FUZZY COMPLEMENTS

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Let A be a fuzzy set on X. Then, by definition, $A(x)$ is interpreted as *the degree to which x belongs to A*. Let cA denote a fuzzy *complement* of A of type c. Then, $cA(x)$ may be interpreted not only as the degree to which x belongs to cA, but also as *the degree to which x does not belong to A*. Similarly, $A(x)$ may also be interpreted as the degree to which x does not belong to cA.

As a notational convention, let a complement cA be defined by a function

$$c : [0, 1] \rightarrow [0, 1],$$

which assigns a value $c(A(x))$ to each membership grade $A(x)$ of any given fuzzy set A. The value $c(A(x))$ is interpreted as the value of $cA(x)$. That is,

$$c(A(x)) = cA(x) \tag{3.4}$$

for all $x \in X$ by definition. Given a fuzzy set A, we obtain cA by applying function c to values $A(x)$ for all $x \in X$.

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To produce meaningful fuzzy complements, function c must satisfy at least the following two axiomatic requirements:

Axiom c1. $c(0) = 1$ and $c(1) = 0$ (*boundary conditions*).

Axiom c2. For all $a, b \in [0, 1]$, if $a \leq b$, then $c(a) \geq c(b)$ (*monotonicity*).

Axioms c1 and c2 be called the *axiomatic skeleton for fuzzy complements*.

In most cases of practical significance, it is desirable to consider various additional requirements for fuzzy complements. Each of them reduces the general class of fuzzy complements to a special subclass. Two of the most desirable requirements, which are usually listed in the literature among axioms of fuzzy complements, are the following:

Axiom c3. c is a continuous function.

Axiom c4. c is *involutive*, which means that $c(c(a)) = a$ for each $a \in [0, 1]$.

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It turns out that the four axioms are not independent, as expressed by the following theorem.

Theorem 3.1. Let a function $c : [0, 1] \to [0, 1]$ satisfy Axioms c2 and c4. Then, c also satisfies Axioms c1 and c3. Moreover, c must be a bijective function.

Proof:

(i) Since the range of c is $[0, 1]$, $c(0) \leq 1$ and $c(1) \geq 0$. By Axiom c2, $c(c(0)) \geq c(1)$; and, by Axiom c4, $0 = c(c(0)) \geq c(1)$. Hence, $c(1) = 0$. Now, again by Axiom c4, we have $c(0) = c(c(1)) = 1$. That is, function c satisfies Axiom c1.

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(ii) To prove that c is a bijective function, we observe that for all $a \in [0, 1]$ there exists $b = c(a) \in [0, 1]$ such that $c(b) = c(c(a)) = a$. Hence, c is an onto function. Assume now that $c(a_1) = c(a_2)$; then, by Axiom $c4$,

$$a_1 = c(c(a_1)) = c(c(a_2)) = a_2.$$

That is, c is also a one-to-one function; consequently, it is a bijective function.

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(iii) Since c is bijective and satisfies Axiom c2, it cannot have any discontinuous points. To show this, assume that c has a discontinuity at a_0, as illustrated in Fig. 3.1. Then, we have

$$b_0 = \lim_{a \to a_0-} c(a) > c(a_0)$$

and, clearly, there must exist $b_1 \in [0, 1]$ such that $b_0 > b_1 > c(a_0)$ for which no $a_1 \in [0, 1]$ exists such that $c(a_1) = b_1$. This contradicts the fact that c is a bijective function. ■

