x}_i \in \Omega$$



- (SC) $\bar{p} = K_*(K\bar{u} - f^\varepsilon) \in C(\Omega)$

$$\arg \max_{v \in \text{Ext}(B)} \langle \bar{p}, v \rangle = \{\delta_{\bar{x}_1}, \dots, \delta_{\bar{x}_M}\}$$

$$\max_{v \in \text{Ext}(B)} \langle \bar{p}, v \rangle = \max_{x \in \Omega} \bar{p}(x) = 1$$

[1] Bredies, Carioni, F., Walter. **Math. Prog.** (2024) [2] Pieper, Walter. **ESAIM: COCV** (2021)

Application: Sparse peak recovery [1, 2]

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2} \|Ku - f^\varepsilon\|_{L^2(\Omega)}^2 + \|u\|_{\mathcal{M}(\Omega)}$$

- **(HP)** \bar{p} strictly concave at x_i

$$\text{sign}(\bar{p}(x_i)) \langle \xi, \nabla^2 \bar{p}(x_i) \xi \rangle \gtrsim |\xi|^2$$

- **Metric:** $g: \text{Ext}(B) \times \text{Ext}(B) \rightarrow [0, \infty)$

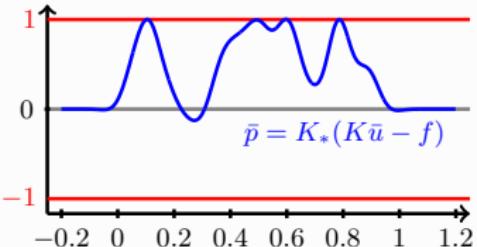
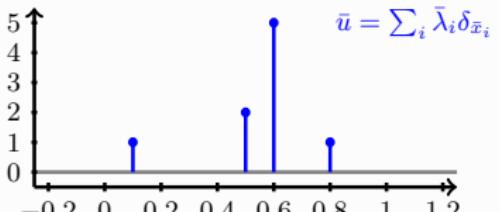
$$g(s_1 \delta_{x_1}, s_2 \delta_{x_2}) := |s_1 - s_2| + |x_1 - x_2|$$

- **(QG)** Quadratic growth of \bar{p} around \bar{u}_i

$$1 - \langle \bar{p}, u \rangle \gtrsim g(u, u_i)^2, \quad u \sim u_i$$

Theorem [1,2]: (HP) \implies (QG)

Gen. Frank-Wolfe converges linearly



- **(SC)** $\bar{p} = K_*(K\bar{u} - f^\varepsilon) \in C(\Omega)$

$$\arg \max_{v \in \text{Ext}(B)} \langle \bar{p}, v \rangle = \{\delta_{\bar{x}_1}, \dots, \delta_{\bar{x}_M}\}$$

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[1] Bredies, Carioni, F., Walter. **Math. Prog.** (2024)

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Application: Sparse peak recovery [1, 2]

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2} \|Ku - f^\varepsilon\|_{L^2(\Omega)}^2 + \|u\|_{\mathcal{M}(\Omega)}$$

Gen. Frank-Wolfe: $\text{Ext}(B) = \{\pm \delta_x : x \in \Omega\}$

Initialize: $u^0 = 0$

Iterate: Given $u^k = \sum_{i=1}^n \lambda_i \delta_{x_i}$

① Insertion Step: $p^k = K_*(Ku^k - f^\varepsilon)$

$$\max_{v \in \text{Ext}(B)} \langle p^k, v \rangle = \max_{x \in \Omega} p^k(x) \rightsquigarrow \hat{x}$$

② Fully-corrective Step: Solve

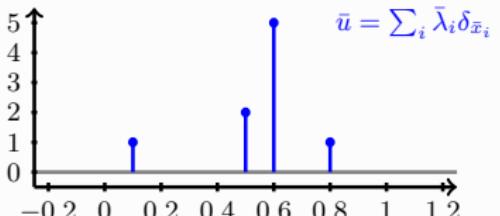
$$(\lambda_1^*, \dots, \lambda_{n+1}^*) \in \arg \min_{\lambda_i \geq 0} G(u^k + \lambda_{n+1} \delta_{\hat{x}}) \rightsquigarrow u^{k+1} := \left(\sum_{i=1}^n \lambda_i^* \delta_{x_i} \right) + \lambda_{n+1}^* \delta_{\hat{x}}$$

Stop: Based on Primal-Dual gap

[1] Bredies, Carioni, F., Walter. **Math. Prog.** (2024) [2] Pieper, Walter. **ESAIM: COCV** (2021)

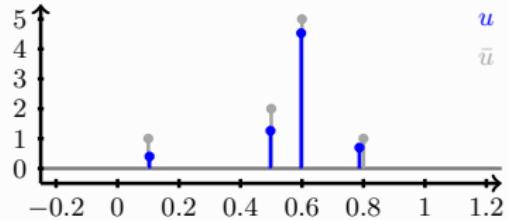
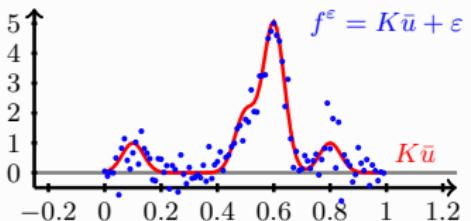
Application: Sparse peak recovery [1, 2]

$$\min_{u \in \mathcal{M}(\Omega)} \frac{1}{2} \|Ku - f^\varepsilon\|_{L^2(\Omega)}^2 + \|u\|_{\mathcal{M}(\Omega)}$$



Numerical experiment:

- ▶ Ground truth \bar{u} with 4 peaks
- ▶ Noiseless data $f = K\bar{u}$
- ▶ Noisy data $f^\varepsilon = K\bar{u} + \varepsilon$
- ▶ Run Gen. Frank-Wolfe $\rightsquigarrow u$
 - ▶ u is minimizer (by Thm)
 - ▶ u correctly has 4 peaks
 - ▶ Weights of peaks are shrunk (effect of regularization)
 - ▶ Empirical **linear convergence**



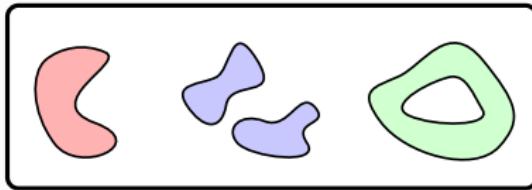
[1] Bredies, Carioni, F., Walter. **Math. Prog.** (2024)

[2] Pieper, Walter. **ESAIM: COCV** (2021)

An open problem

Total variation: $X = \text{BV}(\Omega)$, $\Omega \subset \mathbb{R}^d$

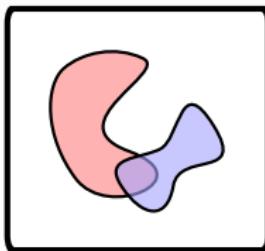
$$G(u) := F(Ku) + \|\nabla u\|_{\mathcal{M}} , \quad \text{Ext}(B) = \left\{ \frac{\chi_E}{\text{Per}(E)} : E \subset \Omega \text{ simple} \right\}$$



Assume: sparse solution $\hat{u} = \sum_{i=1}^M \lambda_i \chi_{E_i}$

Fast convergence: Which “metric”???

$$g(E_i, E_j) := |E_i \triangle E_j|^{-1} \quad ???$$



- **Connected problems:** Exact recovery, Noise Robustness



Outline

1 Introduction to Inverse Problems & Sparsity

2 Algorithm for sparse solution recovery

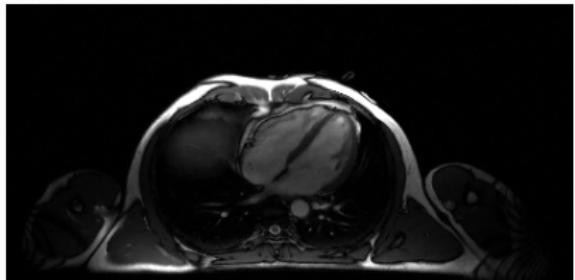
3 Dynamic Inverse Problems

4 Application to Dynamic MRI

Motivation: Magnetic Resonance Imaging (MRI)



MRI Scanner



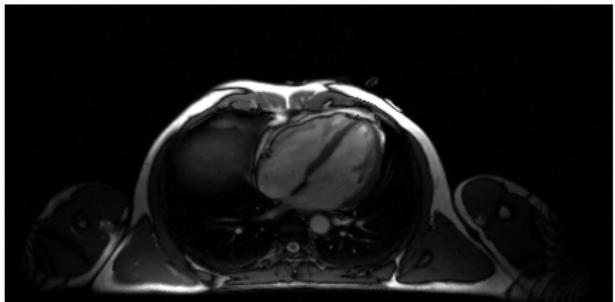
Human Heart

MRI: Medical imaging device, producing gray-scale images

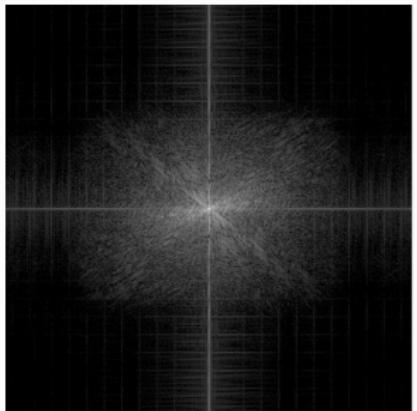
$$u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$$

