1 INTRODUCTION

It is reasonable to describe turbulence in fluids as "random" or "chaotic" behavior (motion) of the fluid that cannot be described exactly. The majority of flows found in nature and in engineering applications are turbulent. Turbulence has a large influence on the transport properties of the flow and other engineering applications, and is thus a subject that has drawn a significant amount of attention. It may therefore seem surprising, that although most flows of practical engineering importance are turbulent, a rigorous definition of turbulence is difficult to formalize. It is sometimes even difficult to agree, particularly when discussing two-dimensional or low-Reynolds number flows, as to whether a particular flow is turbulent or not. Fluid turbulence can be an intimidating subject to get involved with. Discussion range from the very mathematical, to the very qualitative and conceptual - with plenty of disagreement among the "experts".

Despite this, several definitions of turbulence have been formulated. Some are:

"Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or above one another." (G.I. Taylor quoted from von Karman.[1])

"Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned. (Hinze, [2].)

Turbulence is a three-dimensional time-dependent motion in which vortex stretching causes velocity fluctuations to spread to all wavelengths between a minimum determined by the viscous forces and a maximum determined by the boundary conditions of the flow. It is the usual state of fluid motion except at small Reynolds numbers." (Bradshaw, *An Introduction to Turbulence and its Measurement."*)

The study of turbulence can lead to emotional and tense debates. These result from different schools of thought and different emphasis on what is important. Although many of the ideas which lay the foundation for much theoretical and modeling work are not new, advances in modeling and understanding are continually being made, helping to keep debate alive.

In this course we will begin by discussing some general features concerning the nature and physics of turbulent flow. We will first approach this on a very qualitative level where some of the important physical and statistical properties of turbulence will be introduced. A large part of the class will be devoted to studying and discussing models of turbulent *mixing* processes. It will be important to understand exactly what we mean by *mixing* and the distinction we make between the effects of turbulence on *momentum* transport and on *scalar mixing*. These ideas will be emphasized

throughout the class.

We will treat turbulence as a continuum phenomenon - meaning that the length and time scales of the flow field are significantly larger than the length and time scales that describe the microscopic molecular processes. As such, it is widely agreed that the Navier-Stokes equations, along with appropriate equations expressing conservation of mass and energy provide an exact description of the turbulent flow field. When boundary and initial conditions are properly specified, these equations provide an exact description of the flow. In practice, it is not possible to generate solutions to these equations as they are a set of simultaneous, nonlinear partial differential equations to which no solution has been found for the general case. These equations are, however, the starting point for most analytical and numerical approaches to treating turbulence. It is assumed for this course that you are familiar with these equations. Although exact solutions are rare, you should feel comfortable with the physical interpretation of the terms in these equations. After a review of the mathematical notation we will be using, a brief derivation and review of the equations will be provided.

These notes are a supplement to material presented in class and reading out of the assigned text. No single text book can cover the range of current research topics and applications in turbulent flows. The assigned text for this class is "Turbulent Flows," by S. B. Pope.[3]. Several texts, collections, and individual papers must be consulted to obtain an in-depth view of any particular area. Throughout the course a list of such references will be provided. These include some classical texts such as Tennekes and Lumley,[4] and Hinze.[2] Some texts focusing on modeling for applications in CFD include Wilcox[5] and Rodi[6].

1.1 Tensor Notation

To deal with the mathematics associated with turbulent flow, a reasonable working knowledge of Cartesian tensors is necessary. Although this can get to be a pretty messy topic, what you need to know in order to work the equations and relationships encountered in fluid mechanics is pretty easy. Some of the notation is summarized below.

A vector is a quantity having both magnitude and direction. One way of representing a vector or a vector operation is by the use of "Einstein" index notation. For example, the velocity vector, $\mathbf{V} = \vec{v} = (u, v, w)$ can be written as $(u_1, u_2, u_3) = u_i$. Similarly, the coordinate vector $\mathbf{x}$ or $\vec{x}$ can be written as $x_i = (x_1, x_2, x_3)$. This illustrates the first property of index notation: *if an index* (say, i) *appears only once in a term, it is a "free" index and can represent any allowed value for that index.*

A commonly encountered vector quantity is the gradient of a scalar. This is written as

$$\nabla\phi = (\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}) \tag{1.1}$$

In tensor notation this is

$$\nabla\phi = \frac{\partial\phi}{\partial x_i} \tag{1.2}$$

It is important to keep in mind the physical interpretation of these symbols. The gradient simply represents the rate of change of some scalar property (like temperature, pressure, chemical species concentration, etc.) in space. Since it is a vector, the gradient has, of course, both magnitude and direction. Large values represent rapid changes, while small values indicate fairly uniform conditions.

Another property of tensor notation is that repeated indices represent summation over that index. For example the vector dot (or scalar) product is defined as

$$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3 = a_ib_i \tag{1.3}$$

Similarly the divergence of a vector is defined as

$$\nabla\cdot\mathbf{A} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \frac{\partial a_i}{\partial x_i} \tag{1.4}$$

The divergence of a vector has a particularly significant importance in fluid dynamics. The divergence of the velocity field, $\frac{\partial u_i}{\partial x_i}$, represents the volumetric rate of change of a fixed mass of fluid. For an incompressible fluid, $\nabla\cdot\mathbf{V} = 0$.

A tensor quantity, like the strain rate would be represented by a double index:

$$S_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.5}$$

Two commonly used tensors with special significance are the Kronecker delta tensor, δ_{ij}, and the permutation tensor, ϵ_{ijk} These tensor quantities are defined by:

$$\begin{aligned} \delta_{ij} &= 1, \quad \text{if} \quad i = j, \\ &= 0, \quad \text{if} \quad i \neq j. \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} \epsilon_{ijk} &= 0, && \text{if any} \quad i,j,k \text{ are equal} \\ &= 1, && \text{if} \quad\ \ i,j,k \text{ are cyclic clockwise} \\ &= -1, && \text{if} \quad\ \ i,j,k \text{ are cyclic counterclockwise} \end{aligned} \tag{1.7}$$

As an example, $\epsilon_{11k} = 0$ for any k, $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$. The delta tensor is also referred to as the "substitution" tensor as a result of the following property: $a_i\delta_{ij} = a_j$. In general, in an expression operated on by δ_{ij}, the effect is to replace any occurance of i by j (or vice versa). This also is useful in manipulating formulae in tensor notation. Another useful property is the following relationship between the delta tensors and the permutation tensor:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{1.8}$$

This is also very useful in manipulating formula written in tensor notation.

The use of the permutation tensor makes for a compact notation for expressing the vector cross product:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_i & A_j & A_k \\ B_i & B_j & B_k \end{vmatrix} = \mathbf{i}(A_2B_3 - A_3B_2) + \dots \tag{1.9}$$

Using tensor notation and remembering repeated indices indicate summation over that index, Eq. 1.9 can be written as

$$C_i = \epsilon_{ijk} A_j B_k \tag{1.10}$$

The vorticity vector, ω_i, is defined as the curl of the velocity field. Using vector symbols and tensor notation it can be written in either of the two following forms:

$$\omega = \nabla \times \mathbf{V} \tag{1.11}$$

or

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{1.12}$$

Convince yourself that this is so.

This is basically all you will need to know. The notation throughout the class will be somewhat inconsistent in that we will switch back and forth between index notation and the use of symbolic operators (∇, $\nabla\cdot$, $\nabla\times$, etc.). This is just because we will use whatever comes easiest for any particular case. Make sure and familiarize yourself with the equivalent notations given in Eqs. (1.9, 1.10 1.11 and 1.12).

1.1.1 Symmetric and Antisymmetric Tensors

Recall that any second order tensor can be decomposed into a symmetric and antisymmetric component. For example, take the velocity gradient tensor,$\frac{\partial u_i}{\partial x_j}$. It's symmetric S_{ij} and antisymmetric R_{ij} components are:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.13}$$

$$R_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{1.14}$$

S_{ij} is the *rate of strain* tensor and R_{ij} is the rotation tensor. Both will play roles in subsequent developments.

By their nature, the tensor product of a symmetric and antisymmetric tensor is zero:

$$S_{ij} R_{ij} = 0 \tag{1.15}$$

By this definition note that

$$\frac{\partial u_i}{\partial x_j} S_{ij} = S_{ij} S_{ij} \tag{1.16}$$

(Show this)

1.2 Scale Analysis

Since exact solutions to the governing equations of fluid motion exist only in the most simplified cases, various types of approximate analysis is used to study the behavior of turbulent flow. In particular, many relationships can be derived based on order of magnitude estimates. These types of relationships will prove extremely useful in both the interpretation of turbulence phenomena and in the development of models to describe the effects of turbulence. The analysis used to generate the approximate relationships is termed scale analysis. Below we first illustrate its use by an example, then set down a few rules for its application.

A typical example of the use of scale analysis is in determining the time for a point source of a particular chemical constituent to diffuse across a certain distance. (This example and the following discussion is based on a presentation of this by Bejan.[7].

Consider a room with a linear dimension L. At one end of the room, a chemical with molecular diffusivity D is released. We wish to use scale analysis to determine the time for the chemical to diffuse across the room. The equation governing this process is the diffusion equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x_i \partial x_i} \tag{1.17}$$

The first step in the scale analysis process is to estimate the order of magnitude of the terms in Eq. 1.17. First we write

$$\frac{\partial C}{\partial t} \sim \frac{\Delta C}{t_o} \tag{1.18}$$

This expression can be interpreted as "the scale over which the concentration changes in a time t is ΔC. The notation $\sim$ can be read as "scales as" or "scales like," or "is of the same order of magnitude." Similarly, the rhs of Eq. 1.17 can be expressed as

$$D\frac{\partial^2 C}{\partial x_i \partial x_i} \sim D\frac{\Delta C}{L^2} \tag{1.19}$$

If Eq. 1.17 holds, the lhs is equal to the rhs, therefore we have:

$$\frac{\Delta C}{t_o} \sim D\frac{\Delta C}{L^2} \tag{1.20}$$

or

$$t_o \sim \frac{L^2}{D} \tag{1.21}$$

This result will compare well with exact solutions. Although we are not able to get quantitative results with this type of approximate analysis, the trends are correct - if the analysis is applied correctly. In applying this analysis, the following must be kept in mind:

1. Carefully define your spatial domain over which your analysis is to be performed. If you have spatial derivatives in an equation to which you wish to apply scale analysis, clearly state the domain over which that change occurs. On the other hand, if a time scale is given, make sure to carefully specify that. Each application will be different, so carefully specifying these parameters is important. You will see many different examples of this in these notes.

2. Consider an equation that consists of several terms:

$$A + B = C + D \tag{1.22}$$

In this case we obviously have lhs $\sim$ rhs. Now if $o(A) > o(B)$ and $o(C) > o(D)$ then $A \sim C$. In other words, the order of magnitude of a sum is the order of magnitude of the dominant term. The notation $o(A)$ reads "the order of magnitude of A" and $o(A) > o(B)$ reads "the order of magnitude of A is greater than the order of magnitude of B."

3. For a product, $A = BC$, the order of magnitude is equal to the product of the order of magnitude of the individual terms:

$$o(A) \sim o(B)o(C). \tag{1.23}$$

Similarly, for a quotient, we have $o(A) \sim o(B/C) \sim o(B)/o(C)$.

1.3 The Probability Density Function

Since turbulence is often characterized as "random" fluid motions, it is often most useful to characterize a turbulent flow by its statistics, rather than by it's detailed instantaneous structure. In a short while, we will discuss some of the many different statistical properties of turbulent flow. The purpose of this section is simply to introduce the idea of the probability density function.

When treating any function as a random variable, its value can only be specified with a certain probability. The complete statistical description of a random variable is given by its *probability distribution* at n points of space-time. The one point *probability density function*, or pdf, of a random variable ϕ provides the complete statistics of a random variable at an individual point in space. It is defined as

$$p_\phi(x)dx = \text{probability that } \phi \text{ has a value between } x \text{ and } x + dx \tag{1.24}$$

An immediate result of this definition is

$$\int_{-\infty}^{\infty} p(x)\, dx = 1 \tag{1.25}$$

since the probability of ϕ taking on some value between $-\infty$ and ∞ is 1. Since this is a one point pdf, multi-point correlations of information regarding length-scale information cannot be obtained at this level of description.

Figure 1.1 a. A hypothetical probability density distribution of the random variable ϕ. b. The cumulative distribution of ϕ.

The mean or expectation of the random variable is easily expressed in terms of the pdf:

$$\overline{\phi(x)} = \int_{-\infty}^{\infty} xp(x)dx \tag{1.26}$$

while the second central moment is

$$\text{var}(\phi) = \int_{-\infty}^{\infty} (x - \overline{\phi})^2 p(x)dx \tag{1.27}$$

A hierarchy of higher order moments and second moments can be similarly defined.

The cumulative distribution is defined as

$$P(x) = \int_{-\infty}^{x} p(x)\, dx \tag{1.28}$$

(Note that we are using the lowercase letters to denote the pdf, and upper case letters to denote the cumulative distribution.) $P(x)$ is a monotonically increasing function of x. From the definition of $p(x)$, we have $P(-\infty) = 0$ and $P(\infty) = 1$. In Fig. 1.1 the pdf and cumulative are illustrated schematically.

The cumulative distribution will prove to be useful later in the course when we wish to generate a series of random numbers that satisfy a particular pdf. Briefly, this is done as follows: Given a pdf, compute the cdf as described above. If the cdf can be expressed analytically, express the random variable ϕ as a function of the cdf. Then by choosing a random number uniformly distributed between $P(-\infty)$

and $P(\infty)$, and using it in the resulting expression for $\phi = f(P(x))$, a sequence of random numbers can be generated to give a random variable ϕ that satisfies $p_\phi(x)$. If an analytic expression for $P(x)$ cannot be obtained, the procedure to generate the random variable can be accomplished numerically.

For a more thorough discussion, see the text (Pope, pp. 37–53).

References

[1] T. von Karman. Turbulence. *Journal of the Aeronautical Sciences*, 41:1109, 1937.

[2] J. O. Hinze. *Turbulence*. McGraw-Hill, New York, 1959.

[3] S.B. Pope. *Turbulent Flows*. Cambridge University Press, Cambridge, 2000.

[4] H. Tennekes and J. L. Lumley. *A First Course in Turbulence*. Massachusetts Institute of Technology Press, Cambridge, 1972.

[5] D.C. Wilcox. *Turbulence Modeling for CFD*. DWC INdustries, Inc., La Canada, CA, 1998.

[6] W. Rodi. *Turbulence models and their application in hydraulics*. 1979.

[7] Bejan. *Convection*. Wiley.

2 GOVERNING EQUATIONS

For completeness we will take a brief moment to review the governing equations for a turbulent fluid. We will present them both in physical space coordinates and wavenumber (or Fourier) space. The equations describing the motion of a fluid are simply expressions for the conservation of mass, momentum ($F = ma$), and energy. These equations, based on the laws of Newtonian mechanics and Thermodynamics are generally accepted to provide an exact model of turbulent motion.

2.1 Conservation of Mass

Consider an arbitrary fixed volume in space. The rate at which mass accumulates in that volume is simply:

$$\text{Mass accumulation} = \iiint_v \frac{\partial \rho}{\partial t} dv \tag{2.1}$$

where v represents the arbitrary volume. This rate of accumulation must equal the net rate at which mass is flowing into this volume which can be expressed as:

$$-\iint_A \rho \mathbf{V} \cdot d\mathbf{A} \tag{2.2}$$

where A represents the surface area through which mass is flowing. The minus sign in front of the integral represents the notation that the area vector is normal to the surface element and points *outward*. $d\mathbf{A}$ is an elemental area vector, (dA_x, dA_y, dA_z). The **bold** characters indicate vector quantities. Note that we could also express $d\mathbf{A}$ as $\mathbf{n}dA$ where $\mathbf{n}$ is the unit normal (outward pointing) and dA is the magnitude of the element of area.

Equating Eq. 2.1 and 2.2 gives

$$\iiint_v \frac{\partial \rho}{\partial t} dv + \iint_A \rho \mathbf{V} \cdot d\mathbf{A} = 0 \tag{2.3}$$

Using Gauss' divergence theorem (see any text on vector analysis) the surface integral on the rhs of Eq. 2.3 can be expressed as a volume integral. The resulting equation is:

$$\iiint_v \frac{\partial \rho}{\partial t} dv + \iiint_v (\nabla \cdot \rho \mathbf{V})\, dv = 0 \tag{2.4}$$

or

$$\iiint_v \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} \right] dv = 0 \tag{2.5}$$

Eq. 2.5 is valid for any arbitrary volume, v. This can only be true if the integrand is identically zero. Thus, the equation expressing the conservation of mass (Continuity equation) is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \tag{2.6}$$

For an incompressible flow this equation reduces to:

$$\nabla \cdot \mathbf{V} = 0 \tag{2.7}$$

Note that Eq. 2.7 does not imply that the density is uniform throughout the fluid, only that it is not compressible. As an example consider a mixture of helium and oxygen at constant temperature and where the Mach number is small. Although there can be mixing between the two fluid of different density, local volume of fluid elements is conserved.

2.2 Conservation of Momentum

The conservation equation for momentum is simply a mathematical expression of Newton's Second Law, $F = ma$, that is stated here for a fixed amount of mass. When formulated for an arbitrary control volume (allowing for momentum flux across the boundaries and deformation of the boundaries), use of Leibnitz integration rule or the Reynolds Transport Theorem, results in an expression that can be stated as: *The rate of change of momentum in an arbitrary volume of fluid is equal to the net flux of momentum through the surfaces of that volume plus the net forces acting on that volume.* The forces acting will consist both of surface forces and body forces.

Let us consider first the surface forces. The surface forces result from the existence of a stress tensor. For a Newtonian fluid this stress tensor can be expressed as:

$$\sigma_{ij} = -p\delta_{ij} + \mu \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \nabla \cdot \mathbf{V} \delta_{ij} \right] \tag{2.8}$$

For an incompressible flow the stress tensor simplifies to:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu S_{ij} \tag{2.9}$$

where S_{ij} is the strain rate tensor and is given by:

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{2.10}$$

The surface force acting on an element of surface dA in the x direction is

$$df_x = \sigma_x \cdot d\mathbf{A} \tag{2.11}$$

or in Einstein index notation (because it's less ambiguous when dealing with tensors):

$$df_i = \sigma_{ij} dA_j \tag{2.12}$$

where f_i represents the force in the i^{th} direction. Repeated indices represent summation over that index.

A number of body forces can be acting on the fluid. These may include gravity forces, Coriolis force (rotation) or electrical or magnetic forces for appropriate conducting fluids. Considering only the gravity force, we have for an element of volume:

$$df_i = g_i dv \tag{2.13}$$

Putting all this together and integrating over our arbitrary volume and surface of this volume gives:

$$\iiint_v \frac{\partial \rho V_i}{\partial t} dv + \iint_A (\rho V_i) V_j dA_j = \iiint_v g_i dv + \iint_A \sigma_{ij} dA_j \tag{2.14}$$

Again applying Gauss' divergence theorem to convert the surface integrals into volume integrals and noting that the expression must hold for any control surface, we arrive at the following equation for the conservation of momentum:

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x_j} = \rho g_i + \frac{\partial \sigma ij}{\partial x_j} \tag{2.15}$$

For an incompressible fluid with constant viscosity this equation reduces to:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} + \mathbf{g} \tag{2.16}$$

For an incompressible fluid, Eqs. 2.7 and 2.16 describe the evolution of the turbulent velocity field. With appropriate initial and boundary conditions, this set of equations could, in principle, be solved for the time development of the velocity field. In practice, however, there are some obvious difficulties with this. The set of equations given by 2.7 and 2.16 are a set of coupled, nonlinear, partial differential equations, with, in the most general cases, complex initial and boundary conditions. As a result, exact solutions exist only for very simplified conditions, all being for laminar flow. To solve these equations for problems of practical combustion applications, numerical techniques must be used. As we will see later, there are also fundamental difficulties with the numerical solution. Approximate methods for the numerical solution of these equations will be discussed in subsequent lectures.

For the moment let us make some observations about turbulence based on equation Eq. 2.15 (ignore gravity for the moment). The only source of nonlinearity in this equation is the convective term. If we ignore nonlinear convection, Eq 2.15 is a linear equation that simply describes the viscous damping of the fluid. The nonlinear convective term thus provides the mechanism for feeding energy to the various length scales. Turbulence is a nonlinear process, and we must attempt to understand and describe these nonlinear interactions if we are to have any hope of dealing with turbulence.

2.2.1 Fluid Deformation

In the first section of the notes the velocity gradient tensor, $\frac{\partial u_i}{\partial x_j}$, was introduced to illustrate the decomposition of any tensor into its symmetric and antisymmetric components. Repeating that decomposition here:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = S_{ij} + R_{ij} \tag{2.17}$$

S_{ij} represents a pure straining motion and is called the *rate of strain* tensor. The second term, R_{ij}, describes a rigid body rotation. The deformation of a fluid element can therefore be separated into these two distinctly different contributions. Details on the nature of these two mechanisms of fluid deformation can be found in most graduate level fluid mechanics textbooks, e.g.,Panton[1].

2.3 Fourier Transforms

Insight into the behavior and structure of turbulent flow is facilitated by considering several different viewpoints from which to study it. One convenient space is frequency or wavenumber space. One approach is to start a theoretical study of turbulence by transforming the governing equations (Eqs. 2.6, 2.15) to their counterparts in wavenumber, or Fourier space. One thing accomplished by this transformation is that differential operators are turned into algebraic multipliers. From some perspectives, this results in a simplification in the analysis of the governing equations.

Furthermore, it is often more natural to treat many of the turbulence phenomena in terms of frequencies and wavenumbers. This includes important dynamical properties such as the kinetic energy and energy dissipation distribution (spectra). For example, the energy spectra, $E(k)$, yields the energy distribution in the flow as a function of wavenumber, where small wavenumbers represent the large eddies, and large wavenumbers represent the small eddies. The Fourier transform decomposes the velocity (or any dependent variable) field into its component waves of different wavelengths. Understanding the interaction of information in the frequency or wavenumber domain often yields useful information in understanding the physical domain.

The general definition of the Fourier Transform of a function, $f(t)$ is

$$g(\omega) = \Im\{f(t)\} \equiv \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t \tag{2.18}$$

and the inverse transform is

$$f(t) = \Im^{-1}\{g(\omega)\} = 2\pi \int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}\omega \tag{2.19}$$

The above describes the transform from a time domain to a frequency space. Instead of looking at the transform from time to frequency, the Fourier Transform can be defined in terms of a transform from physical space domain to a wavenumber domain:

$$g(\mathbf{k}) = \Im\{f(\mathbf{x})\} \equiv \int_{-\infty}^{\infty} f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x} \tag{2.20}$$

and the inverse transform is

$$f(\mathbf{x}) = \Im^{-1}\{g(\mathbf{k})\} = 2\pi \int_{-\infty}^{\infty} f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k} \tag{2.21}$$

2.3.1 Discrete Fourier Transform

Consider a velocity component in a turbulent flow. This variable is in general a function of both space and time. For a cubic domain with sides of length L, with periodic boundary conditions the discrete Fourier transform (space to wavenumber transformation) is defined as:

$$\mathbf{V}(\mathbf{x}, t) = \left(\frac{2\pi}{L}\right)^3 \sum_{k_1,k_2,k_3=-\infty}^{\infty} \hat{\mathbf{V}}(\mathbf{k}, t) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) \tag{2.22}$$

The inverse transform is given by:

$$\hat{\mathbf{V}}(\mathbf{k}, t) = \sum_{x_1,x_2,x_3=-\infty}^{\infty} \mathbf{V}(\mathbf{x}, t) \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) \tag{2.23}$$

For a general flow where $L \to \infty$, the integral Fourier transform is defined as above:

$$\hat{\mathbf{V}}(\mathbf{x}, t) = \iiint \mathbf{V}(\mathbf{x}, t) \exp(-i\mathbf{k}\cdot\mathbf{x}) d\mathbf{x} \tag{2.24}$$

The derivative of Eq. 2.22 is simply obtained by differentiating the series expansion term by term:

$$\frac{\partial \mathbf{V}(\mathbf{x}, t)}{\partial x_1} = \left(\frac{2\pi}{L}\right)^3 \sum_{k_1,k_2,k_3=-\infty}^{\infty} ik_1 \hat{\mathbf{V}}(\mathbf{k}, t) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) \tag{2.25}$$

or

$$FT\left(\frac{\partial U(x)}{\partial x}\right) = ik\hat{U}(k) \tag{2.26}$$

where FT indicates the Fourier Transform.

2.3.2 Fourier Transform of Governing Equations

The transforms defined above can be applied to the governing equations to give a set of coupled algebraic equations. The transforms of the linear terms are straight forward. The treatment of the nonlinear terms is slightly complicated. To obtain the Fourier transform of the nonlinear convective term, consider a multiplication of the type $W = U_i U_j$ (we will drop the explicit t dependence in the following equations; it will be implicitly assumed):

$$\begin{aligned} W(x) &= \sum_{\|k\|\leq\infty} w(k)\exp(ikx) \\ &= U_iU_j = \left[\sum_{\|k\|\leq\infty}\hat{U}_i(k)\exp(ikx)\right]\left[\sum_{\|k\|\leq K}\hat{U}_j(k)\exp(ikx)\right] \end{aligned} \tag{2.27}$$

In Eq. 2.27 w_k will then be given by the convolution sum:

$$w_k = \sum_{p+q=k,\|k\|\leq\infty}\hat{U}_i(p)\hat{U}_j(q) \tag{2.28}$$

Now Fourier transforming Eqs. 2.7 and 2.16 then gives the following:

$$\mathbf{k}\cdot\hat{\mathbf{V}}(\mathbf{k}) = 0 \tag{2.29}$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\hat{V}_i(\mathbf{k}) = -i\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_i(\mathbf{k_2}) - ik_i\hat{p}(\mathbf{k}) \tag{2.30}$$

At this point a few things can be pointed out from Eqs. 2.29 and 2.30. First, the nonlinear terms give rise to interactions among a triad a wave numbers k, k_1, and k_2 such that $k_1 + k_2 = k$. From Eq. 2.29 it is seen that $\hat{V}(k)$ must lie in a plane perpendicular to $\mathbf{k}$ since the dot product is zero. ($\mathbf{A}\cdot\mathbf{B}$ is identically zero if the vector $\mathbf{A}$ is perpendicular to $\mathbf{B}$) The pressure gradient term $ik_ip(\mathbf{k})$ is parallel to $\mathbf{k}$ since p is a scalar. Pressure can be eliminated from Eq. 2.30 by taking the dot product of 2.30 with $\mathbf{k}$. Using the incompressibility condition 2.29, the left hand side of 2.30 will be zero (after taking the dot product). We are then left with:

$$-ik_i\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_i(\mathbf{k_2}) = ik_ik_i\hat{p}(\mathbf{k}) \tag{2.31}$$

Since $k_ik_i = k^2$ (the magnitude of the wave number), we can write the Fourier transform of the pressure as:

$$\hat{p}(\mathbf{k}) = -\frac{k_\alpha}{k^2}\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_\alpha(\mathbf{k_2}) \tag{2.32}$$

Note that we have change the repeated index i to α. This is O.K since it is just a dummy. Now using Eq. 2.32 in 2.30 gives

$$\begin{aligned} \left(\frac{\partial}{\partial t}+\nu k^2\right)V_i(\mathbf{k}) &= -i\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_i(\mathbf{k_2}) \\ &\quad +ik_i\frac{k_\alpha}{k^2}\sum_{k_1+k_2=k} k_j\hat{V}_j(\mathbf{k_1})\hat{V}_\alpha(\mathbf{k_2}) \end{aligned} \tag{2.33}$$

In most of the literature Eq. 2.33 is usually written in the more compact form:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \hat{V}_i(\mathbf{k}) = -ik_\alpha P_{ij} \sum_{k_1+k_2=k} \hat{V}_j(\mathbf{k_1}) \hat{V}_\alpha(\mathbf{k_2}) \tag{2.34}$$

where

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2} \tag{2.35}$$

The transformed equations are now a set of algebraic equations for the coefficients $\hat{V}(k)$. Starting with prescribed initial conditions, these equations can be integrated in time. As we will see, these equations cannot be solved exactly (due to computer limitations) for general high Reynolds number flow. Although the range of wavenumbers k is finite due to viscosity (the sum in Eq. 2.34 is over finite k), we will see in the next section that the magnitude of the largest wavenumber increases as the Reynolds number increases. Many recent approaches to turbulence modeling attempt to solve Eq. 2.34 exactly only for a small range of k, and model the interactions with the higher wavenumbers. This is the idea of *Large Eddy Simulation* (keep in mind that small k corresponds to large eddies). For now we leave this discussion here.

