VANIER COLLEGE

ÉPREUVE SYNTHÈSE DE PROGRAMME (ESP)

Computing Solutions of Pendulum Systems

Author:
Simon Foo
Rakeeb Hossain

Presented to:
Sandy Mak
Ivan Ivanov

DIFFERENTIAL EQUATIONS - PROBABILITY AND STATISTIC

May 2021

Abstract

The purpose of this research is to further understand the pendulum system. Since pendulums are unsolvable differential equations, we are challenged to understand it using different approaches, such as simple harmonic motion, phase portraits, and Python simulations. These different approaches all bring something to the table to better understand the system. Also, since we used python, our workload was greatly reduced from time in computation. When looking at the probability aspect of the pendulum systems, different distributions are explored in order to further understand its motion. In addition, noise can be introduced in these systems in order to get a better understanding of the chaos and randomness which are seen in pendulums. Due to the consequence of noise, double pendulums should be unpredictable since probability cannot be applied in that specific situation.

Contents

Chapter 1

Introduction

No doubt that you have all seen the video ”Trust in Physics” on YouTube, if you have not or you do not remember, refer to Appendix A. Anyway, the video is about a physics professor, Prof. Walter Lewin, putting his life on the line to show his faith in physics and the math behind it to his students. Prof. Walter Lewin used a single pendulum, a common system, in his demonstration to showcase the law of conservation of energy. He told his student that if he let goes of the bob right at his chin with zero initial velocity, the bob would return below his chin. Obviously in a vacuum, the pendulum would go back to where it started, however with air resistance, some energy is lost and dissipated into the room.

The purpose of this research paper is to better understand the mechanics of pendulum systems without the need to solve its system of differential equations. Often, the goal of finding an analytic solution for a differential equation is to better understand and allow us to make efficient computation. However, we are able to skip the solution step and still able to build understanding and make computations through the use of computers. Accurately, we are going to use python to do the plotting and computation in order to better grasp the pendulum system. This allows us to make changes to initial values without burning too much time. The exponential time saved from computer computation also allows us to notice the changes in the system by varying initial values. We can explore on the basics of the chaos theory, where the slightest changes in the starting state could result in drastic changes in motion.

The chaos theory can be applied when looking at the probability portion of pendulum systems. The randomness in these systems will be explored in order to fully comprehend the idea of chaos, along with probability distributions such as the exponential distribution and the Poisson random variable distribution, and a quick peek at angular distribution. The pendulums' behaviours will also be examined when exposed to noise in the form of air current hitting the pendulum during its motion.

Chapter 2

Single Pendulum

2.1 Simple Harmonic Motion

One way of studying a single pendulum's motion is through Simple Harmonic Motion. In SHM, a pendulum's angle is a function of time, amplitude, and frequency. Therefore, $\theta(t)$ is determined by:

$$\theta(t) = \theta_0 cos(wt) \tag{2.1}$$

In the equation above, θ_0 is the initial angle of the pendulum, which is also the maximum angle. Additionally, the frequency (w) is equal to the square root of gravity over length:

$$w = \sqrt{\frac{g}{L}} \tag{2.2}$$

However, a problem surfaces when trying to fully understand pendulum motion. It prevents accurate calculation when the angles are too wide. Therefore, it can only be used for close proximity of angles. Thus, we need to find a different approach of understanding the pendulum Applying Differential Equation
Let x be the arc length between the equilibrium and the position of the mass. Using geometry, we find that:

$$x = L\theta \ (given\ \theta\ is\ in\ radian) \tag{2.3}$$

$$x'' = L\theta'' \tag{2.4}$$

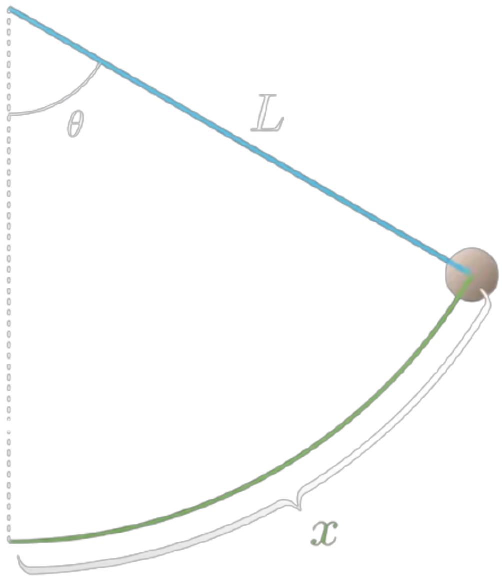


Figure 2.1: Pendulum

Using the figure below, knowing that both angles are equal, we can find the pendulum's acceleration due to gravity.

$$x'' = -gsin(\theta) \tag{2.5}$$

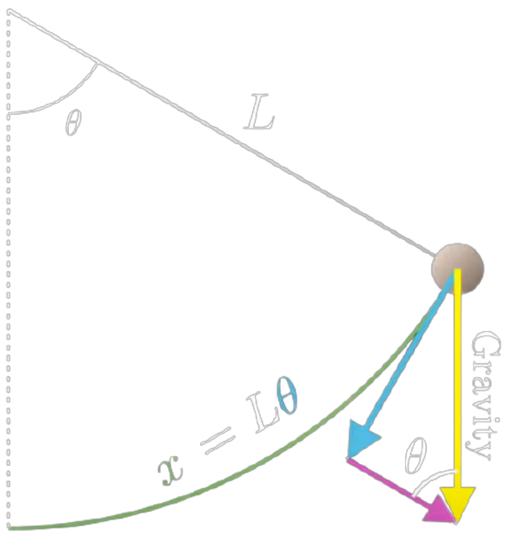


Figure 2.2: Pendulum

We can double check the equation by inserting initial values at:

