Bridging Mathematical Concepts

Set is a well defined collection of distinct objects, usually unordered.

$A = \{1, 2, 2, 3, 5\}$ X

$A = \{1, 2, 3\}$

$B = \{2, 1, 3\}$

here $A = B$

Set of all vowels $\Rightarrow$ {'a', 'e', 'i', 'o', 'u'}

Set of all integers less than 5 and greater than 0

$\Rightarrow \{1, 2, 3, 4\}$

~~Set~~ Collection of famous three malayalam actors

$\Rightarrow$ this is not a set as it is not well-defined.

Collection of five healthy foods $\Rightarrow$ Not a set.

R - Real Numbers

N - Natural Numbers

Z - Integers

Q - Irrational

$\mathbb{C}$ - Complex

Roaster Form

$A = \{2, 4, 6, \dots\}$

Set builder form

$A = \{x \mid x \text{ is a positive even number}\}$

$A = \{x \mid 2 \mid x\}$

$A = \{x \mid x = 2y,\ y \in Z^{+}\}$

Cardinality: No: of elements of a set.
Singleton set : having only one element.
eg: Set of integers that are not +ve or -ve. $\Rightarrow \{0\}$

Null set : having no ~~set~~ elements. Represented as $\phi, \{\}$.

Equal sets: Elements of both set are the same.
$A \subseteq B$ and $B \subseteq A$.
Equivalent sets : Having same no: of elements.

Subset : set having elements from another set.

Proper subset: $A \subset B$ and $A \neq B$.

Set Operations

Union, Intersection, Difference, Cross Difference
Symmetric Difference :-
$(A-B) \cup (B-A)$ or $(A \cup B) - (A \cap B)$

Let 'A' and 'B' be two non-empty sets
$A \times B = \{ (a,b) \mid a \in A$ and $b \in B \}$
Eg: $A = \{1, 2, 3\}$
$B = \{a, b, c\}$

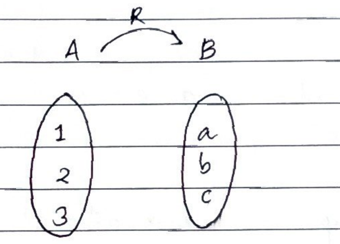
$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$
$|A \times B| = |A| \times |B|$

Relation

A ~~relay~~ relation is a ~~set~~ subset of $A \times B$.
$R : A \rightarrow B$

$R = \{ (a,b) \mid a \in A$ ~~and~~, $b \in B$, and $p(a,b) = $ true$\}$
where p is a predicate.
$(a,b) \in R$; aRb

From the earlier example of $A \times B$
$R = \{ (a,b) \mid a$ is odd and b is a vowel$\}$
$\therefore R = \{ (1,a), (3,a) \}$



Domain Co-Domain

~~A related to B~~ and ~~B related to A~~ → ~~Reflexive~~ Symmetric (aRb bRa)
$(a,a), (b,b)$ → Reflexive
$aRb \Rightarrow b \not R a$ → Asymmetric
$(aRb = bRa) \Rightarrow a = b$ → Antisymmetric
aRb and bRc then aRc → Transitivity

If a relation is an equivalence relation it should be reflexive, symmetric and transitive.

If a relation is reflexive, anti-symmetric and transitive, it is called partial ordering.

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$|A \times B| = |A| \times |B|$

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Algebraic Structure :- A

(S) set $\quad a, b, c \in S$

↓

closure → $(S, *)$ Magma $\quad a * b \in S$

↓

associative → $(S, *)$ Semi Group $\quad (a * b) * c = a * (b * c)$

↓

Identity Element $(S, *)$ Monoid $\quad a * e = a$

↓

Inverse Element $(S, *)$ Group $\quad a * a^{-1} = e$

↓

Commutative $(S, *)$ Abelian Group $\quad a * b = b * a$

Example : ~~$(R, *)$~~ $(R, +)$

$c, a, b \in R$

$a + b \in R$ (Magma) ✓

$(a+b)+c = a+(b+c)$ (Semi group) ✓

$a + 0 = a$ (Monoid) ✓

$a + -a = 0$ (Group) ✓

$a + b = b + a$ (Abelian) ✓

$(R, -)$

$a, b, c \in R$ (~~Magma~~) ✓

$a - b \in R$ (~~Semi group~~ Magma) ✓

$(a-b)-c = a-(b-c)$ (~~Monoid~~ Semi group) ✓

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$a - 0 = a$ (Monoid) ✓

$a - (+a) = 0$ (Group) ✓

$a - b \neq$ ~~a~~ $b - a$ (Abelian) X

Algebraic Structure

A mathematical entity or mathematical structure that is defined using a set and an operation and the operation should satisfy a set of properties ~~on~~ over that set.

x is a string ; $x \in S^*$

$x \longrightarrow \boxed{E} \longrightarrow x'$

eg: $abc \rightarrow xyz$

$x \in S^* \longrightarrow x' \in S^*$

Assignment

Explain the following mathematical structures in a mathematical way & along with proper examples, also discuss about applications of these structures ~~in~~ in computational systems.

1. Lattice
2. Ring
3. Field

Functions

Taking Relations, $R : A \rightarrow B$

$R \subseteq A \times B$

$R = \{ (x, y) \mid P(x, y) = true \}$

Domain : set of all x ; $(x,y) \in R$
Range : set of all y ; $(x,y) \in R$
Co-Domain: set B itself

$A = \{1, 2, 3\}$
$R_1 : A \to A$; $R_2 : A \to A$; $R_3 : A \to \cancel{B}\, A$

$R_1 = \{(x,y) \mid 2 \mid x+y\}$ — Domain $= \{1,2,3\}$
$\Rightarrow \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$ — Range $= \{1,2,3\}$
Co-Domain $= \{1,2,3\}$

$R_2 = \{(x,y) \mid x = y\}$ — Domain $= \{2,2,3\}$
$\Rightarrow \{(1,1), (2,2), (3,3)\}$ — Range $= \{1,2,3\}$
Co-Domain $= \{1,2,3\}$

