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Linear Transformations

Mathematical Foundation

Agenda

- Basic Matrix Transformations
- Determinant
- Matrix Inverse
- Determinant and Inverse for special matrices
- Orthogonal matrix & Gram-Schmidt Process
- Eigen Values and Vectors
- Eigen basis and transformations
- Python Notebook

Basic Matrix Transformations

- Transformation matrix is a matrix that transforms one vector into another vector. The positional vector of a point is changed to another positional vector of a new point, with the help of a transformation matrix.

- Types:
    - Scale Transformation
    - Reflection Transformation
    - Projection Transformation
    - Rotation Transformation

Scale Transformation

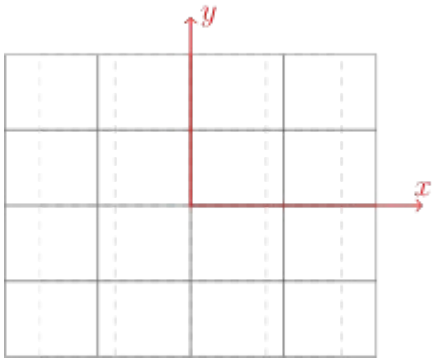
- A scaling transform changes the size of an object by expanding or contracting all vertices along the three axes by three scalar values specified in the matrix.

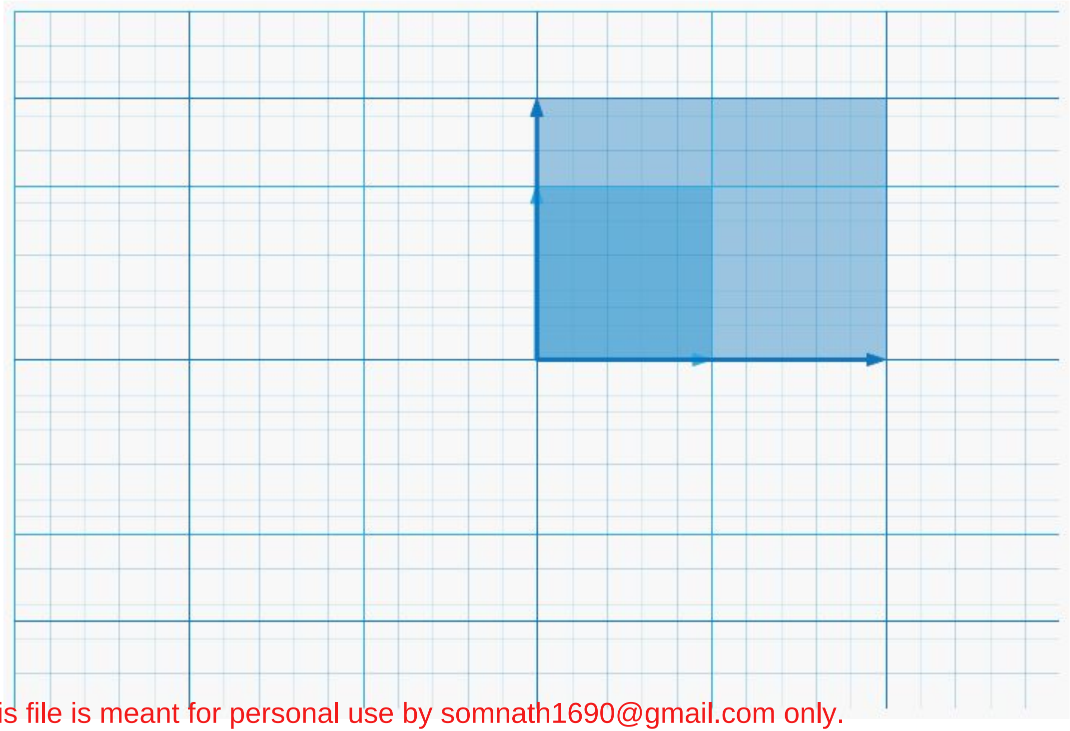
$$\mathbf{M} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The s_x, s_y, and s_z values represent the scaling factor in the X, Y, and Z dimensions, respectively. Applying a scaling matrix to a point v produces an output vector with each component multiplied with the corresponding scaling value.

$$\begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x v_x \\ s_y v_y \\ s_z v_z \\ 1 \end{pmatrix}$$

Scale Transformation





Transform

$$\begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix}$$

Reflection Transformation

- A reflection is a transformation that maps a figure to its reflection image. The figure on the right is the reflection image of a drawing and the point A over the line m. This transformation is called R_m, and we write A = R_m(A).

Reflection about y axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

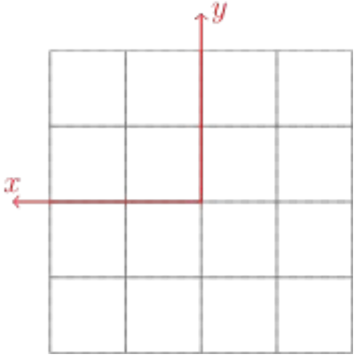
Reflection about x axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection about the origin

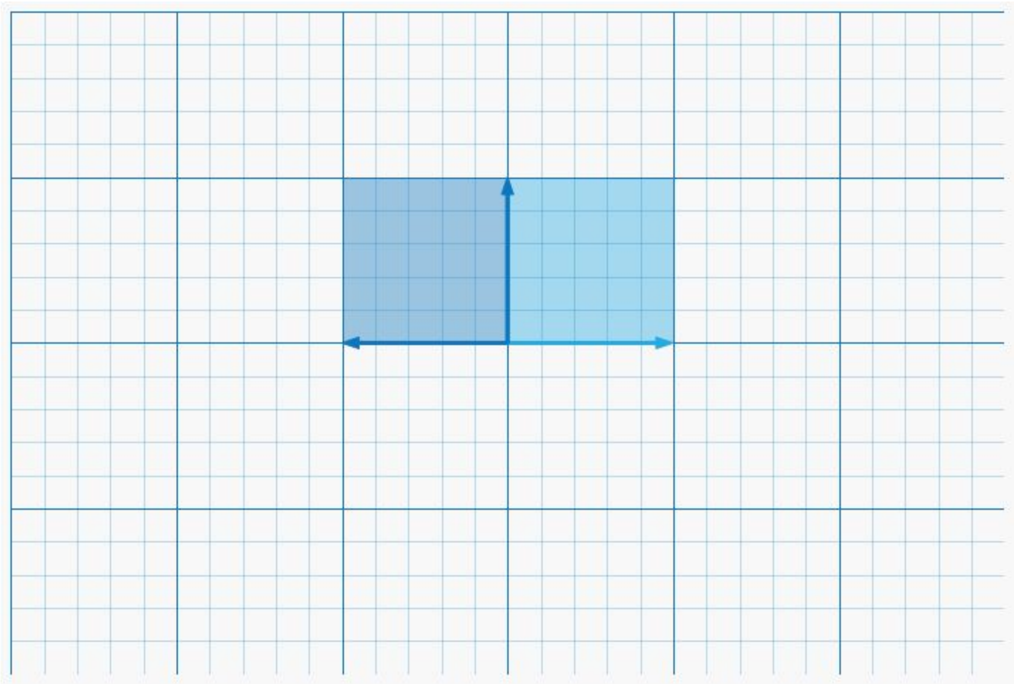
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection Transformation



Transform

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

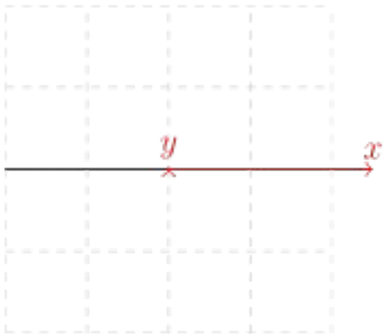


Projection Transformation

- In linear algebra, a projection matrix is a matrix associated to a linear operator that maps vectors into their projections onto a subspace.

