EXTENSION PRINCIPLE FOR FUZZY SETS

EXTENSION PRINCIPLE

We say that a crisp function

$$f : X \to Y$$

is fuzzified when it is extended to act on fuzzy sets defined on X and Y. That is, the fuzzified function, for which the same symbol f is usually used, has the form

$$f : \mathcal{F}(X) \to \mathcal{F}(Y),$$

and its inverse function, f^{-1}, has the form

$$f^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X).$$

EXTENSION PRINCIPLE

$$\chi_{B}(y) = \chi_{f(A)}(y) = \bigvee_{y=f(x)} \chi_{A}(x) \qquad \mu_{\underset{\sim}{B}}(y) = \bigvee_{f(x)=y} \mu_{\underset{\sim}{A}}(x)$$

Example 12.1. Suppose we have a crisp set A = {0, 1}, or, using Zadeh's notation,

$$A = \left\{ \frac{0}{-2} + \frac{0}{-1} + \frac{1}{0} + \frac{1}{1} + \frac{0}{2} \right\}$$

defined on the universe X = {−2, −1, 0, 1, 2} and a simple mapping $y = |4x| + 2$. We wish to find the resulting crisp set B on an output universe Y using the extension principle. From the mapping we can see that the universe Y will be Y = {2, 6, 10}. The mapping described by Eq. (12.2) will yield the following calculations for the membership values of each of the elements in universe Y:

$$\chi_B(2) = \vee\{\chi_A(0)\} = 1$$

$$\chi_B(6) = \vee\{\chi_A(-1), \chi_A(1)\} = \vee\{0, 1\} = 1$$

$$\chi_B(10) = \vee\{\chi_A(-2), \chi_A(2)\} = \vee\{0, 0\} = 0$$

Notice there is only one way to get the element 2 in the universe Y, but there are two ways to get the elements 6 and 10 in Y. Written in Zadeh's notation this mapping results in the output

$$B = \left\{ \frac{1}{2} + \frac{1}{6} + \frac{0}{10} \right\}$$

or, alternatively, B = {2, 6}.

EXTENSION PRINCIPLE

$$\chi_{B}(y) = \chi_{f(A)}(y) = \bigvee_{y=f(x)} \chi_{A}(x) \qquad \mu_{\underset{\sim}{B}}(y) = \bigvee_{f(x)=y} \mu_{\underset{\sim}{A}}(x)$$

Let fuzzy sets $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ be defined on the universes $X_1, X_2, \ldots, X_n$. The mapping for these particular input sets can now be defined as $\underset{\sim}{B} = f(\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n)$, where the membership function of the image $\underset{\sim}{B}$ is given by

$$\mu_{\underset{\sim}{B}}(y) = \max_{y=f(x_1,x_2,\ldots,x_n)} \{\min[\mu_{\underset{\sim}{A}_1}(x_1), \mu_{\underset{\sim}{A}_2}(x_2), \ldots, \mu_{\underset{\sim}{A}_n}(x_n)]\} \qquad (12.7)$$

EXTENSION PRINCIPLE

$$\mu_{\underset{\sim}{B}}(y) = \bigvee_{x \in X} (\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{R}}(x, y)) \tag{12.13}$$

The preceding expression is analogous to a fuzzy composition performed on fuzzy vectors, or $\underset{\sim}{\mathbf{b}} = \underset{\sim}{\mathbf{a}} \circ \underset{\sim}{R}$, or in vector form,

$$\underset{\sim}{\mathbf{b}}_j = \max_i(\min(a_i, r_{ij})) \tag{12.14}$$

where $\underset{\sim}{\mathbf{b}}_j$ is the jth element of the fuzzy image $\underset{\sim}{B}$.

Example 12.2. Suppose we have a fuzzy mapping, $\underset{\sim}{f}$, given by the following fuzzy relation, $\underset{\sim}{R}$:

$$\underset{\sim}{R} = \begin{array}{c} \begin{array}{ccccc} 1.4 & 1.5 & 1.6 & 1.7 & 1.8 \end{array} \quad (\mathrm{m}) \\ \begin{bmatrix} 1 & 0.8 & 0.2 & 0.1 & 0 \\ 0.8 & 1 & 0.8 & 0.2 & 0.1 \\ 0.2 & 0.8 & 1 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.8 & 1 & 0.8 \\ 0 & 0.1 & 0.2 & 0.8 & 1 \end{bmatrix} \begin{array}{l} 40 \quad (\mathrm{kg}) \\ 50 \\ 60 \\ 70 \\ 80 \end{array} \end{array}$$

EXTENSION PRINCIPLE

Example 12.2. Suppose we have a fuzzy mapping, $\underset{\sim}{f}$, given by the following fuzzy relation, $\underset{\sim}{R}$:

$$\underset{\sim}{R} = \begin{array}{c} \\ \\ \end{array}\begin{array}{l} \begin{array}{ccccc} 1.4 & 1.5 & 1.6 & 1.7 & 1.8 \end{array} \ \text{(m)} \\ \begin{bmatrix} 1 & 0.8 & 0.2 & 0.1 & 0 \\ 0.8 & 1 & 0.8 & 0.2 & 0.1 \\ 0.2 & 0.8 & 1 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.8 & 1 & 0.8 \\ 0 & 0.1 & 0.2 & 0.8 & 1 \end{bmatrix} \begin{array}{l} 40 \ \text{(kg)} \\ 50 \\ 60 \\ 70 \\ 80 \end{array} \end{array}$$

which represents a fuzzy mapping between the length and mass of test articles scheduled for flight in a space experiment. The mapping is fuzzy because of the complicated relationship between mass and the cost to send the mass into space, the constraints on length of the test articles fitted into the cargo section of the spacecraft, and the scientific value of the experiment. Suppose a particular experiment is being planned for flight, but specific mass requirements have not been determined. For planning purposes the mass (in kilograms) is presumed to be a fuzzy quantity described by the following membership function:

$$\underset{\sim}{A} = \left\{ \frac{0.8}{40} + \frac{1}{50} + \frac{0.6}{60} + \frac{0.2}{70} + \frac{0}{80} \right\} \text{kg}$$

or as a fuzzy vector $\underset{\sim}{a} = \{0.8, 1, 0.6, 0.2, 0\}$ kg.

The fuzzy image $\underset{\sim}{B}$ can be found using the extension principle (or, equivalently, composition for this fuzzy mapping), $\underset{\sim}{b} = \underset{\sim}{a} \circ \underset{\sim}{R}$ (recall that a set is also a one-dimensional relation). This composition results in a fuzzy output vector describing the fuzziness in the length of the experimental object (in meters), to be used for planning purposes, or $\underset{\sim}{b} = \{0.8, 1, 0.8, 0.6, 0.2\}$ m.