$$\theta = \frac{\pi}{2},\ \theta = 0 \tag{2.6}$$

$$x''(\frac{\pi}{2}) = -g,\ x''(0) = 0 \tag{2.7}$$

Since the length is perpendicular to gravity, there is no tension pulling the mass sideways. Hence, it makes sense that at $\theta = \frac{\pi}{2}$, the force acting on the pendulum equals to the gravity. Additionally, it is reasonable that gravity has no effect on the mass at equilibrium. Lastly, to solve for angular acceleration, we substitute in $L\theta''$. Note that in the real world, there will be air friction against velocity. For the sake of simplicity we will start by neglecting air resistance, meaning the system is in a vacuum.

$$x'' = -gsin(\theta) \tag{2.8}$$

$$L\theta'' = -gsin(\theta) \tag{2.9}$$

$$\theta'' = \frac{-g}{L}sin(\theta) \tag{2.10}$$

2.2 Setting up Simulation

To visualize how the function behave, we will separate this second order differential equation into first order differential equations. As a result, we end up with a matrix system of two first order differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta(t)\\ \theta'(t)\end{pmatrix} = \begin{pmatrix}\theta'(t)\\ \theta''(t)\end{pmatrix} = \begin{pmatrix}\theta'(t)\\ -\frac{g}{L}sin(\theta(t))\end{pmatrix}. \tag{2.11}$$

Then if we are to put in the system of equation into a python code, we can then plot a phase portrait graph of θ' in function of θ. In the phase portrait, we can set initial values to show a few solution curve. (Using g = 9.8 m/s^2 and Length = 15m)

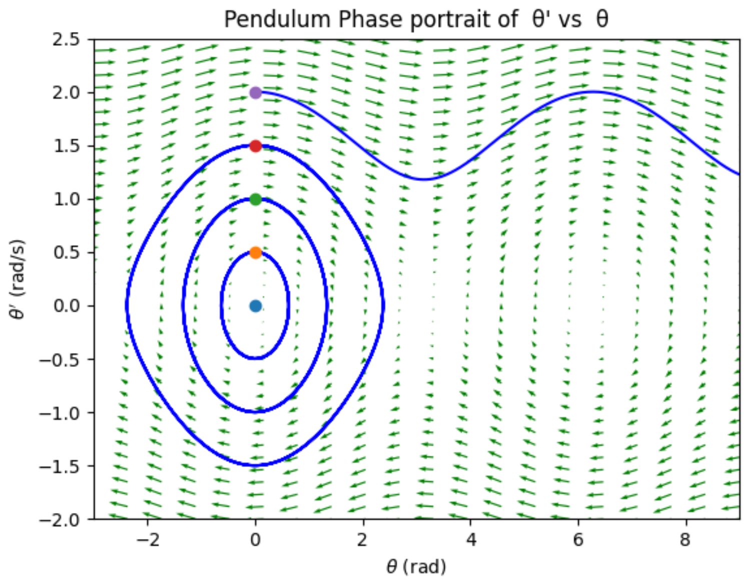


Figure 2.3: Phase Portrait of a pendulum system in vacuum

2.3 Understanding using Phase Portraits

The first law of thermodynamics, the law of conservation of energy, tells us that energy cannot be created or destroyed in an isolated system, which means that the pendulum will keep swinging with the initial energy given into the system. Using the initial conditions in the graph above $(\theta, \theta') = (0, 0), (0, 0.5), (0, 1), (0, 1.5), (0, 2)$, we can get an idea of a pendulum's behavior. Assuming that positive velocity means that the pendulum is pointing right and negative is point left, most solution loops around the origin. Notice that the higher the solution curve starts, the larger its path is going to be around the origin. In terms of real movements, a full rotation around the origin in the phase portrait equates to one period. Recall that $w = \sqrt{\frac{g}{L}}$, since frequency is not determined by initial angular velocity nor initial angle, we can conclude that each solution curve does a full rotation at the same time.

In addition, observe the curve starting from the purple point(we will call it y_5) and how it differs from the others. Unlike the other curves, y_5 does not spin around the origin, but fluctuates in the region of $\theta' = 1$ to $\theta' = 2$. In fact, y_5 never settles around the origin and fluctuates continuously.

The sinusoidal curve y_5 is in fact a representation of a pendulum doing full rotation. As a matter of fact, if the pendulum has enough energy to do a full rotation, it will continuously spin around the pendulum's origin. Therefore, we can assume that there is a critical value that exists between both cases.