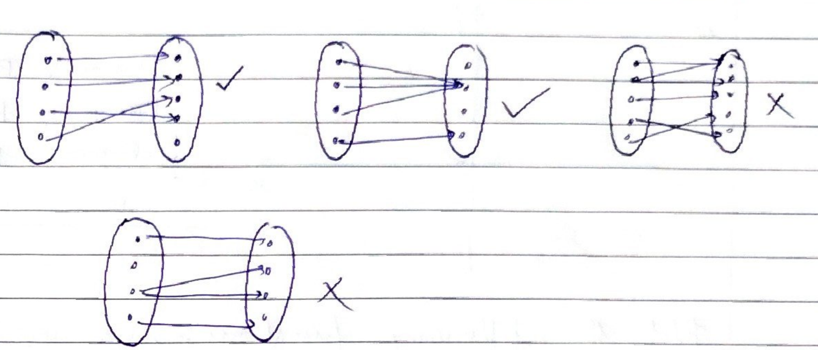
$R_3 = \{(x,y) \mid x \neq y\}$ — Domain $= \{1,2,3\}$
$\Rightarrow \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$ — Range $= \{1,2,3\}$
Co-Domain $= \{1,2,3\}$

$R_4 = \{(x,y) \mid x = y+1\}$ — Domain $= \{2,3\}$
$\Rightarrow \{(2,1), (3,2)\}$ — Range $= \{1,2\}$
Co-Domain $= \{1,2,3\}$

Functions:-

Every ~~th~~ element in the first step should participate the relation. That is, ~~to every element in the~~
i) All elements in set A should have an image in the set B.
ii) Every element should have one and only one ~~element~~ image in B.

A relationship can be considered as a function if every element has a unique image in 2nd set → Function



Real-Valued ~~F~~ Function

$f: A \to R$

consider a complex number z, if a function is used to find its magnitude

eg: $|z| \Rightarrow f(a+ib) = \sqrt{a^2+b^2}$

$z \in \mathbb{C}$ but $|z| \in R$
so $z \to \boxed{f} \to |z|$ is a real-valued fn.

eg: Determinant of a matrix
input: matrix
output: real value

Real Functions

$f: \mathbb{R} \to \mathbb{R}$
$\downarrow \quad \downarrow$
$x \quad y$

we can align x and y perpendicularly to plot the function (by connecting the ...

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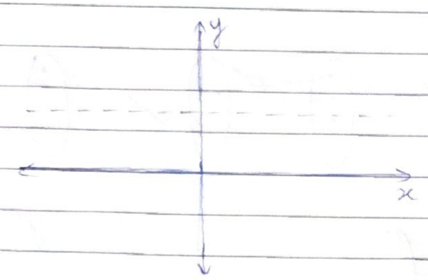
eg:



Domain : R

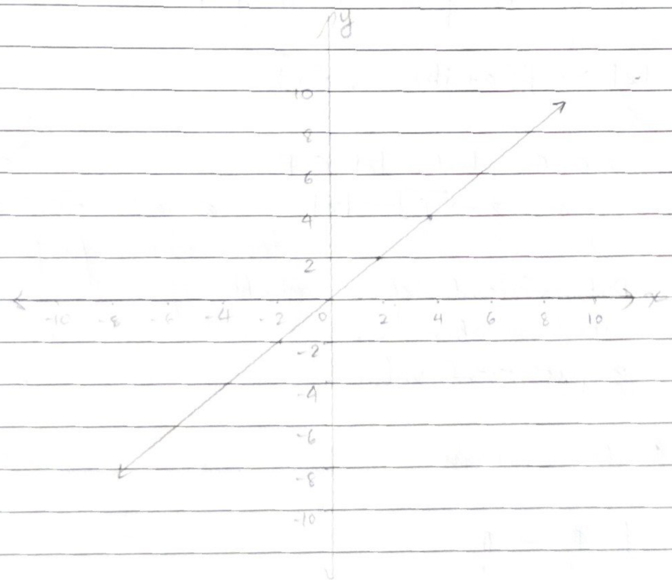
Range : R+

Codomain : R

Plot the following functions :-

1. ~~x = y~~ $y = x$
2. $y = x^2$
3. $y = (x-3)^3 + 1$
4. $y = |x|$
5. ~~$y = \pm\sqrt{1-x^2}$~~
5. 6. $y = [x]$

1)



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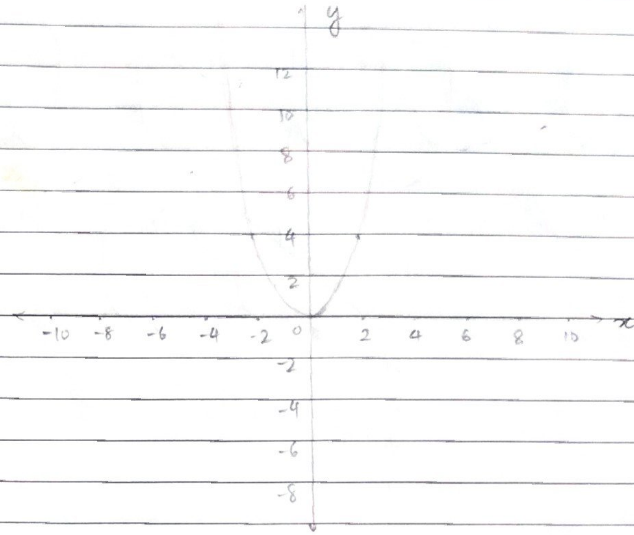
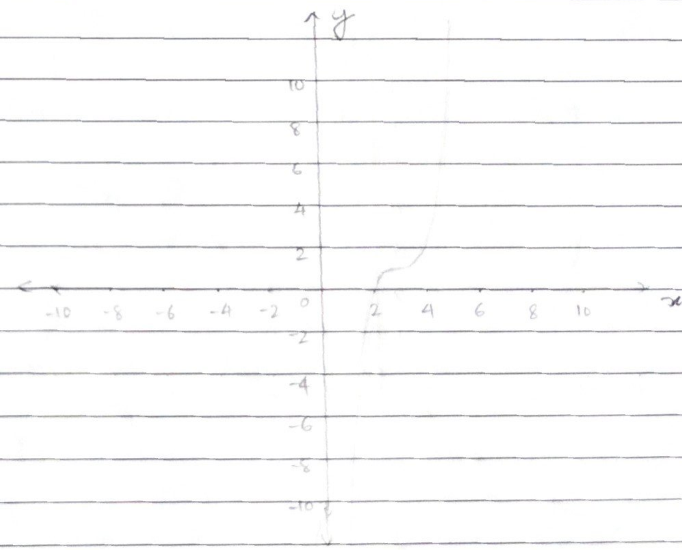
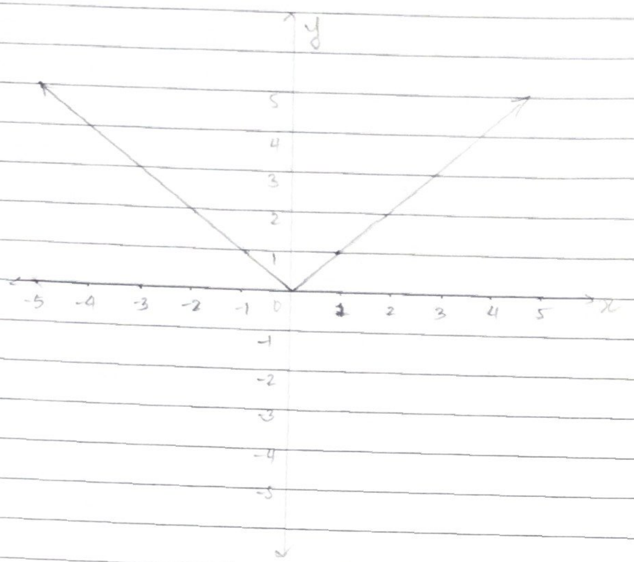
2)