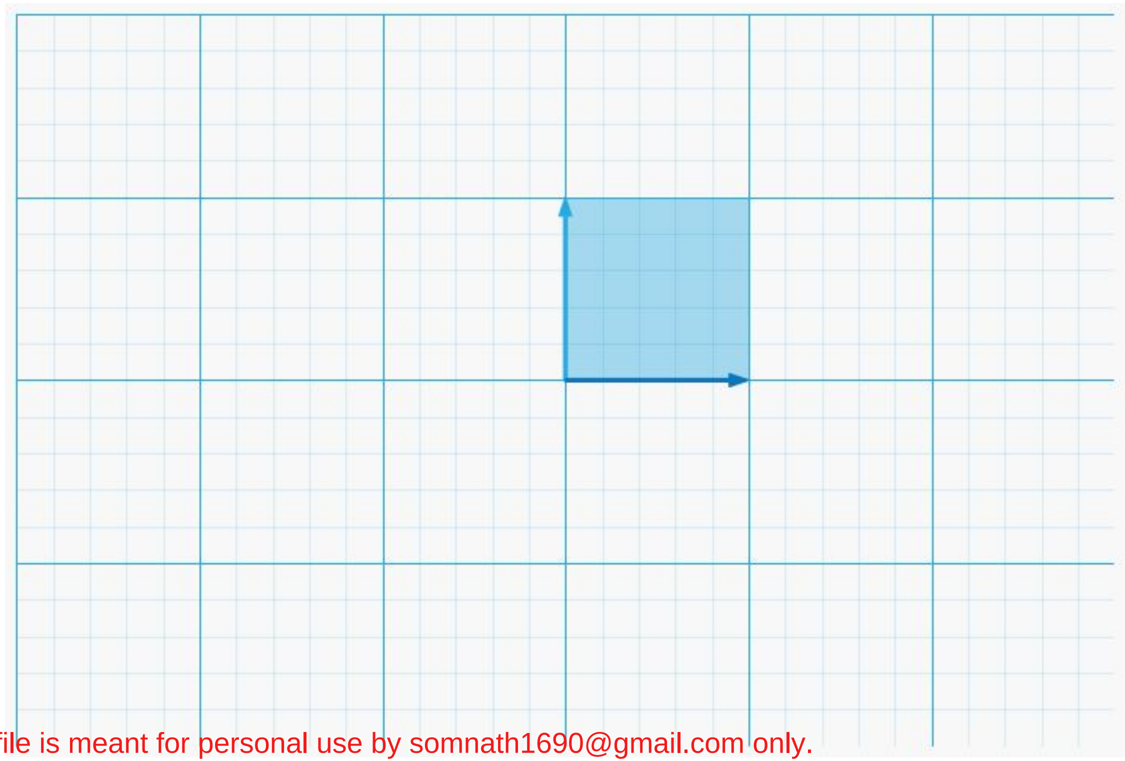
- The rule for this mapping is that every vector v is projected onto a vector T(v) on the line of the projection.

Projection Transformation



Transform

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



Rotation Transformation

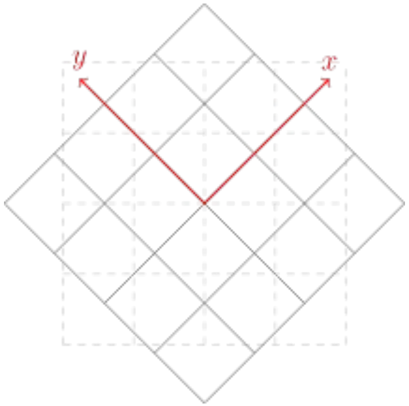
- A rotation matrix rotates an object about one of the three coordinate axes, or any arbitrary vector.
- The following three matrices **R**$_X$, **R**$_Y$ and **R**$_Z$ and represent transformations that rotate points through the angle θ in radians about the coordinate origin.

$$\mathbf{R}_X(\theta)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_Y(\theta)=\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

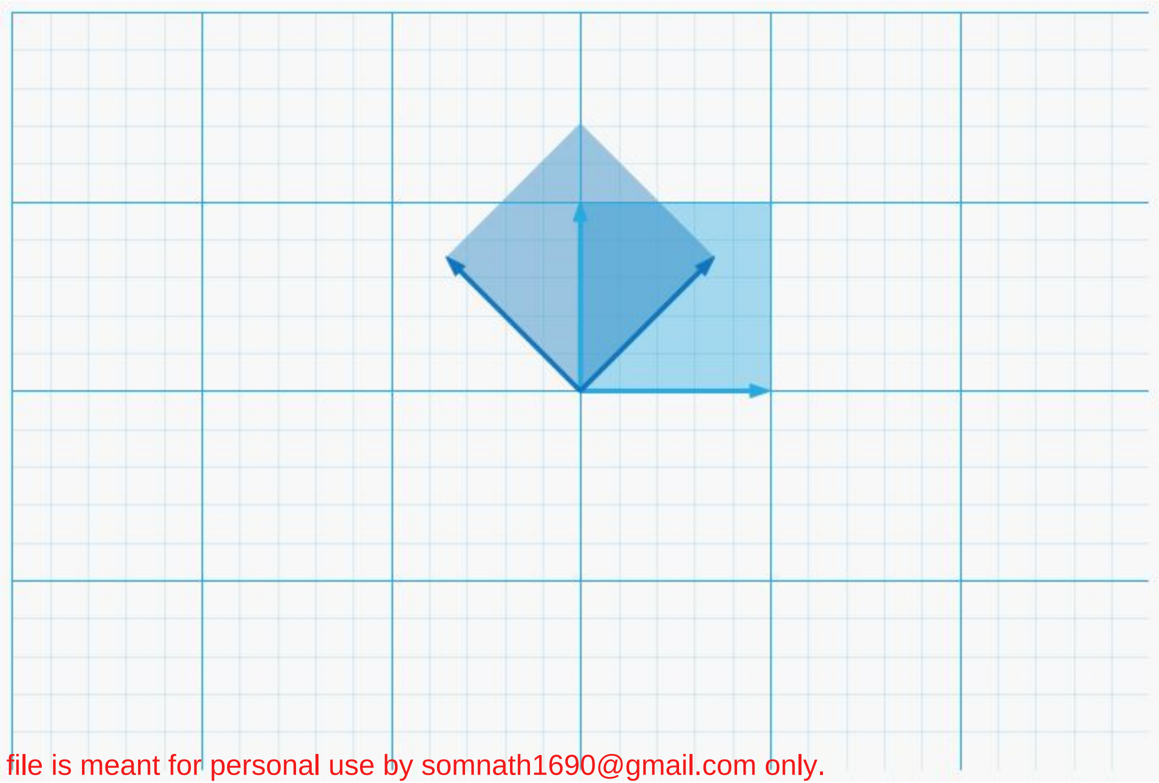
$$\mathbf{R}_Z(\theta)=\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation Transformation



Transform

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$



Determinant

Determinant for 2x2 matrix and 3x3 matrix - explanation with example

Intuition with area of graph transformed

$T: R^n \to R^n$. If you take a figure $S \subseteq R^n$, then $T(S) \subseteq R^n$.