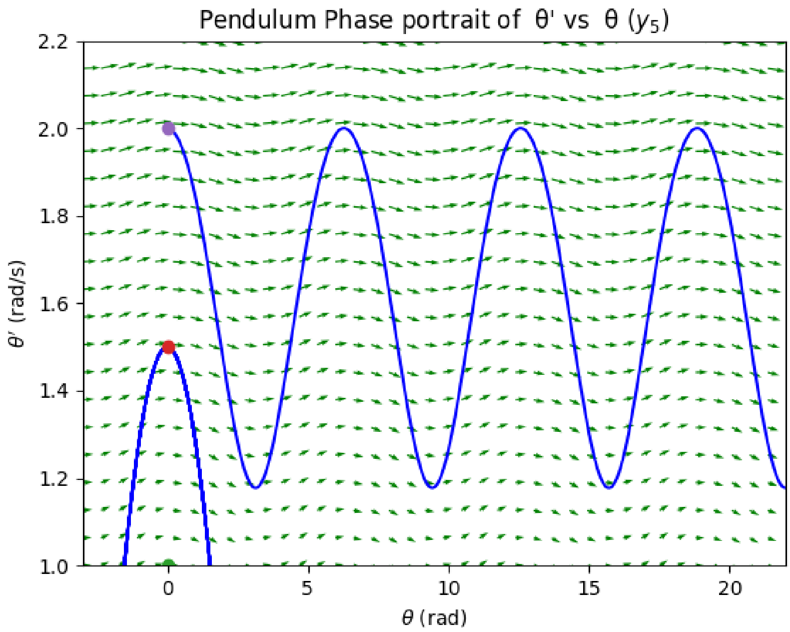
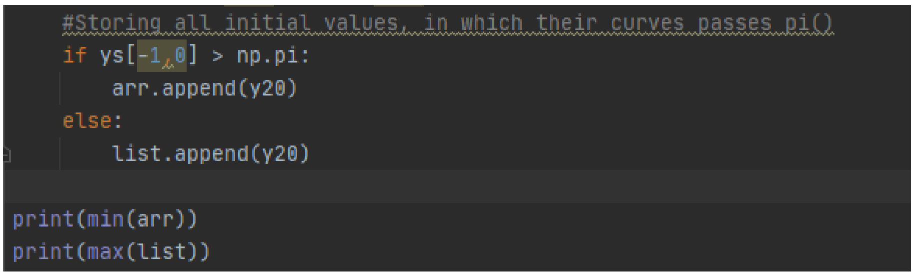


Figure 2.4: y_5 Phase Portrait

Knowing that a full rotation is equal to 2π, half a rotation is π which it is also the critical value if angular velocity is 0. Thus, we can find an estimate critical value using python. To solve for the critical value, we can use brute force, meaning that we set thousands of initial points and figure out which curves will go over $\theta = \pi$ and which will not. Then, we store all initial values in two separate arrays and find the lowest one and the highest one.. As a result, we have 2 values that bounds the critical value. Therefore, we have a general approximation of the critical value using the range.



```
    #Storing all initial values, in which their curves passes pi()
    if ys[-1,0] > np.pi:
        arr.append(y20)
    else:
        list.append(y20)

print(min(arr))
print(max(list))
```

Figure 2.5: Computing Critical Value

Using $\theta = 0$ as initial value, we find that the critical value of angular velocity is between $\theta_0' = 1.61658 rad/s$ and $\theta_0' = 1.61659 rad/s$. The critical value represent a system where the pendulum is sitting perfectly still on top of the origin while starting at $\theta = 0$. If we reflect more about the point $(\pi, 0)$ in Figure 1.3, we can

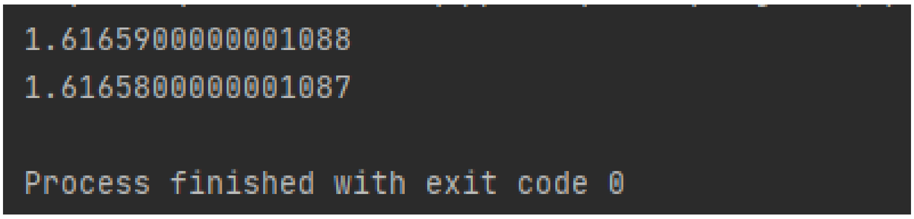


Figure 2.6: Critical value Output

consider it as a source. Think of a pendulum pointing upward, the slightest touch would make it fall over and never return to the point.

2.4 Applying Probability

When applying the knowledge of probability on a single pendulum system, there is a recurrence of chaos and random behaviour. This can occur due the slightest discrepancy in the motion of the pendulum from the initial position which can differ the result of the final position. This characterization can range in several hierarchies of randomness which will be explored when comparing single and double pendulums. These probability distributions can be used to demonstrate a pendulum's chaotic behaviour and to show that noise is induced randomly.

2.5 Introducing air resistance

For an accurate simulation, we will introduce a wind resistance constant (μ). This damping is a scalar of angular velocity(θ'). Thus, our pendulum equation with damping is

$$\theta''(t) = -\mu\theta'(t) - \frac{g}{L}sin(\theta(t)). \tag{2.12}$$

In order to visualize pendulum's movement, we will separate this second order differential equation into first order differential equations. As a result, we end up with a matrix system of two first order differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta(t)\\ \theta'(t)\end{pmatrix} = \begin{pmatrix}\theta'(t)\\ \theta''(t)\end{pmatrix} = \begin{pmatrix}\theta'(t)\\ -\mu\theta'(t) - \frac{g}{L}sin(\theta(t))\end{pmatrix}. \tag{2.13}$$

Now using python to plot the phase portrait, we obtain:

