BINARY FUZZY RELATIONS

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Given a fuzzy relation $R(X, Y)$, its *domain* is a fuzzy set on X, dom R, whose membership function is defined by

$$\text{dom}\, R(x) = \max_{y \in Y} R(x, y) \tag{5.3}$$

for each $x \in X$. That is, each element of set X belongs to the domain of R to the degree equal to the strength of its strongest relation to any member of set Y. The *range* of $R(X, Y)$ is a fuzzy relation on Y, ran R, whose membership function is defined by

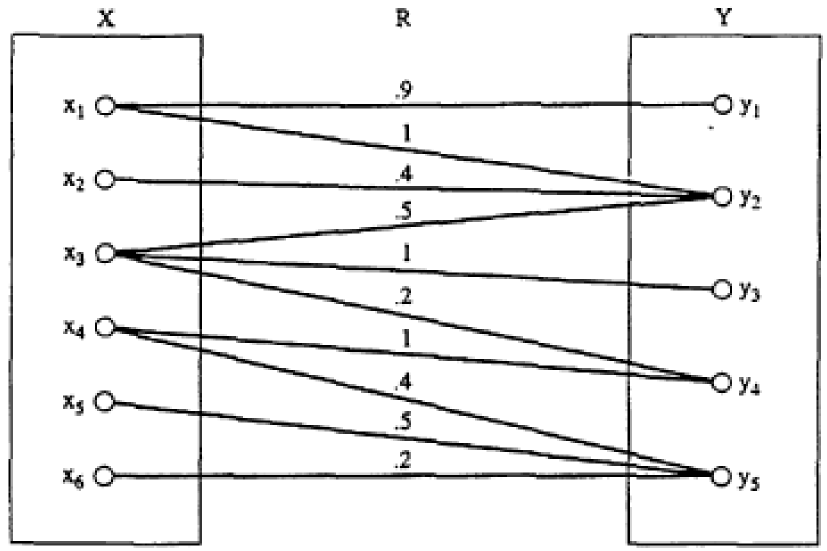
$$\text{ran}\, R(y) = \max_{x \in X} R(x, y) \tag{5.4}$$

for each $y \in Y$. That is, the strength of the strongest relation that each element of Y has to an element of X is equal to the degree of that element's membership in the range of R. In addition, the *height* of a fuzzy relation $R(X, Y)$ is a number, $h(R)$, defined by

$$h(R) = \max_{y \in Y} \max_{x \in X} R(x, y). \tag{5.5}$$

That is, $h(R)$ is the largest membership grade attained by any pair $\langle x, y \rangle$ in R.

BINARY FUZZY RELATIONS



(a)

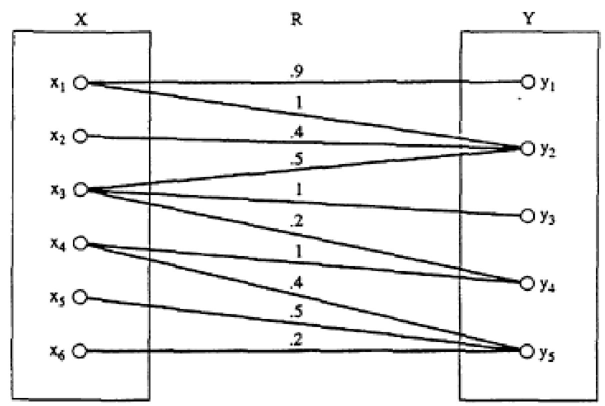
$$
R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array}
\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\
\begin{bmatrix} .9 & 1 & 0 & 0 & 0 \\ 0 & .4 & 0 & 0 & 0 \\ 0 & .5 & 1 & .2 & 0 \\ 0 & 0 & 0 & 1 & .4 \\ 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & .2 \end{bmatrix} \end{array}
$$

(b)

Figure 5.2 Examples of two convenient representations of a fuzzy binary relation: (a) sagittal diagram; (b) membership matrix.