3)



4)



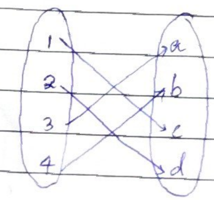
5)

One to one function :- No two ~~images~~ elements in first set should have the same image in 2nd set.

~~Onto~~ Onto function :- All elements in second set ~~is~~ has a preimage.

Bijective function :- Satisfying both above

Bijective Function



$1,2,3,4 \longrightarrow \boxed{f} \longrightarrow c,d,a,b$

$c,d,a,b \longrightarrow \boxed{g = f'} \longrightarrow 1,2,3,4$

this is inverse of function f.

Inverse exists only for bijective functions.

eg: $f(x) = 2x+3$

$y = 2x+3$

$x = \frac{y-3}{2}$

$\therefore$ inverse $= f'(x) = \frac{x-3}{2}$

$f \circ f' = f' \circ f = x$ $\left(\cancel{ie} \because f \circ f' = f(f'(x)) \right)$

proving $f \circ f'$, $f(x) = 2\left(\frac{x-3}{2}\right) + 3 = x-3+3 = x$

now $f' \circ f = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x$ //

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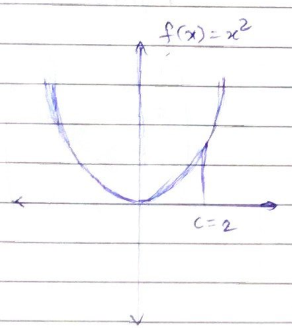
Calculus

Limit of a function

i) ~~Does~~ Is there any limit that exists for the function at a point.

ii) What is the limit value of the function at a point?

$f(x)=x^2$; $c=2$



h	c^-	$f(c^-)$	c^+	$f(c^+)$
2	0	0	4	16
1	1	1	3	9
0.5	1.5	2.25	2.5	6.25
0.2	1.8	3.24	2.2	4.84
0.1	1.9	3.61	2.1	4.41
0.01	1.99	3.9601	2.01	4.0401
0.001	1.999	3.996001	2.001	4.004001
0.0001	1.9999	3.99960001	2.0001	4.00040001
0.00001	1.99999	3.9999600001	2.00001	4.0000400001

$\therefore \lim_{c \to 2} x^2 = 4$

For limit to exist,

~~then~~

$$\lim_{c^- \to c} f(c^-) = \lim_{c^+ \to c} f(c^+)$$

(or)

$$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h)$$

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$f(x) = [x]$; $c=2$; $h = 0.5, 0.1, 0.01, 0.001$

h	c^-	$f(c^-)$	c^+	$f(c^+)$
1	1	1	3	3
0.5	1.5	1	2.5	2
0.1	1.9	1	2.1	2
0.01	1.99	1	2.01	2
0.001	1.999	1	2.001	2

$\therefore$ there is no limit for this function as the converged values are not approaching each other.

HW

Q. $f(x) = \begin{cases} x^2 & \text{if } x<3 \\ 2x+4 & \text{if } x \geq 3 \end{cases}$

i) Plot the function
ii) Does it have an inverse
iii) Check the existence of limit at $x=1$, $x=2$ and $x=3$
iv) Find the limit of $f(x)$ at points where limit exists.

Continuity of a function

If the left hand limit and right hand limit both are equal to the value of the function at that point, then ~~it is~~ the function is continuous at that point.

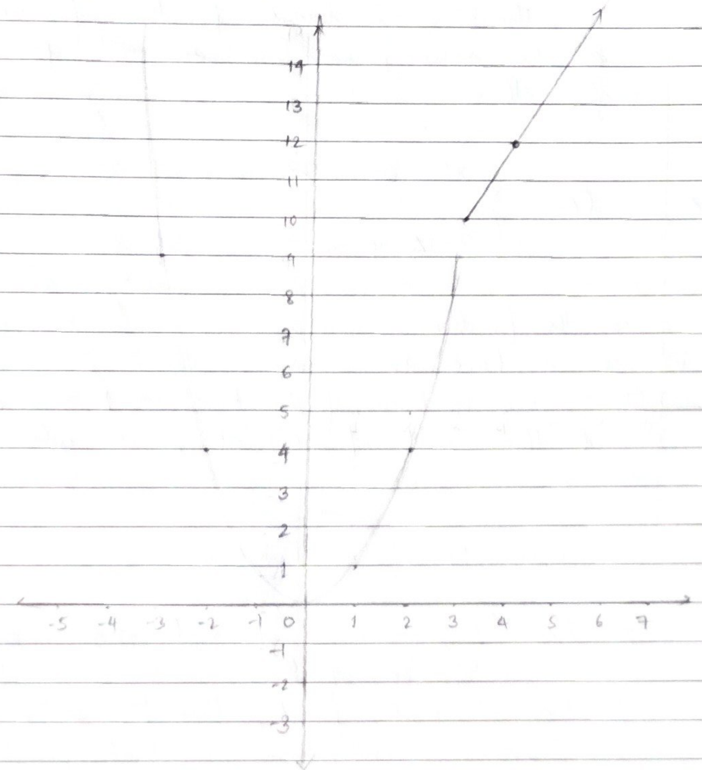
A function is continuous if it is continuous at every element in its domain.