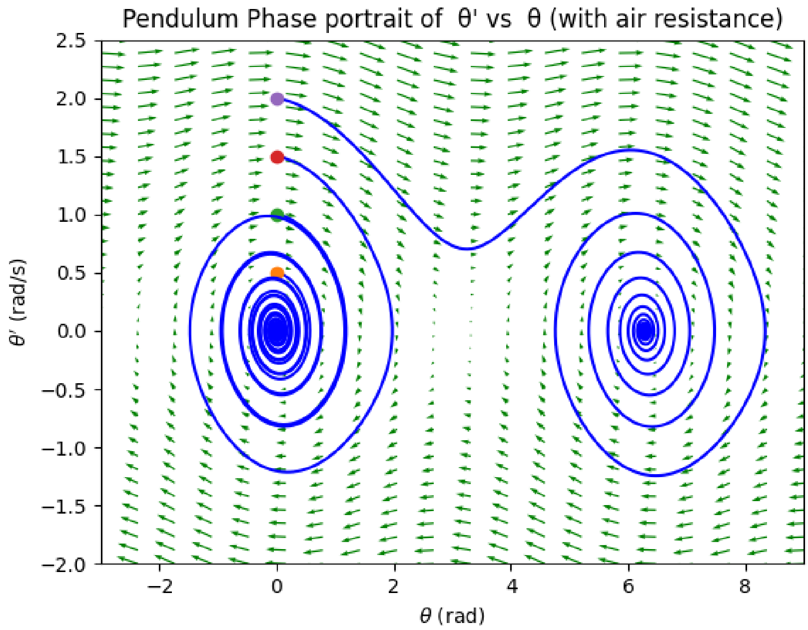


Figure 2.7: Phase Portrait of a Pendulum system with air resistance

With the introduction of air resistance, the pendulum loses energy and comes to rest. Consequently, the point $(0, 0)$ and $(2\pi, 0)$ becomes a sink from the original origin of an ellipse. Also notice that y_5 no longer fluctuates and settles at $\theta = 2\pi$ after a complete rotation.
Moreover, using the same code that found the critical value range in a vacuum, we can obtain the bounds of a damped system containing the critical value ($\theta_0' = 1.8212rad/s$ to $\theta_0' = 1.8211rad/s$).

2.6 Chaos Theory

According to Mark Berliner's paper "Statistics, Probability and Chaos", the chaos theory is, "associated with complex, "random" behavior and forms of unpredictability. Mathematical models and definitions associated with chaos are reviewed. The relationship between the mathematics of chaos and probabilistic notions, including ergodic theory and uncertainty modeling, are emphasized." In simpler words, the chaos theory is the behaviour of unpredictable randomness which can be seen in pendulums, but mostly in double pendulums due to its unpredictable characteristics.

Refer to simulation SimplePendulum.py, and alter any value such as the initial position or angle which will result in a different pattern on the pendulum's movement.

2.7 Probability Distribution

Through probability distributions, we can also find angular displacement in pendulums. Using these distributions, we can find the number of times for which the pendulum returns to an original position. In this scenario, a two state Bernoulli system can be used to display the return time distribution in this chaotic system.

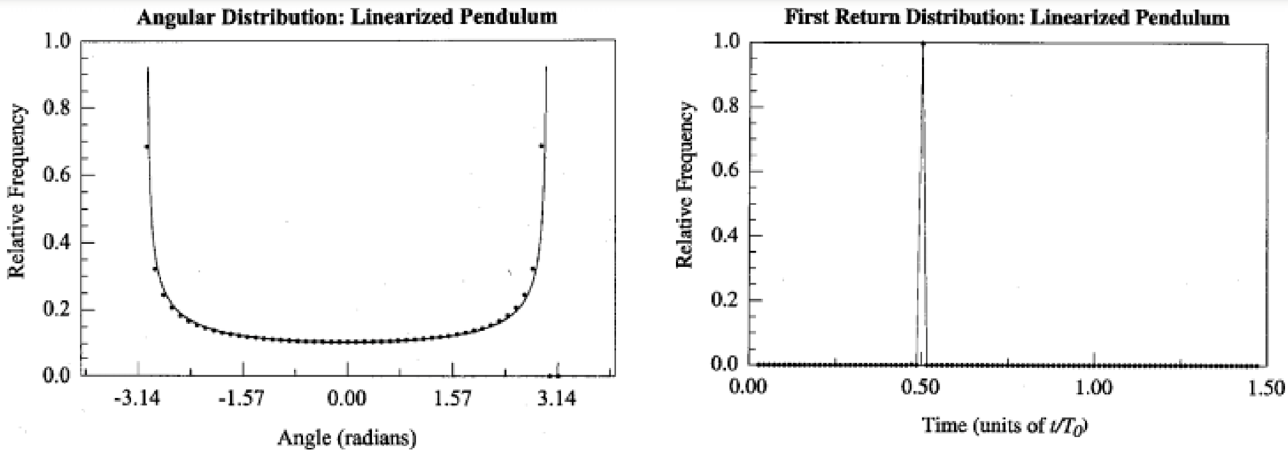


Figure 2.8: Angular Distribution of a single pendulum

The motion of a single pendulum's may be written as:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2 sin\theta = 0$$

$$\theta(0) = \theta_0$$

$$\omega(0) = 0.$$

The amount of time dt spent in d is given by the following equation:

$$dt = \left|\frac{\mathrm{d}t}{\mathrm{d}\theta}\right| d\theta = \left|\frac{1}{\frac{\mathrm{d}\theta}{\mathrm{d}t}}\right| d\theta = \frac{d\theta}{w_0\sqrt{\theta_0^2 - \theta^2}}$$

This expression can be seen as a probability density function. This can only occur if we assume that the amount of the time the pendulum spends at a certain angular position is directly proportional to the probability of that system being in that interval. Thus, for

$$\int_{-\pi}^{\pi} P(\theta) d\theta = 1$$

the function can be written as:

$$P(\theta) = \frac{1}{\pi\sqrt{\theta_0^2 - \theta^2}}.$$

If the function is examined, the only variable that is not constant is the initial and final angle at which the pendulum swings to and from. Knowing this information, it can be said that the results are controlled by the user, therefore making it predictable. Thus, it can be said that the probability density function only indicates the time the pendulum remains at a certain angle interval, and not the probability that the pendulum will follow the given interval, since it is guaranteed to do so.