Mathematical model for MRI

Image $u: \Omega \rightarrow \mathbb{R}$



$$\xrightarrow{\mathfrak{F}}$$



$$(\mathfrak{F}u) [\xi] = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x) e^{i\xi \cdot x} dx, \quad \xi \in \mathbb{R}^2$$

MRI machine measures Fourier coefficients



MRI Inverse Problem

Inverse Problem:

- ▶ Given MRI data y
- ▶ Find image $u: \Omega \rightarrow \mathbb{R}$ s.t.

$$\mathfrak{F}u = y$$



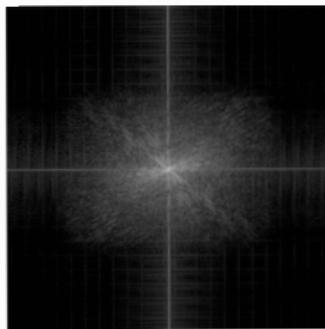
MRI Inverse Problem

Inverse Problem:

- ▶ Given MRI data y
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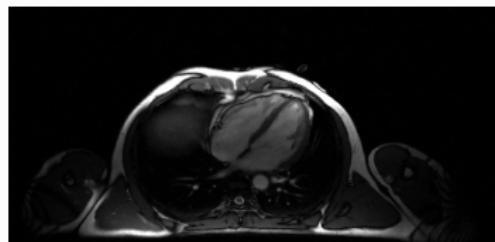
$$\mathfrak{F}u = y$$

Ideal World: Fourier transform is invertible. Unique solution is $u = \mathfrak{F}^{-1}y$



Data y

$$\xrightarrow{\mathfrak{F}^{-1}}$$



Reconstruction u



MRI Inverse Problem

Reality: Things are not straightforward

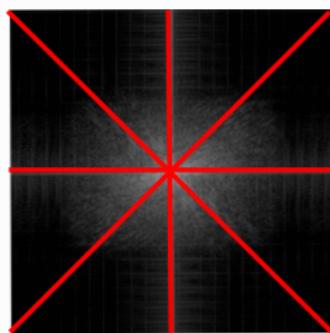
- ▶ Machine is slow in acquiring data \implies can only sample **limited data**
- ▶ Measurement process is inherently **noisy**

MRI Inverse Problem

Reality: Things are not straightforward

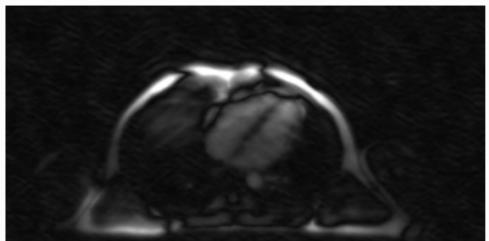
- ▶ Machine is slow in acquiring data \implies can only sample **limited data**
- ▶ Measurement process is inherently **noisy**

Issue: Plain inversion \leadsto poor reconstructions



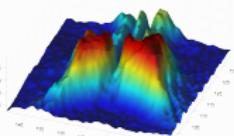
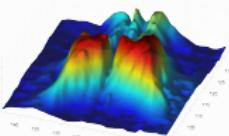
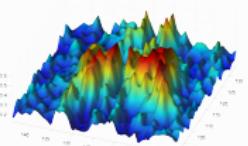
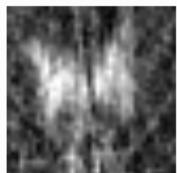
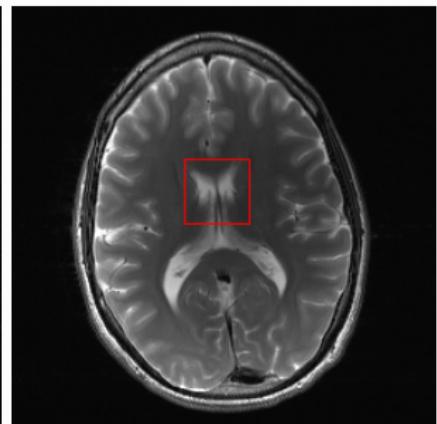
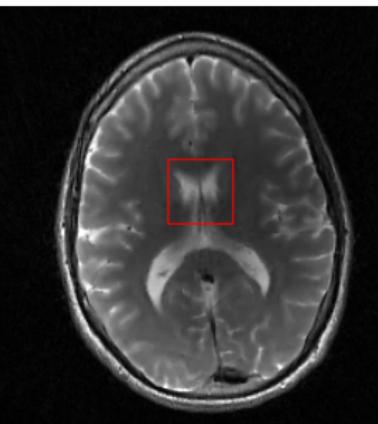
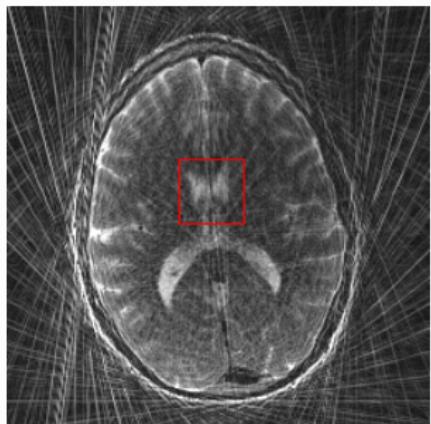
Undersampled noisy data y

$$\xrightarrow{\mathcal{F}^{-1}}$$



Reconstruction u

Benchmark Regularizer: TGV



Unders. Noisy Data
Least Squares

Unders. Noisy Data
Regularized (TGV)

Full Data
Least Squares

Bredies, Kunisch, Pock. **Total Generalized Variation. SIAM Imaging** (2010)



Motivation: Undersampled Dynamic MRI

Goal: Dynamic MRI \rightsquigarrow **Motion** is big challenge to accurate reconstructions

- ▶ High resolution imaging
- ▶ Imaging moving organs

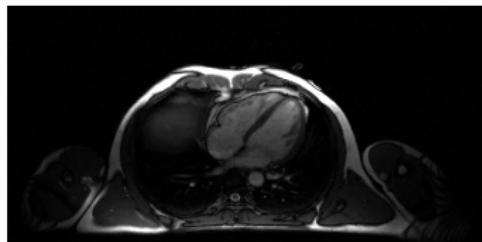
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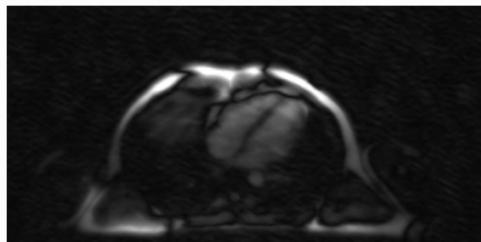
- ▶ High resolution imaging
- ▶ Imaging moving organs

Dynamic IP: Reconstruct movie u_t from undersampled data series y_t

$$\mathfrak{F}(u_t) = y_t \quad \text{for all } t \in [0, 1]$$



Fully sampled data



Undersampled data

Solution: We need regularization for **dynamic inverse problems**



Motivation: Undersampled Dynamic MRI

Goal: Dynamic MRI \rightsquigarrow **Motion** is big challenge to accurate reconstructions

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Fully sampled data

Undersampled data

Solution: We need regularization for **dynamic inverse problems**



Dynamic Inverse Problem

- ▶ **Images:** Radon Measures $\mu \in \mathcal{M}(\Omega)$ ($\Omega \subset \mathbb{R}^N$ bounded closed domain)
- ▶ **Data spaces:** Hilbert spaces H_t for $t \in [0, 1]$
- ▶ **Measurement Operators:** linear continuous maps

$$K_t: \mathcal{M}(\Omega) \rightarrow H_t$$

- ▶ **Data points:** Curve $t \mapsto y_t$ with $y_t \in H_t$

[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)



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Dynamic Inverse Problem: Find **curve** of measures $t \mapsto \mu_t \in \mathcal{M}(\Omega)$ s.t.

$$K_t \mu_t = y_t \quad \text{for all } t \in [0, 1] \tag{P}$$

Assumptions: weak time-regularity for $\{H_t\}_t$ and K_t^* (wk*-measurability)

[2] Bredies, Fanzon. *ESAIM: Mathematical Modelling and Numerical Analysis* (2020)