Area when n = 2

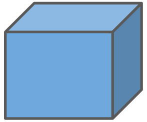
For orientation preserving transformations :

1D - length,

2D - area

3D - volume

Volume when n=3



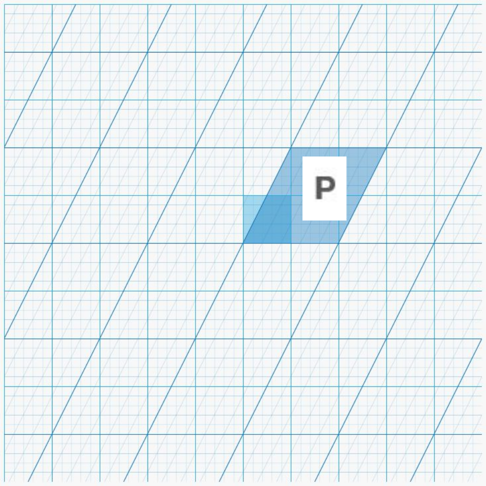
the content of T(S) will be the determinant times the content of S.

For an orientation reversing transformation the factor is the negation of the determinant.

Determinant effect: Scaling of Area

- Consider The Following Transformation Matrix T applied on unit square which transforms unit square to parallelogram . Then The Area of parallelogram is Det(T)* area of unit square

$T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ then Area(P) = Det(T) * area of unit square det(T) = 4



Matrix Multiplication

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$C_{m \times p} = A_{m \times n} \, B_{n \times p}$ $\quad$ $c_{ij} = \Sigma a_{ik} \, b_{kj}$ for k=1 to n for each ij

A columns needs to be equal to B rows, to have a possible multiplication.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \quad \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & & b_{np} \end{pmatrix}$$

$$c_{11} = a_{11} \, b_{11} + a_{12} \, b_{12} + \dots \dots + a_{1n} \, b_{n1}$$

Matrix Properties: Not Commutative

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

AxB need not be equal to BxA

$$\begin{pmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}} \\ a_{21} \quad a_{22} \quad \cdots \quad a_{2n} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1} \quad a_{n2} \quad \quad a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & & b_{nn} \end{pmatrix} \neq \begin{pmatrix} \boxed{b_{11} \quad b_{12} \quad \cdots \quad b_{1n}} \\ b_{21} \quad b_{22} \quad \cdots \quad b_{2n} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ b_{n1} \quad b_{n2} \quad \quad b_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

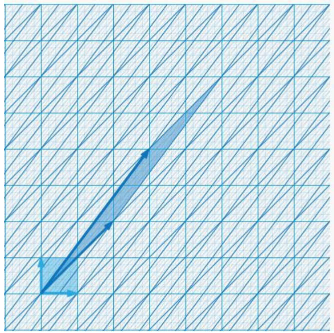
$c_{11}=a_{11}\, b_{11}+a_{12}\, b_{12}+\ldots\ldots \quad +a_{1n}\, b_{n1}$

$c_{11}=b_{11}\, a_{11}+b_{12}\, a_{12}+\ldots\ldots \quad +b_{1n}\, a_{n1}$

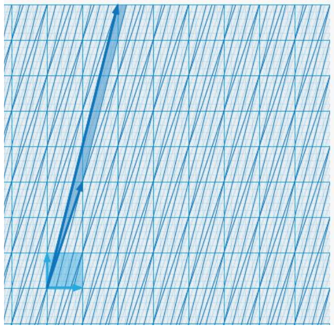
Matrix Properties: Not Commutative

$$A_{n \times n} B_{n \times n} \neq B_{n \times n} A_{n \times n}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 8 & 3 \end{pmatrix}$$



Matrix Operations (Dot/ Inner product)

$$\begin{array}{ccc} A & B & C \\ \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \end{array}$$

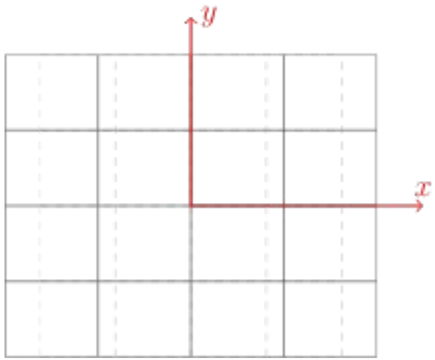
A*(B*C)

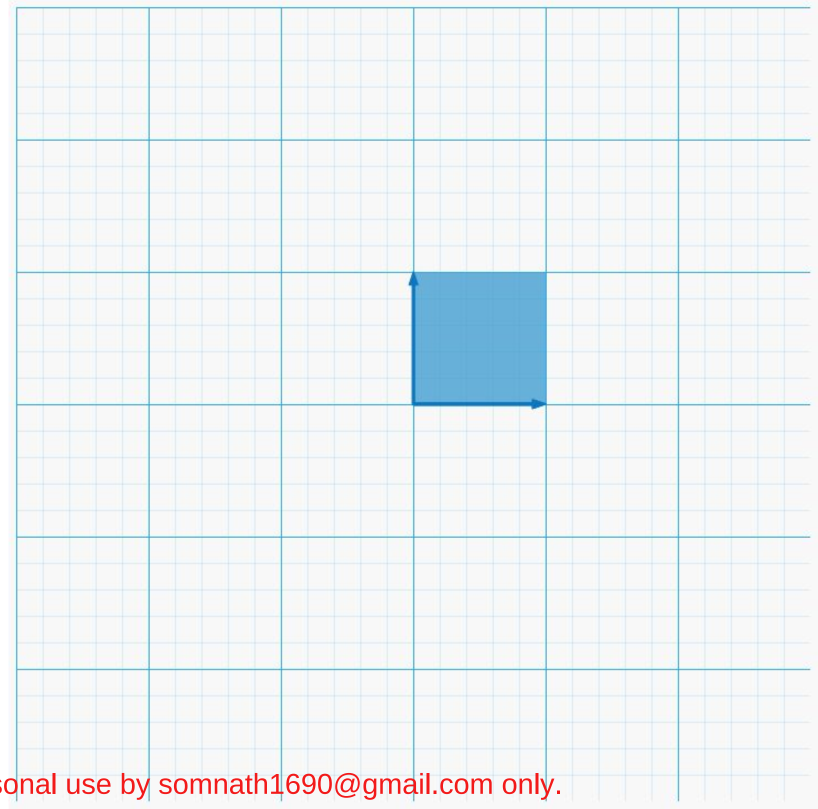
$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 11 \end{pmatrix} = \begin{pmatrix} 7 & 17 \\ 8 & 20 \end{pmatrix}$$

(A*B)*C

$$\begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 17 \\ 8 & 20 \end{pmatrix}$$

Identity Matrix Transformation





Transform

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Identity Matrix & Matrix Inverse