References

[1] R. L. Panton. *Incompressible Flow, 2nd Edition.* Wiley-Interscience, New York, 1996.

3 OBSERVATIONS ABOUT TURBULENT FLOW

Fluid mechanics has been recognized as a topic of engineering importance for most of recorded history. Around 200 B.C. Archimedes formulated principles of buoyancy and floatation that remain essentially unchanged today. Even earlier there were sophisticated aqueduct systems that archaeologists have now discovered all over the world. These aqueduct builders obviously had some working knowledge of fluid mechanics and also probably had some description for the phenomena we now call turbulence.

One of the first descriptions of a turbulent flow by an engineer was by Leonardo da Vinci. Among his many accomplishments in art and science were his detailed studies of fluid motion. As an astute observer of nature and strong proponent of experimental methods, Leonardo studied and recorded flow patterns in many different configurations. As a result of these studies he formulated some principles of fluid motion including the first laws of mass conservation for incompressible flows. It is interesting to note that in his early descriptions, Leonardo identified two types of "eddying" motions: one of which is ordered, and one of which is random. It is now recognized that coherent motions exist even in very high intensity turbulence. This organized structure plays an important role in turbulent mixing and has been the subject of much recent turbulence research.

As already pointed out, turbulence is usually defined in ways that express it as a random, unpredictable fluid motion. Turbulence is also a dissipative process. That is, by the action of viscosity, the fluctuations in a fluid will tend to be damped out. This means that for a flow to remain turbulent, there must be some external source of energy and some mechanisms for that energy to be fed into the flow. The mechanisms of energy transfer within a turbulent flow is a subject we will later treat in some detail.

3.1 Laminar vs. Turbulent Flow

Most fluid flows can be conveniently placed into one of two different categories: Laminar or Turbulent. The qualitative difference between the two is usually (although not always) obvious. This distinction is important as the characteristics of a flow change dramatically as a flow transitions from a laminar to a turbulent state. As already mentioned, mixing of mass, momentum, and energy occur much more rapidly in turbulent flows. Many applications related to such diverse areas as aerodynamics, sediment transport, combustion, acoustics, and the weather are all significantly affected by turbulence.

By laminar flow we are generally referring to a smooth, steady fluid motion, in which any induced perturbations are damped out due to the relatively strong viscous forces. In turbulent flows, other forces may be acting that counteract the action of

viscosity. If such forces are large enough, the equilibrium of the flow is upset and the fluid cannot respond rapidly enough to viscosity. As a result, a complex, rapidly changing flow structure may develop. The forces that upset this equilibrium can include buoyancy, inertia, or rotation to mention only a few. To illustrate this idea consider fluid flowing in a channel. The viscous and inertial forces acting on the fluid are proportional to:

$$F_v \propto \nu L \tag{3.1}$$

$$F_i \propto V L^2 \tag{3.2}$$

where ν is the fluid viscosity, and L and V are characteristic velocity and length scales. (This material can be found in any introductory fluid mechanics text book.) If the viscous forces on the fluid are large compared with others, any disturbances introduced in the flow will tend to be damped out. If, on the other hand, the inertial forces become large, the fluid will tend to break up into eddies. (**NOTE***: We will be using the term "eddy" often in this course. More than anything else, this terminology is used to express a concept rather than a distinct entity in the flow. We want to think of an eddy as an entity associated with a given length and time scale. We often refer to a coherent structure in a flow as an "eddy," and this terminology is OK as long as you keep in mind what your definition is. On the other hand, there are many flows in which the distinction of coherent motions is unclear, but which clearly exhibit a range of length and time scales. The motions associated with these length and time scales can be conceptualized as eddies.*) For greater inertial forces, the eddies will break up into even smaller eddies. This will continue until we reach a small enough length scale (eddy size) on which the viscous forces dominate. From 3.1 and 3.2 we see that the inertial forces are proportional to L^2 while the viscous forces are proportional to L. Therefore, for a small enough L, the viscous forces will eventually dominate and damp out all perturbations. Another way of looking at this is to recognize that this process results in a large distribution of eddy sizes in the flow. The largest of these eddies will be constrained by the physical size constraints on the flow (like channel diameter). The smallest eddies will be constrained by the viscous forces which act strongest at the smallest length scales. One of the difficulties associated with the prediction of turbulent flow is that the range of length scales (eddies) present can be very large. We will quantify this later.

Although I am no fan of cute poems, there is a famous little rhyme that well describes this breakdown process:

> Big whorls have little whorls,
> That feed on their velocity.
> And little whorls have lesser whorls,
> And so on to viscosity.
> (in the molecular sense).
>
> *Richardson, 1922*[1]

Again we have the qualitative description of inertial forces breaking up a flow into eddies of various sizes, with the limiting size of the smallest eddies being determined by the viscosity. Richardson was the first to qualitatively express the idea of an *energy cascade* in which the turbulent kinetic energy passes through the wavenumber spectrum before it is converted to heat by friction. Later, Kolmogorov extended this concept into his two famous hypotheses about the equilibrium range in turbulent flows. We will come back to this many times.

The ratio between the inertial and viscous forces can be cautiously considered to give a numerical estimate of the relative strengths of the two forces. The nondimensional number that results is the Reynolds number, one of the most important parameters in fluid dynamics. For turbulence to develop, the inertial forces must be much larger than the viscous forces:

$$Re = \frac{UL}{\nu} \gg 1 \tag{3.3}$$

This interpretation of the Reynolds number as a ratio of forces is neither unique or specifically useful. Therefore, this interpretation of the Reynolds number must be treated cautiously. For many flows of practical importance, the Reynolds number can be on the order of $Re \sim 10^6$. This does not mean that viscous forces are not important. It is just that the scales over which molecular effects are important are much smaller than in the definition of the length scales given above. As we will see in later sections, it is the viscous decay at the small scales that provides the necessary sink of kinetic energy that makes a statistically steady universal range possible. For high Reynolds number flows, we can assume there are scales over which molecular forces are not significant. However, in any turbulent flow the molecular viscosity is always important at some scale. Turbulence is always a dissipative process, and if there is no source of energy to sustain the turbulence, it will decay as a result of the turbulent motions. Putting this in other wording we can say that the inertial forces result in the spreading of the turbulent kinetic energy over progressively wider ranges of length scales (wave numbers). This process is only stopped when viscous damping becomes important at small length scales (large wavenumbers).

This idea can be related to other fluid mechanical situations. Consider a flow across a flat surface with a no-slip condition at the wall. The viscous forces are confined to a small region near the wall (the *boundary layer*). As the flow Reynolds number is increased, this region decreases in thickness and the velocity of the flow changes very rapidly from zero at the surface to the free-stream velocity at the outer edges of the boundary layer. Again we see the tendency of the nonlinear inertial terms to generate discontinuities in the flow at high Re, and the viscous damping effect to smooth out these discontinuities at high wavenumbers. It is notable that in general turbulent flows these processes occur within the flow.

Other Reynolds Number Interpretations

Since the Reynolds number is a dimensionless number, there are many other interpretations. To be useful, these interpretations must, of course, have physical signifi-

cance. We will see in our subsequent analysis that interpretations in terms of length and time scale ratios will be more useful. For example, consider fluid flowing in a duct of width L, with a mean velocity U. A fluid particle with a transverse velocity u' would cross the duct in a time ("inertial" time) $T_i \sim L/u'$. However, this motion will also be acted on by viscous forces, which have a time scale, $T_v \sim L^2/\nu$. In a turbulent flow, the inertial time-scale will be much less than the diffusive time-scale, giving

$$\frac{T_v}{T_{in}} = \frac{u'L}{\nu} > 1 \tag{3.4}$$

In terms of length scale interpretations, we will see that the Reynolds number can also be used to describe the range of length scales we can expect in a turbulent flow. We will explore these ideas more carefully in future lectures.

3.2 Turbulent Diffusivity

One of the most significant features of turbulence is the effect it has on the transport of mass, momentum, and energy. Turbulent flows all exhibit much higher mixing or diffusion rates than could be accomplished by molecular diffusion alone. This may or may not be a desirable feature, depending on the application. In many applications, such as combustion, where an efficient process depends on thorough mixing of fuel and oxidizer, turbulence is a desired feature. We often speak of the *turbulent diffusivity*. Of course, we must be able to describe and characterize the effects of turbulence.

To illustrate the idea of turbulent diffusivity, consider a "marked" fluid in which a region of the fluid is marked with some contaminant. Assume also that the fluid is not in relative motion with itself (in other words, we are considering a pure diffusion process). Under these conditions, the marked fluid satisfies the diffusion equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x_j x_j} \tag{3.5}$$

D is the molecular diffusivity of the fluid. Dimensionally, Eq. 3.5 can be interpreted as

$$\frac{\Delta C}{t_m} \sim D\frac{\Delta C}{\Delta L^2} \tag{3.6}$$

or

$$t_m \sim \frac{L^2}{D} \tag{3.7}$$

where ΔC is an initial concentration difference, L is a characteristic length scale of the problem, t_m is the time scale for the molecular diffusion, and $\sim$ can be read as "scales as."

Now assume the marked fluid is transported not only by molecular diffusion, but by turbulent velocity fluctuations as well. In this case, the marked fluid satisfies the convection-diffusion equation

$$\frac{\partial C}{\partial t} + \frac{\partial u_j C}{\partial x_j} = D\frac{\partial^2 C}{\partial x_j x_j} \tag{3.8}$$

Performing a scale analysis of the convection and diffusion terms gives

$$\frac{\partial u_j C}{\partial x_j} \sim \frac{u\Delta C}{L} \tag{3.9}$$

$$D\frac{\partial^2 C}{\partial x_j x_j} \sim D\frac{\Delta C}{L^2} \tag{3.10}$$

The ratio of convection to diffusion in Eq. 3.8 can be then expressed as

$$\frac{\text{convection}}{\text{diffusion}} \sim \frac{uL}{D} \tag{3.11}$$

The quantity $\frac{uL}{D}$ is the Peclet number (Pe) and as we will show, usually takes on a large value for turbulent flows. Using our rules for scale analysis, we can express Eq. 3.8 dimensionally as (assuming turbulent flows):

$$t_T \sim \frac{L}{u} \tag{3.12}$$

Here t_T is the turbulent time-scale. It is instructive to plug in some reasonable numbers and look at the ratio of these two time scales expressed in Eq. 3.7 and 3.12.

It will turn out useful to express Eq. 3.12 in terms of a *turbulent diffusivity*. Following the form of Eq. 3.7:

$$t_T \sim \frac{L^2}{D_T} \tag{3.13}$$

D_T is the turbulent diffusivity. Comparison of Eqs. 3.12 and 3.13 gives

$$D_T \sim uL \tag{3.14}$$

Eq. 3.14 is an important relationship that expresses the turbulent diffusivity of the fluid in terms of characteristic turbulent velocity and length scales.

There is another way to look at this issue, and since the idea of a turbulent diffusivity is so important to the material we will be covering and so fundamental to the description of turbulence in general, let us explore this idea from a different point of view. The classical definition of a diffusion coefficient is as a flux divided by a gradient. That is, the diffusion flux vector is driven by local gradients. And this definition is clear in applications in which molecules execute a random walk (Brownian motion) with mean free path much smaller than the distance over which the bulk properties vary. Now looking at turbulence from a macroscopic level, the

idea that fluid elements experience random motions over distances much smaller than distances over which bulk properties vary is not necessarily the case. To define a turbulent diffusion coefficient, we must therefore go back to the micro-picture and reconsider the underlying random processes.

To do this, let us refer to the analysis of one-dimensional Brownian motion. In this case, we are considering the displacement of a particle after n random steps of length l. A one dimensional analysis of the random walk problem (see Chandrasekar[2] for a detailed analysis) gives the following for the probability of having a displacement between some value x and $x + dx$ after n steps:

$$p(x, n)dx = (2\pi nl^2)^{-1/2} \exp\left[\frac{-x^2}{2nl^2}\right] dx \tag{3.15}$$

Where the number of steps taken is proportional to time, $n = Kt$.

Now consider the diffusion of a concentration field, c where $c = c_0$ at $t = 0$ and $x = 0$, and $C = 0$ for all $x \neq 0$ at $t = 0$. The concentration at any time t can be given by

$$C(x, t) = C_0 p(x, t) \tag{3.16}$$

Since the process we are describing satisfies the diffusion equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2,} \tag{3.17}$$

we can substitute Eq. 3.15 and 3.16 into 3.17. This is satisfied if $K = 2D/l^2$, giving

$$C = \frac{C_0}{2\sqrt{\pi Dt}} \exp\left[\frac{-x^2}{4Dt}\right] \tag{3.18}$$

From Eq. 3.18 it is possible to relate the diffusivity to the mean square displacement of fluid particles. Note that finding a particle between x and $x + dx$ can be obtained by multiplying the solution for C/C_0 by the element of area dx:

$$p(x, t)dx = \frac{1}{2\sqrt{\pi Dt}} \exp\left[-x^2/4Dt\right] dx \tag{3.19}$$

The mean square displacement is given by multiplying $p(x, t)dx$ by x^2 and integrating over all possible values of x:

$$\langle x^2 \rangle = \int_0^\infty x^2 2P(x, t)dx \tag{3.20}$$

Substituting Eq. 3.19 into 3.20 and integrating gives:

$$\langle x^2 \rangle = 2Dt \tag{3.21}$$

A similar analysis can be carried out for three-dimensional configurations. The analysis for a three dimensional random walk gives [2]:

$$p(r, n) = \frac{1}{(2\pi nl^2/3)^{3/2}} \exp(-3r^2/2nl^2) \tag{3.22}$$

Continuing with the analysis outlined above for the one dimensional problem yields

$$\langle x^2 \rangle = 6Dt \tag{3.23}$$

Note that there is no prefered direction or biasing of the molecular random walks. The flux arises due to the inhomogeneity of the number density of the diffusing species. The molecular diffusivity D obtained from the classical statistical mechanics/kinetic theory arguments ($\langle x^2 \rangle = 6Dt$) is exactly the bulk property (flux)/(gradient).

To relate this all to turbulence, consider a flow seeded with fluid particles with no inertia of their own so that they follow fluid path lines exactly. These particles experience random fluctuations, so as time progresses, the particles will wander from their initial positions. At some later time, t_2, assume that we can determine the particle displacements from their initial positions, and that the mean square displacement is given as $\langle x^2 \rangle$. In this case, the turbulent diffusivity can be defined from

$$\langle x^2 \rangle = 6D_T t \tag{3.24}$$

Note that the above does not imply any particular mechanism for the transport. It provides a definition of turbulent diffusivity in terms of displacements without any specification about how that displacement occurs. As mentioned above, the molecular diffusivity D obtained from the classical statistical mechanics/kinetic theory arguments ($\langle x^2 \rangle = 6Dt$) is exactly the bulk property (flux)/(gradient). The statistical mechanics definition is the more fundamental, but the (flux)/(gradient) interpretation is more accessible experimentally so is the more widely used in engineering.

In turbulence, the random walk analysis should always give the correct answer for the turbulent diffusivity, while the gradient transport ideas may or may not be applicable.

3.3 3-D Nature of Turbulence

Turbulence is rotational and an inherently three-dimensional phenomena. It is characterized by large fluctuations in vorticity. Most of the important characteristics of turbulent flow, such as vortex stretching and length scale reduction (to be discussed later) are identically zero in two dimensions. This contributes to the difficulties in describing turbulence both analytically and numerically. You will, however, find reference made to "two-dimensional turbulence." There are situations in which the "turbulent" velocity field is, for some length scales confined to two dimensions. Flow in the atmosphere is often treated as two-dimensional as it is confined to a thin layer over the surface of a planet. Although some important dynamical mechanisms are absent in two-dimensional flows, they can be highly complex and nonlinear, and often provide a reasonable approximation to study the features of interest.

Later in the course we will be spending some time discussing different types of "free-shear flows" in which a lot of the statistics are dominated by the influence of large-scale 2-D structures. These structures can play a dominant role in the *transport* of scalar material. However, the smaller-scale, three-dimensional motions come

into play when mixing at molecular scales is important (for example, in combustion problems). The distinction between scalar transport and molecular diffusion will be critical to understanding and predicting turbulent mixing processes.

3.4 Order and Randomness

The idea that turbulence consists of completely random motions must be modified to take into account the coherent motions that are now recognized to be inherent aspects of many turbulent flows. For example, turbulent boundary layers and homogeneous turbulent shear flows exhibit horseshoe, or hairpin vortices that appear to be inherent characteristics. Free shear flows like the mixing layer exhibit coherent vortex structures very clearly, again even for very high turbulence intensities. This realization that coherent structure is a part of turbulence has led to some important developments in the numerical simulations of turbulent flows. These developments include Direct Numerical Simulation (DNS) and Large Eddy Simulation (LES). These methods, and some of what they have told us about the structure and dynamics of turbulent flow will be discussed later.

The concepts of order and randomness have also led to some new analytic approaches and new interpretations in the study of turbulence. The names of these disciplines are familiar to most of us: Chaos, Bifurcation Theory, and Dynamical Systems. Most of the impact these theories have had on the study of turbulence have been in the area of hydrodynamic stability and the transition from laminar flow to turbulence. These approaches have come to the attention of mathematicians, physicists, and engineers as a result of observations that very simple (limited degrees of freedom) nonlinear systems obeying deterministic equations can exhibit very complex, unpredictable behavior. The solution of two problems obeying the same nonlinear equations with only slight differences in initial conditions, will diverge rapidly. This "sensitivity to initial conditions" is a characteristic of the nonlinear systems. Turbulence is a complex nonlinear system with many degrees of freedom. Much simpler systems that have been studied, however, display many of the characteristics of real turbulence. It is hoped that by studying these simpler systems, a better understanding, and possibly some fundamentally different approaches to treating turbulence in engineering applications, can be achieved.

Related to this is the use of fractals to describe turbulence. Here the idea is that uniform scaling laws exist that should allow us to describe a wide class of flows. Unfortunately, these approaches have not had any impact on engineering calculations of the prediction of turbulence.

References

[1] L.F. Richardson. *Weather Prediction by Numerical Process.* Cambridge University Press, Cambridge, 1922.

[2] S. Chandrasekar. Stochastic problems in physics and astronomy. *Reviews of Modern Physics*, 15, 1943.

4 STATISTICS OF THE FLOW

Owing to the complexity of turbulent flows, they are generally studied using tools of statistical analysis. For engineering design purposes there is no conceptual problem with this, as it is seldom necessary to know all the details of the turbulent velocity field. The efficiency of a device such as a heat exchanger, mixing vessel, or combustor will be measured by its average performance over time. For most engineering purposes we therefore generally neglect the details of the turbulence and work only with averages.

From the point of view of gaining physical understanding of turbulent flows, it is also often convenient to describe the flow in terms of it's statistical properties. A description of the flow in wavenumber space is also often more informative than in terms of raw data. In the following, we will define and interpret some of the more commonly used single and multi-point statistics and use these statistics to describe different aspects of the flow.

4.1 Mean Values

The simplest statistical property is the mean, or first moment. In section **??** we defined the expectation of a random variable and the higher-order single point moments by using the probability density function. In laboratory experiments or in the use of numerically obtained data, the mean value of a random variable at a particular spatial location can be obtained by averaging the long time measurement of that variable:

$$\overline{\phi} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \phi(t) dt \tag{4.1}$$

where $\overline{\phi}$ indicates the mean value of the random variable ϕ. This time average only makes sense if Eq. 4.1 is independent of t_0 and independent of T for large T (i.e., the integral converges). In such a case we would call the flow a "statistically steady," or a "stationary" process.

In a flow configuration where Eq 4.1 does not converge, either an "ensemble" or "volume" average must be used to describe the mean flow behavior. The volume average is defined as:

$$\overline{\phi} = \lim_{vol\to\infty} \frac{1}{vol} \int_{vol} \phi(\mathbf{x}) dx dy dz \tag{4.2}$$

where the integration is now performed over a volume, vol, at one instant in time. Equation 4.2 only makes sense when the statistical properties do not depend on spatial

position. The ensemble average is defined as:

$$\overline{\phi} = \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \phi(x) \tag{4.3}$$

In Eq. 4.3 the summation is over a number of samples, N, taken at the same location x, at the same time, t, for N different realizations, or experiments. For a stationary process, the averaging defined by Eq. 4.1 and 4.3 are the same. This is called the "ergodic hypothesis."

4.2 Higher Order Moments

The n^{th} order central moment, defined earlier using the pdf, can also be computed from

$$\overline{(\phi - \overline{\phi})^n} \tag{4.4}$$

Assuming, of course, that the averaging process indicated by the overline is properly defined and converges.

A hierarchy of moments can be defined which describe the statistical state of a random variable (the velocity field being one of the random variables in a turbulent flow). Mathematically, the turbulent velocity field is generally treated as a random variable with a mean and fluctuating component. Letting ϕ' represent the fluctuating component of the random variable, the random variable ϕ can be expressed as

$$\phi = \overline{\phi} + \phi' \tag{4.5}$$

or, for the velocity field

$$u_i = \overline{u_i} + u_i' \tag{4.6}$$

After the mean value, the next most important statistical property of turbulence is its second central moment, or variance.

$$\mathrm{var}(\phi) = \overline{(\phi - \overline{\phi})^2} \tag{4.7}$$

This measure of a random variable gives a quantitative measure of how large the variations from the mean value can be expected to be. In the terminology of turbulence, we often speak of the *turbulence intensity*, which is just the square root of the second central moment, or the root mean square of the velocity fluctuation,$\sqrt{u'^2}$. As it's name implies, the value of this term gives a measure of the intensity of the turbulence.

The third central moment is called the *skewness*. This gives a measure of the symmetry of the probability distribution (see below) of the random variable. For a perfectly symmetric distribution, the skewness is zero. Continuing this hierarchy, the fourth central moment, or kurtosis, gives a measure of how fast the probability distribution goes to zero. A large kurtosis would indicate that values of the random variable, far from the mean value exist with higher probability than lower kurtosis functions.

4.3 Two-Point, Time Statistics

The moments defined above are single point moments. That is, they contain only information about a random variable at a point. In a turbulent flow, it is important to have some statistical measure of spatial information about the flow. For example, to draw conclusions about length scale information, two point statistics are needed. The autocorrelation function is the correlation between velocity components at two different times, defined by $\overline{u(t)u(t+\tau)}$. The normalized correlation function, the *autocorrelation coefficient*,$\rho(\tau)$ is defined as

$$\rho(\tau) = \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2(t)}} \tag{4.8}$$

Note that $\rho(0) = 1$. Also, from Schwartz's inequality, $\rho(\tau) < 1$ for all $\tau \neq 0$.

The correlation tensor is often used to define an *integral scale* of turbulence:

$$L_v = \int_0^\infty \rho(\tau)\, d\tau \tag{4.9}$$

L_v gives an estimate of the time interval over which the velocity component u is correlated.

The spatial correlation tensor, R_{ij} gives the correlation between velocity components at two different spatial locations and has an important interpretation in turbulent flows. It is defined by

$$R_{ij}(\mathbf{r}) = \overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})} \tag{4.10}$$

To describe the various scales of spatial motion in a turbulent flow it is more instructive to work with the Fourier transform of the correlation tensor rather than the correlation tensor itself.

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^2} \iiint R_{ij} \exp(-i\mathbf{k}\cdot\mathbf{r}) d\mathbf{r} \tag{4.11}$$

$\Phi_{ij}(\mathbf{k})$ is appropriately called the *spectrum tensor* or *spectral density* as it represents the contribution of a wavenumber, k, to the value of R_{ij}. In other words, $\Phi_{ij}(\mathbf{k})$ gives wavenumber distribution of the correlation tensor. Each wavenumber k corresponds to a physical space structure with a wavelength of $2\pi/k$.By use of the inverse transform we have:

$$R_{ij}(\mathbf{r}) = \iiint \Phi_{ij} \exp(i\mathbf{k}\cdot\mathbf{r}) d\mathbf{k} \tag{4.12}$$

Of particular significance is the sum of the diagonal components, $R_{ii}(\mathbf{r})$, for $\mathbf{r} = 0$. For this case we have

$$R_{ii}(0) = u_1^2 + u_2^2 + u_3^2 \tag{4.13}$$

which is twice the kinetic energy. Setting $\mathbf{r} = 0$, we can write Eq. 4.12 as

$$\begin{aligned} \frac{1}{2}R_{ii}(0) &= \frac{1}{2}\iiint \Phi(\mathbf{k})d\mathbf{k} \\ &= \int_0^\infty \left[\frac{1}{2}\iint \Phi_{ii}(\mathbf{k})d\sigma\right] dk \\ &= \int_0^\infty E(k)dk \end{aligned} \tag{4.14}$$

where

$$E(k) = \frac{1}{2}\iint \Phi_{ii}(\mathbf{k})d\sigma \tag{4.15}$$

$E(k)dk$ represents the contribution to the kinetic energy at a wavenumber of k in a spherical shell, σ, of thickness dk and is called the three dimensional energy spectrum. Integration of $E(k)$ over all k gives the total kinetic energy. With an eddy of a particular size, l, associated with a wavenumber of certain magnitude, k, the energy spectrum, $E(k)$, can be interpreted to give the distribution of energy among the different eddy sizes. As discussed above, the contribution of a wavenumber k corresponds to a structure with a wavelength of $2\pi/k$. The energy of and eddy of size $2\pi/k$ is therefore proportional to $kE(k)$. A large portion of the theoretical work on turbulent flows (including modeling) is concerned with the description of energy in the wavenumber spectrum and the transfer of energy among the different wavenumbers and frequencies.

4.4 Homogeneous and Isotropic Turbulence

A simplification in the mathematical treatment of turbulence comes about if we consider a flow in which the statistical quantities are independent of space. A turbulent flow with this property is call "homogeneous". In a homogeneous turbulent flow, the correlation tensor given by Eq. 4.10 is independent of position, $\mathbf{x}$, but still depends on the vector $\mathbf{r}$. If, in addition, the statistics are independent of orientation, the flow is considered "isotropic". In an isotropic flow, the correlation tensor depends only on the magnitude of r, and is independent of direction.

Most of the turbulent flows we are interested in are neither completely isotropic or homogeneous. However, much of the analysis and theory of turbulence has been formulated for isotropic turbulence. This results primarily from the complexity of the nonlinear governing equations which prevents detailed analysis to be performed. In the case of homogeneous isotropic turbulence, the equations simplify significantly. As a result, these simplified flows have been studied in some detail. Fortunately, though, there are regimes within the turbulence spectrum where it becomes increasingly likely that the flow can be treated as homogeneous and isotropic. In the next section (5) we will discuss how the turbulence structure is affected by vortex stretching. As the vortex structures (eddies) are broken down to smaller scales, and as the time scales associated with these structures decrease, a loss of any preferred orientations to mean

shear is expected. At small enough length scales, it is then often assumed that the turbulence is isotropic. This is very important for many of the turbulence closures which are presently being studied. Some of the implications of this will be pointed out later.

The terminology *fully developed* turbulence is sometimes used to describe the state of the flow at length scales small enough that the information about the large scale motions or energy input mechanisms have been lost.

Turbulent flows are always unsteady and the flow at a particular point fluctuates intensely. However, the statistical behavior of a turbulent flow often does not change appreciably with time. A *statistically steady* or *stationary* turbulent flow is one who's statistics are approximately constant in time.