Dynamic Inverse Problem

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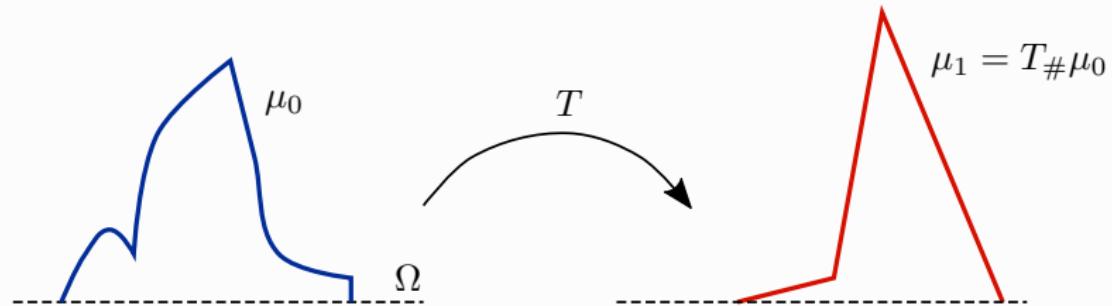
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Proposal: Regularize (P) with an **Optimal Transport Energy**

[2] Bredies, **Fanzon**. *ESAIM: Mathematical Modelling and Numerical Analysis* (2020)

Optimal Transport - Static Formulation

$\Omega \subset \mathbb{R}^d$ bounded domain, $\mu_0, \mu_1 \in \mathcal{P}(\Omega)$, $T: \Omega \rightarrow \Omega$ measurable displacement



Goal: move μ_0 to μ_1 in the cheapest way, with cost of moving mass from x to y

$$c(x, y) := |x - y|^2$$

Optimal Transport: a transport plan \hat{T} solving

$$\hat{T} \in \arg \min \left\{ \int_{\Omega} |T(x) - x|^2 d\mu_0(x) : T: \Omega \rightarrow \Omega, T_{\#}\mu_0 = \mu_1 \right\}$$

Optimal Transport - Dynamic Formulation

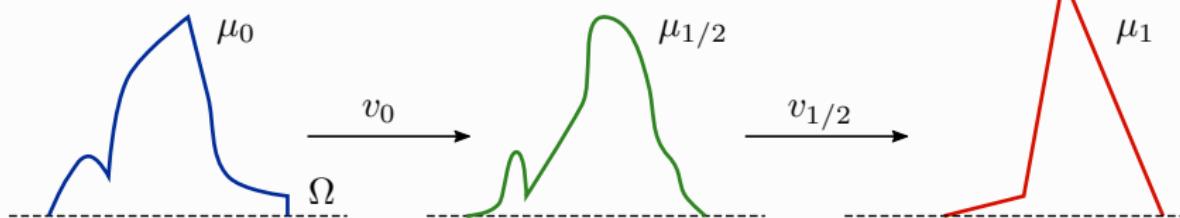
Idea: introduce a time variable $t \in [0, 1]$ and consider the density **evolution**

- ▶ time dependent probability measures

$$t \mapsto \mu_t \in \mathcal{P}(\Omega) \text{ for } t \in [0, 1]$$

- ▶ μ_t is advected by the velocity field

$$v_t(x): [0, 1] \times \Omega \rightarrow \mathbb{R}^d$$



Dynamic model: (μ_t, v_t) solves the **continuity equation** with initial conditions

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\mu_t v_t) = 0 \\ \text{Initial data } \mu_0, \text{ final data } \mu_1 \end{cases} \quad (\text{CE-IC})$$



Benamou-Brenier Formula

Theorem: Benamou-Brenier [1]

$$\min_{\substack{(\mu_t, v_t) \\ \text{solving (CE-IC)}}} \int_0^1 \int_{\Omega} |v_t(x)|^2 \mu_t(x) dx dt = \min_{\substack{T: \Omega \rightarrow \Omega \\ T_{\#} \mu_0 = \mu_1}} \int_{\Omega} |T(x) - x|^2 \mu_0(x) dx$$

Advantages of Dynamic Formulation:

- ① By introducing the momentum $m_t := \rho_t v_t$ we have

$$\int_0^1 \int_{\Omega} |v_t(x)|^2 \mu_t(x) dx dt = \int_0^1 \int_{\Omega} \frac{|m_t(x)|^2}{\mu_t(x)} dx dt$$

which is **convex** in (μ_t, m_t)

- ② The continuity equation becomes **linear**

$$\partial_t \mu_t + \operatorname{div} m_t = 0$$

- ③ We know the full trajectory μ_t and can recover the velocity field v_t from m_t

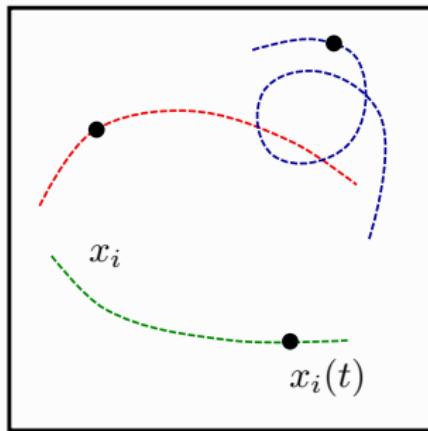
[1] Benamou, Brenier. **Numerische Mathematik** (2000)

Optimal Transport Regularization



Trajectories: Curve of measures

$$t \mapsto \mu_t \in \mathcal{M}(\Omega), \quad t \in [0, 1]$$



$$\mu_t = \sum_i \delta_{x_i(t)}$$

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Optimal Transport Regularization

Trajectories: Curve of measures

$$t \mapsto \mu_t \in \mathcal{M}(\Omega), \quad t \in [0, 1]$$

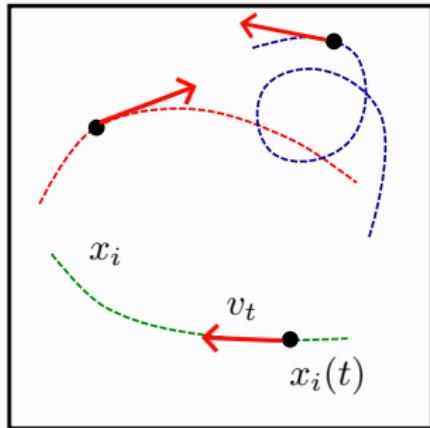
Assumptions:

- μ_t satisfies **Continuity Equation**

$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

for some velocity field (to find)

$$v_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



$$\mu_t = \sum_i \delta_{x_i(t)}$$

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Optimal Transport Regularization

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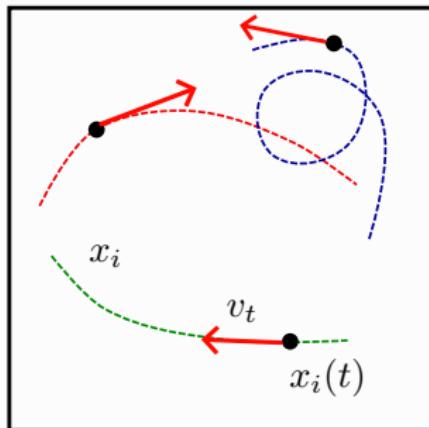
$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

for some velocity field (to find)

$$v_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

- Finite **Kinetic Energy**

$$\int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 d\mu_t(x) dt < \infty$$



$$\mu_t = \sum_i \delta_{x_i(t)}$$

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Optimal Transport Regularization



Minimization Problem: Given data $t \mapsto y_t \in H_t$

$$K_t \mu_t = y_t \quad \leadsto \quad \min_{\mu, v} L(\mu) + R(\mu, v)$$

[2] Bredies, Fanzon. *ESAIM: Mathematical Modelling and Numerical Analysis* (2020)



Optimal Transport Regularization

Minimization Problem: Given data $t \mapsto y_t \in H_t$

$$K_t \mu_t = y_t \quad \leadsto \quad \min_{\mu, v} L(\mu) + R(\mu, v)$$

- $L \rightsquigarrow$ **Loss Function:** Fits $t \mapsto \mu_t$ to data $t \mapsto y_t$ (Generalized Bochner spaces [2])

$$L(\mu) := \int_0^1 \|K_t \mu_t - y_t\|_{H_t}^2 dt$$

[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)



Optimal Transport Regularization

Minimization Problem: Given data $t \mapsto y_t \in H_t$

$$K_t \mu_t = y_t \quad \leadsto \quad \min_{\mu, v} L(\mu) + R(\mu, v)$$

- **$L \sim$ Loss Function:** Fits $t \mapsto \mu_t$ to data $t \mapsto y_t$ (Generalized Bochner spaces [2])

$$L(\mu) := \int_0^1 \|K_t \mu_t - y_t\|_{H_t}^2 dt$$

- **$R \sim$ Regularizer:** Connected to **Optimal Transport** (Benamou-Brenier formula)

$$R(\mu, v) := \underbrace{\int_0^1 \int_{\Omega} |v_t(x)|^2 d\mu_t(x) dt}_{\text{Kinetic Energy}} + \underbrace{\int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} dt}_{\text{Radon Norm}}$$

s.t. $\underbrace{\partial_t \mu_t + \operatorname{div}(v_t \mu_t)}_{\text{Continuity Equation}} = 0$