2.8 Introducing Noise

Another way to introduce probability into this scenario is to bring forward the idea of noise. When noise is introduced in the dynamic equations of the pendulum, random variables are added to the scenario. In our case, it will be air current. Doing so creates random jolts and displacements which interrupts the pendulum's movement. When looking at the displacement of the pendulum's angle:

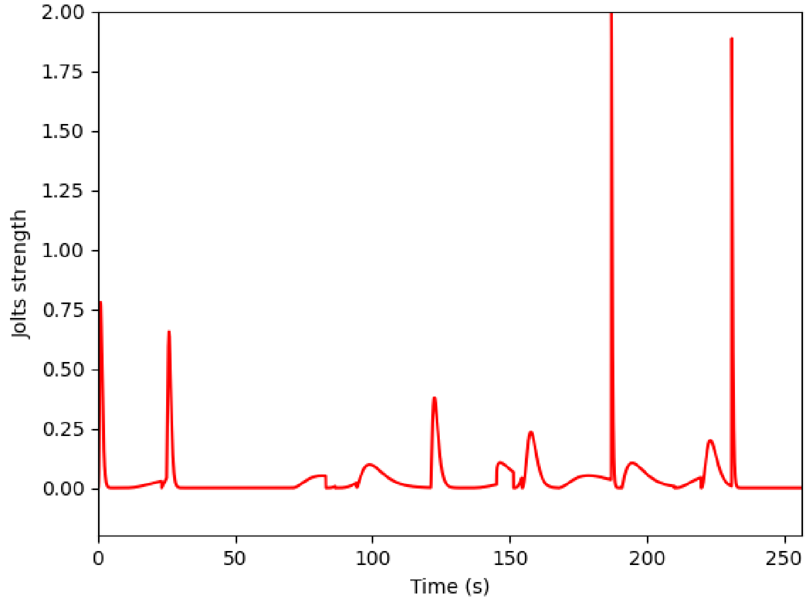


Figure 2.9: Random air current graph for a single pendulum (Jolts)

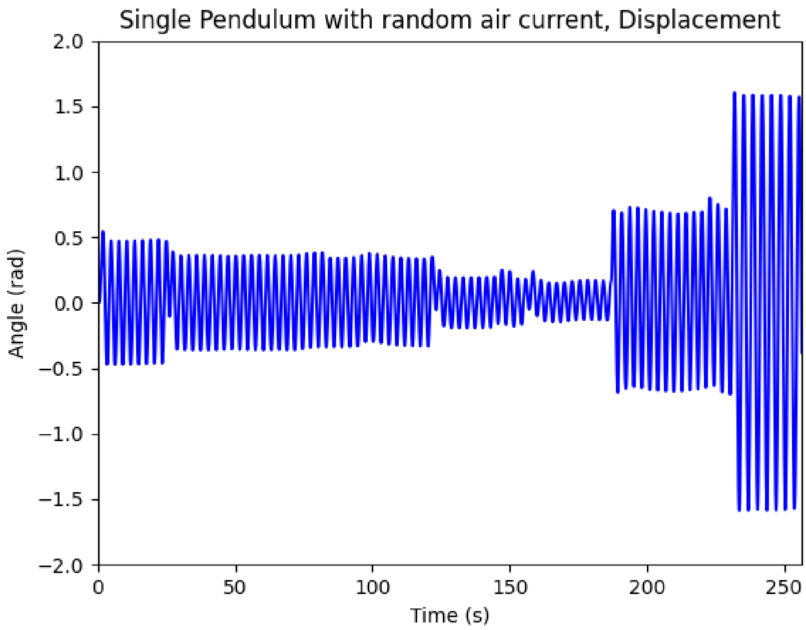


Figure 2.10: Random air current graph for a single pendulum (Displacement)

Notice that the pendulum's angular displacement varies after every jolt. Frequently after jolt, the system gains energy and travels larger distances, however we can see in the graph that random jolts actually reduce the total energy in the system. Refer to the timestamp of t = 125s, we see a small jolt on the jolts graph that translates into smaller amplitude in the displacement graph. Then, refer to around 175s, where we see a massive air current hitting the system, unlike the previous jolt, this one increased the systems total energy, thus increasing the amplitude. The cause of both cases can be explained by the angular velocity at the moment.

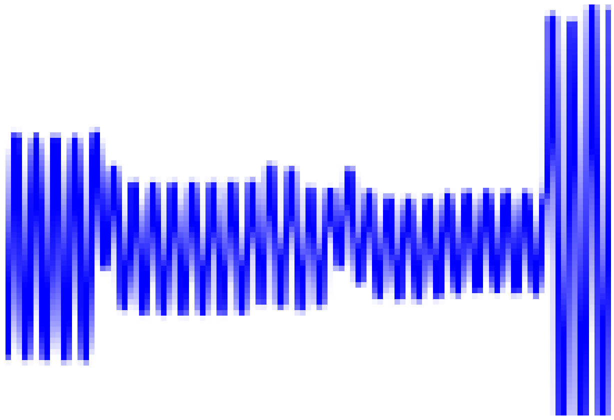


Figure 2.11: Closer view on the jolts

From the graph, we can see that the pendulum is interrupted mid-period. The first jolt on the left is a representation of the first case where the air current oppose the angular velocity. However, it is not strong enough to push it further to increase the maximum angle. The second case is seen on the last jolt, where the pendulum is getting ready to turn direction, then it gets hit by a massive air current. Consequently, the air current increased the maximum angle of the system.

