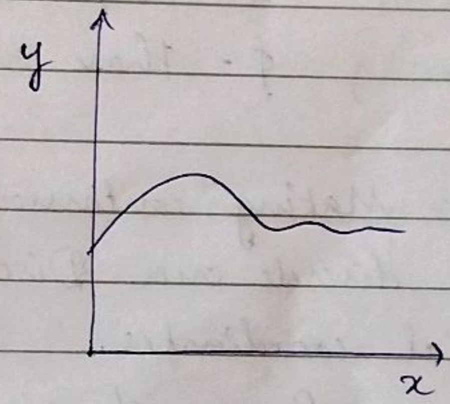
DIGITAL SIGNAL PROCESSING

- Signal- carries information by creating somekind of disturbance. Every signal is represented in the form of mathematical equations.

$y = f(t)$ one dimensional signal

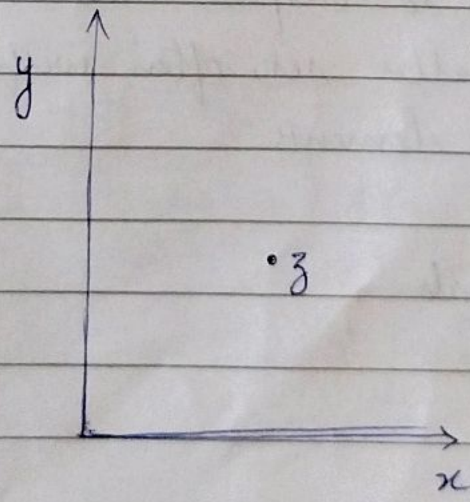
strength of signal/amplitude



$z = f(x,y)$ two dimensional signal

$x,y \rightarrow$ spacial coordinates

$z \rightarrow$ intensity value at x,y



Mathematical Representation

$y = \sin(t)$ but this is incomplete

$\therefore y = A \sin(\omega t + \phi)$

ω - Angular frequency $= 2\pi f$

A - Amplitude

ϕ - Phase angle

Sampling: Making continuous spacial coordinates into discrete ones. Discredization of spacial coordinates.

Quantization:- Process of mapping input values from a large ~~set~~ set (often a continuous set) to output values in a (countable) smaller set, often with a finite ~~set~~ no: of elements.

Signals

functions

1. Impulse Sequence

$$x(n_1, n_2) = \delta(n_1, n_2) = \begin{cases} 1 & , n_1 = n_2 = 0 \\ 0 & , \text{else} \end{cases}$$

$\delta(x) = 0 \; ; \; x \neq 0 \rightarrow$ Dirac delta function

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \delta(x)\, dx = 1$$

$$\delta(n) = \begin{cases} 0 & , n \neq 0 \\ 1 & , n = 0 \end{cases} \Big\} \text{ Kronecker (discrete)}$$

The particular value

eg: $\delta(n) = 1$ for $n = 0$;

is called the impulse

$\Rightarrow$ this is used for sampling.

We further have to show that this function satisfies shifting property

$$\int_{-\infty}^{\infty} f(x')\, \delta(x - x')\, dx' = \int f(x)$$

this gives 1 (as $\frac{\varepsilon}{\varepsilon} \int f(x)\, dx = 1$)

if $x' = 0$

but that is for point 0,

but for $x' = x$

so x' is shifted to x,
so impulse at x $(t-t_0)$

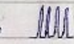
$$\int_{-\infty}^{\infty} f(x')\,\delta(x-x')\,dx' = f(x)$$

Writing it in discrete form,

$$\sum_{m=-\infty}^{\infty} f(m)\,\delta(n-m) = \underline{\underline{f(n)}}$$

this function takes value 1 for $m=n$. So we are defining the impulse for different points.

This is also called comb function

from  to 

we get more than one impulse value by shifting the point (taking different samples). ~~in 2 dimensions~~

In 2 dimension $\rightarrow$ bed of nail function

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

So if we do

$$\sum_{n=-\infty}^{\infty} f(n)\,\delta(x-n) = \underline{\underline{f(x)}}$$

Shifting applied on 2-dimension,

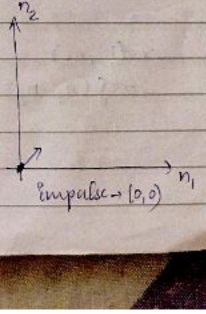
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,\delta(x-x'),(y-y')\,dx'\,dy'$$

$$= \underline{\underline{f(x,y)}} \qquad f(x',y')\,\delta(x-x',y-y')$$

Writing for discrete case

$$\Rightarrow \sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} f(m',n')\,\delta(m-m',n-n')$$

$$= \underline{\underline{f(m,n)}}$$

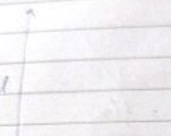


Suppose the first impulse $\rightarrow (0,0) \rightarrow (n_1, n_2)$ we want to shift to (m_1, m_2) then delta fn $\rightarrow \delta(m_1-n_1, m_2-n_2)$

$f(n_1, n_2)$ $= f(m_1, m_2)$

$x(n_1, n_2) = \delta(n_1)$

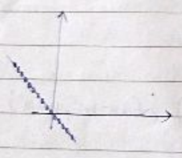
vertical line impulse



this is the graph as delta function is defined in terms of n_1 (when $n_1 = 0$, x-coordinate = 0 ⇒ y-axis)

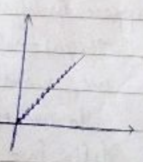
to get horizontal line impulse → $\delta(n_2)$

Suppose we want



$x(n_1, n_2) = \delta(n_1 + n_2)$

for



$x(n_1, n_2) = \delta(n_1 - n_2)$

$x(n_1, n_2) = \delta(2n_1 - n_2)$
$x(n_1, n_2) = \delta(n_1 + n_2 - 1)$ } Sketch in 2 Dimension

2) ~~Dim~~ Exponential Sequence

$x(n_1, n_2) = a^{n_1} b^{n_2}$ $\quad -\infty < n_1, n_2$

$a = e^{j\omega_1}$
$b = e^{j\omega_2}$ } a and b takes unit form →

when a and b takes these values its characteristics change $x(n_1, n_2) = e^{j\omega_1 n_1} e^{j\omega_2 n_2}$ $= e^{j\omega_1 n_1 + j\omega_2 n_2}$	magnitude of complex numbers ⇒ root of (sum of square of real & imaginary part) ⇒ $\sqrt{\cos^2\theta + \sin^2\theta}$ $= \sqrt{1} = 1$

this Exponential transforms to a complex sinusodial sequence → base of Forries transfer when a takes a unit form.

P.T.O.

3) Seperable Sequence

$x(n_1, n_2) = x_1(n_1) \cdot x_2(n_2)$

$\delta(n_1, n_2) = \delta(n_1) \cdot \delta(n_2) \rightarrow$ this is also seperable

4) Periodic Sequence

eg: sin function

Fourier is a periodic sequence

It is repeats itself periodically after certain time intervals.

Problem context requires us to drop the function intensity.

$x(n)$ is periodic, mathematically represent

$x(n) = x(n+N)$

$N \rightarrow$ is the period

If there are 2 functions with different time p

$x(n_1, n_2) = x(n_1 + N_1, n_2)$

$= x(n_1, n_2 + N_2)$

Q. $x(n_1, n_2) = \cos(\frac{\pi}{4} n_1 + \frac{\pi}{2} n_2)$

$x(n_1, n_2) = \cos(n_1 + n_2)$

Check whether the 2 functions are periodic or not

2D System

System → input, process, output

Mathematically,

$y(n_1, n_2) = T(x(n_1, n_2))$

$x(n_1, n_2) \Rightarrow$ input

$T() \Rightarrow$ Transforming process

$y(n_1, n_2) \Rightarrow$ output

IMP Linear System

$x(n_1') = T(x(n_1))$

Linear system satisfies scaling, if multiplying input reflects on output

Super solving must be done by maintaining noise compression

i) any scaling of the input will lead to equal scaling in output.

ii) if input signal is the sum of 2 input signal, then the output signal can also be written as the sum of corresponding output signals.

A linear system follows both the above properties.

Response of the system to a weighted sum of signals should be equal to the corresponding weighted sum of the output of the system to each to the individual input systems. $x_1(n_1, n_2) = y_1(n_1, n_2)$, $x_2(n_1, n_2) = y_2(n_1, n_2)$

$$T[a x_1(n_1, n_2) + b x_2(n_1, n_2)] = [a y_1(n_1, n_2) + b y_2(n_1, n_2)]$$

Superposition principle?

