Fuzzy sets

(CRISP)Set

- Sets are one of the most fundamental concepts in mathematics.
- In mathematics, a **set** is a well-defined collection of distinct objects.

- For example, the numbers 2, 4, and 6 are distinct objects when considered separately, but when they are considered collectively they form a single set of size three, written {2,4,6}.

- CRISP:- Strongly felt and unlikely to change.

Set

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the set of all integers),
$\mathbb{N} = \{1, 2, 3, \ldots\}$ (the set of all positive integers or natural numbers),
$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ (the set of all nonnegative integers),
$\mathbb{N}_n = \{1, 2, \ldots, n\}$,
$\mathbb{N}_{0,n} = \{0, 1, \ldots, n\}$,
$\mathbb{R}$: the set of all real numbers,
$\mathbb{R}^+$: the set of all nonnegative real numbers,

To indicate that an individual object x is a *member* or *element* of a set A, we write

$$x \in A.$$

Whenever x is not an element of a set A, we write

$$x \notin A.$$

Annotations of Crisp Set

There are three basic methods by which sets can be defined within a given universal set X:

1. A set is defined by naming all its members (the list method). This method can be used only for finite sets. Set A, whose members are $a_1, a_2, \ldots, a_n$, is usually written as

$$A = \{a_1, a_2, \ldots, a_n\}.$$

2. A set is defined by a property satisfied by its members (the rule method). A common notation expressing this method is

$$A = \{x | P(x)\},$$

where the symbol | denotes the phrase "such that," and $P(x)$ designates a proposition of the form "x has the property P." That is, A is defined by this notation as the set of all elements of X for which the proposition $P(x)$ is true. It is required that the property P be such that for any given $x \in X$, the proposition $P(x)$ is either true of false.

Annotations of Crisp Set

3. A set is defined by a function, usually called a *characteristic function*, that declares which elements of X are members of the set and which are not. Set A is defined by its characteristic function, χ_A, as follows:

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A. \end{cases}$$

That is, the characteristic function maps elements of X to elements of the set $\{0, 1\}$, which is formally expressed by

$$\chi_A : X \to \{0, 1\}.$$

For each $x \in X$, when $\chi_A(x) = 1$, x is declared to be a member of A; when $\chi_A(x) = 0$, x is declared as a nonmember of A.

Family of Set

A set whose elements are themselves sets is often referred to as a *family of sets*. It can be defined in the form

$$\{A_i | i \in I\},$$

where i and I are called the *set index* and the *index set*, respectively. Because the index i is used to reference the sets A_i, the family of sets is also called an *indexed set*. In this text, families of sets are usually denoted by script capital letters. For example,

$$\mathcal{A} = \{A_1, A_2, \ldots, A_n\}.$$

Subset of set and equal sets

If every member of set A is also a member of set B (i.e., if $x \in A$ implies $x \in B$), then A is called a *subset* of B, and this is written as

$$A \subseteq B.$$

Every set is a subset of itself, and every set is a subset of the universal set. If $A \subseteq B$ and $B \subseteq A$, then A and B contain the same members. They are then called *equal sets*; this is denoted by

$$A = B.$$

Proper subset

To indicate that A and B are not equal, we write

$$A \neq B.$$

If both $A \subseteq B$ and $A \neq B$, then B contains at least one individual that *is* not a member of A. In this case, A is called a *proper subset* of B, which is denoted by

$$A \subset B.$$

When $A \subseteq B$, we also say that A is *included* in B.

Power set and cardinality

The family of *all subsets* of a given set A is called the *power set* of A, and it is usually denoted by $\mathcal{P}(A)$. The family of all subsets of $\mathcal{P}(A)$ is called a *second order power set of* A; it is denoted by $\mathcal{P}^2(A)$, which stands for $\mathcal{P}(\mathcal{P}(A))$. Similarly, *higher order power sets* $\mathcal{P}^3(A)$, $\mathcal{P}^4(A)$, ... can be defined.

The number of members of a *finite set* A is called the *cardinality* of A and is denoted by $|A|$. When A is finite, then

$$|\mathcal{P}(A)| = 2^{|A|}, \ |\mathcal{P}^2(A)| = 2^{2^{|A|}}, \text{ etc.}$$

Compliment Set

The *relative complement* of a set A with respect to set B is the set containing all the members of B that are not also members of A. This can be written $B - A$. Thus,

$$B - A = \{x | x \in B \text{ and } x \notin A\}.$$

If the set B is the universal set, the complement is absolute and is usually denoted by $\overline{A}$. The absolute complement is always *involutive*; that is, taking the complement of a complement yields the original set, or

$$\overline{\overline{A}} = A.$$

The absolute complement of the empty set equals the universal set, and the absolute complement of the universal set equals the empty set. That is,

$$\overline{\varnothing} = X$$

and

$$\overline{X} = \varnothing.$$

Union

The *union* of sets A and B is the set containing all the elements that belong either to set A alone, to set B alone, or to both set A and set B. This is denoted by $A \cup B$. Thus,