BINARY FUZZY RELATIONS

A convenient representation of binary relation $R(X, Y)$ are *membership matrices* $\mathbf{R} = [r_{xy}]$, where $r_{xy} = R(x, y)$. Another useful representation of binary relations is a *sagittal diagram*. Each of the sets X, Y is represented by a set of nodes in the diagram;



(a)

$$\mathbf{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} .9 & 1 & 0 & 0 & 0 \\ 0 & .4 & 0 & 0 & 0 \\ 0 & .5 & 1 & .2 & 0 \\ 0 & 0 & 0 & 1 & .4 \\ 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & .2 \end{bmatrix} \end{array}$$

(b)

Figure 5.2 Examples of two convenient representations of a fuzzy binary relation: (a) sagittal diagram; (b) membership matrix.

BINARY FUZZY RELATIONS

The *inverse* of a fuzzy relation $R(X, Y)$, which is denoted by $R^{-1}(Y, X)$, is a relation on $Y \times X$ defined by

$$R^{-1}(y, x) = R(x, y)$$

for all $x \in X$ and all $y \in Y$. A membership matrix $\mathbf{R}^{-1} = [r_{yx}^{-1}]$ representing $R^{-1}(Y, X)$ is the transpose of the matrix $\mathbf{R}$ for $R(X, Y)$, which means that the rows of $\mathbf{R}^{-1}$ equal the columns of $\mathbf{R}$ and the columns of $\mathbf{R}^{-1}$ equal the rows of $\mathbf{R}$. Clearly,

$$(\mathbf{R}^{-1})^{-1} = \mathbf{R} \tag{5.6}$$

for any binary fuzzy relation.

BINARY FUZZY RELATIONS

Consider now two binary fuzzy relations $P(X, Y)$ and $Q(Y, Z)$ with a common set Y. The *standard composition* of these relations, which is denoted by $P(X, Y) \circ Q(Y, Z)$, produces a binary relation $R(X, Z)$ on $X \times Z$ defined by

$$R(x, z) = [P \circ Q](x, z) = \max_{y \in Y} \min[P(x, y), Q(y, z)] \tag{5.7}$$

for all $x \in X$ and all $z \in Z$.

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix}$$

BINARY FUZZY RELATIONS

Consider now two binary fuzzy relations $P(X, Y)$ and $Q(Y, Z)$ with a common set Y. The *standard composition* of these relations, which is denoted by $P(X, Y) \circ Q(Y, Z)$, produces a binary relation $R(X, Z)$ on $X \times Z$ defined by

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for all $x \in X$ and all $z \in Z$.

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & .5 & .7 \\ .5 & .4 & .5 & .6 \end{bmatrix}$$

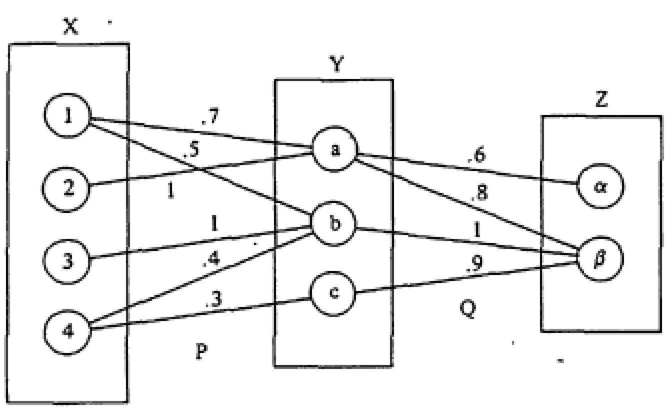
$$\begin{aligned} .8(= r_{11}) &= \max[\min(.3, .9), \min(.5, .3), \min(.8, 1)] \\ &= \max[\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})], \\ .4(= r_{32}) &= \max[\min(.4, .5), \min(.6, .2), \min(.5, 0)] \\ &= \max[\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})]. \end{aligned}$$

BINARY FUZZY RELATIONS

A similar operation on two binary relations, which differs from the composition in that it yields triples instead of pairs, is known as the *relational join*. For fuzzy relations $P(X, Y)$ and $Q(Y, Z)$, the relational join, $P * Q$, corresponding to the standard max-min composition is a ternary relation $R(X, Y, Z)$ defined by

$$R(x, y, z) = [P * Q](x, y, z) = \min[P(x, y), Q(y, z)] \tag{5.9}$$

for each $x \in X$, $y \in Y$, and $z \in Z$.