$$\overline{A}(x) = 1 - A(x)$$

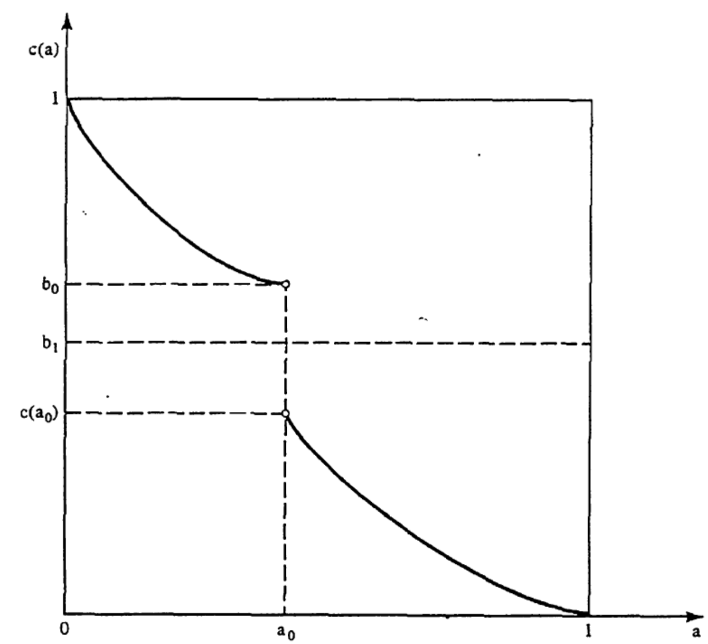

Figure 3.1 Illustration to Theorem 3.1.

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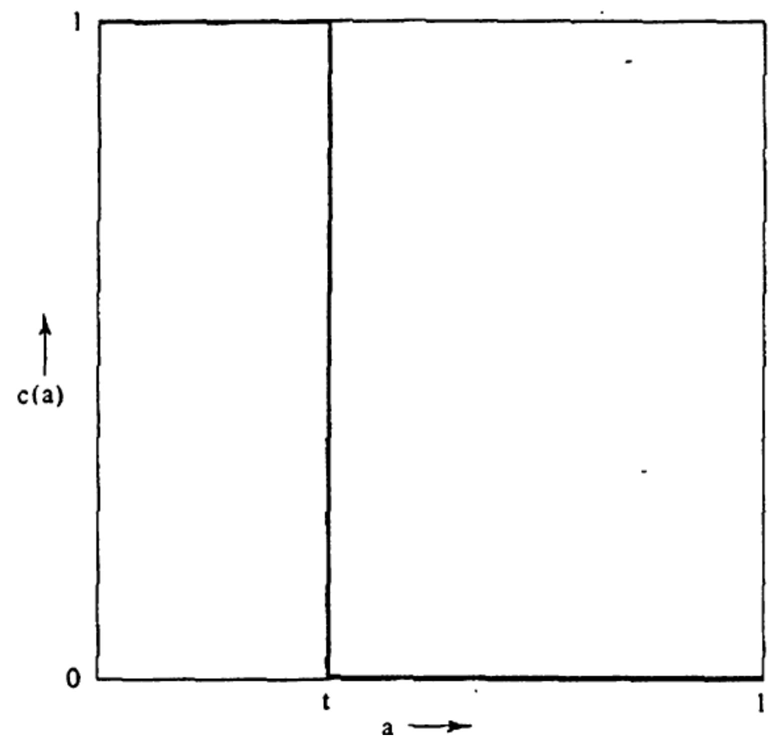
Examples of general fuzzy complements that satisfy only the axiomatic skeleton are the threshold-type complements defined by

$$c(a) = \begin{cases} 1 & \text{for } a \leq t \\ 0 & \text{for } a > t, \end{cases}$$

where $a \in [0, 1]$ and $t \in [0, 1)$; t is called the threshold of c.



