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CENTRE MATHÉMATIQUES LAURENT SCHWARZ

PDE Analysis

ON THE EXISTENCE OF SMOOTH SOLUTIONS TO THE INCOMPRESSIBLE EULER AND NAVIER-STOKES EQUATIONS

BACHELOR OF SCIENCE THESIS - MATHEMATICS

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1 Preface

This thesis was written over the course of eight weeks and is dedicated to an exploration and analysis of aspects of the incompressible Euler equations. It is an early exposure for an undergraduate mathematics and physics student to partial differential equations analysis from a more fundamental and analytic standpoint, as opposed to the more hand-wavy, physics-oriented approaches outlined in the calculus, optics, and mechanics classes offered in the mathematics-physics track of École Polytechnique's Bachelor of Science. This thesis is undertaken after a course on topological and metric spaces and an introduction to functional analysis, a convenient prerequisite to work with regular maps defined on Banach spaces. Earlier exposure to Hilbert spaces and series of functions and thorough exposure to ordinary differential equations theory gives a good contextualization for this thesis. In contrast, the fact that it precedes a course on measure and integration means that some gaps in analysis may cause interruptions in the progress of the thesis's primary goal.

The thesis will be primarily based on the derivations in Jacob Bedrossian and Vlad Vicol's *The Mathematical Analysis of the Incompressible Euler and Navier-Stokes Equations: An Introduction*[1]. Most of the covered analysis will be based on Elliott H. Lieb and Michael Loss's *Analysis*[8]. All other sources will be properly indicated and cited as they appear.

The primary objective of this thesis is to cover the first two chapters of Bedrossian and Vicol's book in the allocated timeframe. These chapters cover the fundamentals of the incompressible Euler theory. The first chapter introduces the postulates and mathematical motivation for studying the incompressible, homogeneous Euler system. The second chapter is dedicated to proving the local-in-time existence of unique H^s-valued solutions and discussing the extension of solutions over a maximal time domain.

The intended structure of this thesis is summarized as follows. It consists of a chronologically arranged account of my exploration of the relevant equations, aspects of the systems of interest, and how we go about deriving solutions to said equations. While the thesis closely follows Bedrossian and Vicol's book to adequately present what is essential, it involves a personalized exploration of all the required analysis notions that are essential to overcome obstacles throughout the subsequent derivations. The challenge will be to have a solid, self-sufficient, complete understanding of the theory involved, without compromising the progress on PDE analysis due to extensive explorations in measure theory and Lebesgue and Sobolev spaces.

2 Ideal, Incompressible Fluids: The Euler Equations

A central assumption in much fluid mechanics theory is that the fluid is a continuum and that we can study its motion from a macroscopic perspective rather than on the molecular level. Roughly speaking, we assume that a fluid point contains a large number of molecules and that its characteristic length is much larger than the distance between neighboring molecules, all while being infinitesimal with respect to the domain's size. We will be working in $d-$ dimensional domains, with $d < +\infty$. A natural choice is $d = 3$, but this is not necessarily the case, especially in peculiar aspect-ratio settings; this is the case in the shallow-water model and its rotating counterpart, where we use $d = 2$. In what follows, we always assume $d \geq 2$.

2.1 The Eulerian and Lagrangian Viewpoints

Suppose the fluid occupies a fixed domain $\Omega \subset \mathbb{R}^d$ - in a lot of what follows we may simply take $\Omega = \mathbb{R}^d$ for simplicity.

Remark. An interesting, equally simple case to consider would be the periodic box, or $d-$ dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Naturally, this choice would bring about more regularity constraints at the boundaries.

There are two fundamental viewpoints through which one considers fluid dynamics, or the motion of a continuum in general.

2.1.1 The Eulerian Viewpoint

Take $T_1, T_2 \in \mathbb{R}$ to be two timestamps and consider a point $(t, x) \in (T_1, T_2) \times \Omega$ representing a fixed location x at which we measure quantities at time t. Such a measurement of hydrodynamic quantities constitutes the Eulerian approach - that is, to to measure, at any time, the quantities at a fluid point of choice. The hydrodynamic quantities of interest in this setting include the following.

- The velocity vector field of the fluid (experienced at a fluid point at a given time):

$$u : \begin{cases} [0, +\infty) \times \Omega & \longrightarrow \mathbb{R}^d \\ (t, x) & \longmapsto u(t, x) = (u_1, ..., u_d)(t, x) \end{cases}$$

  The dimensions of the velocity field are given as follows:

$$[u] = LT^{-1}$$

- The density scalar field (at agiven fluid point at a given time):

$$\rho : \begin{cases} [0, +\infty) \times \Omega & \longrightarrow \mathbb{R} \\ (t, x) & \longmapsto \rho(t, x) \end{cases}$$

$$[\rho] = M L^{-d}$$

- The pressure scalar field:

$$p : \begin{cases} [0, +\infty) \times \Omega & \longrightarrow \mathbb{R} \\ (t, x) & \longmapsto p(t, x) \end{cases}$$

$$[p] = ML^{-d+2}T^{-2}$$

The gradient of this scalar field is a force per unit area acting on fluid particles due to surrounding fluid particles. (Observe that $[F] = MLT^{-2}, [A] = L^{d-1}$ where the second quantity is what we will call the generalized surface area - when we work in 2D, our pressure gradient is a force per unit length,...)

We insist that the central characteristic of the Eulerian viewpoint is that measurements are conducted at fixed (t, x) points in space and time.

2.1.2 The Lagrangian Viewpoint

The Lagrangian viewpoint, rather than recording the hydrodynamic quantities at fixed locations at selected times, consists of working only with the locations of the fluid parcels that move from an initial, reference configuration. In the initial configuration, each fluid parcel is labeled by its location $a \in \Omega \subseteq \mathbb{R}^d$. The particle trajectory function of the fluid parcels is defined as follows.

$$X : \begin{cases} [0, +\infty) \times \Omega & \longrightarrow \mathbb{R}^d \\ (t, a) & \longmapsto X(t, a) \end{cases}$$

The vector $X(t, a)$ is the position, at time t, of the fluid parcel initially located at a. We call the Lagrangian flow map at time t the function that, a fixed time t, retrieves all the fluid parcel locations:

$$X(t, \cdot) : \begin{cases} \Omega & \longrightarrow \mathbb{R}^d \\ a & \longmapsto X(t, a) \end{cases}$$

2.1.3 The Flow-Trajectory Relation

The connection between the Eulerian and Lagrangian viewpoints is encoded in the fact that the trajectory followed by a fluid parcel is directly linked to the velocity of fluid parcels passing through the different fluid points. Mathematically, the relation writes as follows.

$$\partial_t X(t, a) = u(t, X(t, a)) \tag{1}$$

The relation in Equation 1 can be interpreted as follows. The left-hand side is the speed of the fluid parcel initially located at a measured at time t. The right-hand side is the fluid velocity field at time t at the location occupied, at time t, by the fluid parcel initially at a (that is, the fluid parcel of interest.) It isn't shocking, from an intuitive perspective, that the fluid moves with a velocity equal to the fluid velocity at the space it occupies, at all times, in light of the continuum assumption. We endow the ODE (for any fixed $a \in \Omega$) in Equation 1 with the intuitive, initial condition imposed by the reference configuration:

$$X(0, a) = a \tag{2}$$

Then, we get a family of first-order initial condition problems indexed by $a \in \Omega$ and described by Equation 1 and Equation 2. We would like to make claims on solutions to this initial condition problem, so we must impose some regularity on the velocity field. A direct application of the Cauchy-Lipschitz theorem yields the following assertion. If $u \in C_t^0 \mathrm{Lip}_x$, then the system of equations above admits a unique maximal solution for any $a \in \Omega$. Hereafter, we shall assume $u \in C_t^0 \mathrm{Lip}_x$ unless explicitly mentioned otherwise. Under these assumptions, for any chosen $a \in \Omega$, we have a unique maximal solution $X(t, a)$ to the system of equations which yields the particle trajectory. Something a bit less obscure than the application of the Cauchy-Lipschitz theorem is the fact that the Lagrangian flow map turns out to be a C^1 diffeomorphism at all times under the $t-$ continuity of u and its $x-$ Lipschitz continuity - the fact that we choose u to be spatially Lipschitz and not just locally Lipschitz will enable the solution's definition at all times t. This is a crucial aspect of the flow map and our subsequent analysis and we prove the following results inspired by the theory of ordinary differential equations. We first state, without its proof, the Grönwall lemma, which quantitatively illustrates the control of nonlinear distortions by other fields and, more importantly, can reveal functions that cannot feature finite-time blowup.

Lemma 2.1. *(Grönwall lemma) [2] Let $a < b$ and let ϕ, ψ, y be three continuous $[a, b \to [0, +\infty[$ functions such that $\forall t \in [a, b]$, we have:*

$$y(t) \leq \phi(t) + \int_a^t y(s)\psi(s)ds$$

Then, for all $t \in [a, b]$, we have:

$$y(t) \leq \phi(t) + \int_a^t \psi(s)\phi(s) \exp\left(\int_s^t \psi(u)du\right) ds$$

We will use the Grönwall lemma to show that, under the aforementioned regularity, we have a Lagrangian flow map at all times $t \in \mathbb{R}$ (provided that is the time domain of definition of the velocity field).

Lemma 2.2. *If $u \in C_t^0 Lip_x$, then, the system given by Equation 1 and Equation 2 admits a maximal solution defined at all times $t \in \mathbb{R}$ for any $a \in \mathbb{R}^d$.*

Proof. For the sake of simplicity in the proof, we may, throughout this proof, assume some additional regularity on u that is not explicitly mentioned; the purpose of this proof is to illustrate how the characteristics of the flow map come about and not to dedicate, at this stage, an in-depth exploration to the minimal required regularity for this result to hold. Observe that, by Cauchy-Lipschitz, maximal solutions must be defined on open intervals of $\mathbb{R}$. Indeed, if a maximal solution is defined on an interval of the form $]a, b]$, then applying the theorem on a Cauchy problem with an initial value appropriately specified at b yields a valid extension of the domain of the solution, contradicting its maximality. Let $a \in \mathbb{R}^d$ be arbitrary and assume $X(\cdot, a)$ is a maximal solution to our system defined on some open interval $]c, b[$, with $c < b$. Assume, for the sake of contradiction, that $b < +\infty$. Take $t_0 \in]c, b[$. For every $t \in [t_0, b[$, we have the following, where we denote $K > 0$ the Lipschitz constant of the functions $\{u(t, \cdot)\}_{t \in \mathbb{R}}$

$$\partial_t X(t, a) = u(t, X(t, a)) = u(t, X(t_0, a)) + u(t, X(t, a)) - u(t, X(t_0, a))$$

$$\implies ||\partial_t X(t, a)|| \leq ||u(t, X(t_0, a))|| + K||X(t, a)|| + K||X(t_0, a)||$$

Observe that, as a continuous function on $\mathbb{R}$, $u(\cdot, X(t_0, a))$ is bounded on all compact intervals of $\mathbb{R}$. Hence, denote $\beta := \sup_{t \in [t_0, b]} ||u(t, X(t_0, a))|| + K(X(t_0, a)) < +\infty$. It immediately follows that:

$$||\partial_t X(t, a)|| \leq \beta + K||X(t, a)||$$

As a consequence of the Grönwall lemma, Proposition A.1 asserts that the following holds for every $t \in [t_0, b[$.

$$||X(t, a)|| \leq ||X(t_0, a)|| \exp(K(t - t_0)) + \frac{\beta}{K}(\exp(K(t - t_0)) - 1) \tag{3}$$

Since the right-hand side of Equation 3 is bounded as $t \to b$, we conclude that the graph of $X(\cdot, a)$ for $t \in]t_0, b[$ is contained in a compact set, and consequently, by Proposition A.2, we can extend the solution $X(\cdot, a)$ to $]c, b]$, solve the initial value problem at b by specifying $X(b, a)$ as the initial condition, and contradict the assumed maximality $X(\cdot, a)$ thanks to an application of Cauchy-Lipschitz. The same can be done for the extremity c, and we conclude. $\square$

With the above established, we can prove that the Lagrangian flow map is a C^1 diffeomorphism at all times $t \in \mathbb{R}$.

Theorem 2.3. *Under the aforementioned regularity, the Lagrangian flow map is a C^1 diffeomorphism at all times.*

Proof. We first prove it is injective. Let $t_0 \in \mathbb{R}$, and suppose $\exists a_1, a_2 \in \mathbb{R}^d$ such that $X(t_0, a_1) = X(t_0, a_2)$. Define $r := X(t_0, a_1)$. Then, both $X(\cdot, a_1)$ and $X(\cdot, a_2)$ solve the initial value problem given by Equation 1 (as an equation of time and a function of time and its time derivative) endowed with the condition $X(t_0, a) = r$ also interpreted as a condition at time t, while a merely characterizes the function. Consequently, by Cauchy-Lipschitz theorem, $X(\cdot, a_1)$ and $X(\cdot, a_2)$ coincide where they are both defined. As they are initially solutions to the initial value problem with condition $X(0, a) = a$, they are both defined at $t = 0$ and coincide there, so $a_1 = a_2$ and we conclude that the flow map is injective. Let us now show it is subjective. Let $t_0 \in \mathbb{R}$ be arbitrary and fixed. Consider the map $X(t_0, \cdot)$ and let $a \in \mathbb{R}^d$ be arbitrarily chosen. We claim the following.

$$X(t_0, X(-t_0, a)) = a$$

Observe that $X(\cdot, X(-t_0, a))$ solves the initial value problem with initial condition $X(-t_0, a)$ at 0. Define $y(t) = X(t - t_0, a)$. One gets that y solves the same ODE as $X(\cdot, a)$ with initial condition $y(0) = X(-t_0, a)$. Since y and $X(\cdot, X(-t_0, a))$ solve the same initial value problem, they must coincide, and consequently $y(t_0) = X(t_0, X(-t_0, a)) = a$ because $X(t_0 - t_0, a) = X(0, a) = a$. We conclude that the Lagrangian flow map is a bijection. Observe that, using the same approach, we can actually show the following semi-group property:

$$X(t + s, a) = X(t, X(s, a)), \quad \forall t, s, a$$

We also observe that the inverse of the flow map is $X(-t, \cdot)$ and hence satisfies the same properties as the flow map, so that any regularity proven for the flow map holds for its inverse. There remains, at this stage, to establish that it is of class C^1. This is a deeper result whose intricacies go out of the scope and focus of this thesis, so we merely state that the differentiability of the solution to the ODE with respect to the initial condition (that is, the differentiability of the flow map) under sufficient regularity is proven in Cartans' *Differential Calculus* [4]. As mentioned above, assuming marginally more regularity is overlooked at this stage for the sake of simplicity. $\square$

When the Lagrangian flow map is a C^1 diffeomorphism, we define its inverse, the back-to-labels map $A(t, \cdot) : \Omega \longrightarrow \Omega$ that satisfies the following.

$$A(t, x) = a \iff X(t, a) = x$$

In other words, A gives us the label, or initial position, of whichever fluid parcel is located at x at time t. We will interchangeably write A and X^{-1}.

2.2 The Chord-Arc Condition

Using the relation between the particle trajectories and fluid velocity field, we would like to quantitatively characterize the evolution of the distance between two fluid parcels, which we expect to be controlled by the velocity field, which governs the velocity of the fluid parcels as outlined in Equation 1. We first recall an important result on differentiable Lipschitz continuous functions.

Proposition 2.4. *Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is $k-$ Lipschitz continuous (with $k > 0$) and that all of its partial derivatives exist everywhere. Then, all of its partial derivatives are bounded.*

Proof. We will denote $\partial_i f(x)$ the partial derivative of f with respect to the $i-$th variable at x. Take $h \neq 0$ in a neighborhood of 0 (in $\mathbb{R}$) and let $(e_1, ..., e_d)$ denote the canonical basis of $\mathbb{R}^d$. We have the following by the

Lipschitz continuity of f.

$$\left|\frac{f(x+he_i)-f(x)}{h}\right| \leq \frac{1}{|h|}k||he_i|| = k$$

Taking the limit $h \to 0$ on both sides yields:

$$|\partial_i f(x)| \leq k$$

Hence, we've shown that all of f's partial derivatives are bounded. We have even shown that they are all bounded by the Lipschitz constant of f. □

Remark. It is important to note that the spatial partial derivatives of u are bounded over all spatial variables at every fixed time, but the bound is possibly time-dependent; t is fixed when making the manipulations in the above proposition are made.

We are now in a position to synthesize estimates on trajectory and parcel-related distortions and the velocity field governing the fluid's domain. Suppose the velocity field is C^1 in addition to being $C_t^0\text{Lip}_x$, and that the particle trajectories are C^2 so that all of its second-order partial derivatives exist and Schwarz's theorem applies. Differentiating the system given by Equation 1 and Equation 2 with respect to the spatial variables, we get the following system after applying the chain rule and Schwarz's theorem.

$$\forall i,k \in \{1,...,d\}, \quad \frac{\partial^2}{\partial t \partial a_k} X_i(t,a) = \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} u_i(t, X(t,a))\right) \cdot \frac{\partial}{\partial a_k} X_j(t,a) \tag{4}$$

$$\partial_{a_k} X_i(t,a) = \delta_{ki} \tag{5}$$

In Equation 5, δ_{ki} denotes the Kronecker symbol. Observe that Equation 4 can be written in the matrix form by considering the Jacobian matrix of $X(t,a)$ taken over the spatial variables (which is equivalent to taking the Jacobian matrix of the Lagrangian flow map at t) which we will denote $\text{Jac}_x(X)(t,a)$.

$$\partial_t \text{Jac}_x(X)(t,a) = \text{Jac}_x(u)(t, X(t,a)) \cdot \text{Jac}_x(X)(t,a) \tag{6}$$

Remark. We assume all the regularity we need in these a priori estimates. We discuss this aspect of the approach more extensively in later sections.

Observe that Equation 5 is equivalent to $\text{Jac}_x(X)(0,a) = I$, where I denotes the identity matrix. Integrating Equation 6 with respect to time yields the following.

$$\text{Jac}_x(X)(t,a) = I + \int_0^t \text{Jac}_x(u)(s, X(s,a)) \cdot \text{Jac}_x(X)(s,a) ds$$

Taking the norm and applying triangle inequality, we get the following for an appropriately chosen norm (and inspecting the entry-wise time integration).

$$|\text{Jac}_x(X)(t,a)| \leq 1 + \int_0^t |\text{Jac}_x(u)(s, X(s,a))| |\text{Jac}_x(X)(s,a)| ds$$

Observe that, by Proposition 2.4, we have that the maximal entry norm of the matrix is bounded as all the partial derivatives themselves are bounded and finite in number; by the equivalence of norms in finite dimension, it follows that any norm of $\text{Jac}_x(u)(s, X(s,a))$ is bounded at any time over all the spatial variables, so we can define the following quantity, which is a time-dependent function only.

$$||\text{Jac}_x(u)||_{L^\infty}(s) := \sup_{x \in \mathbb{R}^d} |\text{Jac}_x(u)(s,x)| < +\infty$$

It immediately follows that:

$$|\mathrm{Jac}_x(X)(t,a)| \leq 1 + \int_0^t ||\mathrm{Jac}_x(u)||_{L^\infty}(s)|\mathrm{Jac}_x(X)(s,a)|ds$$

Under regularity conditions (spatial derivatives continuous in t) and since the norm is a Lipschitz application due to the (reverse) triangle inequality, we can apply the Grönwall lemma after fixing $a \in \mathbb{R}^d$ to get one-variable functions and we get the following for any time t.

$$|\mathrm{Jac}_x(X)(t,a)| \leq \exp\left(\int_0^t ||\mathrm{Jac}_x(u)||_{L^\infty}(s)ds\right), \quad \forall a \in \mathbb{R}^d \tag{7}$$

Taking the supremum over the fluid parcels and noting that the right-hand side is $a-$ independent, we get the following.

$$\sup_{a\in\mathbb{R}^d} |\mathrm{Jac}_x(X)(t,a)| \leq \exp\left(\int_0^t ||\mathrm{Jac}_x(u)||_{L^\infty}(s)ds\right) \tag{8}$$

Hence, the velocity field and its Fréchet differential modulate the volume distortions due to the motion of fluid parcels. We shift our focus slightly to look at the back-to-labels map and make similar manipulations. Differentiating the equation $x = X(t, A(t,x))$ with respect to time yields the following for all $k \in \{1, ..., d\}$.

$$0 = \partial_t X_k(t, A(t,x)) + \sum_{i=1}^{d} \partial_{a_i} X_k(t, A(t,x))\partial_t A_i(t,x) \tag{9}$$

$$\iff 0 = u_k(t,x) + \sum_{i=1}^{d} \partial_{a_i} X_k(t, A(t,x))\partial_t A_i(t,x) \tag{10}$$

Then, differentiating $a = A(t, X(t,a))$ with respect to the label, we get the following.

$$\delta_{ij} = \sum_{k=1}^{d} \partial_{x_k} A_i(t, X(t,a))\partial_{a_j} X_k(t,a) \tag{11}$$

In light of the spatial derivatives of the flow map appearing in both Equation 10 and Equation 11, we choose to left multiply Equation 10 (in its vectorial form, taken over $k \in \{1, ..., d\}$) by $\mathrm{Jac}_x(A)(t,x)$. We get a vectorial equation, and the $i-$ component of the system writes as follows, where we omit the understood arguments t, x, a for clarity.

$$\sum_{k=1}^{d}\left(\sum_{j=1}^{d} \partial_{a_j} X_k \partial_t A_j\right)\partial_{x_k} A_i + \sum_{k=1}^{d} u_k \partial_{x_k} A_i = 0 \tag{12}$$

Thanks to Equation 11, the first term of the above equation simplifies to $\partial_t A_i$ after eliminating all terms with $i \neq j$ and identifying the remaining sum as $\delta_{ii} = 1$. We get the following.

$$\partial_t A_i + \sum_{k=1}^{d} u_k \partial_{x_k} A_i = 0$$

Representing the system with a vectorial equation and commonly used shorthand notation, we get:

$$\partial_t A + u \cdot \nabla A = 0 \tag{13}$$

In the above, $u \cdot \nabla A$ is a vector and its $i-$th component writes as follows.

$$(u \cdot \nabla A)_i = \sum_{k=1}^{d} u_k \partial_k A_i$$

By differentiating spatially **along the flow map** and assuming the spatial and time derivatives commute, we get the following:

$$\partial_t(\mathrm{Jac}_x(A)(t, X(t,a))) = -\mathrm{Jac}_x(u)(t, X(t,a)) \cdot \mathrm{Jac}_x(A)(t, X(t,a)) \tag{14}$$

Remark. The second-order spatial derivatives of A vanish when evaluating it along the flow map due to Equation 11.