HW. Find example of 5 functions and some points where the function is having a limit but not continuous.

Homework - Answers

1. $f(x) = \begin{cases} x^2 & \text{if } x<3 \\ 2x+4 & \text{if } x \geq 3 \end{cases}$

i)



ii) The function is not one to one as both
eg: Both -2 and 2 have the image 4.
For a function to have an inverse it must be one-to-one and onto.
∴ This function does not have an inverse

iii)

Checking limit at $x=1$; $c=1$

h	c^-	$f(c^-)$	c^+	$f(c^+)$
1	0	0	2	4
0.5	0.5	0.25	1.5	2.25
0.2	0.8	0.64	1.2	1.44
0.1	0.9	0.81	1.1	1.21
0.01	0.99	0.9801	1.01	1.0201
0.001	0.999	0.99801	1.001	1.002001
0.0001	0.9999	0.99980001	1.0001	1.00020001

~~limit~~ ∴ $\lim_{c^- \to c} f(c^-) = \lim_{c^+ \to c} f(c^+)$

∴ limit exists at $x=1$

Checking limit at $x=2$; $c=2$

h	c^-	$f(c^-)$	c^+	$f(c^+)$
1	1	1	3	9
0.5	1.5	2.25	2.5	6.25
0.2	1.8	3.24	2.2	4.84
0.1	1.9	3.61	2.1	4.41
0.01	1.99	3.9601	2.01	4.0401
0.001	1.999	3.996001	2.001	4.004001
0.0001	1.9999	3.99960001	2.0001	4.00040001

$\therefore \lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h)$

so limit exists at $x = 2$

Checking limit at $x=3$; $c=3$

h	c^-	$f(c^-)$	c^+	$f(c^+)$
1	2	4	4	~~16~~ 12
0.5	2.5	6.25	3.5	11
0.2	2.8	7.84	3.2	10.4
0.1	2.9	8.41	3.1	10.2
0.01	2.99	8.9401	3.01	10.02
0.001	2.999	8.994001	3.001	10.002
0.0001	2.9999	8.99940001	3.0001	10.0002

here, $\lim_{h \to 0} f(c-h) \neq \lim_{h \to 0} f(c+h)$

so limit does not exist at $x = 3$

iv) we know,

limit exists at $x=1$ and $x=2$

when $x=1$

limit $= \lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = \underline{\underline{1}}$

when $x=2$

limit $= \lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = \underline{\underline{4}}$

Discontinuity Definition

The function "f" will be discontinuous at $x=a$ in any of the following cases:

- $f(a)$ is not defined
- $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.
- $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$

Few functions that have limits but are not continuous:-

1. Piecewise Function:-

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2 & \text{if } x \geq 1 \end{cases}$$

This function has a limit as x approaches 1, but it is not continuous at $x=1$ because the left-hand ~~limit~~ limit is 1, while the function value is 2.

2. Step Function (Heaviside Function)

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

The limit of $f(x)$ as x approaches 0 exists and is 1, but the function jumps from 0 to 1 at $x=0$, so it is not continuous there.

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3. Sign Function

$$f(x) = sgn(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

The limit of $f(x)$ as x approaches 0 does not exist due to the jump discontinuity at $x=0$. However, if we consider the limit as approaching from either side separately, both side limits are -1 and 1 respectively, which are not the same, thus not continuous at $x=0$.

4. Function with Removable Discontinuity

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

The function $\frac{x^2-1}{x-1}$ simplifies to $x+1$ for $x=1$.

Thus, the limit as x approaches 1 is 2. However, the function is not continuous at $x=1$, since it is defined as 2 instead of 2.

5. Absolute Value Function with a Jump:

$$f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

The limit of $f(x)$ as x approaches 0 is 0, but the function value at 0 is 1. Hence, it is

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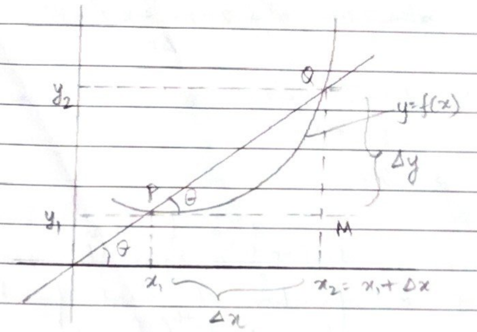
not continuous at $x=0$.

Derivatives:



let $x_1 = x$

$y_1 = y = f(x)$

$y_2 = y + \Delta y$
$= f(x+\Delta x)$

$$\tan\theta = \frac{QM}{PM} = \frac{y_2-y_1}{x_2-x_1} = \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$$

this is called Difference ~~Quotient~~ Coefficient.

It becomes the derivative of the function when points P and Q end up as a single point (that is when the line meets the function only at a single point)

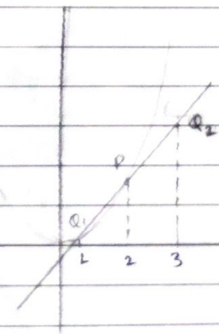
$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$x = 1$

$x + \Delta x = 2$

$f(x+\Delta x) = 4$; $\left.\frac{dy}{dx}\right|_{Q_1} = \frac{4-1}{1} = 3$

$f(\cancel{x+\Delta x}^{x_2} + \Delta x) = 9$; $\left.\frac{dy}{dx}\right|_{Q_2} = \frac{9-4}{1} = 5$



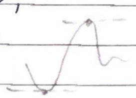
$$\lim_{x \to \Delta x} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$\Rightarrow \cancel{x^2 + 2\Delta x} \quad \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\Rightarrow \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$\Rightarrow \underline{\underline{2x}}$$

When derivative or $\tan\theta$ of a function is equal to zero, the value of the function at that point is an extremum value of ~~that~~ the function. It can be a maxima or a minima



Mathematically representing the trends of a data
→ Regression.
It is also an example of curve fitting (modelling)

$m = f(x)$ → simple / univariate function
$y = f(x_1, x_2, x_3, \dots x_n)$ → multivariate function

$x_1, x_2, \dots x_n$ are independent of each other
y is a dependent variable.

