An Analysis of Chladni plates

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1. ABSTRACT

The resonance frequencies of multidimensional objects are nontrivial to solve. In this project I discuss computational tools that can be used to make tractable complex dynamical systems. This consists of the discretization of partial differential equations, the use of fast Fourier transforms, and the spectral method.

2. INTRO AND BACKGROUND

To discuss the work of this project, a framework must be built that explains the key concepts and ideas needed to understand this work. In that spirit, I will explain the concept of resonance frequencies,partial differential equations, Fourier transforms, and their connections. To begin, Resonance frequencies, which are also referred to as natural frequencies, are a dynamic property of physical systems that result from the geometry and material composition of a physical thing. When a periodic force is applied to an object with a frequency near the resonance frequency of an object an amplification of the periodic force occurs. This occurs because the physical system is moving in phase with the force being applied. A common analogy of this effect is pushing a child on a swing. If one were to push the child when the swing was coming towards them then they would be damping the swing system. However, if one pushes the child right when the swing moves away from them then the force applied is moving in phase with the swing and adding energy to the swing system resulting in a increase in the maximum height of the swing. The same thing occurs with all physical systems , where applying a force at an objects natural frequency or a harmonic of the natural frequency adds energy to the system because the force is moving in phase with the physical system. To solve for these frequencies, one can write down the forces acting on an object from each degree of freedom. Organizing these forces into a matrix form while assuming an oscillatory solution allows one to find the eigen values of the physical system, which are in this case the mathematical equivalent to the natural frequencies. One could then treat a system's geometry as one with infinite degrees of freedom. At a point even relatively simple geometries have complex natural responses and are found in practice by performing a frequency sweep and finding the frequency that produces the maximum amplitude in the object. Something as simple as a square plate has remarkably complex natural frequencies and modes. However, due to various constraints frequency sweeps are not possible for many physical systems. As a result, we can recreate the experiment using a simulation consisting of partial differential equations analogous to the eigen value method used to find the natural frequencies of simple frequencies discussed previously. Fourier transforms can then be used on the dynamic response of the simulation to find the the natural frequencies. Motivation and direction for this work was provided primarily by the paper [1].

3. GOVERNING DIFFERENTIAL EQUATIONS

The differential equations governing our simulation of a square plate comes from the Kirchoff hypothesis a good summary of these assumptions comes from Virginia tech notes on the text [2] it states that in order for kirchoff's hypothesis to hold you must have:

1. The material of the plate is elastic, homogenous, and isotropic.
2. The plate is initially flat.
3. The deflection (the normal component of the displacement vector) of the midplane is small compared with the thickness of the plate. The slope of the deflected surface is therefore very small and the square of the slope is a negligible quantity in comparison with unity.
4. The straight lines, initially normal to the middle plane before bending, remain straight and normal to the middle surface during the deformation, and the length of such elements is not altered. This means that the vertical shear strains $\gamma_{xy}$and γz are negligible and the normal strain ϵzmay also be omitted. This assumption is referred to as the "hypothesis of straight normals."

5. The stress normal to the middle plane,σz, is small compared with the other stress components and may be neglected in the stress-strain relations.

6. Since the displacements of the plate are small, it is assumed that the middle surface remains unstrained after bending.

If this holds you can eliminate shear and normal strains in the z direction reducing the plane deformation to a 2d problem as a result you can write the deformation of the plane in the z direction as

$$\frac{d^4w}{dx^4} + 2\frac{d^4w}{dx^2dx^y} + \frac{d^4w}{dy^4} + \frac{\rho}{D}\frac{d^2w}{dt^2} = \frac{q}{D} \tag{1}$$

where w is deformation in the z direction, ρ is density,q is the force on the plate, and D is the flexural rigidity given by $\frac{Et}{12(1-v^2)}$ where E is younge's modulus,t is thickness, and v is Poisson's ratio.

4. NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

In class we discussed various methods to solve ordinary differential equations. Partial differential equations can be solved in much the same way if you discretize the derivative using finite differences. This discretization is done by using taylor expansions to approximate derivatives given a discrete number of points. Once you discretize the equation you can then solve the initial boundary problem using Euler or Runge-Kutta methods. Let's see an example of this in solving the 2 dimensional burgers equation. The burger equation is regularly used to model convection diffusion processes seen in thermal and fluid dynamic problems.