From this point, we apply the same manipulations as we did earlier for the flow map and Grönwall's lemma yields the following estimate.

$$\sup_{a\in\mathbb{R}^d} |\mathrm{Jac}_x(A)(t, X(t,a))| \leq \exp\left(\int_0^t ||\mathrm{Jac}_x(u)||_{L^\infty}(s)ds\right) \tag{15}$$

We can now prove a quantitative constraint on fluid parcel motion with respect to the velocity field to which the parcels are subjected.

Corollary 2.4.1. *(Chord Arc Condition) Under sufficient regularity, we have the following for any time $t > 0$ and $a, b \in \mathbb{R}^d$.*

$$\exp\left(-\int_0^t ||Jac(u)||_{L^\infty}(s)ds\right) \leq \frac{|a-b|}{|X(t,a) - X(t,b)|} \leq \exp\left(\int_0^t ||Jac(u)||_{L^\infty}(s)ds\right)$$

Proof. From the mean value theorem, there exists some ξ on the line joining $X(t,a)$ and $X(t,b)$ (assuming our domain in convex) such that the following holds.

$$|A(t, X(t,a)) - A(t, X(t,b))| = |\mathrm{Jac}_x(A)(t,\xi) \cdot (X(t,a) - X(t,b))|$$

It follows that, up to a potential multiplicative constant resulting from norm equivalence and the fact that the spectral norm is the one known to satisfy sub-multiplicativity:

$$\begin{aligned}
|A(t, X(t,a)) - A(t, X(t,b))| &\leq |\mathrm{Jac}_x(A)(t,\xi)| \cdot |(X(t,a) - X(t,b))| \\
&\leq \sup_{p\in\mathbb{R}^d} |\mathrm{Jac}_x(A)(t, X(t,p))| \cdot |(X(t,a) - X(t,b))| \\
&\leq \exp\left(\int_0^t ||\mathrm{Jac}(u)||_{L^\infty}(s)ds\right) \cdot |(X(t,a) - X(t,b))|
\end{aligned}$$

In the above, we have used that $X(t,\cdot)$ is a bijection, which follows from Theorem 2.3. We conclude on the upper bound by noting that $|A(t, X(t,a)) - A(t, X(t,b))| = |a-b|$. For the lower bound, we apply the mean value theorem on the Lagrangian flow map. We get the following, again assuming convexity on the domain.

$$|X(t,a) - X(t,b)| \leq \sup_{p\in\mathbb{R}^d} |\mathrm{Jac}_x(X)(t,p)| \cdot |a-b| \leq |a-b| \exp\left(\int_0^t ||\mathrm{Jac}(u)||_{L^\infty}(s)ds\right)$$

It is important that our proof involves the sub-multiplicativity of the spectral norm. □

We state below a proposition involved in the above proof.

Proposition 2.5. *Suppose $f : \mathbb{R}^d \to \mathbb{R}^d$ is differentiable and Lipschitz. Then, we have the following for any norm*

$|\cdot|$.

$$\sup_{x\in\mathbb{R}^d} |Jac(f)(x)| < +\infty$$

Proof. This follows from Proposition 2.4, the fact that $|\cdot|_\infty : A \mapsto \max_{i,j} |A_{ij}|$ is a norm on the $d \times d$ matrices, and that all norms are equivalent in finite dimension. □

The chord arc condition is a quantitative description of how fluctuations in the velocity field modulate how differently the trajectories of two neighboring parcels will evolve with time. This is the spirit of the Grönwall lemma, which allows us to control some forms of nonlinear evolution.

2.3 Incompressibility and Transport

We now shift our focus to the movement of matter and the interaction of the "stationary" velocity field and the Lagrangian flow. First, we define incompressible fluids, which we will be working with for the bulk of this study. Denote our domain $\Omega \subset \mathbb{R}^d$. We are interested in showing that incompressibility, in the intuitive sense that the volume of a collection of parcels is time-invariant, is equivalent to the fact that the velocity field is divergence-free everywhere.

2.3.1 Incompressible Flow

Let us first define incompressible velocity fields and volume-preserving flows. Take any collection of parcels $V \subset \Omega$ in the initial configuration and define the following.

$$V(t) := \{X(t,a) : a \in V\} = X(t,V)$$

Intuitively, $V(t)$ is the collection of positions at time t occupied by the parcels originally located in positions in V. We say that u is incompressible if the Lagrangian flow map (governed by the velocity field, of course) is volume preserving; that is, we have the following for all $t \in \mathbb{R}$ and all $V \subset \Omega$ measurable with respect to the Lebesgue measure:

$$\lambda(V(t)) = \lambda(V)$$

In the above, λ denotes the Lebesgue measure. Without elaborating on its definition, we recall that the Lebesgue measure has the convenient property of giving the Euclidean volume of nice sets. This means that integrating the identity with respect to the Lebesgue measure gives us what we know as a volume, and this makes the definition of incompressible plausibly pertinent. We want to get away from this definition and characterize incompressibility with the velocity field.

Lemma 2.6. *The definition of incompressibility with a volume-preserving flow map is equivalent to a definition requiring the velocity field to have 0 divergence everywhere.*

Proof. We first begin with the following change of variable when computing the volume of $V(t)$ using simple multivariable calculus. For an arbitrary, regular enough function f, we have:

$$\int_{V(t)} f(x)dx = \int_V f(X(t,a)) \det \mathrm{Jac}_x(X)(t,a)da \tag{16}$$

Recall that the Lagrangian is a C^1 diffeomorphism at all times. Hence, its Fréchet derivative is an isomorphism everywhere [3]; in particular, $\mathrm{Jac}_x(X)(t,a)$ is invertible for all the fluid parcels, at any time. We use this fact to compute the time derivative of the Jacobian determinant and then use Equation 1 to create an ODE on this determinant. We have the following where we have used Jacobi's formula.

$$\partial_t \det \mathrm{Jac}_x(X)(t,a) = \det \mathrm{Jac}_x(X)(t,a)\mathrm{Tr}(\mathrm{Jac}_x(X)(t,a)^{-1} \cdot \partial_t \mathrm{Jac}_x(X)(t,a))$$

Using Equation 6 which is based on the relation between the trajectory and velocity, we substitute the expression for $\partial_t \mathrm{Jac}_x(X)(t,a)$ and use the fact that $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$ and get the following.

$$\partial_t \det \mathrm{Jac}_x(X)(t,a) = \det \mathrm{Jac}_x(X)(t,a) \mathrm{Tr}(\mathrm{Jac}_x(u)(t, X(t,a)))$$

By inspection, we identify the trace of the derivative of u as its divergence (spatial) and we conclude:

$$\partial_t \det \mathrm{Jac}_x(X)(t,a) = \det \mathrm{Jac}_x(X)(t,a) \cdot (\nabla \cdot u)(t, X(t,a)) \tag{17}$$

Solving the ODE in Equation 17 yields the following expression for $J(t,a) := \det \mathrm{Jac}_x(X)(t,a)$.

$$J(t,a) = J(0,a) \cdot \exp\left(\int_0^t (\nabla \cdot u)(s, X(s,a)) ds\right) \tag{18}$$

From Equation 5, we know that $J(0,a) = 1$. Hence, we establish the intermediate results that $J(t,a) = 1$ at all times everywhere if and only if the following holds everywhere for all times.

$$\exp\left(\int_0^t (\nabla \cdot u)(s, X(s,a)) ds\right) = 1 \iff \nabla \cdot u \equiv 0$$

On the other hand, we have that the volume is given by:

$$|V(t)| = \int_{V(t)} dx = \int_V J(t,a) da \tag{19}$$

We immediately recover that the flow is incompressible if and only if $J(t,a) = 1$ almost everywhere at all times, which occurs if and only if u is divergence-free. □

2.3.2 Transport and the Convective Derivative

In this section, we emphasize on the different interpretations of the variation of fields with time. Consider a scalar field (such as pressure or density) $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, (t,x) \mapsto f(t,x)$. We can consider the variation of f with time at a fixed location x. We denote it $\partial_t f(t,x)$ and this is the Eulerian view of the time derivative. In contrast, we can measure f by fixing the parcel of fluid at which we measure it, yielding a Lagrangian approach to time evolution. This writes as the time derivative of the entire functional.

$$(D_t f)(t, X(t,a)) = \partial_t (f(t, X(t,a)))$$

The chain rule yields the following.

$$(D_t f)(t, X(t,a)) = \partial_t f(t, X(t,a)) + \sum_{k=1}^{d} \partial_{x_k} f(t, X(t,a)) \cdot \partial_t X_k(t,a)$$

Equivalently, the above writes as follows where the gradient is taken over the spatial part.

$$(D_t f)(t, X(t,a)) = \partial_t f(t, X(t,a)) + \partial_t X(t,a) \cdot \nabla f(t, X(t,a))$$

Using Equation 1, this is also:

$$(D_t f)(t, X(t,a)) = \partial_t f(t, X(t,a)) + u(t, X(t,a)) \cdot \nabla f(t, X(t,a))$$

More compactly, we have:

$$D_t f = \partial_t f + u \cdot \nabla f$$

Remark. We've assumed the functions are C^1.

The convective derivative arises from the need to understand the transport of fluid parcels and quantities. We are generally interested in the fluctuations of scalar fields over transported volumes containing a fixed collection of fluid parcels. From this interest arises the transport theorem.

Theorem 2.7. *(Transport Theorem) Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable scalar field. Assume that u is also C^1 and denote $V(t)$ the pushforward of a volume V under the flow map. We have the following.*

$$\frac{d}{dt}\left(\int_{V(t)} f(t,x)dx\right) = \int_{V(t)} \left(\partial_t f + \nabla \cdot (fu)\right)(t,x)dx$$

Proof. Using the change of coordinate formula - owing to Theorem 2.3 - we get the following, where we pass by an intermediate internal over a fixed volume to interchange integration and differentiation.

$$\begin{aligned}\frac{d}{dt}\left(\int_{V(t)} f(t,x)dx\right) &= \frac{d}{dt}\left(\int_{V} f(t,X(t,a))J(t,a)da\right)\\ &= \int_V ((D_t f)(t,X(t,a))J(t,a) + f(t,X(t,a))\partial_t J(t,a))da\end{aligned}$$

Using the expression of the convective derivative and Equation 13 for the expression of the Jacobian determinant's time derivative, we get the following.

$$\begin{aligned}\frac{d}{dt}\left(\int_{V(t)} f(t,x)dx\right) &= \int_V (\partial_t f + u \cdot \nabla f + f \cdot (\nabla \cdot u))(t,X(t,a)) \cdot J(t,a)da\\ &= \int_V (\partial_t f + \nabla \cdot (fu))(t,X(t,a)) \cdot J(t,a)da\\ &= \int_{V(t)} (\partial_t f + \nabla \cdot (fu))(t,x)dx\end{aligned}$$

□

So far, all our derivations and quantitative characterizations have been consequences of ODE theory due to the relation between the trajectory map and the velocity field as well as general mathematical results that manifest themselves under sufficient regularity. Equipped with these characterizations, we can now extract evolution equations from physical postulates.

2.4 The Incompressible, Homogenous Euler Equations

As Dalton and Lavoisier have long asserted, we assume that mass is neither created nor destroyed, and we apply this principle to a volume element moving in the fluid.

2.4.1 The Conservation of Mass

Let $V \subset \Omega \subset \mathbb{R}^d$ be a volume in the fluid. The mass of this volume is defined as follows.

$$m(t,V) = \int_V \rho(t,x)dx$$

The conservation of mass in our fluids translates to the following equation, where we use the same notation for the pushforward (image) of a volume under the flow map.

$$\frac{d}{dt}m(t, V(t)) = 0, \quad \forall V \subset \Omega \tag{20}$$

One can apply the scalar transport equation to the mass and obtain both an integral and differential form of the conservation law.

Theorem 2.8. *Under sufficient regularity, Equation 20 is equivalent to the following.*

$$\int_{V(t)} (\partial_t \rho + \nabla \cdot (\rho u))(t, x) dx = 0, \quad \forall V \subset \Omega \tag{21}$$

Provided the terms in the above integrated are continuous, the fact that Equation 21 holds for all initial volumes translates to the following differential form.

$$\partial_t \rho + \nabla \cdot (\rho u) = 0 \tag{22}$$

Proof. That Equation 21 holds results from a direct application of Theorem 2.7. As for the differential form, assuming the integrand of Equation 21 is continuous, we have that the following.

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x, \varepsilon)|} \int_{B(x,\varepsilon)} (\partial_t \rho + \nabla \cdot (\rho u))(t, y) dy = (\partial_t \rho + \nabla \cdot (\rho u))(t, x)$$

□

The integrated on the right-hand side of the transport theorem is, in general, similar but not identical to the convective derivative. However, when the flow is incompressible, we have, by Lemma 2.6, the following.

$$\partial_t f + \nabla \cdot (f u) = \partial_t f + u \cdot \nabla f + f \cdot (\nabla \cdot u) = \partial_t f + u \cdot \nabla f = D_t f$$

Hence, in light of Theorem 2.8 and the definition of the convective derivative, our mass conservation writes as follows.

$$\frac{d}{dt} (\rho(t, X(t, a))) = 0, \quad \forall (t, a) \in \mathbb{R} \times \Omega \tag{23}$$

It follows that:

$$\rho(t, X(t, a)) = \rho_0(a) \iff \rho(t, x) = \rho_0(A(t, x))$$

In other words, the density of a moving fluid parcel never changes. When all fluid parcels have the same density, we say the density or the fluid is homogenous and we have $\rho(t, x) = \rho_0$.

2.4.2 Momentum Balance

The second physical postulate we will use is the conservation of momentum on the scale of any volume element. Volume elements are subjected to body forces and traction forces; the former act on every point of the volume while the latter act on the boundaries as a result of inter-volume or inter-layer interactions within the fluid. We note that the following equations are postulates that result from Cauchy's theorem and Newton's second and third laws of motion, the latter of which brings about the conservation of linear momentum in the absence of external forces. Leaving a more extensive discussion on continuum mechanics in A.2, we recall that by Cauchy's stress theorem, there exists a symmetric matrix called the stress tensor $\sigma(t, x)$ that modulates the traction vector on the surface of a fluid element at any point in time and space by the equation $t = \sigma(x, t)n(t, x)$ where n is the outward unit normal of the surface in question. Assuming no body forces act on the fluid elements, Newton's second law of motion applied on a volume element $V(t)$ writes as follows, where we use the fact that the traction forces that do not cancel each other lie on the boundary of the fluid element.

$$\frac{d}{dt}\left(\int_{V(t)} \rho(t,x)u(t,x)dx\right) = \int_{\partial V(t)} \sigma(t,x)\cdot n(t,x)dS(x) \tag{24}$$

As mentioned in A.2, we assume we are working with an ideal fluid and consequently, we have the following.

$$\sigma(t,x) = -p(t,x)\cdot I$$

Assuming the required smoothness and applying the divergence theorem to Equation 24 as well as the ideal fluid assumption, we get the following.

$$\frac{d}{dt}\left(\int_{V(t)} \rho(t,x)u(t,x)dx\right) = -\int_{V(t)} \nabla p(t,x)dx \tag{25}$$

Note that, in the above, the divergence theorem was applied on each row of the stress tensor, and since it is diagonal, the three scalar equations can be written in a vectorial equation using the gradient of the pressure (recall that the velocity is a vector field.) We are interested in having a differential form for the momentum balance, as we proceeded for mass conservation.

Theorem 2.9. *(Density Transport Theorem) Assume f, ρ are C^1 scalar fields. Then, under the conservation of mass, we have the following for all $t \geq 0$ and for any open set $V \subset \Omega$.*

$$\frac{d}{dt}\left(\int_{V(t)} \rho(t,x)f(t,x)dx\right) = \int_{V(t)} \rho(t,x)D_t f(t,x)dx \tag{26}$$

Proof. Appealing to Theorem 2.7, we have the following.

$$\frac{d}{dt}\left(\int_{V(t)} \rho(t,x)f(t,x)dx\right) = \int_{V(t)} (\partial_t(\rho f) + \nabla\cdot(\rho f u))(t,x)dx$$

Applying Equation 22, the right hand side simplifies as follows.

$$\frac{d}{dt}\left(\int_{V(t)} \rho(t,x)f(t,x)dx\right) = \int_{V(t)} (\rho\partial_t f + \rho u\cdot\nabla f)(t,x)dx$$

Factoring out the density and recognizing the convective derivative fo f, we conclude.

$$\frac{d}{dt}\left(\int_{V(t)} \rho(t,x)f(t,x)dx\right) = \int_{V(t)} (\rho D_t f)(t,x)dx$$

□

We extend the definition of convective derivative to vector fields as follows, denoting $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$.

$$D_t f = (D_t f_i)_{1\leq i\leq d} = \partial_t f + (u\cdot\nabla f_i)_{1\leq i\leq d}$$

By abuse of notation, we will denote $(u\cdot\nabla f_i)_{1\leq i\leq d} = (u\cdot\nabla)f$. Applying the density transport to the velocity field as well as Equation 25 yields the following.

$$\int_{V(t)} \rho(t,x)(\partial_t + (u\cdot\nabla)u)(t,x)dx = -\int_{V(t)} \nabla p(t,x)dx \tag{27}$$

Under sufficient regularity, the arbitrariness of the volume element yields the following.

$$\rho \cdot D_t u = \rho \cdot (\partial_t u + (u \cdot \nabla)u) = -\nabla p \tag{28}$$

Finally, the same derivations can be made in the presence of body forces. Newton's second law applied on a volume element writes as follows, where f denotes the body force.

$$\frac{d}{dt}\left(\int_{V(t)} \rho(t,x)u(t,x)dx\right) = -\int_{\partial V(t)} p(t,x) \cdot n(t,x)dS(x) + \int_{V(t)} \rho(t,x)f(t,x)dx \tag{29}$$

Applying the aforementioned derivations, the differential form of the momentum balance postulate becomes:

$$\rho(\partial_t u + (u \cdot \nabla)u) = -\nabla p + \rho f \tag{30}$$

Remark. The body force is a force per unit mass.

2.4.3 The Euler Equations for Ideal, Homogenous, Incompressible Fluids

A close inspection of the premise in Theorem 2.9 shows us that mass conservation is encoded in Equation 30. Hence, our assumptions are summarized in the following system, where the first line includes mass conservation and momentum balance as well the homogeneity of the fluid and the second line includes incompressibility.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\frac{1}{\rho_0}\nabla p + f \\ \nabla \cdot u = 0 \end{cases} \tag{31}$$

The above equations form a Cauchy problem governing the evolution of the velocity. We supplement them with an initial condition on the velocity.

$$u(t=0,x) = u_0(x) \tag{32}$$

For simplicity and without loss of generality, we normalize the density of a homogeneous fluid to $\rho_0 = 1$, in which case the Euler system writes:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p + f \\ \nabla \cdot u = 0 \end{cases} \tag{33}$$

2.4.4 Recovering the Pressure

Observe that we have $d+1$ unknown scalar fields; the d components of the velocity as well as the pressure field. However, we also have $d+1$ differential equations, so we can expect to recover the pressure from the overly specified velocity. Assuming the spatial and time derivatives commute, we differentiate the incompressibility constraint in time.

$$\partial_t \nabla \cdot u = \nabla \cdot \partial_t u = 0$$

Then, we inject the momentum balance equation into the right-hand side of the above.

$$-\nabla \cdot \nabla p = \nabla \cdot ((u \cdot \nabla)u) - \nabla \cdot f$$

Assuming there are no present body forces, we get the Poisson equation. In what follows, we denote Δ the Laplacian operator.

$$-\Delta p = \nabla \cdot ((u \cdot \nabla)u)$$

Evaluating the right-hand side of the above, we get the following.

$$\begin{aligned}\nabla \cdot ((u \cdot \nabla)u) &= \sum_{i=1}^{d} \partial_{x_i} \sum_{j=1}^{d} u_j \partial_{x_j} u_i \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_{x_i} u_j \partial_{x_j} u_i + u_j \partial_{x_i} \partial_{x_j} u_i\end{aligned}$$

We also observe that, if we can invert the Laplacian operator and evaluate it at the velocity-dependent quantity above, we get the pressure, so we are certain that our system is properly specified.

$$p = (-\Delta)^{-1}(\nabla \cdot ((u \cdot \nabla)u))$$

Indeed, solving the Poisson equation enables us to recover the pressure from the velocity field. An extensive discussion on the Poisson equation and solutions to the Laplace and Poisson equations is available in A.3 and is heavily inspired by [5] and [6]. In A.3, we illustrate approaches to solving the Poisson equation including that with the Fourier transform and show how decay - characteristic of Sobolev and Lebesgue spaces - brings about the uniqueness of the solution to the Poisson equation, which is critical to our message in this section. This enables us to assert that the pressure can be well-recovered from the velocity and that our $d + 1$ equations are sufficient to specify the unknowns of our problem.