$y = f(x_1, x_2)$ → bivariate function, with $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$

$(p, q) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

A function is dependent on more than one variables. Then we need to find how much each variable contribute to the value of ~~a~~ the function.
Such a derivative,

$$\lim_{\substack{x \to \Delta x \\ y \to \Delta y}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + \Delta y}$$

but we usually keep ~~one~~ others variables constant while finding derivative ~~a~~ with respect to one variable.

$$\lim_{x \to \Delta x} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{\cancel{\delta f}}{\delta x} \quad \frac{\delta df}{\delta dx}$$

$$\lim_{y \to \Delta y} \frac{\cancel{f(x+}\ f(x, y+\Delta x) - f(x,y)}{\Delta y} = \frac{\delta df}{\delta dy}$$

these are called partial derivatives with respect to x and y.
Let $y = f(x_1, x_2, x_3, \dots, x_n)$

A vector comprising of partial derivates with respect to all variables is called the gradient.

$$\begin{bmatrix} \frac{\delta f}{\delta x_1} \\ \frac{\delta f}{\delta x_2} \\ \vdots \\ \frac{\delta f}{\delta x_n} \end{bmatrix} = \Delta f$$

This can be equated to zero to find extremums.

Write Notes

probability
Inclusion-exclusion principle
Random Variable
Probability Distribution Function
Probability Mass Function
Cumulative Distribution Function
Discrete Random Variables
Continuous Random Variables

1. Probability

It is a measure of the likelihood of an event occuring, ranging from 0 (impossible) to 1 (certain)

Experiment: A procedure with uncertain outcomes.
Sample Space (S): Set of all possible outcomes &
Event (E): A subset of the sample space.

Probability of an Event

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Independent Events: The outcome of one event does not affect another.
Dependent Event: The outcome of one event affects another.
Mutually Exclusive Events: Events that cannot occur at the same time. Total of prob of these events $\rightarrow 1$

Addition Rule: For mutually exclusive events,
$$P(A \cup B) = P(A) + P(B)$$

Multiplication Rule: For independent events,
$$P(A \cap B) = P(A) \times P(B)$$

Conditional Probability
Probability of an event given that another event has occured.
$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

2. Inclusion Exclusion Principle
The inclusion exclusion principle is a method used to calculate the probability or count of the union of multiple overlapping events. It involves adding the probabilities of individual events, then subtracting the probabilities of the intersections of every pair of events to avoid double-counting, adding back the probabilities of the intersections of every triplet of events, and continuing this process of for higher-order intersections. For two events, the formula is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. This principle helps accurately determine the probabilities or count when events are not mutually exclusive.

3. Random Variable
A random variable is a numerical outcome of a random process or experiment, where each possible outcome is associated with a probability. It is a function that assigns a real number to each outcome in a sample space. There are two main types of random variables: discrete and continuous. A discrete random variable

takes on a finite or countable set of values, like the roll of a die, while a continuous random variable takes on an infinite range of values, like the exact height of a person. Random variables are fundamental in probability theory, allowing us to model and analyze random phenomena quantitatively.

4.5 Probability Distribution and Mass Function

A probability distribution function describes how the probabilities are distributed over the values of a random variable. For a discrete random variable, it is called probability mass function (PMF) and gives the probability of each specific value. For a continuous random variable, it is called a probability density function (PDF) and represents the likelihood of the variable taking on a particular value, with the area under the curve equating to probabilities. Probability distribution function are key in understanding the behaviour and characteristics of random variables.

6. Cumulative Distribution function

A cumulative distributive function (CDF) gives the probability that a random variable takes on a value less than or equal to a specific value. For any given value x, the CDF, denoted as $F(x)$, is the sum (for discrete variables) or the integral (for continuous variables) of the probability distribution function up to x. The CDF starts at 0 and increases to 1 as x moves from the smallest possible value to the largest, providing a complete view of the probability distribution by showing the cumulative probability up to any point.

7. Discrete Random Variables

Discrete Random Variables are types of random variables that take on a countable number of distinct values. Each value corresponds to a specific probability, which is described by the probability mass function (PMF). Examples of discrete random variables include the number of heads in a series of coin tosses or the number rolled on a die. Since the outcomes are countable, the sum of the probabilities for all possible values is equal to 1. Discrete random variables are fundamental in scenarios where outcomes are clearly distinct and finite.

8 Continuous Random Variables

Continuous Random Variables can take on an infinite number of values within a given range. Unlike discrete random variables, which have distinct, countable outcomes, continuous random variables are associated with ranges of values, and their probabilities are described by a probability density function (PDF). The probability of a continuous random variable taking on an exact value is zero; instead, probabilities are determined for intervals of values. Examples include measurements like height, weight, or time. The area under the PDF curve over a given interval represents the probability that the variable falls within that interval.

Conditional Probability

Probability of an event when another event has already occurred.

A: Getting an odd number
B: Getting one

$P(A) = 1/2$
$P(B) = 1/6$

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\frac{n(B\cap A)}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{n(B\cap A)}{n(A)}$$

$$= \frac{1}{3}$$

or $P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{1/6}{1/2} = 2\left(\frac{1}{6}\right) = \frac{1}{3}$

Q. What is the probability for getting an even number as a sum of the values of two top faces of two ~~dices.~~ dies, provided the difference between these two faces is exactly two.

A ~~B~~: {(1,3), (2,4), (3,5), (4,6)}
~~A~~ B: {(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6) }

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{P(A)}{P(A)} \quad (\because P(B\cap A) = P(A))$$

$$= 1$$

Q. What is the conditional probability for getting a prime number provided, the dice shows an odd number.

B: {2, 3, 5}
A: {1, 3, 5}

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\frac{n(B\cap A)}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{2}{3}$$

Frame a conditional probability question.

Q. Two coins are tossed at the same time. What is the conditional probability for getting ~~an odd number or 2 coins~~ head on 2 coins provided one of them is head.

B: {(H,H)}
A: {(H,T), (T,H), (H,H)}

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\frac{n(B\cap A)}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{1}{3}$$

Q. ~~In a~~ Given a ~~num~~ set of number cards. What is the conditional probability for getting the sum of 2 cards as 8, ~~one needs to be red and other needs to~~ provided that one of drawn card is red and other is black

B: { (2,6), (6,2), (3,5), (5,3), (4,4), (2,6), (6,2), (3,5), (5,3), (4,4),

A: {

Red (Spade)

—/—/—

Joint Probability

Here we find the likelihood of two events happening together.