5 THE TRANSITION TO TURBULENCE

A study of turbulence is incomplete without some mention of fluid stability and the transition to turbulence. We have previously noted that transition occurs when the fluid equilibrium is upset. Although not a topic that will be explored in this class, here is a brief overview of some ideas and methods used in studying fluid stability.

The study of hydrodynamics stability is a complete area of study and specialization in itself. The main objectives of this branch of fluid mechanics are to understand the conditions under which a given laminar flow becomes unstable, and give some information about the subsequent development of the instability. The development of instabilities in a laminar flow is the first step towards the transition to turbulence. In most fluid mechanics applications the ability to control the transition would greatly increase engineering efficiency and performance. For example, to achieve the lowest drag around aerodynamic bodies such as aircraft or cars, it is most desirable to delay transition[1]Although in the case of airfoil design in extreme operating conditions "vortex generators" are sometimes used to promote a turbulent boundary layer on certain areas of the wing. This is because a turbulent boundary layer will not separate as easily as a laminar boundary layer in an adverse pressure gradient. There is a tradeoff here to obtain the best overall performance.. It is obviously important to know when and where transition occurs. In combustion devices, high turbulence levels promote the mixing of fuel and oxidizer. In supersonic combustion devices, there are difficulties encountered in obtaining well mixed reactants, as a result of the stability of these flows. A better understanding of the transition process is necessary to achieve better combustion performance.

Numerous excellent text books are completely devoted to hydrodynamic stability. See for example, Drazin and Reed[1], Chandrsekar[2], or Lin [3], to mention only a few. Here we will briefly touch upon some of the analytic methods, and the type of information that these analyses yield.

From a historical perspective, the most well known experiment on hydrodynamic stability was carried out by Osborne Reynolds in 1888. He performed a set of experiments in which he carefully studied the behavior of flow in a pipe by varying different flow conditions. Specifically, by varying the diameter of the pipe, the velocity of the fluid, and the viscosity of the fluid, Reynolds found that there was a relationship between these variables that indicated the transition from a smooth laminar flow, to a complex turbulent flow. Namely, when the value of VD/ν (which we now know as the Reynolds number) exceeded a particular value, the perturbations began to grow, and the instantaneous flow structure became very complex. The transition depended

only on the value of the Reynolds number, and not on the values of the individual terms.

About the same time Reynolds was performing his careful experiments on the transition to turbulence in pipe flows, many other investigators were making progress in the study of hydrodynamic stability. Some of the important scientists studying this problem were Helmholtz, Rayleigh, Taylor, and Kelvin, among others. Many types of flow instabilities now carry their names.

Kelvin-Helmholtz Instability Kelvin-Helmholtz instability is the name given to instabilities that occur when two parallel fluid layers, each with a different velocity and density are in contact with each other. For certain values of the velocities and densities the interface between the two fluids will begin to oscillate, indicating the onset of instability. A special case of this is when the density of the two fluid layers is the same.

Taylor-Couette Taylor-Couette flow is the flow that occurs between two concentric cylinders when the outer cylinder is held fixed and the inner cylinder is rotating at some specified frequency. Many interesting phenomena can be observed in this type of flow. As the velocity of the inner cylinder increases, the flow becomes unstable and a new, qualitatively different steady flow arises. Toroidal vortices form down the length of the cylinder. As the velocity increases further, these toroidal vortices themselves become unstable. Eventually, a fully developed turbulent flow results. This type of fluid motion was first studied successfully by Taylor in 1923 assuming a cylinder of infinite length (thus neglecting end effects).

Rayleigh-Benard Convection Rayleigh convection refers to the fluid motion that develops when a stable fluid is heated from below. If we have a fluid of depth H with a higher bottom temperature than surface temperature, a lower density will develop at the bottom. If the temperature difference (density difference) becomes large enough an instability will develop. This particular configuration can also exhibit a sequence of transitions to other stable flow configurations before fully developed turbulence occurs. One of these quasi-steady states displays the formation of counter-rotating convection cells (Benard cells) throughout the fluid.

The Von-Karman Vortex Street Flow around a cylinder displays a very intricate vortex pattern as it becomes unstable in the form of a series of offset, counter-rotating vortex structures. This structure is called the Von Karman vortex street.

The above descriptions were given to provide a flavor of the various types of flow fields that can develop as a fluid undergoes transition to turbulence. A number of tools have been developed to study and predict the behavior of a fluid as it undergoes

transition to turbulence. Three different analytic methods you should have some familiarity with are:

1. Linear (normal mode) analysis
2. Nonlinear Analysis
3. Dynamical Systems Approach

The normal mode approach is the oldest and most applied approach in the study of hydrodynamics stability. It has proven very useful in understanding the initial development of some rather complex flows. In the remainder of this section we will illustrate how this approach is used and what type of information it gives us.

5.1 Linear Analysis

The general idea behind linear analysis is to add a small perturbation to a given, steady flow field. Since the perturbations to the steady flow are assumed small, quadratic terms in the fluctuating variables are eliminated, resulting in a linear equation. The equations for these small perturbations are solved to see if they grow or decay with time. To illustrate this procedure, we first separate the dependent variables into their steady and fluctuating components:

$$u_i = u_{0i} + u_i' \tag{5.1}$$

$$p = p_o + p' \tag{5.2}$$

where u_{0i} is the solution to the steady flow, and u_i' are the perturbations. The expressions for the velocity and pressure are then inserted into the governing equations (mass and momentum). Neglecting the nonlinear terms, we are left with the following linear equations for the velocity perturbations,

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{5.3}$$

$$\frac{\partial u_i'}{\partial t} + u_{0j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial u_{0i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u_i'}{\partial x_j \partial x_j} \tag{5.4}$$

The general approach to handling this equations is the method of "normal modes." Random disturbances to the flow field consist of a superposition of many modes. Because we have linearized the problem, each of these modes, if unstable, will grow at its own rate. The analysis then, consists of assuming a solution of the form $u_i = \sum a_i \exp(\omega_i t)$. This results in an eigenvalue problem for a_i, where ω_i are the eigenvalues. If the real part of ω_i is greater than zero, the perturbations will grow and the flow is considered unstable. The goal of this type of analysis is to determine the parameter range in which the flow becomes unstable, and to get some quantitative

Figure 5.1 Configuration for stability analysis of shear layer.

information about the instability. For example, the flow may be unstable to some perturbations, and stable for others. Also, the unstable modes will have various growth rates. This is the type of information we would like to obtain through our stability analysis.

As an example, let us consider the stability of the parallel shear layer we discussed above (Kelvin-Helmholtz instability). This flow is very unstable to any perturbations. Small disturbances can grow rapidly, leading to a complex flow structure. This particular flow has been studied extensively both in the laboratory, and numerically and analytically. It is a useful flow for analysis because such shear layers are approximated in many real flows, and under proper simplification, this is a flow that can be analyzed in detail. In Fig. 5.1 the configuration for this analysis is shown. The flow field consists of a mean velocity $-1/2\,U$ for $z > 0$, and $1/2\,U$ for $z < 0$. The position of the interface between the two surfaces is given by η. We will neglect viscosity and apply a small disturbance $\mathbf{u}$ to the basic flow. The perturbation velocities in the two streams will be identified by $\mathbf{u_1'}$ and $\mathbf{u_2'}$, where they each satisfy the incompressible Euler equations. The subscripts 1 and 2 refer to the upper and lower region of the domain shown in Fig. 5.1. In these two regions we can also assume the flow is irrotational. With this assumption the velocity can be expressed in terms of the gradient of a scalar function called the velocity potential:

$$\mathbf{u}_{1,2} = \nabla \Phi_{1,2} \tag{5.5}$$

Furthermore, the continuity equation for steady flow becomes:

$$\nabla^2 \Phi_{1,2} = 0 \tag{5.6}$$

where Φ is the velocity potential.The boundary conditions for this problem are

$$\nabla \Phi_1 \to -\frac{1}{2} U \quad \text{as} \quad z \to \infty \tag{5.7}$$

$$\nabla \Phi_2 \to \frac{1}{2} U \quad \text{as} \quad z \to -\infty \tag{5.8}$$

$$\text{Dynamic B.C.} \quad p_1 = p_2 \quad \text{at} \quad z = \eta \tag{5.9}$$

$$\text{Dynamic B.C. :} \qquad \text{Verticalvelocitiesareequalacrosssurface} \tag{5.10}$$

The kinematic condition can be expressed as

$$w_1 = \frac{\partial \Phi_1}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \Phi_1}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \Phi_1}{\partial y}\frac{\partial \eta}{\partial y}\Big|_{z=\eta} \tag{5.11}$$

$$w_2 = \frac{\partial \Phi_2}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \Phi_2}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \Phi_2}{\partial y}\frac{\partial \eta}{\partial y}\Big|_{z=\eta} \tag{5.12}$$

By integrating Euler's equations for this flow we get

$$\frac{p_1}{\rho_1} + \frac{\partial \Phi_1}{\partial t} + \frac{1}{2}\left(\nabla \Phi_1\right)^2 + gz = C_1 \tag{5.13}$$

and

$$\frac{p_2}{\rho_2} + \frac{\partial \Phi_2}{\partial t} + \frac{1}{2}\left(\nabla \Phi_2\right)^2 + gz = C_2 \tag{5.14}$$

Applying the dynamic boundary condition at $z = \eta$ gives

$$\rho_1 \left[C_1 - \frac{\partial \Phi_1}{\partial t} - \frac{1}{2}\left(\nabla \Phi_1\right)^2 - gz \right] = \rho_2 \left[\frac{\partial \Phi_2}{\partial t} - \frac{1}{2}\left(\nabla \Phi_2\right)^2 - gz \right] \Big|_{z=\eta} \tag{5.15}$$

Evaluating the constants at steady state (this relationship must hold for the mean steady flow as well as the disturbed flow) gives:

$$C_1 = \frac{\rho_2}{\rho_1}\left(C_2 - \frac{1}{2} U_2^2 \right) + \frac{1}{2} U_1^2 \tag{5.16}$$

Eqs. 5.5 - 5.15 describe the nonlinear stability of the interface. To look at the linear stability we consider a small perturbation to the mean state:

$$u_i = U + u_i' \tag{5.17}$$

or, in terms of the velocity potential,

$$\Phi_1 = U_1 x + \phi_1 \tag{5.18}$$

and

$$\Phi_2 = U_2 x + \phi_2 \tag{5.19}$$

Linearizing the kinematic boundary condition gives:

$$w_1|_\eta = \frac{\partial \Phi_1}{\partial z}|_\eta = \frac{\partial \eta}{\partial t} + \mathbf{u}_1 \cdot \nabla \eta = \frac{\partial \eta}{\partial t} \tag{5.20}$$

or

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} = \frac{\partial \phi_2}{\partial z} \tag{5.21}$$

The linearized dynamic boundary condition is (using the given condition that $U_1 = -1/2U$ and $u_2 = 1/2U$):

$$\rho_1 \left(-\frac{1}{2} U \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial t} + g\eta \right) = \rho_2 \left(\frac{1}{2} U \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial t} + g\eta \right) \tag{5.22}$$

We also assume that the flow is periodic in space. The solution for the elevation and velocity potential can then be expressed as a superposition of normal modes.

$$\eta(x, y, t) = \sum_{l,m=0}^{\infty} A_{lm}(t) \exp[\sigma t + i(lx + my)] \tag{5.23}$$

$$\phi(x, y, t) = \sum_{l,m=0}^{\infty} B_{lm}(t) \exp(\mp kz) \exp[\sigma t + i(lx + my)] \tag{5.24}$$

These solutions are then substituted into the boundary conditions. The two kinematic boundary conditions give:

$$-kB_1 = \sigma A - \frac{1}{2} UiAl \tag{5.25}$$

and

$$-kB_2 = \sigma A + \frac{1}{2} UiAl \tag{5.26}$$

The dynamics boundary condition gives:

$$\rho_1 \left[-\sigma B_1 - gA + \frac{1}{2} UilB_1 \right] = \rho_2 \left[-\sigma B_2 - gA - \frac{1}{2} UilB_2 \right] \tag{5.27}$$

Eqs. 5.25 - 5.27 provide three equations for the three unknowns, B_1, B_2, and A. These equations can be written in matrix form as:

$$[M]\,[B] = 0 \tag{5.28}$$

where

$$[B] = \begin{bmatrix} A \\ B_1 \\ B_2 \end{bmatrix} \tag{5.29}$$

and

$$[M] = \begin{bmatrix} \sigma - \frac{1}{2}Uil & k & 0 \\ \sigma + \frac{1}{2}Uil & 0 & -k \\ g(\rho_2 - \rho_1) & -\rho_1(\sigma - \frac{1}{2}iUl) & \rho_2(\sigma + \frac{1}{2}iUl) \end{bmatrix} \tag{5.30}$$

For any solution to be possible we must have $\det M = 0$:

$$\det M = \rho_1 \left(\sigma - \frac{1}{2}Uil\right)^2 k + \rho_2 \left(\sigma + \frac{1}{2}Uil\right)^2 k + k^2 g(\rho_2 - \rho_1) = 0 \tag{5.31}$$

This equation gives a dispersion relation for σ:

$$\sigma = f(l,\, m,\, U,\, \rho_1,\, \rho_2) \tag{5.32}$$

In general, σ is a complex number, $\sigma = \sigma_r + i\sigma_I$, the real part giving the growth rate, and the imaginary part giving wave propagation information. If the real part is greater than zero we will get exponential growth and the flow is unstable. For the situation above, we get two modes:

$$\frac{\sigma}{kU} = -\frac{1}{2}i\frac{l}{k}\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \pm \left[\frac{l^2}{k^2}\frac{\rho_1\rho_2}{(\rho_1 + \rho_2)^2} - \frac{g(\rho_2 - \rho_1)}{kU^2(\rho_2 + \rho_1)}\right]^{\frac{1}{2}} \tag{5.33}$$

The flow will be neutrally stable ($\sigma_r = 0$) if the RHS of Eq. 5.33 is purely imaginary. This occurs for:

$$U^2 \leq \frac{(\rho_1 + \rho_2)}{\rho_1\rho_2}\frac{k}{l^2}g(\rho_2 - \rho_1) \tag{5.34}$$

The condition for instability is that the term in brackets in Eq. 5.33 is greater than zero ($\sigma_r > 0$):

$$U^2 > \frac{(\rho_1 + \rho_2)}{\rho_1\rho_2}\frac{k}{l^2}g(\rho_2 - \rho_1) \tag{5.35}$$

Note that σ_r can never be negative due to the square root in Eq.. 5.33. For a homogeneous fluid where $\rho_1 = \rho_2$ we have $\sigma = \pm\frac{1}{2}U$ so the flow is unstable to any disturbances. If $\rho_1 > \rho_2$ the flow is also unstable, whereas for $\rho_2 > \rho_1$, neutral stability is possible. However, for large enough l (short waves), the flow will always be unstable.

The example above illustrates the use of the normal mode approach for treating parallel flows. By assuming linear velocity profiles for the mean flow, more complex

configurations can be studied by applying boundary conditions at each of the interface surfaces. For smoothly varying velocity profiles, the analysis of the differential equations is not so easy and the equations must be integrated numerically.

A more general approach for incompressible free shear flows is describe the velocity field by a stream function.

$$\frac{\partial \Psi}{\partial y} = u \tag{5.36}$$

$$\frac{\partial \Psi}{\partial x} = -v \tag{5.37}$$

where u and v are the two components of the velocity. It is next assumed that the stream function of the disturbance, ψ, can be given by:

$$\psi(x, y, t) = \phi(y) \exp[i(\alpha x - \beta t)] \tag{5.38}$$

where

$$\alpha = \frac{2\pi}{\lambda} \tag{5.39}$$

and

$$\beta = \beta_r + i\beta_I \tag{5.40}$$

Eq. 5.38 can also be written as:

$$\psi(x, y, t) = \phi(y) \exp[i\alpha(x - ct)] \tag{5.41}$$

In the above notation the sign of the imaginary part of c will then determine if the small perturbations will grow or decay with time. Inserting Eq. 5.40 for the stream function in the equation of motion (Eq. 5.4) yields the following equation for the amplitude of the stream function:

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = -\frac{i}{\alpha\, Re}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \tag{5.42}$$

α and Re are assumed known. This is then an eigenvalue problem for the amplitude ϕ with the wave numbers c being the eigenvalues. This famous equation of boundary layer theory and hydrodynamic stability is called the *Orr-Sommerfield* equation and was first derived in 1907. It can generally be solved numerically to give information about when instability will set in. Fig. 5.2 shows a graphical representation of the solution to the above equation. Of particular interest is the region where c is greater than zero. In this region all perturbations will grow in time according to linear theory. Since all wave numbers are usually present in flows of practical interest, instability will generally set in when the critical Reynolds number is exceeded. The actual shape of Fig. 5.2 will of course be different for different flow configurations (flow over a flat plate, flow in a pipe, etc.)

The linear analysis discussed above is very useful in helping us identify flow regimes where instabilities may set in. Unfortunately, it does not tell the whole picture. Linear stability cannot answer questions regarding large amplitude disturbances, nor can it describe the finite amplitude, nonlinear flow that eventually occurs in the transition to turbulence. To address these issues, other approaches must be taken.

Figure 5.2 Stability region for boundary layer.

5.2 Non-Linear Analysis

As the linear instabilities develop, they will eventually undergo nonlinear interactions as the amplitudes of the unstable modes grow. Here we will very superficially mention some aspects of the nonlinear stability analysis. In the weakly nonlinear analysis we consider a stream function for the perturbation quantities of the form:

$$A(t)\psi(y)\exp[i\alpha(x-ct)] \tag{5.43}$$

The difference between this and the nonlinear case is the inclusion of a finite amplitude disturbance, $A(t)$, that takes into account the effects on stability that larger amplitude disturbances can have. When this expression is substituted into the equations of motion, the following equation for the amplitude is obtained:

$$\frac{d|A|^2}{dt} = \alpha c_i|A|^2 - a_1|A|^4 + \text{H.O.T's} \tag{5.44}$$

Eq. 5.44 is known as the Landau equation and a_1 is the Landau constant. αc_i represents the amplifications factor of the linear analysis. Now depending on the value of a_1 the effect of a finite amplitude disturbance can either increase or decrease the growth rate of the linear modes. Eq. 5.44 can be converted to a linear equation for $|A|^{-2}$:

$$\frac{d|A|^{-2}}{dt} + \alpha c_i|A|^{-2} = a_1 \tag{5.45}$$

This equation has the solution:

$$|A|^{-2} = \frac{a_1}{2\alpha c_i} + \left(A_0^{-2} - \frac{a_1}{2\alpha c_i}\right)\exp(-2\alpha c_i t) \tag{5.46}$$

or

$$|A|^2 = \frac{A_0^2}{\frac{a_1}{2\alpha c_i}A_0^2 + \left(1 - \frac{a_1}{2\alpha c_i}A_0^2\right)\exp(-2\alpha c_i t)} \tag{5.47}$$

Consider first the case where αc_i is less than 0. In the linear case this would correspond to the case of exponential growth of the initial disturbance. However, in the weakly nonlinear analysis it can be shown that all perturbations will eventually arrive at some finite amplitude. If the initial perturbation amplitude is greater than this critical value it will decrease until this value is reached. This is the supercritical case. On the other hand, if the value of αc_i is positive, and if the perturbation amplitude is greater than a certain value, perturbations that would be expected to decay in the nonlinear analysis will grow if a_1 is less than zero. The amplitude of the initial disturbances can clearly effect the development of the flow.

5.3 Dynamical Systems

Another approach to studying hydrodynamic stability that has attracted recent attention is the application of the theory of dynamical systems. To introduce this idea it is informative to consider two different ideas concerning the transition to turbulence in hydrodynamic systems. These are the Landau-Hopf and Ruelle-Takens theories on transition. In the Landau-Hopf theory, the transition is seen as a series of bifurcations that lead to an increasingly more complex flow. As the Reynolds number approaches a critical value for the flow under consideration, a periodic flow develops. As the Reynolds number is increased further, this periodic flow itself becomes unstable, giving rise to additional periodic components of the flow. This process continues until a very complex, quasi-periodic flow develops.

In the Ruelle-Takens theory, totally chaotic motions are assumed to arise after only a few bifurcations. Beyond this point the dynamics of the flow are considered inherently chaotic. An important distinction between the quasi-periodic and chaotic flows has to do with their dependence on initial conditions. Chaotic (although deterministic) motion is very sensitive on the initial conditions. Two identical flow fields with only infinitesimal differences in their initial states can exhibit solutions that rapidly diverge from one another. In quasi-periodic flow, however, a slight change in the initial conditions would not be expected to have a great effect on the subsequent fluid motions. Spectral analysis of turbulent flows appears to support the ideas of Ruelle and Takens.

The dynamical systems approach to studying fluid stability began with the discovery that very simple, low order nonlinear equations can show remarkable properties of chaotic motions that bear a resemblance to fluid turbulence. In 1963, Lorentz[4] obtained a set of ordinary differential equations by severely truncating a set of equations

that describe thermal convection. Although having much fewer degrees of freedom than real turbulence, these equations were parameterized by the important nondimensional numbers that characterize Rayleigh convection. By varying these parameters in the truncated system of equations, the bifurcation of the solutions to other periodic solutions, and then totally chaotic motion could be studied.

Although the equations are deterministic, the development of a physical system governed by them is essentially unpredictable. This is because of their extreme sensitivity to initial conditions. Real phenomena, like the weather, also appear to have this character. This is why it is essentially impossible to predict the weather more than a few days in advance. Just pay attention to the weather reports for a while before you debate this.

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6 VORTICITY DYNAMICS

As mentioned in the introduction, turbulence is rotational and characterized by large fluctuations in vorticity. In this section we would like to identify some of the mechanisms of vorticity transport and highlight their effects on the dynamics of turbulent flows. This will lead into a more general discussion of the energy cascade and some famous hypotheses concerning the behavior of turbulence.

6.1 Vorticity Equation

The equation for the transport of vorticity can be obtained by taking the curl of the momentum equation. The result of this operation is:

$$\frac{D\omega}{Dt} = -\omega(\nabla \cdot \mathbf{V}) + \frac{(\nabla\rho \times \nabla p)}{\rho^2} + (\omega \cdot \nabla)\mathbf{V} + \frac{1}{Re}(\nabla^2\omega) \tag{6.1}$$

or, in Cartesian tensor notation

$$\frac{\partial \omega_i}{\partial t} + \frac{\partial u_j \omega_i}{\partial x_i} = -\omega_i \frac{\partial u_j}{\partial x_j} + \epsilon_{ijk}\frac{\partial \rho}{\partial x_j}\frac{\partial p}{\partial x_k}/\rho^2 + \omega_j \frac{\partial}{\partial x_j}u_i + \frac{1}{Re}\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \tag{6.2}$$

Eq. 6.1 or 6.2 describes the transport and generation of vorticity for a general three-dimensional flow. Four different physical mechanisms can be identified above that affect the vorticity transport. Below each of them is briefly discussed.

Expansion The first term on the rhs of Eq. 6.1 represents the effects of expansion on the vorticity field. In a flow that is expanding, $\nabla \cdot \mathbf{V}$ is a positive quantity. This term would then result in a decrease in the magnitude of vorticity due to the minus sign in front of this term. This term can play a major role in the vorticity dynamics of combustion or reacting flows, as the combustion heat release can result in large changes in the density of a fluid (high expansion rates). If the fluid is under compression, the magnitude of the vorticity will increase. This can be easily seen from conservation of angular momentum principles. This is like the ice-skater spiraling around on one skate. As the skater arms are brought in close to the body, the rotation rate increases, whereas as the arms are extended, the rotation rate decreases. Note that this is not a generation or destruction term in the sense of creating or destroying vorticity. It acts to redistribute existing vorticity. In this course we will be dealing only with incompressible flows so that this term will not play a role. It is however, a very important mechanism in combustion and nonreacting compressible flows.

Baroclinic Torque The second term on the rhs is called the baroclinic torque. This term results in a generation of vorticity from unequal acceleration as a result of nonaligned density and pressure gradients. If the density gradient and pressure gradient are aligned, this term is zero. Say the pressure gradient is perpendicular to the density gradient (the case in which the baroclinic torque is largest. It is easy in this case to see how vorticity in generated. The lighter density fluid will be accelerated faster than the high density fluid, resulting in a shear layer, thus the generation of vorticity. Obviously, this term is only non zero in a variable density flow. variable density flows. Again, in reacting flows and compressible (reacting or nonreacting) flows where large density gradients can occur, this term can play a significant role in the development of the flow field. It is not necessary for the fluid to be compressible. In a variable density, but incompressible fluid like the atmosphere, this mechanism often affects the dynamics.

Viscous Diffusion The last term in Eq. 6.1 simply describes the effects of viscous diffusion on the vorticity distribution. As a result of viscosity, the vorticity in a flow tends to diffuse in space. In high Reynolds number turbulent flow, viscous diffusion of vorticity will be dominated by the other mechanisms in the vorticity transport equations. This will be the case unless the length scales of the turbulence are small enough where the contributions of viscosity can be important. The effects of viscosity on the large scale vortex structures in a turbulent flow are generally small. The development of the large scale features is therefore independent of viscosity. This realization is very important for the justification of "Direct Numerical Simulations" which we will discuss towards the end of this course.

Consider a constant density flow for a moment. If the flow is also two-dimensional, the vorticity equation reduces to

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \tag{6.3}$$

Under this restriction, the vorticity simply acts as a passive scalar that follows fluid particle paths (except for the influence of viscosity, which is small for high Reynolds number flow). The vorticity vector is confined to a plane perpendicular to the flow, and no enhancement of vorticity of transport to smaller scales by vortex stretching mechanisms is possible.

Vortex Stretching The term $(\omega \cdot \nabla)\mathbf{V}$ is called the vortex stretching term and can be argued to be the most important mechanism in the turbulence dynamics. It represents the enhancement of vorticity by stretching and is the mechanism by which the turbulent energy is transferred to smaller scales. In two dimensional flow, this term is identically zero. Although two-dimensional flows can exhibit highly random character, they lack this distinctive mechanism and their

development is qualitatively different from three-dimensional flows. This is just another example of the difficulty that has been encountered in trying to give an operational definition of turbulence.

6.2 Qualitative Aspects of the Energy Cascade

The vortex stretching term is essential to the 3-D structure of turbulence and to the development and amplification of turbulence as we will see below.

In section 3 we discussed characteristic length scales associated with the kinetic energy and viscous dissipation. Most of the kinetic energy being associated with the largest structures (small wavenumbers), and the dissipation of this energy into heat occurring at the smallest scales (large wavenumbers). With this as the case, there must be a mechanism for the transfer of energy across these scales. This mechanism is "vortex stretching." Let us look at this from two points of view.

Consider first an element of fluid in a turbulent flow having a vorticity vector ω. Denote the two ends of this element as 1 and 2. Since we are considering a turbulent flow, each end of this element will be subject to random (no preferred orientation) velocity perturbations. Making an analogy from the "random walk," points one and two will tend to grow further and further apart with time . This increasing separation of the two ends of this element results in a stretching of the element (hence the term vortex stretching), and a decrease in the diameter of this element (length scale reduction). Conservation of angular momentum considerations now complete the picture.

The angular momentum is proportional to ωr^2. Decreases in r therefore imply an increase in ω if angular momentum is conserved. The effects of vortex stretching are now seen to reduce the length scales of the turbulence in the two directions perpendicular to the turbulence, while intensifying the vorticity, yielding the transfer of energy to smaller scales.

Effects of vortex stretching on the kinetic energy of the turbulence can also be addressed with this simple model. As stated above, the angular momentum is conserved and is proportional to ωr^2. The kinetic energy on the other hand, is proportional to $\omega^2 r^2$. If angular momentum is conserved as we reduce length scales, the kinetic energy must increase (at the expense of the kinetic energy that does the stretching. The source of this energy is larger scale motions (as creatively put by Richardson (section 1)). At the largest scales of the flow, there must be some external input energy mechanism, or else the total kinetic energy would eventually decay by the actions of viscosity. This energy source can be of many forms. In a boundary layer, or free-shear layer, the energy source is the mean shear (we will illustrate this by analyzing equations for the mean turbulent kinetic energy later in the course). It may also be due to heating. A heated plate will result in a decrease in density near the plate, which results in an unstable stratification and drives the flow. The energy source can take many other forms as well (magnetic, electrical, chemical, etc.).