The nonlinear 2d coupled burger equations we will solve are given by the following equations where α is the diffusive term, the first order derivatives are convection terms, and u and v are the velocities in the x and y directions

$$\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} = \alpha(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}) \tag{2}$$

$$\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} = \alpha(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}) \tag{3}$$

We can solve for time derivative in these coupled equations

$$\frac{du}{dt} = -u\frac{du}{dx} - v\frac{du}{dy} + \alpha(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}) \tag{4}$$

$$\frac{dv}{dt} = -u\frac{dv}{dx} - v\frac{dv}{dy} + \alpha(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}) \tag{5}$$

Discretization of each spatial and time derivative using second order approximations gives

$$u_{i,j}^{n+1} = u_{i,j}^n + \Delta t(-u_{i,j}^n\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x} - v_{i,j}^n\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2\Delta y} + \alpha(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2})) \tag{6}$$

$$v_{i,j}^{n+1} = v_{i,j}^n + \Delta t(-u_{i,j}^n\frac{v_{i+1,j}^n - v_{i-1,j}^n}{2\Delta x} - v_{i,j}^n\frac{v_{i,j+1}^n - v_{i,j-1}^n}{2\Delta y} + \alpha(\frac{v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n}{\Delta x^2} + \frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{\Delta y^2})) \tag{7}$$

$$\tag{8}$$

Iterating these discretization with the boundary conditions $u(0,y) = u(1,y) = v(x,1) = 0$,$u(x,0) = u(x,1) = sin(2\pi x)$,$v(x,0) = 1$, and $v(0,y) = v(1,y) = 1 - y$ gives the steady state solutions seen in 1 and 2. Something one has to be careful about when dealing with pde's is that the discretization in the time domain is at least half of the spacial domain else the pde will become unstable and blow up a proof for this can be found in [3].

