FUZZY AGGREGATION OPERATIONS

AGGREGATION OPERATIONS

Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set. Assume, for example, that a student's performance (expressed in %) in four courses taken in a particular semester is described as high, very high, medium, and very low, and each of these linguistic labels is captured by an appropriate fuzzy set defined on the interval [0, 100]. Then, an appropriate aggregation operation would produce a meaningful expression, in terms of a single fuzzy set, of the overall performance of the student in the given semester.

Formally, any *aggregation operation* on n fuzzy sets ($n \geq 2$) is defined by a function

$$h : [0, 1]^n \to [0, 1].$$

When applied to fuzzy sets $A_1, A_2, \ldots, A_n$ defined on X, function h produces an aggregate fuzzy set A by operating on the membership grades of these sets for each $x \in X$. Thus,

$$A(x) = h(A_1(x), A_2(x), \ldots, A_n(x))$$

for each $x \in X$.

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In order to qualify as an intuitively meaningful aggregation function, h must satisfy at least the following three axiomatic requirements, which express the essence of the notion of aggregation:

Axiom h1. $h(0, 0, \ldots, 0) = 0$ and $h(1, 1, \ldots, 1) = 1$ (*boundary conditions*).

Axiom h2. For any pair $\langle a_1, a_2, \ldots, a_n \rangle$ and $\langle b_1, b_2, \ldots, b_n \rangle$ of n-tuples such that $a_i, b_i \in [0, 1]$ for all $i \in \mathbb{N}_n$, if $a_i \leq b_i$ for all $i \in \mathbb{N}_n$, then

$$h(a_1, a_2, \ldots, a_n) \leq h(b_1, b_2, \ldots, b_n);$$

that is, h is *monotonic increasing* in all its arguments.

Axiom h3. h is a *continuous* function.

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Besides these essential and easily understood requirements, aggregating operations on fuzzy sets are usually expected to satisfy two additional axiomatic requirements.

Axiom h4. h is a *symmetric* function in all its arguments; that is,

$$h(a_1, a_2, \ldots, a_n) = h(a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)})$$

for any permutation p on $\mathbb{N}_n$.

Axiom h5. h is an *idempotent* function; that is,

$$h(a, a, \ldots, a) = a$$

for all $a \in [0, 1]$.

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We can easily see that fuzzy intersections and unions qualify as aggregation operations on fuzzy sets. Although they are defined for only two arguments, their property of associativity provides a mechanism for extending their definitions to any number of arguments. However, fuzzy intersections and unions are not idempotent, with the exception of the standard min and max operations.

AVERAGE OPERATIONS

It is significant that any aggregation operation h that satisfies Axioms h2 and h5 satisfies also the inequalities

$$\min(a_1, a_2, \ldots, a_n) \leq h(a_1, a_2, \ldots, a_n) \leq \max(a_1, a_2, \ldots, a_n) \tag{3.31}$$

for all n-tuples $\langle a_1, a_2, \ldots, a_n \rangle \in [0, 1]^n$. To see this, let

$$a_* = \min(a_1, a_2, \ldots, a_n) \text{ and } a^* = \max(a_1, a_2, \ldots, a_n).$$

If h satisfies Axioms h2 and h5, then

$$a_* = h(a_*, a_*, \ldots, a_*) \leq h(a_1, a_2, \ldots, a_n) \leq h(a^*, a^*, \ldots, a^*) = a^*.$$

Conversely, if h satisfies (3.31), it must satisfy Axiom h5, since

$$a = \min(a, a, \ldots, a) \leq h(a, a, \ldots, a) \leq \max(a, a, \ldots, a) = a$$

for all $a \in [0, 1]$. That is, all aggregation operations between the standard fuzzy intersection and the standard fuzzy union are idempotent. Moreover, by Theorems 3.9 and 3.14, we may conclude that functions h that satisfy (3.31) are the only aggregation operations that are idempotent. These aggregation operations are usually called *averaging operations*.

Theorem 3.9. The standard fuzzy intersection is the only idempotent t-norm.

Theorem 3.14. The standard fuzzy union is the only idempotent t-conorm.

AVERAGE OPERATIONS

Theorem 3.9. The standard fuzzy intersection is the only idempotent t-norm.

Proof: Clearly, $\min(a, a) = a$ for all $a \in [0, 1]$. Assume that there exists a t-norm such that $i(a, a) = a$ for all $a \in [0, 1]$. Then, for any $a, b \in [0, 1]$, if $a \le b$, then

$$a = i(a, a) \le i(a, b) \le i(a, 1) = a$$

by monotonicity and the boundary condition. Hence, $i(a, b) = a = \min(a, b)$. Similarly, if $a \ge b$, then

$$b = i(b, b) \le i(a, b) \le i(1, b) = b$$

and, consequently, $i(a, b) = b = \min(a, b)$. Hence, $i(a, b) = \min(a, b)$ for all $a, b \in [0, 1]$. ∎

AVERAGE OPERATIONS

One class of averaging operations that covers the entire interval between the min and max operations consists of *generalized means*. These are defined by the formula

$$h_\alpha(a_1, a_2, \ldots, a_n) = \left(\frac{a_1^\alpha + a_2^\alpha + \ldots + a_n^\alpha}{n}\right)^{1/\alpha}, \tag{3.32}$$

where $\alpha \in \mathbb{R}$ $(\alpha \neq 0)$ and $a_i \neq 0$ for all $i \in \mathbb{N}_n$ when $\alpha < 0$;

Aggregation operations that, for
any given membership grades $a_1, a_2, \ldots, a_n$, produce a membership grade that lies between $\min(a_1, a_2, \ldots, a_n)$ and $\max(a_1, a_2, \ldots, a_n)$ are called *averaging operations*. For any given fuzzy sets, each of the averaging operations produces a fuzzy set that is larger than any fuzzy intersection and smaller than any fuzzy union.

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for $\alpha = 1$,

$$h_1(a_1, a_2, \ldots, a_n) = \frac{1}{n}(a_1 + a_2 + \ldots + a_n),$$

which is the *arithmetic mean*.

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For $\alpha = -1$,

$$h_{-1}(a_1, a_2, \ldots, a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}},$$

which is the *harmonic mean*;

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For $\alpha \to 0$,

$$\lim_{\alpha \to 0} h_\alpha = (a_1 \cdot a_2 \ldots a_n)^{1/n}.$$

geometric mean,

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$$h_{-\infty}(a_1, a_2, \ldots, a_n) = \lim_{\alpha \to -\infty} h_\alpha(a_1, a_2, \ldots, a_n) = \min(a_1, a_2, \ldots, a_n)$$

and its upper bound

$$h_{\infty}(a_1, a_2, \ldots, a_n) = \lim_{\alpha \to \infty} h_\alpha(a_1, a_2, \ldots, a_n) = \max(a_1, a_2, \ldots, a_n).$$