Q. Determine whether the following systems are linear

i) $y(n_1, n_2) = n\, x(n_1, n_2)$

ii) $y(n_1, n_2) = e^{x(n_1, n_2)}$

iii) $y(n_1, n_2) = A\, x(n_1, n_2)$

07.01.24

Shift Invariant System

One dimension → time invariant

The input - output characteristic do not change with time.

Spacially invariant (two dimensional) :- An input gives the same characteristic output even when the space is shifted.

$$T[x(n_1 - m_1, n_2 - m_2)] = y(n_1 - m_1, n_2 - m_2)$$

LSI (Linear Shift Invariant)

If two conditions is satisfied then system is LSI.

Q. $y(n) = x(-n)$

check whether the given system is LSI or not.

$$y(n, t) = T[x(n-t)] \quad \text{(shift in input)}$$

$$= x(-n-t) \quad \text{(shift in output)}$$

if $y(n-t) = \cancel{x(-(n+t))}\ x(-(n-t))$

$= x(-n+t)$

∴ the values are not invariant
So the given system is not Linear
Shift Invariant.

Q $y(n_1, n_2) = 5x(n_1, n_2)$

$y(n_1, n_2) = n\,x(n_1, n_2)$

If a system is LSI, given an impulse and its response, we can find the output corresponding to different inputs

Static vs Dynamic Systems

Static (Memory-less System) - No need for buffering or saving. So the output of the system depends only on the current input, at that point of time, the inputs of past and future doesn't matter.

Stable system or stability of an LSI System

BIBO (Bounded Input - Bounded Output)
When you add all the impulse response it is less than infinity.

$$\sum_{n_1}\sum_{n_2} |f_n(n_1, n_2)| < \infty$$

2D Kronecker delta

Assume this point 2D Kronecker delta function is defined at location (m', n').
We want to get output at location (m, n) the output,

$$\underbrace{h(m, n, m', n')}_{\text{impulse-response of the systems (PSF)}} = \underbrace{H}_{\text{filter}}\left[\underbrace{\delta(m-m', n-n')}_{\text{input}}\right]$$

we only need this to find unique output to given input as it characterises the system. ~~FH~~ Also called PSF (Point Spread Function)
Light is filter that scans through the signal.
It spread from a point of light, the area this light covers → Region of support.

The smallest closed region in m', n' plane outside which the impulse response is zero - Region of support
FIR - Finite Impulse Response (

But when feedback is introduced it becomes infinite (IIR)

$y(m,n) = H[x(m,n)]$

$= H\left[\sum_{m'} \sum_{n'} x(m',n') \delta(m-m', n-n')\right]$

$= \sum_{m'} \sum_{n'} x(m',n') H[\delta(m-m', n-n')]$

$= \sum_{m'} \sum_{n'} x(m',n') h(m,n,m',n')$

$\boxed{\text{for } H[\delta(m,n)] = h(m,n,0,0)}$

So $H[\delta(m-m', n-n')] = h(m-m', n-n', 0, 0)$

therefore,

$y(m,n) = \sum_{m'} \sum_{n'} x(m',n') h(m-m', n-n', 0, 0)$

$= \sum_{m'} \sum_{n'} x(m',n') H[\delta(m-m', n-n')]$

$= \sum_{m'} \sum_{n'} x(m',n') h(m-m', n-n')$

(no need for 0)

30.01.24

Spatial Correlation

$y(m,n) = x(m,n) \circledast h(m,n)$ (convolution)

where convolving $x(m,n)$ and $h(m,n)$ gives $y(m,n)$

$y(m,n) = f(m,n) \star h(m,n)$ (correlation)

measuring similarity or pattern between $f(m,n)$ and $h(m,n)$. [Correlation doesn't satisfy Associativity]

$(f \odot h_1) \odot h_2 = f \odot (h_1 \odot h_2)$ (Associativity property over convolution)

f is the whole input set, so first calculating $(h_1 \odot h_2)$ is more easier. h_1 and h_2 are point spread functions (with small values or dimensions)

$y(m,n) = f(m,n) \star h(m,n)$

$\Rightarrow \sum_{m'} \sum_{n'} f(m+m', n+n') h(m',n')$

Q. Take an Input $x(m,n)$

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

the impulse response is given $h(m,n)$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Toeplez matrix makes this computation easier.

First element in $x(m,n) \rightarrow 4$
its index $(0,0)$

so $x(m,n)$

$$\begin{bmatrix} 4_{(0,0)} & 5_{(0,1)} & 6_{(0,2)} \\ 7_{(1,0)} & 8_{(1,1)} & 9_{(1,2)} \end{bmatrix}$$

$$h(m,n) \Rightarrow \begin{bmatrix} 1_{(0,0)} \\ 1_{(1,0)} \\ 1_{(2,0)} \end{bmatrix}$$

dimension of
The resultant matrix

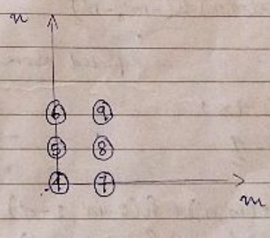
$$\begin{bmatrix} 00 & 01 & 02 \\ 10 & & \\ 20 & & \\ 30 & & \end{bmatrix}$$

$$\frac{\text{The no: of rows } x(m,n) + \text{The no: of rows of } h(m,n) - 1}{\text{no: of columns of } x(m,n) + \text{no: of columns of } h(m,n) - 1}$$

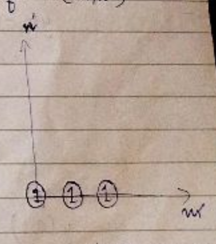
$$\Rightarrow \frac{2+3-1}{3+1-1} \Rightarrow \frac{4}{3} \quad (\text{i.e } 4 \times 3)$$

Plot in $m \times n$ space

for $x(m,n)$



for $h(m',n')$

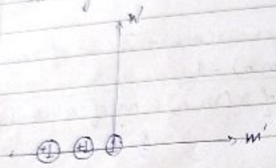


(0,0) (1,0) (2,0)

(0,0) (-1,0) (-2,0)

$y(0,0) =$

drawing $h(-m',-n')$



$$y(m,n) = \sum_{m'}\sum_{n'} x(m',n')\, h(m-m', n-n')$$

$(m,n) = (0,0)$

$$y(0,0) = \sum_{m'}\sum_{n'} x(m',n')\, \underbrace{h(-m',-n')}_{\text{as depicted above}}$$

$= 4 \times 1$

$= \underline{4}$

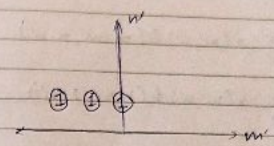
$-n'+1$

$= 1-n'$

$$y(0,1) = \sum_{m'}\sum_{n'} x(m',n')\, h(0-m', 1-n') \quad -n'+1$$

$$= \sum_{m'}\sum_{n'} x(m',n')\, h(-m', 1-n')$$

Plotting on graph $\rightarrow h(-m', 1-n')$

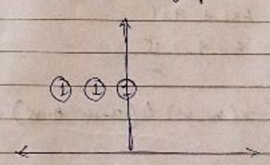


$y(0,1) = 5 \times 1 = \underline{5}$

$$y(0,2) = \sum_{m'}\sum_{n'} x(m',n')\, h(m-m', n-n')$$

$$= \sum_{m'}\sum_{n'} x(m',n')\, h(-m', 2-n')$$

~~Rott~~ Plotting on graph



$y(0,2) = 6 \times 1 = \underline{6}$

no use in going to $y(0,3)$

$$y(1,0) = \sum_{m'}\sum_{n'} x(m',n')\, h(1-m', -n')$$

$= 4\times 1 + 7 \times 1 = 4+7 = \underline{11}$

$$y(1,1) = \sum_{m'}\sum_{n'} x(m',n')\, h(m-m', n-n')$$

$$= \sum_{m'}\sum_{n'} x(m',n')\, h(1-m', 1-n')$$

$$= 5\times1 + 8\times1$$

$$= 5+8 = \underline{\underline{13}}$$

$$y(1,2) = \sum_{m'}\sum_{n'} x(m',n')\, h(m-m', n-n')$$

$$= \sum_{m'}\sum_{n'} x(m',n')\, h(1-m', 2-n')$$

$$= 6\times1 + 9\times1$$

$$= 6+9 = \underline{\underline{15}}$$

$$y(2,0) = \sum_{m'}\sum_{n'} x(m',n')\, h(2-m', 0-n')$$

$$= 4+7 = \underline{\underline{11}}$$

$$y(2,1) = \sum_{m'}\sum_{n'} x(m',n')\, h(2-m', 1-n')$$

$$= 5+8 = \underline{\underline{13}}$$

$$y(2,2) = \sum_{m'}\sum_{n'} x(m',n')\, h(2-m', 2-m')$$

$$= 6+9 = \underline{\underline{15}}$$

$$y(3,0) = \sum_{m'}\sum_{n'} x(m',n')\, h(3-m', 0-n')$$

$$= \underline{\underline{7}}$$

$$y(3,1) = \sum_{m'}\sum_{n'} x(m',n')\, h(3-m', 1-n')$$

$$= \underline{\underline{8}}$$

$$h(3,2) = \sum_{m'}\sum_{n'} x(m',n')\, h(3-m', 2-n')$$

$$= \underline{\underline{9}}$$

The resultant matrix

$$\begin{bmatrix} 4 & 5 & 6 \\ 11 & 13 & 15 \\ 11 & 13 & 15 \\ 7 & 8 & 9 \end{bmatrix}$$