2.4.5 The Leray Projector

In the previous section, we showed that the pressure can be recovered from the velocity field. In doing so, we showed that the pressure writes as the convolution of the fundamental solution of the Laplace equation with the function of the velocity that equals its Laplacian. Hence, through such an integration, $\partial_t u(t, x)$ depends on the velocity field at all points of Ω, rather than just velocity-derived quantities (itself and its derivatives) at (t, x). This makes the Euler equation non-local. To this end, we shift our lens and try to treat the nonlocal spatial operation as a projection, the residual of which is encoded in the pressure gradient. Denote $L^2_\sigma(\mathbb{R}^d)$ the closed, linear space of divergence-free $L^2(\mathbb{R}^d)^d$ operators; see A.5 for the definition of this space and its relevant properties. Recalling that $L^2(\mathbb{R}^d)^d$ is a Hilbert space, we are certain of the existence of an orthogonal projection $\mathbb{P} : L^2(\mathbb{R}^d)^d \to L^2(\mathbb{R}^d)^d$ onto $L^2_\sigma(\mathbb{R}^d)$ and we call it the Leray projector. In what follows, we take p to lie in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, where the projector is defined before using the unique extension of continuous liner maps on dense subspaces of Banach spaces. We require the following to hold for some vector field φ that is regular enough. Suppose the following holds for some regular vector field φ.

$$\varphi = \mathbb{P}\varphi + \nabla p \tag{34}$$

Taking the divergence of the above yields the following.

$$\nabla \cdot \varphi = \Delta p$$

Thanks to the decay of Schwartz functions, the (negative) Laplacian is invertible and we can write the following.

$$\nabla p = -\nabla(-\Delta)^{-1}\nabla \cdot \varphi$$

Hence, we get the following.

$$\mathbb{P}\varphi = \varphi - \nabla p = \varphi + \nabla(-\Delta)^{-1}\nabla \cdot \varphi$$

Identifying the action of the projector, we propose the following definition for the Leray projector.

$$\mathbb{P} = I + \nabla(-\Delta)^{-1}\nabla\cdot$$

That the Leray projector is linear results from the linearity of the differential operators - the linearity of the Laplacian's inverse comes from its own linearity. One easily checks that the operator $\nabla(-\Delta)^{-1}\nabla\cdot$ acts as the negative identity map on gradients, so that the kernel of the Leray projector includes gradients. Taking the Leray projection of Equation 33, we get the following in the absence of body forces. Observe

$$\partial_t u + \mathbb{P}((u \cdot \nabla)u) = 0 \tag{35}$$

The initial data becomes a divergence-free u_0. We claim that the system in Equation 33 is equivalent to the above (in the absence of body forces). The fact that the above follows from the original system is obvious. On the other hand, Equation 35 forces the velocity to remain divergence-free since we have the following under regularity.

$$\nabla \cdot \left(\partial_t u + \mathbb{P}((u \cdot \nabla)u)\right) = 0$$

$$\iff \partial_t \nabla \cdot u = 0, \quad \text{with } \nabla \cdot u_0 = 0$$

Furthermore, setting $\nabla p = (I - \mathbb{P})((u \cdot \nabla)u)$ and substituting for $\mathbb{P}((u \cdot \nabla)u)$, we conclude that Equation 35 is equivalent to the original Euler system with an application of the Leray projection on the initial data.

Remark. Observe that the above definition of the Leray projector requires u to be in $\mathcal{S}(\mathbb{R}^d)^d$. We show, however, in A.6 that this projector can be extended to $L^2(\mathbb{R}^d)^d$ using the density of Schwartz functions in L^2, and this effect is better illustrated by its representation as a Fourier multiplier; this phenomenon is common when dealing with differential operators. It is important to note, among the results presented in A.6, that the Leray projector is a bounded $L^2(\mathbb{R}^d)^d \to L^2(\mathbb{R}^d)^d$ and $H^s(\mathbb{R}^d)^d \to H^s(\mathbb{R}^d)^d$.

2.5 Symmetries and Conservation Laws

In what follows, consider the Euler equations for incompressible flows, homogeneous fluids, and in the absence of body forces. Suppose u (from which p follows by the Poisson equation's unique solution) solves Equation 35 and suppose p is the corresponding pressure field. It can be shown that the following also solves Equation 35 for any orthogonal matrix O, any $t_0 \in \mathbb{R}$, any $x_0 \in \mathbb{R}^d$, and any $\lambda, \mu > 0$.

$$v(t,x) = \frac{\lambda}{\mu} O^T u\left(\frac{t - t_0}{\mu}, \frac{O(x - x_0)}{\lambda}\right)$$

The associated pressure is given by the following.

$$q(t,x) = \frac{\lambda^2}{\mu^2} p\left(\frac{t - t_0}{\mu}, \frac{O(x - x_0)}{\lambda}\right)$$

One can show this easily by direct injection in Equation 33. Hence, the Euler equations feature translational, rotational, and scaling symmetries. These symmetries could help look in the appropriate spaces for solutions depending on the sought characterizations. Solutions to the Euler equations also feature the following conservation law, provided they are regular enough.

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} u(t,x)dx\right) = 0 \tag{36}$$

The above can be shown using the transport theorem on $\mathbb{R}^d$, whose pushforward is itself at all times since the flow map is subjective. Using incompressibility, we get the following.

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} u(t,x)dx\right) = -\int_{\mathbb{R}^d} \nabla p(t,x)dx$$

We choose to represent the pressure gradient as follows, using the canonical basis of $\mathbb{R}^d$.

$$\begin{aligned}\nabla p &= \sum_{i=1}^{d} \partial_i p\, e_i \\ &= \sum_{i=1}^{d} e_i \cdot (\nabla \cdot (pe_i))\end{aligned}$$

This allows us to use the divergence theorem as follows.

$$\begin{aligned}-\int_{\mathbb{R}^d} \nabla p(x)dx &= \lim_{R\to+\infty} -\sum_{i=1}^{d} e_i \int_{B(0,R)} \nabla \cdot (pe_i)dx \\ &= \lim_{R\to+\infty} -\sum_{i=1}^{d} e_i \int_{\partial B(0,R)} p(n(y)\cdot e_i)dS(y)\end{aligned}$$

Provided the pressure vanishes faster than the measure of $\partial B(0,R)$ with increasing R, we conclude:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} u(t,x)dx\right) = -\int_{\mathbb{R}^d} \nabla p(t,x)dx = 0$$

Another conserved quantity is the kinetic energy - we expect this in the absence of body forces as well due to work-energy theorem.

Proposition 2.10. *Under sufficient regularity, we have the following.*

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^d} ||u||^2 dx\right) = 0$$

Proof. Define $E(t) = \frac{1}{2}\int_{\mathbb{R}^d} ||u||^2 dx$. We have the following using the transport theorem.

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_{\mathbb{R}^d} (\partial_t(u\cdot u) + \nabla\cdot(||u||^2 u))dx$$

Using the chain rule, the first term writes as follows.

$$\frac{1}{2}\int_{\mathbb{R}^d} \partial_t(u\cdot u)dx = \int_{\mathbb{R}^d} u\cdot\partial_t u\, dx$$

Using the incompressibility of the flow, the second term writes as follows.

$$\frac{1}{2}\int_{\mathbb{R}^d} \nabla\cdot(||u||^2 u)dx = \frac{1}{2}\int_{\mathbb{R}^d} u\cdot\nabla||u||^2 dx$$

Computing the Fréchet derivative:

$$\nabla||u||^2 = 2(u\cdot\nabla)u$$

We get:

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^d} u\cdot(\partial_t u + (u\cdot\nabla)u)dx = -\int_{\mathbb{R}^d} u\cdot\nabla p\, dx$$

Using $\nabla \cdot u = 0$, we write:

$$\frac{d}{dt}E(t) = -\int_{\mathbb{R}^d} \nabla \cdot (pu)\, dx$$

Using the divergence theorem and a decaying argument, we conclude. $\square$

We now conclude this preliminary section of derivations associated to the foundations of the Euler equations.

2.6 Discussion

Throughout this section, we've connected the Lagrangian and Eulerian viewpoints using the family of systems given by Equation 1 and Equation 2, which impose some structural constraints, an example of which is the chord-arc condition. Under regularity, we get that the Lagrangian flow map is a C^1 diffeomorphism, which in turn allows us to use it to change variables and establish the relations due to the transport theorem. These tools combined with the physical postulates of the conservation of mass and linear momentum allow us to establish the momentum equation, to which we add incompressibility and homogeneity and form the Euler system. Endowing the system with a decay requirement, the solution to the Poisson equation relating the pressure and velocity becomes unique, and the $d + 1$ Euler equations prove to be sufficient to specify our system with smooth initial data. We tackle the nonlocality of the equation due to the convolution involved in the Laplacian's inversion by defining the Leray projector. Throughout our manipulations, we were not too concerned with the smoothness required to apply some theorems, because the goal of this section was to contextualize and illustrate how our description of continua comes together with physical postulates and kinematic constraints to inspire the following governing equation for the dynamics of an appropriate fluid at hand.

$$\partial_t u + \mathbb{P}((u \cdot \nabla)u) = 0$$

The above equation can be proposed without discussing regularity at all; we only mention it to apply theorems that justify our choice of such an equation based on well-established heuristics and mechanics postulates. In the following part of this thesis, the above equation will be considered a given, and we will be concerned with its solutions and their characteristics. Only then will we be diligent about the regularity and characteristics of its respective solutions.

3 The Existence of Regular Solutions to Euler's Equations

The goal of this section is to establish the local existence of unique solutions to the Euler equations and then show that under certain criteria that we will define, these solutions do not blow up in finite time so that they can be extended. We will consider $\Omega = \mathbb{R}^d$ and impose decay as $|x| \to +\infty$ as a boundary condition. We will construct what we later describe as *strong solutions* to the Euler system within Lebesgue and Sobolev spaces (maps into these spaces to be precise). The required preliminary theory on these spaces is elaborated in A.5, where the notation is also explicitly specified. We tackle the following Euler system with some initial data $u_0 : \mathbb{R}^d \to \mathbb{R}^d$ on which we later impose the regularity.

$$\partial_t u + \mathbb{P}((u \cdot \nabla)u) = 0, \quad u(t = 0, \cdot) = u_0 \tag{37}$$

We will state the theorem asserting the local existence of unique, strong, and regular solutions to the system represented by Equation 37, make estimates on supposed solutions to gauge their potential for extension of the real line of time, enumerate the steps of the proof of this central theorem and its aspects, and finally proceed with the proof as outlined. First, however, we must describe what it means to solve Equation 37.

3.1 The Space of Solutions and the Main Result

In ODE theory, a solution to an initial value problem or a Cauchy problem is a map defined on an interval within the domain of the definition of the ODE (a subset of the real numbers) that is regular enough to be differentiated as required and that satisfies the pointwise equality defined by the ODE. We can interpret our PDE as the time evolution of the velocity field rather than an intertwining of spatial and time evolution. With this in mind, a solution would be a map that maps a time t to the adequate vector field $u_t : \mathbb{R}^d \to \mathbb{R}^d$. In particular, we look for solutions $u(t, \cdot)$ to the Cauchy problem on functions of time taking values in the space of $\mathbb{R}^d \to \mathbb{R}^d$ functions. The question that remains is our potential control of the regularity of the space in which solutions that behave nicely with time will yield spatial maps; we seek a smooth, natural evolution of the field of velocity with time while staying in a space of well-behaved spatial maps. This interpretation of what we call solutions also makes sense in light of the nature of the initial data, which is a spatial vector field; we thus present our PDE as an ODE whose solutions take values in an infinite-dimensional function space. In the above, we have used the classical sense of solution. Should the need arise, we may introduce definitions of "solving the equation" that are perhaps a bit more peculiar.

Remark. We do not use the word Banach yet because we have no a priori guarantee that we can find a $\mathbb{R} \to (\mathbb{R}^d)^{\mathbb{R}^d}$ solution that takes values in a Banach space (this is the goal of the next section).

Observe that, from the form of the momentum equation in Equation 37, the regularity of the velocity depends on that of the Leray projection of the velocity differential quantity involved. For this, we recall the results in A.6.
We finally introduce the central theorem of this section. In what follows, we interchangeably use $u(t) = u(t, \cdot)$ to emphasize that the values are taken in a function space. We also mention that in this section, locality typically refers to time locality, which is further supported by the previously introduced interpretation of the Euler system.

Theorem 3.1. *(Local, Strong Solutions and the Propagation of Regularity) Suppose the initial data satisfies $u_0 \in H^s_\sigma(\mathbb{R}^d)^d$ for some $s > \frac{d}{2} + 1$. Then, there exists $C \geq 1$ such that, defining $T_0 := \frac{1}{C||u_0||_{H^s}} \in]0, +\infty]$, we have that there exists a unique solution $u : t \in \mathbb{R} \mapsto u(t, \cdot)$ to the time evolution Cauchy problem defined by 37 satisfying the following.*

$$u \in C^0(] - T_o, T_o[; H^s_\sigma(\mathbb{R}^d)^d) \cap Lip(] - T_o, T_o[; H^{s-1}_\sigma(\mathbb{R}^d)^d)$$

Moreover, if $\alpha > s$ and $u_0 \in H^\alpha_\sigma$, then, with T_0 still defined as above (dependent only on s and not α), we have the following.

$$u \in C^0(] - T_o, T_o[; H^\alpha_\sigma(\mathbb{R}^d)^d) \cap Lip(] - T_o, T_o[; H^{\alpha-1}_\sigma(\mathbb{R}^d)^d)$$

In this case, there exists $D \geq 1$ (dependent on both s, α) such that, for every $0 \leq t < T_0$, we have the following (see A.5 for the norm definitions for the vector-valued functions).

$$||u(t, \cdot)||_{H^\alpha} \leq ||u_0||_{H^\alpha} \exp\left(D \int_0^t ||u(\tau, \cdot)||_{H^s} d\tau\right)$$

We will tackle the proof in a series of steps we enumerate below. The approach is to solve an easier problem before considering its convergence to the initial problem and evaluating the properties that it is able to transfer.

1. Find and extend, over the real line, solutions to the *mollified* Euler equations that we denote $\{u^\varepsilon\}_{\varepsilon>0}$.

2. Show that the family of mollified solutions lie in a same space that is not parametrized by the index ε and outline some uniform-in-ε bounds for the time of definition and solution magnitude.

3. Show that the family of mollified solutions is compact in a particular, complete space so that it admits a limit object.

4. Show that said limit object is the unique solution to our problem where it is defined.

3.2 The Mollified Euler Equations

We look to solve a simpler problem than Equation 37 before arguing by convergence. To this end, we consider a standard family of compactly supported mollifiers $\{\phi_\varepsilon\}_{\varepsilon>0}$ defined in A.7. We also define a new system parametrized by $\varepsilon > 0$. We call the following system the mollified Euler equations, with solutions still taking values in function space.

$$\partial_t u^\varepsilon + J_{\phi,\varepsilon}\mathbb{P}\left((J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon\right) = 0 \tag{38}$$

$$u^\varepsilon(t=0) = u_0^\varepsilon := J_{\phi.\varepsilon}u_0 \tag{39}$$

The initial data of the new system is defined with respect to that in the initial system, assumed to lie in $H^s_\sigma(\mathbb{R}^d)$. Using the Fourier representation, we have the following, where we use notation introduced in A.6 for the Fourier multiplier of the Leray projector, and where the hat and $\mathcal{F}$ are interchangeably used for the Fourier transform.

$$\mathcal{F}\left[\mathbb{P}J_{\phi,\varepsilon}u_0\right](\xi) = m(\xi)\cdot\widehat{\phi}(\varepsilon\xi)\ \widehat{u_0}(\xi)$$

Since $\widehat{\phi}(\varepsilon\xi)$ is a scalar, it commutes with $m(\xi)$ and we have the following, where we use that $\mathbb{P}u_0 = u_0$ by assumption.

$$\mathcal{F}\left[\mathbb{P}J_{\phi,\varepsilon}u_0\right](\xi) = \mathcal{F}\left[J_{\phi,\varepsilon}u_0\right](\xi)$$

Since the Fourier transform is an isomorphism, we conclude that $u_0^\varepsilon \in H^s_\sigma(\mathbb{R}^d)$ since the convolution does not inhibit the regularity of the initial data and keeps it divergence free as shown above. We define the nonlinearity F_ε as follows on the space $H^s(\mathbb{R}^d)^d$.

$$F_\varepsilon : u \in H^s(\mathbb{R}^d)^d \longmapsto -J_{\phi,\varepsilon}\mathbb{P}\left((J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u\right)$$

We want to establish the space in which F_ε takes values and, of course, that it is well-defined.

Proposition 3.2. *We have that the following nonlinearity is well-defined and takes values in $H^s(\mathbb{R}^d)^d$.*

$$F_\varepsilon : u \in H^s(\mathbb{R})^d \longmapsto -J_{\phi,\varepsilon}\mathbb{P}\left((J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u\right)$$

Proof. Let $k \in \{1,...,d\}$. We have the following for $u \in H^s(\mathbb{R}^d)^d$ where the index j implicitly represents a summation.

$$\left[(J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u\right]_k = (J_{\phi,\varepsilon}u)_j\partial_j(J_{\phi,\varepsilon}u)_k$$

By Lemma A.10, we have that $\partial_j(J_{\phi,\varepsilon}u)_k = J_{\phi,\varepsilon}\partial_j u_k$, and by assumption, $\partial_j u_k$ is square integrable, so $J_{\phi,\varepsilon}\partial_j u_k$ lies in $H^s(\mathbb{R}^d)$ by A.7. For $s > \frac{1}{2}$, the Sobolev space is stable under pointwise multiplication (we admit it is an algebra due to Young's inequality), so $\left[(J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u\right]_k$ lies in $H^s(\mathbb{R}^d)$. We conclude by the boundedness of the Leray projector and Proposition A.9 to conclude. □

We've shown, when handling the new initial data, that the mollification operator and the Leray projection commute. Hence, we can further restrict the codomain of the nonlinearity as outlined in the trivial corollary below.

Corollary 3.2.1. *We have that the following nonlinearity takes values in $H^s_\sigma(\mathbb{R}^d)$.*

$$F_\varepsilon : u \in H^s(\mathbb{R})^d \longmapsto -J_{\phi,\varepsilon}\mathbb{P}\left((J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u\right)$$

Remark. It is understood that the mollification of a vector field is done component-wise.

In light of the above, we have that Equation 38 writes as an ODE system with values in the Banach space $H^s_\sigma(\mathbb{R}^d)$.

$$\frac{d\,u^\varepsilon}{dt} = F_\varepsilon(u^\varepsilon), \quad u(0) = u_0^\varepsilon \in H^s_\sigma(\mathbb{R}^d) \tag{40}$$

This ODE formulation and the fact that F_ε takes values in a Banach space motivate an approach similar to that used to prove the Cauchy-Lipschitz theorem, in which the key ingredient is the Banach fixed-point theorem. The definition of d/dt is a bit ambiguous on the function spaces we consider. We will call a strong solution to the problem a $I \subset \mathbb{R} \to H^s_\sigma(\mathbb{R}^d)$ map that satisfies the following (and, of course, is such that the following quantities are defined).

$$u(t) = u_0^\varepsilon + \int_0^t F_\varepsilon(u(\tau))d\tau, \quad \forall t \in I$$

Let $T_\varepsilon > 0$. We define the following operator, whose range will be inspected more closely later.

$$\Phi_\varepsilon : C([-T_\varepsilon, T_\varepsilon]; H^s_\sigma(\mathbb{R}^d)) \ni u \mapsto \Phi_\varepsilon(u) : \begin{cases} [-T_\varepsilon, T_\varepsilon \quad] \to H^s_\sigma(\mathbb{R}^d) \\ \qquad\qquad t \mapsto u_0^\varepsilon + \int_0^t F_\varepsilon(u(\tau))\, d\tau \end{cases}$$

Observe that Φ_ε is well defined since $F_\varepsilon \circ u$ is a regulated function when u is continuous as a result of the continuity of the mollification and Leray operators (and the continuity of the differentiation of a mollified u.) We can also see from now that $\Phi_\varepsilon(u)$ takes values in $H^s_\sigma(\mathbb{R}^d)$ by the properties of the integral operator and the fact that the initial data and nonlinearity are divergence free. The goal will be to restrict this operator so as to get a contraction and conclude on the existence of a unique fixed point. Let us first bound the Sobolev norm of the $F_\varepsilon(u)$. To this end, we need to understand how the H^s norm of mollified functions scale when the latter are differentiated - which can equivalently be described as being dropped to H^{s-1} space using the Fourier representation. Suppose that $f \in L^2(\mathbb{R}^d)$. Then, outside of a certain compact set (only dependent on the mollifier), we have that $|\widehat{\phi}(\xi)| \leq D_{\phi,s}\, (\varepsilon^2 + |\xi|^2)^{-\frac{s}{2}}$ where $D_{\phi,s} > 0$ depends on both the mollifier and s. This is because the mollifier and its Fourier transform are Schwartz and decay faster than any polynomial. We have the previous inequality hold on $\mathbb{R}^d$ by taking $D_{\phi,s}$ big enough. Consequently, we can show that $J_{\phi,\varepsilon} : L^2 \to H^s$ is well-defined and bounded.

$$\begin{aligned} ||J_{\phi,\varepsilon} f||^2_{H^s} &= \int_{\mathbb{R}^d} (1 + |\xi|^2)^s\, |\widehat{\phi}(\varepsilon\xi)|^2\, |\widehat{f}(\xi)|^2\, d\xi \\ &\leq D^2_{\phi,s} \int_{\mathbb{R}^d} \left(\frac{1 + |\xi|^2}{\varepsilon^2 + |\varepsilon\xi|^2} \right)^s |\widehat{f}(\xi)|^2\, d\xi \\ &\leq \frac{D^2_{\phi,s}}{\varepsilon^{2s}} ||f||^2_{L^2} \end{aligned} \tag{41}$$

The factor ε^{-2s} illustrates the cost to bound the H^s norm of the mollified function with its H^0 norm. We generalize this phenomenon in A.7 to H^r functions with $r < s$. In fact, a direct application of Proposition A.11 yields the following for $f \in H^s(\mathbb{R}^d)$, whose partial derivatives necessarily lie in $H^{s-1}(\mathbb{R}^d)$.

$$||J_{\phi,\varepsilon} \partial_i f||_{H^s} \lesssim_{d,s,\phi} \varepsilon^{-1} ||\partial_i f||_{H^{s-1}} \leq \varepsilon^{-1} ||f||_{H^s} \tag{42}$$

The last inequality above comes from the Fourier representation of the partial derivatives of f. We now have the tools to bound the nonlinearity, knowing it takes values in H^s by Proposition 3.2. Due to A.6, the Leray projector is bounded as an $H^s \to H^s$ operator - this is where its argument in the nonlinearity expression lies - and obeys $||\mathbb{P}\varphi||_{H^s} \leq ||\varphi||_{H^s}$. Hence, we have the following, where we have shown that all the quantities are well-defined.

$$||F_\varepsilon||_{H^s} \leq ||(J_{\phi,\varepsilon} u \cdot \nabla) J_{\phi,\varepsilon} u||_{H^s}$$

Observe that, for $s > d/2$, H^s is an algebra. Hence, for any $f, g \in H^s(\mathbb{R}^d)$, we have the following for a constant C that only depends on s (and d, but we take this as a problem constant).

$$||fg||_{H^s} \leq C ||f||_{H^s} ||g||_{H^s}$$

Hence, for $(i, j) \in \{1, ..., d\}^2$, we have the following.

$$||(J_{\phi,\varepsilon}u_j)\partial_j(J_{\phi,\varepsilon}u_i)||_{H^s} \le C\ ||u_j||_{H^s}\ ||J_{\phi,\varepsilon}\partial_j u_i||_{H^s}$$

We have appealed to Lemma A.10 and Proposition A.9. Using Equation 42, we get the following.

$$||(J_{\phi,\varepsilon}u_j)\partial_j(J_{\phi,\varepsilon}u_i)||_{H^s} \lesssim_{s,d,\phi} \varepsilon^{-1}||u_j||_{H^s}||u_i||_{H^s}$$

Let be a constant dependent on s, d, ϕ only that satisfies the following.

$$||(J_{\phi,\varepsilon}u_j)\partial_j(J_{\phi,\varepsilon}u_i)||_{H^s} \le \gamma\varepsilon^{-1}||u_j||_{H^s}||u_i||_{H^s}$$

We have the following for any $u \in H^s(\mathbb{R}^d)^d$ where we combine the above results and take the appropriate summation.

$$||F_\varepsilon(u)||_{H^s} \le \frac{\gamma}{\varepsilon}||u||_{H^s}^2 \tag{43}$$

Remark. It is important to note that γ is $\varepsilon-$independent.