As seen earlier, say for a 3 X 3 Matrix, Identity Matrix would be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix Inverse (A^{-1} is a Matrix which reverses/ nullifies the transformations caused by Matrix A

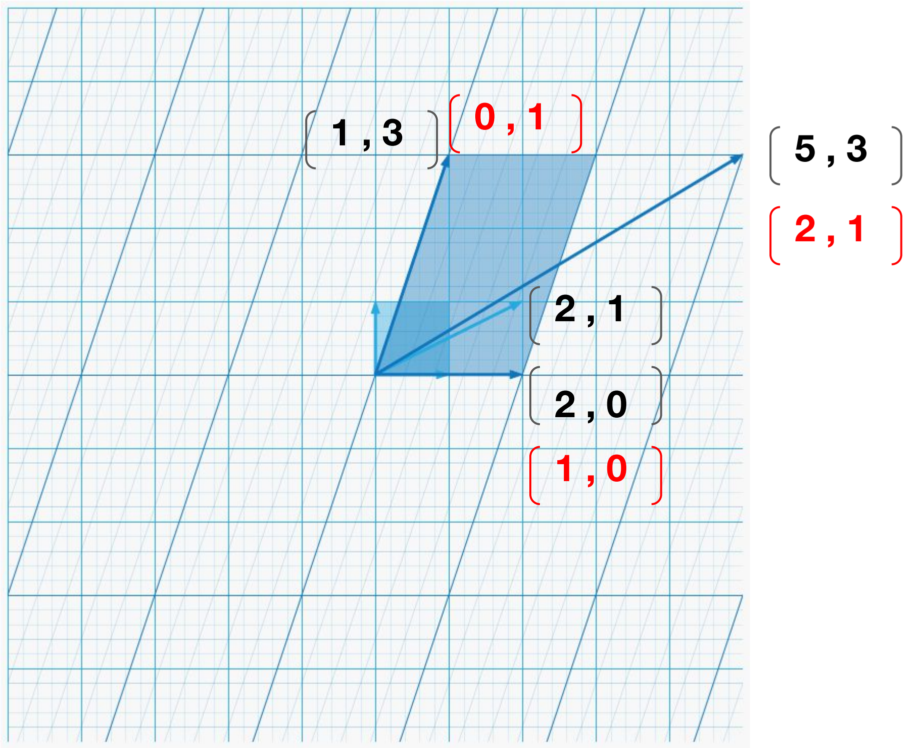
$$A \; I = A$$

$$A \; A^{-1} = I$$

Matrix Transform can be viewed as Changing basis

New Basis

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



Revisiting the Simultaneous Equations

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 5 & 6 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 11 \\ 15 \\ 19 \end{pmatrix}$$

$$A \qquad X = B$$

$$A\ X = B$$

$$A^{-1}(A\ X) = A^{-1}B$$

$$(A^{-1}A)\ X = A^{-1}B$$

2D example for easy visualization

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{A} \qquad \mathbf{X} = \mathbf{B}$$

$$\mathbf{A\ X = B}$$

$$\mathbf{A^{-1}(A\ X) = A^{-1}B}$$

$$\mathbf{(A^{-1}A)\,X = A^{-1}B}$$

Physical Meaning of Matrix Inverse

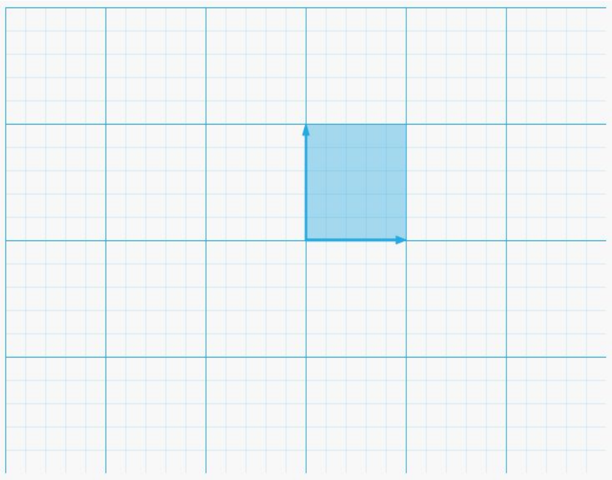
$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

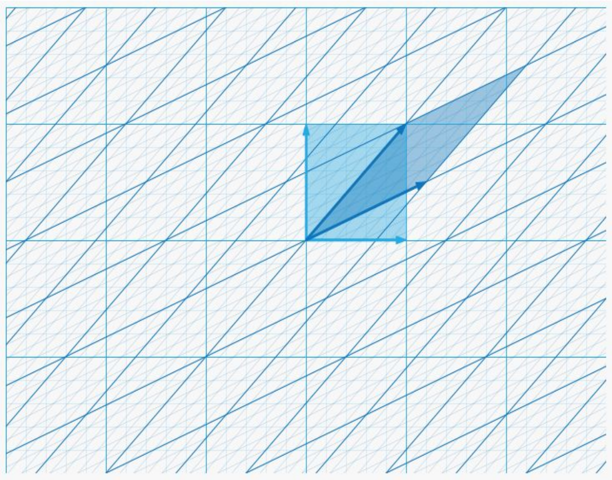
$$\mathbf{A} \quad \mathbf{X} = \mathbf{B}$$

- Applying a transformation of A on the vector X results in the vector B
- To find X, we need to apply a transformation on B which would inverse/reverse the effect of the transformation of A
- The $\mathbf{A}^{-1}$ transforms a inverse of the transformation introduced by A
- If we know the vector (B) in the transformed space, we can get the vector X by removing the effect of the transform

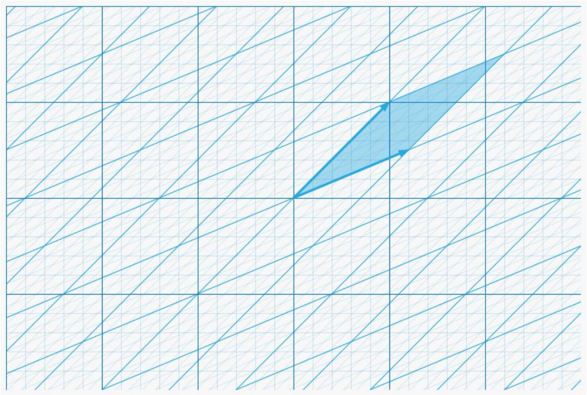
Physical Meaning of Matrix Inverse on Space

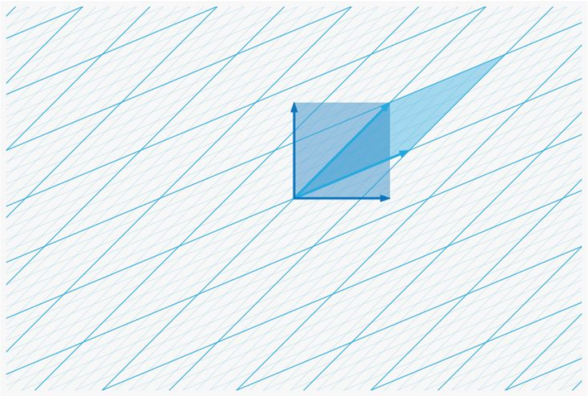
- Applying a transformation of A on the the space (Light blue) results in the space (Dark blue)