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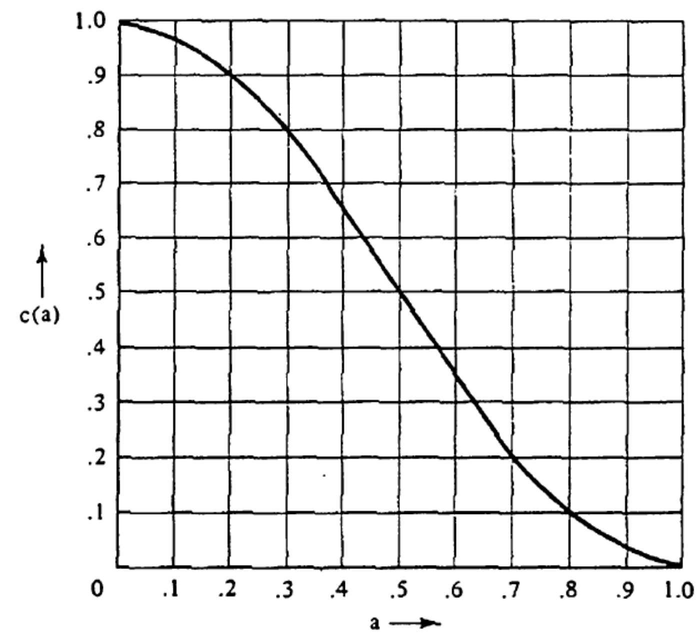
$$c(a) = \frac{1}{2}(1 + \cos \pi a)$$



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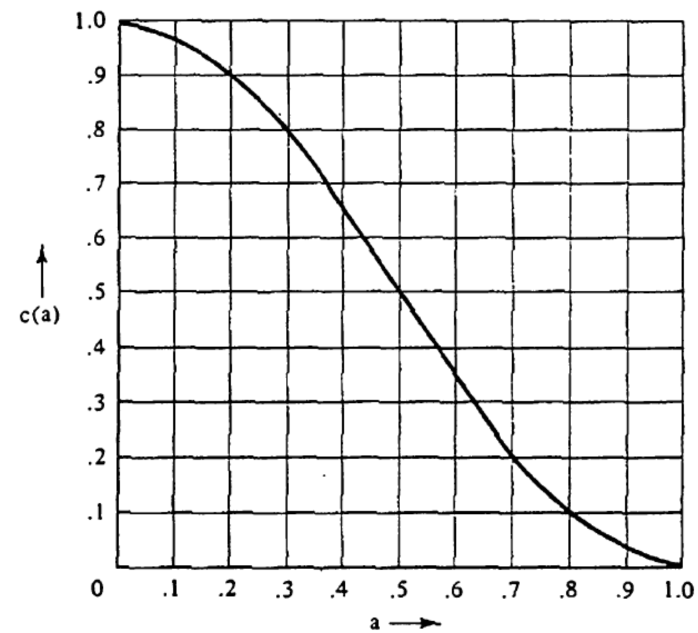
An example of a fuzzy complement that is continuous (Axiom c3) but not involutive (Axiom c4) is the function

$$c(a) = \frac{1}{2}(1 + \cos \pi a)$$

$c(.33) = .75$ but $c(.75) = .15 \neq .33.$



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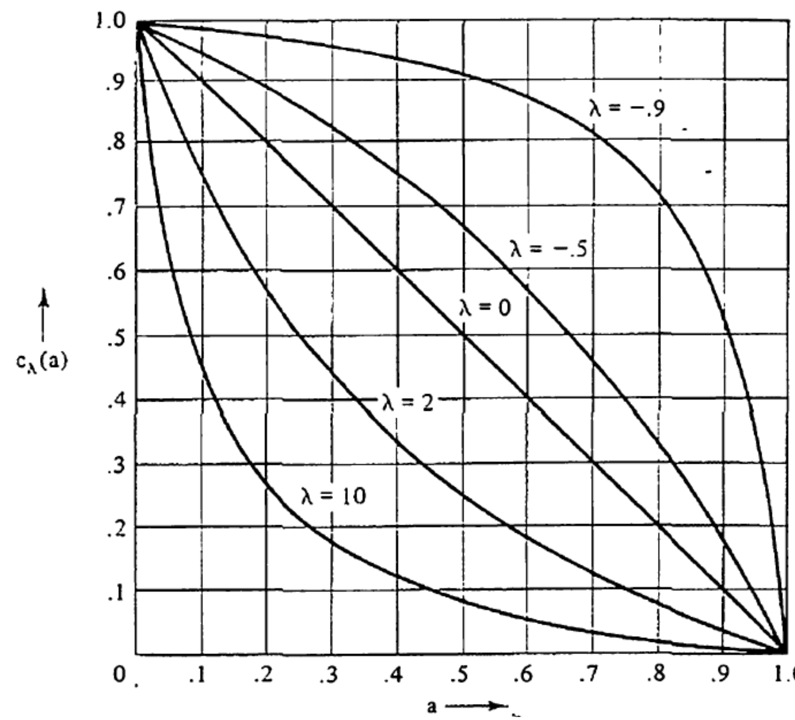
One class of involutive fuzzy complements is the *Sugeno class* defined by

