

Hence,

$$\begin{aligned}
\nabla_{\theta} J(\theta) &= \nabla_{\theta} \frac{1}{2} (\vec{X}\theta - \vec{y})^T (\vec{X}\theta - \vec{y}) \\
&= \frac{1}{2} \nabla_{\theta} (\theta^T X^T X \theta - \theta^T X^T \vec{y} - \vec{y}^T X \theta + \vec{y}^T \vec{y}) \\
&= \frac{1}{2} \nabla_{\theta} \text{tr} (\theta^T X^T X \theta - \theta^T X^T \vec{y} - \vec{y}^T X \theta + \vec{y}^T \vec{y}) \\
&= \frac{1}{2} \nabla_{\theta} (\text{tr} \theta^T X^T X \theta - 2 \text{tr} \vec{y}^T X \theta) \\
&= \frac{1}{2} (X^T X \theta + X^T X \theta - 2 X^T \vec{y}) \\
&= X^T X \theta - X^T \vec{y}
\end{aligned}$$

In the third step, we used the fact that the trace of a real number is just the real number; the fourth step used the fact that  $\text{tr}A = \text{tr}A^T$ , and the fifth step used Equation (5) with  $A^T = \theta$ ,  $B = B^T = X^T X$ , and  $C = I$ , and Equation (1). To minimize  $J$ , we set its derivatives to zero, and obtain the **normal equations**:

$$X^T X \theta = X^T \vec{y}$$

Thus, the value of  $\theta$  that minimizes  $J(\theta)$  is given in closed form by the equation

$$\theta = (X^T X)^{-1} X^T \vec{y}.$$

### 3 Probabilistic interpretation

When faced with a regression problem, why might linear regression, and specifically why might the least-squares cost function  $J$ , be a reasonable choice? In this section, we will give a set of probabilistic assumptions, under which least-squares regression is derived as a very natural algorithm.

Let us assume that the target variables and the inputs are related via the equation

$$y^{(i)} = \theta^T x^{(i)} + \epsilon^{(i)},$$

where  $\epsilon^{(i)}$  is an error term that captures either unmodeled effects (such as if there are some features very pertinent to predicting housing price, but that we'd left out of the regression), or random noise. Let us further assume that the  $\epsilon^{(i)}$  are distributed IID (independently and identically distributed) according to a Gaussian distribution (also called a Normal distribution) with

mean zero and some variance  $\sigma^2$ . We can write this assumption as “ $\epsilon^{(i)} \sim \mathcal{N}(0, \sigma^2)$ .” I.e., the density of  $\epsilon^{(i)}$  is given by

$$p(\epsilon^{(i)}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\epsilon^{(i)})^2}{2\sigma^2}\right).$$

This implies that

$$p(y^{(i)}|x^{(i)}; \theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right).$$

The notation “ $p(y^{(i)}|x^{(i)}; \theta)$ ” indicates that this is the distribution of  $y^{(i)}$  given  $x^{(i)}$  and parameterized by  $\theta$ . Note that we should not condition on  $\theta$  (“ $p(y^{(i)}|x^{(i)}, \theta)$ ”), since  $\theta$  is not a random variable. We can also write the distribution of  $y^{(i)}$  as  $y^{(i)} | x^{(i)}; \theta \sim \mathcal{N}(\theta^T x^{(i)}, \sigma^2)$ .

Given  $X$  (the design matrix, which contains all the  $x^{(i)}$ 's) and  $\theta$ , what is the distribution of the  $y^{(i)}$ 's? The probability of the data is given by  $p(\vec{y}|X; \theta)$ . This quantity is typically viewed a function of  $\vec{y}$  (and perhaps  $X$ ), for a fixed value of  $\theta$ . When we wish to explicitly view this as a function of  $\theta$ , we will instead call it the **likelihood** function:

$$L(\theta) = L(\theta; X, \vec{y}) = p(\vec{y}|X; \theta).$$

Note that by the independence assumption on the  $\epsilon^{(i)}$ 's (and hence also the  $y^{(i)}$ 's given the  $x^{(i)}$ 's), this can also be written

$$\begin{aligned} L(\theta) &= \prod_{i=1}^m p(y^{(i)} | x^{(i)}; \theta) \\ &= \prod_{i=1}^m \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right). \end{aligned}$$

Now, given this probabilistic model relating the  $y^{(i)}$ 's and the  $x^{(i)}$ 's, what is a reasonable way of choosing our best guess of the parameters  $\theta$ ? The principle of **maximum likelihood** says that we should choose  $\theta$  so as to make the data as high probability as possible. I.e., we should choose  $\theta$  to maximize  $L(\theta)$ .

Instead of maximizing  $L(\theta)$ , we can also maximize any strictly increasing function of  $L(\theta)$ . In particular, the derivations will be a bit simpler if we

instead maximize the **log likelihood**  $\ell(\theta)$ :

$$\begin{aligned}\ell(\theta) &= \log L(\theta) \\ &= \log \prod_{i=1}^m \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\ &= \sum_{i=1}^m \log \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\ &= m \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{\sigma^2} \cdot \frac{1}{2} \sum_{i=1}^m (y^{(i)} - \theta^T x^{(i)})^2.\end{aligned}$$

Hence, maximizing  $\ell(\theta)$  gives the same answer as minimizing

$$\frac{1}{2} \sum_{i=1}^m (y^{(i)} - \theta^T x^{(i)})^2,$$

which we recognize to be  $J(\theta)$ , our original least-squares cost function.

To summarize: Under the previous probabilistic assumptions on the data, least-squares regression corresponds to finding the maximum likelihood estimate of  $\theta$ . This is thus one set of assumptions under which least-squares regression can be justified as a very natural method that's just doing maximum likelihood estimation. (Note however that the probabilistic assumptions are by no means *necessary* for least-squares to be a perfectly good and rational procedure, and there may—and indeed there are—other natural assumptions that can also be used to justify it.)

Note also that, in our previous discussion, our final choice of  $\theta$  did not depend on what was  $\sigma^2$ , and indeed we'd have arrived at the same result even if  $\sigma^2$  were unknown. We will use this fact again later, when we talk about the exponential family and generalized linear models.

## 4 Locally weighted linear regression

Consider the problem of predicting  $y$  from  $x \in \mathbb{R}$ . The leftmost figure below shows the result of fitting a  $y = \theta_0 + \theta_1 x$  to a dataset. We see that the data doesn't really lie on straight line, and so the fit is not very good.