Physical Meaning of Matrix Inverse on Space





- Say, we have the transformed space, we need to apply a transformation on the same, which would inverse/reverse the effect of the transformation of A
- The **A**$^{-1}$ transforms a inverse of the transformation introduced by A
- If we know the vector (B) in the transformed space, we can get the vector X by removing the effect of the transform

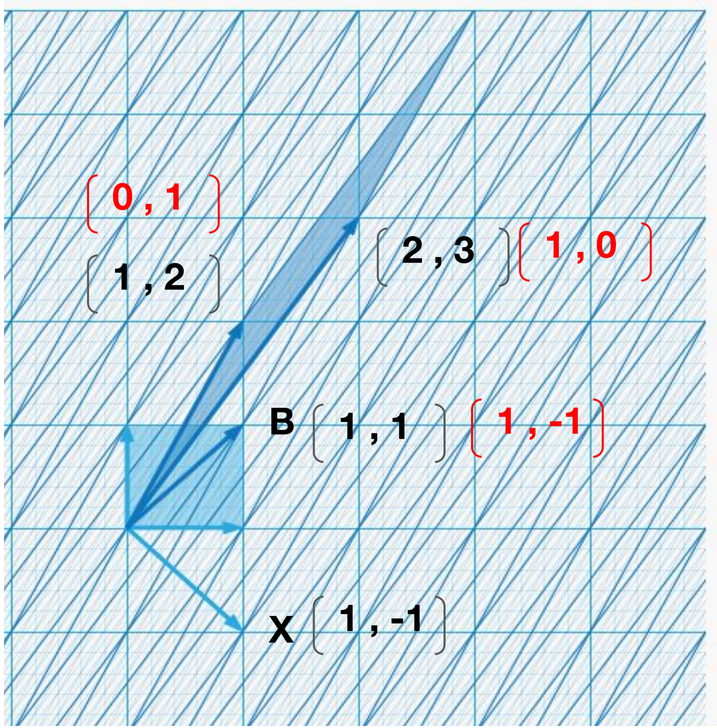
Physical Meaning of Matrix Inverse (Example)

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{A} \quad \mathbf{X} = \mathbf{B}$$

$$\mathbf{A}^{-1} \quad \mathbf{B} = \mathbf{X}$$

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



Gaussian Elimination to get Matrix Inverses (1/5)

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 5 & 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 11 \\ 15 \\ 19 \end{pmatrix}$$

$R_3 \text{ -> } R_3 - R_2 - R_1$

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 0 & -1 & -6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 11 \\ 15 \\ -7 \end{pmatrix}$$

Gaussian Elimination to get Matrix Inverses (2/5)

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 0 & -1 & -6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 11 \\ 15 \\ -7 \end{pmatrix}$$

$R_2 \rightarrow (2/3) R_2 - R_1$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1/3 & -2/3 \\ 0 & -1 & -6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2/3 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ -1 \\ -7 \end{pmatrix}$$

$R_3 \rightarrow R_3 - 3R_2$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1/3 & -2/3 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2/3 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ -1 \\ -4 \end{pmatrix}$$

Gaussian Elimination to get Matrix Inverses(3/5)

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1/3 & -2/3 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2/3 & 0 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 11 \\ -1 \\ -4 \end{pmatrix}$$

$R_3 \to (-1/4)\, R_3$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1/3 & -2/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2/3 & 0 \\ -1/2 & 3/4 & -1/4 \end{pmatrix} \begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix}$$

$R_2 \to R_2 + (2/3)R_3$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -4/3 & 7/6 & -1/6 \\ -1/2 & 3/4 & -1/4 \end{pmatrix} \begin{pmatrix} 11 \\ -1/3 \\ 1 \end{pmatrix}$$

Gaussian Elimination to get Matrix Inverses (4/5)

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -4/3 & 7/6 & -1/6 \\ -1/2 & 3/4 & -1/4 \end{pmatrix}$$

$$\begin{pmatrix} 11 \\ -1/3 \\ 1 \end{pmatrix}$$

R_2 -> $-3\ R_2$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & -7/2 & 1/2 \\ -1/2 & 3/4 & -1/4 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \\ 1 \end{pmatrix}$$

R_1 -> R_1 $-3R_2$$-4R_3$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -9 & 15/2 & -1/2 \\ -4 & -7/2 & 1/2 \\ -1/2 & 3/4 & -1/4 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

Gaussian Elimination to get Matrix Inverses (5/5)

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -9 & 15/2 & -1/2 \\ -4 & -7/2 & 1/2 \\ -1/2 & 3/4 & -1/4 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$R_1 \rightarrow -1/2\, R_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -9/2 & 15/4 & -1/4 \\ -4 & -7/2 & 1/2 \\ -1/2 & 3/4 & -1/4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Orthogonal Matrix

A matrix with orthonormal row and column vectors is called an orthogonal matrix.

Some useful properties:

- An orthogonal matrix Q is necessarily invertible (with inverse $Q^{-1} = Q^{T}$),
- The determinant of any orthogonal matrix is either +1 or −1.
- As a linear transformation, an orthogonal matrix preserves the inner product of vectors, such as a rotation, reflection or rotoreflection. In other words, it is a unitary transformation.

Orthogonal Matrix (Examples)

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$v_1.v_2=v_2.v_3=v_1.v_3=0$

$|v_1|=|v_2|=|v_3|=0$

Det=1

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$
$$\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$v_1.v_2=v_2.v_3=v_1.v_3=0$

$|v_1|=|v_2|=|v_3|=0$

Det=-1

Gram - Schmidt Process

Converts a matrix of column vectors $u_1, u_2 \ldots u_n$ to an orthogonal matrix v1,v2…. v_n

The steps in Gram-Schmidt process

- Normalize the first column vectors (u_1) of the matrix to compute v_1
- Compute w_2 by removing the vector projection of u_2 in v_1 from u_2 and normalizing the vector w_2 shall provide v_2
- Compute w_3 by removing the vector projection of u_3 in v_1 and v_2 from u_3 and normalizing the vector w_3 shall provide v_3
- ……..
- Compute w_n by removing the vector projection of u_n in v_1, v_2 to v_{n-1} from u_n and normalize the vector w_n shall provide v_n

Gram-Schmidt Process- Explanation

$$\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{pmatrix} 2 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 4 & 5 \end{pmatrix} \end{array}$$

$u_1 = (2, 0, 0)$

$w_1 = (2, 0, 0)$

$|w_1| = \sqrt{(4+0+0)} = 2$

$v_1 = w_1/|w_1|$ $=\mathbf{(1, 0,0)}$

$u_2 = (2, 3, 4)$

$w_2 = u_2-(u_2.v_1)v_1$

$= (2, 3, 4) - 2(1, 0, 0)$

$= (0, 3,4)$

$v_2 = w_2/|w_2|$ $=\mathbf{(0,3/5, 4/5)}$

Gram-Schmidt Process- Explanation (Continued)

$$\begin{array}{ccc} u_1 & u_2 & u_3 \\ \end{array}$$
$$\begin{pmatrix} 2 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 4 & 5 \end{pmatrix}$$