- **Theorem [2]:** R provides stable regularization for the dynamic inverse problem

[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)



Extremal Points

$$R(\mu, v) := \int_0^1 \int_{\Omega} |v_t(x)|^2 d\mu_t(x) dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} dt \quad \text{s.t. } \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

Superposition Principle [1,2]. $\Gamma := C([0, 1]; \Omega)$ with L^∞ norm. Equivalently:

- ▶ μ_t solves (CE) with $\int_0^1 \int_{\Omega} |v_t(x)|^2 d\mu_t(x) < \infty$
- ▶ $\exists \sigma \in P(\Gamma)$ concentrated on curves $\operatorname{AC}^2([0, 1]; \Omega)$ solutions to

$$\dot{\gamma}(t) = v_t(\gamma(t)) \text{ and s.t. } \int_{\Omega} \varphi d\mu_t = \int_{\Gamma} \varphi(\gamma(t)) d\sigma(\gamma), \quad \forall \varphi \in C(\Omega)$$

Theorem [2]: $\operatorname{Ext}(\{R \leq 1\})$ are measures $t \mapsto \mu_t$ supported on AC^2 curves

$$t \mapsto \mu_t = \delta_{\gamma(t)}, \quad \gamma \in \operatorname{AC}^2([0, 1]; \Omega)$$

Proof Idea: μ_t extr. for $R \iff \sigma$ extr. for $\|\cdot\|_{\mathcal{M}(\Gamma)}$ $\iff \sigma = \delta_{\gamma} \iff \mu_t = \delta_{\gamma(t)}$

[1] Bredies, Carioni, **Fanzon**, Romero. **Bull. LMS** (2021)

[2] Ambrosio. **Inv. Math.** (2004)



Non-homogeneous case

- Homogeneous continuity equation implies mass preservation
 $\mu_t(\Omega)$ is constant for all t
- Restrictive in applications – e.g. contrast agent in MRI
- Modify the regularizer to allow change of mass
- Based on **Unbalanced OT** – a.k.a. Hellinger-Kantorovich distance [2,3]

$$R(\mu, v) := \int_0^1 \int_{\Omega} |v_t(x)|^2 + |g_t(x)|^2 d\mu_t(x) dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} dt$$
$$\text{s.t. } \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = g_t \mu_t \quad (\text{CE})$$

Theorem [1]: R is stable regularizer for the dynamic inverse problem

$$K_t \mu_t = y_t$$

[1] Bredies, Fanzon. **ESAIM: M2AN** (2020)

[2] Chizat, Peyré, Schmitzer, Vialard. **Found. of Comp. Math.** (2018)

[3] Liero, Mielke, Savaré. **Inv. Math.** (2018)



Theorem: Superposition principle for (CE)

$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = g_t \mu_t \quad (\text{CE})$$

$$\mathcal{C}_\Omega = \{h\delta_\gamma \in \mathcal{M}(\Omega) : h \geq 0, \gamma \in \Omega\} \quad (\text{flat topology})$$

$$\mathcal{S}_\Omega = \{t \rightarrow \mu_t : \text{narrowly continuous}, \mu_t \in \mathcal{C}_\Omega\} \quad (\text{sup distance})$$

① Assume μ_t solves (CE) with

$$\int_0^1 \int_\Omega |v_t(x)|^2 + |g_t(x)|^2 d\mu_t(x) < \infty$$

$\exists \sigma \in \mathcal{M}^+(\mathcal{S}_\Omega)$ concentrated on $t \mapsto h(t)\delta_{\gamma(t)}$ such that

$$\dot{\gamma}(t) = v_t(\gamma(t)) \text{ a.e. in } \{h > 0\}, \quad \dot{h}(t) = g_t(\gamma(t))h(t) \text{ a.e. in } (0, 1) \quad (\text{ODE})$$

$$\int_\Omega \varphi(x) d\mu_t(x) = \int_{\mathcal{S}_\Omega} h(t)\varphi(\gamma(t)) d\sigma(\gamma, h) \quad \forall \varphi \in C(\Omega), t \in [0, 1] \quad (\text{R})$$

② Conversely, assume $\sigma \in \mathcal{M}^+(\mathcal{S}_\Omega)$ concentrated on solutions to (ODE) and s.t.

$$\int_0^1 \int_{\mathcal{S}_\Omega} h(t) (1 + |v_t(\gamma(t))| + |g_t(\gamma(t))|) d\sigma(\gamma, h) dt < \infty.$$

Then (R) defines $t \rightarrow \mu_t$ solution of (CE)



Extremal Points – Non-homogeneous case

$$R(\mu, v) := \int_0^1 \int_{\Omega} |v_t(x)|^2 + |g_t(x)|^2 d\mu_t(x) dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} dt$$
$$\text{s.t. } \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = g_t \mu_t \quad (\text{CE})$$

Theorem [1]

Let $B = \{R \leq 1\}$. Then $\operatorname{Ext}(B)$ are curves of measures of the form

$$t \mapsto \mu_t = h(t) \delta_{\gamma(t)}$$

- ▶ $h, \sqrt{h} \in \operatorname{AC}^2(0, 1)$, $\gamma \in C(\{h > 0\}; \Omega)$, $\sqrt{h}\gamma \in \operatorname{AC}^2([0, 1]; \mathbb{R}^d)$
- ▶ $\{h > 0\}$ is connected

Proof Idea: Novel Probabilistic Superposition Principle to (CE)

[1] Bredies, Carioni, **Fanzon**. Communications in PDEs (2022)



Numerical optimization

Dynamic IP: Given $t \mapsto y_t \in H_t$ find $t \mapsto \mu_t \in \mathcal{M}(\Omega)$ s.t.

$$K_t \mu_t = y_t \quad \text{for all } t \in [0, 1]$$

Optimal Transport Regularization: $\min_{\mu, v} L(\mu) + R(\mu, v)$

$$L = \int_0^1 \|K_t \mu_t - y_t\|_{H_t}^2 , \quad R = \int_0^1 \int_{\Omega} |v_t(x)|^2 dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} dt$$

s.t. $\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$

- **Given:** $\Omega \subset \mathbb{R}^d$ and $t \mapsto y_t \in H_t$ data
- **Forw. operator:** $K: \mathcal{M}(\Omega) \rightarrow H_t$
- **Ext. points:** $B = \{R \leq 1\}$

$$\operatorname{Ext}(B) = \left\{ t \mapsto \delta_{\gamma(t)} : \gamma \in H^1([0, 1]; \Omega) \right\}$$



Generalized Frank-Wolfe [1]

$$\min_{\mu, v} \int_0^1 \|K_t \mu_t - y_t\|_{H_t}^2 + \int_0^1 \int_{\Omega} |v_t(x)|^2 dt + \int_0^1 \|\mu_t\|_{\mathcal{M}(\Omega)} dt$$

s.t. $\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$

Algorithm: **Initialize:** $\mu^0 = 0$ **Iterate:** Given $\mu^k = \sum_{i=1}^n \lambda_i \delta_{\gamma_i}$

① Insertion Step: $p_t^k = K_*(K \mu_t^k - y_t)$ $p_t^k \in L^\infty([0, 1]; C(\Omega))$

$$\max_{w \in \operatorname{Ext}(B)} \langle p^k, w \rangle = \max_{\gamma \in H^1([0, 1]; \Omega)} \left(\int_0^1 |\dot{\gamma}(t)|^2 dt + 1 \right)^{-1} \int_0^1 p_t^k(\gamma(t)) dt \quad \leadsto \quad \hat{\gamma}$$

② Fully-corrective Step: Solve

$$\lambda_i^* \in \arg \min_{\lambda_i \geq 0} G \left(\mu^k + \lambda_{n+1} \delta_{\hat{\gamma}} \right) \quad \leadsto \quad \mu^{k+1} := \left(\sum_{i=1}^n \lambda_i^* \delta_{\gamma_i} \right) + \lambda_{n+1}^* \delta_{\hat{\gamma}}$$

[1] Bredies, Carioni, **Fanzon**, Romero. **Found. of Computational Mathematics** (2023)



Convergence Analysis

Theorem [1]

μ^k sparse iterate from the Generalized Frank-Wolfe Algorithm. Then

$$\mu^k \xrightarrow{*} \bar{\mu}, \quad \bar{\mu} \in \arg \min G, \quad G := L + R$$

General convergence rate is **sublinear**

(expected for gradient methods)

$$G(\mu^k) - \min G \lesssim \frac{1}{k}$$

Work in Progress: $\bar{\mu}$ **sparse** + “**Source Condition**” + “**Quadratic Growth**”

$$\implies \text{linear convergence:} \quad G(\mu^k) - \min G \lesssim \frac{1}{2^k}$$

[1] Bredies, Carioni, **Fanzon**, Romero. **Found. of Computational Mathematics** (2023)



Details and additional tweaks

- ▶ Solve the curve insertion problem

$$\hat{\gamma} \in \arg \max_{\gamma \in H^1([0,1];\Omega)} \left(\int_0^1 |\dot{\gamma}(t)|^2 dt + 1 \right)^{-1} \int_0^1 p_t^k(\gamma(t)) dt$$

via gradient descent with suitable stepsize rule

Theorem [1]

Under suitable regularity assumptions, the gradient descent procedure converges subsequentially to stationary points and strongly in $\text{AC}^2([0, 1]; \Omega)$.

