PERMUTATION ANALYTICAL METHOD

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1. Concordance Matrix

The concordance matrix is a key component in multi-criteria decision-making methods, such as the ELECTRE family. It evaluates the level of agreement between pairs of alternatives based on the given criteria. The concordance matrix C is constructed as follows:

1.1. **Construction of the Concordance Matrix.** We start with the normalized decision matrix N, which is derived from the original decision matrix D using an appropriate normalization technique (e.g., max-min normalization). The concordance matrix C is then created based on N following these steps:

(1) **Normalize the Decision Matrix**: Compute the normalized matrix $N_{m\times n}$, where m is the number of alternatives and n is the number of criteria.
(2) **Assign Weights**: Assign weights w_j to each criterion j, ensuring that $\sum_{j=1}^{n} w_j = 1$.
(3) **Pairwise Comparisons**: For each pair of alternatives A_i and A_k:
   (a) Identify the set of criteria S_{ik} for which A_i is at least as good as A_k:
$$S_{ik} = \{j \mid n_{ij} \geq n_{kj},\ j = 1, 2, \ldots, n\}.$$
   (b) Compute the concordance index C_{ik} as the sum of the weights of the criteria in S_{ik}:
$$C_{ik} = \sum_{j\in S_{ik}} w_j.$$

1.2. **Interpretation of the Concordance Matrix.** Each row i of the concordance matrix C represents the degree to which alternative A_i dominates the other alternatives. Each column k represents how the corresponding alternative A_k is dominated by others.

$$C_{ik} = \begin{cases} C_{ik}, & \text{if } i \neq k, \\ 0, & \text{if } i = k. \end{cases}$$

1.3. **Example Concordance Matrix and Calculations.** Consider three alternatives (A_1, A_2, A_3) evaluated on two criteria (C_1, C_2) with equal weights $(w_1 = w_2 = 0.5)$. The normalized matrix N is:

$$N = \begin{bmatrix} 0.8 & 0.6 \\ 0.7 & 0.8 \\ 0.9 & 0.5 \end{bmatrix}.$$

The concordance matrix C is computed as follows:

$$C = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 1 \\ 1 & 0.5 & 0 \end{bmatrix}.$$

Here are the detailed calculations for three elements of C:

- C_{12}: Comparing A_1 with A_2, for C_1, $n_{11} > n_{21}$, and for C_2, $n_{12} < n_{22}$. Thus:
$$S_{12} = \{C_1\}, \quad C_{12} = w_1 = 0.5.$$
- C_{23}: Comparing A_2 with A_3, for C_1, $n_{21} < n_{31}$, and for C_2, $n_{22} > n_{32}$. Thus:
$$S_{23} = \{C_2\}, \quad C_{23} = w_2 = 0.5.$$
- C_{31}: Comparing A_3 with A_1, for C_1, $n_{31} > n_{11}$, and for C_2, $n_{32} < n_{12}$. Thus:
$$S_{31} = \{C_1\}, \quad C_{31} = w_1 = 0.5.$$

1.4. **Scoring Permutations and Determining the Winner.** After constructing the concordance matrix C, we can evaluate the performance of each alternative across all pairwise comparisons. This process involves calculating a score for each permutation of alternatives and identifying the best-performing one.

1.5. **Scoring Permutations and Determining the Winner.** After constructing the concordance matrix C, we can evaluate the performance of each alternative by scoring each permutation of alternatives. This process involves calculating a **net score** for each permutation and identifying the best-ranking one.

1.5.1. *Scoring a Permutation.* A permutation of the alternatives represents a ranking order. For example, $\pi = (A_1, A_2, A_3)$ implies A_1 is ranked higher than A_2, which is ranked higher than A_3. The scoring involves:

(1) For a given permutation π, consider the concordance indices C_{ik} where A_i precedes A_k in π.
(2) Compute the **upper triangle sum**:
$$S^+(\pi) = \sum_{i<k} C_{ik}, \quad \text{where } A_i \text{ precedes } A_k \text{ in } \pi.$$
(3) Compute the **lower triangle sum**:
$$S^-(\pi) = \sum_{i>k} C_{ik}, \quad \text{where } A_k \text{ precedes } A_i \text{ in } \pi.$$
(4) Calculate the **net score** for the permutation:
$$\text{Net Score}(\pi) = S^+(\pi) - S^-(\pi).$$

1.5.2. *Example of Scoring a Permutation.* Consider the concordance matrix:
$$C = \begin{bmatrix} 0 & 0.6 & 0.8 \\ 0.4 & 0 & 0.7 \\ 0.2 & 0.3 & 0 \end{bmatrix}.$$

For the permutation $\pi = (A_1, A_2, A_3)$:

- Upper triangle sum:
$$S^+(\pi) = C_{12} + C_{13} + C_{23} = 0.6 + 0.8 + 0.7 = 2.1.$$
- Lower triangle sum:
$$S^-(\pi) = C_{21} + C_{31} + C_{32} = 0.4 + 0.2 + 0.3 = 0.9.$$

- Net score:

$$\text{Net Score}(\pi) = S^{+}(\pi) - S^{-}(\pi) = 2.1 - 0.9 = 1.2.$$

1.5.3. *Determining the Winning Permutation.* To find the best-ranking permutation:

(1) Calculate the **net score** for all possible permutations of alternatives.
(2) The permutation with the **highest net score** is selected as the best-ranking permutation.

1.6. **Applications of Permutation Scoring.** The permutation scoring method, particularly with net scoring, is applied in:

- Ranking alternatives in multi-criteria decision-making problems.
- Selecting the most consistent and preferred alternative when both positive and negative preferences are considered.
- Supporting decision-making in areas like supplier evaluation, project prioritization, and resource allocation.