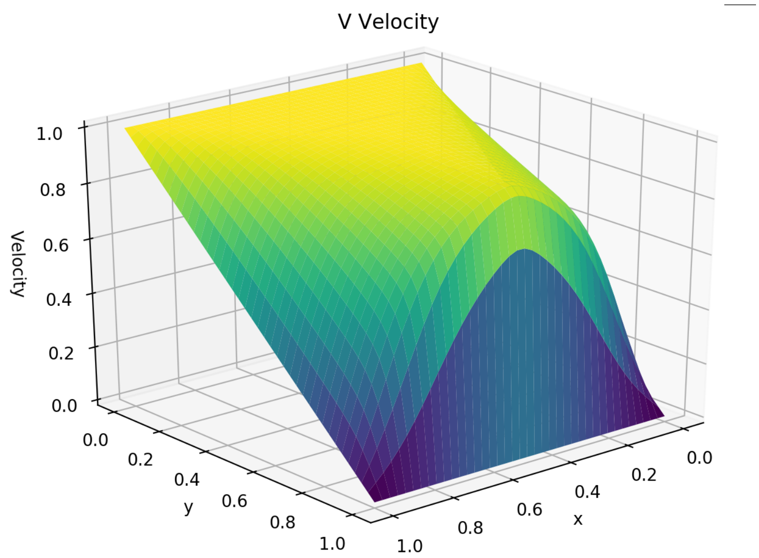

FIG. 1: Steady state velocity magnitude in the y direction

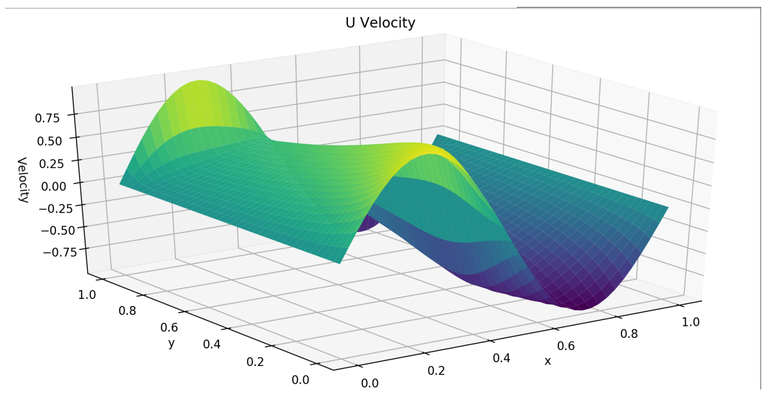

FIG. 2: Steady state velocity in the x direction

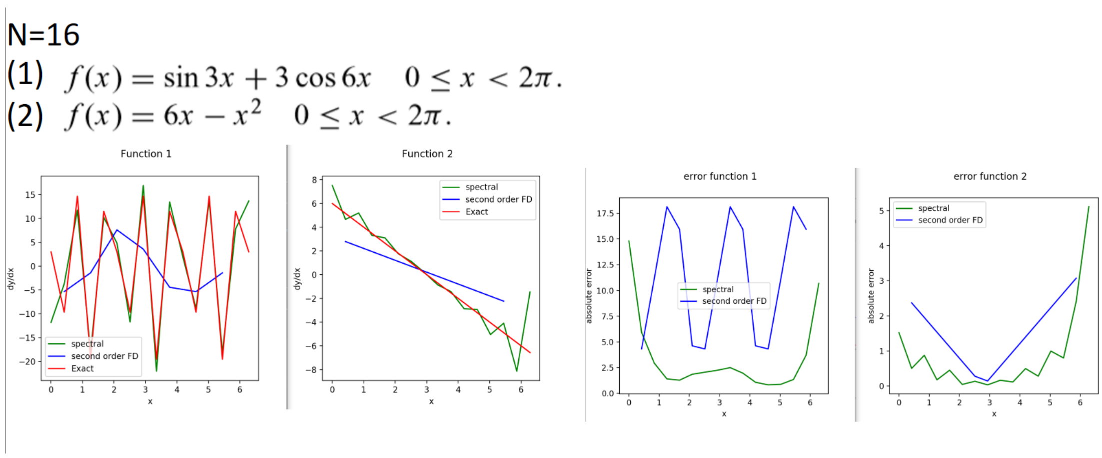

FIG. 3

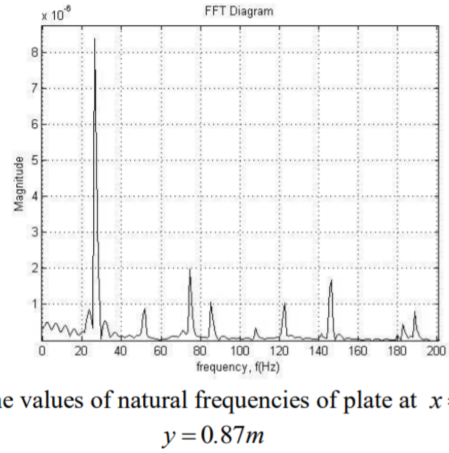

FIG. 4: Fourier transform of the dynamic response at x=0.85 meters and y=0.25 meters found in [1]

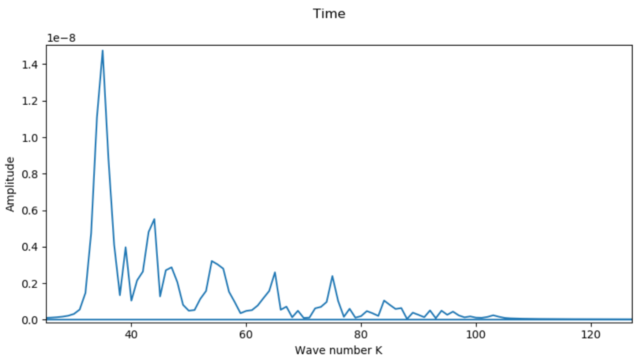

FIG. 5: Fourier transform of the dynamic response at x=0.85 meters and y=0.25 meters

