

CS310: Homework 1

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Exercise (1). Solve the sum of a geometric series: $\sum_{i=1}^{\infty} (2/5)^i$

Definition. Equation to solve for a infinite geometric series given a special case $|r| < 1$:

$$\sum_{i=1}^{\infty} ar^n = \frac{a_0}{1-r} \quad (a_0 \text{ is first term})$$

Solution (1). $\sum_{i=1}^{\infty} (2/5)^i = \frac{2}{3}$

Work. Solving for sum using $s = \frac{a}{1-r}$

$$\frac{2}{5} < 1 \quad (\text{Convergence Test}) \quad (1)$$

$$s = \frac{a}{1-r} \quad (2)$$

$$= \frac{.4}{1-.4} \quad (3)$$

$$= \frac{.4}{.6} \quad (4)$$

$$= \frac{2}{3} \quad (5)$$

□

Exercise (2a). How many binary digits are there in 2^{50} and 10^{50} ? How are the two numbers related? Hint: This is a question about logarithm.

Definition. 1.1 Base Conversion formula

$$\log_a x = \frac{\log_b x}{\log_b a} \quad (\text{convert from base a to base b})$$

Solution (2a). There are 50 binary digits in 2^{50} and about 166 binary digits in 10^{50} . We can see this relationship expressed in the base conversion formula, when base 10 is converted into base 2. This same formula could be applied to find how many hex or octal digits there are in these numbers.

Work. Utilizing base conversion formula

$$\begin{aligned} \log_2 10^{50} &= \frac{\log_{10} 10^{50}}{\log_{10} 2} && (\text{from def. 1.1}) \\ &= \frac{50}{.301} \\ &= 166.096 \\ &= 166 && (\text{round down}) \end{aligned}$$

□

Exercise (2b). Show that $\log_a x = c * \log_b x$ for some constant c expressed only in terms of constants a and b .

Solution (2b). We can see that $c = \frac{1}{\log_b a}$ from the formula for base conversion (Definition 1.1)

Work.

$$\begin{aligned}\log_a x &= \frac{\log_b x}{\log_b a} && \text{(from definition 1.1)} \\ \log_a x &= c * \log_b x \\ c &= \frac{1}{\log_b a} \\ \log_a x &= \frac{\log_b x}{\log_b a}\end{aligned}$$

□

Exercise (3a). Problem 5.19 of the textbook.

Solution (3a). The order of each algorithm from slowest growth to largest growth is $\frac{2}{n}, 37, \sqrt{n}, n \log \log n, n \log n, n \log n^2, n \log^2 n, n^{1.5}, n^2, n^2 \log n, n^3, 2^{n/2}, 2^n$. Two algorithms grow at the same rate, $n \log n$ and $n \log n^2$. We can see that as the Limit of n goes to infinity, for $\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)}$, if the limit equals 0, then $f(x)$ grows slower than $g(x)$. if the limit is equal to some constant c , then the growth rate is the same.

1. $\frac{2}{n}$, As n gets larger, runtime approaches zero. $\lim_{n \rightarrow \infty} \frac{2}{37} = 0$
2. 37, Constant time. $\lim_{n \rightarrow \infty} \frac{37}{\sqrt{n}} = 0$
3. \sqrt{n} , Grows slower than n , but faster than constant time. $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n \log \log n} = 0$
4. $n \log \log n$, Grows slower than $n \log n$ because it takes the log twice. $\lim_{n \rightarrow \infty} \frac{n \log \log n}{n \log n} = 0$
5. $n \log n$, Same growth rate as $n \log n^2$. $\lim_{n \rightarrow \infty} \frac{n \log n}{n \log n^2} = .5$
6. $n \log n^2$, Same growth rate as $n \log n$. $\lim_{n \rightarrow \infty} \frac{n \log n^2}{n \log n} = 2$
7. $n \log^2 n$, Doubles log value, grows faster than only doubling inner value. $\lim_{n \rightarrow \infty} \frac{n \log^2 n}{n^{1.5}} = 0$
8. $n^{1.5}$, Faster than linear time, but slower growth than quadratic. $\lim_{n \rightarrow \infty} \frac{n^{1.5}}{n^2} = 0$
9. n^2 , Quadratic time. $\lim_{n \rightarrow \infty} \frac{n^2}{n^2 \log n} = 0$
10. $n^2 \log n$, Greater than quadratic, slower than cubic time. $\lim_{n \rightarrow \infty} \frac{n^2 \log n}{n^3} = 0$
11. n^3 , Cubic time. $\lim_{n \rightarrow \infty} \frac{n^3}{2^{n/2}} = 0$
12. $2^{n/2}$, exponential time, but has a lower exponent than 2^n . $\lim_{n \rightarrow \infty} \frac{2^{n/2}}{2^n} = 0$
13. 2^n , exponential.