Q. $x(m,n)$ is a 3x3 matrix ; $h(m,n)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Q.2) $x(m,n)$ is a 3X3 matrix ; $h(m,n)$ is 1x3

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$$

01.02.24

Transforms

We are moving the data from one domain to another domain ~~in this~~ using Transforms (a mathematical tool)

The information content of the data should not change after the process.

The purpose lies in the fact that certain characteristics are attainable only when we ~~change~~ move the data to another domain.

After transforming, certain processes like filtering & other alterations takes very less time, it also provides more information compared to the actual domain.

~~11-08-24~~

Transform are projection of data into basis functions (these are orthogonal).

The need for transforms - 2 aspects

1) The mathematical convenience
   We can perform operations very easily
2) To extract more information.
   The domain change can provide more information.

Given an audio signal, in one domain we can only ~~all~~ know the ~~fre~~ amplitude at each time point. But transforming can give us a lot more properties like range of frequencies etc.

Classification of Transforms

1) Orthogonal Sinusoidal, eg: ~~Fourier~~ Fourier Transform, DCT
2) Orthogonal Non-sinusoidal (basis fn are orthogonal)
   eg: ~~B~~ HARR transform, Wavelet
3) ~~Non-orthogonal Non sinusoidal~~ Transform based on Statistics of the Data input Signal
   leads to a very compact representation (energy compactation) eg: SVD, KL transform

(based on autocorrelation method)

4) Directional Transform (Medical ~~Imaging~~ Imaging)
   eg: Radon Transform, Hough transform
5) ~~Hough Transform~~
   ~~Detecting shapes in signal (or image)~~

Fourier Theorem

We project input signal through set of basis functions where each basis functions are sinusoidal and orthogonal and can be represented in exponential form.

Complex sinusoidal sequence is used.

$$F(\xi) = \int_{-\infty}^{\infty} f(x) \cdot \underbrace{e^{-j2\pi\xi x}}_{\text{set of basis function}} \, dx$$

in terms of frequency

this is in terms of time

$\xi \rightarrow$ frequency information

$x \rightarrow$ time

$f(x) \rightarrow$ input signal

$2\pi\xi \rightarrow$ angular frequency

~~Inverse ~~Fast~~ Fourier Transform,~~

~~$$F(\xi) = \int_{-\infty}^{\infty} f(x)$$~~

Inverse Fourier Transform :-

$$f(x) = \int_{-\infty}^{\infty} F(\xi)\, e^{j2\pi\xi x}\, d\xi$$

here exponential term (as sin + cos ?.

to get scalar values ⟶ we need magnitude spectrum

That's why DCT became popular

For 2-dimension,

$$F(\xi_1, \xi_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi[\xi_1 x + \xi_2 y]}\, dx\, dy$$

inverse,

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi_1, \xi_2)\, e^{j2\pi[\xi_1 x + \xi_2 y]}\, d\xi_1\, d\xi_2$$

Homework

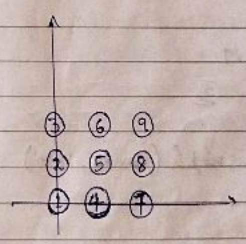
Q. $x(m,n)$ is a 3×3 matrix ; $h(m,n)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

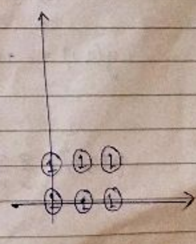
$$\begin{bmatrix} 1_{(0,0)} & 2_{(0,1)} & 3_{(0,2)} \\ 4_{(1,0)} & 5_{(1,1)} & 6_{(1,2)} \\ 7_{(2,0)} & 8_{(2,1)} & 9_{(2,2)} \end{bmatrix} \qquad \begin{bmatrix} 1_{(0,0)} & 1_{(0,1)} \\ 1_{(1,0)} & 1_{(1,1)} \\ 1_{(2,0)} & 1_{(2,1)} \end{bmatrix}$$

Plotting,

for $x(m,n)$



for $h(m',n')$



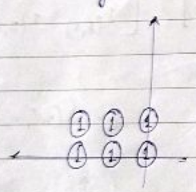
Finding dimension of resultant matrix,

$$\Rightarrow \frac{3+3-1}{3+2-1} = \frac{5}{4} \quad (\text{i.e } 5\times 4)$$

$$y(0,0) = \sum_{m'}\sum_{n'} x(m',n')\, h(m-m', n-n')$$

$$\Rightarrow \sum_{m'}\sum_{n'} x(m',n')\, h(-m',-n')$$

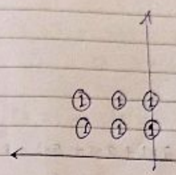
~~Plotting~~
Plotting $h(-m',-n')$,



$$y(0,0) = 1\times 1 + 2\times 1$$
$$= 1 + 2 = 3$$

$$y(0,1) = \sum_{m'}\sum_{n'} x(m',n')\, h(-m', 1-n')$$

Plotting, $h(-m', 1-n')$



$$y(0,1) = 2\times 1 + 3\times 1$$
$$= 2+3 = 5$$

$$y(0,2) = 3\times 1 = 3$$

$$y(0,3) = 0$$

$$y(0,4) = 0$$

$$y(1,0) = 1\times 1 + 4\times 1 + 2\times 1 + 5\times 1$$
$$\Rightarrow 1 + 4 + 2 + 5 = 12$$

$$y(1,1) = 2\times 1 + 5\times 1 + 3\times 1 + 6\times 1$$
$$\Rightarrow 2 + 5 + 3 + 6 = 16$$

$$y(1,2) \Rightarrow 3\times 1 + 6\times 1 = \cancel{3+6}$$
$$\Rightarrow 3+6 = 9$$

$y(1,3) = 0$

$y(1,4) = 0$

$y(2,0) = 1\times1 + 4\times1 + 7\times1 + 2\times1 + 5\times1 + 8\times1$

$\Rightarrow 1 + 4 + 7 + 2 + 5 + 8$

$\Rightarrow 27$

$y(2,1) = 2\times1 + 5\times1 + 8\times1 + 3\times1 + 6\times1 + 9\times1$

$\Rightarrow 2 + 5 + 8 + 3 + 6 + 9$

$\Rightarrow 33$

$y(2,2) = 3\times1 + 6\times1 + 9\times1$

$\Rightarrow 3 + 6 + 9 \Rightarrow 18$

$y(2,3) \Rightarrow 0$

$y(2,4) \Rightarrow 0$

$y(3,0) = 4\times1 + 7\times1 + 5\times1 + 8\times1$

$\Rightarrow 4 + 7 + 5 + 8$

$\Rightarrow 24$

$y(3,1) = 5\times1 + 8\times1 + 6\times1 + 9\times1$

$\Rightarrow 5 + 8 + 6 + 9 \Rightarrow 28$

$y(3,2) \Rightarrow 6\times1 + 9\times1$

$\Rightarrow 6 + 9 = 15$

$y(3,3) \Rightarrow 0$

$y(3,4) \Rightarrow 0$

$y(4,0) \Rightarrow 7\times1 + 8\times1 \Rightarrow 15$

$y(4,1) \Rightarrow 8\times1 + 9\times1 \Rightarrow 17$

$y(4,2) \Rightarrow 9\times1 = 9$

$y(4,3) \Rightarrow 0$

$y(4,4) \Rightarrow 0$

$y(5,0) \Rightarrow 0$

$y(5,1) \Rightarrow 0$

~~$y(5,2) = 0$~~

~~$y(5,3) = 0$~~

~~$y(5,4) = 0$~~

~~$y(5,$~~ The resultant matrix,

$$\begin{bmatrix} 3 & 5 & 3 & 0 & 0 \\ 12 & 16 & 9 & 0 & 0 \\ 27 & 33 & 18 & 0 & 0 \\ 24 & 28 & 15 & 0 & 0 \\ \cancel{0}15 & \cancel{0}17 & \cancel{0}9 & 0 & 0 \end{bmatrix}$$