2.9 Simulation of a Pendulum system

2.9.1 ODEINT

First, in order to simulate a pendulum using python, we would use the function odeint from the scipy.integrate library. odeint is a linear multistep method that maps out the solutions to a differential equation. A multistep method differs from for example, Eulor's single-step method, by storing the previous value and calculating from there rather than starting from scratch at each step. Using a multistep method to compute solutions to a differential equations is comparable to dynamic program-

ming.
Moreover, by importing the scipy library, we import different linear multistep method such as fourth order Runge-Kutta. In fact, the scipy library uses RK23, Radau IIA's Runge-Kutta, backward-differentiation formulas, LSODA... For the odeint function, we will focus on the LSODA method. LSODA is a combination of Adams-Bashforth methods and Backward Differential Formulas with the addition of automatic stiffness detection for the inputted differential equation. Stiffness is a measure of how efficiently a differential equation can be solved. For instance, a stiff differential equation is stiff if a solution is changing slowly at a particular place, but the surrounding solutions vary rapidly, therefore the steps needed to compute all solutions would vary too much for the same precision.

2.9.2 Initial Values

In the simulation SimplePendulum.py, we used scipy.integrate's odeint to simulate a moving pendulum. Using the simulation, we can vary the initial conditions and see how it affects the behavior. First, by varying the initial angle, we see that maximum angle changes, however the period stays constant if θ_0 is small. Same with initial angular velocity, it defines the maximum angle along with initial angle. Furthermore, when we increase gravitational acceleration, we the pendulum speed up and completing a full period in shorter period of time. Whilst if we increase pendulum length, we see it slow the period down. Lastly, by increasing the air resistance constant, we see the pendulum slowly settle down to $\theta = 0$.

2.10 Types of Distributions

2.10.1 Exponential Distribution

The chaos in pendulums can be explained through the exponential distribution since the motion's path keep changing due to the air current. Therefore, we can say that they are at different orbits, for different periods of times, and at different angles; thus, we can that the motion is unstable, sometimes more in one orbit than the other. Because it is unstable, the motion of irregular random spikes can have an exponential distribution. This can be written as:

$$f(x; \lambda) =$$

$$\begin{cases} \lambda^{-\lambda} & 0 \leq x \\ 0 & x \leq 0 \end{cases}$$

where $f(x;\lambda)$ is the probability density function, is the rate, and x is the random variable.

The exponential distribution is used in pendulums to analyze the distance between each jolt as Poisson. Thus, it adds to the random behaviour the noise brings forward in the pendulum system.

2.10.2 Poisson Distribution

As seen in the graph, the pendulum experiences random jolts when hit by the current of wind. This can be explained through the Poisson Distribution and its process. According to towardsdatascience.com, a Poisson Distribution is a discrete probability distribution that finds the probability of the number of events which can occur in a certain time period, or the waiting time between two random events. This distribution has a probability mass function that gives the probability of observing x events in a certain period of given time and an approximate number of events can be written as:

According to towardsdatascience.com, "A Poisson Process is a model for a series of discrete events where the average time between events is known, but the exact timing of events is random." This can be used in a pendulum system since the random jolts of air current is unknown and random. There are certain conditions which must be met in order to see if pendulums respect the Poisson Process:

1. Events are independent of each other. Thus, the probabilities of multiple events do not affect each other's.

2. The average rate (λ) is constant.

3. Events cannot occur at the same time, meaning they must be successive.

Because the pendulum gets hit with multiple events of air currents that are independent from each too other and occur at different times, we can say that conditions 1 and 3 are met. The average rate is also constant because the number of events in each period does not change. The use of this process demonstrates that the timing between the jolts are random and irregular due to the independence of the events.

This can be used to demonstrate the time elapsed between the random events through the exponential random variable.

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

As mentioned previously, x is the events in a certain time period, and is the rate of events per time.

Chapter 3

Double Pendulum

3.1 Initializing a Double Pendulum

3.1.1 Position, Velocity and Acceleration

To fully grasp the functioning of a double pendulum, we must first start from scratch. Therefore, we will assign a few initial variables for the position of the tip of both pendulums.By using trigonometry, we find that the position of the first pendulum is a dependent variable of pendulum length and its angle compared to the y-axis. Therefore, we can conclude that:

$$x_1 = L_1 sin(\theta_1) \tag{3.1}$$

$$y_1 = L_1 cos(\theta_1) \tag{3.2}$$

Furthermore, to find the position of the second pendulum, we simply add the position of the first pendulum on top of the second one. Then we obtain:

$$x_2 = x_1 + L_2 sin(\theta_1) \tag{3.3}$$

$$y_2 = x_2 + L_2 cos(\theta_1) \tag{3.4}$$

Note that y_1 and y_2 are negative values, since a pendulum's stable equilibrium state is straight below its pivot, assuming that the pivot is the origin.
We differentiate our position function to obtain our velocity functions which are:

$$x_1' = L_1 cos(\theta_1)(\theta_1') \tag{3.5}$$

$$y_1' = L_1 sin(\theta_1)(\theta_1') \tag{3.6}$$

$$x_2' = x_1' + L_2cos(\theta_2)(\theta_2') \tag{3.7}$$

$$y_2' = y_1' + L_2sin(\theta_2)(\theta_2') \tag{3.8}$$

Thus, the acceleration functions are:

$$x_1'' = L_1(\theta_1''cos(\theta_1) - \theta_1'^2 sin(\theta_1)) \tag{3.9}$$

$$y_1'' = L_1(\theta_1''sin(\theta_1) + \theta_1'^2 cos(\theta_1)) \tag{3.10}$$

$$x_2'' = x_1'' + L_2(\theta_2''cos(\theta_2) - \theta_2'^2 sin(\theta_2)) \tag{3.11}$$

$$y_2'' = y_1'' + L_2(\theta_2''sin(\theta_2) + \theta_2'^2 cos(\theta_2)) \tag{3.12}$$