(a)

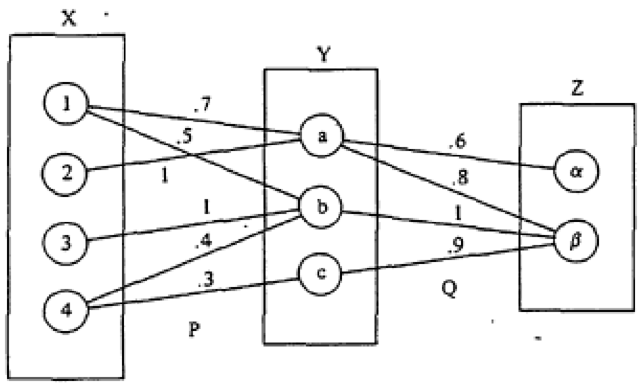
Join: $S = P * Q$			
x	y	z	$\mu_S(x, y, z)$
1	a	α	.6
1	a	β	.7
1	b	β	.5
2	a	α	.6
2	a	β	.8
3	b	β	1
4	b	β	.4
4	c	β	.3

(b)

BINARY FUZZY RELATIONS

$$[P \circ Q](x, z) = \max_{y \in Y}[P * Q](x, y, z) \tag{5.10}$$

for each $x \in X$ and $z \in Z$.



(a)

$$R(1, \beta) = \max[S(1, a, \beta), S(1, b, \beta)]$$
$$= \max[.7, .5] = .7.$$

Join: S = P * Q

x	y	z	$\mu_S(x, y, z)$
1	a	α	.6
1	a	β	.7
1	b	β	.5
2	a	α	.6
2	a	β	.8
3	b	β	1
4	b	β	.4
4	c	β	.3

(b)

Composition: R = P ∘ Q

x	z	$\mu_R(x, z)$
1	α	.6
1	β	.7
2	α	.6
2	β	.8
3	β	1
4	β	.4

(c)

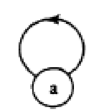
Figure 5.3 Composition and join of binary relation.

BINARY RELATIONS ON A SINGLE SET

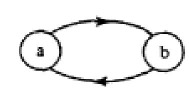
In addition to defining a binary relation that exists between two different sets, it is also possible to define a crisp or fuzzy binary relation among the elements of a single set X. A binary relation of this type can be denoted by $R(X, X)$ or $R(X^2)$ and is a subset of $X \times X = X^2$. These relations are often referred to as *directed graphs* or *digraphs*.

A crisp relation $R(X, X)$ is *reflexive* iff $\langle x, x \rangle \in R$ for each $x \in X$, that is, if every element of X is related to itself. Otherwise, $R(X, X)$ is called *irreflexive*. If $\langle x, x \rangle \notin R$ for every $x \in X$, the relation is called *antireflexive*.

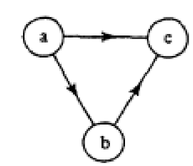
A crisp relation $R(X, X)$ is *symmetric* iff for every $\langle x, y \rangle \in R$, it is also the case that $\langle y, x \rangle \in R$, where $x, y, \in X$. Thus, whenever an element x is related to an element y through a symmetric relation, y is also related to x. If this is not the case for some x, y, then the relation is called *asymmetric*. If both $\langle x, y \rangle \in R$ and $\langle y, x \rangle \in R$ implies $x = y$, then the relation is called *antisymmetric*. If either $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$, whenever $x \neq y$, then the relation is called *strictly antisymmetric*.



Reflexivity



Symmetry



Transitivity

BINARY RELATIONS ON A SINGLE SET

These three properties can be extended for fuzzy relations $R(X, X)$, by defining them in terms of the membership function of the relation. Thus, $R(X, X)$ is *reflexive* iff

$$R(x, x) = 1$$

for all $x \in X$. If this is not the case for some $x \in X$, the relation is called *irreflexive*; if it is not satisfied for all $x \in X$, the relation is called *antireflexive*. A weaker form of reflexivity, referred to as *ε-reflexivity*, is sometimes defined by requiring that

$$R(x, x) \geq \varepsilon,$$

where $0 < \varepsilon < 1$.