Next, we would like to relate the qualitative discussion above to the vorticity trans-

port equation. In Cartesian tensor notation the vortex stretching term is expressed as

$$\omega_j \frac{\partial u_i}{\partial x_j} \tag{6.4}$$

If we decompose the deformation tensor that appears in Eq. into its symmetric (strain, S_{ij}) and antisymmetric (rotation, R_{ij}) components, we obtain:

$$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j S_{ij} + \omega_j R_{ij} \tag{6.5}$$

By expanding out the antisymmetric component, it can be shown to be identically zero so that

$$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j S_{ij} \tag{6.6}$$

The vortex stretching term is then seen to represent the amplification of vorticity by the local strain rate. A positive strain will result in an increase in the magnitude of vorticity, a negative strain gives a decrease. Similarly, straining motions aligned with a vortex element will result in a length scale reduction and intensification of vorticity as described above.

A Definition of Turbulence There are no general agreements on a definition of turbulence. We have already discussed that it is usually described in terms of unpredictability and randomness. An inherent aspect of turbulence, and what gives it much of its distinctive character, however, is the three-dimensional vortex stretching. With this as one of the defining characteristics, Bradshaw[1] has come up with the following definition: *Turbulence is a three-dimensional time-dependent motion in which vortex stretching causes velocity fluctuations to spread to all wavelengths between a minimum determined by the viscous forces and a maximum determined by the boundary conditions of the flow. It is the usual state of fluid motion except at low Reynolds numbers.*

In the following subsection we will apply these ideas of vortex stretching to describe the dynamics of the flow in various wavenumber regimes.

6.3 Equilibrium Range Theories

In the previous section we stated that no real flow is completely isotropic, but alluded to the idea that there are regimes where the assumption of homogeneity and isotropy may be valid. We also discussed the idea that a wide range of temporal and spatial scales characterize turbulent flows. In each of these ranges the flow has different characteristics. At the largest scales (in the range of the integral scales),

[1] In his book *An Introduction to Turbulence and Its Measurement.*

the fluctuations draw their energy directly from the mean motion. The turbulence is highly anisotropic, and its statistical behavior can vary significantly from flow to flow. The behavior of the flow at these large scales is strongly dependent on the geometry of the flow field, and the physical processes that the flow may be undergoing. At scales much smaller than this, but still larger than the dissipation lengths ($L \gg l_i \gg l$), the dynamics take on a more universal behavior independent of the large scale, anisotropic flow field. This is a result of the random, three-dimensional vortex stretching which reduces length and time scales. As we approach scales significantly smaller than the large energy containing eddies, the length and time scales are reduced to such an extent that they respond very rapidly relative to the large scale motions. As the energy cascades through the spectrum, directional preferences that the large scale motions exhibit are lost.

A way of looking at this is to visualize physically what effects the vortex stretching is producing in the flow. Assume we have a mean flow that is producing a stretching only in one direction, say z. This will intensify any motions in the x and y directions. The intensification of motion in the x direction will cause the same in the y and z directions, while the increased y motions cause higher levels of fluctuation in z and x. This process continues so that the fluctuations in x, y, and z eventually become isotropic at the small scales (if viscosity doesn't damp out the fluctuations before this occurs).

The smallest scales in the flow will exhibit the highest velocity gradients. (Small length scales and intense vorticity). As a result of the high gradients, kinetic energy is efficiently converted into heat (internal energy). This is the dissipation range. Over length scales smaller than this, no spatial variation can be maintained due to the dominating influence of viscosity which yields a hydrodynamically stable flow at these scales.

In the high wavenumber range (small length scales), the character of the turbulence can therefore be argued to be independent of the external flow conditions. Throughout this wavenumber range, the eddies obtain their energy by inertial transfer from larger eddies. There is a steady flux of energy across the spectrum, and the eddies are approximately in equilibrium with each other. This range of length scales is called the *equilibrium range*. In this range, there are only two parameters that determine the character of the turbulence, the dissipation and viscosity.

These ideas were first put forth by Kolmogorov[1], in his two famous hypotheses. These hypotheses, along with what they imply about the structure of turbulent flow, rank among the greatest contributions to turbulence theory.

Kolmogorov's 1^{st} Similarity Hypothesis: At sufficiently high Reynolds numbers there is a range of high wavenumbers where the turbulence is statistically in equilibrium and uniquely determined by the parameters ϵ and ν. This state of equilibrium is universal.

The term universal is used here to emphasize that the character of the turbulence in this range does not depend on any specific mechanisms of the mean flow. The only

parameters that play any role in the description of the turbulence are the dissipation and viscosity. From this we can derive characteristic length, time and velocity scales of the small scales. The Kolmogorov length scale is defined using dimensional analysis as:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \tag{6.7}$$

Using ν and ϵ to form a velocity scale gives:

$$v_\eta = (\nu\epsilon)^{1/4} \tag{6.8}$$

It is interesting to compute the Reynolds number based on these scales:

$$Re_\eta = \frac{\eta v_\eta}{\nu} = \nu^{-1}\left(\frac{\nu^3}{\epsilon}\right)^{1/4}(\nu\epsilon)^{1/4} = 1 \tag{6.9}$$

The value of $Re_\eta = 1$ is a constant independent of any flow parameters. This is not unexpected as at the small scales viscosity dominates and the relative Reynolds number will be very small. This Reynolds number is characteristic only of the strong viscous region (dissipation range) and is not characteristic of the turbulence throughout the equilibrium range.

For completeness, it is noted that a characteristic time scale can be defined:

$$\tau_\eta = \frac{\eta}{v_\eta} = \left(\frac{\nu}{\epsilon^2}\right)^{1/4} \tag{6.10}$$

If the Reynolds number of the flow is very high, it can be expected that there are values of l_i much larger than η, but still within the equilibrium range, where the dissipation is very small compared with the energy flux through this region. Kolmogorov, in his second hypothesis specifically addressed this regime where $L \gg l_i \gg \eta$, where L is the integral scale and η is the Kolmogorov scale. In terms of wavenumbers, this region corresponds to:

$$k_L \ll k_i \ll k_\eta \tag{6.11}$$

Kolmogorov's 2^{nd} Similarity Hypothesis: If the Reynolds number is sufficiently large, there exists a range of wavenumbers, $k_L \ll k_i \ll k_\eta$, where the turbulence is independent of ν and is unambiguously defined by the value of the dissipation, ϵ.

In this range the inertial transfer of energy is the primary parameter characterizing the turbulence. Hence, this range (given by Eq. 6.11 is called the *inertial subrange*.

For another, qualitative way, to look at the ideas that may have led to the formalization of these hypotheses, consider a flow field consisting of large eddies whose size and development is determined by the geometry and forcing conditions of the flow. Imbedded within the large eddies, are smaller eddies which, as we have qualitatively

sketched out, acquire their energy from the large eddies. But because the small eddies have time scales much smaller than the large eddies (we will show this to be a factor of $Re^{1/2}$), the small scales respond rapidly to any attempt by the mean motion and large eddies to order their structure. At small enough length scales the dynamics will then be independent of the large scale motion, as Kolmogorov has formally hypothesized.

6.3.1 Energy Spectrum in the Equilibrium Range

Because the character of the turbulence in the equilibrium range depends only on a small number of parameters, it is possible to derive relations for the form of the energy spectrum. In the equilibrium range the turbulence depends only on the wavenumber k, the dissipation ϵ, and the viscosity, ν. Dimensional analysis then gives the following form for the energy spectrum in this range.

$$E(k) = g\left(\nu^{3/4}\epsilon^{-1/4}k\right)\epsilon^{2/3}k^{-5/3} \tag{6.12}$$

Within the inertial subrange (in the equilibrium range) the effects of viscosity go to zero. In this subrange the energy spectrum becomes:

$$E(k) = C\epsilon^{2/3}k^{-5/3} \tag{6.13}$$

where $C = g(0)$. Eq. 6.13 is called the *Kolmogorov* spectrum and C the Kolmogorov constant. C is a "universal" constant who's experimentally determined value lies in the range of 1.5-2.5. A number of famous experiments have verified the functional form given by Eq. 5.12. Most notable were the measurements of turbulence spectra in Knights Inlet, (a tidal channel in British Columbia) by Grant et al.[2].

In other regions of the flow, the form of the spectrum cannot be determined from dimensional arguments alone. However, a differential equation for the energy spectrum can be derived (e.g., Hinze[3], Chapter 3). With appropriate approximation and simplification the form of the energy spectrum has been computed in the various wavenumber regimes of the flow.

A Final Note

The random energy transfer (vortex stretching) throughout the wavenumber spectrum is seen to homogenize the small scale disturbances. In any real flow then, it seems reasonable to assume that there are regions of the flow that can be treated as isotropic and homogeneous. With this justification, the study of isotropic flow is not just a mathematical necessity, but an approximation that should reasonably describe the properties of certain turbulent flows.

References

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[2] H. L. Grant, R. W. Stewart, and A. Molliet. *Journal of Fluid Mechanics*, 12:241, 1962.

[3] J. O. Hinze. *Turbulence*. McGraw-Hill, New York, 1959.

7 LENGTH AND TIME SCALES IN TURBULENT FLOWS

Turbulent motions occur over a wide range of length and time scales. For example, consider the growth of a cumulus cloud. The large scale of the cloud can be of the order of kilometers and may grow or persist over long periods of time. Within the cloud, mixing of dry "external" air and moist air within the cloud may occur over scales on the order of millimeters. In more down to earth applications consider a utility boiler. Large eddies transport fuel and oxidizer throughout the combustion chamber, while mixing and chemical reaction ultimately occur at the very small scales with very short time scales compared with the large scale motions. This variation in length and time scales is an important characteristic of turbulent flows, and a characteristic that is in part responsible for the difficulty encountered in the numerical and theoretical analysis of turbulent flows. Here we take some time to discuss some of the features associated with the length and time scales of turbulence.

We have already seen that scaling laws are an important tool for describing turbulence. Much of what we say and conclude about turbulence is based on order of magnitude estimates that follow from logical applications of scaling laws and dimensional analysis. Also identifying the appropriate scalings using scale analysis along with an understanding of how different lengths and times scale has very practical uses. Within the class lectures a lot of more particular examples and uses will be discussed.

7.1 Turbulent Length Scales

First let us consider the range of length scales (eddy sizes) that one may expect to encounter in turbulent flows. The size of the largest eddies in the flow will be given by L, the smallest eddies by η. As previously discussed, the largest eddies in the flow account for most of the transport of momentum and energy. The size of these eddies is only constrained by the physical boundaries of the flow. We will refer to L as the *integral* length scale. The size of the smallest scales of the flow will be determined by viscosity. We have already discussed the idea that as we approach smaller and smaller length scales, the effects of viscosity become more important. The smallest length scales existing in a turbulent flow are those where the kinetic energy is dissipated into heat. For very high Reynolds number flows, the viscous forces become increasingly small with respect to the inertial forces. Smaller scale motions are then necessarily generated until the effects of viscosity become important and energy is dissipated.

For a statistically steady turbulent flow, the energy dissipated at the small scales must equal the energy supplied by the large scales. From the arguments leading to

Kolmogorov's 1st similarity hypothesis, the only factors influencing the behavior of the small scale motions are the overall kinetic energy production rate (which equals the dissipation rate) and the viscosity. The dissipation rate will be independent of viscosity, *but the scales at which this energy is dissipated will depend on both the dissipation rate and viscosity.*

To arrive at an estimate for the scales at which the energy is dissipated we must then form a length scale based only on the dissipation rate and viscosity. The dissipation rate per unit mass (ϵ) has dimensions (m^2/sec^3) and viscosity, ν has dimension (m^2/sec). The length scale formed from these quantities is:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \tag{7.1}$$

This length scale is called the *Kolmogorov* length scale and is the smallest hydrodynamic scale in turbulent flows.

To relate this length scale to the largest length scales in the flow we need an estimation for the dissipation rate in terms of the large scale flow features. Since the dissipation rate is equal to the kinetic energy production rate, we need to obtain an approximation for the rate at which kinetic energy is supplied to the small scales. The kinetic energy of the flow is proportional to U^2. The time scale of the large eddies (commonly referred to as the large eddy "turnover" time) can be estimated as L/U. It is reasonable to assume that the kinetic energy supply rate will be related to the inverse of this time scale. The dissipation rate can now be estimated by the relation

$$\epsilon \sim \frac{UU}{L/U} \sim \frac{U^3}{L} \tag{7.2}$$

With this estimate for ϵ, Eq. 7.1 becomes:

$$\eta = \left(\frac{\nu^3 L}{U^3}\right)^{1/4} \tag{7.3}$$

To repeat what we said earlier, we can see from Eq. 7.2 and 7.3 that the dissipation does not depend on the viscosity. Viscosity serves only to determine at what length scale the dissipation occurs. This now immediately gives an estimate for the ratio of the largest to smallest length scales in the flow:

$$\frac{L}{\eta} \sim \left(\frac{UL}{\nu}\right)^{3/4} = Re^{3/4} \tag{7.4}$$

where Re is the Reynolds number based on the large scale flow features. As we should intuitively expect, the separation of the largest and smallest length scales increases as the Reynolds number is increased. This is a widely used relationship.

Another commonly encountered length scale in turbulence is the *Taylor* microscale. This length scale does not have the same easily understood physical significance as

the Kolmogorov or integral length scales but provides a convenient estimate for the fluctuating strain rate field. The Taylor microscale, λ is defined through the relation:

$$\left(\frac{\partial u'}{\partial x}\right)^2 = \frac{u'^2}{\lambda^2} \tag{7.5}$$

u' is the rms of the fluctuating velocity field. We will later talk about some features of turbulence where the Taylor Microscale is the appropriate length scale.

Since the Taylor microscale is related to the turbulence fluctuations, it is sometimes called the *turbulence length scale*. A turbulence Reynolds number can be computed based on the Taylor microscale and the rms velocity fluctuations:

$$Re_\lambda = \frac{u'\lambda}{\nu} \tag{7.6}$$

The Taylor microscale, λ, has a historical significance as well as it was the first length scale derived to describe the turbulence.

7.2 Time Scales

In the previous subsection we have already referred to the "large eddy turnover" time defined by

$$t_L = \frac{L}{U} \tag{7.7}$$

From the information we have already presented, we can also generate a time scale for the small eddies using the viscosity and the dissipation:

$$t_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2} \tag{7.8}$$

Using our previous estimate for the dissipation rate we obtain

$$t_\eta = \left(\frac{\nu L}{U^3}\right) \tag{7.9}$$

The ratio of time scales is therefore:

$$\frac{t_L}{t_\eta} = \left(\frac{UL}{\nu}\right)^{1/2} = Re_L^{1/2} \tag{7.10}$$

The large scale structures in the flow are seen to have a much larger time scale (duration) than the smallest energy dissipating eddies. As the Reynolds number of the flow increases, the magnitude of the separation between both time and length scales increases.

It is worth taking some time to look at the definition of the large eddy turnover time in a little more detail and how it relates to some physical processes. The eddy turnover time of a size l eddy (τ_l) can be related to the time it takes for that size eddy

to traverse the inertial range (t_l) (i.e., be reduced to the Kolmogorov scale). From dimensional analysis we get

$$\frac{dl}{dt} \sim -\frac{l}{t_l} \tag{7.11}$$

where the characteristic time scale of a size l eddy can be related to the large eddy turnover time, τ_L using Kolmogorov scalings applicable to the inertial range:

$$t_l \sim \left(\frac{l}{L}\right)^{2/3} \tau_L \tag{7.12}$$

Now substituting the value of t_l in Eq. 7.12 into Eq. 7.11 and then integrating from the integral scale to the Kolmogorov scale, η gives the time it takes for an integral scale eddy to be reduced to the Kolmogorov scale:

$$\frac{t_L}{\tau_L} = 1 - \left(\frac{\eta}{L}\right)^{2/3} = 1 - Re^{-1/2} \tag{7.13}$$

where the final equality is obtained using the relationship $L/\eta \sim Re^{3/4}$ for inertial range turbulence. Note from Eq. 7.13 that the large eddy turnover time is the time scale for an eddy to traverse the inertial range for high Re flows. Equation 7.13 illustrates the Re dependence of this process, which vanishes for high-Re flows.

In the previous subsection it was shown how the separation of length scales effects the ability to numerically predict a turbulent flow. A similar argument holds for the separation of time scales. In a numerical calculation of the time development of the velocity field, the time step must be small enough to resolve the fast time scales of the small scale motion. Implicit algorithms could be used to avoid this time stepping restriction, but time accurate solutions require the small time steps.

6.3 Length and Time Scales of the Scalar Field

A scalar in a turbulent flow field will experience stretching and straining due to the turbulent motions, as well as diffuse under the action of molecular diffusivity. We have shown previously that the effects of the turbulence on the transport of the scalar field dominate over the effects of diffusion. However, molecular diffusion plays an important role in turbulent transport, mixing and reaction. For example, mixing between two species can only occur under the action of molecular diffusivity. If the molecular diffusivity was absent, the two constituents could be well stirred and distributed by the turbulence, but they would never see each other at a molecular level. Under this condition the scalar length scale will be reduced to a molecular scale. In reality, diffusion acts over the scalar field, most effective at the smallest length scales, so that the scalar field is homogenized at length scales much larger than the molecular scales. This smallest length scale of the scalar field where this homogenization occurs is called the *Batchelor* scale if it is smaller than the Kolmogorov scale, and the Obukov-Corrsin scale if it is larger than the Kolmogorov scale. The relative size of the smallest scalar length scale to the Kolmogorov scale depends on the relative

magnitudes of the kinematic viscosity and molecular diffusivity. A major focus of this course is to study the details of the overall turbulent mixing process. A derivation of the scalar length scales and a discussion of the scalar energy spectrum will be deferred until we specifically treat the mixing process.

7.3 Implications on the Numerical Solution of Turbulent Flows

The ratio of largest to smallest length scales in the flow has just been shown to be proportional the Reynolds number raised to the three-quarters power. This result has important implications concerning the numerical solution of a turbulent flow. For example, in a flow with a Reynolds number of 10^5, the ratio L/l is proportional to $10^{15/4}$. However, turbulence is three-dimensional and to resolve the entire range of length scales in a turbulent flow simulation we would need a computational domain that consisted of at least 10^{10} grid points. Considering that this resolution would be needed for each of the dependent variables, it is easy to see that this amount of information far exceeds the capacity of any existing computer. This becomes even clearer when the unsteady, transient nature of turbulence is considered. In an accurate numerical solution, time must be discretized and the governing equations must be integrated in time using small time steps. Even if computer memory was large enough to contain all the information needed to describe the turbulent flow field in detail, the time needed to integrate the equations in both space and time would be completely unpractical. Even though 40 something years have passed since this type of argument was first made and computing power has grown by orders of magnitude, the argument is still valid today.

Since full numerical simulations of turbulent flow are impossible, what implications does this have regarding our ability to predict the behavior of turbulent flows in applications of practical importance? Since the flow field cannot be predicted exactly, the general approach is to use approximate methods and statistical methods that give us the desired information about the flow. In the absence of the needed technology to solve the exact governing equations, much of the effort in turbulence research is directed at developing *models* that can be used in conventional solution procedures. These models take on a wide variety of forms, ranging from the very empirical to some that are very complex and physically based. At some level, however, all models rely on empirical data. The goal of the models is to provide as good as description as possible of the turbulence within a framework that allows for solutions of practical problems. As we will see, there have been many successes along with many failures in this venture. The current and continued research in the turbulence modeling area indicates the difficulty in describing the turbulence. The next several sections will discuss some of the modeling approaches used to treat the effects of turbulence.

8 TURBULENT TRANSPORT

This section will provide an introduction to one of the most common approaches by which turbulence is treated and studied from a statistical viewpoint. This will serve as an introduction and background for some of the modeling approaches to be studied in the next section. We begin this discussion by applying the decomposition of the flow into its mean and fluctuating components as alluded to earlier. By applying this decomposition to the transport equations of momentum and kinetic energy we will be able to isolate some of the mechanisms by which the turbulence affects the mean flow. Mechanisms of energy exchange between the mean flow and the turbulence will be pointed out. Also, we can approximate the magnitude of terms appearing in these equations to determine which mechanisms are most important in certain flow regimes.

8.1 Reynolds Averaged Equations

To obtain an equation for the mean values of the dependent variables, we first decompose the variables into their mean and fluctuating components:

$$\begin{aligned} u_i &= \overline{u_i} + u_i' \\ p &= \bar{p} + p' \end{aligned} \tag{8.1}$$

Similarly, the stress tensor can be decomposed as

$$\sigma_{ij} = \overline{\sigma_{ij}} + \sigma_{ij}' \tag{8.2}$$

where

$$\overline{\sigma_{ij}} = \bar{p}\delta_{ij} + \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{8.3}$$

and

$$\sigma_{ij}' = p'\delta_{ij} + \mu\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{8.4}$$

The mean and fluctuating rate of strain tensors are given by:

$$\frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) = \bar{S}_{ij} \tag{8.5}$$

and

$$\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) = s_{ij} \tag{8.6}$$

(For an incompressible fluid.) From the definition of our mean values we have

$$\overline{u_i'} = \overline{p'} = \overline{\sigma_{ij}'} = 0 \tag{8.7}$$

We then substitute Eq. 8.1 - 8.4 into the Navier-Stokes equations and average the entire equation. This gives

$$\rho\frac{\partial\overline{\bar{u}_i + u_i'}}{\partial t} + \rho\frac{\partial\overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\overline{\sigma_{ij}} + \sigma_{ij}'}\right) \tag{8.8}$$

Applying Eq. 8.7 and noting that $\overline{\bar{u}_i} = \bar{u}_i$, yields the final equation for the average momentum transport:

$$\rho\frac{\partial\bar{u}_i}{\partial t} + \rho\frac{\partial(\bar{u}_i\bar{u}_j + \overline{u_i'u_j'})}{\partial x_j} = \frac{\partial}{\partial x_j}\overline{\sigma_{ij}} \tag{8.9}$$

Eq. 8.9 for the mean momentum resembles the full momentum equation except for the term $\partial/\partial x_j(\overline{u_i'u_j'})$. If u_i' and u_j' are uncorrelated, this term is zero. However, this is not generally the case. Physically, this correlation represents the mean value of the transport of fluctuating momentum by the fluctuating velocity field. From Eq. 8.9 it is clear that this transport of fluctuating momentum influences the transport of the mean momentum. This term can therefore be interpreted as a mechanism for momentum exchange between the mean flow and the turbulence. In other words, the velocity fluctuations produce an additional mean momentum flux that would not appear in laminar flow.

Momentum flux can be associated with a stress in the fluid. The turbulent momentum exchange mechanism appears in Eq. 8.9 is in the form of a divergence of a quantity we can thus associate with a stress. Turbulence can therefore be interpreted to produce additional stress in the fluid. As a result of this behavior, the correlation $\overline{u_i'u_j'}$ is called the *Reynolds stress tensor*, in honor of O. Reynolds who first applied this type of decomposition. With this interpretation for the velocity correlation, Eq. 8.9 can be rearranged and written as:

$$\rho\frac{\partial\bar{u}_i}{\partial t} + \rho\frac{\partial\bar{u}_i\bar{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\sigma_{ij}} - \overline{u_i'u_j'}\right) \tag{8.10}$$

Arranging the Reynolds stress tensor in the momentum equation in this manner also has bearing on how we will model the effects of turbulence a little later. The mean stress tensor in the turbulent flow can then be expressed as:

$$T_{ij} = \overline{\sigma_{ij}} - \overline{u_i'u_j'} = -\bar{p}\delta_{ij} + 2\nu S_{ij} - \overline{u_i'u_j'} \tag{8.11}$$

8.2 An Estimate For the Reynolds Stress

As you should now be becoming familiar with, dimensional analysis and scale analysis play an important role in the analysis of turbulent flows. In many cases, order of magnitude estimates are the only way of providing information about the turbulence

terms, and play a major role in the formulation of turbulence models (to be discussed in soon). Let us assume the velocity fluctuations are characterized by $\sqrt{u'^2} \equiv u_t$. The integral scale of the flow (characterizing the large eddies) is given by L. With these scale relations, the mean stress tensor can be estimated as

$$\nu \bar{S}_{ij} \sim \nu \frac{u_t}{L} \tag{8.12}$$

and the Reynolds stress tensor estimated as u_t^2. The ratio of these two is

$$\frac{u_i' u_j'}{\nu \bar{S}_{ij}} \sim \frac{u_t^2}{\nu u_t / L} = \frac{u_t L}{\nu} = Re \tag{8.13}$$

The ratio of these two terms is proportional to the Reynolds number. Therefore, for high Reynolds number flow, we have $u_i' u_j' \gg \overline{\sigma_{ij}}$. This implies that viscous stresses have a lower order effect on the mean flow than the induced turbulent stresses. The viscosity thereby plays a small role in the transport of the mean momentum.

It is interesting to note that the Reynolds stress itself is usually a negative quantity if the mean velocity gradient, $\partial \bar{u}_1 / \partial x_2$, is greater than zero. In this case positive values of u_2' tend to convect fluid lower absolute values of u_1', resulting in a decrease in u_1', yielding a negative value for $\overline{u_1' u_2'}$.

In the introduction to this class we stated one of the characteristics of turbulent flow was the enhanced transport of mass momentum and energy. In the case of momentum transport we see this turbulent diffusivity results from the presence of the Reynolds stress tensor, which generally makes a much larger contribution to the overall momentum transport than the viscous stress tensor.

8.3 Turbulent Kinetic Energy

In § 8.1 the effects of turbulence on the mean momentum of the flow was illustrated. We would like to repeat that here for the kinetic energy. (We will soon see that the turbulent kinetic energy plays an important role in developing turbulence models. Also, it is useful to understand how the mean flow feeds kinetic energy to the turbulence.) For an incompressible fluid with constant transport coefficients, the transport equation for the kinetic energy is:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) + \frac{\partial}{\partial x_j}\left(u_j \frac{u_i u_i}{2}\right) = & - \frac{1}{\rho}\frac{\partial u_j p}{\partial x_j} + \frac{\partial}{\partial x_j} u_i 2\nu S_{ij} \\ & - 2\nu S_{ij}\frac{\partial u_i}{\partial x_j} \end{aligned} \tag{8.14}$$

This can be derived directly from the Navier-Stokes equations with a little manipulation. The terms on the RHS of Eq. 8.14 represent the work done by pressure forces, the work done by viscous stresses, and the dissipation, respectively. This equation

will sometimes be seen written as:

$$\frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) = - \frac{\partial}{\partial x_j} u_j \left(\frac{p}{\rho} + \frac{u_i u_i}{2}\right) + \frac{\partial}{\partial x_j} u_i 2\nu S_{ij} - 2\nu S_{ij}\frac{\partial u_i}{\partial x_j} \tag{8.15}$$

The first term on the RHS now represents the work done by the total dynamic pressure.