3.1.2 Force and Tension

Treating the double pendulum system in a free body diagram, we can set a few more initial values. A double pendulum is composed of two mass, m_1 and m_2, each separated by a distance of L_1 and L_2. Furthermore, both mass experiences constant gravitational pull(g). Newton's third law, action and reaction, tells us that there will a force countering gravity's force, in our case it is Tension(T). Each rod holding a mass has a Tension pointing against gravity and always parallel to the rod.
From the figure above, we can first find the tension equation for the second rod using Newton's second law (F = ma). Knowing that tension is a force meant to counter the gravitational pull, we then know that

$$m_2y_2'' = T_2cos(\theta_2) - m_2g \tag{3.13}$$

$$m_2x_2'' = -T_2sin(\theta_2). \tag{3.14}$$

Further reflection on the fact that tension is a counteractive force and knowing that T_2 and m_1 will pull on T_1, we find that

$$m_1y_1'' = T_1cos(\theta_1) - T_2cos(\theta_2) - m_1g \tag{3.15}$$

$$m_2x_2'' = -T_1sin(\theta_1) + T_2sin(\theta_2). \tag{3.16}$$

Finally, using equation (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) and (2.16), we solve for θ_1'' and θ_2'' in terms of θ and θ'' obtaining:

$$\theta_1'' = \frac{-g(2m_1 + m_2)sin(\theta_1) - m_2gsin(\theta_1 - 2\theta_2) - 2sin(\theta_1 - \theta_2)m_2(\theta_2'^2 L_2 + \theta_1'^2 L_1cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2cos(2\theta_1 - 2\theta_2))} \tag{3.17}$$

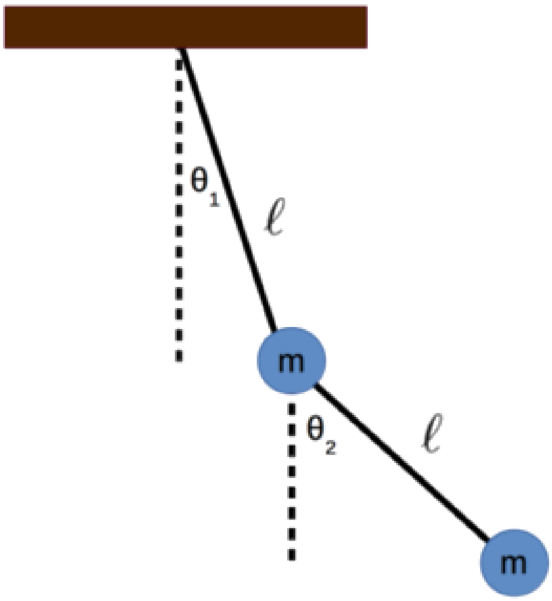


Figure 3.1: Double Pendulum Free Body Diagram
(from https://blog.cupcakephysics.com/classical20mechanics/2015/08/09/small-angle-oscillations-of-the-double-pendulum.html)

$$\theta_2'' = \frac{2sin(\theta_1 - \theta_2)(\theta_1'^2 L_1(m_1 + m_2) + g(m_1 + m_2)cos(\theta_1) + \theta_2'^2 L_2 m_2 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))} \tag{3.18}$$

3.2 Setting up Simulation

Doing the same prep work as a single pendulum, we separate the angular acceleration function into two matrix system.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta_1(t)\\ \theta_1'(t)\end{pmatrix} = \begin{pmatrix}\theta_1'(t)\\ \theta_1''(t)\end{pmatrix} = \begin{pmatrix}\theta_1'(t)\\ \theta_1''(3.17)\end{pmatrix}. \tag{3.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\theta_2(t)\\ \theta_2'(t)\end{pmatrix} = \begin{pmatrix}\theta_2'(t)\\ \theta_2''(t)\end{pmatrix} = \begin{pmatrix}\theta_2'(t)\\ \theta_2''(3.18)\end{pmatrix}. \tag{3.20}$$

3.3 Simulation of a Double Pendulum system

3.3.1 Initial Values

When looking at double pendulums, the motion becomes more complex, therefore meaning it behaves more randomly than a single system. Thus, chaotic data delivers an idea of the use of probability in an inevitable, random system.

Refer to doublependulum.py for the simulation of this chaotic event.

Although single pendulums' motions are somewhat predictable, double pendulums are a completely different case. As seen in the simulation, there are no patterns to the motion since everything is completely random and chaotic.

If noise were to be added to a double pendulum system, randomness would just be added to the already high level of chaos. This means that events are still unpredictable, with or without noise. Thus, it can be concluded that double pendulums are completely unpredictable and full of chaos, therefore meaning that probability cannot be applied.

