PROJECTIONS AND CYLINDRIC EXTENSIONS

PROJECTIONS AND CYLINDRIC EXTENSIONS

Consider the Cartesian product of all sets in the family $\mathcal{X} = \{X_i | i \in \mathbb{N}_n\}$. For each sequence ($n$-tuple)

$$\mathbf{x} = \langle x_i | i \in \mathbb{N}_n \rangle \in \bigtimes_{i \in \mathbb{N}_n} X_i$$

and each sequence (r-tuple, $r \leq n$)

$$\mathbf{y} = \langle y_j | j \in J \rangle \in \bigtimes_{j \in J} X_j,$$

where $J \subseteq \mathbb{N}_n$ and $|J| = r$, let $\mathbf{y}$ be called a *subsequence* of $\mathbf{x}$ iff $y_j = x_j$ for all $j \in J$.

$\mathbf{y} \prec \mathbf{x}$ denote that $\mathbf{y}$ is a subsequence of $\mathbf{x}$.

For example, if n-tuples $\mathbf{x}$ represent states of a system, $\mathbf{y}$ may be called a

substate of $\mathbf{x}$; if they are strings of symbols in a formal language, $\mathbf{y}$ may be called a *substring* of $\mathbf{x}$.

PROJECTIONS AND CYLINDRIC EXTENSIONS

Given a relation $R(X_1, X_2, \ldots, X_n)$, let $[R \downarrow \mathcal{Y}]$ denote the *projection* of R on $\mathcal{Y}$ that disregards all sets in X except those in the family

$$\mathcal{Y} = \{X_j \mid j \in J \subseteq \mathbb{N}_n\}.$$

Then, $[R \downarrow \mathcal{Y}]$ is a fuzzy relation whose membership function is defined on the Cartesian product of sets in $\mathcal{Y}$ by the equation

$$[R \downarrow \mathcal{Y}](\mathbf{y}) = \max_{\mathbf{x} \succ \mathbf{y}} R(\mathbf{x}). \tag{5.1}$$

PROJECTIONS AND CYLINDRIC EXTENSIONS

Consider the sets $X_1 = \{0, 1\}$, $X_2 = \{0, 1\}$, $X_3 = \{0, 1, 2\}$ and the ternary fuzzy relation on $X_1 \times X_2 \times X_3$ defined in Table 5.1. Let $R_{ij} = [R \downarrow \{X_i, X_j\}]$ and $R_i = [R \downarrow \{X_i\}]$ for all $i, j \in \{1, 2, 3\}$. Using this notation, all possible projections of R are given in Table 5.1. A detailed calculation of one of these projections, R_{12}, is shown in Table 5.2.

TABLE 5.1 TERNARY FUZZY RELATION AND ITS PROJECTIONS

$\langle x_1,$	$x_2,$	$x_3\rangle$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$	$R_{13}(x_1, x_3)$	$R_{23}(x_2, x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

PROJECTIONS AND CYLINDRIC EXTENSIONS

TABLE 5.1 TERNARY FUZZY RELATION AND ITS PROJECTIONS

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$	$R_{13}(x_1, x_3)$	$R_{23}(x_2, x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

TABLE 5.2 CALCULATION OF THE PROJECTION R_{12} IN EXAMPLE 5.3

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$
0	0	0	0.4	
0	0	1	0.9	$\max[R(0,0,0), R(0,0,1), R(0,0,2)] = 0.9$
0	0	2	0.2	
0	1	0	1.0	
0	1	1	0.0	$\max[R(0,1,0), R(0,1,1), R(0,1,2)] = 1.0$
0	1	2	0.8	
1	0	0	0.5	
1	0	1	0.3	$\max[R(1,0,0), R(1,0,1), R(1,0,2)] = 0.5$
1	0	2	0.1	
1	1	0	0.0	
1	1	1	0.5	$\max[R(1,1,0), R(1,1,1), R(1,1,2)] = 1.0$
1	1	2	1.0	

PROJECTIONS AND CYLINDRIC EXTENSIONS

Another operation on relations, which is in some sense an inverse to the projection, is called a *cylindric extension*. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the same families of sets as employed in the definition of projection. Let R be a relation defined on the Cartesian product of sets in the family $\mathcal{Y}$, and let $[R \uparrow \mathcal{X} - \mathcal{Y}]$ denote the cylindric extension of R into sets $X_i (i \in \mathbb{N}_n)$ that are in $\mathcal{X}$ but are not in $\mathcal{Y}$. Then,

$$[R \uparrow \mathcal{X} - \mathcal{Y}](\mathbf{x}) = R(\mathbf{y}) \tag{5.2}$$

for each $\mathbf{x}$ such that $\mathbf{x} \succ \mathbf{y}$.

Membership functions of cylindric extensions of all the projections in Example 5.3 are actually those shown in Table 5.1 under the assumption that their arguments are extended to $\langle x_1, x_2, x_3 \rangle$. For instance:

$$[R_{23} \uparrow \{X_1\}](0, 0, 2) = [R_{23} \uparrow \{X_1\}](1, 0, 2) = R_{23}(0, 2) = 0.2.$$

We can see that none of the cylindric extensions (identical with the respective projections in Table 5.1) are equal to the original fuzzy relation from which the projections involved in the cylindric extensions were determined. This means that some information was lost when the given relation was replaced by any one of its projections in this example.

PROJECTIONS AND CYLINDRIC EXTENSIONS

Relations that can be reconstructed from one of their projections by the cylindric extension exist, but they are rather rare. It is more common that a relation can be exactly reconstructed from several of its projections by taking the set intersection of their cylindric extensions. The resulting relation is usually called a *cylindric closure*. When projections are determined by the max operator (see Sec. 3.1), the min operator is normally used for the set intersection. Hence, given a set of projections $\{P_i | i \in I\}$ of a relation on $\mathcal{X}$, the cylindric closure, cyl $\{P_i\}$, based on these projections is defined by the equation

$$\text{cyl}\{P_i\}(\mathbf{x}) = \min_{i \in I}[P_i \uparrow \mathcal{X} - \mathcal{Y}_i](\mathbf{x})$$

TABLE 5.3 CYLINDRIC CLOSURES OF THREE FAMILIES OF PROJECTIONS CALCULATED IN EXAMPLE 5.3

$\langle x_1,$	$x_2,$	$x_3\rangle$	$\text{cyl}\{R_{12}, R_{13}, R_{23}\}$	$\text{cyl}\{R_1, R_2, R_3\}$	$\text{cyl}\{R_{12}, R_3\}$
0	0	0	0.5	0.9	0.9
0	0	1	0.9	0.9	0.9
0	0	2	0.2	0.9	0.9
0	1	0	1.0	1.0	1.0
0	1	1	0.5	0.9	0.9
0	1	2	0.8	1.0	1.0
1	0	0	0.5	0.9	0.5
1	0	1	0.5	0.9	0.5
1	0	2	0.2	0.9	0.5
1	1	0	0.5	1.0	1.0
1	1	1	0.5	0.9	0.9
1	1	2	1.0	1.0	1.0