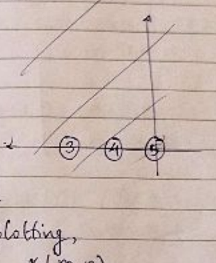
Q $x(m,n)$ is ~~a~~ 3×3 matrix; $h(m',n')$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad [\, 3, \ 4 \ \ 5 \,]$$

The dimensions of the resultant matrix,

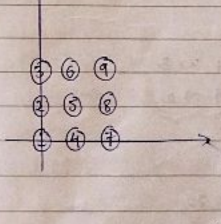
$$\Rightarrow \frac{3+1-1}{3+3-1} = \frac{3}{5} \ (\text{i.e. } 3\times5)$$

~~plotting $h(-m',-n')$~~

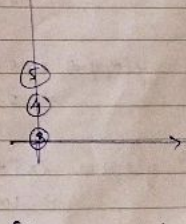


plotting,

$x(m,n)$



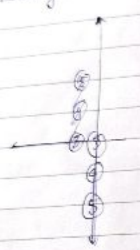
$h(m',n')$



$$y(0,0) = \sum_{m'}\sum_{n'} x(m',n')\, h(m-m', n-n')$$

$$= \sum_{m'}\sum_{n'} x(m',n')\, h(-m',-n')$$

Plotting $h(-m,-n)$.



$y(0,0) = 3\times 1 = 3$

$y(0,1) = 1\times 4 + 2\times 3$
$\Rightarrow 4+6 = \underline{10}$

$y(0,2) \Rightarrow 1\times 5 + 2\times 4 + 3\times 3$
$\Rightarrow 5+8+9 \Rightarrow \underline{22}$

$y(0,3) \Rightarrow 2\times 5 + 3\times 4$
$\Rightarrow 10+12 = \underline{22}$

$y(0,4) \Rightarrow 3\times 5 \Rightarrow \underline{15}$

$y(1,0) = 4\times 3 = \underline{12}$

$y(1,1) = 4\times 4 + 5\times 3$
$\Rightarrow 16+15 = \underline{31}$

$y(1,2) = 4\times 5 + 5\times 4 + 6\times 3$
$\Rightarrow 20+20+18 = \underline{58}$

$y(1,3) = 5\times 5 + 6\times 4$
$\Rightarrow 25+24 = \underline{49}$

$y(1,4) = 6\times 5 = \underline{30}$

$y(2,0) = 7\times 3 = \underline{21}$

$y(2,1) = 7\times 4 + 8\times 3$
$\Rightarrow 28+24 \Rightarrow \underline{52}$

$y(2,2) \Rightarrow 7\times 5 + 8\times 4 + 9\times 3$
$\Rightarrow 35+32+27 \Rightarrow \underline{94}$

$y(2,3) = 8\times 5 + 9\times 4$
$\Rightarrow 40+36 \Rightarrow \underline{76}$

$y(2,4) = 9 \times 5 = \underline{\underline{45}}$

The resultant matrix,

$$\begin{bmatrix} 3 & 10 & 22 & 22 & 15 \\ 12 & 31 & 58 & 49 & 30 \\ 21 & 52 & 94 & 76 & 45 \end{bmatrix}$$

06-02-24

Properties of Fourier Transform

i) Spatial Frequencies

$f(x,y)$ - intensity spacial co-ordinance of x,y then

$\xi_1, \xi_2 \Rightarrow$ spacial frequencies with respect to spacial distance in spacial domain.

ii) Uniqueness :-

If there is $f(x,y)$ & $f(\xi_1, \xi_2)$, then inverse exists and there is no loss of information. Can be constructed back.

iii) Separability Property:

$f(\xi_1, \xi_2)$ can be written as a succession of 1 dimensional transformation along each of the spacial coordinates.

iv) Frequency Response and Eigen function of the shift invariant system.

Eigen f^n - function will be reproduced as such that the output input will be reproduced at the output but with a change in the amplitude.

For any fixed ξ_1, ξ_2 the output of a linear shift invariant system.

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-x', y-y')\, e^{(j2\pi(\xi_1 x' + \xi_2 y')}\, dx'dy'$$

Assume $\tilde{x} = x - x'$

$\tilde{y} = y - y'$

$$g(x,y) = \iint h(\tilde{x}, \tilde{y})\, e^{-j2\pi(\xi_1 \tilde{x} + \xi_2 \tilde{y})}$$

$\Rightarrow$ Fourier Transform of impulse response

$$g(x,y) = H(\xi_1, \xi_2)$$

$\rightarrow$ Frequency Response

v) Convolution Theorem:-

$$g(x,y) = f(x,y) \circledast h(x,y)$$

$$\Leftrightarrow \underset{\text{Fourier domain}}{G(\xi_1, \xi_2)} = \underbrace{F(\xi_1, \xi_2) \cdot H(\xi_1, \xi_2)}_{\text{Element by element multiplication}}$$

Convolution in spacial domain ~~in~~ is equal to multiplication in the fourier Domain.

Fourier Transform of the convolution 2 functions is the product of their fourier transforms.

Correlation can be written as:-

$$C(\xi_1, \xi_2) = F(\xi_1, \xi_2) \cdot H(-\xi_1, -\xi_2)$$

$$C(x,y) = f(x,y) * h(x,y)$$

vi) Inner product Preservation

$$\int_{-\infty}^{\infty} |f(x,y)|^2 \, dx \, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(\xi_1, \xi_2)|^2 \, d\xi_1 \, d\xi_2$$

Inner product of the signal in spatial domain is same as inner product of the signal in fourier domain

vii) Parseval Energy Conservation

Energy of the signal in space/time domain and the energy of the signal in frequency domain is same.

We have different frequency components which we perform sampling on.

Discrete Fourier Transform

$$v(k) = \sum_{n=0}^{N-1} u(n) \, e^{-\frac{j2\pi kn}{N}} \longrightarrow (1)$$

where k varies from 0, 1, ..., N-1.

Inverse fourier transform

$$u(n) = \sum_{k=0}^{N-1} v(k) \, e^{\frac{j2\pi kn}{N}}$$

where n = 0, 1, ..., N-1

as we perform it N times, an additional component appears, to eradicate it

$$u(n) = \frac{1}{N} \sum_{k=0}^{N-1} v(k) \, e^{\frac{j2\pi kn}{N}} \longrightarrow (2)$$

These are unitary transforms (as complex nos are ~~inform~~ involved).

Otherwise we could call it orthogonal transform where,

$AA^T = I$

So, $A^{-1} = A^T$

$\therefore$ the matrix formed is invertible.

But here ~~we~~ in unitary transform

$AA^{*T} = I$

So, $A^{-1} = A^{*T}$

$\downarrow$

complex conjugate transpose
(hermission matrix)

The equations can also be written as.

$$v(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u(n)\, e^{\frac{-j2\pi kn}{N}} \longrightarrow ①$$

and

$$u(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k)\, e^{\frac{j2\pi kn}{N}} \longrightarrow ②$$

$\star$ $F = \frac{1}{\sqrt{N}} e^{\frac{-j2\pi kn}{N}}$; $0 \le k, n \le N-1$

This F is the unitary matrix (all columns are basis vectors) used in Fourier Transform. We project input vector 'u' onto F to obtain output vector 'v'.