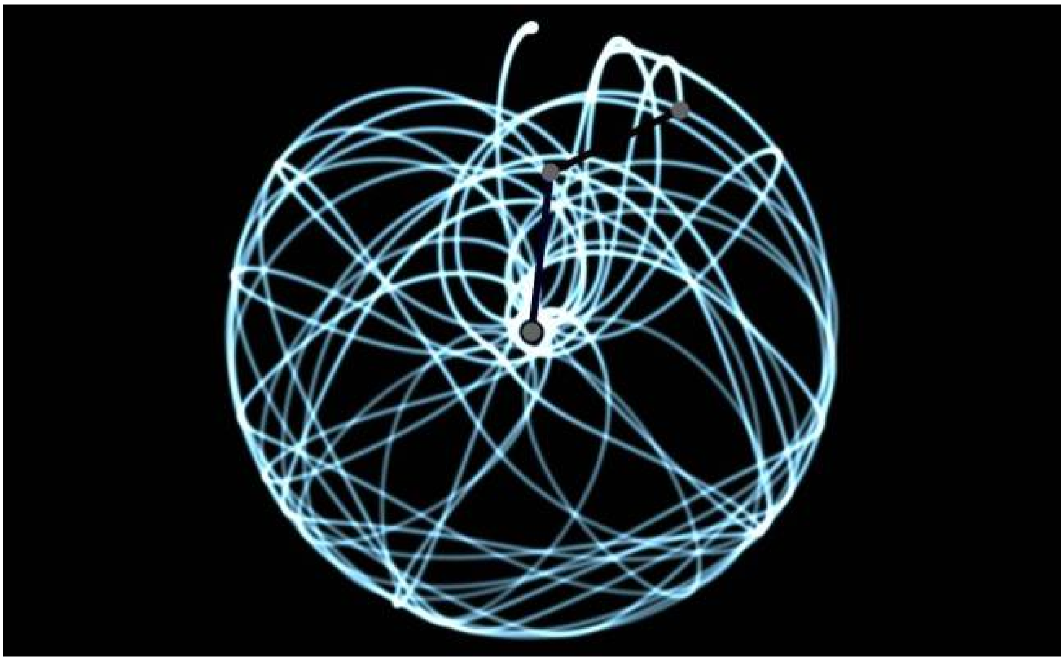


Figure 3.2: Chaos of Double Pendulum (from https://cute766.info/the-chaos-of-double-pendulum/)

Chapter 4

Conclusion

Pendulum systems is a widely studied topic in the field of physics and mathematics. Using the laws of physics and theorems in mathematics, new theory forms to create solution. That being said, examining a pendulum system or a double pendulum system through differential equation is a more challenging approach. Since it is almost impossible to solve and find a general solution describing pendulum position. Thus, some has used the close angle approximation to form a solution, known as simple harmonic motion. Studying the pendulum system through simple harmonic motion, one can predict its motion and position at any given time with great precision. Nonetheless, studying with SHM comes with its downfall, it only allows us to study the system in a limited domain. If we go beyond a small angle, the precision falls drastically. Moreover, another approach to learn about the pendulum is to use phase portraits. Using a phase portrait of angle vs angular velocity, we can notice how the general system moves by the effect of the initial velocity. We can also perceive the speed at every angle, given an initial state. This allows us to get a picture of how it spins around the origin. Furthermore, by plotting a few solution curve by inserting initial conditions, we find that two different case comes to light, one that rotate in a circular motion and the other that travels on a sinusoidal wave. The existence of two completely different case informs us that there could be a critical value that lies between both. Using python, air resistance and more phase portraits, we can conclude that the critical value that lies between both case is a source or an unstable point. Lastly, by creating a simulation, it was made possible to visualize the movement of a pendulum. Simultaneously, we were able to verify our critical value. We can confidently say that the critical value was computed correctly because in the simulation, we can see the pendulum standing vertically still for a long period of time. Nevertheless, it eventually falls back down gaining back its kinetic energy. The pendulum is not able to be completely still at the top as time approaches infinity, since we used an estimation to find the value. Meaning that there is a large room for error that is defined by the precision. Therefore, the higher the precision, the more accurate our value is going to be and the longer the pendulum will stay still.

Applying what we learned from the single pendulum, we are able to find the system of equation for a double pendulum after deriving the formula for the angular acceleration. Figuring that solving for the general solution for a double pendulum is impossible, we used the last approach to understand the double pendulum. Using a combination of linear multistep methods from scipy's odeint, we are able to simulate a double pendulum's movement.

When examining the probability portion of pendulums, different distributions such as Poisson random variable and exponential distribution were used in order to further understand the causes of chaos and randomness in the motion. It can be seen that when noise is introduced in the system, chaos is at its peak because of the increase in random behaviour caused by the air current. The motion of the system is seen through Poisson and exponential distributions since it demonstrates the effects of noise and how they are random. When looking at the double pendulum system, it was concluded that probability cannot be applied because of the heavy chaotic behaviour it contains. Even without noise, the motion of the system cannot be predicted since is truly is random. If air current were to be added, it would make the system even more unpredictable. Although multiple methods were used to examine pendulum systems, this paper only scratched the surface of the information in order to peak one's interest in the topic of pendulums, and its mathematical theory.

Chapter 5

References

[1] Neumann, Erik. Double Pendulum. myPhysicsLab, 2002.

[2] Moler, Cleve. Stiff Differential Equations. MathWorks, 2003.

[3] Langton Baker, Gregory. Probability. pendulums, and pedagogy. American Journal of Physics, 2006.

[4] Beliner, L. Mark. Statistics, Probability and Chaos. Institute of mathematical statistics, 1992.

[5] Koehrsen, Will. The Poisson Distribution and the Poisson process explained. Towards Data Science, 2019.

5.1 Appendix A

https://www.youtube.com/watch?v=xXXF2C-vrQE

5.2 Appendix B

https://github.com/sfhelmet/System-of-Pendulums-/blob/main/PhasePortrait.py
https://github.com/sfhelmet/System-of-Pendulums-/blob/main/SimplePendulum.py

This research paper was written in

LaTeX

on the IDE

Overleaf

The phase portrait and simulations was written in

python