$u_3 = (4,5, 5)$

$w_3 = u_3-(u_3.v_1)v_1-(u_3.v_2)v_2$

$= (4, 5,3) - 4 (1, 0,0)- 7(0,3/5, 4/5)$

$= (0, 4/5,-3/5)$

$v_3= w_3/|w_3| =(0,4/5, -3/5)$

$\mathbf{v_1 = (1, 0, 0)}$

$\mathbf{v_2 = (0,3/5, 4/5)}$

$\mathbf{v_2 = (0,4/5, -3/5)}$

$$\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix} \end{array}$$

Gram-Schmidt Process (through python code)

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 5 & 6 & 5 \end{pmatrix}$$

```
array([[ 0.32444284,  0.78039897,  0.57601367],
       [ 0.48666426,  0.34684399,  0.57851808],
       [ 0.81110711, -0.52026598, -0.57751631]])
```

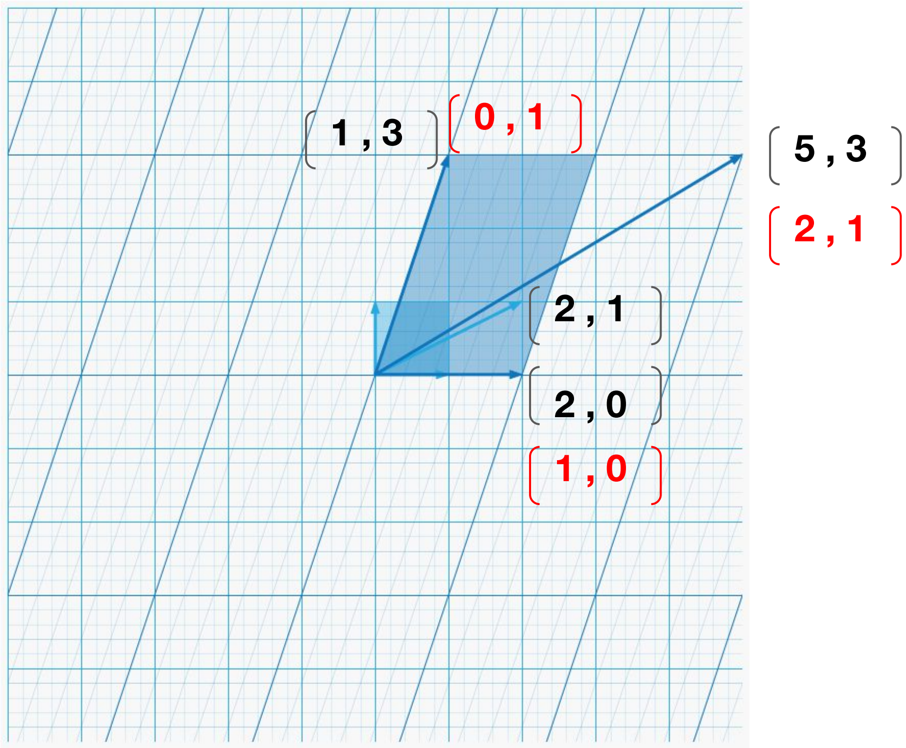
Revisiting the Changing basis through transform

- Lets assume that $\{u_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$ be basis vector
- Let $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ be vector with respect to $\{u_1, u_2\}$
- Then How to rotate by 45 degree $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in $\{u_1, u_2\}$
- First transform $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ to with respect to standard basis $\Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- Then rotate the result by 45 degree $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- Again Transform result to $\{u_1, u_2\}$ i.e $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Revisiting the Changing basis through transform

New Basis

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



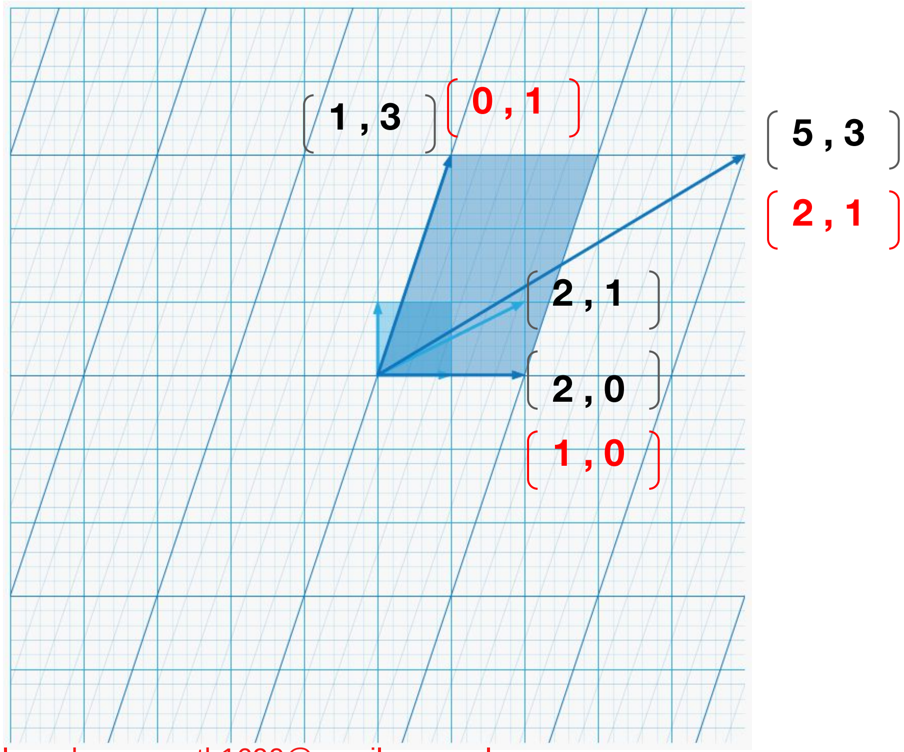
Rotation in the New Basis (1/2)

New Basis (B)

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

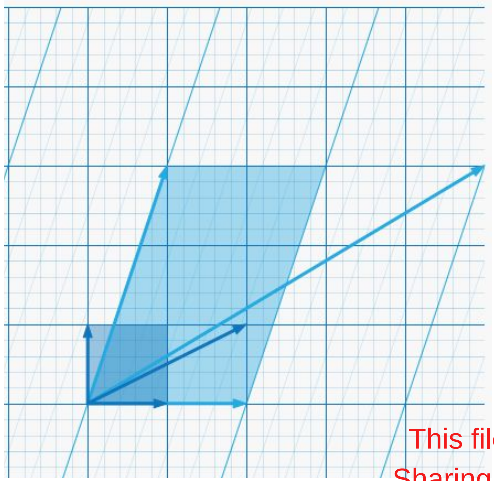
$$B^{-1} \quad \begin{pmatrix} 3/6 & -1/6 \\ 0 & 2/6 \end{pmatrix}$$



Rotation in the New Basis (2/2)

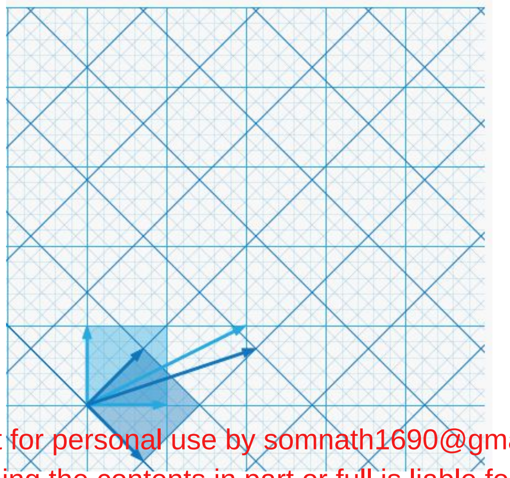
B^{-1} of the transformed space and Vector (5,3) of new space