So, $v = Fu$

and, $u = F^{*T}v$

Q. Construct a 4 point DFT matrix where N = 4 and check whether it is unitary matrix or not.

$u(n) \rightarrow$ time domain

$v(k) \rightarrow$ frequency domain

$$v(k) = \sum_{n=0}^{3} u(n)\, e^{\frac{-j2\pi kn}{4}} \;;\; k = 0,1,2,3$$

$$v(0) = u(0) + u(1) + u(2) + u(3)$$

$$v(1) = \sum_{n=0}^{3} u(n)\, e^{\frac{-j2\pi kn}{4}}$$

$$= u(0) + u(1)\, e^{\frac{-j2\pi}{4}} + u(2)\, e^{\frac{-j2\pi(2)}{4}}$$

$$+ u(3)\, e^{-j\frac{2\pi 3}{4}}$$

$$\Rightarrow u(0) + u(1)\, e^{-\frac{j\pi}{2}} + u(2)\, e^{-j\pi} + u(3)\, e^{-j\frac{3\pi}{2}}$$

$$\cancel{\Rightarrow u(0) - j\,u(1)}$$

$$e^{-j\left(\frac{\pi}{2}\right)} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2}$$

$$\Rightarrow 0 - j\,(1)$$

~~∴ ans,~~

$$\cancel{\Rightarrow u(0) + u(1)}$$

$$e^{-j\pi} = \cos\pi - j \sin\pi$$

$$= -1 + 0 = -1$$

$$e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j \sin\frac{3\pi}{2}$$

$$\Rightarrow 0 - j(-1) = j$$

∴ ans

$$\Rightarrow u(0) - j\,u(1) - u(2) + j\,u(3)$$

$$V(2) = \sum_{n=0}^{3} u(n)\, e^{-\frac{j2\pi kn}{4}}$$

$$= u(0) + u(1)\, e^{-\frac{j(2)\pi(2)(1)}{4}} + u(2)\, e^{-\frac{j(2)\pi(2)(2)}{4}}$$

$$+ u(3)\, e^{-\frac{j(2)\pi(2)(3)}{4}}$$

$$\Rightarrow u(0) + u(1)\, e^{-j\pi} + u(2)\, e^{-j2\pi}$$

$$+ u(3)\, e^{-j3\pi}$$

$$e^{-j\pi} = \cos\pi - j \sin\pi$$

$$= -1 - 0 = -1$$

$$e^{-j2\pi} = \cos 2\pi - j \sin 2\pi$$

$$= 1 - j(0) = 1$$

$$e^{-j3\pi} = \cos 3\pi - j \sin 3\pi$$

$$= -1 - j(0) = -1$$

$$V(2) = u(0) - u(1) + u(2) - u(3)$$

$$v(3) = \sum_{n=0}^{3} u(n)\, e^{-j\frac{2\pi(3)n}{4}}$$

$$= u(0) + u(1)\, e^{-j\frac{(2)\pi(3)(1)}{4}} + u(2)\, e^{-j\frac{(2)\pi(3)(2)}{4}}$$

$$\cancel{+\, u(3)\, e}$$

$$+ u(3)\, e^{-j\frac{(2)\pi(3)(3)}{4}}$$

$$\Rightarrow u(0) + u(1)\, e^{-j\frac{3\pi}{2}} + u(2)\, e^{-j(3\pi)} + u(3)\, e^{-j\left(\frac{9\pi}{2}\right)}$$

$$e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j \sin\frac{3\pi}{2}$$

$$\Rightarrow 0 - j(-1) = j$$

$$e^{-j3\pi} = \cos 3\pi - j \sin 3\pi$$

$$\Rightarrow -1 - j(0) = -1$$

$$e^{-j\frac{9\pi}{2}} = \cos\frac{9\pi}{2} - j \sin\frac{9\pi}{2}$$

$$\Rightarrow 0 - j(1) = -j$$

$$\therefore v(3) = u(0) + j\,u(1) - u(2) - j\,u(3)$$

$\cancel{v = Fu}$

$$\cancel{\begin{bmatrix} v(0) \\ v(1) \\ v(2) \\ v(3) \end{bmatrix}} \cancel{=}$$

$$v(0) = u(0) + u(1) + u(2) + u(3)$$
$$v(1) = u(0) - j\,u(1) - u(2) + j\,u(3)$$
$$v(2) = u(0) - u(1) + u(2) - u(3)$$
$$v(3) = u(0) + j\,u(1) - u(2) - j\,u(3)$$

$$v = Fu$$

$$V = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}}_{\text{basis vectors}} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix}$$

1st row ⇒ no sign change

so low frequency component

(DC ~~[illegible]~~ signal → Direct current)

2nd row ⇒ 2 sign change
3rd row ⇒ 3 sign change

} high frequency components.

4th row ⇒ 1 sign change

Now to check whether the obtained F matrix is unitary, we need to prove
$AA^{*T} = I$

To find A^{*T} we find transpose of A (which is same as A) and swap the signs of j element.

$$A^{*T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

AA^{*T}

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\begin{matrix} 4 & 0 & 0 & 0 \end{matrix}$$
$\Rightarrow$ 0

The 4 came as we didn't eradicate additional term.

2-dimensional Fourier Transform

$$v(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} u(m,n)\, e^{\frac{-j2\pi mk}{M}}\, e^{\frac{-j2\pi nl}{N}}$$

$$u(m,n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} v(k,l)\, e^{\frac{j2\pi mk}{M}}\, e^{\frac{j2\pi nl}{N}} \times \frac{1}{MN}$$

$v = \mathcal{F} u$

$\mathcal{F} \rightarrow$ its a block matrix (each element of this matrix is a matrix)

if we want to use F (the matrix containing basis vectors used in 1D Fourier)

$v = F u F^T$

and $u = F^{*T} v F^*$

where u is assumed to be an $n \times n$ image.

Suppose it is an $M \times N$ image

$$V = F_M U F_N$$

$$U = F_M^{*T} V F_N^{*T}$$

Properties of DFT

1. Seperable property

$$F(k,l) = \sum_{m=0}^{M-1} \left[\sum_{n=0}^{N-1} f(m,n) e^{-j\frac{2\pi nl}{N}} \right] e^{-j\frac{2\pi mk}{M}}$$

2. Spacial Shift Property

Assume,

$$f(m,n) \longrightarrow F(k,l)$$

when there is a shift in dimension

$$DFT\{f(m-m_0,n)\} \rightarrow F(k,l)\, e^{-j\frac{2\pi m_0 k}{M}}$$

$$f(x,y) e^{j2\pi(k_0 x + l_0 y)/N} \Leftrightarrow F(k-k_0, l-l_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(k,l)\, e^{-j2\pi(kx_0 + ly_0)/N}$$

3. Convolution Property

Suppose 2 functions are convolved,

$$DFT\{f(m,n) \odot g(m,n)\} = F(k,l) \cdot G(k,l)$$

The DFT of ~~2~~ the convolution of 2 functions is equal to the product by product multiplication of their Fouriers.

Similarly for correlation,

$$DFT\{f(m,n) \circledast g(m,n)\} = F(k,l) \cdot G(-k,-l)$$
$$= F(-k,-l) \cdot G(k,l)$$

4. Periodicity property

$$F(k,l) = \cancel{F(k+pN, l+q}$$
$$= F(k+pN, l+qM)$$

5. Scaling Property

$$DFT\{f(am,bn)\} = \frac{1}{|ab|} F(k/a, l/b)$$

The scaling in spacial domain) leads to ~~or the~~ the compression of frequency in Forier Domain.

~~expan~~ Expansion of signal in one domain is equal to the compression of signal in ~~as~~ the other domain

6. Conjugate Symmetric Property

$$DFT\{f^*(m,n)\} = F^*(-k,-l)$$

↳ just rotate and take conjugate.

7. Rotation Property

If there is a rotation in original domain then ~~it~~ there will be a corresponding rotation in the Fourier domain.

$$F(k,l) = \underbrace{|F(k,l)|}_{\text{we need magnitude to display the } F(k,l)} e^{j\phi(k,l)}$$

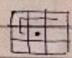
$$|F(k,l)| = \sqrt{(\text{real part of } F(k,l))^2 + (\text{i part of } F(k,l))^2}$$

$$= \sqrt{R^2\{F(k,l)\} + I^2\{F(k,l)\}}$$

To, find ϕ (phase)

$$\cancel{\tan^{-1} \sqrt{R\{F(k,l)\} - I\{F(k,l)\}}}$$

$$\phi(k,l) = \tan^{-1}\left(\frac{I\{F(k,l)\}}{R\{F(k,l)\}}\right) \Rightarrow \text{angle}$$

Causal System 

The output of these systems that is not affected by future inputs but only the past inputs. (Real time calculation is possible)

Anti-Causal System 

The output of such system depends only on the future inputs (but not real time; we can update the values based on future inputs).

Markov Process

HMM Model ?

$$\left.\begin{aligned} &P[u(n) \mid u(n-1), u(n-2), \ldots] \\ &= P[u(n) \mid u(n-1), u(n-2), \ldots u(n-p)] \end{aligned}\right\} \text{Markov-}p$$

If the condition is satisfied, ~~theuor~~ then only a few values need to be buffered for the computation.