A consequence of Equation 43 is the following for $R > 0$.

$$F_\varepsilon\left(B_{H^s_\sigma(\mathbb{R}^d)}(0, R)\right) \subset B_{H^s_\sigma(\mathbb{R}^d)}\left(0, \frac{\gamma R^2}{\varepsilon}\right)$$

Before showing Φ_ε is a contraction, we show the following.

Proposition 3.3. *We have that F_ε is locally Lipschitz continuous on $H^s_\sigma(\mathbb{R}^d)$.*

Proof. For $u, v \in H^s_\sigma(\mathbb{R}^d)$, we have the following.

$$\begin{aligned}
||F_\varepsilon(u) - F_\varepsilon(v)||_{H^s} &\le ||(J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u - (J_{\phi,\varepsilon}v\cdot\nabla)J_{\phi,\varepsilon}v||_{H^s} \\
&\le ||(J_{\phi,\varepsilon}(u-v)\cdot\nabla)J_{\phi,\varepsilon}u + (J_{\phi,\varepsilon}v\cdot\nabla)J_{\phi,\varepsilon}(v-u)||_{H^s} \\
&\le ||(J_{\phi,\varepsilon}(u-v)\cdot\nabla)J_{\phi,\varepsilon}u||_{H^s} + ||(J_{\phi,\varepsilon}v\cdot\nabla)J_{\phi,\varepsilon}(v-u)||_{H^s} \\
&\le \gamma\varepsilon^{-1}||u-v||_{H^s}(||u||_{H^s} + ||v||_{H^s})
\end{aligned}$$

Since $(||u||_{H^s} + ||v||_{H^s})$ is locally bounded, we conclude. □

We now want to apply the fixed point theorem on a restriction of Φ_ε.

Proposition 3.4. *Let $R := 2||u_0||_{H^s}$ and $T_\varepsilon := \frac{\varepsilon}{4\gamma R}$. Then, define the following subset $\mathcal{B}$ of the domain of Φ_ε.*

$$\mathcal{B} := \mathcal{C}\left([-T_\varepsilon, T_\varepsilon]; \overline{B}_{H^s_\sigma(\mathbb{R}^d)}(0, R)\right)$$

Then, $\Phi_{\varepsilon|\mathcal{B}}$ takes values in $\mathcal{B}$ and is a $\mathcal{B} \to \mathcal{B}$ contraction.

Proof. Observe that Φ_ε takes values in $\mathcal{C}([-T_\varepsilon, T_\varepsilon]; H^s_\sigma(\mathbb{R}^d))$ by a component-wise application of the first fundamental theorem of calculus on the continuous vector field F_ε (whose continuity results from its definition as a composition of continuous operators). Then, we have the following for $u \in \mathcal{B}$.

$$\begin{aligned}
||\Phi_\varepsilon(u)|| &= \sup_{t\in[-T_\varepsilon,T_\varepsilon]} ||\Phi_\varepsilon(u)(t)|| \\
&\leq ||u_0^\varepsilon||_{H^s} + \sup_{t\in[-T_\varepsilon,T_\varepsilon]} \int_0^t ||F_\varepsilon(u(\tau))||_{H^s} d\tau \\
&\leq ||u_0||_{H^s} + \sup_{t\in[-T_\varepsilon,T_\varepsilon]} \int_0^t ||F_\varepsilon(u(\tau))||_{H^s} d
\end{aligned}$$

Appealing to Equation 43 and that u takes values in a ball of radius R centered at the origin, we have the following.

$$\begin{aligned}
||\Phi_\varepsilon(u)|| &\leq ||u_0||_{H^s} + T_\varepsilon \cdot \gamma\varepsilon^{-1}||u||_{H^s}^2 \\
&\leq ||u_0||_{H^s} + T_\varepsilon \cdot \gamma\varepsilon^{-1} \cdot R^2 \\
&\leq \frac{R}{2} + \frac{R}{4} \leq R
\end{aligned}$$

Hence, $\Phi_\varepsilon(\mathcal{B}) \subseteq \mathcal{B}$. To show that its restriction to $\mathcal{B}$ is a contraction, we appeal to Proposition 3.3 in what follows. Let $u, v \in \mathcal{B}$.

$$\begin{aligned}
||\Phi(u) - \Phi(v)|| &\leq \sup_{t\in[-T_\varepsilon,T_\varepsilon]} \int_0^t ||F(u(\tau)) - F(v(\tau))||_{H^s} d\tau \\
&\leq T_\varepsilon\gamma\varepsilon^{-1}||u - v||_{\mathcal{B}}(||u||_{\mathcal{B}} + ||v||_{\mathcal{B}}) \\
&\leq \frac{1}{4R}||u - v||(2R) \\
&\leq \frac{1}{2}||u - v||
\end{aligned}$$

□

Corollary 3.4.1. *We have that Φ_ε admits a unique fixed point in $\mathcal{B}$.*

Proof. Observe that $H^s_\sigma(\mathbb{R}^d)$ is a Banach space. As any closed subset of a complete space is complete, $\overline{B}_{H^s_\sigma(\mathbb{R}^d)}(0, R)$ is complete, and immediately, we get that $\mathcal{B}$ is complete [3]. The result follows from applying the Banach-Picard fixed point theorem. □

Remark. All derivations in this section are made under a fixed ε; all the uniform-in-ε estimates will be made at the next step of the proof of the theorem.

We thus recover a unique strong solution u^ε to Equation 38 in $\mathcal{C}([-T_\varepsilon, T_\varepsilon]; H^s_\sigma(\mathbb{R}^d))$. By unique, we mean that any two solutions will coincide where they are both defined. Finally, assert below that our solutions are Lipschitz when taking their values in a slightly less regular Sobolev space. This is an important aspect of the solutions that the limiting object will inherit from to satisfy the theorem's claim.

Proposition 3.5. *Using the notation and results above, we have the following.*

$$u^\varepsilon \in Lip([-T_\varepsilon, T_\varepsilon]; H^s_\sigma(\mathbb{R}^d))$$

In particular, we have the following, weaker result.

$$u^\varepsilon \in Lip([-T_\varepsilon, T_\varepsilon]; H^{s-1}_\sigma(\mathbb{R}^d))$$

Proof. Let $t_1, t_2 \in [-T_\varepsilon, T_\varepsilon]$. We have the following, by definition of the strong solution u^ε.

$$\begin{aligned}||u^\varepsilon(t_2) - u^\varepsilon(t_1)||_{H^s} &= \left|\left| \int_{t_1}^{t_2} F_\varepsilon(u^\varepsilon(t))dt \right|\right|_{H^s} \\ &\leq |t_2 - t_1| \sup_{t\in[T_\varepsilon, T_\varepsilon]} ||F_\varepsilon(u^\varepsilon(t))||_{H^s}\end{aligned}$$

For any $t \in [T_\varepsilon, T_\varepsilon]$, we have, by construction, that $u^\varepsilon(t) \in \overline{B}_{H^s_\sigma(\mathbb{R}^d)}(0, 2||u_0||_{H^s})$. Consequently, by Equation 43, we have the following.

$$||u^\varepsilon(t_2) - u^\varepsilon(t_1)||_{H^s} \leq |t_2 - t_1| 4\gamma\varepsilon^{-1}||u_0||^2_{H^s}$$

The weaker result follows from the fact that $||\cdot||_{H^{s-1}} \leq C||\cdot||_{H^s}$ on $H^s(\mathbb{R}^d)$ for some $C > 0$. □

As our arguments are based on the classic ODE approach outlined in the proof of the Cauchy-Lipschitz theorem, one can formulate an analog of Lemma A.2 - which is proven in [2] - for Banach space-valued ODEs. We present the statement below without its proof to avoid redundancy. This will be important when checking for uniform-in-time existence of solutions to the mollified Euler equations.

Lemma 3.6. *The solution u^ε can be uniquely extended to a maximal, open time interval $(T^-_\varepsilon, T^+_\varepsilon)$. Either the extremity is infinite or $||u^\varepsilon(t)||_{H^s}$ blows up at the extremity. This applies to both extremities $T^-_\varepsilon, T^+_\varepsilon$.*

3.3 Uniform Estimates on the Mollified Euler Solutions

The subsequent parts of the proof consist of claiming the family $\{u^\varepsilon\}_{\varepsilon>0}$ is Cauchy is a space that is not parametrized by ε. This requires uniform-in-ε time definition. Let us formulate a bound on the norm of $u^\varepsilon(t)$ before finding a uniform one. We appeal to the transport theorem, which takes a simpler form owing to the fact that u^ε is divergence-free and that the Lagrangian flow map is a C^1-diffeomorphism. In what follows, $\langle\nabla\rangle^s$ is identified by its Fourier multiplier $(1 + |\xi|^2)^{s/2}$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u^\varepsilon(t)||^2_{H^s} &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} (\langle\nabla\rangle^s u^\varepsilon)^2 \ dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} [2\,\langle\nabla\rangle^s u^\varepsilon \cdot \partial_t \langle\nabla\rangle^s u^\varepsilon + u^\varepsilon \cdot \nabla((\langle\nabla\rangle^s u^\varepsilon)^2] dx\end{aligned}$$

The second term in the integrand can be rewritten as a factor of the divergence of $u^\varepsilon(t)$ so it vanishes. Since the spatial (fractional) derivatives write as time-independent Fourier multipliers and since the time derivative of u^ε takes values in the required Sobolev space from our analysis of the nonlinearity, we can interchange $\langle\nabla\rangle^s$ and time differentiation in the first term of the integrand. We get the following.

$$\frac{d}{dt}||u^\varepsilon(t)||^2_{H^s} = \frac{1}{2}\int_{\mathbb{R}^d} \langle\nabla\rangle^s u^\varepsilon \cdot \langle\nabla\rangle^s(\partial_t u^\varepsilon) dx$$

Injecting the substitution from the mollified Euler equations, we get the following.

$$\frac{d}{dt}||u^\varepsilon(t)||^2_{H^s} = -\int_{\mathbb{R}^d} \langle\nabla\rangle^s u^\varepsilon \cdot \langle\nabla\rangle^s (J_{\phi,\varepsilon}\mathbb{P}((J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon))\, dx$$

Observe that, in light of the matrix representation of $\mathbb{P}$ shown in A.6, the Leray projector is self adjoint, and Fourier multipliers commute. The mollification operator is also self-adjoint and we have the following.

$$\frac{d}{dt}||u^\varepsilon(t)||^2_{H^s} = -\int_{\mathbb{R}^d} \langle\nabla\rangle^s J_{\phi,\varepsilon}u^\varepsilon \cdot \langle\nabla\rangle^s((J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon)\, dx$$

Lemma 3.7. *With the above assumptions and denoting $v_\varepsilon := J_{\phi,\varepsilon}u$, we have the following.*

$$\int_{\mathbb{R}^d} \langle\nabla\rangle^s v_\varepsilon \cdot (v_\varepsilon \cdot \nabla)\langle\nabla\rangle^s v_\varepsilon \, dx = 0$$

Proof. By the chain rule, we have that the integrand writes as $(v_\varepsilon \cdot \nabla)\left(\langle\nabla\rangle^s v_\varepsilon\right)^2$. Since the flow is incompressible, this is equal to $\nabla \cdot (\left(\langle\nabla\rangle^s v_\varepsilon\right)^2 v_\varepsilon)$. The result follows from the divergence theorem. □

Using Lemma 3.7, we have the following.

$$\frac{d}{dt}||u^\varepsilon(t)||^2_{H^s} = -\int_{\mathbb{R}^d} \langle\nabla\rangle^s J_{\phi,\varepsilon}u^\varepsilon \cdot [\langle\nabla\rangle^s, J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla] J_{\phi,\varepsilon}u^\varepsilon \, dx$$

Recalling that the Fourier transform of the product of two functions is the convolution of the transforms, we write the integrand of the right-hand side of the above using its Fourier transform and get the following, where we set $f := J_{\phi,\varepsilon}u^\varepsilon(t)$.

$$\frac{d}{dt}||u^\varepsilon(t)||^2_{H^s} = -(2\pi)^{d/2}\mathrm{Re}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \langle\eta\rangle^s \overline{\widehat{f}(\eta)} \cdot (\langle\eta\rangle^s - \langle\xi\rangle^s)\widehat{f}(\eta-\xi) i\xi \widehat{f}(\xi) d\xi d\eta$$

Lemma 3.8. *We have the following.*

$$|\langle\eta\rangle^s - \langle\xi\rangle^s| \lesssim_s (\langle\xi\rangle^{s-1} + \langle\eta-\xi\rangle^{s-1})|\eta-\xi|$$

Proof. Using the fundamental theorem of calculus, we have the following.

$$\langle\eta\rangle^s - \langle\xi\rangle^s = \int_0^1 \frac{d}{dt}\langle\xi + t(\eta-\xi)\rangle^s dt$$

Computing the derivative yields the following.

$$\partial_t(\langle\xi + t(\eta-\xi)\rangle^s) = s\langle\xi + t(\eta-\xi)\rangle^{s-2}(\xi + t(\eta-\xi)) \cdot (\eta-\xi)$$

Trivially, we have $\langle x\rangle^1 \geq |x|$. It immediately follows that we have the following.

$$|\langle\xi + t(\eta-\xi)\rangle^{s-2}(\xi + t(\eta-\xi))| \leq \langle\xi + t(\eta-\xi)\rangle^{s-1}$$

Consequently, we get the following, where we use the positivity of the integrand.

$$|\langle\eta\rangle^s - \langle\xi\rangle^s| \leq s|\eta-\xi|\int_0^1 \langle\xi + t(\eta-\xi)\rangle^{s-1} dt$$

Since $s-1 \geq 0$, the Japanese bracket raise to $s-1$ satisfies the triangle inequality and by positivity of the Japanese bracket, we conclude with the following.

$$|\langle\eta\rangle^s - \langle\xi\rangle^s| \leq s|\eta-\xi|(\langle\xi\rangle^{s-1} + \langle\eta-\xi\rangle^{s-1})\int_0^1 dt$$

□

Applying Lemma 3.8, we get the following, where we still use $f := J_{\phi,\varepsilon}u^\varepsilon(t)$.

$$\frac{d}{dt}||u^\varepsilon(t)||^2_{H^s} \lesssim_s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \langle\eta\rangle^s|\widehat{f}(\eta)||\eta-\xi|(\langle\xi\rangle^{s-1} + \langle\eta-\xi\rangle^{s-1})|\xi||\widehat{f}(\eta-\xi)||\widehat{f}(\xi)|d\xi d\eta$$

We apply the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^d)$ with the functions $\eta \mapsto \langle\eta\rangle^s|\widehat{f}(\eta)|$ and $\eta \mapsto \int_{\mathbb{R}^d} |\eta-\xi|(\langle\xi\rangle^{s-1}+\langle\eta-\xi\rangle^{s-1})|\xi||\widehat{f}(\eta-\xi)||\widehat{f}(\xi)|d\xi$ and identify the Sobolev norm of f.

$$\frac{d}{dt}||u^\varepsilon(t)||_{H^s}^2 \lesssim_s ||\widehat{f}||_{H^s}\left|\left|\eta \mapsto \int_{\mathbb{R}^d} |\eta-\xi|(\langle\xi\rangle^{s-1}+\langle\eta-\xi\rangle^{s-1})|\xi||\widehat{f}(\eta-\xi)||\widehat{f}(\xi)|d\xi\right|\right|_{L^2}$$

By the triangle inequality and that $\langle x\rangle^s|x| \leq \langle x\rangle^{s+1}$ for $s > 0$, we have the following.

$$\begin{aligned}\left|\left|\int_{\mathbb{R}^d} |\eta-\xi|(\langle\xi\rangle^{s-1}+\langle\eta-\xi\rangle^{s-1})|\xi||\widehat{f}(\eta-\xi)||\widehat{f}(\xi)|d\xi\right|\right|_{L^2} &\leq \left|\left|\int_{\mathbb{R}^d} |\eta-\xi|\langle\xi\rangle^s|\widehat{f}(\eta-\xi)||\widehat{f}(\xi)|d\xi\right|\right|_{L^2} \\ &+ \left|\left|\int_{\mathbb{R}^d} \langle\eta-\xi\rangle^s|\xi||\widehat{f}(\eta-\xi)||\widehat{f}(\xi)|d\xi\right|\right|_{L^2}\end{aligned}$$

The two integrals are the same up to a translational coordinate change. In any case, they feature a convolution involving the multiplier for the Sobolev norm and the Fourier transform of f's gradient. We use Young's convolution inequality to get the following.

$$\left|\left|\int_{\mathbb{R}^d} |\eta-\xi|\langle\xi\rangle^s|\widehat{f}(\eta-\xi)||\widehat{f}(\xi)|d\xi\right|\right|_{L^2} \leq ||\langle\nabla\rangle^s f||_{L^2}||\widehat{\nabla f}||_{L^1}$$

Whether the last factor is interpreted as a component-wise gradient or Jacobian norm is exactly the same thing (in the worst case up to a constant). Identifying the Sobolev norm of f in the above, we have the following.

$$\frac{d}{dt}||u^\varepsilon(t)||_{H^s}^2 \lesssim_s ||f||_{H^s}^2||\widehat{\nabla f}||_{L^1}$$

By Cauchy-Schwarz, we have the following where we needed $s-1 > d/2$ to ensure the convergence of the integral of $\langle\xi\rangle^{-(s-1)}$.

$$\int_{\mathbb{R}^d} |\xi||\widehat{f}(\xi)|d\xi \leq \int_{\mathbb{R}^d} \langle\xi\rangle^s|\widehat{f}(\xi)|\cdot\langle\xi\rangle^{-(s-1)}d\xi \lesssim_s ||f||_{H^s}$$

Using the chain rule and $||f||_{H^s} \leq ||u^\varepsilon(t)||_{H^s}$ due the mollification, we conclude the following.

$$\frac{d}{dt}||u^\varepsilon(t)||_{H^s} \lesssim_s ||u^\varepsilon(t)||_{H^s}^2 \tag{44}$$

Let $C_s > 0$ be an $s-$dependent constant that turns the above to a proper inequality. We have the following.

$$\begin{aligned}&\frac{1}{||u^\varepsilon(t)||_{H^s}^2}\cdot\frac{d}{dt}||u^\varepsilon(t)||_{H^s} = -\frac{d}{dt}\left(\frac{1}{||u^\varepsilon||_{H^s}}\right) \\ &\implies \frac{1}{||u^\varepsilon(t)||_{H^s}} - \frac{1}{||u_0^\varepsilon||_{H^s}} \geq -C_s t \\ &\implies ||u^\varepsilon||_{H^s} \leq \frac{||u_0^\varepsilon||_{H^s}}{1-C_s||u_0^\varepsilon||_{H^s}t} \leq \frac{||u_0||_{H^s}}{1-C_s||u_0||_{H^s}t}\end{aligned}$$

Remark. The solution to the mollified Euler equations is continuously differentiable by the continuity of the nonlinearity and the fundamental theorem of calculus. Furthermore, the above estimate is valid for $t \geq 0$ so that the integration in Equation 44 doesn't reverse the inequality. A similar (symmetric) calculation holds for $t < 0$.