We next want to apply the Reynolds decomposition to Eq. 8.14. The decomposition of the kinetic energy gives:

$$u_i u_i = \bar{u}_i \bar{u}_i + 2\bar{u}_i u_i' + u_i' u_i' \tag{8.16}$$

Substituting Eq. 8.16 and Eqs. 8.1 - 8.7 in Eq. 8.14 and averaging the entire equation gives:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i' u_i'}\right) = & - \frac{\partial}{\partial x_i}\bar{u}_i\left(\frac{\bar{p}}{\rho} + \frac{1}{2}\bar{u}_j\bar{u}_j\right) + \nu\frac{\partial}{\partial x_i}\bar{u}_j 2\bar{S}_{ij} \\ & - \nu 2\bar{S}_{ij}\frac{\partial \bar{u}_j}{\partial x_i} - \frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j' u_j'\right)} \\ & - \frac{\partial}{\partial x_i}\bar{u}_j\overline{u_i' u_j'} - \frac{1}{2}\frac{\partial}{\partial x_i}\bar{u}_i\overline{u_j' u_j'} \\ & + \nu\frac{\partial}{\partial x_i}\overline{u_j' 2 s_{ij}} - \overline{\nu 2 s_{ij}\frac{\partial u_j'}{\partial x_i}}\end{aligned} \tag{8.17}$$

This equation contains both the mean and turbulent kinetic energy. To obtain an equation for the mean turbulent kinetic energy equation alone we multiply Eq. 8.10 by $\bar{u}_i$. The resulting equation is:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) + \frac{\partial}{\partial x_i}\bar{u}_i\left(\frac{\bar{p}}{\rho} + \frac{1}{2}\bar{u}_j\bar{u}_j\right) = & - \overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(-\overline{u_i' u_j'}\bar{u}_i) \\ & + \nu\frac{\partial}{\partial x_i}\bar{u}_j 2\bar{S}_{ij} - \nu 2\bar{S}_{ij}\frac{\partial \bar{u}_i}{\partial x_j}\end{aligned} \tag{8.18}$$

The last term on the LHS of Eq. 8.18 and the last two terms on the RHS have the same physical interpretation as their counterparts in Eq. 8.14. As was the case in the equation for the mean momentum, there now appear additional terms due to the turbulence that effect the mean kinetic energy. The first term on the RHS of Eq. 8.18 represents the deformation work by the turbulent stresses. Because $\overline{u_i' u_j'}$ is usually negative, the action of this term tends to decrease, or take away kinetic energy from the mean motion. We will see shortly that this same term results in an increase in the turbulent kinetic energy. The second term on the RHS is the work by the turbulent stresses.

A similar order of magnitude scaling analysis as was done in **§8.1** for the Reynolds stresses shows that the viscous terms in Eq. 8.18 have a lower order effect than

the other (turbulence) terms on the RHS of Eq. 8.18. Viscosity does not have a significant impact on the mean kinetic energy transport. This again shows that the large structures in turbulent flows are relatively independent of viscosity.

More information about the turbulence and its interaction with the mean flow can be obtained from the transport equation for the turbulent kinetic energy. This is now easily obtained by subtracting Eq. 8.18 from Eq. 8.17:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'u_i'}\right)+\bar{u}_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'}\right) = & - \frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho}+\frac{1}{2}u_j'u_j'\right)}-\overline{u_i'u_j'}\frac{\partial\bar{u}_j}{\partial x_i} \\ & + \nu\frac{\partial}{\partial x_i}\overline{u_j'2s_{ij}}-\nu\overline{2s_{ij}\frac{\partial u_j'}{\partial x_i}} \end{aligned} \tag{8.19}$$

The mechanisms that affect the turbulent kinetic energy on the RHS of Eq. 8.19 are the turbulent diffusion of mechanical energy, deformation work on the mean flow by the turbulent stresses, the viscous work by the turbulent shear stresses, and the viscous dissipation of the turbulent kinetic energy.

As opposed to the case for the mean flow kinetic energy equation, the viscous terms are important to the turbulent kinetic energy balance. The last term in Eq. 8.19 has a particular significance. It is always positive and therefore indicates a drain of energy. This is the viscous dissipation term.

For a statistically steady flow, the time derivatives in the equations for the mean and turbulent kinetic energy are zero. Also, $s_{ij}s_{ij} = s_{ij}\partial u_j'/\partial x_i$ (convince yourself this is so). Applying these two conditions, the equations for the mean and turbulent kinetic energy can be written as:

$$\begin{aligned}\bar{u}_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) = & \frac{\partial}{\partial x_j}\left(-\frac{\bar{p}}{\rho}\bar{u}_j+2\nu\bar{u}_i\bar{S}_{ij}-\overline{u_i'u_j'}\bar{u}_i\right) \\ & +2\nu\bar{S}_{ij}\bar{S}_{ij}+\overline{u_j'u_i'}\bar{S}_{ij}\end{aligned} \tag{8.20}$$

and

$$\begin{aligned}\bar{u}_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_j'}\right) = & - \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j'p'}+\frac{1}{2}\overline{u_i'u_i'u_j'}-2\nu\overline{u_i's_{ij}}\right) \\ & - \overline{u_i'u_j'}\bar{S}_{ij}-2\nu\overline{s_{ij}s_{ij}}\end{aligned} \tag{8.21}$$

These are the same equations as (3.1.11) and (3.2.1) given in Tennekes and Lumley[1]. Note that the deformation work term $-\overline{u_i'u_j'}\bar{S}_{ij}$ (or, equivalently $-\overline{u_i'u_j'}\frac{\partial\bar{u}_j}{\partial x_i}$) appears in both the equations for the mean and turbulent kinetic energy equation, but with an opposite sign. We also call this the production term as the effect of deformation work is to exchange energy between the mean flow and the turbulence. Usually, this term is negative (the Reynolds stress is usually negative) so that there is a net flow of kinetic energy from the mean flow to the turbulence. This is not, however strictly the case, as a *countergradient flux* of momentum and energy is observed in certain situations. In a purely laminar flow, viscous stresses dissipate the

kinetic energy directly into heat. In turbulent flow, however, the deformation work caused by the turbulent stresses first converts mean flow kinetic energy into turbulent kinetic energy. It then cascades through the wavenumber spectrum (as explained in **§6**) before it is finally dissipated into heat.

The terms in parentheses on the RHS of Eq. 8.21 are in the form of a divergence of a vector quantity. These are *conservative* terms, and when integrated over an infinite flow field volume will vanish. These inertial terms do not create or destroy energy, they simply redistribute the turbulent kinetic energy from place to place in the flow. In wavenumber space (the Fourier transformed equations) the effect of theses nonlinear inertial terms is to redistribute the energy among the different frequencies. Without this, the energy in each mode would decay, independent of all other modes. Again, we see it is the nonlinearities that give turbulence its interesting characteristics.

In a statistically steady homogeneous turbulent flow a further simplification of Eq. 8.21 can be made. In this case, all spatial gradients of the turbulent transport quantities will be zero. Under this condition, the only remaining terms give:

$$-\overline{u_i' u_j'}\bar{S}_{ij} - 2\nu\overline{s_{ij}s_{ij}} = 0 \tag{8.22}$$

or

$$-\overline{u_i' u_j'}\bar{S}_{ij} = 2\nu\overline{s_{ij}s_{ij}} \tag{8.23}$$

In words, Eq. 8.23 is saying that the rate of production of the turbulent kinetic energy is balanced by the rate of viscous dissipation.

8.4 Turbulent Transport of a Passive Scalar

One of the main points of emphasis of this course is to study in detail the mixing of both passive and reacting scalars in turbulent flows. In particular we will be discussing the physical mechanisms acting in the overall mixing process, and discuss many of the methods used to model mixing. Subsequent notes will be devoted to this topic. For completeness of this section, however, a short discussion of the mean transport equation for a passive scalar in a turbulent flow is given here.

The "exact" equation for the transport of a scalar variable, ϕ is (assuming a constant diffusivity D):

$$\frac{\partial\phi}{\partial t} + \frac{\partial u_j\phi}{\partial x_j} = D\frac{\partial^2\phi}{\partial x_j\partial x_j} \tag{8.24}$$

In Eq. 8.24 ϕ could correspond to any passive scalar such as temperature or a chemical species or contaminant in the flow with D as its appropriate diffusion coefficient. Decomposing the variables into their mean and fluctuating components and averaging the equation gives the transport for the mean value of ϕ.

$$\frac{\partial\bar{\phi}}{\partial t} + \frac{\partial\bar{u}_j\bar{\phi}}{\partial x_j} = -\frac{\partial\phi' u_j'}{\partial x_j} + D\frac{\partial^2\phi}{\partial x_j\partial x_j} \tag{8.25}$$

or

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{u}_j \bar{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D\frac{\partial \bar{\phi}}{\partial x_j} - \overline{\phi' u_j'} \right) \tag{8.26}$$

Similar to the Reynolds stress term in Eq. 8.10, the correlation $\overline{\phi' u_j'}$ represents the turbulent transport of ϕ in the x_j direction. As written in Eq. 8.26 this can again be interpreted as an enhanced diffusivity due to the turbulence.

Note that in § 8.1 we did not derive an equation for the mean transport of fluctuating momentum as $\rho\overline{u_i'}$ is identically equal to zero. We can derive an equation for the *nonaveraged* fluctuating momentum equation by subtracting the averaged Navier-Stokes equation from the non-averaged equation. We will see that the resulting equation can be used to derive an equation for the dissipation rate (next section).

9 TURBULENCE MODELING

The statistical treatment for turbulent fluid flow is based on a decomposition of the random fluctuating variables (velocity, pressure, temperature, etc.) into a mean component, and a fluctuating component, as first suggested by O. Reynolds. From an engineering point of view, this idea is entirely satisfactory as we are rarely interested in the details of the turbulence. Usually, it is only a knowledge of the mean values (mean transport rates) that is wanted for most design purposes. The turbulence, however, can significantly alter the mean behavior of the flow (as seen in the previous section), and these effects must be accounted for in any description of turbulent flow.

It has been previously demonstrated that the exact solution to the equations of motion is not possible for most turbulent flows of interest. This is true for both analytic and numerical approaches. In order to provide tools for engineering use, it is generally necessary to resort to some type of approximate analysis in which the effects of turbulence are not treated exactly, but are handled by means of turbulence *models*. By resorting to treating turbulence by means of models, we are sacrificing an ability to study the dynamics of turbulence from the governing equations. Instead, we are assuming something about the behavior of the turbulence, and incorporating this expected behavior into our models. It is of course the hope that the effects of turbulence are faithfully incorporated into these models.

The earliest turbulence models were simply correlations of experimental results. These types of methods are not very useful as they do not contain much physics and can only be used to predict the flow that generated the experimental results. More general methods have therefore been adopted which are based on the conservation equations. Owing to the hopelessness of obtaining exact solutions to these equations, statistical approaches have been developed for their solution. Unfortunately, our lack of understanding of turbulence still requires that all models presently used in practical engineering applications reduce to correlations with experimental data at some level. However, by building as much physics as possible into the models it is hoped that the models will have applicability to a wider range of problems and that the exact functional relationships that need to be specified by experimental data will be more universal.

9.1 The Closure Problem

The equations for the mean flow (Eq. 8.9) derived previously are exact as no approximations have been made. (The equations were derived subject to our definitions regarding decomposing a variable into mean and fluctuating components and the definition of our averaging process.) We have used these equations to provide information on how the turbulence affects the mean flow and to identify some of the mechanisms of

turbulent transport. However, a fundamental problem associated with the statistical treatment of turbulence is revealed in the equations for the mean properties derived in **§8**. Through the process of decomposing the flow into its mean and fluctuating components, and averaging the governing equations, we have ended up with equations that contain more unknowns than before the averaging procedure was applied. No additional equations have been added so the equations no longer constitute a closed set. This "closure" problem is the central stumbling block in most all predictive methods for turbulent flow. As seen in Eq. 8.10, these additional unknowns (the Reynolds stresses) take the form of correlations between the fluctuating quantities. In order to close the set of equations given by Eq. 8.10, it is necessary to introduce empirical models to describe the behavior and effects of the Reynolds stresses.

One potential solution to the problem is to derive a set of transport equations that describe the time evolution of the Reynolds Stresses. This can be done (we will derive these equations later in this section), but higher order moments appear in the equations for the Reynolds Stresses. A hierarchy of equations can be derived for the unclosed terms, but closure is never achieved. Equations for the second order moments contain third order moments; equations for the third order moments contain forth order moments, and so on. At some level in this hierarchy, approximations must be introduced to close the equations. Providing a realistic closure has been a dominant motivation in turbulence research.

Hundreds of turbulence models have been suggested in the literature over the past 50 years. We will obviously not try to discuss each of these. Rather, we will attempt to clarify the physical insight that has (or has not) gone into the development of the models. Picking the best turbulence model for a particular application is not a simple matter. The complexity of the model and the amount of computer time required will vary from application to application. Here we hope to provide a flavor of the different turbulence models and point out their strengths and weaknesses.

For those interested in reviewing a rather comprehensive discussion on a wide range of turbulence models used in CFD applications, the text of Wilcox [2] is recommended.

9.2 Reynolds Stress Modeling

Let us first consider the equation for the mean motion derived in **§8**:

$$\rho\frac{\partial \bar{u}_i}{\partial t} + \rho\frac{\partial \bar{u}_i\bar{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\sigma_{ij}} - \overline{u_i'u_j'}\right) \tag{8.10}$$

As just discussed, the presence of the Reynolds stress makes the set of equations described by Eq. 8.10 unsolvable in their present form. Although this equation is exact, there is no way of determining the value of the Reynolds Stresses.

The earliest ideas for modeling the Reynolds stresses that appear in Eq. 8.10 still make up a big part of most turbulence models in use today. In an analogy to the viscous stresses in laminar flows, Boussinesq [3] suggested that the turbulent stresses

should be represented as a function of the mean velocity gradients. The "constant" (or more appropriately, function) of proportionality has been called the *eddy viscosity*. In general, the eddy viscosity representation of the Reynolds stress is

$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{9.27}$$

where k represents the total turbulent kinetic energy. The last term on the RHS of Eq. 9.27 is to ensure that the sum of the diagonal components of the Reynolds Stress tensor equal the turbulent kinetic energy. The components $\overline{u_1'^2}$, $\overline{u_2'^2}$, and $\overline{u_3'^2}$ act as normal stresses and thus act like a pressure force. Substituting Eq. 9.27 into the equation of motion gives:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{9.28}$$

Because the turbulent kinetic energy acts like a normal stress in this application, we can absorb it in the pressure term in Eq. 9.28:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial (\bar{p} + 2/3k)}{\partial x_i} + \frac{\partial}{\partial x_i} \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \tag{9.29}$$

Because k can be absorbed in p, it is not necessary to compute it explicitly. You will look at this more carefully in a homework problem.

At first glance, it appears that using an eddy viscosity should be a reasonable approach to modeling the turbulent stresses. One of the major characteristics of turbulence which we have discussed earlier is its enhanced mixing (or diffusive) properties. Applying an *apparent* turbulent viscosity given in Eq. 9.27 by ν_t seems to make good sense on physical grounds. Determining this turbulent, or eddy viscosity is then an interesting problem and one of crucial importance.

9.3 Modeling the Eddy Viscosity

By modeling the effects of turbulence by an eddy viscosity we are sacrificing any ability to study fundamental dynamics of turbulent flow using this model. The effects of turbulence are treated by a model that has an assumed physical behavior and whose "correct" quantitative behavior will ultimately rely on empirical correlations with experimental data. However, the goal is to build as much physics as possible into the models and thereby hopefully achieve a more general applicability.

The earliest models used constant values for the eddy viscosity. Models based on that assumption do not perform well and won't be discussed here.

The next step in complexity was achieved by further extensions of the analogy between molecular viscosity and eddy viscosity. Based on our knowledge of kinetic theory of gasses, we will derive some fairly simple expressions for the eddy viscosity (first put forth by Prandtl [4]). Then we will show why the arguments that lead to these expressions can't be precisely valid, but will point out situations where they still

seem to work quite well. Finally, alternatives leading to more advanced models will be derived in following subsections. It is not our goal in this class to take the "best" available technology in turbulence modeling and apply it to problems of engineering interest, but to study a variety of developments in turbulence modeling as a learning tool to try to understand more about fluid mechanics and turbulence.

The basic idea behind early expressions for the eddy viscosity was to consider the eddies in a turbulent flow to behave similarly to molecules in a gas. A fluid exhibits viscous effects as a result of molecular collisions which result in an exchange of momentum. Let us now think of turbulent eddies as individual entities which also collide with each other and exchange momentum. The molecular viscosity in a gas is proportional to the speed of the molecules (which is a function of temperature) and the mean free path between molecular collision. Relating these same concepts to the eddy viscosity, we will assume that the eddy viscosity is going to be proportional to a characteristic velocity of the turbulence, and some length scale, termed the mixing length by Prandtl, L_m:

$$\nu_t \propto V L_m \tag{9.30}$$

Next an estimate for V and L_m are still needed. When Prandtl formulated this "mixing length" model he considered a shear layer with only one mean velocity component, $\bar{u}_1$. The only mean velocity gradient is then $\partial\bar{u}_1/\partial x_2$ and the only contribution to the Reynolds stress tensor in this case is $\overline{u_1' u_2'}$. With this simplified configuration Prandtl then suggested that the mean turbulent velocity fluctuation could be represented by:

$$V = L_m \left|\frac{\partial \bar{u}_1}{\partial x_2}\right| \tag{9.31}$$

Using Eq. 9.31, the eddy viscosity can be written as:

$$\nu_t = L_m^2 \left|\frac{\partial \bar{u}_1}{\partial x_2}\right| \tag{9.32}$$

The Prandtl mixing length theory thus relates the eddy viscosity to the mean velocity gradient and the mixing length. The determination of the mixing length still remains.

The determination of the mixing length relies on experimental measurements. Because of the original assumptions of a shear layer with only one nonzero mean velocity component, the mixing length models have been most successful for thin shear layers where the mixing length can often be treated as constant across the shear layer and proportional to the width of the layer at any location. For various flow configurations, the values of the mixing length is given in Table 9.1. In this table δ is the local width or thickness of the shear layer.

For a wall bounded flow such as a boundary layer on a flat plate or the walls of a duct, the expression for the mixing length has more complexity and must be modified. This is because there are different regions in the boundary layer where the mixing length will take on different values. For example, in a typical boundary

Table 9.1: Mixing Lengths For Various Configurations

Flow Configuration	L_m
Plane Mixing Layer	0.07δ
Plane Jet	0.09δ
Round Jet	0.075δ
Plane Wake	0.07δ

layer, three different regions are usually identified: An inner layer where viscous shear dominates, an outer layer where turbulent shear dominates, and an intermediate, or overlap region where both effects can play a role. If you have had a boundary layer or viscous flow course, this identification should be familiar to you. In order to account for this, different expression for the mixing length are usually used in each region. The inner layer is usually very thin and in this region the mixing length is given by the distance from the wall, $L_m = y$. In the intermediate region, a value for L of $L_m = cy$ can be used where c is a constant that must be determined. Finally, in the outer region, where turbulence dominates, a constant value for L_m is used.

The mixing length models are not very successful for numerous types of flows encountered in practice. For complex flows, this is the case mainly because of the difficulty in determining the mixing length.

9.3.1 Conceptual Difficulties With The Eddy Viscosity - Molecular Viscosity Analogy

The eddy viscosity concept cannot be fundamentally correct. Models for the molecular viscosity based on the kinetic theory of gasses have two important assumptions: the molecules retain their shape (the collisions are elastic), and the mean free path between particle collisions is large compared with the molecular size. Neither of these statements holds true for turbulent eddies. The largest eddies in the flow are of the same dimensions as the flow geometry, so that their mean free paths cannot be large. Also, the interaction between eddies causes their structure to change. Since the molecular collisions are elastic, they do not loose energy during the collisions. The eddies in a turbulent flow, however, require a steady input of energy to maintain themselves. Never the less, the eddy viscosity models are the most popular in use and their performance has been OK. The eddies do exchange momentum with each other, and instead of treating Eq. 9.27 as a physically derived law, it should can be considered simply a *definition* of the eddy viscosity.

9.4 One Equation Models

In order to improve on the mixing length model discussed above, it is necessary include more physics into our models. More sophistication can be put into a turbulence model by including a transport equation for some of the turbulence models. We will still be working within the eddy viscosity concept.

If we are to describe the velocity fluctuations by a single scale, the most realist velocity scale is $k^{\frac{1}{2}}$ where $k = \frac{1}{2}\overline{u_i' u_i'}$ is the turbulent kinetic energy. Using $k^{\frac{1}{2}}$ as a velocity scale the following expression for the eddy viscosity is obtained:

$$\nu_t = C_\mu k^{\frac{1}{2}} L \tag{9.33}$$

C_μ is a "constant" whose value must be determined experimentally. The value of k throughout the flow can be determined by a transport equation for k. We derived such an equation earlier (Eq. 8.19). For completeness we present this equation below:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i' u_i'}\right) + \bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i' u_i'}\right) = & - \frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j' u_j'\right)} - \overline{u_i' u_j'}\frac{\partial \bar{u}_j}{\partial x_i} \\ & + \nu \frac{\partial}{\partial x_i}\overline{u_j' 2 s_{ij}} - \nu \overline{2 s_{ij}\frac{\partial u_j'}{\partial x_i}} \end{aligned} \tag{8.19}$$

The physical interpretation of the various terms in this equation are given in **§8.3**. In its present form, Eq. 8.19 is not of much use to us as there are terms containing triple moments that we do not have any information on. Neither do we have a transport equation (at this point) to compute their behavior. This is just another manifestation of the closure problem. Namely, the turbulent diffusion (the first term on the RHS of 8.19) and the dissipation (the last term on the RHS of 8.19) must be modeled. So by introducing an equation to describe the transport of the turbulent kinetic energy, it is necessary to model additional terms in order to obtain closure.

The last two terms in Eq. 8.19 (the viscous terms) are often written in a different form. Expanding out s_{ij} we can write:

$$\begin{aligned} 2\overline{\frac{\partial}{\partial x_i}u_j' s_{ij}} - \overline{2\nu s_{ij}\frac{\partial u_j'}{\partial x_i}} &= \overline{\frac{\partial}{\partial x_i}u_j'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} - \overline{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\frac{\partial u_j'}{\partial x_i}} \\ &= \frac{\partial}{\partial x_i}\overline{u_j'\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_i}\frac{1}{2}\frac{\partial}{\partial x_i}\overline{k^2} - \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}} - \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}} \end{aligned} \tag{9.34}$$

For an incompressible fluid we have

$$\overline{u_j\left(\frac{\partial^2 u_i}{\partial x_i \partial x_j}\right)} = 0 \tag{9.35}$$

So the first and third terms on the RHS of Eq. 9.34 cancel out.

$$2\overline{\frac{\partial}{\partial x_i}u_j' s_{ij}} - \overline{2\nu s_{ij}\frac{\partial u_j'}{\partial x_i}} = \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}\overline{k^2} - \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}} \tag{9.36}$$

Upon substitution into Eq. 8.19, we are left with:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i' u_i'}\right) + \bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i' u_i'}\right) = & - \frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j' u_j'\right)} - \overline{u_i' u_j'}\frac{\partial \bar{u}_j}{\partial x_i} \\ & + \frac{1}{2}\nu\frac{\partial^2}{\partial x_i \partial x_i}\overline{k^2} - \nu\overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}} \end{aligned} \tag{9.37}$$

In most of the literature the viscous dissipation is written as

$$\epsilon = \nu \overline{\frac{\partial u_j'}{\partial x_i} \frac{\partial u_j'}{\partial x_i}} \tag{9.38}$$

This is not exactly true as can be seen by comparing Eq. 8.19 and Eq. 9.37. Note that although neither equation involves any approximations (except for incompressibility), the correct form for the dissipation rate is

$$\epsilon = 2\nu \overline{\frac{\partial u_j'}{\partial x_i} s_{ij}'} = 2\nu \overline{s_{ij}' s_{ij}'} \tag{9.39}$$

9.4.1 Closure of the diffusive transport

The diffusive transport term is usually taken to be proportional to the gradient of the turbulent kinetic energy:

$$-\overline{u_i' \left(\frac{p'}{\rho} + \frac{1}{2} u_j' u_j' \right)} = C_2 \nu_t \frac{\partial k}{\partial x_i} \tag{9.40}$$

The constant of proportionality has been written as proportional to the eddy viscosity. This constant, C_2 is determined from data and can be interpreted as a diffusion coefficient. The motivation here is that the turbulent transport acts as an enhanced diffusivity. Modeling it as such makes sense on physical grounds.

The dissipation term must also be modeled. For high Reynolds number flow we have discussed previously that the dissipation rate is approximately independent of viscosity and depends only on the kinetic energy supply rate determined by the large eddies. From dimensional reasoning we showed that $\epsilon \sim U^3/l$. We then model the dissipation term as

$$\overline{\nu 2 s_{ij} \frac{\partial u_j'}{\partial x_i}} = C_3 \frac{k^{\frac{3}{2}}}{L} \tag{9.41}$$

C_3 is another empirical diffusion coefficient that must be determined by comparison with experimental data.

To complete the one-equation turbulence model formulation it is necessary to determine the length scale, L. It is the determination of L that provides the primary distinction among the various one equation models. Unfortunately, L is no easier to determine for the one equation model than for the mixing length model itself. The length scale L varies widely from flow to flow and except for thin shear layers, little data exists on its values. The following are a few references that discuss some of the ways the length scale has been determined. In practice however, the one equation models are not used very much. For more discussion on these models and the computation of the length scales two good references are Rodi[5] and Reynolds and Cebeci [6]. The most widely used models are ones in which two differential equations are solved to model the eddy viscosity. We now move on to discuss these types of models.

9.4.2 Two Equation Models

To alleviate the difficulty (and arbitrariness) of specifying the length scale, it has been suggested that the length scale be determined by a transport equation of its own. This is a reasonable approach as the length scale in a turbulent flow will be effected by many of the turbulent transport processes that affect the transport of mass, momentum, and energy. For example, if the flow is at some initial time, t_0, characterized by a length scale distribution, $L(x)$, it will at a later time have been convected to provide a different distribution. Furthermore, vortex stretching will tend to decrease length scales where this mechanisms is strong, and viscous dissipation will destroy the smallest length scales, leading to an increase in the length scale. Describing these processes by means of a differential transport equation is the goal of the two equation models.

A number of different equations have been derived that give us length scale information. Most of these equations do not solve for the length scale itself, but the length scale is easily obtainable. For example, any function of the form $Z = k^n L^m$ will give the length scale L as a modeled kinetic energy equation can be solved for k (see [5]). The equations that have been derived are mostly of the form:

$$\begin{aligned} \frac{\partial Z}{\partial t} + \bar{u}_i \frac{\partial Z}{\partial x_i} = & \quad \frac{\partial}{\partial x_j}\left(\frac{\sqrt{k}L}{\sigma_z}\right) \\ & - \quad c_{z1}\frac{Z}{k}\overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - c_{z2} Z \frac{\sqrt{k}}{L} + S \end{aligned} \tag{9.42}$$

The important thing about this equation is that in order to have a realistic representation of a turbulence length scale ($L = (Z/k^{-n})^{-m}$), the quantity Z that is solved for must be a property that is representative of the turbulence. In the above equation there are also a number of constants that must again be determined from data. These are represented by σ_z, c_{z1}, and c_{z2}. The first term on the RHS represents a diffusion of L, the second term a production (lengthening) of L, and the last two terms a loss in L. The last source term is represented only by S to illustrate the differences between the various forms of Z (or equivalently, the various ways of determining the length scale L).

By far the most popular of the two equation models is the $k - \epsilon$ model. This is partly a result of the ease in deriving an equation for the dissipation. From dimensional reasoning, the length scale L is determined from k and ϵ as:

$$L \propto \frac{k^{3/2}}{\epsilon} \tag{9.43}$$

Plugging this expression for L into Eq. 9.33 gives the following for the eddy viscosity:

$$\nu_t = C_\mu \frac{k^2}{\epsilon} \tag{9.44}$$

The full transport equation for the dissipation is obtained by manipulating the equation for the fluctuating momentum field. For the homogeneous dissipation, the equation can be written as :

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_j \frac{\partial \epsilon}{\partial x_j} = - 2\nu \frac{1}{\rho} \frac{\partial}{\partial x_i} \overline{\left(\frac{\partial u_i'}{\partial x_j} \frac{\partial p'}{\partial x_j} \right)} - \nu \frac{\partial}{\partial x_k} \overline{\left(u_k' \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \right)} - 2\nu \overline{u_k' \frac{\partial u_i'}{\partial x_j}} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k}$$

$$- 2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_k}} \frac{\partial \bar{u}_k}{\partial x_j} - 2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} \frac{\partial \bar{u}_i}{\partial x_k} \tag{9.45}$$

$$- 2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_k'}{\partial x_j}} - 2\nu^2 \overline{\left(\frac{\partial^2 u_i'}{\partial x_j \partial x_k} \right)^2} + \mu \frac{\partial^2 \epsilon}{\partial x_k \partial x_k} \tag{9.46}$$

This equation is also not closed (due to the unknown triple moments) and must be modeled in order to obtain a solution. The modeling of this equation for the dissipation is simplified by the following observations.