A fuzzy relation is *symmetric* iff

$$R(x, y) = R(y, x)$$

for all $x, y \in X$. Whenever this equality is not satisfied for some $x, y \in X$, the relation is called *asymmetric*. Furthermore, when $R(x, y) > 0$ and $R(y, x) > 0$ implies that $x = y$ for all $x, y \in X$, the relation R is called *antisymmetric*.

BINARY RELATIONS ON A SINGLE SET

A fuzzy relation $R(X, X)$ is *transitive* (or, more specifically, *max-min transitive*) if

$$R(x, z) \geq \max_{y \in Y} \min[R(x, y), R(y, z)] \tag{5.11}$$

is satisfied for each pair $\langle x, z \rangle \in X^2$. A relation failing to satisfy this inequality for some members of X is called *nontransitive*, and if

$$R(x, z) < \max_{y \in Y} \min[R(x, y), R(y, z)],$$

for all $\langle x, z \rangle \in X^2$, then the relation is called *antitransitive*.

The *transitive closure* of a crisp relation $R(X, X)$ is defined as the relation that is transitive, contains $R(X, X)$, and has the fewest possible members.

BINARY RELATIONS ON A SINGLE SET

Given a relation $R(X, X)$, its transitive closure $R_T(X, X)$ can be determined by a simple algorithm that consists of the following three steps:

1. $R' = R \cup (R \circ R)$.
2. If $R' \neq R$, make $R = R'$ and go to Step 1.
3. Stop: $R' = R_T$.

$$\mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .4 & 0 & 0 \\ 0 & 0 & .8 & 0 \end{bmatrix}.$$

BINARY RELATIONS ON A SINGLE SET

$$\mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .4 & 0 & 0 \\ 0 & 0 & .8 & 0 \end{bmatrix}.$$

1. $R' = R \cup (R \circ R)$.

BINARY RELATIONS ON A SINGLE SET

$$\mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .4 & 0 & 0 \\ 0 & 0 & .8 & 0 \end{bmatrix}.$$

1. $R' = R \cup (R \circ R)$.

Applying Step 1 of the algorithm, we obtain

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & .4 \\ 0 & .4 & 0 & 0 \end{bmatrix} \quad \mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 1 \\ 0 & .4 & 0 & .4 \\ 0 & .4 & .8 & 0 \end{bmatrix} = \mathbf{R}'.$$

BINARY RELATIONS ON A SINGLE SET

$$\mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .4 & 0 & 0 \\ 0 & 0 & .8 & 0 \end{bmatrix}.$$

1. $R' = R \cup (R \circ R).$

Applying Step 1 of the algorithm, we obtain

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & .4 \\ 0 & .4 & 0 & 0 \end{bmatrix} \quad \mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 1 \\ 0 & .4 & 0 & .4 \\ 0 & .4 & .8 & 0 \end{bmatrix} = \mathbf{R}'.$$

Since $\mathbf{R}' \neq R$, we take $\mathbf{R}'$ as a new matrix $\mathbf{R}$ and, repeating the previous procedure, we obtain

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & .4 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .4 & .4 \end{bmatrix} \quad \mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & .1 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .8 & .4 \end{bmatrix} = \mathbf{R}'.$$

BINARY RELATIONS ON A SINGLE SET

$$\mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .4 & 0 & 0 \\ 0 & 0 & .8 & 0 \end{bmatrix}.$$

1. $R' = R \cup (R \circ R)$.

Applying Step 1 of the algorithm, we obtain

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & .4 \\ 0 & .4 & 0 & 0 \end{bmatrix} \quad \mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 1 \\ 0 & .4 & 0 & .4 \\ 0 & .4 & .8 & 0 \end{bmatrix} = \mathbf{R}'.$$

Since $\mathbf{R}' \neq R$, we take $\mathbf{R}'$ as a new matrix $\mathbf{R}$ and, repeating the previous procedure, we obtain

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & .4 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .4 & .4 \end{bmatrix} \quad \mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & .1 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .8 & .4 \end{bmatrix} = \mathbf{R}'.$$

Since $\mathbf{R}' \neq \mathbf{R}$ at this stage, we must again repeat the procedure with the new relation. If we do this, however, the last matrix does not change. Thus,

$$\begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & 1 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .8 & .4 \end{bmatrix}$$

is the membership matrix of the transitive closure R_T corresponding to the given relation $R(X, X)$.