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

The union operation can be generalized for any number of sets. For a family of sets $\{A_i | i \in I\}$, this is defined as

$$\bigcup_{i \in I} A_i = \{x | x \in A_i \text{ for some } i \in I\}.$$

Intersection

The *intersection* of sets A and B is the set containing all the elements belonging to both set A and set B. It is denoted by $A \cap B$. Thus,

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

The generalization of the intersection for a family of sets $\{A_i | i \in I\}$ is defined as

$$\bigcap_{i \in I} A_i = \{x | x \in A_i \text{ for all } i \in I\}.$$

Disjoint Set

Any two sets that have no common members are called *disjoint*. That is, every pair of disjoint sets, A and B, satisfies the equation

$$A \cap B = \varnothing.$$

Partition of on Set

A family of pairwise disjoint nonempty subsets of a set A is called a *partition* on A if the union of these subsets yields the original set A. We denote a partition on A by the symbol $\pi(A)$. Formally,

$$\pi(A) = \{A_i | i \in I, A_i \subseteq A\},$$

where $A_i \neq \varnothing$, is a partition on A iff

$$A_i \cap A_j = \varnothing$$

for each pair $i, j \in I, i \neq j$, and

$$\bigcup_{i \in I} A_i = A.$$

Members of a partition $\pi(A)$, which are subsets of A, are usually referred to as *blocks* of the partition. Each member of A belongs to one and only one block of $\pi(A)$.

Cartesian product

The *Cartesian product* of two sets—say, A and B (in this order)—is the set of all ordered pairs such that the first element in each pair is a member of A, and the second element is a member of B. Formally,

$$A \times B = \{\langle a, b \rangle | a \in A, b \in B\},$$

where $A \times B$ denotes the Cartesian product. Clearly, if $A \neq B$ and A, B are nonempty, then $A \times B \neq B \times A$.

Relations

The Cartesian product of a family $\{A_1, A_2, \ldots, A_n\}$ of sets is the set of all n-tuples $\langle a_1, a_2, \ldots, a_n\rangle$ such that $a_i \in A_i (i = 1, 2, \ldots, n)$. It is written as either $A_1 \times A_2 \times \ldots \times A_n$ or $\bigtimes_{1 \le i \le n} A_i$. Thus,

$$\bigtimes_{1 \le i \le n} A_i = \{\langle a_1, a_2, \ldots, a_n\rangle \mid a_i \in A_i \text{ for every } i = 1, 2, \ldots, n\}.$$

The Cartesian products $A \times A, A \times A \times A, \ldots$ are denoted by $A^2, A^3, \ldots,$ respectively.

Subsets of Cartesian products are called *relations*.

Properties of Crisp set

TABLE 1.1 FUNDAMENTAL PROPERTIES OF CRISP SET OPERATIONS

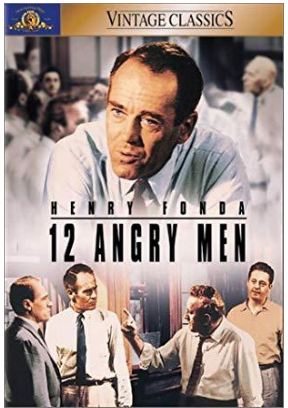
Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and $\varnothing$	$A \cup X = X$
	$A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$
	$A \cap X = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$
Law of excluded middle	$A \cup \overline{A} = X$
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Uncertainty

- The state of being uncertain.
- Imprecision:- lacking exactness and accuracy of expression or detail. (accuracy of measure)
- Non-specificity:- clearly defined or identified.
  - *Every woman talked to* ***a student (particular student or any student)****.*
- Vagueness:- Indefinite or indistinct in nature or character, as ideas or feelings.
- Inconsistency:-not staying the same throughout.

Uncertainty

- Vagueness :- Indefinite or indistinct in nature or character, as ideas or feelings
  - Fuzzy sets
    - The elements in a set will give different values
      - Eg: White color car in a campus.
- Ambiguity :- The quality of being open to more than one interpretation; inexactness.
  - Fuzzy measurements
    - The set itself will give different values
      - Eg: Criminal trial



Fuzzy sets: Basic concepts

- The characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set.
- This function is can be generalized such that the values assigned to the elements of the universal sets fall within a specific range and indicate the membership grade of these elements in the set.
- Larger values denote higher degrees of set membership. Such a function is called a membership function and the set defined by it a fuzzy set.

Fuzzy sets: Basic concepts

Two distinct notations are most commonly employed in the literature to denote membership functions. In one of them, the membership function of a fuzzy set A is denoted by μ_A; that is,

$$\mu_A : X \to [0, 1].$$

In the other one, the function is denoted by A and has, of course, the same form:

$$A : X \to [0, 1].$$