Exercise (3b). Rank the following functions: $\log n, \log n^2, \log \log n$, and $\log^2 n$. Explain reasons for your ranking.

Solution (3b). The ranking from slowest to greatest growth is $\log \log n, \log n, \log n^2$, and $\log^2 n$. The growth rates of $\log n$ and $\log n^2$ are the same.

1. $\log \log n$, Grows slower than $n \log n$ because it takes the log twice. $\lim_{n \rightarrow \infty} \frac{\log \log n}{\log n} = 0$
2. $\log n$, Same growth rate as $\log n^2$ $\lim_{n \rightarrow \infty} \frac{n \log n^2}{n \log n} = 2$
3. $\log n^2$, Same growth rate as $\log n^2$ $\lim_{n \rightarrow \infty} \frac{n \log n}{n \log n^2} = .5$
4. $\log^2 n$, grows faster than $\log n^2$ because it doubles the total value, not the inner value.

Exercise (4). Problem 5.26 of the textbook: Analyze the cost of an average successful search for the binary search algorithm.

Solution (4). The average successful search is done in $O(\log n)$ time.

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^{\log n+1} i 2^{i-1} && \text{(Represents the nodes at each level)} \\
& \sum_{i=1}^{\log n+1} i 2^{i-1} && \text{(expand summation)} \\
& = 1 * 2^0 + 2 * 2^1 + 3 * 2^2 + 4 * 2 + \dots + \log(n+1) * 2^{\log(n+1)-1} \\
& = 1 * 2^{\log(n+1)-1} \\
& \sum_{i=1}^{\log(n+1)} ((n+1) - 2^{i-1}) \\
& = \sum_{i=1}^{\log(n+1)} ((n+1) - \sum_{i=1}^{\log(n+1)} 2^{i-1}) \\
& = (n+1) * \log(n+1) - n \\
& = \frac{(n+1) + \log(n+1) - 1}{n} && (1/n \text{ comes from original equation}) \\
& = O(\log n)
\end{aligned}$$

Exercise (5). Use the telescoping technique to derive this equation:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Definition. 1.2

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \text{(formula for summation of i)}$$

Solution (5). Prove $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

(Original Problem)

$$\sum_{i=1}^n i^3 - (i-1)^3 = n^3 + 3n^2 + 3n$$

(expand telescoping sum)

$$\sum_{i=1}^n i^3 - (i-1)^3$$

(Insert telescoping sum for i^2)

$$= i^3 - i^3 + 3i^2 + 3i + 1$$

$$= 3i^2 + 3i + 1$$

$$3 \sum_{i=1}^n i^2 + 3 \sum_{i=1}^n i + \sum_{i=1}^n 1$$

$$3 \sum_{i=1}^n i^2 + 3 \frac{n(n+1)}{2} + n$$

(Insert definition 1.2, sum of 1 goes to n)

$$3 \sum_{i=1}^n i^2 + 3 \frac{n(n+1)}{2} + n = n^3 + 3n^2 + 3n$$

(Set equal to expanded summation)

$$\sum_{i=1}^n i^2 = \frac{1}{3} (n^3 + 3n^2 + 3n - 3 \frac{n(n+1)}{2} - n)$$

$$\sum_{i=1}^n i^2 = \frac{1}{6} n(2n^2 + 3n + 1)$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$