From Logistic Regression to Neural Networks

CS114 Lab 5

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February 14, 2020

Logistic Regression

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  - Value x_i
  - Weight w_i

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- Cross-entropy loss $L = -[y \log \hat{y} + (1 - y) \log(1 - \hat{y})]$

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- Compute gradient $\nabla L = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \vdots \\ \frac{\partial L}{\partial w_n} \\ \frac{\partial L}{\partial b} \end{bmatrix}$, where

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  - $\dfrac{\partial L}{\partial b} = \hat{y} - y$

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  - $\dfrac{\partial L}{\partial w_{ic}} = (\hat{y}_c - y_c)x_i$
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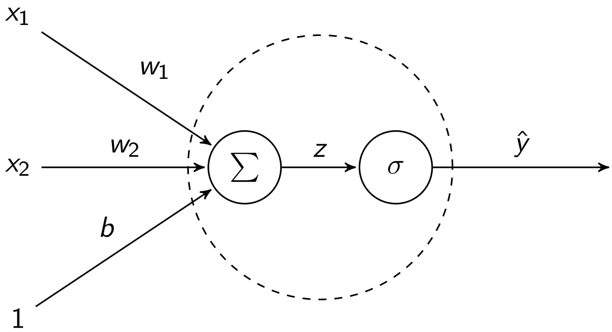
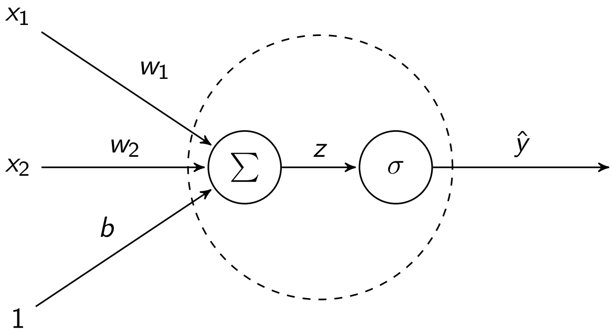
Example

- Python time!

Graphical Representation of Logistic Regression



Graphical Representation of a Neuron



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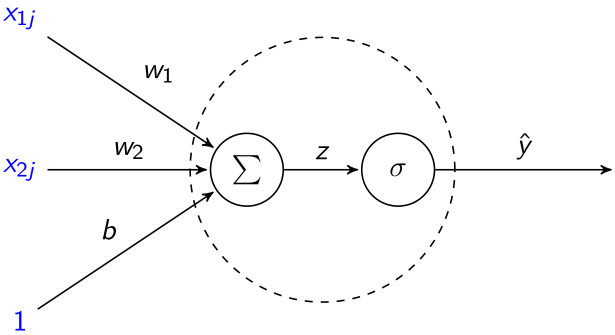

- Inputs (to neuron j)

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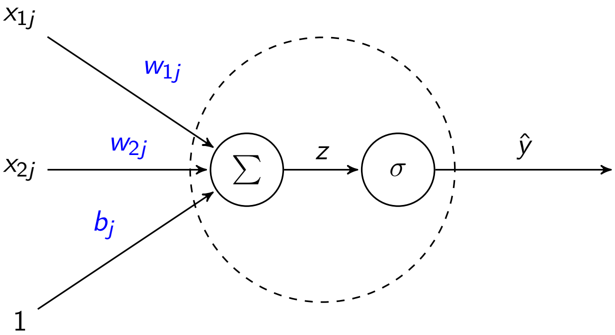

- Weights (and bias term)

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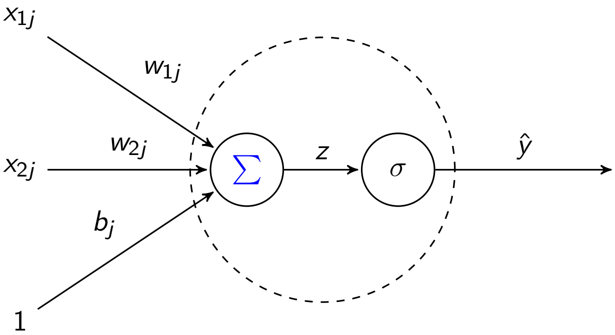

- Input function (almost always $\sum$)

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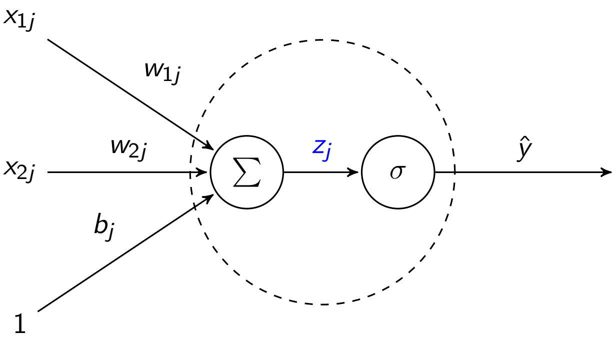

- “Score”

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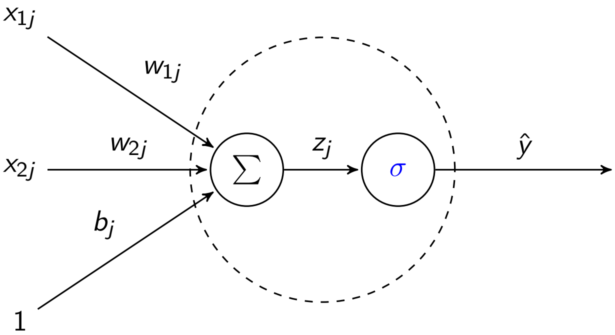

- Activation function (logistic, softmax, tanh, ReLU, etc.)