Stationary Process

* ~~Stat~~ Strict sense Stationary
* Wide sense Stationary

If the variance remain constant throughout ~~living for~~ → wide sense
If the covariance matrix is same or approximately same as the toplex matrix its ~~an~~ wide sense stationary.

The joint probability of the original signal sequence and the shifted signal sequence should approximately same for Strict sense Stationary.

Stationary Process

Wide sense stationary :- variation is constant at any range.
Toeplitz matrix of a covariance matrix is wide sense stationary
Strict sense stationary - joint probability of shifted sequence and actual sequence should be the same

~~Kashu Karhun~~
Karhunen - Loeve (KL) Transform:-
Does not have fixed basis vectors. It is extracted from the ~~sight~~ signal itself. Represents a compact version of signal.

Given $x(n)$ where $1 \leq n \leq N$
Correlation between the same variable but in shifted point of time. Auto correlation matrix (R) is used for transformation.

$$y = \phi^{*T} x \qquad \phi \text{ is a unitary matrix}$$

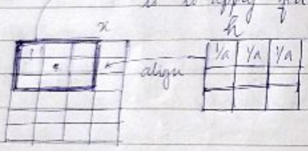
$$E[yy^{*T}] = \phi^{*T} \{ \underbrace{E(xx^{*T})}_{\text{auto-correlation matrix}} \} \phi$$

$= \phi^{*T} R \phi = \Lambda \rightarrow$ diagonalization
$\therefore$ Eigen value decomposition

$\phi \rightarrow$ contains eigen vectors of R

Complex conjugate transform of eigen vectors
- KL Transform

* Spatial Filtering
Neighbourhood - The points in the data that is to apply filter.



Kernel - Impulse response (h)
Filter mask - Impulse response
Template - Impulse response

} same thing and size is same as neighborhood

The values inside the 'h' matrix is called filter coefficients.
The center values of 'h' alligns with center value of neighbourhood. Thats how it should move.
It satisfies linearity property.

Average filter:- reduces noise, smoothens the Eg images i.e. loses high frequency components.

Eg: $\frac{1}{9} \times x_1 + \frac{1}{9} \times x_2 + \frac{1}{a} x_3 + \ldots$

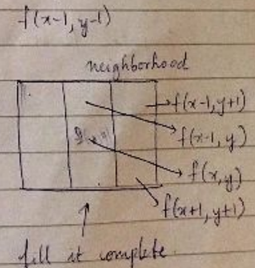
$\frac{1}{9}(x_1 + x_2 + x_3 + \ldots + x_a)$

'h' 'x'

Parameters - Sliding, padding

(h)

	-1	0	1
-1	$\omega(-1,-1)$	$\omega(-1,0)$	$\omega(-1,1)$
0		$\omega(0,0)$	
1			



$\therefore R = w(-1,-1)f(x-1,y-1) + w(-1,0)f(x-1,y) + \ldots + w(1,1)f(x+1,y+1)$

General form:

mask size $= m \times n$ where $\left.\begin{array}{l} m = 2a+1 \\ n = 2b+1 \end{array}\right\}$ mask of odd size

$$g(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x+s, y+t) \Rightarrow \text{General form}$$

Padding depends on a, b

Noises can be additive or impulse.

Additive Noise ⇒ make pixels are corrupted by some value.

Impulse Noise ⇒ corruption ~~at~~ only at a certain location.

19.03.'24

Smoothing Spatial Filters

1. **Averaging filters** (for additive noises)

A simple filter used for smoothe~~ning~~ning the noises present in the images. (its a linear filter)

$$\begin{array}{|c|c|c|} \hline 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \\ \hline \end{array} \Rightarrow 1/9 \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

There are also weighted filters where we can give weights of our choice. ~~that gives more priority to central weight and least priority to corner cells.~~

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \Rightarrow 1/16 \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array}$$

Gaussian averaging filters :- high value to the central ~~pix~~ element and the value decreases as we move towards the corner. It is also considered a linear filter.

2) Order Statistics Filter (for impulse noise)
(~~Median filters~~) Here we arrange the weights in order and make filters based on ~~the~~ certain weights.
eg: Min filter, max filter, median filter.

If the ~~noise~~ image is known to have noise of ~~to~~ having values in lower range, we apply Max filter.

Similarly for min filter.
Sometimes, we take min and max and then take average.

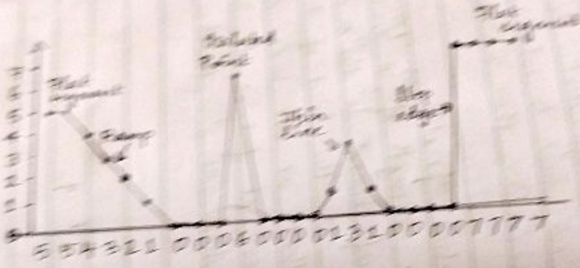
If the noise is having range in both lower and higher values, median filter is used.

Median filter is the ~~be~~ better filter for impulse noise (as it covers ~~both~~ noises in both higher and lower range)

Sharpening Filters (Spatial)

These are filters that are used to enhance the fine details of an image.
While we used integration based operation for smoothening, ~~we~~ here in Sharpening we used differentiation based operation.
(As we need to enhance ~~the~~ in the area where there is change in ...

Derivative Operators
The strength of the response of a derivative operator is proportional to the ~~ex~~ degree of the image at the point of discontinuity.
For that we use image differentiation.
Emphasis at fast variation in frequency.
De-emphasis at slow variation in frequency.



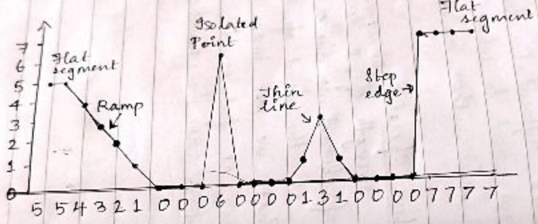
From here, we need to achieve the ones having higher rate of change of frequency (isolated point, Transfer of step edge)

1st order derivative and second order derivative can be found directly

First order derivative → 0 -1 -1 -1 -1 -1 0 0 6 -6 0 0 0 1 2 -2 -1 0 0 0 7 0 0 0

Second order derivative → -1 0 0 0 0 1 0 6 -12 6 0 0 1 1 -4 1 1 0 0 7 -7 0 0

In first order deriv



From here, we need to enhance the areas having higher rate of change of frequency. (isolated point, ~~To for~~ ~~ft~~ step edge)

1^{st} order derivatives and second order derivatives can be found directly.

First order derivatives → 0 -1 -1 -1 -1 -1 0 0 6 -6 0 0 0 1 2 -2 -1 0 0 0 7 0 0 0

Second order derivatives → -1 0 0 0 0 1 0 6 -12 6 0 0 1 1 -4 1 1 0 0 7 -7 0 0

In first order der

11.03.'24

How 2^{nd} order derivative can be used for image enhancement.

→ Laplacian Filter (its a 2^{nd} order derivative operator)

It ~~filte~~ satisfies isotropic property (i.e. its independent of direction of discontinuity, its rotation invariant)

So if we rotate a filtered image it will be same as filter applied on a rotated image.

$f(x, y)$

$$\nabla^2 f = \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2}$$

$$\frac{\partial^2 f}{\delta x^2} = [(f(x+1, y) - f(x, y)) - (f(x, y) - f(x-1, y))]$$

$$= f(x+1, y) - 2f(x, y) + f(x-1, y)$$

Similarly,

$$\frac{\delta^2 f}{\delta y^2} = [(f(x, y+1) - f(x, y)) - (f(x, y) - f(x, y-1))]$$

$$= f(x, y+1) - 2f(x, y) + f(x, y-1)$$

$$\nabla^2 f = \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2}$$

$$\Rightarrow f(x+1, y) + f(x, y+1) + f(x-1, y) + f(x, y-1) - 4f(x, y)$$

Here the Laplacian matrix is 3x3, we can extract it from the equation

0	1	0
1	-4	1
0	1	0

here the diagonal values not present

If we add the filter coefficients, we get zero. (So applying on a flat surface it gives zero)

Another mask (Laplacian)

1	1	1
1	-8	1
1	1	1

$\nabla^2 f$ only gives the discontinuity map, which can be further used for extraction, sharpening etc.