B^{-1} $\begin{pmatrix} 3/6 & -1/6 \\ 0 & 2/6 \end{pmatrix}$



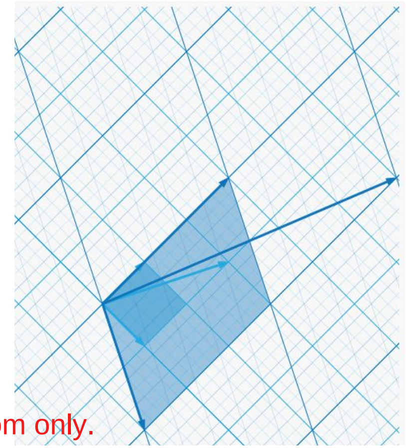
Rotation in the normal space and Vector (2,1) in normal space

R $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

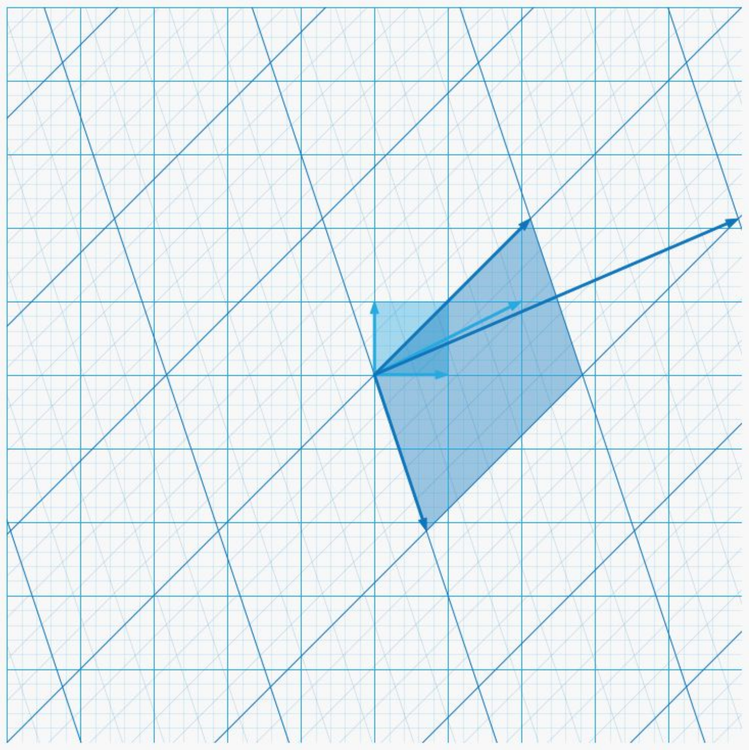


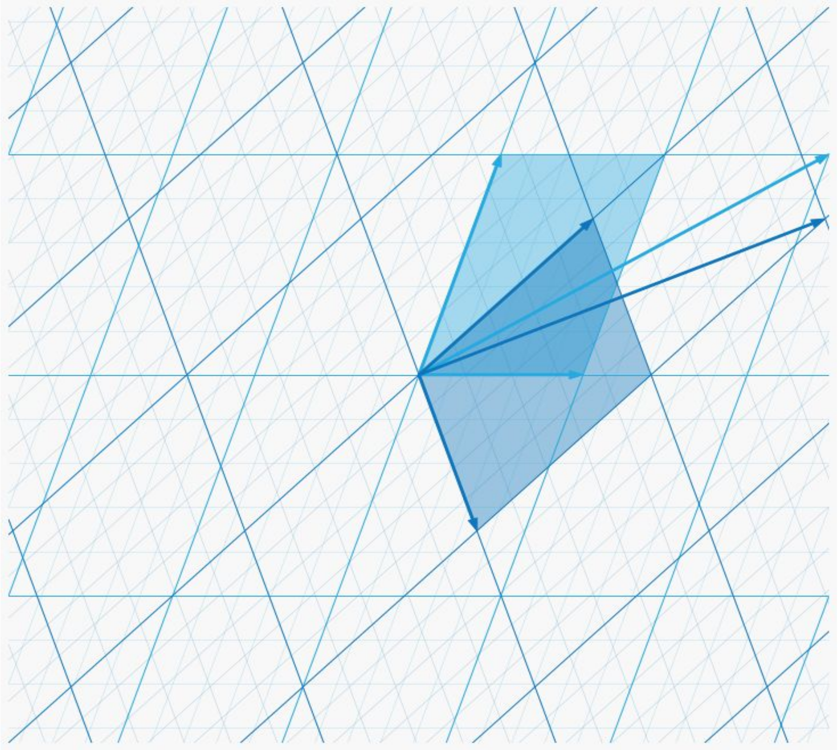
B of the normal space and the rotated vector in the new space

B $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$



Rotation(in old basis) in the New Basis





Eigen Vector and Eigen Values- Physical Significance

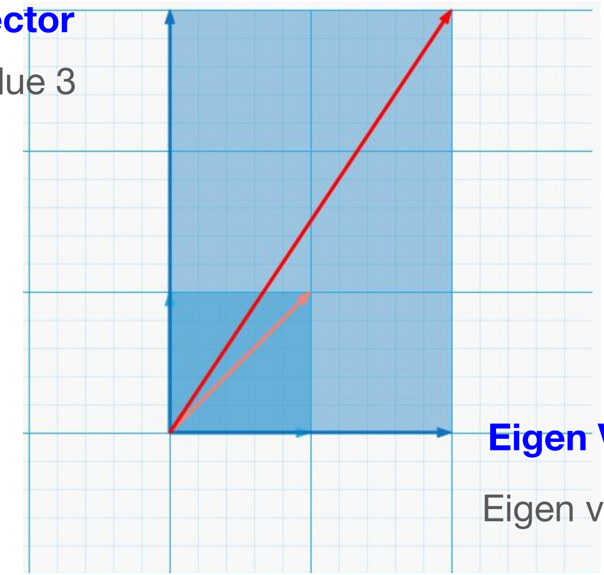
Eigen - means ‘Characteristic’ or Special

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

For this Transformation, The horizontal and vertical vectors are the Eigen vectors.