- ▶ Multiple starts with suitable initial guess (crossovers, random curves, etc.) to increase chance to obtain global minimizer
- ▶ Multiple insertion \leadsto insert all obtained stationary points

[1] Bredies, Carioni, **Fanzon**, Romero. **Found. of Computational Mathematics** (2023)



Outline

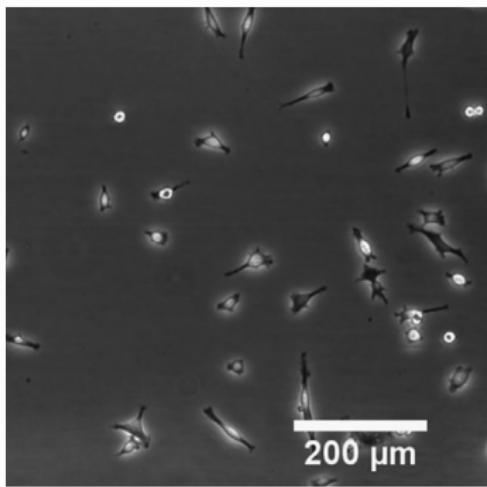
- 1 Introduction to Inverse Problems & Sparsity
- 2 Algorithm for sparse solution recovery
- 3 Dynamic Inverse Problems
- 4 Application to Dynamic MRI

Motivation: Particle Tracking

- ▶ Stars from ground-based telescope

Imaging Method \Rightarrow

- ▶ Microbubbles in blood vessels
- ▶ Cell migration



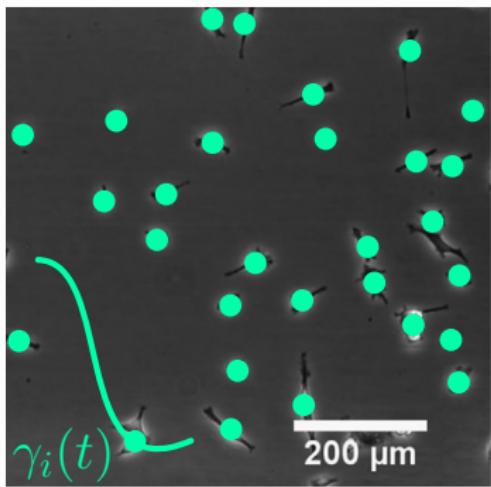
Microscopy image of cells

Image from: Yang, Venkataraman, Styles, et al. **Journal of Biomechanics** (2016)

Motivation: Particle Tracking

Imaging Method \implies

- ▶ Stars from ground-based telescope
- ▶ Microbubbles in blood vessels
- ▶ Cell migration



Microscopy image of cells

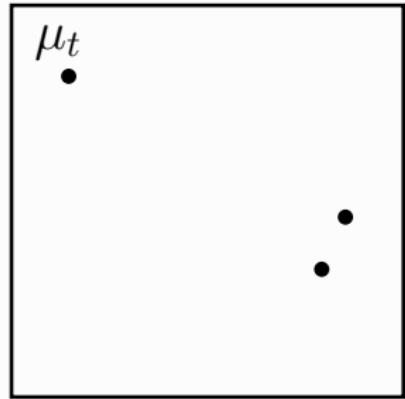
$$\mu_t = \sum_{i=1}^M \delta_{\gamma_i(t)}$$

Image from: Yang, Venkataraman, Styles, et al. **Journal of Biomechanics** (2016)

Application: Peak tracking for Dynamic MRI



y_t



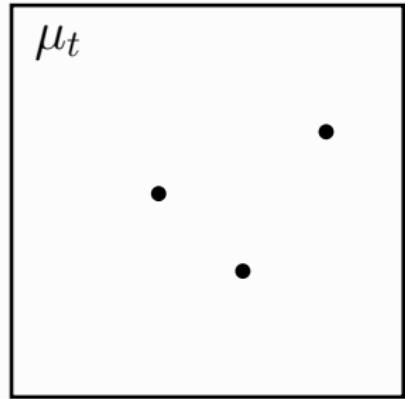
$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

Application: Peak tracking for Dynamic MRI



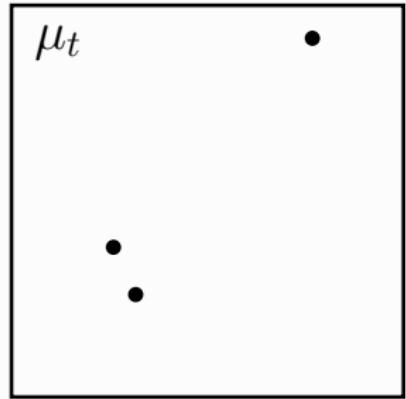
y_t



$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

Application: Peak tracking for Dynamic MRI



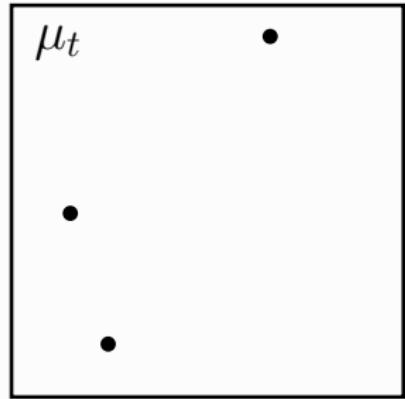
$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

Application: Peak tracking for Dynamic MRI



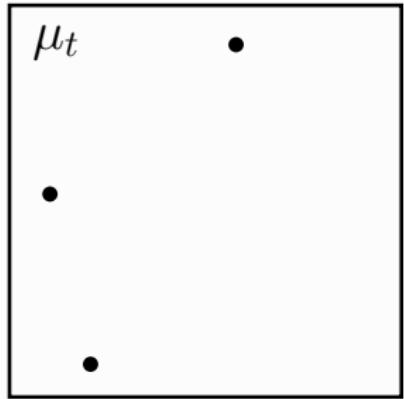
y_t



$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

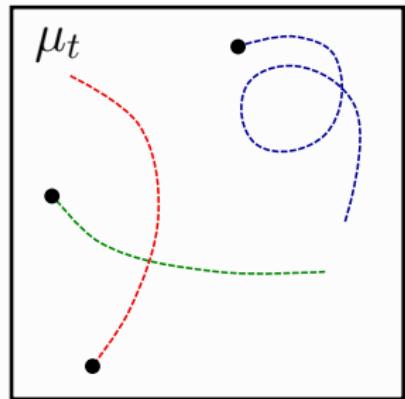
Application: Peak tracking for Dynamic MRI



$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

Application: Peak tracking for Dynamic MRI



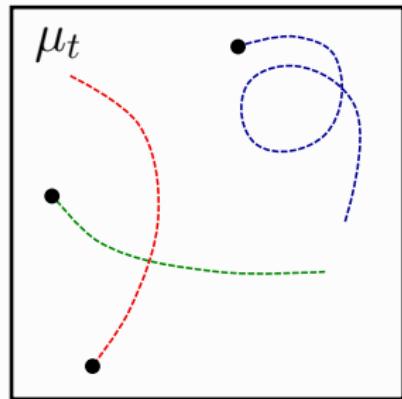
$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image $\mu_t \implies$ Interpolate Trajectories

Application: Peak tracking for Dynamic MRI



y_t



$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

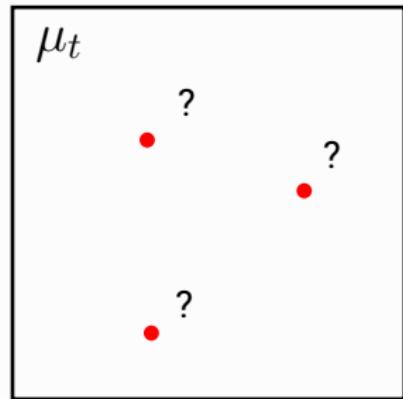
Frame-by-Frame: MRI Data $y_t \sim$ Image $\mu_t \implies$ Interpolate Trajectories