5. SPECTRAL METHODS

Discretization is one way of solving Partial differential equations another is by utilizing the derivative form of Fourier transform. In short, one can think of a Fourier transform as being a transformation where $f(x) = \Sigma f_k(A)e^{ikx}$ we can use the unique properties of e^{ikx} to find the derivative of the transform to be $\Sigma ik * f_k(A)e^{ikx}$. Inverse transforming back to the x domain gives the derivative of $f(x)$. This can be utilized to give a surprisingly clean way of numerically approximating partial differential equations. All one has to do is transform the spacial dimensions using fft and then multiply the fourier components by a factor of the corresponding ik until you get the desired derivative, transforming back using ifft gives the results. It's worth noting that spectral methods are rarely used to approximate time derivatives you instead discretize the time derivative and use a time integrator like Euler, Adam Bashforth, or Runge Kutta to solve the differential equation. An example of the numerical derivative and spectral derivative of 2 functions can be seen in figure 3. From this figure, one can see both the power and fault of the spectral method. While yes, the spectral method can find the derivative of oscillatory functions exactly with relatively few grid points it does poorly with non oscillating function causing the error to blow up at the end points. This same effect carries over when spectral methods are used to approximate spatial derivatives in partial differential equations. Another downside to the spectral method is that you must have periodic boundary conditions in the differential equation because you cannot set the boundary conditions trivially as you can with complete discretization. As a result, this method is not very useful for our purposes of calculating the resonance but for fun we can assume periodic boundary conditions resulting in a new physical system to apply the spectral method upon.

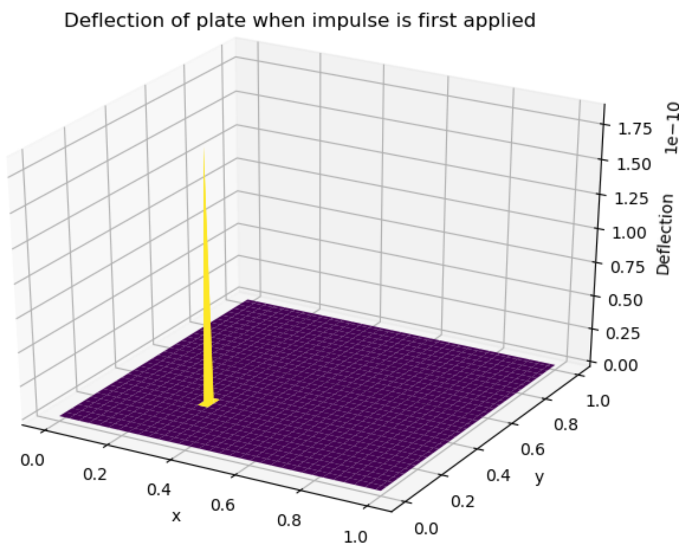

FIG. 6: Fourier transform of the dynamic response at x=0.85 meters and y=0.25 meters