$$c_\lambda(a) = \frac{1-a}{1+\lambda a} \qquad \text{where } \lambda \in (-1, \infty)$$



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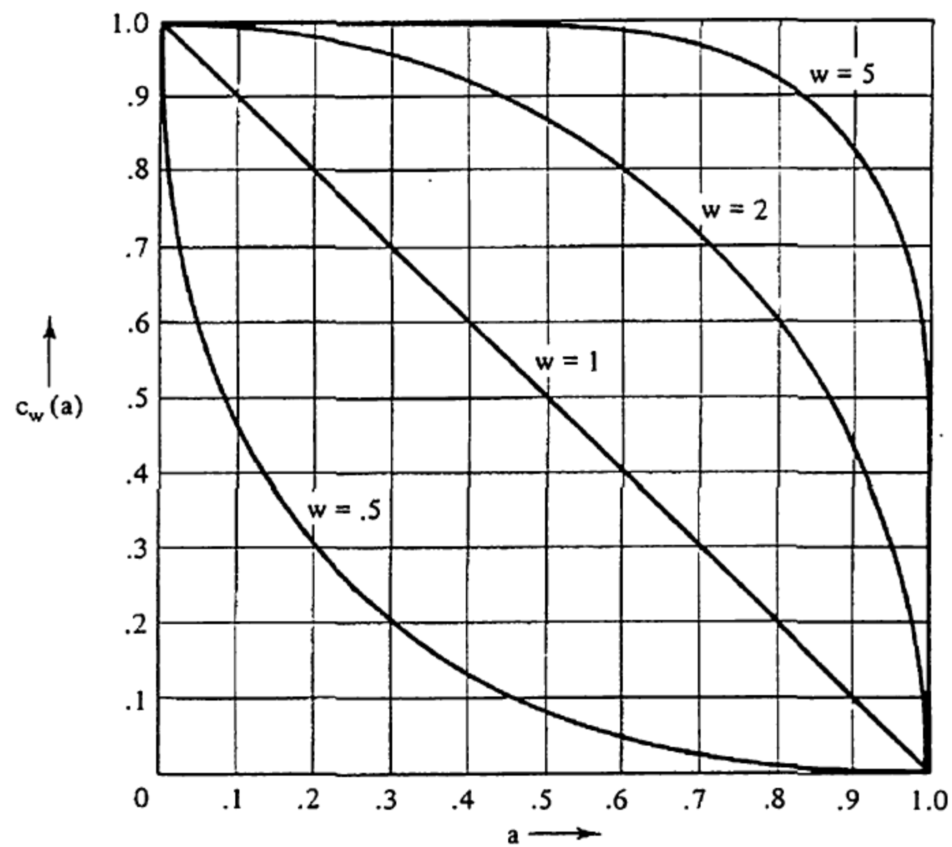
Another example of a class of involutive fuzzy complements is defined by

$$c_w(a) = (1 - a^w)^{1/w}, \quad \text{where } w \in (0, \infty);$$

Yager class of fuzzy complements.



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equilibrium of a fuzzy complement c, which is defined as any value a for which $c(a) = a$. In other words, the equilibrium of a complement c is that degree of membership in a fuzzy set A which equals the degree of membership in the complement cA.

for the classical fuzzy complement is .5, which is the solution of the equation $1 - a = a$.