Hence, by Lemma 3.6, we get a uniform-in-ε time of definition $T_0 := \frac{1}{2C_s||u_0||_{H^s}}$ such that, for any $\varepsilon > 0$, we have $u^\varepsilon \in C([-T_0, T_0]; H_\sigma^s(\mathbb{R}^d))$. On this time interval, we are sure to have the following uniform bound.

$$\sup_{t\in[-T_0,T_0]} ||u^\varepsilon(t)||_{H^s} \leq 2||u_0||_{H^s} \tag{45}$$

Furthermore, suppose $u_0 \in H_\sigma^\alpha(\mathbb{R}^d)$ for some $\alpha > s > 1 + d/2$. Repeating the above calculations yields the following.

$$\frac{d}{dt}||u^{\varepsilon}(t)||^2_{H^\alpha} \lesssim_s ||u^{\varepsilon}(t)||^2_{H^\alpha}||\widehat{\nabla f}||_{L^1} \lesssim_s ||u^{\varepsilon}(t)||^2_{H^\alpha}||u^{\varepsilon}(t)||_{H^s}$$

Applying the Grönwall lemma, we get the following.

$$||u^{\varepsilon}(t)||^2_{H^\alpha} \leq ||u_0||_{H^\alpha} \exp\left(C_{\alpha,s}\int_0^t ||u^{\varepsilon}(s)||_{H^s}\, ds\right)$$

Hence, we retrieve that T_0 is a uniform time of existence for solutions for both H^s and H^α solutions.

Remark. At this stage having established that the uniform time of existence for H^s solutions is also uniform for H^α solutions, if we can prove the first part of the theorem - the global existence of unique H^s solutions on said time interval - then the assertion on H^α solutions follows. Furthermore, repeating the above calculations as a priori estimates on solutions u to the Euler system yields the last part of the theorem. Hence, from this point on, we are only concerned with proving the existence of unique H^s solutions. We insist that the a priori estimates are important as it is these calculations that motivate the above procedure on the mollified Euler equations; they helped us, though we did illustrate this effect, where to look for the uniform time of existence for the mollified Euler equations.

We now want a uniform-in-ε Lipschitz constant for our solutions. This will be important to later assert that the limit object of the solutions to Equation 38 is Lipschitz. To this end, we need a refinement of Equation 43.

Lemma 3.9. *We have the following for any $u \in H^s(\mathbb{R}^d)^d$.*

$$||F_\varepsilon(u)||_{H^{s-1}} \lesssim_{s,\phi} ||u||^2_{H^s}$$

Proof. We have the following thanks to the boundedness of the Leray projector and mollification.

$$||F_\varepsilon(u)||_{H^{s-1}} \leq ||(J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u||_{H^{s-1}}$$

Using summation (from the outside) over index j, we have the following for any $i \in \{1,...,d\}$ owing to the fact that $s-1 > d/2$ implies H^{s-1} spaces are algebras.

$$||J_{\phi,\varepsilon}u_j \cdot \partial_j J_{\phi,\varepsilon}u_i||_{H^{s-1}} \lesssim_s ||u_j||_{H^{s-1}}||J_{\phi,\varepsilon}\partial_j u_i||_{H^{s-1}}$$

Furthermore, we have the following, where we have summed from the outside over j.

$$\begin{aligned}||J_{\phi,\varepsilon}\partial_j u_i||^2_{H^{s-1}} &= \int_{\mathbb{R}^d}(\langle\xi\rangle^{s-1})^2|\xi|^2|\widehat{\phi}(\varepsilon\xi)|^2|\widehat{u_i}(\xi)|^2\, d\xi \\ &\leq \int_{\mathbb{R}^d}(\langle\xi\rangle^{s})^2|\widehat{\phi}(\varepsilon\xi)|^2|\widehat{u_i}(\xi)|^2\, d\xi \\ &\leq ||u_i||^2_{H^s}\end{aligned}$$

As outlined above, we do not need to use the decay of the mollifier to make the Japanese bracket decay since u lies in H^s. We conclude the following.

$$||J_{\phi,\varepsilon}u_j \cdot \partial_j J_{\phi,\varepsilon}u_i||_{H^{s-1}} \lesssim_s ||u_j||_{H^{s-1}}||u_i||_{H^s}$$

Applying the inequality on the square of the quantities and summing over i yields the following.

$$||(J_{\phi,\varepsilon}u\cdot\nabla)J_{\phi,\varepsilon}u||_{H^{s-1}} \lesssim_s ||u||_{H^{s-1}}||u||_{H^s} \leq ||u||^2_{H^s}$$

We conclude by injecting this in the first inequality. □

In light of the above lemma and the uniform bound in Equation 45, we conclude that for any $\varepsilon > 0$, $u^\varepsilon \in \mathrm{Lip}([-T_0, T_0]; H^{s-1}_\sigma(\mathbb{R}^d))$ and there exists a uniform-in-ε Lipschitz constant.

3.4 Compactness of the Solutions to Mollified Euler

Thanks to the above derivations, we can assert the solutions to the mollified Euler equations lie in the same space that is not parametrized by ε. We would like to prove a compactness result in order to ensure the existence of a limiting object, which would then be used as a candidate for the solution to the Euler equations. To start with, we claim, by abuse of definition, that $\{u^\varepsilon\}_{\varepsilon>0}$ is Cauchy in $C([-T_0, T_0]; L^2(\mathbb{R}^d)^d)$. We claim that, for any $\gamma > 0$ there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon, \delta < \varepsilon_0$, there holds the following.

$$||u^\varepsilon - u^\delta||_{C([-T_0,T_0];L^2(\mathbb{R}^d)^d)} \leq \gamma$$

The first step is to prove compactness in the space of bounded, measurable functions. As a family of continuous functions on a compact domain, we have that $u^\varepsilon \in L^\infty([-T_0, T_0]; L^2(\mathbb{R}^d)^d)$ in addition to the continuous and Lipschitz spaces. Using the estimate in Corollary A.12.1, we can prove the first important step in this section.

Proposition 3.10. *For $0 < \varepsilon$ and $0 < \delta \leq 1$, we have the following where C is independent of ε and δ.*

$$||u^\varepsilon - u^\delta||_{L^\infty([-T_0,T_0];L^2)} \leq C||u_0||_{H^s} \max(\varepsilon, \delta)$$

Proof. Assume, without loss of generality, that $\varepsilon \geq \delta$. We have the following, where we omit the time argument for clarity.

$$F_\varepsilon(u^\varepsilon) - F_\delta(u^\delta) = \mathbb{P}J_{\phi,\varepsilon}((J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon) - \mathbb{P}J_{\phi,\delta}((J_{\phi,\delta}u^\delta \cdot \nabla)J_{\phi,\delta}u^\delta)$$

By expanding the right-hand side of what follows, one can check that the below equality holds.

$$\begin{aligned}F_\varepsilon(u^\varepsilon) - F_\delta(u^\delta) &= \mathbb{P}(J_{\phi,\varepsilon} - J_{\phi,\delta})((J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon) + \mathbb{P}J_{\phi,\delta}(((J_{\phi,\varepsilon} - J_{\phi,\delta})u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon) \\ &+ \mathbb{P}J_{\phi,\delta}((J_{\phi,\delta}(u^\varepsilon - u^\delta) \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon) + \mathbb{P}J_{\phi,\delta}((J_{\phi,\delta}u^\delta \cdot \nabla)(J_{\phi,\varepsilon} - J_{\phi,\delta})u^\varepsilon) \\ &+ \mathbb{P}J_{\phi,\delta}((J_{\phi,\delta}u^\delta \cdot \nabla)J_{\phi,\delta}(u^\varepsilon - u^\delta))\end{aligned}$$

Recall that the solutions to mollified Euler are continuously differentiable. Furthermore, by the chain rule, we have the following.

$$\partial_t||u||^2_{L^2} = 2\langle u, \partial_t u\rangle$$

Noting that $\partial_t u^\varepsilon(t) = F_\varepsilon(u^\varepsilon(t))$, we take a scalar product in L^2 of $F_\varepsilon(t) - F_\delta(t)$ with $u^\varepsilon(t) - u^\delta(t)$. We use Corollary A.12.1 and the Cauchy-Schwarz inequality to make the following estimates of the terms on the right hand-side of the expression of $F_\varepsilon(u^\varepsilon) - F_\delta(u^\delta)$. All norms are taken on the evaluation of the below maps at some time t, though we omit the arguments for clarity. We recall that $s > d/2 + 1 \geq 2/2 + 1 \geq 2$. Furthermore, we note that, by the Sobolev embedding theorem and $s > d/2 + 1$, we have that $u^\varepsilon(t)$, $u^\delta(t)$ and their spatial derivatives lie in $L^\infty(\mathbb{R}^d)$ (and, even more, that their L^∞ norms are controlled by their Sobolev norms).

$$\begin{aligned}|\langle\mathbb{P}(J_{\phi,\varepsilon} - J_{\phi,\delta})((J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon), u^\varepsilon - u^\delta\rangle_{L^2}| &\leq ||u^\varepsilon - u^\delta||_{L^2} \cdot ||(J_{\phi,\varepsilon} - J_{\phi,\delta})(J_{\phi,\varepsilon}u^\varepsilon \cdot \nabla)J_{\phi,\varepsilon}u^\varepsilon||_{L^2} \\ &\leq ||u^\varepsilon - u^\delta||_{L^2} \cdot |u^\varepsilon||_{L^\infty}||\nabla(J_{\phi,\varepsilon} - J_{\phi,\delta})u^\varepsilon||_{L^2} \\ &\leq ||u^\varepsilon - u^\delta||_{L^2} \cdot ||u^\varepsilon||_{L^\infty}||(J_{\phi,\varepsilon} - J_{\phi,\delta})u^\varepsilon||_{H^1} \\ &\lesssim \varepsilon||u^\varepsilon - u^\delta||_{L^2}||u^\varepsilon||_{L^\infty}||u^\varepsilon||_{H^2} \qquad (46)\end{aligned}$$

$$\begin{aligned}|\langle \mathbb{P}J_{\phi,\delta}(((J_{\phi,\varepsilon}-J_{\phi,\delta})u^\varepsilon\cdot\nabla)J_{\phi,\varepsilon}u^\varepsilon),u^\varepsilon-u^\delta\rangle_{L^2}| &\le ||u^\varepsilon-u^\delta||_{L^2}||((J_{\phi,\varepsilon}-J_{\phi,\delta})u^\varepsilon\cdot\nabla)J_{\phi,\varepsilon}u^\varepsilon||_{L^2}\\ &\le ||u^\varepsilon-u^\delta||_{L^2}||\nabla u^\varepsilon||_{L^\infty}||||(J_{\phi,\varepsilon}-J_{\phi,\delta})u^\varepsilon||_{L^2}\\ &\lesssim \varepsilon||u^\varepsilon-u^\delta||_{L^2}||\nabla u^\varepsilon||_{L^\infty}||||u^\varepsilon||_{H^1} \qquad (47)\end{aligned}$$

The apparition of the $||\cdot||_{L^\infty}$ norm's evaluations comes from knowing it exists for the relevant arguments due to the Sobolev embedding theorem and writing the integrals corresponding to the considered norms. We proceed for the last three terms.

$$|\langle \mathbb{P}J_{\phi,\delta}((J_{\phi,\delta}(u^\varepsilon-u^\delta)\cdot\nabla)J_{\phi,\varepsilon}u^\varepsilon),u^\varepsilon-u^\delta\rangle| \lesssim ||u^\varepsilon-u^\delta||^2_{L^2}\cdot||\nabla u^\varepsilon||_{L^\infty} \qquad (48)$$

$$\begin{aligned}|\langle \mathbb{P}J_{\phi,\delta}((J_{\phi,\delta}u^\delta\cdot\nabla)(J_{\phi,\varepsilon}-J_{\phi,\delta})u^\varepsilon),u^\varepsilon-u^\delta\rangle_{L^2}| &\lesssim ||u^\varepsilon-u^\delta||_{L^2}||u^\delta||_{L^\infty}||\nabla(J_{\phi,\varepsilon}-J_{\phi,\delta})u^\varepsilon||_{L^2}\\ &\lesssim \varepsilon||u^\varepsilon-u^\delta||_{L^2}||u^\delta||_{L^\infty}||||\nabla u^\varepsilon||_{H^1}\\ &\lesssim \varepsilon||u^\varepsilon-u^\delta||_{L^2}||u^\delta||_{L^\infty}||||u^\varepsilon||_{H^2} \qquad (49)\end{aligned}$$

$$\begin{aligned}\langle \mathbb{P}J_{\phi,\delta}((J_{\phi,\delta}u^\delta\cdot\nabla)J_{\phi,\delta}(u^\varepsilon-u^\delta)),u^\varepsilon-u^\delta\rangle_{L^2} &= \langle (J_{\phi,\delta}u^\delta\cdot\nabla)J_{\phi,\delta}(u^\varepsilon-u^\delta),J_{\phi,\delta}(u^\varepsilon-u^\delta)\rangle_{L^2}\\ &= \frac{1}{2}\int_{\mathbb{R}^d}\nabla\cdot(J_{\phi,\delta}u^\delta|J_{\phi,\delta}(u^\varepsilon-u^\delta)|^2)dx \qquad (50)\\ &= 0 \qquad (51)\end{aligned}$$

We have appealed to the divergence free condition in Equation 50 and the divergence theorem (with a decay argument) in Equation 51. Putting it all together, we obtain the following.

$$\frac{d}{dt}||u^\varepsilon-u^\delta||^2_{L^2} \lesssim \varepsilon||u^\varepsilon-u^\delta||_{L^2}(||u^\varepsilon||_{L^\infty}||u^\varepsilon||_{H^2}+||\nabla u^\varepsilon||_{L^\infty}||u^\varepsilon||_{H^1}+||u^\delta||_{L^\infty}||u^\varepsilon||_{H^2})+||u^\varepsilon-u^\delta||^2_{L^2}||\nabla u^\varepsilon||_{L^\infty}$$

Using the uniform-in-ε bound in Equation 45 and the fact that since $s>2$, the H^s norm of u^ε controls the H^p norms of u^ε for $p<s$ and that of its gradients for $p<s-1$, and by the Sobolev embedding theorem of H^s into L^∞ for $s>d/2$, we conclude the following for any $t\in[-T_0,T_0]$ as defined earlier.

$$\frac{d}{dt}||u^\varepsilon-u^\delta||^2_{L^2} \lesssim \varepsilon||u^\varepsilon-u^\delta||_{L^2}||u_0||^2_{H^s}+||u^\varepsilon-u^\delta||^2_{L^2}||u_0||_{H^s}$$

Define $f(t):=||u^\varepsilon-u^\delta||_{L^2}$. For some $\varepsilon,\delta-$independent constant C, we can write the following.

$$\frac{d}{dt}(f(t))^2 \le C\varepsilon||u_0||^2_{H^s}f(t)+C||u_0||_{H^s}f(t)^2$$

Using the chain rule, we have the following.

$$2f(t)\frac{d}{dt}f(t) \le C\varepsilon||u_0||^2_{H^s}f(t)+C||u_0||_{H^s}f(t)^2$$

Where f doesn't vanish (we don't expect it to vanish outside a negligible set when $\delta\neq\varepsilon$), we can write the following by using $2C$ instead of C.

$$\begin{aligned}&\frac{f'(t)}{\varepsilon||u_0||_{H^s}+f(t)} \le C||u_0||_{H^S}\\ \iff\ &\frac{d}{dt}\left(\ln(\varepsilon||u_0||_{H^s}+f(t))\right) \le C||u_0||_{H^S}\end{aligned}$$

Integrating from 0 to $t \in [-T_0, T_0]$, we get the following.

$$\varepsilon||u_0||_{H^s} + f(t) \leq (\varepsilon||u_0||_{H^s} + f(0)) \exp\left(C||u_0||_{H^s}T_0\right)$$

Observe that $f(0) \lesssim \varepsilon||u_0||_{H^1} \leq \varepsilon||u_0||_{H^s}$. Hence, for some constant D, we conclude the following.

$$f(t) \leq \varepsilon||u_0||_{H^s}(1+D)\exp\left(C||u_0||_{H^s}T_0\right)$$

Since the bound on the right is uniform in time, we conclude. $\square$

By definition of the uniform metric, Proposition 3.10 implies that $\{u_\varepsilon\}_{\varepsilon>0}$ is Cauchy in $C([-T_0,T_0];L^2(\mathbb{R}^d)^d)$. Since $L^2(\mathbb{R}^d)$ is complete, the product space $L^2(\mathbb{R}^d)^d$ is complete, and it follows that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ that converges to 0 such that $(u^{\varepsilon_n})_{n\in\mathbb{N}}$ converges strongly in $C([-T_0,T_0];L^2(\mathbb{R}^d)^d)$. Observe that the uniform bound Equation 45 holds for the sequence $(u^{\varepsilon_n})_{n\in\mathbb{N}}$, so we can apply the Banach-Alaoglu theorem Lemma A.21 to formulate the following result.

Proposition 3.11. *There exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_+$ that converges to 0 and a map u that satisfy the following.*

$$u \in C([-T_0,T_0];L^2(\mathbb{R}^d)^d) \cap Lip([-T_0,T_0];H_\sigma^{s-1}(\mathbb{R}^d)) \cap L^\infty([-T_0,T_0];H_\sigma^s(\mathbb{R}^d))$$

$$\begin{aligned} u^{\varepsilon_n} &\to u \quad \textit{in} \quad C([-T_0,T_0];L^2(\mathbb{R}^d)^d) \textit{ as } n \to +\infty \\ u^{\varepsilon_n} &\overset{*}{\rightharpoonup} u \quad \textit{in} \quad Lip([-T_0,T_0];H^{s-1}(\mathbb{R}^d)^d) \textit{ as } n \to +\infty \\ u^{\varepsilon_n} &\overset{*}{\rightharpoonup} u \quad \textit{in} \quad L^\infty([-T_0,T_0];H^s(\mathbb{R}^d)^d) \textit{ as } n \to +\infty \end{aligned}$$

Remark. The weak-* convergence implicitly requires us to admit that L^∞ is the dual of L^1. The intricacies of this assumption are not important and are encoded in the fact that we admit Lemma A.21. A similar explanation goes for the space of Lipschitz maps, which can be identified as the space $W^{1,\infty}$ (the square integrable functions admitting first-order distributional derivatives that lie in L^∞).

Proof. Observe first that the following holds for any $\varepsilon > 0$ due to previously shown results.

$$u^\varepsilon \in C([-T_0,T_0];L^2(\mathbb{R}^d)^d) \cap \mathrm{Lip}([-T_0,T_0];H_\sigma^{s-1}(\mathbb{R}^d)) \cap L^\infty([-T_0,T_0];H_\sigma^s(\mathbb{R}^d))$$

Since L^2 spaces are complete, $C([-T_0,T_0];L^2(\mathbb{R}^d)^d)$ is complete, and by Proposition 3.10, there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_+$ that converges to 0 such that the following holds for some $u \in C([-T_0,T_0];L^2(\mathbb{R}^d)^d)$.

$$u^{\delta_n} \to u \quad \text{in} \quad C([-T_0,T_0];L^2(\mathbb{R}^d)^d) \text{ as } n \to +\infty$$

By Equation 45 and Lemma A.21, there exists a strictly increasing map $\psi : \mathbb{N} \to \mathbb{N}$ and a $v \in L^\infty([-T_0,T_0];H^s(\mathbb{R}^d)^d)$ such that the following holds.

$$u^{\delta_{\psi(n)}} \overset{*}{\rightharpoonup} v \quad \text{in} \quad L^\infty([-T_0,T_0];H^s(\mathbb{R}^d)^d) \text{ as } n \to +\infty$$

There remains to show weak-* convergence in the Lipschitz space and show that the limiting objects coincide. It can be shown that $\mathrm{Lip}([-T_0,T_0];H^{s-1}(\mathbb{R}^d)^d)$ is the dual of a separable Sobolev (normed and linear) space. Hence, it is sequentially weakly-* compact (this separability is necessary because in the non-metrizable weak-* topology, compactness certainly doesn't imply sequential compactness). Hence, there exists a strictly increasing map $\zeta : \mathbb{N} \to \mathbb{N}$ such that the following holds for some $w \in \mathrm{Lip}([-T_0, T_0]; H^{s-1}(\mathbb{R}^d)^d)$.

$$u^{\delta_{\zeta\circ\psi(n)}} \overset{*}{\rightharpoonup} w \quad \text{in} \quad \mathrm{Lip}([-T_0, T_0]; H^{s-1}(\mathbb{R}^d)^d) \text{ as } n \to +\infty$$

Observe that the three convergence statements imply the weak-* convergence of $(u^{\delta_{\zeta\circ\psi(n)}})$ to u, v and w in the space $L^\infty([-T_0, T_0]; L^2(\mathbb{R}^d)^d)$ by the inclusion of the continuous, Lipschitz, and L^∞ spaces in this common space. By Proposition A.19, we conclude on the uniqueness of the limit object. $\square$

3.5 Solving the Euler System with the Limit Object of Mollified Solutions

Let u denote the limit object described by Proposition 3.11. In this section, we show that it is a solution to the incompressible, homogeneous Euler equations. To start, we claim that u is a strong solution to Equation 37. In other words, u satisfies that the following integral form of the problem and its definition ensures all the below quantities are well-defined.

$$u(t) = u_0 - \int_0^t \mathbb{P}((u(\tau) \cdot \nabla)u(\tau))\, d\tau\,, \quad \forall t \in [-T_0, T_0]$$

In order to see that the above quantities are well-defined, we make the following observation. For any $\tau \in [0, t]$ ($[t, 0]$ if $t \le 0$), we have, by Proposition 3.11, that $u(\tau)$ lies in $H^s(\mathbb{R}^d)^d$. Since $s > d/2+1$, $H^{s-1}(\mathbb{R}^d)$ is an algebra and consequently the argument of the Leray projection above lies in $H^{s-1}(\mathbb{R}^d)^d$ and the Leray projection is well-defined. The Leray projection takes values in a Banach space, and provided that the integrand is regulated - for this, it suffices to show it is continuous - we conclude that the right-hand side of the above is well-defined. Observe that u is continuous from time to $L^2(\mathbb{R}^d)^d$, the operation corresponding to spatial differentiation and multiplication is also continuous into $L^2(\mathbb{R}^d)^d$ since u takes values in $H^s(\mathbb{R}^d)^d$. Finally, we conclude by the boundedness of the Leray projector that the right-hand side of the above equation is well-defined as the integrand is a continuous function from time into the space of square integrable functions.