To find sharpened $g(x,y)$ [we add]

$$\underbrace{g(x,y)}_{\text{sharpened}} = \underbrace{f(x,y)}_{\text{original}} - \overbrace{\nabla^2 f(x,y)}^{\text{discontinuity map}} \quad \text{(as centre value } -4)$$

if Laplacian mask

0	-1	0
-1	4	-1
0	-1	0

$$g(x,y) = f(x,y) + \nabla^2 f(x,y)$$

We can also find $g(x,y)$ directly

$$g(x,y) = f(x,y) - f(x+1,y) - f(x,y+1) - f(x-1,y) - f(x,y-1) + 4f(x,y)$$

$$\Rightarrow 5f(x,y) - f(x+1,y) - f(x,y+1) - f(x-1,y) - f(x,y-1)$$

So mask,

0	-1	0
-1	5	-1
0	-1	0

So we can just directly apply this mask

If we considered it containing diagonal values,

-1	-1	-1
-1	9	-1
-1	-1	-1

High Boost filter

0	-1	0
-1	5	-1
0	-1	0

initial

$\Rightarrow$

0	-1	0
-1	A+4	-1
0	-1	0

The high boost filter

When A takes value 1, it becomes a laplacian mask. As we increase A value, the average grey value increase, i.e. the image becomes more bright.

03.24 Gradient (1st order derivative)

It is used for enhancement (edges extraction)

$$\nabla f = \begin{bmatrix} G_x \\ G_y \end{bmatrix} = \begin{bmatrix} \frac{\delta f}{\delta x} \\ \frac{\delta f}{\delta y} \end{bmatrix}$$

Gradient can also be used as a feature vector of the image.

On applying the ~~f~~ gradient to the image we only get the discontinuity map which ~~must~~ is then ~~be~~ threshold ~~to~~ with the image to ~~f~~ get the edge enhanced image

$$\cancel{mg}\ mag(\nabla f) = \left(G_x^2 + G_y^2\right)^{1/2}$$

Robert's Cross Difference (Robert's Filter)

z_1	z_2	z_3	$G_x = z_8 - z_5$
z_4	z_5	z_6	$G_y = z_6 - z_5$
z_7	z_8	z_9	

This is not Robert's Filter

In Robert's filter we take the cross difference

z_1	z_2	z_3	$G_x = z_9 - z_5$
z_4	z_5	z_6	$G_y = z_8 - z_6$
z_7	z_8	z_9	

The masks come as,

$$\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$$

for G_x for G_y

~~If we take 3x3 horizontal mask~~

$$\cancel{\begin{matrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{matrix}}$$

Sobel Filter

It has 3x3 filters.

horizontal difference

-1	0	1
-2	0	2
-1	0	1

vertical difference

-1	-2	-1
0	0	0
1	2	1

The weight is given in the centre values to give a particular highlight or importance to centre position.

Prewitt Filter

It is same as Sobel, but doesn't give importance to centre values.

-1	0	1
-1	0	1
-1	0	1

-1	-1	-1
0	0	0
1	1	1

Taking the diagonal case of sobel filter

0	1	2
-1	0	1
-2	-1	0

-2	-1	0
-1	0	1
0	1	2

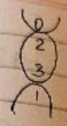
These are linear filters. Non-linear filters (like mean, median) are more popular.

Up until now we did convolution of the filter on the image.
Now we will apply the filter on the transform of the image.

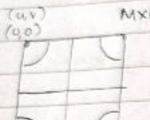
Filtering in Frequency Domain

Frequency in frequency domain is same as the rate of intensity values in spatial domain.
For ~~the~~ Function in frequency domain
$F(u, v)$,
if $u = v = 0$,
it means there is minute or zero intensity variation.

$u = v = 0 \Rightarrow$ DC part
all the other part AC

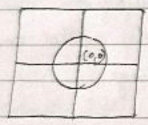


MxN dimension



The low frequency components are at the corners.

Now ~~so~~ we perform ~~swapping~~ shifting which is same as swapping



now (0,0) is at $(M/2, N/2)$

Now the frequency domain is designed perfectly, as at the origin (centre) we have the low frequency component and as we go further from the centre, ~~we got~~ the frequency values increase. This makes the filter designing much easier and efficient.

suppose we are given $f(x,y)$, which has to be converted into $F(u,v)$.

(function) (filter)

$g(x,y) = f(x,y) \odot h(x,y)$ } in spacial domain

In frequency domain

$G(u,v) = F(u,v) \cdot H(u,v)$

(frequency response of $h(x,y)$)

to get $g(x,y)$

$$g(x,y) = \chi^{-1}[G(u,v)]$$

$$\underbrace{g(x,y)}_{\text{filtered output image.}} = \chi^{-1}[F(u,v) \cdot H(u,v)]$$

$\chi^{-1} \Rightarrow$ inverse fourier transform

The dimensions of $f(x,y)$, $h(x,y)$, $F(u,v)$ and $H(u,v)$ are the same.

19.03.24

Frequency Domain Filtering

$$g(x,y) = \mathcal{F}^{-1}[\underbrace{F(u,v)}_{\text{directly computable from } F(x,y)} . \underbrace{H(u,v)}_{\text{computing this is hard}}]$$

centering in another method,

$$\underbrace{f(x,y)(-1)^{x+y}}_{\text{another method}} \iff \underbrace{F(u - M/2, v - N/2)}_{\text{first find } F(u,V) \text{ from } f(x,y) \text{ and then shift.}}$$

N ____ Filter

A constant function with a hole or zero (frequency) at the centre.

The 0,0 at $(M/2, N/2)$ will have frequency of zero and all the other points will have frequency 1.

The centre usually contains the ~~average~~ avg frequency of the image. ~~Upon~~ (after centering?) Then we can apply N filter on it, the image will be

Low Pass Filter (LPF) and High Pass Filter (HPF)

LPF only ~~on~~ passes through low frequency components and it will   the high frequency components.
HPF does the opposite.

Padding in Fourier Domain

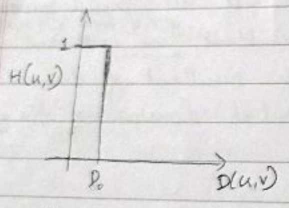
Here both $F(u,v)$ and $H(u,v)$ have the same dimensions. ~~So~~ Due to the periodicity property of the Fourier ~~Trans~~ Domain the resultant $g(x,y)$ from $\mathcal{F}^{-1}[F(u,v).H(u,v)]$ ~~etc~~ will not be the same as the resultant of the convolution in spatial domain of $f(x,y)$ and $h(x,y)$.

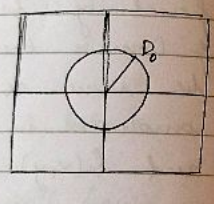
Given an input signal and an impulse response.
If the input signal is not periodic the convolved result will be different from the case if input signal is periodic.

To avoid this phenomenon, we pad in the fourier domain to eradicate the periodicity

Ideal Low Pass Filters

The cut off is very sharply defined. $D(u,v)$ represents different points in the frequency rectangle, it describes the distance from the $(0,0)$ that is now centered.





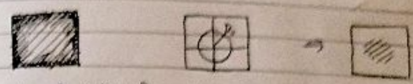
this is done to define $H(u,v)$.

$$H(u,v) = \begin{cases} 1, & \text{if } D(u,v) \le D_0 \\ 0, & \text{if } D(u,v) > D_0 \end{cases}$$

The distance $D(u,v)$ can be found,

$$D(u,v) = \left((u - M/2)^2 + (v - N/2)^2 \right)^{1/2}$$

i.e.



centered $F(U,V)$ $H(u,V)$

So only low frequency ~~points~~ points were passed through

Disadvantage

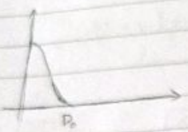
It causes a ring effect, ~~likeara~~ so its not directly used.

The ringing effect is inversely proportional to the selected D_0 value.

In spatial domain, ~~on~~ convolving with sinc function we get the corresponding result of Ideal LPF in Fourier Domain.

Gaussian Low Pass Filter

It overcomes the disadvantage of Ideal LPF. Graphically,



Its Gaussian in the spatial domain as well.

$$H(u) = A\,e^{-u^2/2\sigma^2}$$

~~H(u,v) = e~~

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

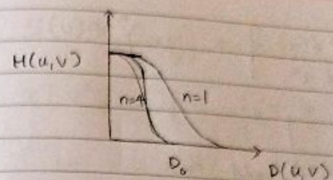
$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

$D_0 \Rightarrow$ cut off frequency.

No ringing artifacts, but it is more smoothened, unrequired components can come in due to this.