Q. What is the probability that when we pick a card, it is ~~Joint probability~~ a red king.

There are 4 kings

$$P(\text{kings}) = \frac{4}{52}$$

There are 36 red cards

$$P(\text{red}) = \frac{26}{52} = \frac{1}{2}$$

$$\therefore P(\text{king} \cap \text{red}) = \frac{4}{52} \times \frac{1}{2} = \frac{2}{52} = \frac{1}{26}$$

$\boxed{\text{OR}}$

$$P = \frac{\text{no: of red kings}}{\text{total no: of cards}} = \frac{2}{52} = \frac{1}{26}$$

$\boxed{\text{OR}}$

using conditional probability,

$$P(\text{king} \mid \text{red}) = \frac{P(\text{king} \cap \text{red})}{P(\text{red})} = \frac{\frac{n(\text{king} \cap \text{red})}{n(S)}}{\frac{n(\text{red})}{n(S)}}$$

$$= \frac{2}{26}$$ (this is a different value)

—/—/—

So a relation,

$$P(A \cdot B) = P(A \mid B) \times P(B)$$

So ~~ske~~ checking for card case

$$P(\text{king} \cdot \text{red}) = P(\text{king} \mid \text{red}) \cdot \cancel{P(B)}\, P(\text{red})$$

$$= \frac{2}{26} \times \frac{1}{2} = \frac{1}{26}$$

Similarly checking,

$$P(\text{king} \cdot \text{red}) = P(\text{red} \mid \text{king}) \cdot P(\text{king})$$

$$= \frac{P(\text{red} \cap \text{king})}{P(\text{king})} \cdot P(\text{king})$$

$$= \frac{\frac{n(\text{red} \cap \text{king})}{n(S)}}{\frac{n(\text{king})}{n(S)}} \cdot P(\text{king})$$

$$\Rightarrow \frac{2}{\cancel{4}} \times \frac{\cancel{4}}{52} = \frac{2}{52} = \frac{1}{26}$$

Random Variable

$S = \{HH, HT, TH, TT\}$

$X : S \to \mathbb{R}$

$\boxed{f} \to \mathbb{R}$

$X(HH) = 2$

$X(HT) = 1$

$X(TH) = 1$

$X(TT) = 0$

Usually to determine the dependent variable y ~~we~~ we find probability of y with respect to independent variable x. But we can also find probabilities with respect to ~~those new~~ random variables.

$$\therefore P(X=0) = P(\{TT\}) = \frac{1}{4}$$

$$P(X=1) = P(\{HT, TH\}) = \frac{2}{4} = \frac{1}{2}$$

$$P(X=2) = P(\{HH\}) = \frac{1}{4}$$

x	0	1	2
$P(X)$	1/4	1/2	1/4

$$0 \le P(x_i) \le 1$$

$$\sum_i P(x_i) = 1$$

Q. Construct the probability distribution table when ~~is~~ for the experiment throwing 2 ~~dices~~ die.
A random variable maps each outcome to the sum of the face values.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

The above (random variable) case, the r.v is ~~distinct~~ discrete. But it can also be ~~dive~~ continuous.

eg: $[-5,5] \xrightarrow[x]{} \boxed{\epsilon} \longrightarrow x^2+1$

Mapping back,

$$S = [1, 26] \xrightarrow[\lfloor\sqrt{x-1}\rfloor]{} [0, 5]$$

So the random variable distribution is continuous and uniform.

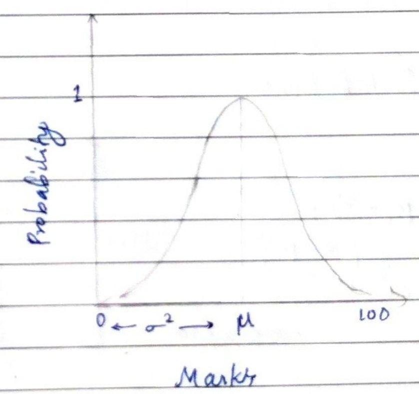
Probability Distribution Function (PDF) is usually used to represent continuous random variables.

~~Suppose~~ Consider, marks of 500 students, total marks = 100

~~P(0)~~ $P(X=0) = 3/500$

$P(X=10) = 20/500$

$P(X=100) = 2/500$



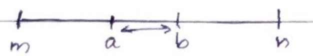
Probability ~~Distribut~~ Distribution Table (PDT) is used to represent discrete random variables.

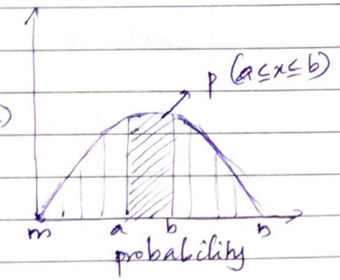
Continuous Probability Distribution

- A continuous random distribution does not have a PMF but has probability density function (PDF)
- It does not have PMF.

PDF

$m \le x \le n$





$f_x(x) = \text{PDF}$

$p < a \le x \le b = \int_a^b f_x(x)\,dx$

$f_x(x)$ satisfies $f_x(x) > 0$

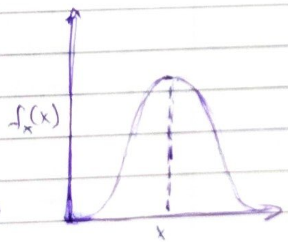
$$\boxed{\int_m^n f_x(dx) = 1}$$

Three Continuous Distributions

1. Normal Distribution

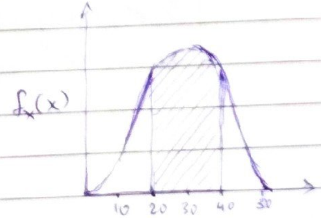
PDF of $x - N(\mu, \sigma^2)$:

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{1/2\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \le x < +\infty$$



μ = Mean, σ^2 = variance, σ = standard deviation

highest value when $x = \mu$



$p(20 \le x \le 30) = \int_{20}^{30} f_x(x)\,dx$

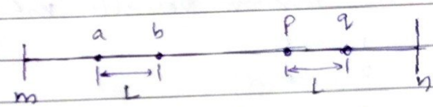
area under graph $\Rightarrow$ probability

$p(\mu - \sigma \le x \le x + \sigma) = 0.68$
$p(\mu - 2\sigma \le x < \mu + 2\sigma) = 0.98$
$p(\mu - 3\sigma < x \le \mu + 3\sigma) = 0.997$

2. Standard Normal Distribution $N(0,1)$ or z distribution

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{1/2\, x^2}, \quad \mu = 0,\ \sigma = 1$$