An order of magnitude analysis can be carried out to determine what terms in Eq. 9.45 dominate the transport for high Reynolds number flows. Representative length and velocity scales for the turbulence are the Kolmogorov scales, η and v ($\eta = (\nu^3/\epsilon)^{1/4}$, $v = (\nu\epsilon^{1/4})$). The velocity and length scales of the large scales (the energy containing eddies) are u' and L (the integral length scale). Using this, we can write:

$$u_i' u_j' \sim u'^2 \tag{9.47}$$

$$\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \sim \left(\frac{v}{\eta} \right)^3 \tag{9.48}$$

$$\overline{u_k' \left(\frac{\partial u_i'}{\partial x_j'} \right)^2} \sim u' \left(\frac{v}{\eta} \right)^2 \tag{9.49}$$

Using relations we derived earlier, namely:

$$\frac{\eta}{L} \propto Re_l^{-3/4} \tag{9.50}$$

and

$$\frac{v}{u'} \propto Re_L^{-1/4} \tag{9.51}$$

where $Re_L = u'L/\nu$, we can see that for large Re, a number of terms can be dropped out. The first two terms on the RHS of Eq. 8.19 describe the turbulent diffusion transport. Of these two terms, the second term dominates over the first term. The third, fourth, fifth, and sixth terms represent the generation of dissipation rate. Order of magnitude analysis using Eqs. 9.47 - 9.49 reveals that the third fourth and fifth terms involving the gradients of the mean velocity are dominated by term six. Finally,

the destruction of the dissipation by term eight is negligible compared with term seven. The reduced form of Eq. 9.45 then becomes:

$$\begin{aligned}\frac{\partial \epsilon}{\partial t} + \bar{u}_j \frac{\partial \epsilon}{\partial x_j} = &- \nu \frac{\partial}{\partial x_k} \overline{\left(u'_k \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} \right)} \\ &- 2\nu \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_k}{\partial x_j}} - 2\nu^2 \overline{\left(\frac{\partial^2 u'_i}{\partial x_j \partial x_k} \right)^2}\end{aligned} \tag{9.52}$$

The terms that are remaining are modeled in a fairly standard manner. The modeled equation for the dissipation is:

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_i \frac{\partial \epsilon}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_i} \right) + c_{1e} 2k \bar{S}_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - c_{2e} \frac{\epsilon^2}{k} \tag{9.53}$$

The constants in these equations must be determined by fits to experimental data. The complete $k-\epsilon$ model includes the solution of Eq. 9.53 and the k equation, (8.19) who's model form is (using 9.40 and 9.41):

$$\frac{\partial k}{\partial t} + \bar{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_i} \right) + 2\nu_t \bar{S}_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon \tag{9.54}$$

It should be noted that the modeling of the kinetic energy equation has a stronger physical justification than the equation for the dissipation. The above discussion should not be interpreted as placing too much mathematical significance on the modeled equations. In fact, it is probably best to interpret the modeled equation as the starting point for the mathematical representation as opposed to viewing the modeling as an attempt to accurately mimic each of the terms in the dissipation equation.

The model is completed by use of Eq. 9.44 to give the eddy viscosity, ν_t. From grid turbulence measurements, the constant $c_{2\epsilon}$ has been measured in the range of 1.8-2.0. $c_\mu$has been determined from measurements in shear layers to be approximately 0.09. The values for all constants in the $k-\epsilon$ model as suggested by Launder and Spalding [7] are:

$$\begin{aligned} c_\mu &= 0.09 \\ c_{1\epsilon} &= 1.44 \\ c_{2\epsilon} &= 1.92 \\ \sigma_k &= 1.0 \\ \sigma_\epsilon &= 1.3 \end{aligned} \tag{9.55}$$

The problem of determining these "constants" is not an easy one. We have stressed earlier that turbulence is not a fluid property, but a flow property that varies widely from application to application. The "constants" listed above are not generally either constant of universal. It is likely that for various applications the best choice for these "constants" has not yet been determined. Even when studying fairly simple flows, the values given in Eq. 9.55 can vary significantly. This again points to the *postdictive*

nature of turbulence models: Experiments are needed to tune models before they can be applied, and then they can only be used with confidence to predict flows similar to the validating flows. Improvements to the $k-\epsilon$ models have been attained in some applications by replacing the constants in the models with functional relationships. This can yield a wider range of validity. The functional relationships, however, are themselves only empirical fits to experimental data.

Besides the somewhat suspect modeling assumptions in the dissipation equation, more fundamental limitations are present in the $k-\epsilon$ model. Eqs. 9.27 and 9.33) show that the eddy viscosity is assumed isotropic. That is, the same value is given for each of the stress components. Since the turbulent stresses can develop and interact separately, this assumption cannot be expected to be too valid. Also, the assumption of gradient transport does not always hold. In fact, we will see later some very common mixing configurations in which the mixing process appears to be quite different from that described by gradient transport. Another difficulty with the $k-\epsilon$ formulation is that it is limited to a single mixing frequency (ϵ/k) or time scale (k/ϵ). Some multiple scale models have been suggested. Again, some references will be provided at the end of this section.

In variable density flows the situation becomes even more complicated due to the fluctuating density terms and addition terms must be modeled. In combustion, for example, other generation terms due to combustion heat release and expansion must be considered. The physics here is much different than in constant density flows and should be modeled in a way that allows this physics to be described in a realistic manner. Buoyancy also can lead to turbulence production. Some references to some fairly recent papers and review articles on the modeling of both constant density and variable density turbulent flows is given at the end of this section.

9.4.3 What's Really Going On Here

Besides attempting to provide a physical understanding of what is going on in turbulent flows and repeating what is known about how turbulence effects are treated in engineering studies, another goal of this class is to provide you with the necessary understanding to intelligently apply existing models - or at least understand what the models really mean. One way to interpret the whole idea of the $k-\epsilon$ model is to realize that what it is attempting to do is provide reasonable descriptions of the diffusive nature of turbulence in terms of an eddy viscosity. This is done by deriving a form for the eddy viscosity based on some key features of the turbulence, i.e., the length, time, and velocity scales of the turbulence. Recall that the viscosity is a length times a velocity scale. If this viscosity, or "eddy" viscosity for the turbulence is formed using "turbulence" length and time scales, it means that all features of the flow smaller than these scales are really being modeled. That is, no information, say below $L = k^{3/2}/\epsilon$ is retained - it is all "smeared out" by the eddy viscosity. Now there is nothing wrong with this provided the gradient transport formulation that has been used is a reasonable representation of the physics. In many cases it is. Note that in

a numerical simulation if you increase the resolution, you may get better results in terms of minimizing discretization errors, but you won't be able to explicitly simulate structure at smaller scales as everything below the $k-\epsilon$ length scale is modeled by a viscous diffusion.

9.5 Reynolds Stress Transport Equations

The problems with the eddy viscosity models could in theory be avoided by solving a transport equation for the Reynolds stresses directly. This idea was suggested back as long ago as the 1920's. The next step in sophistication of the turbulence models using Reynolds averaging techniques is in the development of a modeled equation for the Reynolds stresses.

To obtain an equation for the Reynolds stresses one can first multiply the Navier-Stokes equation for the V_i component of velocity by u_i'. Next reverse the subscripts i and j in the N-S equation and multiply by u_j'. Adding these two resulting equations together and then averaging will give the appropriate equation for the Reynolds stresses. The exact equation for the transport of the Reynolds stresses can be expressed as:

$$\frac{\partial}{\partial t}R_{ij} + \bar{u}_k\frac{\partial}{\partial x_k}R_{ij} = P_{ij} + T_{ij} - D_{ij} - \frac{\partial}{\partial x_k}J_{ijk} \tag{9.56}$$

The terms appearing in Eq. 9.56 are:

$$R_{ij} = \overline{u_i'u_j'} \tag{9.57}$$

$$P_{ij} = -\left(R_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + R_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) \tag{9.58}$$

$$T_{ij} = \overline{\frac{1}{\rho}p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \tag{9.59}$$

$$D_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{9.60}$$

$$J_{ijk} = -\nu\frac{\partial}{\partial x_k}R_{ij} + \overline{u_i'u_j'u_k'} + \frac{1}{\rho}\left(\overline{u_j'p'}\delta_{ik} + \overline{u_i'p'}\delta_{jk}\right) \tag{9.61}$$

R_{ij} is the Reynolds stress tensor, P_{ij} is the production term, T_{ij} is called the pressure - strain term, D_{ij} is the dissipation term, and J_{ijk} represents the flux of the Reynolds stress along the k direction. This flux is due to viscous diffusion, turbulent diffusion, and pressure. The separation of the terms into the above contributions is not unique. In particular, the decomposition of the pressure term into the pressure-strain-correlation and diffusive transport has important implications in the modeling of Eq. 9.56. It is not necessarily clear what form of Eq. 9.56 is the most conducive to modeling. Different choices could lead to very different results for the modeled equations. See Speziale [8] for other treatments.

Although the need to model the Reynolds stresses has been eliminated in this approach, the task of modeling the other terms that appear in Eq. 9.56 has greatly increased. In fact, the terms represented by Eqs. 9.59 - 9.61 must all be modeled.

The pressure-strain term has no counterpart in the kinetic energy equation. The tendency of this term is to enhance isotropy. It has no effect on the total energy balance, as it only *redistributes* energy among its various components. The name "pressure-strain" arises because this term is a correlation between the pressure and the fluctuating strain rate tensor. Before we discuss the modeling of this crucial term, let us first observe the other terms that must be modeled.

The belief that the dissipation for high Reynolds number flow is isotropic leads to

$$D_{ij} = \frac{2}{3}\epsilon\delta_{ij} \tag{9.62}$$

where ϵ is the isotropic dissipation discussed extensively earlier. The flux term, J_{ijk} is usually modeled using a gradient transport assumption. One such form was proposed by Daly and Harlow [9]:

$$J_{ijk} = c_s \frac{\partial}{\partial x_l}\left(\frac{k}{\epsilon}\overline{u_k' u_l'}\frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right) \tag{9.63}$$

A more general model is given by Hanjalic and Launder (1972) and Launder et al. [10] is:

$$J_{ijk} = C\frac{k}{\epsilon}\left(R_{il}\frac{\partial}{\partial x_l}R_{jk} + R_{jl}\frac{\partial}{\partial x_l}R_{ik} + R_{kl}\frac{\partial}{\partial x_l}R_{ij}\right) \tag{9.64}$$

See also Lumley [11].

The pressure-strain term is unique to the Reynolds stress equations and its accurate specification is crucial to the overall performance of the modeling. Some insight into modeling the pressure-strain redistribution term can be obtained by looking into the exact equation for the pressure fluctuation. Taking the divergence of the full momentum equation, and then separating the variables into their mean and fluctuating components. The result of this operation (assuming an incompressible fluid and neglecting gravity forces) is:

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_i}p' = -\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_i'}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{9.65}$$

This Poisson equation for the pressure has two terms on the RHS that will effect the pressure field. The first only contains turbulence terms, the second term also contains the mean deformation tensor. Each of these two contributions to the pressure-strain correlation is usually modeled separately. In the absence of any mean shear, the pressure-strain terms acts to returns an anisotropic flow to isotropic. The "return to isotropy" contribution has been modeled by Rotta (1972)(and discussed by Gibson et al. [12]; Rodi [5]) as:

$$T_{ij}^1 = -c_1\frac{\epsilon}{k}\left(\overline{u_i' u_j'} - \frac{2}{3}\delta_{ij}k\right) \tag{9.66}$$

where the superscript 1 has been used to indicate that this is the first part of the pressure-strain correlation.

The other contribution to the pressure-strain correlation is called the "mean shear" or "rapid" part. This term has been modeled by Launder et al. [10] as:

$$T_{ij}^2 = -\frac{c_2+8}{11}\left(P_{ij}-\frac{2}{3}\delta_{ij}P\right)-\frac{30c_2-2}{55}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)k-\frac{8c_2-2}{11}\left(D_{ij}-\frac{2}{3}\delta_{ij}P\right) \quad (9.67)$$

where

$$D_{ij} = -\left(\overline{u_i'u_k'}\frac{\partial\bar{u}_k}{\partial x_j}+\overline{u_j'u_k'}\frac{\partial\bar{u}_k}{\partial x_i}\right) \quad (9.68)$$

P is the kinetic energy production rate as given by Eqs. 8.18 and 8.19.

A simpler more widely used model is a simplification of the above model:

$$T_{ij}^2 = -c_2'\left(P_{ij}-\frac{2}{3}\delta_{ij}P\right) \quad (9.69)$$

This model for the mean shear contribution to the pressure-strain correlation has provided a reasonable approximation as the first term in Eq. 9.68 was shown by Launder et al.[10] to make the major contribution. In summary, the complete pressure-strain model can be expressed as

$$\begin{aligned} T_{ij} = \; & -\; c_1\frac{\epsilon}{k}\left(\overline{u_i'u_j'}-\frac{2}{3}\delta_{ij}k\right) \\ & -\; c_2\left(P_{ij}-\frac{2}{3}\delta_{ij}P\right) \end{aligned} \quad (9.70)$$

In general, there is also a contribution due to buoyancy. This is described in [12]. A more general pressure-strain model can be found in Jones and Musogne [13]

The above is a very abbreviated introduction into the closure of the Reynolds stress equations. A set of notes prepared by Gibson [12] for a short course on turbulence modeling contains a relatively recent and up to date collection of references on Reynolds transport modeling. It is a good source for those wishing to use turbulence models in their work. One of the biggest shortcomings of Reynolds stress transport modeling is the lack of data, or the inability to obtain data to study and evaluate the performance of the proposed models for the various terms. Many of the higher order correlations, especially those involving pressure perturbations are very difficult to measure.

9.6 Recapitulation

It's clear from the above discussions that the modeling of turbulent flows is a difficult task. There also is no general agreement concerning the best turbulence models to use for a particular application. The use of turbulence models is the only way in which flows of practical engineering application can be solved. This, as we have pointed

out many times, is due to the extreme range of time and space scales that exist in turbulent flows.

The type of modeling we have discussed in this section is all based upon the Reynolds averaged form of the governing equations for mass, momentum, kinetic energy, dissipation, and the Reynolds stresses. A difficulty with this approach is that this "global" type of averaging procedure results in the loss of information about *all* scales of the flow. One of the effects of this is the closure problem. In our discussions of the various length scale regimes in turbulent flows we have shown that the small scale structure of turbulent flows can be expected to behave in a relatively universal way. The large scale motions, on the other hand are highly anisotropic, and their development is highly dependent on the operating conditions and geometry of the flow. Furthermore, it is the large scale motions that are responsible for the major contribution to the turbulent transport of mass, momentum, and energy. It is very unlikely then that a universal model can be expected to reproduce the large scale structure for a wide class of flows.

We discussed two basic approaches to modeling the Reynolds stresses that appear in the averaged momentum equation. One based on introducing an eddy viscosity to model the diffusive properties of turbulence, and another based on the solution of the transport equations for the Reynolds stresses. The eddy viscosity models include three basic types: Zero equation, one equation, and two equation models. The zero equation models require the specification of a velocity scale and a mixing length scale. The one equation models solve a transport equation for the velocity scale, generally the turbulent kinetic energy. The specification of a length scale is still required. The two equation models include an additional transport equation to give the mixing length information. The most popular two equation model is the $k\epsilon$ model which includes a transport equation for both the kinetic energy and the dissipation. This is probably the most widely used turbulence model in use today. These equations are themselves unclosed and modeling assumptions must be made to achieve their solution. From a comparison with the k and ϵ equation in their exact and modeled form, it is seen that the modeled dissipation equation appears to be on a shakier foundation than the kinetic energy equation.

A further degree of sophistication is achieved by solving for the Reynolds stresses by their own transport equations. These equations also are not closed due to the appearance of pressure-strain correlation and triple moments of the fluctuating velocities. This is debatably considered the highest degree of complexity that can be realistically used in practical applications. Although with the increasing capabilities of computing power and the continued development of large eddy simulation (LES) and refinement of subgrid models, more detailed approaches to simulating turbulent flow are gaining popularity. (LES will be discussed later in the course.) The addition of the Reynolds transport equations implies that much more computing time is necessary due to the additional transport equations that must be solved. Unfortunately, the uncertainty in some of the modeling assumptions detracts from the advantages of solving for the Reynolds stresses directly. For many complex flows, however, the

lower order models are fundamentally incapable of reproducing the correct behavior. In such cases it should be expected that better performance could be obtained by the modeled Reynolds stress equations. Unfortunately, these models are expensive to evaluate and are still under development. As a result, most engineering calculations of turbulent flows use the $k - \epsilon$ model. All these models are far from perfect. Much more work needs to be done before the Reynolds stress models will consistently outperform the lower order $k - \epsilon$ models.

It must be emphasized that numerical calculations employing turbulence models of the type discussed in this section do not directly enhance our fundamental understanding of turbulence. Eventually, they all reduce to a correlation of with experimental data at some point. A well developed turbulence model must contain a high degree of physical reality in its formulation. The developer of a turbulence model attempts to capture the physics in a simplified representation. Any results obtained from a turbulence model therefore only describe the physics built into it, and cannot be used to identify physical mechanisms directly. Indirectly, however, comparison of model performance with real physical data can point to shortcomings in the models. Furthermore, most turbulence models are developed with a conscientious effort towards understanding the appropriate physics and how to best model it. Physical understanding will therefore be a bi-product of the search for a more universal model.

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10 TURBULENT SCALAR MIXING PROCESSES

Consider a configuration that is initially composed of two distinct and segregated scalar constituents. These scalar constituents may be differentiated by different chemical composition, different temperatures, contain different trace elements, or any other scalar marker. The description and ideas of scalar mixing to be discussed in the next few sections are concerned with the process by which these different constituents mix at the *molecular* level to produce a homogeneous mixture and how we attempt to describe the statistical state of the scalar field throughout its development.

Accurately predicting the amount of mixing that will occur at the molecular level in a turbulent flow is important in many applications. For example, in combustion applications, chemical reactions can only take place when the the fluid is mixed at the molecular level. In waste processing, constituents must interact at a molecular level. In atmospheric applications, the forces that drive the turbulence can be dramatically affected as cool and warm air mix at the molecular level. The mechanisms by which this mixing takes place, and approaches to modeling this process are the subject of this section. Unfortunately, accurately predicting the amount of molecular mixing that occurs in turbulent flows is a difficult task. Most models currently being used simply do not adequately treat this process. As a result, the reliability of predictions of turbulent flow that require a description of the molecular mixing process must be assessed on an individual basis. In the following we will discuss some of the physical mechanisms of turbulent mixing to illustrate the difficulties in accurately modeling this process.

10.1 Physical Mechanisms of Turbulent Mixing

As mentioned above, by turbulent mixing, we are referring to the process by which two separate constituents of the flow (different chemical species or scalar constituents with different values, i.e. temperature, etc.) eventually are brought together and interact at the molecular level. The governing equation that describes this process is the convection diffusion equation

$$\frac{\partial \phi_\alpha}{\partial t} + \frac{\partial u_j \phi_\alpha}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_\alpha \frac{\partial \phi_\alpha}{\partial x_j} \right) \tag{10.1}$$

In Eq. 10.1, ϕ_α represents scalar constituent α and D_α is the molecular diffusion coefficient of constituent α. In this equation, there is no summation over α (only j). Equation 10.1 describes the two distinctly different physical processes involved in

turbulent mixing: turbulent stirring (convection), and molecular diffusion. If chemical conversions are involved, a third process, the chemical reaction rate, involving yet another physical process must be accounted for. Chemical reaction, of course, can occur only when molecules interact act the molecular level. Below, we discuss these different processes.

10.1.1 Turbulent Convection

Consider a turbulent flow field containing two different constituents, initially unmixed. The action of this flow field on the scalar field is to distort and increase the surface area of the interface between the two constituents. As a result, the gradients of the scalar constituents will be increased and the scalar length scale will be decreased. As a result of this "stirring" process a complex structure of the scalar field evolves. Stirring has the effect of redistributing the scalar field throughout the flow field.

10.1.2 Molecular Diffusion

Mixing at the molecular level is a diffusion process. The turbulent stirring process described above acts only to redistribute or convect the scalar throughout the flow domain. Without molecular diffusion intermixing of separate constituents will not occur. Communication at the molecular level can only occur through the action of molecular diffusion. Our macroscopic view of molecular diffusion is a results of the random motion (Brownian motion) of fluid particles at the molecular level. This process is most effective in regions of high gradients and acts most effectively at the smallest scales of the flow. The enhanced mixing properties of turbulence are thus due to the action of turbulent stirring, which increases the effectiveness of molecular diffusion by increasing the scalar gradient and increasing the surface area over which the separate constituents can diffuse.

A little time spent thinking about these two processes will reveal the difficulties in modeling the mixing process. Namely, the actual diffusion is a small-scale process which depends critically on the small-scale structure of the scalar field. It cannot be parametrized easily in terms of the large scale features of the flow. As a result, a detailed description of the scalar field at the smallest length scales of the flow is, in principle, necessary if a reliable prediction of turbulent mixing is to be achieved.

10.2 Spectrum of the Scalar Field

Previously we discussed some properties of the turbulence energy spectrum. In particular, the energy spectrum can be roughly looked at consisting of a wave number regime characterizing the energy containing eddies, a universal equilibrium regime (the inertial range) described by the $k^{-5/3}$ law, and a dissipation range, below the Kolmogorov wave number. In discussing the spectrum of the scalar field, let us first define the scalar analogy to the Kolmogorov scale. Recall that the Kolmogorov scale is the smallest velocity length scale in the flow. Below this scale, viscosity effectively

damps out inhomogenieties in the flow. Analogously, there is a smallest scalar length of the flow which can be larger, smaller, or of the same scale as the Kolmogorov scale, depending on the relative magnitude of the kinematic viscosity to molecular diffusivity. This is parametrized by the Schmidt number, Sc, defined as $Sc \equiv \frac{\nu}{D}$, where ν is the kinematic viscosity of the fluid, and D is its molecular diffusivity. Below are some comments regarding the size of the scalar length scale.

For a given flow configuration (i.e., where the integral length and velocity scales, U and L, are specified), the size of the Kolmogorov scale will depend on the viscosity. (Recall that earlier we discussed that the viscosity does not determine the amount of dissipation in a flow, but the scale at which that dissipation occurs.) Similarly, the smallest scalar length scale will be determined by the molecular diffusivity, D. Intuitively, for $\nu \sim D$, it should be expected that the two scales (velocity and scalar) will be of the same order of magnitude. For $\nu > D$, velocity fluctuations will be damped out at scales larger than the scalar fluctuations so $\eta > l_s$ for $\nu > D$. Similarly for $\nu < D$ we can argue that $\eta < l_s$.

To arrive at an estimate for the scalar length scale consider the scalings implied by the diffusion equation, $l^2 \sim Dt$. First consider the case where $Sc > 1$ ($\nu > D$). Our region over which we are applying the scale analysis is the smallest scalar length scale, l_s. In this region, the scalar field is subject to the complete range of strain-rate fluctuations (since $l_s < \eta$) and the appropriate time scale is the Kolmogorov time scale, $\tau = (\nu/\epsilon)^{1/2}$. This gives $l_s^2 \sim D(\nu/\epsilon)^{1/2}$. In this regime of $Sc > 1$ the smallest scalar length scale is referred to as the Batchelor scale, l_B. Then using the expression for the Kolmogorov scale, $\eta = (\nu^3/\epsilon)^{1/4}$ gives

$$\frac{l_B}{\eta} \sim (D/\nu)^{1/2} = Sc^{-1/2} \tag{10.2}$$

Now for small $Sc < 1$, the above scaling will not be correct. In this case, the smallest scalar length scale will be greater than the Kolmogorov scale and may extend into the inertial subrange. The scalar length scale in this case is termed the Obukov-Corsin scale, l_C. Under this condition, the only parameters describing the scalar field will be the dissipation and the molecular diffusivity (recall that the viscosity does not play a role in the inertial subrange). The appropriate time scale in this case is $\tau = (D/\epsilon)^{1/2}$. Using this in our scale analysis results in $l_C \sim (D^3/\epsilon)^{1/4}$. Multiplying and dividing this equation by $\eta = (\nu^3/\epsilon)^{1/4}$ then gives:

$$\frac{l_C}{\eta} \sim (D/\nu)^{3/4} = Sc^{-3/4} \tag{10.3}$$

Besides providing extremely useful information on the structure of the scalar field (it has implications in numerical simulation, among many other things), this scale analysis illustrates the importance of carefully specifying the region over which the scale analysis is conducted and correctly defining the appropriate length, time, and velocity scales in that domain.

For most gases, the Schmidt number is approximately one, so the smallest scalar length scales are approximately equal to the Kolmogorov scale. For liquids, Sc can

be on the order of 10^3 or greater. In liquids then, the scales at which diffusion occurs is over an order of magnitude smaller than the smallest hydrodynamic scales. This illustrates an additional complication associated with predicting mixing and reaction in liquids. Namely, since the scales at which diffusion is occurring are so much smaller, the computational requirements to numerically resolve these scales, and thus accurately describe the mixing process are correspondingly increased.

10.2.1 Spectrum of the Scalar Field, $Sc > 1$

For Schmidt number greater than 1, the Batchelor scale will be smaller than the Kolmogorov scale. In the inertial subrange, the velocity and scalar spectrum will behave similarly. Beyond the Kolmogorov wave number cut-off, velocity fluctuations cease. Scalar fluctuations beyond this range will be reduced by the strain field, which below the Kolmogorov scale is $(\epsilon/\nu)^{1/2}$. This reduction occurs until the Batchelor scale is reached. This region of wavenumbers for $k_\eta > k > k_{l_B}$ is termed the *viscous–convective* subrange. (The velocity scales are in the viscous range, but the scalar scales are convective.)

For wave numbers $k > k_{l_B}$, the scalar fluctuations are effectively damped by molecular diffusion. In this region the scalar fluctuations are rapidly dissipated. This is called the *viscous–diffusive* subrange.

10.2.2 Spectrum of the Scalar Field, $Sc < 1$

For $Sc < 1$, we have $l_C > \eta$ or $k_{l_C} < k_\eta$. In this case, the diffusive cut-off for the scalar field will appear in the inertial subrange. This subrange is referred to as the *inertial–diffusive* subrange. (Velocity field inertial, scalar field diffusive.)

10.3 Mixing Configurations

In this section we wish to put some of the discussion above into a clearer perspective by discussing the mixing process in two specific configurations: the planar mixing layer, and mixing in a homogeneous turbulent flow.

10.3.1 The Planar Mixing Layer

The planar mixing layer is generated when two parallel flow streams of different velocity come into contact. In the laboratory, this flow is generated by allowing two streams to come into contact at the end of a splitter plate. This configuration is approximated in practice when any stream of fluid is injected into another. As a result of the shear generated at the interface of the two fluids, the flow becomes characterized by large scale vortex structures that grow and interact with each other as the flow develops in the streamwise direction. This has been an intensely studied flow both because it is relatively easy to study in the laboratory, as well as being a generic configuration for mixing between two streams. We discuss this flow here as

it illustrates quite clearly the mechanisms involved in the overall mixing process as outlined above in section 10.1. For this configuration, the mixing process involves: 1) The entrainment of fluid from the two free streams into the mixing region, and 2), mixing of the fluids between the two streams as they come into contact at the molecular level.

In the laboratory, a useful measure of molecular mixing has been to measure the amount of chemical product formed when the fluids in the two feed streams consist of initially nonpremixed chemically reacting species. If the rate of chemical reaction is fast compared to the local fluid time scales, and since reaction can only occur at the molecular level, the amount of product formed will give a direct indication of the extent of molecular mixing.