Butterworth LPF

Very popular. It aims at achieving advantage of both Ideal LPF and Gaussian LPF.



we get multiple curves for different n values

As n value increases we get Ideal LPF, if n value decreases we get Gaussian.

$$H(u,v) = \frac{1}{1 + \left[\frac{D(u,v)}{D_0}\right]^{2n}}$$

The high pass filter

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

Laplacian in Frequency Domain

When we take fourier transform of the n^{th} order derivative of a function,

$$\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$$

$$\left.\begin{array}{c}\dfrac{\delta^2 f(x,y)}{\delta x^2} \\[2em] \dfrac{\delta^2 f(x,y)}{\delta y^2}\end{array}\right\} \text{add these}$$

$$\frac{\delta^2 f(x,y)}{\delta x^2} + \frac{\delta^2 f(x,y)}{\delta y^2} = (ju)^2 F(u,v) + (jv)^2 F(u,v)$$

fourier of

$$\left[\frac{\delta^2 f(x,y)}{\delta x^2} + \frac{\delta^2 f(x,y)}{\delta y^2}\right]$$

$$= (ju)^2 F(u,v) + (jv)^2 F(u,v)$$

$$\mathcal{F}[\nabla^2 f(x,y)] = -(u^2+v^2)\, F(u,v)$$

(when $F(u,v)$ is not centred)

so, $H(u,v) = -(u^2+v^2)$

If $F(u,v)$ is centred,

$$\mathcal{F}[\nabla^2 f(x,y)] = -\left[\left(u-\frac{M}{2}\right)^2 + \left(v-\frac{N}{2}\right)^2\right] F(u,v)$$

we will only get discontinuity map (not sharpened) on taking inverse fourier ~~not~~

To get sharpened image $g(x,y)$

$$\Rightarrow \underbrace{\left[1+(u^2+v^2)\right]}_{\text{i.e. } [1-(-(u^2+v^2))]} F(u,v)$$

Other filters in Spacial Domain

$$\underbrace{g(x,y)}_{\text{(output)}} \underset{\text{we assume}}{=} \underbrace{f(x,y)}_{\text{(original image)}} + \underbrace{\eta(x,y)}_{\text{(some random noise)}}$$

$$\left.\hat{f}(x,y) = \frac{1}{mn} \sum_{(s,t)\in S_{xy}} g(s,t)\right\} \text{Arithmetic Mean filter.}$$

We can also apply Geometric ~~mor~~ mean, which gives a better result compared to of arithmetic mean.

$$\hat{f}(x,y) = \left(\prod_{(s,t) \in S_{xy}} g(s,t) \right)^{\frac{1}{mn}}$$

it will not blur the image as much as the arithmetic mean.

Harmonic Mean

It is very good for salt and pepper noise. (that is, only ~~some~~ ~~values~~ some parts have noise) especially at regions with low intensity or high intensity (0 or 180)). Salt and pepper noise is subset of Random Value impulse noise.

$$\hat{f}(x,y) = \frac{mn}{\sum_{(s,t) \in S_{xy}} \frac{1}{g(s,t)}}$$

It is faster than geometric.

It can be used for image with Gaussian noises as well.

Contra Harmonic Mean

$$\hat{f}(x,y) = \frac{\sum_{(s,t) \in S_{xy}} g(s,t)^{Q+1}}{\sum_{(s,t) \in S_{xy}} g(s,t)^{Q}}$$

where Q is called the order of the filter
when $Q = 0$, it becomes Arithmetic mean filter
when $Q = -1$, it becomes Harmonic ~~av~~ mean filter.

Order Statistics - Non Linear Filters

Min, Max, Median, two more:-

Midpoint - Filter

Using a combination of Min and Max filter. It takes the average of the Min and Max result (Min-max average (Midpoint) can be used in case of additive noises and not in the case of impulse noises).

~~Alpha Trimmed Mean Filter~~ Midpoint ~~filter~~ filter,

$$\hat{f}(x,y) = \frac{1}{2}\left[\max_{(s,t)\in S_{xy}}\{g(s,t)\} + \min_{(s,t)\in S_{xy}}\{g(s,t)\}\right]$$

Alpha - Trimmed Mean Filter

Delete $\frac{\alpha}{2}$ lowest and $\frac{\alpha}{2}$ highest intensity values of $g(s,t)$ in the neighborhood of S_{xy} $(s,t) \in S_{xy}$.

Remaining filters $\Rightarrow mn - \alpha$

and $g_r(s,t)$ is a subset of $g(x,y)$.

$$\hat{f}(x,y) = \frac{1}{mn-\alpha} \sum_{(s,t)\in S_{xy}} g_r(s,t)$$

α can take value between 0 and $mn-1$.

If $\alpha = 0$, the filter gets reduced to arithmetic mean filter

if $\alpha = mn-1$, the filter gets reduced to median filter.

Very popular because as it can become has arithmetic filter and the median filter. So it has averaging (linear) and order-statistics (non-linear) nature.

So it can process images having realistic noise (mixture of different types of mean).

Adaptive filters (Most popular)

It considers and depends on the local statistics. The values of ~~the~~ change with windows. In each window, depending on its characteristics the values change.

-03.24

Periodic Noise Reduction

Some recurrent ~~information~~ pattern, we can remove it or extract it according to requirement. First we need to find band.

- Band Rejection Filter

To reject some bands and turn other values to 1. It is of different types:-

1. Ideal Band Reject Filter

We define a set of bands, which is then ~~all~~ attenuated (its frequency is turned to zero) It is sharp

Usually we only need the distance from the centre but now we also need the width; called bandwidth (The radial centric distance $\Rightarrow d_0$)

$$H(u,v) \Rightarrow \begin{cases} 1 \text{ (the area inside the band)} \\ 0 \text{ (area within the band)} \\ 1 \text{ (area outside the band)} \end{cases}$$

that is,

$$H(u,v) = \begin{cases} 1, & D(u,v) < D_0 - \frac{W}{2} \\ 0, & D_0 - \frac{W}{2} \le D(u,v) \le D_0 + \frac{W}{2} \\ 1, & D(u,v) > D_0 + \frac{W}{2} \end{cases}$$

2) Gaussian Band Reject Filter

It is not sharp, helps avoid ring effect.

$$H(u,v) = 1 - e^{-\frac{1}{2}\left[\frac{D^2(u,v) - D_0^2}{D(u,v)W}\right]}$$

3. Butterworth Band Reject Filter

Advantage of Ideal and Gaussian

$$H(u,v) = \frac{1}{1 + \left[\frac{D(u,v)W}{D^2(u,v) - D_0^2}\right]^{2n}}$$

Considering Band Pass Filters

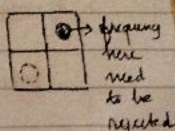
$$H_{BP}(u,v) = 1 - H(u,v)$$

Notch Filter

We can define it at any point, it is not based on taking distance from centre but rather it is defined on the quadrants.

1. Ideal Notch Reject

Because of the symmetric property of DFT, to achieve the frequency rejection, we would also need to attenuate the area that symmetrically corresponds our area of preference



The centre of the area ⇒ (u_0, v_0)
Then centre of symmetric region ⇒ $(-u_0, -v_0)$
the radius ⇒ D_0
We need to find $D_1(U,V)$ and $D_2(U,V)$

We can shift (u_0, v_0) to $\left(\frac{M}{2}, \frac{N}{2}\right)$ origin
and the origin must be shifted to (u_0, v_0)

$$D_1(U,V) = \left[\left(U - \frac{M}{2} - u_0\right)^2 + \left(V - \frac{N}{2} - v_0\right)^2\right]^{1/2}$$

this is if,

and for symmetric region

$$D_2(U,V) = \left[\left(U - \frac{M}{2} + u_0\right)^2 + \left(V - \frac{N}{2} + v_0\right)^2\right]^{1/2}$$

this is if,

$$H(U,V) = \begin{cases} 0 \; ; & \text{if } D_1(U,V) \le D_0 \text{ or } D_2(U,V) \le D_0 \\ 1 \; ; & \text{else} \end{cases}$$

2. Gaussian Notch Reject Filter

$$H(U,V) = 1 - e^{-\frac{1}{2}\left[\frac{D_1(U,V)\, D_2(U,V)}{D_0^2}\right]}$$

It is for reducing the sharpness.

3. Butterworth Notch Reject Filter

$$H(U,V) = \frac{1}{1 + \left[\frac{D_0^2}{D_1(U,V)\, D_2(U,V)}\right]^n}$$

if (u_0, v_0) is zero, this notch reject filter becomes high pass filter, as the ~~sym~~ it is naturally at center, so symmetric feature disappears, ∴ only high frequency passes.

$$H_{NP}(U,V) = 1 - H_{NR}(U,V)$$

(Notch pass) (Notch Reject)