Theorem 3.2. Every fuzzy complement has at most one equilibrium.

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Theorem 3.2. Every fuzzy complement has at most one equilibrium.

Proof: Let c be an arbitrary fuzzy complement. An equilibrium of c is a solution of the equation

$$c(a) - a = 0,$$

where $a \in [0, 1]$. We can demonstrate that any equation $c(a) - a = b$, where b is a real constant, must have at most one solution, thus proving the theorem. In order to do so, we assume that a_1 and a_2 are two different solutions of the equation $c(a) - a = b$ such that $a_1 < a_2$. Then, since $c(a_1) - a_1 = b$ and $c(a_2) - a_2 = b$, we get

$$c(a_1) - a_1 = c(a_2) - a_2. \tag{3.7}$$

However, because c is monotonic nonincreasing (by Axiom c2), $c(a_1) \geq c(a_2)$ and, since $a_1 < a_2$,

$$c(a_1) - a_1 > c(a_2) - a_2.$$

This inequality contradicts (3.7), thus demonstrating that the equation must have at most one solution. ■

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Theorem 3.3. Assume that a given fuzzy complement c has an equilibrium e_c, which by Theorem 3.2 is unique. Then

$$a \leq c(a) \text{ iff } a \leq e_c$$

and

$$a \geq c(a) \text{ iff } a \geq e_c.$$

Proof: Let us assume that $a < e_c, a = e_c$, and $a > e_c$, in turn. Then, since c is monotonic nonincreasing by Axiom c2, $c(a) \geq c(e_c)$ for $a < e_c, c(a) = c(e_c)$ for $a = e_c$, and $c(a) \leq c(e_c)$ for $a > e_c$. Because $c(e_c) = e_c$, we can rewrite these expressions as $c(a) \geq e_c, c(a) = e_c$, and $c(a) \leq e_c$, respectively. In fact, due to our initial assumption we can further rewrite these as $c(a) > a, c(a) = a$, and $c(a) < a$, respectively. Thus, $a \leq e_c$ implies $c(a) \geq a$ and $a \geq e_c$ implies $c(a) \leq a$. The inverse implications can be shown in a similar manner. ■

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The equilibrium for each individual fuzzy complement c_λ of the Sugeno class is given by

$$e_{c_\lambda} = \begin{cases} ((1+\lambda)^{1/2} - 1)/\lambda & \text{for } \lambda \neq 0, \\ 1/2 & \text{for } \lambda = 0 \end{cases}$$

This is clearly obtained by selecting the positive solution of the equation

$$\frac{1 - e_{c_\lambda}}{1 + \lambda e_{c_\lambda}} = e_{c_\lambda}.$$

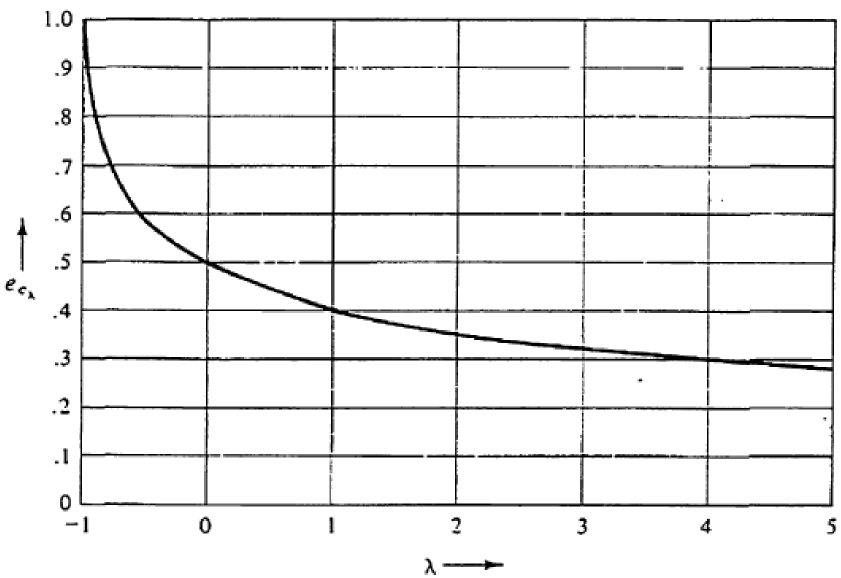

Figure 3.5 Equilibria for the Sugeno class of fuzzy complements.