Eigen Vector

Eigen value 3

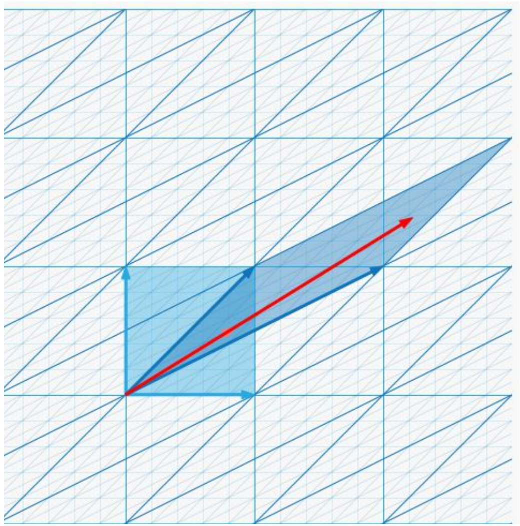


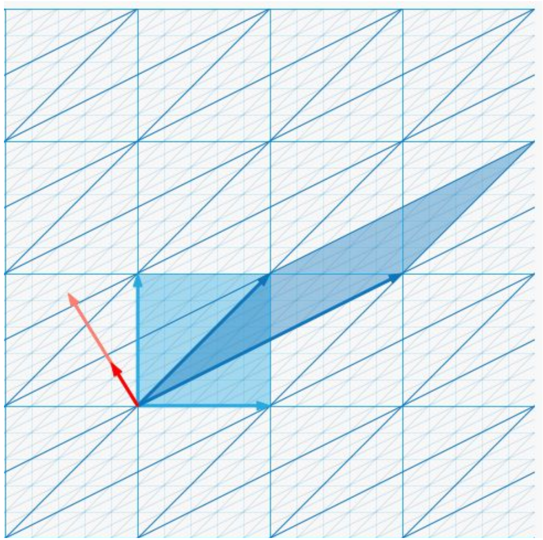
Eigen Vector

Eigen value of 2

Eigen Vector and Eigen Values- Physical Significance

```
array([2.61803399+0.j,          0.38196601+0.j])
```





T

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

For this Transformation,
The light red vectors
are the Eigen vectors

```
array([[ 0.85065081,          -0.52573111],
       [ 0.52573111,           0.85065081]])
```

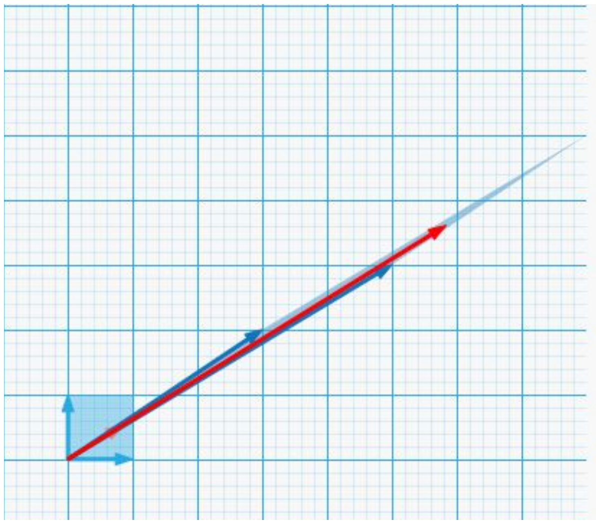
Eigen Vector and Eigen Values- Physical Significance

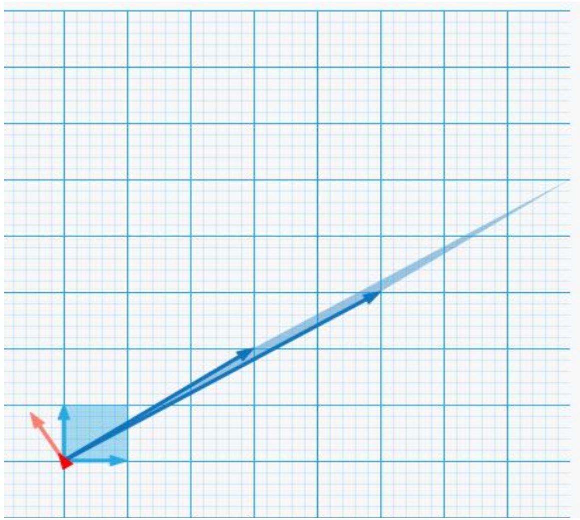
T*T

$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

The eigen vectors and values remain the same for any Transformation

```
array([2.61803399+0.j,          0.38196601+0.j])
```





```
array([[ 0.85065081,          -0.52573111],
       [ 0.52573111,           0.85065081]])
```

Eigen Vector and Eigen Values- Derivation

$A X = \lambda X$

$A X - \lambda X = 0$

$(A - \lambda I) X = 0$

$$\underset{A}{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}} = \lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\underset{(A-\lambda I)}{\left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)} \underset{X}{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}} = \underset{0}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

Eigen Vector and Eigen Values

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Determinant of (A-λI) = 0

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)-1=0$$

$$\lambda = 2.618 \text{ or } 0.3813$$

Eigen Vector and Eigen Values

$\lambda = 2.618$

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(2-2.618)\ x_1+y_1=0$

$0.618\ x_1= y_1$

$x_1/y_1=1/(0.618)$

$$\begin{pmatrix} 1.000 \\ 0.618 \end{pmatrix}$$

$$\begin{pmatrix} 0.851 \\ 0.526 \end{pmatrix}$$

$\lambda = 0.3813$

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(2-0.3813)\ x_1+y_1=0$

$1.618x_1= -y_1$

$x_1/y_1=-1/(1.618)$

$$\begin{pmatrix} -1.000 \\ 1.618 \end{pmatrix}$$

$$\begin{pmatrix} -0.526 \\ 0.851 \end{pmatrix}$$

Significance of Eigen values and Eigen vectors

- Eigen vectors represent those axes of perception/learning along which we can know/understand/perceive things around us in very effective way(s).
- For example, a 2D image captures most of the information captured by 3D image. The front facing 2D place with the X and Y axis is sufficient to understand the 3D object.
- This helps in differentiating different objects in 3D with just its 2D representation (a 2D image).
- The magnitude of change in the eigen space is the eigen value which helps in differentiation. Hence eigen vectors with higher values are significant and lower ones are discarded.

Eigen basis and Transformations

$$E^{-1}TE = D$$

$$T = EDE^{-1}$$

Where, E is the Eigen vectors matrix of T and D is the Diagonal matrix of corresponding eigen values

$$T^*T = EDE^{-1}EDE^{-1}$$

$$= E\ D\ I\ D\ E^{-1}$$

$$= ED^2E^{-1}$$

$$T^n = ED^nE^{-1}$$

Eigen basis and Transformations

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 0.85065081 & -0.52573111 \\ 0.52573111 & 0.85065081 \end{pmatrix} \quad D = \begin{pmatrix} 2.61803399 & 0 \\ 0 & 0.38196601 \end{pmatrix} \quad E^{-1} = \begin{pmatrix} 0.85065081 & 0.52573111 \\ -0.52573111 & 0.85065081 \end{pmatrix}$$

Eigen basis and Transformations

T^*T

$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$\underset{E}{\begin{pmatrix} 0.85065081 & -0.52573111 \\ 0.52573111 & 0.85065081 \end{pmatrix}} \underset{D^2}{\begin{pmatrix} 6.85410197 & 0 \\ 0 & 0.14589803 \end{pmatrix}} \underset{E^{-1}}{\begin{pmatrix} 0.85065081 & 0.52573111 \\ -0.52573111 & 0.85065081 \end{pmatrix}}$$

Appendix

- Rotation transformation:

```
np.array([
        [np.cos(angle), -np.sin(angle), 0],
        [np.sin(angle), np.cos(angle), 0],
        [   0  ,   0  ,   1  ]    ])
```

- Determinant of matrix A: **B = det(A)**

- To calculate rank : **from numpy.linalg import matrix_rank**
  rank = matrix_rank(A)

- Inverse of matrix: **from numpy.linalg import inv**
  inv_A = inv(A)

- Eigen Values and Eigen Vectors: **import scipy.linalg as la**
  eigvals, eigvecs = la.eig(T) *#T is a Matrix*

Thank You