Remark. We do not know that u is continuous from time into H^s at this stage. This is checked after checking that u is a $\mathcal{C}([-T_0, T_0]; L^2(\mathbb{R}^d)^d)$ solution.

We check the following for $t \in [-T_0, T_0]$ by expansion on the right-hand side.

$$\begin{aligned} u(t) - u_0 + \int_0^t \mathbb{P}((u(\tau) \cdot \nabla)u(\tau))\, d\tau = u(t) &- u^\varepsilon(t) - (\mathbb{I} - \mathbb{J}_{\phi,\varepsilon})u_0 \\ &+ \int_0^t \mathbb{P}(((u(\tau) - u^\varepsilon(\tau)) \cdot \nabla)u(\tau))\, d\tau \\ &+ \int_0^t \mathbb{P}((u^\varepsilon(\tau) \cdot \nabla)(u(\tau) - u^\varepsilon(\tau)))\, d\tau \\ &+ \int_0^t (\mathbb{I} - J_{\phi,\varepsilon})\mathbb{P}((u^\varepsilon(\tau) \cdot \nabla)u^\varepsilon(\tau))\, d\tau \\ &+ \int_0^t \mathbb{P}(((\mathbb{I} - J_{\phi,\varepsilon})u^\varepsilon(\tau) \cdot \nabla)u^\varepsilon(\tau))\, d\tau \\ &+ \int_0^t \mathbb{P}((u^\varepsilon(\tau) \cdot \nabla)(\mathbb{I} - J_{\phi,\varepsilon})u^\varepsilon(\tau))\, d\tau \end{aligned}$$

Since the integrand on the left-hand-side of the above is continuous and takes values in L^2, the integral takes values in L^2, so we can take the L^2 norm of the above and apply the triangle inequality on the right-hand side. We treat the terms on the right-hand side one by one. Appealing to Lemma A.12, we have the following.

$$||(\mathbb{I} - J_{\phi,\varepsilon})u_0||_{L^2} \lesssim \varepsilon ||u_0||_{H^1}$$

For any $\tau \in [0,t]$ (or $[t,0]$ if $t < 0$), we have that $u(\tau) \in H^s(\mathbb{R}^d)^d$ so its first-order derivatives lie in $H^{s-1}(\mathbb{R}^d)^d$. Since $s-1 > d/2$, we have that H^{s-1} embeds into L^∞ and we have the following.

$$\begin{aligned}
||\mathbb{P}(((u(\tau)-u^\varepsilon(\tau))\cdot\nabla)u(\tau))||_{L^2} &\le ||((u(\tau)-u^\varepsilon(\tau))\cdot\nabla)u(\tau)||_{L^2} \\
&\le \left(\sum_{i=1}^{d} ||(u(\tau)_j - u^\varepsilon(\tau)_j)\cdot\partial_j u(\tau)_i||_{L^2}^2\right)^{1/2} \\
&\le ||\nabla u(\tau)||_{L^\infty}||u(\tau)-u^\varepsilon(\tau)||_{L^2}
\end{aligned}$$

Proceeding similarly, we have the following.

$$||\mathbb{P}((u^\varepsilon(\tau)\cdot\nabla)(u(\tau)-u^\varepsilon(\tau)))||_{L^2} \le ||u^\varepsilon(\tau)||_{L^2}||\nabla u(\tau)-\nabla u^\varepsilon(\tau)||_{L^\infty} \lesssim ||u_0||_{H^s}||\nabla u(\tau)-\nabla u^\varepsilon(\tau)||_{L^\infty}$$

$$\begin{aligned}
||(\mathbb{I}-J_{\phi,\varepsilon})\mathbb{P}((u^\varepsilon(\tau)\cdot\nabla)u^\varepsilon(\tau))||_{L^2} &\lesssim \varepsilon||u^\varepsilon(\tau)\cdot\nabla u^\varepsilon(\tau)||_{H^1} \\
&\lesssim \varepsilon||u^\varepsilon(\tau)\cdot\nabla u^\varepsilon(\tau)||_{H^{s-1}} \\
&\lesssim \varepsilon||u^\varepsilon(\tau)||_{H^{s-1}}||\nabla u^\varepsilon(\tau)||_{H^{s-1}} \\
&\lesssim \varepsilon||u^\varepsilon(\tau)||_{H^s}^2
\end{aligned}$$

$$\begin{aligned}
||\mathbb{P}((((\mathbb{I}-J_{\phi,\varepsilon})u^\varepsilon(\tau)\cdot\nabla)u^\varepsilon(\tau))||_{L^2} &\le ||\nabla u^\varepsilon(\tau)||_{L^\infty}||(\mathbb{I}-J_{\phi,\varepsilon})u^\varepsilon(\tau)||_{L^2} \\
&\lesssim \varepsilon||\nabla u^\varepsilon(\tau)||_{L^\infty}||u^\varepsilon(\tau)||_{H^1}
\end{aligned}$$

$$\begin{aligned}
||\mathbb{P}((u^\varepsilon(\tau)\cdot\nabla)(\mathbb{I}-J_{\phi,\varepsilon})u^\varepsilon(\tau))||_{L^2} &\le ||u^\varepsilon(\tau)||_{L^\infty}||(\mathbb{I}-J_{\phi,\varepsilon})u^\varepsilon(\tau)||_{H^1} \\
&\lesssim \varepsilon||u^\varepsilon(\tau)||_{L^\infty}||u^\varepsilon(\tau)||_{H^1}
\end{aligned}$$

Putting it all together, we observe that, thanks to the uniform-in-ε and uniform-in-time bound on the norm in Equation 45 and thanks to the strong convergence part of the statement in Proposition 3.11 as well as the Sobolev embedding theorem and $s > s-1 > d/2$, the following condition will suffice to conclude on the proposed expression for $u(t)$ by sending $\varepsilon \to 0$.

$$\sup_{\tau\in[-T_0;T_0]} ||\nabla u(\tau)-\nabla u^\varepsilon(\tau)||_{L^\infty} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Observe that $\nabla u(\tau)$ is well-defined and lies in $H^{s-1}(\mathbb{R}^d)^{d\times d}$ as $u(\tau) \in H^s(\mathbb{R}^d)^d$. The same holds for ∇u^ε. Furthermore, both spatial derivatives are bounded since H^{s-1} embeds in L^∞ for $s-1 > d/2$. For fixed τ, we consequently have that $||\nabla u(\tau)-\nabla u^\varepsilon(\tau)||_{L^\infty}$ is well-defined. There remains to check that the supremum over time is finite and tends to 0. For this, we consider refining our argument in the compactness section by admitting the Sobolev interpolation inequality.

Lemma 3.12. *The limit object described in Proposition 3.11 satisfies the following.*

$$\nabla u^{\varepsilon_n} \to \nabla u \quad \textit{as } n \to +\infty \quad \textit{in } L^\infty([-T_0,T_0];L^\infty(\mathbb{R}^d)^{d\times d})$$

Proof. By the Sobolev interpolation inequality, we have the following for any $0 \le r \le s$ and for any $t \in [-T_0,T_0]$.

$$||u^{\varepsilon_n}(t)-u(t)||_{H^r} \le ||u^{\varepsilon_n}(t)-u(t)||_{L^2}^{1-r/s}||u^{\varepsilon_n}(t)-u(t)||_{H^s}^{r/s}$$

We've already shown the following.

$$\sup_{t\in[-T_0,T_0]} ||u^{\varepsilon_n}(t)-u(t)||_{L^2} \to 0 \quad \text{as} \quad n\to+\infty$$

We claim that $\sup_{t\in[-T_0,T_0]} ||u^{\varepsilon_n}(t)-u(t)||_{H^s} < +\infty$. We have the following.

$$||u^{\varepsilon_n}(t)-u(t)||_{H^s} \le ||u^{\varepsilon_n}(t)||_{H^s} + ||u(t)||_{H^s}$$

The first term of the right-hand side of the above is bounded uniformly-in-n and uniformly-in-t by Equation 45. We admit that weak-* convergence is lower-semi continuous - this result is described in Lemma A.18. This immediately implies that the second term is uniformly-in-t bounded. Hence, provided $1-r/s$ doesn't vanish - which is the case for $r<s$, we have the following.

$$\sup_{t\in[-T_0,T_0]} ||u^{\varepsilon_n}(t)-u(t)||_{H^r} \underset{n\to+\infty}{\to} 0$$

Since $s > 1+d/2$, we can take $s > r > 1+d/2$ and apply the above. Recalling the definition of the Sobolev norm, one easily checks the following.

$$||\nabla u(t)-\nabla u^{\varepsilon_n}(t)||_{H^{r-1}} \le ||u(t)-u^{\varepsilon_n}(t)||_r$$

Thus, we recover the following.

$$\nabla u^{\varepsilon_n} \to \nabla u \quad \text{as} \quad n\to+\infty \quad \text{in} \quad L^\infty([-T_0,T_0];H^{r-1}(\mathbb{R}^d)^{d\times d})$$

Since $r-1>d/2$, H^{r-1} embeds in L^∞ and we conclude. □

The above lemma allows us to conclude the following.

$$u(t) = u_0 - \int_0^t \mathbb{P}((u(\tau)\cdot\nabla)u(\tau))\,d\tau \quad \forall t\in[-T_0,T_0]$$

The above equality is, of course, almost everywhere due to the structure of L^2. Thus, we have shown that u is a strong solution to Equation 37. We can refine this result.

Theorem 3.13. *The constructed solution u is a classical solution to Equation 37 in the sense that it can be interpreted as a map of (t,x) into $\mathbb{R}^d$ and differentiated so as to satisfy the system.*

Proof. The result follows if everywhere equality holds, which in turn holds if the integrand is continuous and takes values in a space of continuous-in-space functions. Using the Sobolev interpolation inequality and the lower semi-continuity of weak convergence, our argument in Lemma 3.12 shows that $(u^{\varepsilon_n})_{n\in\mathbb{N}}$ is Cauchy in $C([-T_0,T_0];H^r(\mathbb{R}^d)^d)$ for $0\le r<s$. It follows, from arguments similar to those in Proposition 3.11, that u lies in $C([-T_0,T_0];H^r(\mathbb{R}^d)^d)$. We insist that $r<s$ so that the exponent $1-r/s$ doesn't vanish in the interpolation. Fix $r\in]1+d/2,s[$. Then, both H^r and H^{r-1} embed respectively in the Hölder spaces $C^{1,\alpha}$. and $C^{0,\alpha}$ for some $0<\alpha<1$. This ensures that $(u\cdot\nabla)u$ is continuous in time (as a composition of continuous functions) and takes values in $L^2\cap C^{0,\alpha}$. We admit that the Leray projector is stable on the intersection of these spaces and conclude that u and its time derivative are continuous and take values in a continuous function space. Hence, u can be written as a continuous function in (t,x). □

To conclude on the theorem, there remains to check that $u\in C([-T_0,T_0];H^s_\sigma(\mathbb{R}^d))$ and that it is unique. That u takes values in a divergence-free space follows from its construction using the integral form. Hence, we must only check that u is continuous as an H^s-valued function of time; this doesn't result from Sobolev interpolation because the exponent of the L^2 norm vanishes for $r=s$. We will proceed using the parallelogram law in inner product space (more precisely, Hilbert spaces) and two characterizations of continuity, which individually are easier to check than the continuity of u. We will appeal to the theorem below.

Theorem 3.14. *(Weak-Strong Continuity) Let $(X, ||\cdot||)$ be a separable Hilbert space. Suppose $u : [-T_0, T_0] \to X$ satisfies the following.*

1. *For any test function $\varphi \in X^*$, the following map is continuous.*

$$[-T_0, T_0] \ni t \mapsto ev_u(t)(\varphi) = \langle u(t), \varphi \rangle \in \mathbb{K}$$

2. *The map $t \mapsto ||u(t)||$ is continuous.*

Then, u is continuous.

Proof. We will use the sequential definition of continuity in normed vector spaces (owing to their first-countability). Let $t_0 \in [-T_0, T_0]$. Then, for any $t \in [-T_0, T_0]$, we have the following using the inner product structure of Hilbert spaces (the parallelogram law).

$$||u(t) - u(t_0)||^2 = 2||u(t)||^2 + 2||u(t_0)||^2 - ||u(t) + u(t_0)||^2$$

Let $(t_n)_n \in \mathbb{N}$ be a sequence convergent to t_0. By continuity of the norm, we have the following.

$$||u(t_n)||^2 \underset{n\to+\infty}{\longrightarrow} ||u(t_0)||^2$$

Since Hilbert spaces are reflexive, we appeal to Lemma A.18 and we have the following.

$$\liminf_{n\to+\infty} ||u(t_n) + u(t_0)||^2 \geq 4||u(t_0)||^2$$

Hence, we get the following.

$$0 \leq \limsup_{n\to+\infty} ||u(t_n) - u(t_0)||^2 \leq 2||u(t_0)||^2 + 2||u(t_0)||^2 - 4||u(t_0)||^2 = 0$$

We conclude on the strong continuity of u. □

We know H^s is a Hilbert space and we know it is separable, so we check the weak continuity of u and the continuity of its norm to conclude on the central theorem of this section. We identify the dual of H^s with H^{-s} - we admit this result. Let $\varphi \in H^{-s}(\mathbb{R}^d)$ have unit norm in H^{-s}. Let $\delta > 0$ be arbitrary. Observe that H^{-s+1} is dense in H^{-s} and is the dual of H^{s-1}. By the density, we can find $\psi \in H^{-s+1}(\mathbb{R}^d)$ that satisfies the following.

$$||\varphi - \psi||_{H^{-s}} \leq \delta \tag{52}$$

Remark. We identify the dual of H^s with H^{-s}, and the subsequently used duality brackets behave as an inner product in L^2 as described below.

$$\langle f, g \rangle_{H^r \times H^{-r}} = \langle \langle \nabla \rangle^r f, \langle \nabla \rangle^{-r} g \rangle_{L^2}$$

These duality relations are all admitted as their analysis goes beyond the scope of this thesis.

Recall that, thanks to the Sobolev interpolation inequality, for $0 \leq r < s$, we have the strong convergence of u^ε to u in $C([-T_0, T_0]; H^r(\mathbb{R}^d)^d)$. Hence, we have the following for $t \in [-T_0, T_0]$ where we appeal to the Cauchy-Schwarz inequality in L^2

$$\begin{aligned}|\langle u^\varepsilon(t)-u(t),\psi\rangle_{H^{s-1}\times H^{1-s}}| &= \left|\int_{\mathbb{R}^d}\langle\nabla\rangle^{s-1}(u^\varepsilon(t)-u(t))(x)\langle\nabla\rangle^{1-s}\psi(x)dx\right|\\ &\leq ||u^\varepsilon(t)-u(t)||_{H^{s-1}}||\psi||_{H^{1-s}}\\ &\leq ||u^\varepsilon-u||_{C([-T_0,T_0];H^{s-1})}||\psi||_{H^{1-s}}\end{aligned}$$

Using the strong convergence in $C([-T_0,T_0];H^{s-1})$ since $s-1\in]0,s[$, we the following for sufficiently small ε.

$$\sup_{t\in[-T_0,T_0]}|\langle u^\varepsilon(t)-u(t),\psi\rangle_{H^{s-1}\times H^{1-s}}|\leq\delta \tag{53}$$

We need more relation to conclude on the weak continuity. Using the fact that u^ε is Lipschitz for the H^{s-1} norm and because we can take an $\varepsilon-$independent Lipschitz constant, we have the following for small enough $\tau>0$ and for any $\varepsilon>0$

$$\sup_{t,s\in[-T_0,T_0],|t-s|<\tau}||u^\varepsilon(t)-u^\varepsilon(s)||_{H^{s-1}}\leq\delta$$

Finally, we recall that u is bounded on $[-T_0,T_0]$ in H^s, so we conclude the following. For any $|t-s|<\tau$, we have the following for small enough ε.

$$\begin{aligned}|\langle u(t)-u(s),\varphi\rangle| &\leq |\langle u(t)-u(s),\varphi-\psi\rangle|+|\langle u(s)-u(t),\psi\rangle|\\ &\leq |\langle u(t)-u(s),\varphi-\psi\rangle|\\ &+|\langle u(s)-u^\varepsilon(s),\psi\rangle|+|\langle u^\varepsilon(s)-u^\varepsilon(t),\psi\rangle|+|\langle u^\varepsilon(t)-u(t),\psi\rangle|\\ &\leq ||\varphi-\psi||_{H^{-s}}||\left(2\sup_{t\in[T_0,T_0]}||u(t)||_{H^s}\right)+3\delta\\ &\lesssim\delta\end{aligned}$$

We have appealed to the fact that the uniform bound in Equation 45 applied to $||u||_{L^\infty(t;H^s)}$ due to the lower semi-continuity of the weak convergence described in Proposition 3.11. Hence, we've shown that $t\mapsto\langle u(t),\varphi\rangle$ is uniformly continuous, so it must be continuous.

Remark. We have abused notation considering the fact that u is a vector-field, but we assert that a component-wise interpretation of the above results is sufficient for the results to hold.

There remains to check that $t\mapsto||u(t)||_{H^s}$ is continuous and that u is unique as a solution to Equation 37. In light of Proposition 3.11, we only know that the aforementioned map is bounded. We will show that it is continuous at $t=0$ and then argue by the uniqueness of the solution that this is sufficient to assert its continuity, which would conclude the proof of Theorem 3.1 at once. Firstly, as a result of the weak continuity shown above, we have the following.

$$\liminf_{t\to0}||u(t)||_{H^s}\geq||u_0||_{H^s}$$

We must now appeal to an important result describing the relationship between $u(t)$ and $\{u^\varepsilon(t)\}_{\varepsilon>0}$ for fixed $t\in[-T_0,T_0]$. We write for any $t\in[-T_0,T_0]$, $\varepsilon>0$ and $\phi\in H^{-s}$

$$\langle u(t),\phi\rangle=\langle u^\varepsilon(t),\phi\rangle_{H^s,H^{-s}}+\langle u(t)-u^\varepsilon(t),\phi\rangle_{H^s,H^{-s}}$$

Using triangle inequality and Cauchy-Schwarz in L^2, we have the following.

$$\langle u(t),\phi\rangle\leq\|u^\varepsilon(t)\|_{H^s}\|\phi\|_{H^{-s}}+|\langle u(t)-u^\varepsilon(t),\phi\rangle|.$$

Since the right-hand side does not depend on ε, we can write the following.

$$\langle u(t), \phi \rangle \leq \limsup_{\varepsilon \to 0} ||u^\varepsilon(t)||_{H^s} ||\phi||_{H^{-s}} + \limsup_{\varepsilon \to 0} |\langle u(t) - u^\varepsilon(t), \phi \rangle|$$

The second term at the right-hand side of the above inequality goes to 0 as $\varepsilon \to 0$ by the weak-* convergence of u^ε to u in $L^\infty([-T_0, T_0]; H^s(\mathbb{R}^d)^d)$. Indeed, let $\delta > 0$. By density of L^2 in H^{-s}, take $\psi \in L^2$ satisfy $||\psi - \phi||_{H^{-s}} \leq \delta$. We have the following.

$$\begin{aligned} |\langle u(t) - u^\varepsilon(t), \phi \rangle| &\leq |\langle u(t) - u^\varepsilon(t), \psi \rangle| + |\langle u(t) - u^\varepsilon(t), \phi - \psi \rangle| \\ &\leq ||u(t) - u^\varepsilon(t)||_{L^2} |||\psi||_{L^2} + ||u(t) - u^\varepsilon(t)||_{H^s} ||\phi - \psi||_{H^{-s}} \end{aligned}$$

By strong convergence in the space of continuous functions into L^2, we can take ε sufficiently small so that the first term is less than δ. Observe that by Equation 45, we have that $||u(t) - u^\varepsilon(t)||_{H^s}$ is bounded uniformly in time and ε. Hence, the second term is, up to a constant, less than δ. Hence, for small enough ε, we have the following.

$$\limsup_{\varepsilon \to 0} |\langle u(t) - u^\varepsilon(t), \phi \rangle| = 0$$

Finally, we observe that, by the construction of the dual, we have the following.

$$\|u(t)\|_{H^s} = \sup_{\|\phi\|_{H^{-s}} \leq 1} \langle u(t), \phi \rangle$$

Taking the required supremum, we get the following.

$$||u(t)||_{H^s} \leq \limsup_{\varepsilon \to 0} ||u^\varepsilon(t)||_{H^s} \cdot \sup_{\|\phi\|_{H^{-s}} \leq 1} ||\phi||_{H^{-s}} \leq \limsup_{\varepsilon \to 0} ||u^\varepsilon(t)||_{H^s}$$

Consider $t > 0$. We apply the estimate derived from the forward integration of Equation 44. We get the following.

$$||u(t)||_{H^s} \leq \frac{||u_0||_{H^s}}{1 - C_s ||u_0||_{H^s} t}$$

Taking a limit superior as $t \to 0^+$ yields the following.

$$\limsup_{t \to 0^+} ||u(t)||_{H^s} \leq ||u_0||_{H^s}$$

A similar calculation holds for $t < 0$ by integrating backwards in time Equation 44. We conclude that $t \mapsto ||u(t)||_{H^s}$ is continuous at 0. Before concluding on the continuity everywhere, we show that u is the unique solution to Equation 37 in its regularity class.