Issue: Motion \implies Low Scan Time \implies **Low Data** y_t

Application: Peak tracking for Dynamic MRI



y_t

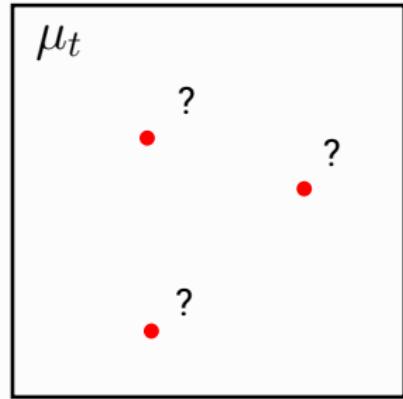


$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \sim$ Image $\mu_t \implies$ Interpolate Trajectories

Issue: Motion \implies Low Scan Time \implies **Low Data** $y_t \sim$ **Particles?**

Application: Peak tracking for Dynamic MRI



$$\mu_t = \sum_{i=1}^3 \delta_{\gamma_i(t)}$$

Frame-by-Frame: MRI Data $y_t \sim$ Image $\mu_t \implies$ Interpolate Trajectories

Issue: Motion \implies Low Scan Time \implies **Low Data** $y_t \sim$ **Particles?**

Global-in-Time: Full Time-Series $t \mapsto y_t \sim$ Trajectories $t \mapsto \mu_t$

The Dynamic Undersampled MRI problem



Dynamic IP MRI: Given $t \mapsto y_t \in \mathbb{C}^{M_t}$ find $t \mapsto \mu_t \in \mathcal{M}(\Omega)$ s.t.

$$K_t \mu_t = y_t \quad \text{for all } t \in [0, 1]$$

Fourier Transform: For $\mu \in \mathcal{M}(\Omega)$

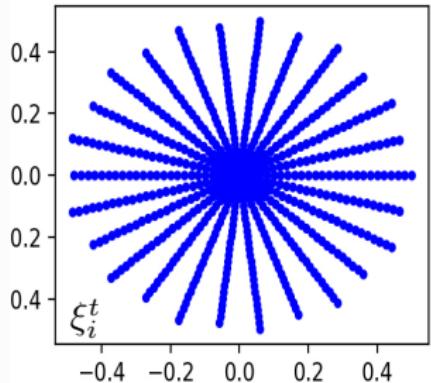
$$\hat{\mu}: \mathbb{C} \rightarrow \mathbb{C}, \quad \hat{\mu}[\xi] := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\xi \cdot x} d\mu(x)$$

Sampling Frequencies: M_t points

$$\xi_1^t, \dots, \xi_{M_t}^t \in \mathbb{C}$$

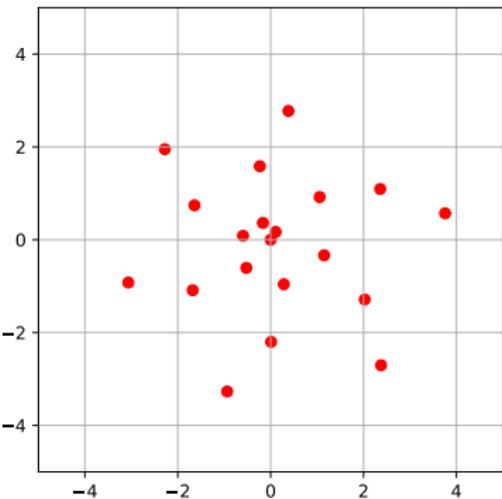
Forward operators: linear cont. $K_t: \mathcal{M}(\Omega) \rightarrow \mathbb{C}^{M_t}$

$$K_t \mu := (\hat{\mu}[\xi_1^t], \dots, \hat{\mu}[\xi_{M_t}^t])$$



Example: Radial Sampling

Experiment 1: Spiral sampling (static)



Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Sampling Freq: ξ_1, \dots, ξ_{20}

Data: Defined by

$$y_t := K\bar{\mu}_t + 20\% \text{ Noise}$$



Experiment 1: Spiral sampling (static)

Algorithm: Generalized Frank-Wolfe $\leadsto t \mapsto \mu_t^k = \sum_{i=1}^M \lambda_i \delta_{\gamma_i(t)}$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Reconstruction: from data

$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

(Thresholded at 0.05)



Experiment 1: Spiral sampling (static)

Algorithm: Generalized Frank-Wolfe $\leadsto t \mapsto \mu_t^k = \sum_{i=1}^M \lambda_i \delta_{\gamma_i(t)}$

Ground truth: Curve of measures

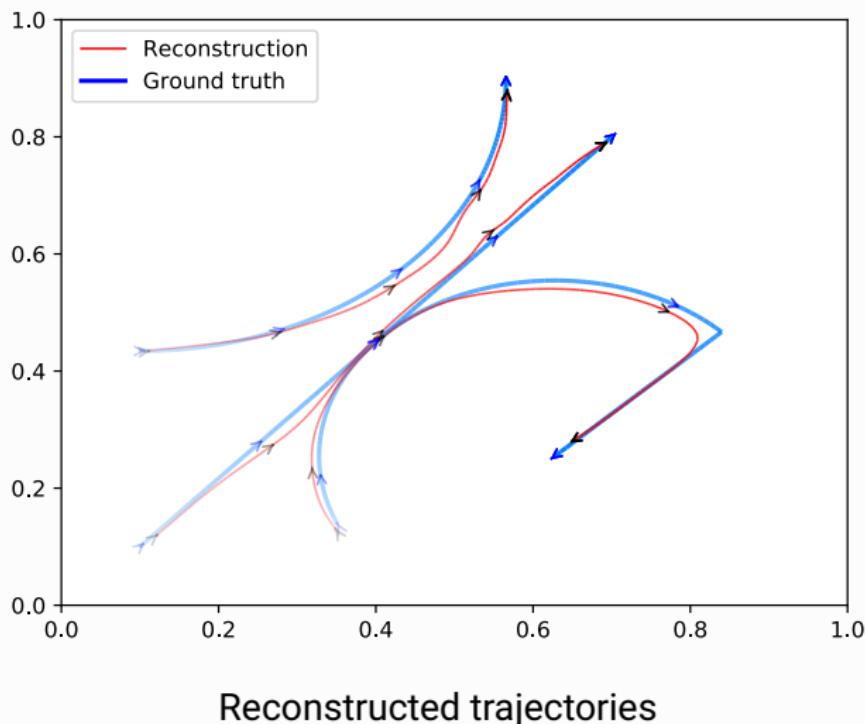
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Reconstruction: from data

$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

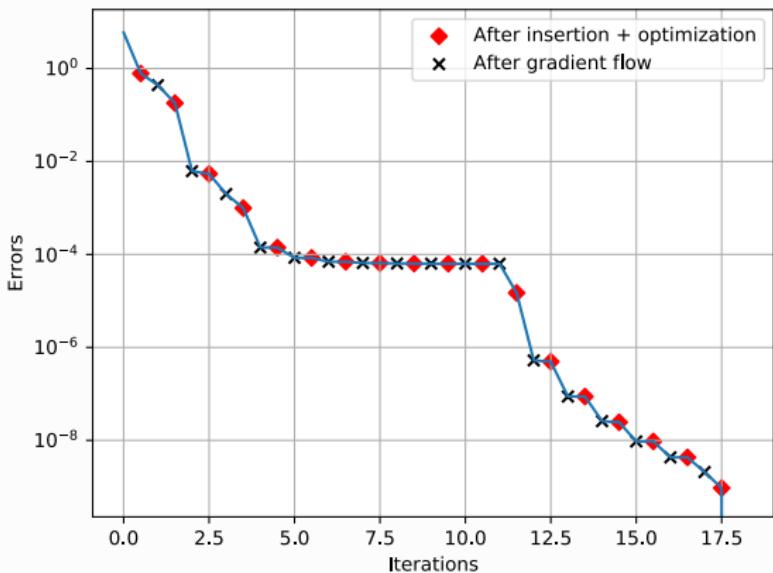
(No Thresholding)

Experiment 1: Spiral sampling (static)



Reconstructed trajectories

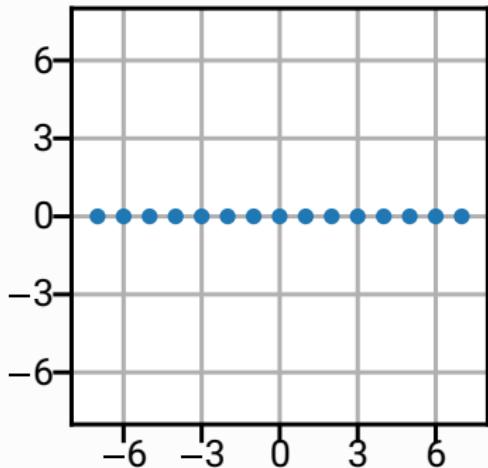
Experiment 1: Spiral sampling (static)