Uniform Distribution



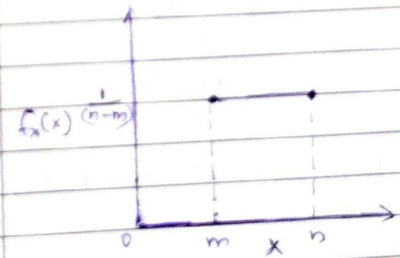
$(b-a) = (q-p) = L$

$p(a \le x \le b) = p(p \le x \le q)$

Uniform Distribution $= x \sim U(m, n)$

PDF of $x \sim U(m,n)$:

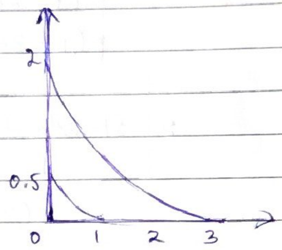
$$f_x(x) = \begin{cases} \frac{1}{n-m} & \text{for } m \le x \le n \\ 0 & \text{otherwise} \end{cases}$$



3. Exponential Distribution (Continuous Distribution)

PDF of $x \sim E(\lambda)$

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & \text{when } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Cumulative Distribution Function

x is a continuous random variable $m \leq x \leq n$

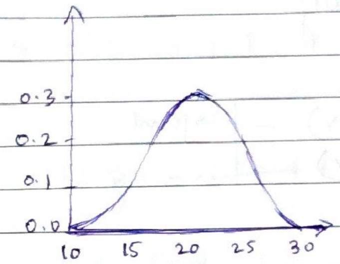
$p(x \leq z) = ?$

$$F_x(z) = p(x \leq z) = \int_m^z f_x(x)\, dx$$

Cumulative Distribution Function for Discrete Random Variable

$$\underset{\uparrow \text{ CDF}}{f_x(k)} = p(x \leq k) = \sum_{i=a}^{k} \underset{\uparrow \text{ PMF}}{p(x=1)}$$

Weight Distribution follows normal distribution
$N(21, 9)$
$p(\text{weight} > 27 \text{ kg}) = ?$



Baye's Theorem

Conditional Probability of an event = probability of that event given that a condition is met.
Conditional Probability $= p(x|y)$

Gene B \ Gene A	X	N
Y	50	30
N	20	100

X : Mutated
Y : Not Mutated
$n = 200$

$P(A=Y) = \frac{50+20}{200} = 0.35$
$p(A=N) = \frac{30+100}{200} = 0.65$
Marginal Probability

$$P(A=Y \mid B=Y) = \frac{p(A=Y, B=Y)}{p(B=Y)}$$

$$P(A=Y \mid B=N) = \frac{P(A=Y, B=N)}{P(B=N)}$$

$$P(B=N \mid A=Y) = \frac{P(A=Y, B=N)}{P(A=Y)}$$

Conditional Probability

$$\boxed{P(X|Y) = \frac{P(X,Y)}{P(Y)}}$$

$P(X,Y) \rightarrow$ joint

$P(Y) \rightarrow$ marginal

$A = Y \Rightarrow X$

$B = N \Rightarrow Y$

$P(A=Y | B=N) = P(X|Y)$

Marginal Probability

$$P(A=Y) = \frac{50+20}{200} = \frac{50}{200} + \frac{20}{200}$$

$$= P(A=Y, B=Y) + P(A=Y, B=N)$$

$$= P(A=Y | B=Y)\, P(B=Y) + P(A=Y | B=N)\, P(B=N)$$

$$\boxed{P(X) = P(X|Y)\,P(Y) + P(X|\neg Y)\,P(\neg Y)}$$

Conditional Probability

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}$$

$$P(Y|X) = \frac{P(X,Y)}{P(X)}$$

$$P(Y|X)\,P(X) = P(X|Y)\,P(Y)$$

$$\boxed{P(Y|X) = \frac{P(X|Y)\,.\,P(Y)}{P(X)}}$$ Baye's Theorem

$P(Y|X) \Rightarrow$ Posterior Probability of Y

$P(X|Y) \Rightarrow$ Likelihood of Y

$P(Y) \Rightarrow$ Prior of Y

$P(X) \Rightarrow$ Marginal Probability of X

Eg: rapid diagnostic test

Sensitivity (True positive rate) of the test = 99%

(True negative rate) of the test = 98%

Prevalence of disease = 0.1%

Q. What is the chance that a person has the ~~the~~ disease if he/she tests positive?

$P(T=+ | D=+) = 99\% = 0.99$

$P(T=- | D=-) = 98\% = 0.98$

$P(D=+) = 0.001$

$P(D=+ | T=+) = ?$

Using Baye's Theorem,

$$P(Y|X) = \frac{P(X|Y)\,.\,P(Y)}{P(X)}$$

$P(Y|X) \rightarrow (D=+)$

$$P(D=+ | T=+) = \frac{\overset{0.99}{P(T=+ | D=+)}\ \overset{0.001}{P(D=+)}}{P(T=+)}$$

$P(T=+) = P(T=+ \mid D=+)\,P(D=+) + P(T=+ \mid D=-)\,P(D=-)$

$= 0.99(0.001) + 0.02(0.999)$

$= \underline{0.047}$

$$P(D=+ \mid T=+) = \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.02 \times 0.999}$$

Data of 5 coin tosses

D: H T H T T

Probability of H, $p(H) = ?$

One model, $p(H) = \theta$; $0 \le \theta \le 1$

Probability of the data given a value of θ

$P(D|\theta) = P(H)\,P(T)\,P(H)\,P(T)\,P(T) = \theta^2(1-\theta)^3$

data model

likelihood

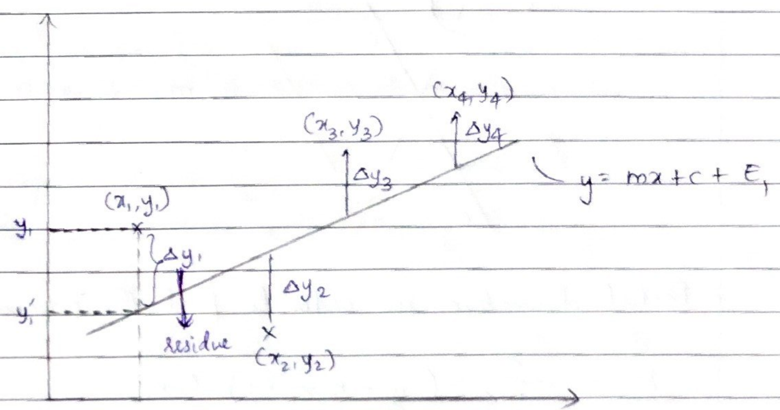
$$\boxed{P(D/\theta) = L(\theta|D)}$$

You can also use Bayes Theorem,

$$P(\theta|D) = \frac{P(D|\theta)\,P(\theta)}{P(D)}$$

29.08.2024

Linear Regression



$\Delta y_i = |y_i - y_i'| = \begin{cases} <0, & y_i < y_i' \\ >0, & y_i > y_i' \end{cases}$