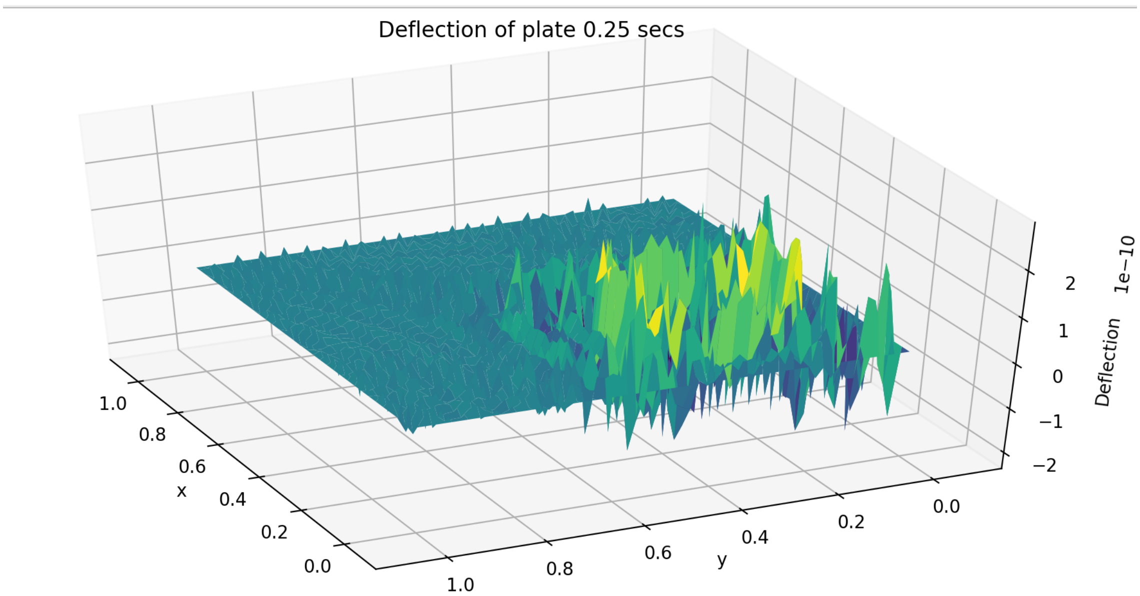

FIG. 7: Response of plate at 0.25 seconds

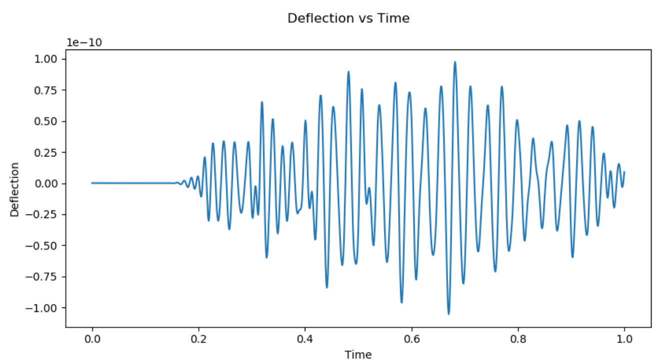

FIG. 8: Deflection vs time at x=0.85 meters and y=0.25 meters

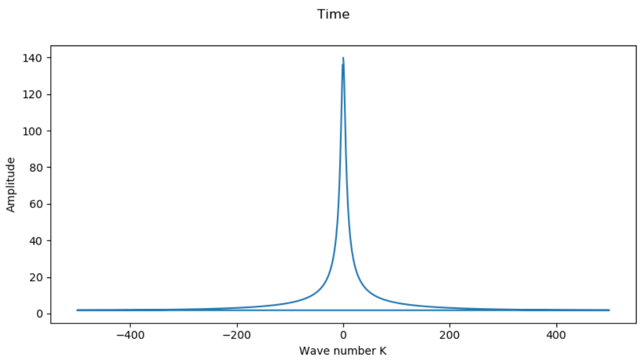

FIG. 9: Fourier transform of the dynamic response at x=0.85 meters and y=0.25 meters

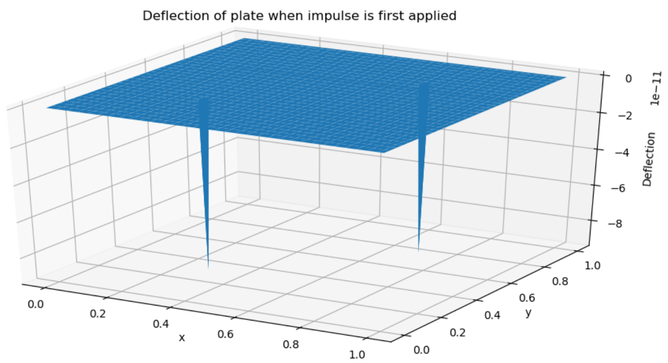

FIG. 10: Fourier transform of the dynamic response at x=0.85 meters and y=0.25 meters notice the two peaks that occured instead of the one, this is the result of the enforced periodic boundary conditions.

6. RESULTS

In much the same way I solved the nonlinear burger equation [1] discretized equation 1 using a Dirichlet boundary condition of 0 at all endpoints. An impulse load was then applied at a $x = 0.3w$ and $y = 0.3l$ in a 64X64 grid as can be seen in fig 6. The deflection of this grid at $x = 0.85m$ and $y = 0.25m$ is then recorded and results can be seen in figure 8.A fast fourier transform is then used to find the natural frequencies in the response of the system. High amplitudes for a given frequencies in the transform denote the natural frequencies of the square plate as can be seen in figure 5. I found nice agreement between my results and [1] for a number of natural frequencies but my results do not exactly match [1]. A comparison between figure 4 and 5 confirms that while my method is close there are some inconsistencies. I also didn't get the same shape at t=0.25 as the paper did which I can't explain beyond differences in computational power. Perhaps my solution would converge more smoothly to the results if I could take even smaller step sizes in time.