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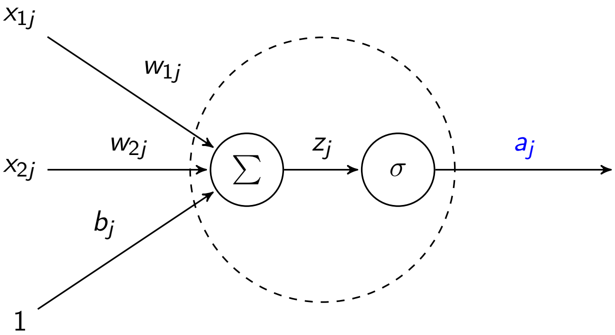

- Activation (output of neuron j)

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- Chain Rule of calculus: $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$
- Looking at the graph: $\frac{\partial L}{\partial w_i} = \frac{\partial L}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial z}\frac{\partial z}{\partial w_i}$

Gradients in Logistic Regression

- $$\frac{\partial L}{\partial w_i} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z} \frac{\partial z}{\partial w_i}$$

Gradients in Logistic Regression

- $\frac{\partial L}{\partial w_i} = \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z} x_i$
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- $\dfrac{\partial L}{\partial w_i} = \dfrac{\partial L}{\partial \hat{y}} \hat{y}(1 - \hat{y}) x_i$
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- $$\frac{\partial L}{\partial w_i} = \frac{\hat{y} - y}{\hat{y}(1 - \hat{y})} \hat{y}(1 - \hat{y}) x_i$$
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    - $\dfrac{\partial z_j}{\partial w_{ij}} = x_{ij}$
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  - Note that for an output neuron, $a_j = \hat{y}$

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  - $$\frac{\partial L}{\partial w_{ij}} = \frac{\partial L}{\partial a_j}\frac{\partial a_j}{\partial z_j}\frac{\partial z_j}{\partial w_{ij}}$$

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- For a non-output neuron:
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    - $\dfrac{\partial L}{\partial a_j} = ?$
  - Note that for a non-output neuron, $a_j \neq \hat{y}$

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- Chain Rule of multivariable calculus:

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- Express L as a function of z_ℓ: $\dfrac{\partial L}{\partial a_j} = \sum_{\ell \in \mathcal{L}} \dfrac{\partial L}{\partial z_\ell} \dfrac{\partial z_\ell}{\partial x_{j\ell}}$

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    - $\dfrac{\partial z_\ell}{\partial x_{j\ell}} = w_{j\ell}$
  - What about $\dfrac{\partial L}{\partial z_\ell}$?

Backpropagation

- We can compute $\dfrac{\partial L}{\partial w_{j\ell}} = \dfrac{\partial L}{\partial a_\ell}\dfrac{\partial a_\ell}{\partial z_\ell}\dfrac{\partial z_\ell}{\partial w_{j\ell}}$ for an output neuron ℓ

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- If we have already computed $\frac{\partial L}{\partial w_{j\ell}}$ for some neuron ℓ, then we have also computed $\frac{\partial L}{\partial z_\ell} \left(= \frac{\partial L}{\partial a_\ell} \frac{\partial a_\ell}{\partial z_\ell} \right)$

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- We can then use $\dfrac{\partial L}{\partial z_\ell}$ to calculate $\dfrac{\partial L}{\partial w_{ij}} = \left(\sum_{\ell \in \mathcal{L}} \dfrac{\partial L}{\partial z_\ell} w_{j\ell}\right)\dfrac{\partial a_j}{\partial z_j}\dfrac{\partial z_j}{\partial w_{ij}}$ for the previous neuron(s) j

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  - Suppose that our neurons are grouped into a sequence of layers
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- Then we can use matrix multiplication

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  - Let δ_l be the vector of $\dfrac{\partial L}{\partial z_i}$ for all neurons i in layer l
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- For an output layer $\mathcal{L}$:
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- For a non-output layer J (with next layer K):
  - $\delta_J = (\mathbf{W}_K^T \cdot \delta_K) \odot \sigma'(\mathbf{z}_J)$

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  - For each (layer of) neuron(s) j, starting from the output and working backwards:
    - Compute gradient $\nabla_j L$

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- Because L is not necessarily convex anymore, we are not guaranteed to reach a global minimum
  - But it works well enough in practice

Example

- Python time (again)!