A series of laboratory studies conducted at Caltech has had as their focus the mechanisms of mixing in this configuration. In our previous discussions on turbulent mixing, we have used order of magnitude scaling arguments and shown that for "high" Reynolds number flows (those commonly encountered in practice) that turbulent diffusion effects dominated over molecular viscosity. However, experiments directed at studying the mixing in fluids of different molecular diffusivities (D), or flows with different Schmidt numbers ($Sc = \frac{\nu}{D}$), have shown important differences.

Comparison of the results of two separate experiments illustrate the unexpected behavior. In the first of these experiments, Mungal and Dimotakis [1] measured the chemical product formed in a fast reaction between hydrogen and fluorine. The hydrogen and fluorine were carried separately in dilute concentrations in the two streams. Nitrogen was used as the carrier gas and the Reynolds number of the flow was approximately 10^5. Chemical reaction between the two occurred as they mixed at the molecular level within the shear layer.

In a similar experiment at about the same Reynolds number, Koochesfahani and Dimotakis [2] measured the amount of product formed for a fast chemical reaction in water. (Here and in the above paragraph "fast" means that the reaction rate is essentially instantaneous once the fluids have mixed at the molecular level). The only substantial difference between these two experiments was that the Schmidt number of water is approximately a factor of 1000 times greater than that of the nitrogen. If turbulent diffusion is the dominant factor in the mixing process, this variation in Schmidt number would be expected to have little effect on the total amount of product formation. However, it was found that the amount of mixing in the gas experiments was about a factor of two greater than in the water experiments. Furthermore, the concentration of the mixed fluid in the water experiments was uniform across the layer, even though the average concentration of the fluids from the two streams varied across the mixing layer.

10.3.2 The Broadwell–Breidenthal model for shear layer mixing.

A model developed by Broadwell and Breidenthal [3] provides an interpretation of these observations. As mentioned above, the path to molecular mixedness can be

characterized by two processes. In the first stage of the mixing process, fluid is entrained into the layer by a process characterized by the large scale features of the flow: an integral length scale L (the thickness of the layer), and a time scale, T_L, given by $L/(U_1 - U_2)$. T_L is the "eddy turnover" time, taken to be the time to reduce length scales to the Kolmogorov scale, η. Once the length scales have been reduced to this scale, molecular diffusion (the second step in the process) occurs in a time scale negligible compared to T_L (i.e., the time to reduce the length scale from L to η). In this scenario, the time to mix the fluid depends only on T_L and is independent of Reynolds number or Schmidt number.

During the mixing process, however, diffusion layers will form between the fluid from the two streams. The thickness of these layers will scale with the diffusion coefficients, and the strain rate. Dimensional analysis gives $w = (D/s)^{1/2}$, where s is the strain rate. Assuming that the strain rate is characterized by $1/T_L$, we have

$$w = (DT_L)^{1/2} \tag{10.4}$$

This result is also obtained from a scaling analysis of the diffusion equation. The time scale for this process is a large eddy turnover time, T_L. The contribution of the diffusion layers to the total mixed fluid concentration will then be given by the surface area of the layer per unit volume multiplied by the layer thickness. If the surface layer/unit volume is assumed to scale as $1/L$, the contribution of the diffusion layer to the total mixing is proportional to

$$P \sim (DT_L/L^2)^{1/2} = Pe^{-1/2} = Re^{-1/2}Sc^{-1/2} \tag{10.5}$$

where Pe is the Peclet number.

After a time $t + T_L$, all fluid that was entrained into the layer prior to t will have been reduced to the Kolmogorov scale. However, a certain amount of fluid entrained between t and $t + T_L$ may attain molecular mixing based on Eq. 10.5. The Broadwell-Breidenthal model can then be expressed as

$$\delta_p = \phi_d Re^{-1/2}Sc^{-1/2} + \phi_m \tag{10.6}$$

where δ_p is a normalized measure of the mixed fluid concentration, ϕ_m is a constant that describes the mixed fluid at the Kolmogorov scale, and $\phi_d Re^{-1/2}Sc^{-1/2}$ represents the contribution from the diffusion sheets.

A Reynolds number and Schmidt number dependency is apparent in Eq. 10.6 and explains the difference between the mixing experiments in water and air. In the experiments in air, the Schmidt number is about 1000 times smaller than that in water, resulting in the higher amounts of mixing observed. In the experiments of Koochesfahani and Dimotakis the condition $ReSc \to \infty$ is apparently met, and mixing is delayed until the entrained fluid reaches the Kolmogorov scale and is then subject to uniform straining.

The lack of lateral variation in the pdfs of the experiments of Koochesfahani (see notes from lecture) can also be explained by the arguments given above. For a fluid

with a high Schmidt number, the amount of mixing in the diffusion sheets will be negligible ($ScRe \to \infty$). Under these conditions, significant mixing only takes place at beyond the Kolmogorov scale. Because there is a delay of $t = T_L$ in mixing once the fluid is entrained into the layer, it has a chance to become distributed, which can account for the more uniform peak concentration across the layer. Under conditions of low Schmidt number, measurable diffusion may take place in the diffusion layers, allowing for lateral variation in the pdf's.

The type of behavior we have described here is not consistent with simple gradient transport, and therefore will not be reproduced by turbulence models based on gradient diffusion. Mixing layer type configurations appear in many applications. Any time turbulent mixing takes place between two initially segregated fluids, or heat (or any other scalar) mixes with the surrounding environment, mechanisms as discussed here can influence the rate of molecular mixing. As we have seen earlier, even boundary layers show regimes where entrainment and mixing occur as described above. Later we will discuss a new model that has been developed to try to describe and provide explanations for the various anomalous features observed in mixing experiments. The key feature of this model is to make an explicit distinction between the effects of molecular diffusion and turbulent transport.

10.3.3 Mixing in Statistically Steady Homogeneous Turbulence

Another configuration in which the mixing process has been extensively studied is the decay of scalar fluctuations in a statistically steady, homogeneous turbulent flow. The initial scalar field consists of initially segregated "marked" and "unmarked" fluid. As mixing proceeds, the scalar concentration throughout the domain evolves to a constant value at the mixed fluid concentration. Of interest here are the statistical details of the scalar field during its evolution from the unmixed to mixed state.

This configuration of a stationary homogeneous turbulent flow is not easy to set up in the laboratory. As a result most of the data available for mixing in this configuration has been obtained from DNS [4, 5]. In interpreting these data, it is important to keep in mind that the DNS data is limited to flows with relatively low Reynolds numbers and Schmidt numbers of order unity. This is a constraint resulting from finite computational resources. None-the-less, the results have proven very useful in helping understand the mixing process and in stimulating further studies. Extrapolation of these results to high high Reynolds number flows must be made with caution however. Some implications will be discussed later.

Results of these simulations have provided interesting information for both theoreticians and modelers. Details of the evolution are, of course, important as it is the goal of model development efforts to incorporate as much of the known physics and behavior as possible into models that are intended to describe the mixing.

The simulations of Eswaran and Pope focused on two main issues: the effects of the initial scalar length scale on the evolution of the scalar statistics, and the functional form of the single-point scalar pdf of the scalar field during its evolution.

As discussed earlier, the single point pdf provides a complete statistical description of the one-point scalar statistics. It was observed that the scalar field evolved from an initially bimodal distribution (representing the two initial unmixed concentration) to a form approximating a Gaussian distribution centered on the mean mixture fraction. The asymptotic form of the pdf is important from a modeling point of view as this behavior should be reproduced by models expecting to provide accurate predictions of this process.

Although a large amount of data has been produced for this specific configuration, there is some evidence that the results may not be easily extendable to mixing in high-Reynolds number flows. [6, 7]. Discussion of this issue is deferred until later.

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11 PREDICTING THE SCALAR MIXING PROCESS

The types of scalar mixing processes (and reaction) that we are concerned with here are completely described by the conservation equations of overall mass continuity, momentum transport, and chemical species conservation. Unfortunately, the wide range of length and time scales place impossible computational resource demands for numerical solutions of these equations. The computer time and memory requirements for full simulations will be beyond feasibility for the foreseeable future. On the other hand, analytic solutions to these equations exist for only a few simplified cases due to the inherent nonlinearity of the multi-dimensional coupled equations. As a result the researcher or engineer must resort to the use of models to provide predictions of the mixing and reaction process. Although essentially all models resort to use of empirical correlations at some level, it is hoped that the models are constructed so that they mimic as much of the known physical behavior of turbulent mixing as possible.

11.1 Moment methods applied to scalar mixing.

For engineering purposes, it is rarely, if ever, necessary to have the complete description of the flow field and scalar field, as is in theory possible from a full numerical simulation of the governing equations. In fact, knowledge of the mean values and a few of the higher order moments is usually sufficient for design purposes. Unfortunately, this information too is difficult to obtain in practice.

Among the most common approaches used to study scalar mixing in realistic configurations are methods based upon a decomposition of the fluctuating quantities into a mean component and a component which is the variation about the mean.

To obtain an equation for the mean values of the dependent variables, we first decompose the variables into their mean and fluctuating components, e.g.:

$$\begin{aligned} u_i &= \overline{u_i} + u_i' \\ \phi &= \bar{\phi} + \phi' \end{aligned} \tag{11.1}$$

Decomposing the variables into their mean and fluctuating components and averaging the equation gives the transport for the mean value of ϕ.

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{u}_j \bar{\phi}}{\partial x_j} = -\frac{\partial \overline{\phi' u_j'}}{\partial x_j} + D\frac{\partial^2 \bar{\phi}}{\partial x_j \partial x_j} \tag{11.2}$$

or

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{u}_j \bar{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D\frac{\partial \bar{\phi}}{\partial x_j} - \overline{\phi' u_j'}\right) \tag{11.3}$$

This equation is very similar in appearance to the exact scalar transport equation. However, note the appearance of the term $\overline{\phi' u_j'}$. This term describes the effects of turbulence on the development of the mean scalar field.

If a simple chemical reaction of the form $r_\alpha = [\phi_\alpha][\phi_\beta]$ is included in the species conservation equation, the equation for the mean scalar transport takes the following form:

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{u}_j \bar{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D\frac{\partial \bar{\phi}}{\partial x_j} - \overline{\phi' u_j'}\right) - \overline{\phi_\alpha}\,\overline{\phi_\beta} - \overline{\phi_\alpha' \phi_\beta'} \tag{11.4}$$

In this case an additional correlation between the two reacting species shows up which represents the effects of scalar fluctuations on the mean reaction rate. For more realistic situations in which the reaction has an explicit temperature dependence, the expression for the mean reaction rate can become extremely complicated.

Another point of importance with respect to scalar mixing is that mean values are often not the primary quantity of interest. In many mixing applications it is the variance that is the main indicator of the amount of mixing that has occured. You derived an equation for this in class and have discussed its importance. The proper characterization of scalar fluctuations is often most important.

11.1.1 The closure problem

The equations for the mean scalar field derived above are exact as no approximations have been made. We have used these equations to conceptualize how the turbulence affects the mean flow and to identify some of the mechanisms of turbulent transport. However, a fundamental problem associated with the statistical treatment of turbulent mixing (and the statistical treatment of turbulent flows in general) is revealed in the equations for the mean properties derived above. Just as in the equations for the mean momentum transport, through the process of decomposing the flow into its mean and fluctuating components and averaging the governing equations, we have ended up with equations that contain more unknowns than before the averaging procedure was applied. No additional equations have been added so the equations no longer constitute a closed set. This "closure" problem is the central stumbling block in most all predictive methods for turbulent flow. These additional unknowns take the form of correlations between the fluctuating quantities. In order to close the equations above, it is necessary to introduce empirical models to describe the behavior and effects of the turbulent transport terms.

One potential solution to the problem is to derive a set of transport equations that describe the time evolution of these unknown terms. This can be done (we will derive some of these equations later, in particular, equations for $\overline{\phi_a' \phi_b'}$), but higher order moments appear in the equations for the lower order moments. A hierarchy of equations can be derived for the unclosed terms, but closure is never achieved. Equations for the second order moments contain third order moments; equations for the third order moments contain forth order moments, and so on. At some level in

this hierarchy, approximations must be introduced to close the equations. Providing a realistic closure has been a dominant motivation in turbulence research.

Therefore, to solve Eq. 11.4, the unknown scalar-scalar and scalar-velocity correlations must be modeled. In practice, this is usually accomplished by making gradient-diffusion assumptions and handling the effects of turbulence on the transport by using an *enhanced*, or *eddy diffusivity*. In such a formulation, the turbulent scalar flux can be modeled as

$$\overline{\phi' u_j'} = \frac{\mu_t}{\sigma_t} \frac{\partial \overline{\phi}}{\partial x_j} \tag{11.5}$$

where μ_t is the eddy viscosity and σ_t is a constant to be determined.

Although we will see that there are problems with this modeling approach, even more severe difficulties arise when one is concerned with modeling the reaction rate term. In addition, modeling the eddy viscosity itself remains a challenging problem as already described in §**??**.

11.2 Modeling the Mean Reaction Rate

Although this course is not directed at reacting flows, taking a look at issues with modeling reaction rates illustrates some exceptionally challenging issues in turbulent flows. Below, some of this will be discussed.

11.2.1 Modeling Difficulties

Appearing in the transport equation for the average species concentration is the mean reaction rate. For a binary reaction, say

$$Y_1 + Y_2 \rightarrow P, \tag{11.6}$$

a realistic form of the reaction rate is

$$\dot{w} = k Y_1 Y_2 \tag{11.7}$$

Using Favre averaging, the mean reaction term can be written as

$$\tilde{\dot{w}} = \tilde{k}(\tilde{Y}_1 \tilde{Y}_2 + \widetilde{Y_1'' Y_2''}) + \widetilde{k'' Y_2''} \tilde{Y}_1 + \widetilde{k'' Y_1''} \tilde{Y}_2 + \widetilde{k'' Y_2'' Y_1''} \tag{11.8}$$

where Y_1 and Y_2 are the concentrations of the two reacting species, and k is a reaction rate coefficient. If the reaction rate coefficient remains constant the reaction rate term reduces to

$$\tilde{\dot{w}} = \tilde{k}(\tilde{Y}_1 \tilde{Y}_2 + \widetilde{Y_1'' Y_2''}) \tag{11.9}$$

A closure problem is evident, because the correlation between the fluctuating scalar fields must be supplied. Even more complexity arises owing to the fact that

the reaction rate coefficients cannot, in most applications, be treated as constant. For a rate coefficient of the form

$$k = k_o \exp(-T_a/T), \tag{11.10}$$

k clearly strongly depends on temperature. Decomposing the temperature, T into its Favre averaged mean and fluctuating components and then expanding the exponential gives

$$k = k_o \exp[-T_a/\tilde{T}] \left[1 + \frac{T_a}{\tilde{T}^2} T'' + \left(\frac{T_a^2}{2\tilde{T}^4} - \frac{T_a}{\tilde{T}^3}\right) T''^2 + ...\right] \tag{11.11}$$

Assuming that $T_a/\tilde{T}$ is large, the Favre average of Eq. 11.11 can be written as

$$\tilde{k} = k_0 \exp\left[-T_a/\tilde{T}\right] \left[1 + \left(\frac{T_a^2}{2\tilde{T}^4}\right) \frac{\overline{\rho T''^2}}{\bar{\rho}} + ...\right] \tag{11.12}$$

Unfortunately, the higher order terms in the expansion above are not generally insignificant. This makes the above formulation impractical for computing mean reaction rates, because a large number of terms in the expansion must be retained to give meaningful results. Fortunately, approximations can be invoked to compute major species reactions rates in many applications.

The most significant simplification in modeling the reaction rate can be achieved if the chemical reactions are fast compared to fluid mixing rates. In this case, the assumption of chemical equilibrium can be applied, and the rate of reaction is determined from chemical equilibrium considerations, and not from detailed kinetics. In many hydrocarbon flames, this assumption works reasonably well for the major species. However, the formation of some pollutants, such as nitric oxide, proceeds relatively slowly. The formation of NO has little effect on the heat generation or major species concentrations, but it cannot be predicted from equilibrium considerations because its formation is kinetically limited. A method that can effectively be used in the limit of fast chemistry is discussed below.

11.2.2 The Mixture Fraction.

In many combustion applications, the fuel and oxidizer are introduced into the combustion chamber in separate feeds, 1 and 2. Under these conditions, it is convenient to define a conserved scalar, f, called the mixture fraction,

$$f = \frac{Y_1}{Y_1 + Y_2} \tag{11.13}$$

f is the mass fraction of fluid that originated in stream 1. Since the mixture fraction is a conserved variable, it obeys the transport equation

$$\frac{\partial \rho f}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k f) = \frac{\partial}{\partial x_k}\left(\rho D \frac{\partial f}{\partial x_k}\right) \tag{11.14}$$

For high Reynolds number flows, the Favre averaged form of Eq. 11.14 is

$$\frac{\partial \bar{\rho}\tilde{f}}{\partial t} + \frac{\partial}{\partial x_k}(\bar{\rho}\tilde{u}_k\tilde{f}) = -\frac{\partial}{\partial x_k}\overline{(\rho u_k'' f'')} \tag{11.15}$$

The last term in Eq. 11.15 must be modeled. Within the $k-\epsilon$ formulation, the model described in Eq. 11.5 is often used. If the fluids in the two feeds have the same diffusivities and their boundary conditions are the same, then the mixture fraction can be used to reduce reacting flow problems to nonreacting turbulent flow problems. Of particular significance is the fast chemistry limit. For a fast, single-step irreversible reaction, fuel and oxidizer cannot coexist. If a reaction of the form

$$Y_f + rY_o \rightarrow Y_p \tag{11.16}$$

is considered, the stoichiometric mixture fraction, f_s, is

$$f_s = \frac{Y_{o_2}}{rY_{f_1} - Y_{o_2}} \tag{11.17}$$

where the subscripts 1 and 2 refer to the mass fractions in the two feed streams. These assumptions allow Y_o and Y_f to be computed directly from the mixture fraction:

$$\begin{aligned} f \leq f_s \qquad Y_f &= 0 \\ Y_o &= rY_b(f_s - f) \end{aligned} \tag{11.18}$$

$$\begin{aligned} f \geq f_s \qquad Y_f &= Y_b(f - f_s) \\ Y_o &= 0 \end{aligned} \tag{11.19}$$

where $Y_b = Y_{f_1}/(1-f_s)$. For a fast reaction of the form given in Eq. 11.16, it can be seen that if the mixture fraction is known, the complete mixture composition can be obtained. The chemistry problem has been reduced to one of solving for a nonreacting scalar in a turbulent flow.

If back reactions occur, the above formulation can still be applied with some modification. In this case, the fast chemistry assumption implies that the forward and backward reactions have shorter time scales than the turbulence. The complete composition is thus again obtainable from the mixture fraction f.

As stated earlier, the major heat-generating reactions in the combustion process of most hydrocarbon fuels occur rapidly with respect to fluid mixing times. An approach based on the above mixture fraction formulation is often used in these conditions. Finite rate kinetic models must be applied to compute slower reactions that are characteristic of other aspects of the combustion process, particularly pollutant formation. Modeling finite rate chemistry in turbulent flows becomes a much more difficult problem. Will discuss issues related to this more extensively later.

11.3 Probability Density Function Methods

In general, the most convenient way to treat the mean reaction rate is in terms of probability density functions (pdf's). The pdf methods have many advantages over mean flow and Reynolds stress closures when applied to reacting flows. Given the one point pdf, all single point statistics can be determined. Most important to reactive flow calculations, however, is the treatment of the mean reaction rate.

Since the mean reaction rate is a function of n scalars ψ_n, the joint pdf $p(\psi_n; x)$ is needed, where ψ_n are the values that the scalar, ϕ_n can take. If the kinetic mechanisms are known, the mean reaction rate can be expressed in terms of the joint pdf of the scalar variables as

$$\tilde{\dot{w}}_\alpha = \int\int\int \dot{w}_\alpha \tilde{p}(\psi_n; x) d\psi_1 \, d\psi_2 \ldots d\psi_n \tag{11.20}$$

where $\tilde{p}$ is a density averaged pdf given by

$$\tilde{p}(\psi_n; x) = \frac{\rho(\psi)}{\bar{\rho}} p(\psi; x) \tag{11.21}$$

The task now is focused on specifying the pdf. One of two approaches is generally used to determine the pdf. One involves solving a transport equation for the pdf. This will be briefly discussed shortly.

The other method, called the assumed pdf technique, is to assume a form of the pdf based on information of the particular flow. For example, if a jet of fuel F is issuing into air, the pdf of F will consist of a delta function that represents the fluid in the jet, another delta function that represents the fluid outside the jet, and another function that represents the fluid mixture in the turbulent zone

$$p(Z) = c_1 \delta(Z) + c_2 \delta(1 - Z) + c_3 p_m(Z) \tag{11.22}$$

where c_1, c_2, and c_3 are constants, and $p_m(Z)$ is the pdf in the turbulent region. Calculations of mixing and reaction have been made using a number of shapes for the pdf $P_m(Z)$. These include Gaussian, clipped Gaussian, and beta functions. Once the form of the pdf is assumed, it is generally characterized by its first and second moments. Most approaches that use this method solve modeled equations for the first two moments. The equation for the mean value is given by Eq. 11.15. A modeled form of the second moment equation is (the notation f' is used to the variance, $\overline{f'^2}$) [1]

$$\frac{\partial \bar{\rho} f'}{\partial t} + \frac{\partial}{\partial x_j} \bar{\rho} \tilde{u}_j f' = \frac{\partial}{\partial x_j} \left[\frac{\mu_t}{\sigma_t} \frac{\partial f'}{\partial x_j} \right] - c_1 \frac{\mu_t}{\sigma_t} \left[\frac{\partial \tilde{f}}{\partial x_j} \frac{\partial \tilde{f}}{\partial x_j} \right] - c_2 \frac{\bar{\rho} f'}{k} \tag{11.23}$$

σ_t, c_1, and c_2 are constants. This reflects standard modeling for turbulent transport, production, and dissipation - like the homework problem you were assigned.

As multiple species equations are encountered, the assumed pdf technique becomes limited by computational resources and uncertainties in modeling. This is due in part

to the large number of second moment equations that must be solved for $(\overline{\phi'_\alpha \phi'_\beta})$ to specify the joint pdf of many species. Furthermore, numerical evaluation of the multivariable integral for the reaction rate given by Eq. 11.20 for multiple species is not reliable [2]. More details on the implementation of this method can be found in Bilger[3].

11.3.1 Evolution Equation for the PDF

From Eq. 11.20 it is apparent that the joint pdf of the species mass fraction can be used to obtain the mean reaction rate. A more rigorous approach than outlined above involves solving a transport equation for the joint pdf. This equation can be derived in a relatively straightforward manner by algebraic manipulation of the governing equations and application of the properties of the pdf. For example, if the flow field is homogeneous the pdf evolution equation for a single random variable is [4],

$$\frac{\partial p(\psi,t)}{\partial t} = -\frac{\partial}{\partial \psi}\left[p(\psi,t)\left(\langle D\nabla^2\phi|\psi\rangle + \dot{w}(\psi)\right)\right] \tag{11.24}$$

In this equation, the reaction rate term appears in closed form. The molecular diffusion term appears as a conditional expectation and must be modeled. If the flow field were not statistically homogeneous, a term describing the convective transport of the pdf would appear. In the transport equation for the composition pdfs, this term is not closed and must also be modeled. In such cases, the information about the flow field is usually obtained by other means, such as the $k-\epsilon$ model. The treatment of convective transport is eliminated if the joint velocity-composition pdf is considered.

In general, the velocity-scalar joint pdf, $P(V_i,\psi_n;\mathbf{x},t)$, provides a complete statistical description of the flow (velocity and scalar fields). Once the joint pdf equation is known it can, in theory, be integrated in time to give the temporal development of the statistical behavior of the flow. However, a number of difficult realities must be dealt with.

The joint pdf transport equation for velocity and scalar fields can be expressed as [4]

$$\begin{aligned}\rho(\psi_n)\frac{\partial P}{\partial t} &+ \rho(\psi_n)V_j\frac{\partial P}{\partial x_j} + \left(\rho(\psi_n)g_j - \frac{\partial < p >}{\partial x_j}\right)\frac{\partial P}{\partial V_j} + \frac{\partial}{\partial \psi_\alpha}\left[\rho(\psi_n)\dot{w}_\alpha(\psi_n)P\right] \\ &= \frac{\partial}{\partial V_j}\left[\left\langle -\frac{\partial \tau_{ij}}{\partial x_i} + \frac{\partial p'}{\partial x_j}\middle| V_i,\psi_n\right\rangle P\right] + \frac{\partial}{\partial \psi_\alpha}\left[\left\langle -\frac{\partial J_i^\alpha}{\partial x_i}\middle| V_i,\psi_n\right\rangle P\right]\end{aligned} \tag{11.25}$$

All terms on the left hand side of Eq. 11.25 are in closed form. This includes effects of convective transport, density differences, and the chemical reaction term $\dot{w}_\alpha$. In this formulation it is seen that all the difficulties in other modeling techniques with the nonlinear reaction term simply do not exist. The conditional expectations that appear on the right-hand side of Eq. 11.25 are not known and must be modeled. These terms each represent diffusive processes. The first term includes the effects of viscous stresses and pressure gradients; the last term represents the molecular mixing.

Even considering that the closures for the diffusion term are far from satisfactory, it is remarkable that all other terms in the joint velocity-composition pdf appear in closed form.

Aside from the difficulties associated with modeling the unclosed term, another problem is readily apparent upon inspection. This is associated with the high dimensionality of the equation. For three-dimensional flows, the pdf equation is a 7+n dimensional equation (three spatial coordinates, time, three velocity components, and n scalar variables). From an Eulerian standpoint, the computational costs of constructing a numerical solution are prohibitive. Fortunately, such a system is conveniently handled by Monte-Carlo techniques.

In methods developed by Pope[5], the flow field (or, the pdf) at any time t is represented by N stochastic particles, which are intended to simulate fluid particles. Each of these particles is characterized by its spatial position, velocity, and composition, which are described by

$$\frac{\partial x_i^n}{\partial t} = U^n \tag{11.26}$$

$$\frac{\partial U_i^n}{\partial t} = \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i \tag{11.27}$$

$$\frac{\partial \phi^n}{\partial t} = \frac{1}{\rho}\frac{\partial J_j^\alpha}{\partial x_j} + S_\alpha \tag{11.28}$$

where x^n, U^n, and ϕ^n are the individual particle position, velocity, and composition. Stochastic models are used to model the right hand side of Eqs. 11.27 and 11.28. An outline of the solution procedure and the stochastic models is given in a comprehensive review of the pdf method [4]. This review also provides a detailed account of the underlying theory and formulation of the pdf technique in reactive flow problems.

Although the equations above can be treated numerically, the question of modeling the molecular mixing term remains. Let us consider again the pdf transport equation in a homogeneous turbulent flow. Note that there is no convective term in this equation. This makes sense, because in probability space, convection acts to transport the probability spatially. Since the flow is assumed homogeneous, this term then plays no role (by definition, the statistics are space independent in a homogeneous flow.) To model the molecular mixing term, many models have been developed. Most of these are variants of Curl's [6] *coalescence-dispersion model*.

11.4 Coalescence-Dispersion Modeling

The Coalescence-Dispersion (C-D) model has been utilized extensively in many different forms for predicting mixing processes in turbulent flows. As such, it is worth taking some time to discuss some of the features and implementation of the process.

The model was originally introduced by Curl[6] to describe the mixing and reaction process that occurred in a mixed phase process involving interaction among droplets.

Mixing is assumed to take place randomly between individual droplets. Individual droplets also are allowed to redisperse into an identical pairs of droplets. The coalescence between two particles is assumed to result in instantaneous mixing between the two and the rates of coalescence and dispersion are determined by characteristic mixing frequencies and number of particles of and in the system. The model has been extended to treat gaseous phase mixing by considering the fluid to be composed of a large number of Lagrangian fluid elements which collectively describe the statistical state of the fluid. Many variants of this model have appeared, e.g. [7, 8].