Approximating the derivatives using the spectral method gave a unique result. 2 peaks occurred on the grid in figure 11 likely a result of the assumed periodicity in the boundary conditions. The dynamic response of the system was very strange as well resembling a exponential in figure 12 instead of a sum of sines and cosines as it did in figure 8.

7. CONCLUSION

My deviation from [1] could be attributed to computational limits of my computer perhaps with a smaller step size I could get all the natural frequencies of the square plate. Overall though it seems that the dynamic response of a square plate can be analyzed efficiently using discretization.A result I wish I could spend more time analyzing is the response of the assumed periodic boundary conditions on the system. I expected that I would get the response of a closed cylinder, or something similar despite this breaking the kirchoff assumptions. Instead, I got a responses that seemed to show the impulse being mirrored diagonally across the square. This was an interesting result and

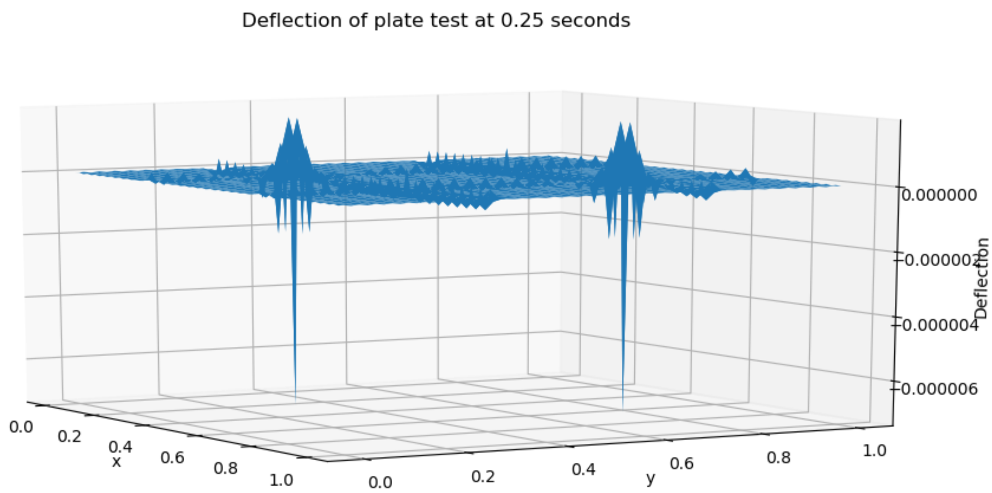

FIG. 11: Response of plate at 0.25 seconds of the spectral method notice the two peaks that occured instead of the one, this is the result of the enforced periodic boundary conditions.

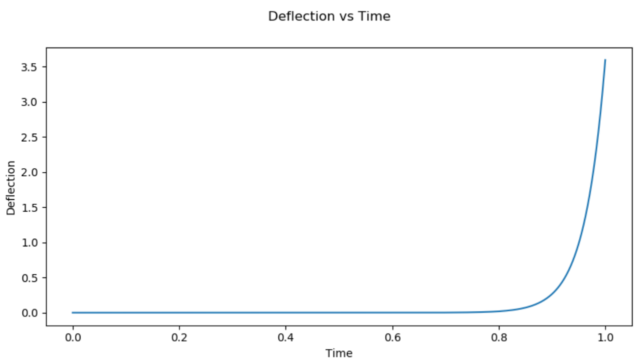

FIG. 12: Deflection vs time at x=0.85 meters and y=0.25 meters of the spectral method

further analysis is needed. It's worth mentioning that covid damped the progress of this project significantly, but I am pleased with what was able to be accomplished despite these tribulations. I would also like to acknowledge Dan Grin in providing direction in how I should pursue this project and guidance in completing it.Without his help none of this would be possible.

[1] Safizadeh, M Reza,Mat Darus, Intan, Mailah, Musa. *Calculating the frequency modes of flexible square plate using Finite Element and Finite Difference Methods.*. 2010 International Conference on Intelligent and Advanced Systems, 1-4. 10.1109/ICIAS.2010.5716169, 2010.
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[4] Singiresus S. Rao *Mechanical Vibrations fifth edition.* University of Miami, 2011.
[5] Mark Newman *Computational Physics.* University of Michigan, 2012.