CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani Discussion 4M

1. Build-up error

What is wrong with the following "proof"?

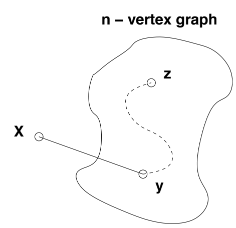
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x, y\}$ to the path from y to z. This proves the claim for $n+1$. □

Answer: The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex."* Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$- vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck —— and properly so, since the claim is false!

2. **Odd degree vertices**

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Induction on $m = |E|$ (number of edges)

(ii) Induction on $n = |V|$ (number of vertices)

(iii) Well-ordering principle

(iv) Direct proof (e.g., counting the number of edges in G)

Answer: Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(i) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G, then all vertices have degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u, v\}$ from G, so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u, v\}$ to get back the original graph G. Note that u has one more edge in G than it does in G', so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G. Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u, v\} & \text{if } u, v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u, v\} & \text{if } u, v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(ii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n+1$ vertices. Remove a vertex v and all edges adjacent to it from G. The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G. Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v, the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G', so they now have even degree in G. On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^{\complement}$ had even degree in G', and they now have odd degree in G. The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v, so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\mathrm{odd}}(G)$. Then

$$V_{\mathrm{odd}}(G) = (V_{\mathrm{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\mathrm{odd}}(G)| = |V_{\mathrm{odd}}(G')| - |A| + |B| + 1 = |V_{\mathrm{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\mathrm{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (i), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

(iii) Here we give a well-ordering proof using the number of edges m as the notion of "size" of G, so this is equivalent to the proof in part (i) using induction on m. (You can also try to give a well-ordering proof using n as the size of G.)

Suppose the contrary that the claim is false for some graphs. This means the set M is not empty, where M is the set of $m \in \mathbb{N}$ for which there exists a graph G with m edges that is a counterexample to the claim. Thus, we have a nonempty subset M of $\mathbb{N}$, so by the well-ordering principle, M has a smallest element m'. Note that $m' > 0$, since the claim is true for all graphs with 0 edges.

Let G be a graph with m' edges for which the claim is false, i.e., $|V_{\mathrm{odd}}(G)|$ is odd (here we know such a G must exist from the definition of $m' \in M$). Remove one edge from G to obtain a smaller graph G' with $m' - 1$ edges (here we need $m' \geq 1$, which we have seen above). By our choice of m' as the smallest element of M, we know that $m' - 1 \notin M$, so the claim holds for G', namely, $|V_{\mathrm{odd}}(G')|$ is even. Now add the removed edge to get back G. By the same argument as in the inductive step in part (i), this implies that $|V_{\mathrm{odd}}(G)|$ is also even, a contradiction.

(iv) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\mathrm{odd}}(G)$ and the even degree vertices $V_{\mathrm{odd}}(G)^{\complement}$, so we can write

$$\sum_{v \in V_{\mathrm{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\mathrm{odd}}(G)} d_v.$$

Both terms in the righthand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\mathrm{odd}}(G)$). So for the lefthand side $\sum_{v \in V_{\mathrm{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\mathrm{odd}}(G)|$ is even.

3. Minimum connectivity

Suppose you have n nodes, and you want to put edges between them to make the resulting graph connected. What is the minimum number of edges that you need? In this problem, we show that the answer is $n-1$. (*Note:* The case when G has $n-1$ edges is called a *tree*, which is a minimally connected graph on n vertices. Trees have many useful properties that we will explore further in tomorrow's lecture.)

Prove that if G is a connected graph on n vertices, then G has at least $n-1$ edges.

Answer: We prove the contraposition that if G has at most $n-2$ edges, then G is not connected. We do so by proving the following stronger claim. The contraposition above is the case $m = n-2$, in which case G has at least $n-(n-2) = 2$ connected components, which means G is not connected.

Claim: If G has $0 \le m \le n-1$ edges, then G has at least $n-m$ connected components.

Proof of claim: Fix $n \ge 1$. We prove the claim using induction on m. The base case $m = 0$ is true because if G has no edges, then it has n connected components. Assume the claim holds for some $0 \le m \le n-2$. Now for the inductive step, let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u, v\}$ from G. The resulting graph G' has m edges, so by inductive hypothesis it has at least $n-m$ connected components. Now add the edge $\{u, v\}$ to get back G.

Note that adding the edge $\{u, v\}$ connects at most two connected components together (namely, the component where u lies, and the component where v lies, but these may be the same component). Therefore, letting $\#\text{Connected}(G)$ denote the number of connected components of G, we conclude that

$$\#\text{Connected}(G) \;\ge\; \#\text{Connected}(G') - 1 \;\ge\; n-m-1 \;=\; n-(m+1)$$

where in the second inequality above we have applied the inductive hypothesis. This completes the proof. $\square$