# CS310: Homework 1

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**Exercise (1).** Solve the sum of a geometric series: $\sum_{i=1}^{\infty}(2/5)^i$

**Definition.** Equation to solve for a infinite geometric series given a special case $|r| < 1$:

$$\sum_{i=1}^{\infty} ar^n = \frac{a_0}{1-r} \qquad (a_0 \text{ is first term})$$

**Solution (1).** $\sum_{i=1}^{\infty}(2/5)^i = \frac{2}{3}$

*Work.* Solving for sum using $s = \frac{a}{1-r}$

$$\begin{aligned}
\frac{2}{5} &< 1 && \text{(Convergence Test)} && (1)\\
s &= \frac{a}{1-r} && && (2)\\
&= \frac{.4}{1-.4} && && (3)\\
&= \frac{.4}{.6} && && (4)\\
&= \frac{2}{3} && && (5)
\end{aligned}$$

□

**Exercise (2a).** How many binary digits are there in $2^{50}$ and $10^{50}$? How are the two numbers related? Hint: This is a question about logarithm.

**Definition. 1.1** Base Conversion formula

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{(convert from base a to base b)}$$

**Solution (2a).** There are 50 binary digits in $2^{50}$ and about 166 binary digits in $10^{50}$. We can see this relationship expressed in the base conversion formula, when base 10 is converted into base 2. This same formula could be applied to find how many hex or octal digits there are in these numbers.

*Work.* Utilizing base conversion formula

$$\begin{aligned}
\log_2 10^{50} &= \frac{\log_{10} 10^{50}}{\log_{10} 2} && \text{(from def. 1.1)}\\
&= \frac{50}{.301}\\
&= 166.096\\
&= 166 && \text{(round down)}
\end{aligned}$$

$\square$

**Exercise (2b).** Show that $log_a x = c * log_b x$ for some constant c expressed only in terms of constants a and b.

**Solution (2b).** We can see that $c = \frac{1}{\log_b a}$ from the formula for base conversion (Definition 1.1)

*Work.*

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{(from defintion 1.1)}$$
$$log_a x = c * log_b x$$
$$c = \frac{1}{log_b a}$$
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$\square$

**Exercise (3a).** Problem 5.19 of the textbook.

**Solution (3a).** The order of each algorith from slowest growth to largest growth is $\frac{2}{n}, 37, \sqrt{n}, n \log \log n$, $n \log n, nlogn^2, n \log^2 n, n^{1.5}, n^2$.

**Exercise (3b).** Rank the following functions: log n, log(n2), log log n, and log2 n. Explain reasons for your ranking.

**Solution (3b).** The ranking from slowest to greatest growth is $\log \log n, \log n, \log n^2, and \log^2 n$. The growth rates of log n and log $n^2$ are the same.

**Exercise (4).** Problem 5.26 of the textbook: Analyze the cost of an average successful search for the binary search algorithm.

**Solution (4).** The average successful search is done in $O(\log n)$ time.

$$\frac{1}{n} \sum_{i=1}^{\log n+1} i2^{i-1} \qquad \text{(Represents the nodes at each level)}$$
$$\sum_{i=1}^{\log n+1} i2^{i-1} \qquad \text{(expand summation)}$$
$$= 1 * 2^0 + 2 * 2^1 + 3 * 2^2 + 4 * 2 + ... + \log(n+1) * 2^{\log(n+1)-1}$$
$$= 1 * 2^{\log(n+1)-1}$$
$$\sum_{i=1}^{\log(n+1)} ((n+1) - 2^{i-1}$$
$$= \sum_{i=1}^{\log(n+1)} ((n+1) - \sum_{i=1}^{\log(n+1)} 2^{i-1}$$
$$= (n+1) * \log(n+1) - n$$
$$= \frac{(n+1) + \log(n+1) - 1}{n} \qquad \text{(1/n comes from original equation)}$$
$$= O(\log n)$$

**Exercise (5).** Use the telescoping technique to derive this equation:
$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

**Definition. 1.2**

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{(formula for summation of i)}$$

**Solution (5).** Prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{(Original Problem)}$$

$$\sum_{i=1}^{n} i^3 - (i-1)^3 = n^3 + 3n^2 + 3n \qquad \text{(expand telescoping sum)}$$

$$\sum_{i=1}^{n} i^3 - (i-1)^3 \qquad \text{(Insert telescoping sum for } i^2\text{)}$$

$$= i^3 - i^3 + 3i^2 + 3i + 1$$

$$= 3i^2 + 3i + 1$$

$$3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n \qquad \text{(Insert definition 1.2, sum of 1 goes to n)}$$

$$3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n = n^3 + 3n^2 + 3n \qquad \text{(Set equal to expanded summation)}$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}(n^3 + 3n^2 + 3n - 3\frac{n(n+1)}{2} - n)$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(2n^2 + 3n + 1)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$