$y_i' = mx_i + c$

$\left[\text{Total error} = \sum_{i=1}^{n} |\Delta y_i|^2\right]$

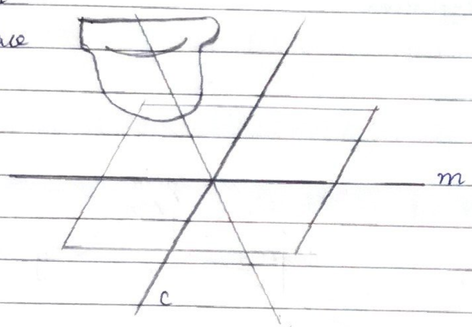
Total error $= \cancel{\sum_{i=1}^{n} [\Delta y_i]^2}\ \sum_{i=1}^{n} |\Delta y_i|^2$

$= \sum_{i=1}^{n} (y_i - (mx_i + c))^2$

$S(m, c) = \sum_{i=1}^{n} (y_i - mx_i - c)^2$

This error should be minimum.

Error surface



Partial derivative in order to find the extreme points.

$$\frac{\delta s}{\partial m} = \sum_{i=1}^{n} 2(y_i - mx_i - c)\cdot(-x_i)$$

$$= -2\sum x_i y_i + 2m\sum x_i^2 + 2c\sum x_i \longrightarrow ①$$

$$\frac{\delta s}{\delta c} = \sum_{i=1}^{n} 2(y_i - mx_i - c)\cdot(-1)$$

$$= -2\sum y_i + 2m\sum x_i + 2nc \longrightarrow ②$$

Now we have to equate these two to zero

$$① \Rightarrow -2\sum x_i y_i + 2m\sum x_i^2 + 2c\sum x_i = 0$$

$$m\sum x_i^2 + c\sum x_i = \sum x_i y_i \longrightarrow ③$$

$$② \Rightarrow -2\sum y_i + 2m\sum x_i + 2nc = 0$$

$$m\sum x_i + nc = \sum y_i \longrightarrow ④$$

$p = \sum x_i^2$; $q = \sum x_i$

$r = \sum y_i$; $s = \sum x_i y_i$

$$\therefore ③ \Rightarrow mp + cq = s$$

$$④ \Rightarrow mq + cn = r$$

Now we have to make the coefficients of ③ and ④ the same $\rightarrow$ ③ $\times q$, ④ $\times p$

$$mpq + cq^2 = sq$$

$$(-)\,mpq + (-)\,cpn = (-)\,rp$$

$$\Rightarrow sq - rp = cq^2 - cpn$$

$$\therefore c = \frac{sq - rp}{q^2 - pn}$$

$$\boxed{c = \frac{\sum x_i y_i \cdot \sum x_i - \sum y_i \sum x_i^2}{(\sum x_i)^2 - n\sum x_i^2}}$$

Now make coefficient of c same, ③ $\times n$ and ④ $\times q$

$$mpn + cqn = sn$$

$$(-)\,mq^2 + (-)\,cqn = (-)\,rq$$

$$\Rightarrow sn - rq = m(np - q^2)$$

$$\therefore m = \frac{sn - rq}{np - q^2}$$

$$\boxed{m = \frac{n\sum x_i y_i - \sum y_i \sum x_i}{n\sum x_i^2 - (\sum x_i)^2}}$$

'c' can be written in terms of m

$$\boxed{c = \frac{\sum y_i - m\sum x_i}{n}}$$

_/_/_

HW Develop an algorithm to optimize the value of m and c using gradient descent and implement it using python.
Also illustrate the impact of L_1 and L_2 regularization on the error surface by plotting it as a 3D surface plot.

03.09.24

Q) One-Dimensional Linear Regression

Hours Studied (x)	Score (Y)
1	52
2	55
3	60
4	63
5	68

1) Perform a simple linear regression analysis on ~~the~~ the dataset to determine the relationship b/w hours studied and score achieved. Find slope (m) and y intercept (b) of the regression line $y = mx + b$

$$y = mx + b$$

$$m = \frac{n\Sigma(xy) - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

$\Sigma x = 1 + 2 + 3 + 4 + 5 = 15$

$\Sigma y = 52 + 55 + 60 + 63 + 63 = 298$

$\Sigma xy = (1 \times 52) + (2 \times 55) + (3(60)) + (4 \times 63) + (5 \times 68) = 934$

$\Sigma x^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

_/_/_

$$\Rightarrow m = \frac{5 \times 934 - 15 \times 298}{5 \times 55 - 15^2} = \frac{200}{50} = 4$$

$$b = \frac{\Sigma y - m\Sigma x}{n}$$

$$b = \frac{298 - 4 \times 15}{5} = \frac{298 - 60}{5} = 47.6$$

$\therefore$ slope $= 4$

y intercept $= 47.6$

2) Predict the score of student who studied for 6 hours.

$y = mx + b$

$y = 4(6) + 47.6$

$= 24 + 47.6$

$= 71.6$

3) Calculate the mean ~~sq~~ squared error (MSE)

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

when $x = 1$ $\quad y = 4(1) + 47.6 = 51.6$

$x = 2$ $\quad y = 4(2) + 47.6 = 55.6$

$x = 3$ $\quad y = 4(3) + 47.6 = 59.6$

$x = 4$ $\quad y = 4(4) + 47.6 = 63.6$

$x = 5$ $\quad y = 4(5) + 47.6 = 67.6$

Now, using eqn,

$(y_i - \hat{y}_i)^2 \Rightarrow (52 - 51.6)^2 = 0.16$

$(55 - 55.6)^2 = 0.36$

$(60 - 59.6)^2 = 0.16$

$(63 - 63.6)^2 = 0.36$

$(68 - 67.6)^2 = 0.16$

$$MSE = \frac{0.16 + 0.36 + 0.16 + 0.36 + 0.16}{5}$$

$$= \frac{1.2}{5} = 0.24$$

2D