Let us consider the case of a passive scalar in a homogeneous turbulence flow. Neglecting the reaction term is not a significant omission here since it appears in closed form in the pdf transport equation. The C-D model can be described within an evolution equation for the pdf. Pope[5] showed a general form of the C-D model could be expressed as

$$\begin{aligned} \tfrac{\partial p(\psi,t)}{\partial t} &= -2\beta\omega p(\psi,t) \\ + \quad & 2\beta\omega \int\int_{-\infty}^{\infty} p(\psi',t)p(\psi'',t)K(\psi,\,\psi',\,\psi'')\mathrm{d}\psi'\,\mathrm{d}\psi'' \end{aligned} \tag{11.29}$$

where a general form for K is

$$K(\psi,\,\psi',\,\psi'') = \int_0^1 A(\alpha)\delta\left[\psi - (1-\psi)\psi' - \frac{1}{2}\alpha(\psi'-\psi'')\right]\mathrm{d}\alpha \tag{11.30}$$

No convection term appears because of the assumption of spatial homogeneity. $A(\alpha)$ is the pdf of α where α is a random variable and a measure of the mixing. $A(\alpha)$ is zero outside the interval $[0,1]$ and is nonnegative and normalized to unity within $[0,1]$. β is a parameter defined by Pope as

$$\beta = \frac{1}{a_1 - \frac{1}{2}a_2} \tag{11.31}$$

where

$$a_m = \int_0^1 \alpha^m A(\alpha)d\alpha \tag{11.32}$$

ω is the mixing frequency and must be supplied by an external source.

The various choices of $A(\alpha)$ define the difference variants of the C-D models. For example, Curls original model is recovered by setting $A(\alpha) = \delta(\alpha - 1)$. The closure used by Janicka et al.[7] is obtained by setting $A(\alpha) = 1$, while Kosaly[9] showed that the Dopazo-O'Brien[8] model of mixing can be recovered from this formulation by setting $A(\alpha) = \delta(\alpha - \epsilon)$ in the limit as $\epsilon \to 0$. During the class lecture we will provide a more physical based interpretation of this and discuss the relation between the stochastic Lagrangian implementation of the CD model to its pdf representation.

In practice, the pdf equation is solved in the stochastic Monte-Carlo approach mentioned above. The mixing by the C-D model involves randomly selecting pairs of elements for mixing. In the original version of Curl's model, the two elements

are assumed to mix completely, and redisperse into two identical particles with the same concentration. The choice of the different models (different $A(\alpha)$'s) simply determine the extent of mixing between the two particles before they redisperse. For example, in the model of Janicka et al., the extent of mixing between the two particles is determined by sampling the pdf(uniform) $A(\alpha)$. So as can be seen, there are basically an infinite number of variant of the C-D model, depending on how the mixing between the chosen particles is handled.

C-D modeling has been used by several researchers to predict experimental data. A discussion of some of this can be found in the paper by Kosaly and Givi[10]. A comparison of the C-D results for scalar mixing in a homogeneous turbulent flow with results of DNS has revealed some of the shortcomings of the C-D modeling approach. In particular none of the variants of the model tested correctly predict the evolution of the higher moments of the scalar field. Inspection of the model reveals the source of some of its difficulties. There is not a lot of physical justification for choosing one form of $A(\alpha)$ over any other, except that one may give a better representation of the evolution of the scalar field. In particular there is no small-scale structural information regarding behavior of the scalar field. Fluid elements, once selected for mixing, mix instantaneously. The combined physics of turbulent convection and molecular diffusion is not accounted for. In cases where this is important, the C-D models will not perform well.

11.5 Other Mixing Models

There are, of course, many other mixing models being used in turbulent flow situations. Most of them suffer from similar difficulties which stem from their failure to realistically represent the physics of the small scales. Here we briefly discuss a few others that are regularly used.

11.5.1 Interaction By Exchange With the Mean (IEM)

The IEM model is also a Lagrangian model where the scalar field is described by eddies or fluid elements with particular scalar values. However, the fluid elements do not interact directly with each other. They instead interact, or exchange information only with the mean. Consider a flow consisting of a single scalar constituent that takes on values between 0 and 1 (a mixture fraction, for example). The complete scalar field is described by a number of fluid elements with concentration c_i. The mean scalar concentration is given by $\bar{c}=1/N\sum_{i=1}^{N} c_i$, where N is the number of elements. In the most general case a different weighting can be given to the different eddies. The governing equation for the concentration of each fluid element in this model formulation is

$$\frac{dc_i}{dt} = h(\bar{c} - c_i) \tag{11.33}$$

h is a frequency describing the rate of mixing between different constituents.

To apply the model, consider a homogeneous, statistically steady turbulent flow consisting of an initial scalar distribution described by a double delta distribution at two extreme values of the scalar (say, -1 and 1; i.e., half the fluid is at a concentration of -1 and half has a concentration of 1). The turbulent mixing process will eventually result in a mixed fluid concentration of 0. The mean remains at $c = 0$ throughout the mixing process. The scalar field can be initialized with two eddies at the different scalar concentrations. Given a particular mixing frequency, $h = 1/\tau_m$, Each fluid element evolves according to Eq. 11.33:

$$\frac{dc}{dt} = -\frac{1}{\tau_m} c \tag{11.34}$$

or

$$c = c_o \exp[-t\tau_m] \tag{11.35}$$

This example serves to show a deficiency of this model. Note that the pdf of the scalar field will evolve as two marching delta functions, approaching each other to the final mixed fluid concentration. This is clearly an unphysical situation. There is no variation of the scalar field in the domain as described by this version of the model. In more general, inhomogeneous flow situations, the flow will be described by a large number of particles. Each element will evolve according to Eq. 11.33 where the local mixing frequency is obtained from the fluid mechanical calculation

Some randomness can be added to the process to produce a more physically realistic evolution of the scalar pdf. For the homogeneous mixing problem described above, now consider initializing the scalar field with a large number of fluid elements, half at a concentration of -1, the other half at a concentration of 1. Fluid elements will interact with the mean, but randomness can be added by randomly selecting the time for exchange based on the mixing frequency, and also selecting the mixing frequency to randomly distributed about the mean mixing frequency. That is, let each element evolve according to

$$\frac{dc}{dt} = -r\, c \tag{11.36}$$

where r is sampled from a pdf $g(r)$. The standard IEM model is obtained by taking $g(r) = \delta(r - 1/\tau_m)$.

Although this *randomized exchange with the mean* (REM) produces a more realistic scalar distribution during the time evolution, there are still problems with this approach. An analysis of the evolution of the higher order moments reveals that the moments diverge in the long time limit, at least for the forms of $g(r)$ that have been studied so far. This is a similar behavior to many of the C-D type models in use. This is an indication of a more fundamental shortcoming of these modeling approaches. Namely, the inability of the models to realistically describe and distinguish between the effects of the important physical processes in the mixing evolution.

11.5.2 Three and four environment models

Of the many different types of mixing models, here we mention two others. The two models mentioned here were basically developed for reactors with two separate, nonpremixed feed stream. In the three environment model the flow is assumed to be described by three different parcels of fluids: the two entering parcels (environments), and the leaving environment. The transfer (or mixing) from the entering environments to the leaving environments is given by a transfer function that depends of the *age* of the material. For a pure mixing problems with no chemical reaction the pdf of the scalar field will be represented initially by a double delta distribution. As time evolves, another delta distribution will develop at the mixed fluid concentration (the concentration of the exiting environment), and the delta distributions representing the entering environments will decrease in magnitude. So at any time only three concentration values exist. Although a calculation using this technique will be much more affordable than a monte-carlo simulation involving many parcels, there are clearly more unphysical aspects of this model.

An extension of this model is the four environment model, which is represented by two leaving environments. In this case each of the entering parcels exchanges mass with only one of the leaving environments. An addition transfer between leaving environments results is the pdf described by a quadruple delta distribution throughout its mixing time: The double delta representing the two incoming feed streams, and two other delta distributions. These are rich in one of the feed stream concentrations, but march in towards the mixed fluid concentration as time progresses.

A Final Comment

To repeat again, although the models that we have mentioned briefly above mimic some features of the real turbulent mixing process, there are clearly deficiencies associated with them. Mainly, the real physics of the mixing process becomes lost. This is not to say that these models are not useful. Many of them, in particular the variants of coalescence-dispersion are established components of predictive numerical codes. However, it is clear that all of the models suffer from deficiencies that make the search for a better mixing model a high priority in turbulent combustion research. Many of the details of the models briefly discussed above can be found in throughout the literature. Other useful references in addition to those already mentioned include: Brodkey, Mehta, and Tarbell[11] [12] [13].

Many variants of these mixing models, as well as different models have been proposed and used. The discussion in this section is by no means complete. It is meant as an introduction and background for further study.

In the next section we will discuss in detail a different model that is currently being used to model the mixing process. This model, the *linear eddy model* has some unique features that have proven to have made it a rather remarkable model of the mixing process. Although the model is not without flaws and drawbacks, it treats the mixing from a most physical viewpoint, and is therefore also a good tool not only to provide predictions of the mixing, but as means to actually study and try to

understand the fundamental behavior of the turbulent mixing process.

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12 LINEAR EDDY MODEL OF TURBULENT MIXING

Of all the different mixing models that have been used in conserved scalar mixing, reacting flow, or microphysical applications, none have been successful in providing accurate and reliable predictions for a wide range of applications. Models are very often *tuned*, and therefore applicable to limited configurations and applications. This is particularly true of finite rate chemical reactions in turbulent environments. Unfortunately, many of the slower, secondary reactions that occur in hydrocarbon combustion are critical to the generation of unwanted byproducts (pollutants). In many other cases the physics are parameterized down to such an extent that little of the real physical processes that are occuring in a turbulent mixing environment are realistically treated.

Because the underlying physics of the mixing process is rarely realistically represented, and because interesting processes are occurring over a wide range of length scales, models cannot predict the wide range of physical processes that occur. In this section we will discuss a relatively new approach that has been used with success to model the mixing and reaction of a scalar quantity (like a chemical species) in a turbulent flow field. This technique is the *Linear Eddy Model* and was developed by Kerstein[1, 2]. It is not a self contained model of turbulence in general, but provides a very useful and insightful way to look at the turbulent mixing of passive or reacting scalars. The model as discussed here is strictly a mixing model, in that the statistics of the velocity field are inputs into the linear eddy model. In the following we will first discuss basic philosophy of the linear eddy model and the formulation of the model. We will follow this up with specific applications of the model and discuss how the model can be set up and implemented to discuss specific issues of technological relevance. This model, of course, has its own set of limitations which restrict its use in complex engineering simulations.

12.1 Linear Eddy Modeling Philosophy

The transport of a scalar in a turbulent flow field is governed by the equation:

$$\frac{\partial \phi}{\partial t} + \frac{\partial u_i \phi}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \tag{12.1}$$

In the formulation of the linear eddy model it is recognized that convection and diffusion which act on the scalar field are two distinctly different physical processes. As such, they should be modeled separately, with an approach that realistically represents the physics. These two processes (the third physical process we wish to consider,

chemical reaction, will be discussed soon) are acting and interacting at all scales of the flow. Fluctuations in the velocity field cover a range from the integral scale to the Kolmogorov scale, while the scalar fluctuations span the integral scale to the Batchelor scale (or Corrsin scale for Sc$<$ 1). The relationship between the Kolmogorov scale and the Batchelor (Corrsin) scale is determined by the Schmidt number of the flow as described earlier. To account for the effects of both stirring and diffusion (and reaction) in a rigorous manner, their effects at all scales of the flow must be realistically represented. This can be achieved only by resolving all important scales of the flow. In traditional solution and modeling approaches, this is of course computationally prohibitive as the computer resources are severely inadequate to treat the full physics in multiple spatial dimensions.

Now, among the unique features of the linear eddy model is that it's description of the scalar field is a high-resolution, one-dimensional representation. Using a one-dimensional representation then allows for the full range of length and time scales to be resolved, even for flows with relatively high Reynolds and Schmidt numbers. The challenge, then, is to develop a model which provides a statistical description of the scalar field in one-spatial dimension, but is representative of the scalar statistics in a real three-dimensional flow. This is achieved by developing the model based on scaling laws representative of high-Reynolds number, three-dimensional flows. This formulation is discussed next.

12.2 Linear Eddy Model Formulation

As discussed above, the basic idea of the linear eddy approach is to treat separately the two different mechanisms acting to describe the evolution of a scalar (chemical species) over a linear domain. The first mechanism we discuss here is molecular diffusion (the last term in Eq. 12.1). Given a scalar field described on a one-dimensional space, diffusion can be implemented explicitly and essentially exactly the numerical time integration of the diffusion equation along the linear domain.

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} \tag{12.2}$$

When chemical reactions are considered, treatment of their source term effect can be implemented explicitly with the diffusion equation:

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} + \dot{w}_\phi \tag{12.3}$$

where the functional dependence of the reaction rate, $\dot{w}$, on the various chemical species is assumed known.

The real key to the model is the manner in which turbulent convection is treated. This constitutes another of the unique features of the model. Convection, or stirring is modeled in a stochastic manner by randomly occurring (subject to certain rules, of course) *rearrangement events* of the scalar field along the linear domain at

intervals dependent upon the flow field conditions. The size of the rearranged region is also randomly chosen based on the eddy size distribution within the flow. The determination of these parameters will be discussed in detail.

To help achieve a qualitative and conceptual feel for the rearrangement process it is instructive to make an analogy of the rearrangement process with the action of individual eddies acting on the scalar field. The rearrangement events are formulated so that they reproduce the same effects of eddies acting on the flow. These effects include a spatial redistribution of the scalar field, an increase in scalar gradients, and an increase in the surface area differentiating scalar values. The size of the rearranged domain represents the eddy size, and the distribution of the eddy sizes and frequency of the events are obtained by using Kolmogorov scaling laws for high Reynolds number turbulent flows.

The rearrangement events involve the following: 1) the location x is chosen within the domain and is dependent on the turbulence intensity distribution in the domain. For a homogeneous turbulence, the location is randomly chosen. 2) the "eddy" size l, is selected (according to scaling laws, which will be discussed in detail shortly) over which the rearrangement will occur. 3) A time is selected for the next "event." 4) The species distribution in the chosen domain is rearranged.

The particular mapping used is termed the "triplet" map. The details of this will be described in lecture. Note that this choice of a rearrangement map is not unique. See class handouts for details. As a side note, it should be recognized that the functions that are derived below which parameterize the mixing process in the model must be consistent with whatever mapping process is selected (if something other than the triplet map may be desired).

In brief summary then, the two steps in the process for the evolution and evaluation of the species field are diffusion, which is implemented explicitly:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \tag{12.4}$$

and the rearrangement events, as described above.

To carry out the complete model, the frequency of the inversion events must be specified, along with the distribution of the eddy sizes. The frequency of the rearrangement events is accomplished by realizing that the random rearrangement events produce a random walk of a marker particle on a line. It can be shown that the diffusivity associated with a random walk is

$$D_T = \frac{1}{2} N \langle x^2 \rangle \tag{12.5}$$

where N is the frequency of the events and x^2 is the mean-square displacement associated with each event. For a classical discussion of the statistics of random walks see Chandrasekar[3]. In its association with turbulent transport, D_T can be interpreted as the turbulent diffusivity. The rearrangement frequency must thus be chosen so that the diffusivity associated with the random rearrangement process (the triplet map in the case considered here) is equal to the physical turbulent diffusivity.

Consider the motion of a particular particle. If an eddy of size l is to displace this particle, the "center" of the rearrangement event must lie within a distance $l/2$ of that particle. The frequency of such a rearrangement event is simply $N = \lambda l$, where λ is a rate parameter. The displacement of any particle by the triplet map will depend on the location of the particle within the eddy. The mean square displacement of all particles within the rearranged domain can be shown to be equal to

$$\langle x^2 \rangle = \frac{4}{27} l^2 \tag{12.6}$$

Plugging expression 12.6 into 12.5 and using $N = \lambda l$, we arrive at

$$D_t = \frac{2}{27} \lambda l^3 \tag{12.7}$$

In a general turbulent flow there will not be just one length scale, but a wide range of scales (eddies) ranging from the integral scale , L, down to the Kolmogorov scale, η. If $f(l)dl$ is the fraction of blocks (eddies) in the range $(l,\, l + dl)$ then the total diffusivity associated with an eddy of size l is

$$D_t = \frac{2}{27} \lambda l^3 f(l) dl \tag{12.8}$$

The total diffusivity associated with all eddies up to size l will then be given by

$$D_t(l) = \int_\eta^l \frac{2}{27} \lambda l^3 f(l) dl \tag{12.9}$$

To complete the description of the model it is necessary to determine the eddy size distribution, $f(l)$, and the rate parameter, λ under these more general conditions. This is accomplished by making use of some additional scalings for the diffusivity. Dimensional analysis yields

$$D_t(L) \sim \frac{L^2}{T_t} \tag{12.10}$$

T_t is a characteristic time scale. This time scale of the turbulence is

$$T_t \sim \frac{L}{u'} \tag{12.11}$$

giving

$$D_t \sim u' L = \nu Re_L \tag{12.12}$$

The l dependence of $D_t(l)$ in Eq. 12.9 is assumed to scale as the Reynolds number based on l:

$$\int_\eta^l l^3 f(l) dl \propto Re_l \sim \left(\frac{l}{\eta} \right)^{4/3} \tag{12.13}$$

or

$$\int l^3 f(l) dl \propto l^{4/3} \quad \Rightarrow f(l) = c l^{-8/3} \tag{12.14}$$

Using the fact that the fraction of block sizes (eddies) between L and η is one, i.e,

$$\int_\eta^L f(l) dl = 1 \tag{12.15}$$

allows the determination of c in Eq. 12.14. Substituting Eq. 12.14 into 12.15 and solving for c gives

$$c = \frac{5}{3} \frac{1}{\eta^{-5/3} - L^{-5/3}} \tag{12.16}$$

or

$$f(l) = \frac{5}{3} \frac{l^{-8/3}}{\eta^{-5/3} - L^{-5/3}} \tag{12.17}$$

Eq. 12.17 specifies $f(l)$. The value of λ can now be determined by equating Eq. refdtint with 12.12:

$$\begin{aligned} \nu Re_L &= \int_\eta^L \frac{2}{27} \lambda l^3 \frac{l^{-8/3}}{\eta^{-5/3} - L^{-5/3}} dl \\ &= \frac{\lambda 5}{54} \frac{\mathrm{L}^{4/3} - \eta^{4/3}}{\eta^{-5/3} - L^{-5/3}} \end{aligned} \tag{12.18}$$

Solving for λ:

$$\lambda = \frac{54}{5} \frac{\nu Re_L}{L^3} \frac{(L/\eta)^{5/3} - 1}{1 - (\eta/L)^{4/3}} \tag{12.19}$$

For high Reynolds number flow, $L \gg \eta$. The leading order approximation to Eq. 12.19 is

$$\lambda = \frac{54}{5} \frac{\nu Re_L}{L^3} \left(\frac{L}{\eta} \right)^{5/3} \tag{12.20}$$

Note that all order one constants that appear in the previous scaling relations are set equal to one. This completes the basic description of the model.

The model applications to date have strictly been as a one-dimensional description of the mixing process. This allows one to resolve *all* relevant length and time scales for flows of practical interest. The model was originally developed to investigate the qualitative mixing properties of turbulent flows. Some important qualities of this model stem from the fact that the basic physics of turbulent mixing are explicitly incorporated. Molecular diffusion is treated exactly, and turbulent convection is modeled in a physically reasonable way by the rearrangement event, or eddy turn-over events. The distinction between molecular diffusion and turbulent convection, even at

the subgrid scales is crucial to the accurate description of the species field, especially when chemical reactions or nonlinear microphysics are involved. This distinction is not made in most turbulent mixing models. This model shares a property with some other commonly used mixing models such as coalescence-dispersion models, in that the hydrodynamic field is assumed to be specified in advance. In other words, the flow field structure is an input to the model, rather than a result of prediction. Even with this specification, the prediction concerning the mixing of a scalar variable transported in the turbulent flow field have been difficult to achieve with existing methods. However, the linear eddy model has the distinction in that the effects of molecular diffusion and turbulent transport are treated separately and explicitly.

Another important aspect of the linear eddy model is that in its implementation as described below, it contains no adjustable parameters. All order unity factors in the scaling relations have been set to one. (Although these can be adjusted to give the best quantitative fit to data.)

The linear eddy model has recently been generalized to treat a variety of flow configurations. Below we discuss how the model is implemented and illustrate its application in studying mixing in various flow configurations.

12.3 Implementation of Linear Eddy

The total number of computational elements along a line must be chosen to resolve the largest and smallest scales in the flow. We will assume that the domain under consideration is on the order of the integral scale, L. From Kolmogorov scaling the ratio of the largest to smallest length scales in the flow is approximately $L/\eta = (Re)^{3/4}$. If the Reynolds number is 10^4, then this ratio is 1000. By taking six computational elements to resolve the eddies at the Kolmogorov scale (by use of the triplet map), then at least 6000 elements are needed to resolve the scalar distribution. Let us chose an initial scalar field that is equal to 1 in half of the domain, and 0 in the other half of the domain. The rate of inversion events is given by λL where L is the domain size. For this example we will also assume periodic boundary conditions in this linear domain. With time discretized according to the fastest time scales of the flow (remember we want to explicitly account for turbulent convection by the rearrangement process for all scales), diffusion is implemented by regularly advancing Eq. 12.4 numerically.

To implement the triplet map, a location for inversion is randomly selected within the domain. The block size is also randomly chosen, but in such a way as to satisfy the probability distribution given by $f(l)$. Inversion takes place at intervals determined by λL. This process is repeated until a desired time has elapsed.

To satisfy the given distribution for l, the block size is chosen as follows: First we form the cumulative of l. The cumulative, denoted by $F(l)$, is the probability that a given block size (eddy) will have a linear dimension less than l. Obviously, $F(\eta) = 0$ since η is the smallest length scale in the flow, and $F(L) = 1$ since all eddies have

$l \leq L$. The cumulative is given by

$$\begin{aligned} F(l) = \int_\eta^l f(l)dl &= \frac{5}{3}\frac{1}{\eta^{-5/3} - L^{-5/3}} \int_\eta^l l^{-8/3} dl \\ &= \frac{1}{L^{-5/3} - \eta^{-5/3}} \left(l^{-5/3} - \eta^{-5/3} \right) \end{aligned} \tag{12.21}$$

Solving for l:

$$l = \left[F(l) \left(L^{-5/3} - \eta^{-5/3} \right) + \eta^{-5/3} \right]^{-3/5} \tag{12.22}$$

Numerically this is implemented by selecting a random number uniformly distributed in $x \in [0, 1]$ and using this as a value for $F(l)$. l is then computed from Eq. 12.22. Choosing a large quantity of random numbers for $F(l)$ and using them to determine l will give the proper distribution for l.

The complete model is implemented as a Monte-Carlo simulation of many individual flow field realizations. The statistics are then computed by averaging over the different realizations. The accuracy of the statistics will of course increase as the number of realizations is increased.

12.4 Applications of Linear Eddy

The model has been successfully implemented in a number of different configurations including grid turbulence, planar mixing layers, and axisymmetric jets. By varying the spatial domain over which the inversion events occur and changing the model inputs (Reynolds number and diffusion coefficient) the mechanisms of mixing in the various configurations can be studied.

The first application of linear eddy was to study mixing in grid turbulence. The laboratory equivalent of this simulation has been provided by Warhaft [4] who used a single heated wire to provide a heat source. The downstream turbulent mixing and measurement of the temperature statistics were the objective of his experiments. In the linear eddy simulations, the initial conditions were arranged by setting the value of the scalar at one grid point to 100 (the point source), and zero else where. Many of the features of the downstream distribution were predicted well. An interesting observation of this experiment (and simulation) is that the peak rms scalar fluctuation did not appear along the centerline of the flow (with respect to the heat source). The lower fluctuations along the centerline where shown by Kerstein to be due to the enhanced mixing due to "eddy-diffusion" which is accounted for in the linear eddy model.

The prediction of the linear eddy model in the early stages of development were less successful. This was partly attributed to the discontinuous nature of the fluid motion, necessitated by the one-dimensional model. Never the less, the linear eddy formulation provides a physically sound description of turbulent mixing. Namely, molecular diffusion is accounted for explicitly, turbulent mixing is treated by rearrangements (triplet map), and by limiting application to one dimension, all relevant

length and time scales can be resolved. Furthermore, the model contains no adjustable parameters. Order one coefficients that appear in the scaling relations are set equal to one. The purpose of this is at present to study the qualitative features of turbulent mixing and to provide a mechanistically sound description of turbulent mixing. In future applications, linear eddy type models will likely find application as a closure to Navier-Stokes based equations used in predicting turbulent mixing and reaction in engineering application. In such uses, adjustments of constants may be necessary to give correct quantitative description.

A second application of the linear eddy model was in studying shear layer mixing. Experimental observations show the thickness of a planar mixing layer (δ) grows linearly in the stream wise direction. The growth of the mixing layer is commonly characterized by a spreading angle, α, such that $\alpha = \tan^{-1}\delta/x$. The linear eddy calculation is performed along a transverse line who's spatial extent then grows linearly with time. (This spatial growth is an input to the model, not a prediction from the model - it is determined from the development of the hydrodynamic field.) Another important observations about shear layer mixing is that the entrainment ratio, E, is not equal to one. That is, the amount of fluid entrained into the shear layer from each of the two streams depends on the velocity difference between them. Within the linear eddy model, the boundaries of the flow in the transverse direction are specified such that $Y_1 + Y_2 = \delta$ and $Y_1/Y_2 = E$.

A distinctive character of mixing in shear layers is the development of a preferred mixed fluid concentration that is independent of the spatial location within the mixing layer (we have discussed this in lecture previously). The mechanisms of mixing by which this peak occurs was explained in the previous section. The linear eddy model has been successful in reproducing this character, and in providing a means of understanding the differences in the pdf concentrations that develop for different flows and similar flows with different transport coefficients.

The linear eddy model can also be extended in a straight forward manner to account for chemical reactions. In this case a number of scalar values corresponding to the different reacting species and their reaction products must be specified. For the reaction $A + B \rightarrow C$, the evolution of the scalars A, B, and, if desired, C. Within each linear eddy cell the species A will evolve governed by:

$$\frac{\partial A}{\partial t} = -kAB + \frac{\partial^2 A}{\partial x^2} \tag{11.23}$$

k is a reaction rate coefficient with units $(\frac{1}{t})$. Turbulent mixing is again accounted for by random rearrangements. An additional input in the reacting case is the Damköhler number, Da, a nondimensional number characterizing the ratio of the reaction frequency to the mixing frequency. The mixing frequency based on the large scale t_L is U/L, so $Da = kL/U$.

Subsequent to these first two applications, the model has been applied in a wide variety of applications including radial and axial descriptions of reacting and non-reacting jets [5, 6, 7], and homogeneous turbulence with and without mean scalar

gradient [2, 8, 9]. The model has reproduced most of the known spectral scaling properties of the scalar field as well as reproduced statistical properties of the flow as measured from both experiments and direct numerical simulation.

Although the model has been mainly used as a tool to study physics of turbulent mixing,[1] it is now being used in works to make predictions and provide design guidelines for real engineering problems [7]. Projects include atmospheric mixing with application to droplet formation, emission production from combustion processes, in particular NO production in hydrogen-air combustion, and pollutant formation in turbulent plumes (smoke stacks). It is interesting to note that these applications all involve microphysics or chemical reactions where the rate of mixing is crucial to the determine the physics of chemistry of the process. Also in these applications, many of the unwanted byproducts (pollution) involve chemical reactions that are slower than the energy releasing reactions, and don't participate much in the overall energetics of the process. In cases where the details of the chemical rates are important, all existing mixing models fail to provide accurate predictions. This is one area in which the linear eddy model is expected to make some significant contributions.

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[1] This statement reveals a feature that sets the linear eddy model apart from all other mixing models. The fact that the model incorporates enough of the appropriate physics of the flow to be used as a tool to *study* physics, as opposed to give predictions is a unique feature and tremendous asset of this model.

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