Proposition 3.15. *The above constructed u is the unique solution to Equation 37 in its regularity class.*

Proof. Suppose v satisfies the regulairty constraints of u and solves Equation 37. Then, $u - v$ satisfies the following system.

$$\partial_t(u - v) + \mathbb{P}((u - v) \cdot \nabla u) + \mathbb{P}(v \cdot \nabla (u - v)) = 0, \quad (u - v)_{t=0} = 0$$

Using the chain rule, taking an L^2 inner product with $u(t) - v(t)$ f the above yields the time derivative of the norm on the left-hand side and the following, where we use that $\mathbb{P}$ is self-adjoint (by Fourier multiplier representation) and that u, v are diveregnce free simply by taking the divergence of Equation 37.

$$\frac{1}{2} \partial_t ||u - v||_{L^2}^2 = \langle v(t) - u(t), ((u - v) \cdot \nabla u \rangle_{L^2} + \langle v(t) - u(t), v \cdot \nabla (u - v) \rangle_{L^2}$$

Observe that the integrand in the expansion of the second term writes as follows by appealing to the divergence free condition.

$$\frac{1}{2}\nabla \cdot (v \cdot |(u(t) - v(t))(x)|^2)$$

Then, by a decay argument, the second term vanishes and we have the following.

$$\begin{aligned}\frac{1}{2}\partial_t ||u - v||_{L^2}^2 &= \langle v(t) - u(t), ((u - v) \cdot \nabla u\rangle_{L^2} \\ &\leq ||u(t) - v(t)||_{L^2}^2 ||\nabla u(t)||_{L^\infty}\end{aligned}$$

Since $s - 1 > 1/2$, we can control $||\nabla u(t)||_{L^\infty}$ by the H^s norm of u, which is uniformly-in-t bounded. Then, noting that $||u(0) - v(0)||_{L^2}^2 = 0$, the Grönwall inequality yields $||u(t) - v(t)||_{L^2} = 0$ for all time $t \in [-T_0, T_0]$. □

Now, not only have we proved the uniqueness of u, but applying all the above derivations on Equation 37 and shifting the system in time (composing u with $t \mapsto t - t_0$ in Equation 37), we get that the continuity at $t = 0$ is enough to assert the continuity of the norm at $t_0 \in [-T_0, T_0]$ arbitrarily chosen, since the solution to the shifted equation coincides with u on $[-T_0, T_0]$. Thus we complete the proof of Theorem 3.1 with the weak continuity and norm continuity and uniqueness which show u is continuous into H^s (and, of course, unique as required by the theorem). We mention that, by the unique construction of u and because its time of definition depends on that of the mollified solutions which in turn are built using arguments related to Cauchy-Lipschitz, we have the following.

Lemma 3.16. *The solution from Theorem 3.1 can be uniquely extended to a maximal interval $]T_-, T_+[$ for $T_- < -T_0$ and $T_+ > T_0$. Furthermore, either these time bounds are infinity or we have the following, where we consider a finite T_+ without loss of generality.*

$$\limsup_{t \nearrow T_+} ||u(t)||_{H^s} = +\infty$$

This results from the construction of u^ϵ and the finite time blowup condition on this family as well as the fact that u is uniquely determined by it.

3.6 Discussion

We conclude on the local existence and extension criterion of a unique, continuous H^s-valued and Lipschitz H^{s-1} valued solution provided the initial data is H^s. The approach included solving an easier problem - the mollified equations - leveraging the smoothening aspect of mollification and how it allows to move quantities into H^s at the cost of factors indexed by the mollifiers. We then establish the common function space of existence of the family of mollified solutions and argue by weak compactness that a unique limit object exists. We then use this limit object as our candidate to solve the Euler equations. We do this by first showing it equals a strong solution thanks to the strong convergence in L^2 and H^r (for $r < s$) of the mollified solutions. By continuity in L^2 we show that u can also be treated as a classical solution since itself and its derivative are continuous in time and valued in a continuous function space thanks to the Sobolev embedding theorem and that $s > d/2 + 1$. Finally, we appeal to the weak continuity of $u(t)$ and the continuity of its norm (which in turn requires pointwise weak convergence in H^s) to assert that u is the desired class, continuous from time with values in H^s_σ. We mention that though it isn't explicitly elaborated beyond the mollification step, that u is divergence-free is an evident consequence of the Leray projector - this is actually the purpose of the design of this projector as outlined in the first chapter, where we demonstrated that the Leray projected equation is equivalent to the system with the divergence-free equation. While the time extension follows naturally, it is currently an open-ended question whether the phenomenon of finite-time solution blow-up occurs at all, and whether it can be predicted for some initial data in some specific regularity classes. For $d = 2$ it can be shown that solutions can be extended over the real line, the question remains unanswered in $d = 3$. A logical continuation to this thesis would an exploration of the Navier-Stokes equations and ultimately this phenomenon of finite blow-up in certain spaces of spatial functions.

A Appendix

A.1 The Grönwall Lemma and Maximal Solutions

The following results are invoked in Lemma 2.2 to show that maximal solutions to Equation 1 and the initial label condition are defined at all times $t \in \mathbb{R}$. The first result is a consequence of the Grönwall lemma and its proof is given in Beuzart-Plessis's book [2].

Proposition A.1. *Let $y : [a, b] \to \mathbb{R}^d$ be of class C^1 and suppose $\alpha, \beta > 0$ satisfy the following for all $t \in [a.b]$.*

$$||y'(t)|| \leq \beta + \alpha||y(t)||$$

Then, the following holds for $t \in [a, b]$.

$$||y(t)|| \leq ||y(a)|| \exp((t-a)\alpha) + \frac{\beta}{\alpha}(\exp((t-a)\alpha) - 1)$$

In the above result, the fact that the right-hand side of the consequent inequality is bounded at t goes to either extremity allows us to assert that y is bounded in a neighborhood of the extremity, which subsequently prevents it from being a maximal solution as it cannot blow up in finite time. The following result, also proven in Beuzart-Plessis's book [2], formalizes the statement.

Proposition A.2. *Let $F : \mathbb{R} \times \mathbb{R}^d$ be continuous, locally Lipschitz in the second variable. Let $\phi :]a, b[\to \mathbb{R}^d$ be a solution to $y' = F(t, y)$. Assume there exists a compact set of $\mathbb{R} \times \mathbb{R}^d$ such that for all $t \in]a, b[$, $(t, \phi(t)) \in C$. Then, ϕ can be extended to a solution on a strictly bigger interval i.e. it isn't maximal.*

A.2 Cauchy's Theorem and Continuum Mechanics

This section is dedicated to briefly presenting some results and notions used to apply the conservation of linear momentum in our derivation of the incompressible, homogeneous Euler equations. We recall that stress is defined as the internal force per unit area exerted by a material in response to an applied load. It is directly linked to the traction forces to which our fluid parcels are subjected. Cauchy's theorem states that, at any point in time, the traction force acting on a parcel in a certain direction is directly linked to the vector normal to the surface by a stress tensor, which could be time-dependent and specific to each fluid parcel.

Theorem A.3. *(Cauchy Stress Theorem) [7] The stress vector $\mathbf{t}$ on a surface through a particle P is uniquely determined by the second-order stress tensor (which can be represented by a matrix) $\mathbf{T}$ in the particle and the unit normal $\mathbf{n}$ to the surface through the following relation.*

$$\mathbf{t} = \mathbf{T}\mathbf{n} \iff t_i = \sum_k T_{ik} n_k, \quad \forall i$$

This statement is powerful as it shows that the traction will only depend on the normal to the surface of the element considered, and not on other shape characteristics.

Remark. By the linearity of the equation in Theorem A.3, one easily recovers Newton's third law of motion - the action-reaction principle.

As a consequence of the conservation of angular momentum, the tensor features the property of being symmetric. We state the result below without its proof, which can be found in Irgens's *Continuum Mechanics* [7].

Theorem A.4. *(Cauchy's Second Law of Motion) The stress tensor, when represented as a matrix, is symmetric. In other words, we have the following $\forall i, j$.*

$$T_{ij} = T_{ji}$$

We will assume we are working with ideal fluids, which only transfer normal stresses across surfaces. This and the fact that pressure is assumed to be isotropic results in a stress tensor, which is the identity matrix multiplied by the scalar pressure.

A.3 The Laplace and Poisson Equations

The following analysis is heavily based on [5], [6]. Denoting Δ the Laplacian operator and assuming all the subsequent functions on which it operates admit second-order partial derivatives, the Laplace equation writes as follows for some maps f, g.

$$\Delta f = 0 \tag{54}$$

The Poisson equation, more general and pertinent in our case, writes as follows.

$$-\Delta f = g \tag{55}$$

To solve said equations amounts to finding, under varying levels of regularity, the maps that satisfy the desired equality, where the the unknown in the above is the map f. In order to solve the Poisson equation, we first consider solutions to the Laplace equation. In order to choose the type of solutions to look for, we make the following observation. In what follows, we fix time and consider scalar fields of the form $\mathbb{R}^d \to \mathbb{R}$. Suppose f is Fréchet differentiable and admits second-order partial derivatives. We observe that the Laplacian can be identified as follows, where $H(f)(x)$ denotes the Hessian of f at x.

$$\Delta f(x) = \mathrm{Tr}(H(f)(x)) = \mathrm{Tr}(\nabla f \cdot (\nabla f)^T)$$

Let Q be a rotation in $\mathbb{R}^d$ - it is orthogonal. Define $g = f \circ Q$. Immediately, g is as regular as f by the properties of linear transformations in finite dimension. Computing the Fréchet derivative of g yields the following.

$$\nabla g(x) = Q^T \nabla f(Qx)$$

Consequently, we have.

$$\begin{aligned}
\Delta(f \circ Q)(x) = \Delta g(x) &= \mathrm{Tr}(H(g)(x)) \\
&= \mathrm{Tr}(\nabla g(x) \nabla g(x)^T) \\
&= \mathrm{Tr}(Q^T \nabla f(Qx) \nabla f(Qx)^T Q) \\
&= \mathrm{Tr}(H(f)(Qx) \cdot QQ^T) \\
&= (\Delta f) \circ Q(x)
\end{aligned}$$

Hence, the Laplacian is invariant under rotational change of coordinates. In particular, if f satisfies the Laplace equation, so does its composition with any rotation since the zero function is invariant under any transformation. In other words, we must solve the same equation up to any rotation on a sphere, which suggests looking for solutions with radial dependence only (when solving the Laplace equation). Denote $r = |x|$ and we propose, as a solution, $v(r) = f(x)$. Note that we use the standard Euclidean norm - owing to the use of orthogonal matrices in the orthonormal basis of $\mathbb{R}^d$ in the above. Assuming all the regularity we need for these a priori estimates of the radial solution, we have the following away from the origin (where $|\cdot|$ is not differentiable).

$$\partial_{x_i} r = \partial_{x_i} \left(\left(\sum_{j=1}^{d} x_i^2 \right)^{-1/2} \right) = \frac{x_i}{r}$$

Consequently, the chain rule yields:

$$\frac{\partial^2}{\partial^2_{x_i}} f(x) = \frac{\partial^2}{\partial^2_{x_i}} v(r) = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

Summing over all variables leads to the following.

$$\Delta f = 0 \iff v'' + \frac{n-1}{r} v' = 0$$

Where v' doesn't vanish, we solve the first-order ODE on v' and we get the following on $\mathbb{R}^d \backslash \{0\}$ for some arbitrary constant b, c which depend on the initial conditions.

$$v(r) = \begin{cases} b \ln r + c & \text{if } d = 2 \\ \frac{b}{r^{d-2}} + c & \text{if } d \geq 3 \end{cases}$$

We normalize v to propose the fundamental solution to the Laplace equation, defined everywhere away from the origin and where $\alpha(d)$ denotes the volume of the $d-$ dimensional unit ball.

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & \text{if } d = 2 \\ \frac{1}{d(d-2)\alpha(d)} \frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases} \tag{56}$$

Using this fundamental solution, we can attempt to solve the Poisson equation. For simplicity, consider Equation 55 with $g \in C_c^2$, twice differentiable with compact support. The following theorem and its proof are taken as is from Evans's book [5].

Theorem A.5. *(Solving Poisson's Equation) Suppose $g \in C_c^2$, denote Φ the fundamental solution to the Laplace equation, and define f as follows on $\mathbb{R}^d$.*

$$f(x) = \int_{\mathbb{R}^d} \Delta\Phi(x-y)g(y)dy$$

Then, f is twice continuously differentiable on $\mathbb{R}^d$ and satisfies $-\Delta f = g$ everywhere.

Proof. By a translational coordinate change, we have the following.

$$f(x) = \int_{\mathbb{R}^d} \Phi(x-y)g(y)dy = \int_{\mathbb{R}^d} \Phi(y)g(x-y)dy$$

Hence, for h is a punctured neighborhood of 0 and denoting $\{e_1, ..., e_d\}$ the canonical basis of $\mathbb{R}^d$, we have the following for any $i \in \{1, ..., d\}$.

$$\frac{f(x+he_i) - f(x)}{h} = \int_{\mathbb{R}^d} \Phi(y) \left[\frac{g(x+he_i-y) - g(x-y)}{h}\right] dy$$

Taking the limit $h \to 0$ and using the uniform convergence arising from g's compact support, we get the following.

$$\partial_{x_i} f(x) = \int_{\mathbb{R}^d} \Phi(y)\partial_{x_i} g(x-y)dy$$

Iterating the process for $j \in \{1, ..., d\}$, we get the following.

$$\frac{\partial^2 f}{\partial_{x_j}\partial_{x_i}}(x) = \int_{\mathbb{R}^d} \Phi(y) \frac{\partial^2 g}{\partial_{x_j}\partial_{x_i}}(x-y)dy$$

Since the right-hand side is continuous, we conclude that f is twice continuously differentiable. Let us now prove that it satisfies the Laplace equation. By partitioning our space to treat the blow-up of Φ at 0 alone, we have the following, where we used the above expression of the second-order partial derivatives and we fix $\varepsilon > 0$

$$\Delta f(x) = \int_{B(0,\varepsilon)} \Phi(y)\Delta g(x-y)dy + \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \Phi(y)\Delta g(x-y)dy$$

Using the compactness of g's support and bounding the integral fo the norm of the fundamental solution, it is easy to check, that for some constant $C, C' > 0$, we have the following.

$$\begin{aligned} \left| \int_{B(0,\varepsilon)} \Phi(y)\Delta g(x-y)dy \right| &\leq C \sup_{x\in\mathbb{R}^d} ||H(g)(x)|| \int_{B(0,\varepsilon)} |\Phi(y)|dy \\ &\leq \begin{cases} C\varepsilon^2 |\ln \varepsilon| & \text{if} \quad d = 2 \\ C\varepsilon^2 & \text{if} \quad d \geq 3 \end{cases} \end{aligned}$$

In the limit $\varepsilon \to 0$, this term vanishes. We turn our attention to the integration away from the ball centered at the origin. An integration by parts (in several dimensions) and noting that by the continuity of the derivatives of g, we have that the boundary term vanishes, we get the following.

$$\Delta f = \lim_{\varepsilon\to 0} - \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \nabla\phi(y) \cdot \nabla(g(x-y))dy$$

Integrating by parts by parts again to differentiate Φ and noting that it is harmonic away from the origin, we get the following, where ν denotes the inward unit normal along the sphere centered at 0.

$$\Delta f = \lim_{\varepsilon\to 0} - \int_{\partial B(0,\varepsilon)} \nu \cdot \nabla\Phi(y) g(x-y)dy$$

By computation, we have $\nu = -y \cdot \varepsilon^{-1}$ on the boundary of $B(0, \varepsilon)$ and $\nabla\Phi(y) = \frac{1}{d\alpha(d)\varepsilon^{d-1}}$ on said boundary. Consequently, we get the following.

$$\Delta f = \lim_{\varepsilon\to 0} \frac{1}{d\alpha(d)\varepsilon^{d-1}} \int_{\partial B(0,\varepsilon)} g(x-y)dS(y)$$

Recognizing the above as the spatial average of the field and sending the radius of the ball to 0 yields the result. □

The result can be extended to $L^p(\mathbb{R}^d)$ functions owing to the density of $C_c^\infty(\mathbb{R}^d; \mathbb{R})$ in $L^p(\mathbb{R}^d)$ for any $1 \leq p \leq \infty$. We mention below a lemma that was critical to the above conclusions.

Lemma A.6. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous for some $n, m \in \mathbb{N}^*$. Then, we have the following, where the balls are implicitly taken over the appropriate spaces and where the dash through the integral denotes an average over the area of integration.*

$$\lim_{\varepsilon\to 0, \varepsilon>0} ⨍_{\partial B(x,\varepsilon)} f(y)dS(y) = f(x), \quad \forall x \in \mathbb{R}^n$$

Proof. Let $x \in \mathbb{R}^n$ and let $\varepsilon > 0$. We have the following.

$$\left| ⨍_{\partial B(x,\varepsilon)} f(y)dS(y) - f(x) \right| = \left| ⨍_{\partial B(x,\varepsilon)} (f(y) - f(x))dS(y) \right|$$

Let $\delta > 0$. Then, by continuity of f, we have that there exists $r > 0$ such that for every $||x - y|| < r$, we have $|f(x) - f(y)| < \delta$. Then, for $0 < \varepsilon < r$, we have the following.

$$\left| \fint_{\partial B(x,\varepsilon)} (f(y) - f(x)) dS(y) \right| \leq \delta$$

□

We could have also proceeded otherwise on spaces regular enough for the Fourier transform to be defined. Taking the Fourier transform of Equation 55 and replacing g with h yields the following equation.

$$-|\xi|^2 \widehat{f} = \widehat{h}$$

Taking an inverse Fourier transform yields the unique f up to a harmonic function. Indeed, in regular enough spaces (a set of spaces that contain all our spaces of interest), the Fourier transform as we define it is a bijection (among other more powerful descriptions) so that the Fourier transform of the Laplacian of f is uniquely associated to it. In the presence of a decay condition, we can even show that f is unique. Indeed, suppose f_1 and f_2 solve the Poisson equation. Then, their difference solves the Laplace equation. In particular, we have the following.

$$(f_1 - f_2)\Delta(f_1 - f_2) = 0$$

$$\implies \int_{\mathbb{R}^d} (f_1 - f_2)\Delta(f_1 - f_2) dx = 0$$

Integrating by parts and using the fact that decay induces a null boundary term yields the following.

$$\int_{\mathbb{R}^d} |\nabla(f_1 - f_2)|^2 dx = 0$$

Hence, the gradient vanishes identically and $f_1 - f_2$ is a constant. Decay forces this constant to be zero.

A.4 Euler's Equations: The Vorticity Formulation

Suppose $d = 3$. The vorticity is defined as follows with respect to the velocity field.

$$\omega(t, x) = \nabla \times u(t, x) \tag{57}$$

Using results from vector calculus, one checks that the following is equivalent to momentum balance.

$$\partial_t u + \omega \times u + \nabla \left(p + \frac{|u|^2}{2} \right) = 0$$

Taking the curl of the equation yields the following (and assuming time and spatial derivatives commute).

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u$$

To treat the incompressibility, we solve the following system.

$$\begin{cases} \nabla \cdot u = 0 \\ \nabla \times u = \omega \end{cases}$$

We do not elaborate too much, but one can show that treating the above results in the following, equivalent formulation of the Euler equations for incompressible flows.

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u \tag{58}$$

$$u = \nabla \times (-\Delta)^{-1}\omega \tag{59}$$

The above equations form a nonlocal system and it is not hard to show that they are equivalent to Equation 35. We only hint at the fact that Equation 59 results from taking the curl of $\omega = \nabla \times u$ and using the relation between the Laplacian and the double curl operation:

$$\Delta = \nabla(\nabla \cdot) - \nabla \times \nabla \times$$

A.5 Lebesgue and Sobolev Spaces

In what follows, we implicitly use the Lebesgue measure and the Borel sigma-algebra.

The Space $L^p(\mathbb{R}^d)$.

Let $1 \leq p < +\infty$. Denote $\mathcal{L}^p(\mathbb{R}^d)$ the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\int_{\mathbb{R}^d} |f|^p < +\infty$. Taking the quotient of this set with the equivalence relation of being equal almost everywhere and norming it with $||f||_{L^p} = \left(\int_{\mathbb{R}^d} |f|\right)^{1/p}$ yields the $L^p(\mathbb{R}^d)$ space. The L^∞ is constructed analogously taking the bounded measurable functions (this definition requires the notion of essential supremum.) Observe that $L^p(\mathbb{R}^d)$ is a Banach space and $L^2(\mathbb{R}^d)$ is a Hilbert space.

Weak Derivatives

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ lies in $L^1_{\text{loc}}(\mathbb{R}^d)$ (a fairly broad set, containing $L^p(\mathbb{R}^d)$ for any $p \geq 1$) - that is, it is L^1 on any compact set of $\mathbb{R}^d$. We say it is weakly differentiable with respect to the $i-$th variable if there exists a function $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that, for any $\phi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ (infinitely many times differentiable real-valued functions on $\mathbb{R}^d$ with compact support/test functions), we have the following.

$$\int_{\mathbb{R}^d} f \partial_i \phi \, dx = -\int_{\mathbb{R}^d} g\phi \, dx$$

We call g the weak derivative of f with respect to the $i-$th variable, and we denote it $\partial_i f = g$. This definition is motivated by the observation that, if f is differentiable, an integration by parts and an annihilation of the boundary term yields the right-hand side of the above. We can generalize this definition to higher-order derivatives, keeping in mind that our intuition holds thanks to the application of the Schwarz theorem to the test functions. Let $\alpha = (\alpha_1, ..., \alpha_k)$ be a multi-index for $k \in \mathbb{N}^*$. Denote $\partial_\alpha := \partial_{\alpha_1} \cdots \partial_{\alpha_k}$. We say f has a α weak derivative (of order $|\alpha| = k$) if there exists a locally integrable g satisfying the following.

$$\int_{\mathbb{R}^d} f \partial_\alpha \phi \, dx = (-1)^k \int_{\mathbb{R}^d} g\phi \, dx$$

We denote $\partial_\alpha f = g$. It can be shown, thanks to the test functions, that the weak derivative is unique up to a negligible set. In particular, it coincides almost everywhere with the (strong) derivative when the latter exists.