Convergence plot: exhibits linear rate

$$\text{Error} = G(\mu^k) - G(\mu^{k+1})$$

Experiment 2: Dynamic sampling on lines



$$t = 0$$

Ground truth: Curve of measures

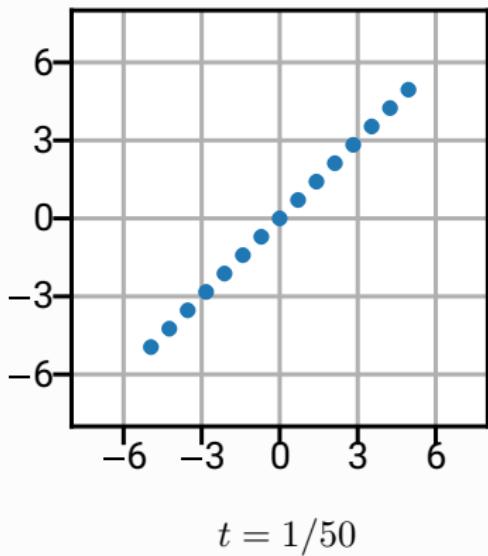
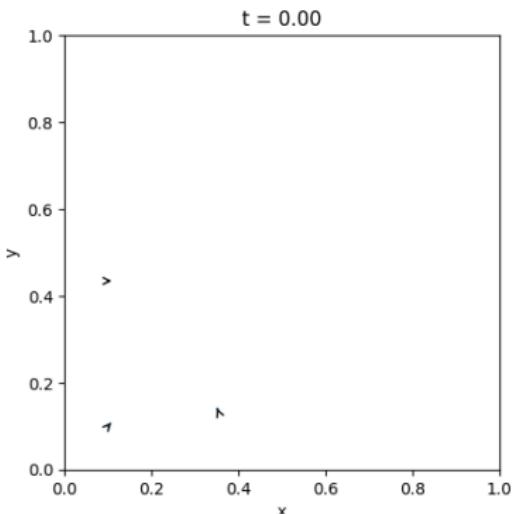
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines



Ground truth: Curve of measures

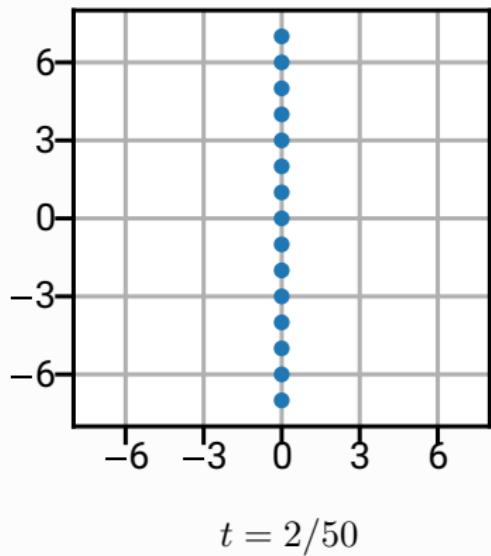
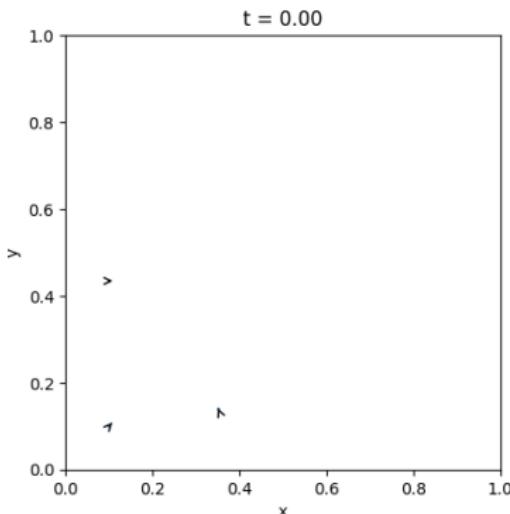
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Experiment 2: Dynamic sampling on lines



Ground truth: Curve of measures

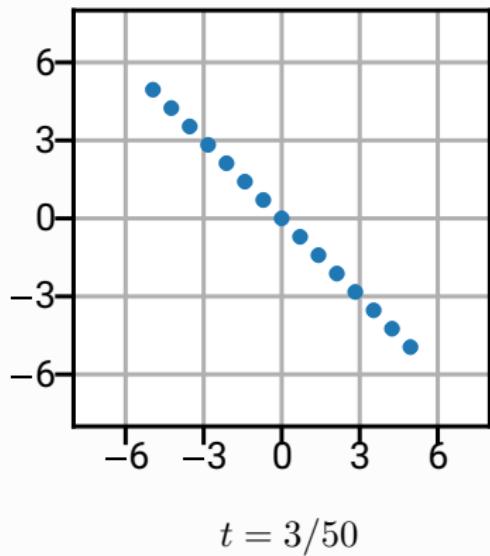
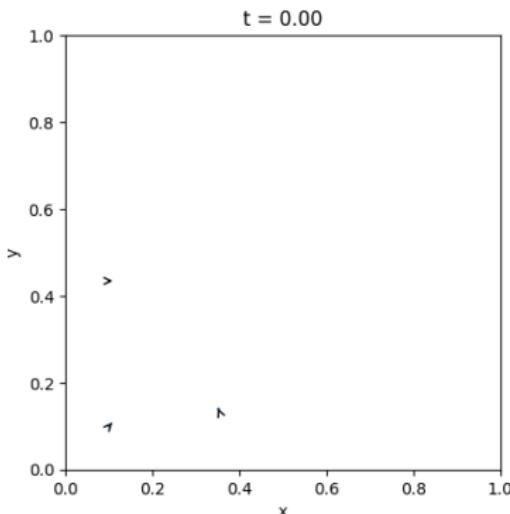
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Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines



Ground truth: Curve of measures

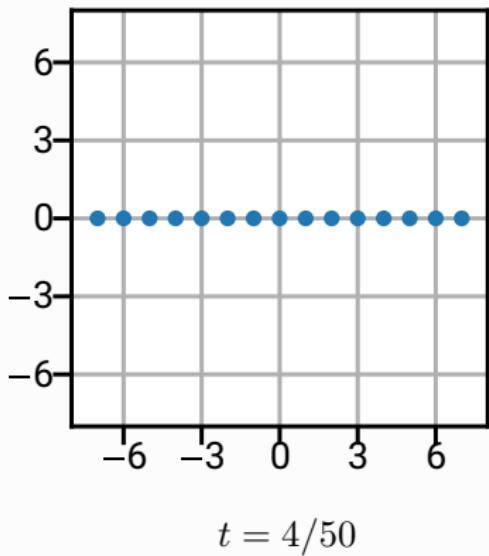
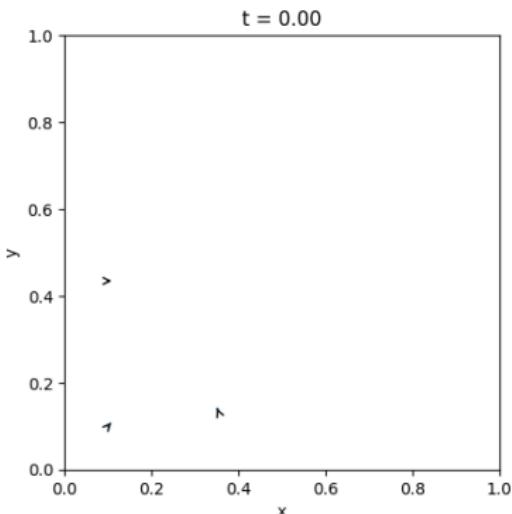
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Data: Defined by

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Experiment 2: Dynamic sampling on lines



Ground truth: Curve of measures

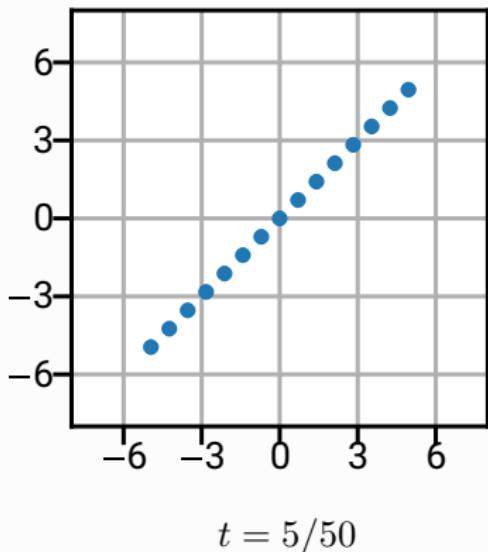
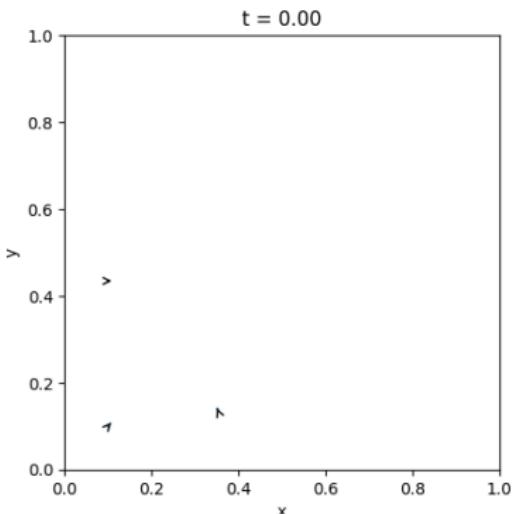
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines



Ground truth: Curve of measures

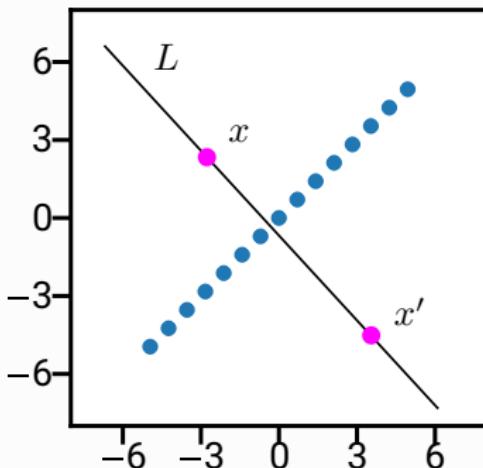
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Sampling Freq: $\xi_1^t, \dots, \xi_{15}^t$

Data: Defined by

$$y_t := K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Experiment 2: Dynamic sampling on lines



- L line orthogonal to the line of sampling frequencies at time t

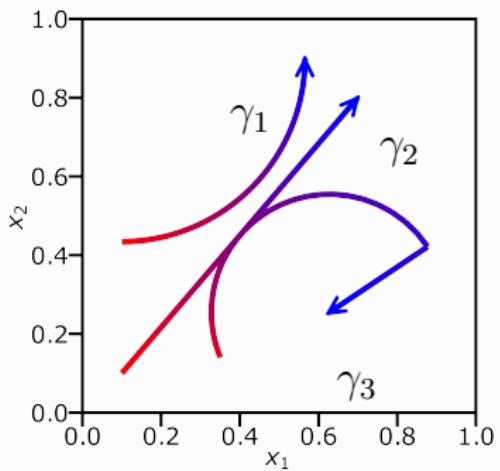
$$x, x' \in L \quad \Rightarrow \quad K_t \delta_x = K_t \delta_{x'}$$

- Impossible to locate source along L at time t
- Only way to locate source is to enforce time regularity
- Example showcases need for time regularization

Experiment 2: Dynamic sampling on lines

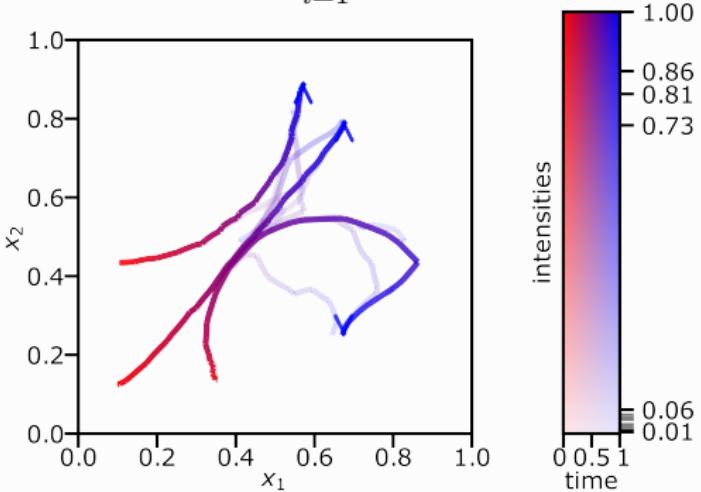
Algorithm: Generalized Frank-Wolfe

$$\sim t \mapsto \mu_t^k = \sum_{i=1}^M \lambda_i \delta_{\gamma_i(t)}$$



Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

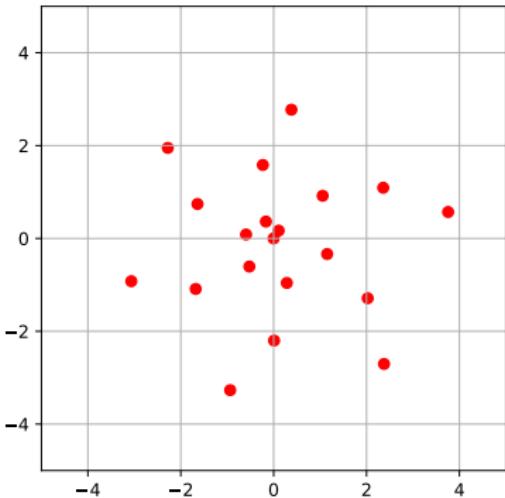


Reconstruction: from data

$$y_t = K_t \bar{\mu}_t + 20\% \text{ Noise}$$

Remarkable reconstructions – considering unfavorable sampling pattern

Experiment 3: Crossing



Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$$

Sampling Freq: ξ_1, \dots, ξ_{20}

Data: Defined by

$$y_t := K\bar{\mu}_t + 20\% \text{ Noise}$$



Experiment 3: Crossing

Algorithm: Generalized Frank-Wolfe $\leadsto t \mapsto \mu_t^k = \sum_{i=1}^M \lambda_i \delta_{\gamma_i(t)}$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$$

Reconstruction: from data

$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

(Thresholded at 0.01)



Experiment 3: Crossing

Algorithm: Generalized Frank-Wolfe $\leadsto t \mapsto \mu_t^k = \sum_{i=1}^M \lambda_i \delta_{\gamma_i(t)}$

Ground truth: Curve of measures

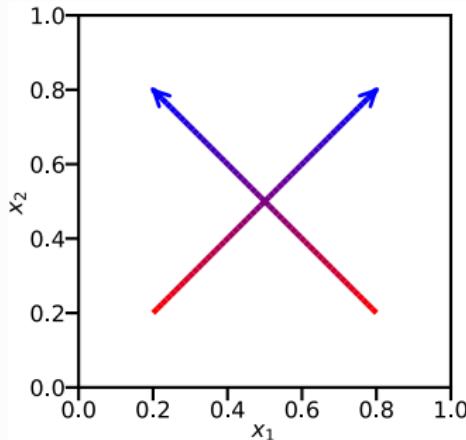
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$$

Reconstruction: from data

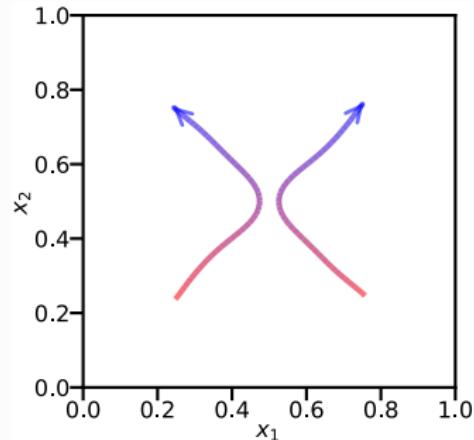
$$y_t = K\bar{\mu}_t + 20\% \text{ Noise}$$

(No Thresholding)

Experiment 3: Crossing



Ground truth: $\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)}$



Reconstruction: $\tilde{\mu}_t := \delta_{\tilde{\gamma}_1(t)} + \delta_{\tilde{\gamma}_2(t)}$

Question: Why do reconstructed trajectories differ from ground truth ones?

Answer: They don't! When regarded as measures they are basically the same

$$dt \otimes \bar{\mu}_t \approx dt \otimes \tilde{\mu}_t$$

Regularizer is Dynamic OT \implies Particles chose shortest path

What to do? Maybe could include curvature penalization



Conclusion

- ① Algorithm for computing **sparse** solutions to

$$\min_{u \in X} F(Ku) + R(u)$$

in **Banach** space



Conclusion

- ① Algorithm for computing **sparse** solutions to

$$\min_{u \in X} F(Ku) + R(u)$$

in **Banach** space

- ② Linear convergence if solution is **Sparse** + “**Source Condition**” + “**Quadratic Growth**”



Conclusion

- ① Algorithm for computing **sparse** solutions to

$$\min_{u \in X} F(Ku) + R(u)$$

in **Banach** space

- ② Linear convergence if solution is **Sparse** + “**Source Condition**” + “**Quadratic Growth**”
- ③ General framework for **dynamic** inverse problems



Conclusion

- ① Algorithm for computing **sparse** solutions to

$$\min_{u \in X} F(Ku) + R(u)$$

in **Banach** space

- ② Linear convergence if solution is **Sparse** + “**Source Condition**” + “**Quadratic Growth**”
- ③ General framework for **dynamic** inverse problems
- ④ Application to **Dynamic MRI**

Thank You!



References

Generalized Frank-Wolfe Algorithm

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Supported by Austrian Science Fund (FWF) and
Christian Doppler Research Society (CDG) Project PIR27
“Mathematical methods for motion-aware medical imaging”