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Theorem 3.4. If c is a continuous fuzzy complement, then c has a unique equilibrium.

Proof: The equilibrium e_c of a fuzzy complement c is the solution of the equation $c(a) - a = 0$. This is a special case of the more general equation $c(a) - a = b$, where $b \in [-1, 1]$ is a constant. By Axiom c1, $c(0) - 0 = 1$ and $c(1) - 1 = -1$. Since c is a continuous complement, it follows from the intermediate value theorem for continuous functions that for each $b \in [-1, 1]$, there exists at least one a such that $c(a) - a = b$. This demonstrates the necessary existence of an equilibrium value for a continuous function, and Theorem 3.2 guarantees its uniqueness. ■

Theorem 3.2. Every fuzzy complement has at most one equilibrium.

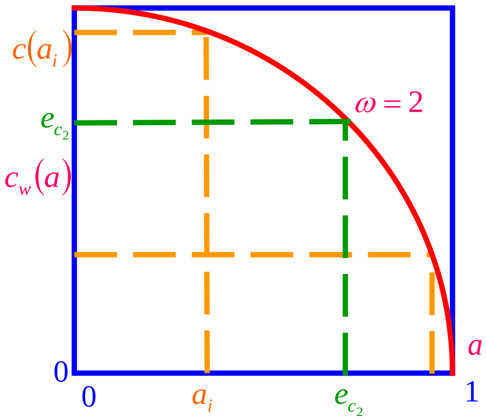
Dual point

If we are given a fuzzy complement c and a membership grade whose value is represented by a real number $a \in [0, 1]$, then any membership grade represented by the real number ${}^d a \in [0, 1]$ such that

$$c({}^d a) - {}^d a = a - c(a) \tag{3.8}$$

is called a *dual point* of a with respect to c.

It follows directly from the proof of Theorem 3.2 that (3.8) has at most one solution for ${}^d a$ given c and a. There is, therefore, at most one dual point for each particular fuzzy complement c and membership grade of value a. Moreover, it follows from the proof of Theorem 3.4 that a dual point exists for each $a \in [0, 1]$ when c is a continuous complement.



Dual point

Theorem 3.5. If a complement c has an equilibrium e_c, then

$$^{d}e_c = e_c.$$

Proof: If $a = e_c$, then by our definition of equilibrium, $c(a) = a$ and thus $a - c(a) = 0$. Additionally, if $^{d}a = e_c$, then $c(^{d}a) = {}^{d}a$ and $c(^{d}a) - {}^{d}a = 0$. Therefore,

$$c(^{d}a) - {}^{d}a = a - c(a).$$

This satisfies (3.8) when $a = {}^{d}a = e_c$. Hence, the equilibrium of any complement is its own dual point. ∎

Dual point

Theorem 3.6. For each $a \in [0,1]$, ${}^d a = c(a)$ iff $c(c(a)) = a$, that is, when the complement is involutive.

Proof: Let ${}^d a = c(a)$. Then, substitution of $c(a)$ for ${}^d a$ in (3.8) produces

$$c(c(a)) - c(a) = a - c(a).$$

Therefore, $c(c(a)) = a$. For the reverse implication, let $c(c(a)) = a$. Then substitution of $c(c(a))$ for a in (3.8) yields the functional equation

$$c({}^d a) - {}^d a = c(c(a)) - c(a).$$

for ${}^d a$ whose solution is ${}^d a = c(a)$. ■