Sobolev Spaces

We restrict our focus to $L^2(\mathbb{R}^d)$. Let $s \in \mathbb{N}^*$. We define $H^s(\mathbb{R}^d)$ the set of $L^2(\mathbb{R}^d)$ maps that admit an α weak derivative for any multi-index α satisfying $0 \leq |\alpha| \leq s$ and which lies in $L^2(\mathbb{R}^d)$. This notion can be characterized and even extended using the Fourier transform of L^2 functions. Suppose $u \in L^2(\mathbb{R}^d)$ is Fréchet-differentiable with partial derivatives in $L^2(\mathbb{R}^d)$, and recall that the Fourier transform is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Then, its Fourier transform is defined, and that of its partial derivatives is also defined and satisfies the following.

$$\mathcal{F}[\partial_k u] = i\xi_k \mathcal{F}[u] \iff \widehat{\partial_k u} = i\xi_k \hat{u}$$

It can be shown that $u \in L^2(\mathbb{R}^d)$ (not necessarily differentiable) admits a weak derivative $\partial_k u$ if and only if $\xi_k \hat{u} \in L^2(\mathbb{R}^d)$, in which case the above equality holds as well. Hence, we can define $H^1(\mathbb{R}^d)$ as follows. A map u is in $H^1(\mathbb{R}^d)$ if and only if the following holds, where the equivalences are enabled by the Fourier transform being an isometry.

$$\begin{aligned} &\begin{cases} u \in L^2(\mathbb{R}^d) \\ \partial_k u \text{ exists for every } k = 1, ..., d \end{cases} \\ \iff &\begin{cases} \hat{u} \in L^2(\mathbb{R}^d) \\ i\xi_k \hat{u} \in L^2(\mathbb{R}^d) \text{ exists for every } k = 1, ..., d \end{cases} \\ \iff &\begin{cases} \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 d\xi < +\infty \\ \int_{\mathbb{R}^d} \xi_k^2 |\hat{u}(\xi)|^2 d\xi < +\infty \text{ for every } k = 1, ..., d \end{cases} \end{aligned}$$

Summing the second line over k and adding the two lines, we define $H^1(\mathbb{R}^d)$ as follows.

$$H^1(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \middle| \int_{\mathbb{R}^d} (1 + |\xi|^2) |\hat{u}(\xi)|^2 d\xi < +\infty \right\}$$

Iterating the process over higher derivatives for H^s with $s \geq 1$, we get the following, which makes sense for $s \in \mathbb{R}$ rather than just $\mathbb{N}^*$.

$$H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 d\xi < +\infty \right\}$$

Thus, we have used the Fourier transform to define fractional derivatives and Sobolev spaces. We expect higher regularity with higher s. In fact, it can be shown that high enough s brings about classical regularity; that is, differentiability and continuity in the classical sense. This is elaborated in Sobolev's embedding theorem, a particular case of which we summarize as follows. If $s - \frac{d}{2} > 0$ for some integer s, there exists an embedding of H^s into the the $\alpha-$Hölder space (more than continuous functions.) Observe that for any real s, H^s a Hilbert space with the following inner product, defined using the Fourier transform.

$$\langle f, h \rangle = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \overline{\hat{f}(\xi)} \hat{h}(\xi) \, d\xi$$

Maps with Values in $\mathbb{R}^d$

In section 3, we will choose the velocity field $u \in L^2(\mathbb{R}^d)^d$, the space of $\mathbb{R}^d \to \mathbb{R}^d$ vector fields with $L^2(\mathbb{R}^d)$ components. In fact, we can show that, if a measurable map u is said to be in $L^2(\mathbb{R}^d; \mathbb{R}^d)$ when $x \mapsto ||u(x)|| \in L^2(\mathbb{R}^d)$, taking $u \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ is equivalent to taking all of its components in $L^2(\mathbb{R}^d)$. We apply the same logic to the notation $u \in H^s(\mathbb{R}^d)^d$ and thus justify our component-wise reasoning. This equivalence is enabled by the fact that $\mathbb{R}^d$ is finite-dimensional and that all norms on this space are equivalent. Recall that the product of Hilbert spaces and the inner product on the product space inherit the properties of the inner product on the individual spaces. Hence, we know what our inner products and norms look like on $L^2(\mathbb{R}^d)^d$ and $H^s(\mathbb{R}^d)^d$, though we may denote them $|| \cdot ||_{L^2}$ when it is understood that the argument is in $L^2(\mathbb{R}^d)^d$ and likewise for the H^s product space.

Finally, we present below the Sobolev embedding theorem, which is critical to a number of results presented above.

Theorem A.7. *Suppose $d < 2s$ and $k + \alpha = s - d/2$. Then we have the following.*

$$H^s \subset C^{k,\alpha}$$

The Divergence-Free Spaces

We have shown that incompressibility can be formulated at $\nabla \cdot u = 0$. We want to extend the definition of divergence to $L^2(\mathbb{R}^d)^d$ fields. Consider $C_c^\infty(\mathbb{R}^d;\mathbb{R})$ and define the set of divergence-free operators as follows, where the divergence is classical and well-defined.

$$C_\sigma := \{u \in C_c^\infty(\mathbb{R}^d;\mathbb{R})^d \mid \nabla \cdot u = 0\} \subset L^2(\mathbb{R}^d)^d$$

Then, $L^2_\sigma(\mathbb{R}^d)$ is defined as follows, where the closure of C_σ is taken in $L^2(\mathbb{R}^d)^d$.

$$L^2_\sigma(\mathbb{R}^d) := \overline{C_\sigma}$$

It can be shown that this formulation is equivalent to defining the divergence in the distributional sense as follows.

$$L^2_\sigma(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d)^d \,\middle|\, \int_{\mathbb{R}^d} u \cdot \nabla\phi \, dx = 0, \forall \phi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}) \right\}$$

It is apparent that the first definition is motivated by the density of the test functions in L^2 spaces. As the closure of a vector space, we have that $L^2_\sigma(\mathbb{R}^d)$ is a closed linear space.

A.6 The Leray Projector

The Leray projector is well-defined on the Schwartz product space.

$$\mathbb{P} : \begin{cases} \mathcal{S}(\mathbb{R}^d)^d & \longrightarrow L^2(\mathbb{R}^d)^d \\ \varphi & \longmapsto \varphi + \left(\nabla(-\Delta)^{-1}\nabla\cdot\right)\varphi \end{cases}$$

It is also evident, from the linearity of the classical differential operators above, that it is a linear operator. Recall that the Fourier transform is an isomorphism on $L^2(\mathbb{R}^d)$ and recall that the Schwartz space lies in $L^2(\mathbb{R}^d)$. We have the following for any $k \in \{1, ..., d\}$, where the hat denotes the Fourier transform and where we have used the relation between Fourier transformation and differentiation (at $\xi \to 0$, the finite limit of the second term is taken).

$$\widehat{(\mathbb{P}\varphi)_k}(\xi) = \widehat{\varphi_k}(\xi) - \frac{\xi_k \sum_{j=1}^d \xi_j \widehat{\varphi_j}(\xi)}{|\xi|^2}$$

We choose to represent this action with a multiplier $m(\xi) \in \mathbb{R}^{d\times d}$.

$$\widehat{(\mathbb{P}\varphi)}(\xi) = m(\xi)\,\widehat{\varphi}(\xi), \quad m(\xi) := I - \frac{\xi\xi^T}{|\xi|^2}$$

Observe that $\frac{\xi\xi^T}{|\xi|^2}$ is symmetric and idempotent, so its spectrum lies in $\{0, 1\}$ and it trivially follows that $m(\xi)$ has its spectrum in $\{0, 1\}$ so its operator norm is bounded by 1 for any $\xi \in \mathbb{R}^d$. Consequently, we immediately get the following (where one can refer to A.5 to understand the norm on $L^2(\mathbb{R}^d)^d$).

$$||\widehat{(\mathbb{P}\varphi)}||_{L^2}^2 \leq ||\widehat{\varphi}||_{L^2}^2$$

We conclude that $\mathbb{P}$ is bounded for the L^2 norm on the Schwartz space by recalling that the Fourier transform is an isometry on $L^2(\mathbb{R}^d)$. Furthermore, since the closure of the product of finitely many Banach spaces is the same as the product of their closures, we get that $\mathcal{S}(\mathbb{R}^d)^d$ is dense in $L^2(\mathbb{R}^d)^d$, so we can uniquely extend the Leray projector to a continuous linear operator on $L^2(\mathbb{R}^d)^d$ [3]. The same can be done for the definition of $\mathbb{P}$ as an $H^s \to H^s$ operator.

A.7 The Standard Families of Mollifiers

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be of class C^∞ be nonnegative. Furthermore, assume it is radial and radially non-increasing; that is, there exists some $\psi : \mathbb{R} \to \mathbb{R}$ that is non-increasing such that $\phi(x) = \psi(|x|)$ for any $x \in \mathbb{R}^d$. Furthermore, suppose it has finite moments and unit mass. Define, for any $\varepsilon > 0$, the following map.

$$\phi_\varepsilon : x \in \mathbb{R}^d \mapsto \frac{1}{\varepsilon^d}\phi\left(\frac{x}{\varepsilon}\right)$$

We call $\{\phi_\varepsilon\}_{\varepsilon>0}$ a standard family of mollifiers. Intuitively, this family can be thought of as maps that have unit mass and that gradually get more and more concentrated at the origin. We typically take ϕ to be compactly supported to enjoy the decay properties and that Schwartz functions are dense in Lebesgue and Sobolev spaces. We define the associated mollification operator as follows.

$$J_{\phi,\varepsilon} : f \in L^1_{\text{loc}}(\mathbb{R}^d) \mapsto (\phi_\varepsilon * f)$$

Observe that, on $L^2(\mathbb{R}^d)^d$, the mollification operator behaves as follows in Fourier space.

$$\widehat{J_{\phi,\varepsilon} f}(\xi) = \widehat{\phi}(\varepsilon\xi)\widehat{f}(\xi)$$

By component-wise distribution of the operator, we can naturally define the mollification operator on $L^1_{\text{loc}}(\mathbb{R}^d)^d$ vector fields. Furthermore, by integration by parts, we have that $J_{\phi,\varepsilon} f$ is infinitely many times differentiable for any $f \in L^2(\mathbb{R}^d)$. The following results highlight the decay of mollified functions.

Proposition A.8. *For any $J_{\phi,\varepsilon} : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ is well defined and bounded, and the following holds.*

$$||J_{\phi,\varepsilon} f||_{H^s} \leq ||f||_{H^s}$$

Proof. As ϕ is nonnegative and has unit mass, we have that $|\widehat{\phi}(\eta)| \leq 1$ for any $\eta \in \mathbb{R}^d$. We conclude using the Fourier representation of the mollification. □

Proposition A.9. *For any $f \in L^2(\mathbb{R}^d)$, we have that $J_{\phi,\varepsilon} f \in H^s(\mathbb{R}^d)$.*

Proof. Observe that, by assumption, we have $\phi \in \mathcal{S}(\mathbb{R}^d)$, so its Fourier transform decays faster than any polynomial, and it is consequently evident that the following integral exists and is finite for any $\varepsilon > 0$.

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{\phi}(\varepsilon\xi)|^2 \, |\widehat{f}(\xi)|^2 \, d\xi$$

□

Lemma A.10. *Suppose $f \in H^s(\mathbb{R}^d)$. Then, the following quantities are well defined, and the following holds:*

$$\partial_i J_{\phi,\varepsilon} f = J_{\phi,\varepsilon} \partial_i f$$

Proof. The left-hand side of the equality is well-defined by Proposition A.9. That the equality holds results from the following and the fact that the Fourier transform is bijective.

$$\mathcal{F}[\partial_i J_{\phi,\varepsilon} f](\xi) = i\xi_i \widehat{J_{\phi,\varepsilon} f}(\xi) = i\xi_i \hat{\phi}(\varepsilon\xi)\hat{f}(\xi) = \mathcal{F}[J_{\phi,\varepsilon}\partial_i f](\xi)$$

□

We can refine our results in Proposition A.9.

Proposition A.11. *Let $0 \le r \le s$ and suppose $f \in H^r(\mathbb{R}^d)$. Then, we have that $J_{\phi,\varepsilon} : H^r \to H^s$ is well-defined as bounded and the following holds.*

$$||J_{\phi,\varepsilon} f||_{H^s} \lesssim_{s,\phi,d} \varepsilon^{r-s}||f||_{H^r}$$

Proof. Observe that since ϕ is Schwartz, its Fourier transform is Schwartz and we have the following for any η in $\mathbb{R}^d$ and any $\varepsilon > 0$.

$$|\hat{\phi}(\eta)| \lesssim_{r,\phi,d} |\eta|^{r-s}$$

Then, we get the following.

$$\begin{aligned}\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{\phi}(\varepsilon\xi)|^2|\hat{f}(\xi)|^2\,d\xi &\lesssim_{s,\phi,d} \int_{\mathbb{R}^d}\frac{(1+|\xi|^2)^s}{\varepsilon^{2(s-r)}\,|\xi|^{2(s-r)}}|\hat{f}(\xi)|^2\,d\xi \\ &\lesssim_{s\phi,d} \varepsilon^{-2(s-r)}||f||_{s-(s-r)} = \varepsilon^{-2(s-r)}||f||_r^2\end{aligned}$$

□

The next two results are critical in proving the compactness of the family of solutions to mollified Euler.

Lemma A.12. *Let $r \ge 0$ and suppose $f \in H^{r+1}(\mathbb{R}^d)$. Then, the following holds, where* **C** *depends on the mollifier and not ε.*

$$||J_{\phi,\varepsilon} f - f||_{H^r} \le C\varepsilon||f||_{r+1}$$

Proof. We compute.

$$\begin{aligned}||J_{\phi,\varepsilon} f - f||_{H^r}^2 &= \int_{\mathbb{R}^d}\langle\xi\rangle^{2r}|\hat{\phi}(\varepsilon\xi)-1|^2|\hat{f}(\xi)|^2\,d\xi \\ &\le \varepsilon^2\int_{\mathbb{R}^d}\langle\xi\rangle^{2(r+1)}|\hat{f}(\xi)|^2\cdot\frac{|\hat{\phi}(\varepsilon\xi)-1|^2}{|\varepsilon\xi|^2}d\xi\end{aligned}$$

Near the origin, $\hat{\phi}$ admits a Taylor expansion with constant term 1, so that $|\hat{\phi}(\varepsilon\xi)-1|^2 = \mathcal{O}(|\varepsilon\xi|^2)$ as $\varepsilon\xi \to 0$. This ensures that, in a small enough ball near the origin, $\frac{|\hat{\phi}(\varepsilon\xi)-1|^2}{|\varepsilon\xi|^2}$ is bounded. Outside of said sphere, the fraction decays very fast, so we conclude that it is bounded on $\mathbb{R}^d$. This bound depends on the mollifier only (the size of the ball centered at the origin might depend on ε but this doesn't affect the upper bound). We denote said bound C and conclude.

$$||J_{\phi,\varepsilon} f - f||_{H^r} \le C\varepsilon||f||_{H^{r+1}}$$

□

Corollary A.12.1. *Let $\varepsilon > 0$ and take $0 < \delta \le \varepsilon$. Let $r \ge 0$ and let $f \in H^{r+1}(\mathbb{R}^d)$. We have the following.*

$$||(J_{\phi,\varepsilon} - J_{\phi,\delta})f||_{H^r} \lesssim \varepsilon||f||_{H^{r+1}}$$

Proof. Using the linearity of the mollification and triangle inequality, we have the following.

$$||(J_{\phi,\varepsilon} - J_{\phi,\delta})f - f||_{H^r}||_{H^r} \le ||J_{\phi,\varepsilon} f - f||_{H^r} + ||J_{\phi,\delta} f - f||_{H^r}$$

Using the previous lemma, we have the following.

$$||(J_{\phi,\varepsilon} - J_{\phi,\delta})f - f||_{H^r}||_{H^r} \leq C||f||_{H^{r+1}}(\varepsilon + \delta) \leq 2\varepsilon C||f||_{H^{r+1}}$$

□

A.8 Weak Compactness

This section is dedicated to introducing reminders and results regarding the dual of Banach spaces and the topologies they induce, the latter of which bring about the notions of weak and weak-* convergence.

Definition A.13. Let $(X, ||\cdot||_X)$ be a normed $\mathbb{K}$-vector space, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then, the dual of X is denoted X^* and is the set of bounded linear functionals.

$$X^* := \mathcal{L}(X, \mathbb{K})$$

It is naturally normed with the operator norm $||\cdot||_{X^*} := ||\cdot||_{\mathcal{L}(X,\mathbb{K})}$.

Using the topological dual, we can define a coarser topology on X.

Definition A.14. Recall that X^* is defined by $||\cdot||_X$ and the topology this norm induces. We call the weak topology on X the initial topology on X defined by X^*. It is the topology generated by the following family (where generated means created by taking the finite intersections and then the arbitrary unions).

$$\{f^{-1}(U) : U \text{ open in } \mathbb{K}, f \in X^*\}$$

Consequently, we say a sequence $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to $x \in X$ if it converges for the weak topology. Equivalently, it is weakly convergent to x if $Lx_n \to x$ as $n \to +\infty$ for every $L \in X^*$.

Definition A.15. Consider the continuous embedding given by the evaluation map defined below.

$$\text{ev} : \begin{cases} X & \longrightarrow X^{**} \\ x & \longmapsto \text{ev}_x : \begin{cases} X^* & \to \mathbb{K} \\ l & \mapsto lx \end{cases} \end{cases}$$

We call the weak-* topology on X^* the initial topology defined by $\text{ev}(X)$. We thus naturally define weak-* convergence in X^*. It can be shown that the weak-* topology is coarser than the weak topology on X^*.

We recall two more definitions before introducing the critical result of this section.

Definition A.16. We say a topological space $(X, \mathcal{O})$ is separable if there exists a dense, countable subset of X.

Definition A.17. We say a Banach space X is reflexive if it is isomorphic to X^{**}.

Remark. It is evident that convergence in a topology implies convergence in any coarser topology thanks to the availability of all required open sets in the finer one. The same logic can be applied to the compactness of topological spaces.

We admit that L^p is reflexive for every $1 < p < \infty$. We also admit that L^∞ is the dual of L^1, which allows us to define the weak-* topology on L^∞. An important result of weak-* convergence is that it can preserve uniform bounds.

Lemma A.18. *Let X be a separable Banach space. Suppose $(x_n)_{n\in\mathbb{N}}$ be a sequence of X that converges weakly-* to $x \in X$. Then, the following holds.*

$$||x||_X \leq \liminf_{n\to+\infty} ||x_n||_X$$

The same holds for weak convergence if X is reflexive.

The following result is important to ensure the uniqueness of weak-* limits, a critical result in showing the compactness of the solutions to mollified Euler.

Proposition A.19. *Let $(X, ||\cdot||)$ be a normed vector space and denote $\mathcal{O}$ the weak-* topology on X^*. Then, $(X^*, \mathcal{O})$ is Hausdorff.*

Proof. Let $f, g \in X^*$ be such that $f \neq g$. Then, necessarily, there exists $x \in X$ such that $f(x) \neq g(x)$ in $\mathbb{K}$ (the field over which the vector space is taken - either the real or complex numbers). By definition, we have the following.

$$\mathrm{ev}_x(f) \neq \mathrm{ev}_x(g)$$

Since $\mathbb{K}$ is Hausdorff, we can take V, W disjoint neighborhoods of, respectively, $\mathrm{ev}_x(f)$ and $\mathrm{ev}_x(g)$. By definition of the weak-* topology, ev_x is continuous and so $\mathrm{ev}_x^{-1}(V)$ and $\mathrm{ev}_x^{-1}(W)$ are neighborhoods of f and g. That they are disjoint results from the fact that the preimage and intersection commute. □

We present the powerful Banach-Alaoglu theorem and a specific case in which it writes in a simpler form.

Theorem A.20. *(Banach-Alaoglu) Let X be a normed vector space (NVS). Then, the closed unit ball*

$$B^* = \{f \in X^* : \|f\|_{X^*} \leq 1\} \subset X^*$$

is compact in the weak- topology. Moreover, if X is a separable NVS, then B^* is sequentially compact.*

Lemma A.21. *Let $-\infty < T_1 < T_2 < +\infty$, and suppose X is a Banach space that is reflexive and separable. Let $(x_n)_{n\in\mathbb{N}} \in L^\infty([-T_1, T_2]; X)^{\mathbb{N}}$ be a bounded sequence - that is, we have $||x_n||_{L^\infty_{t;X}}$. Then, $(x_n)_{n\in\mathbb{N}}$ admits a weakly-* convergent subsequence that converges in $L^\infty([-T_1, T_2]; X)$.*

An important consequence of this theorem is weak subsequential convergence of sequences that are uniformly bounded in H^s (or other suitable spaces).

Theorem A.22. *Let X be a reflexive, separable Banach spacee. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of X. Then, there exists $(y_n)_{n\in\mathbb{N}}$ a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges weakly to some $y \in X$.*

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