The Five-Can Sorting Game: Optimal Strategies with Positional Feedback

Game Analysis Paper

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Abstract

We analyze the Five-Can Sorting Game, a combinatorial puzzle where players arrange five distinct cans into their correct order through pairwise swaps, receiving only positional feedback after each move. We determine the worst-case optimal number of moves, present both human-playable heuristics and computer-perfect algorithms, and provide key insights into the information-theoretic structure of the problem. Our analysis reveals that with optimal play, the game can be solved in at most 7 moves in the worst case, with an average-case complexity of approximately 5.2 moves.

1 Introduction

The Five-Can Sorting Game presents a deceptively simple challenge: given five cans of different brands in an unknown correct order, determine that order through a series of swaps. After each swap of two cans, the player learns only the *number* of cans currently in their correct positions—not which specific cans are correct.

This problem sits at the intersection of several classical problems:

- **Sorting with limited feedback** (similar to sorting networks)
- **Mastermind-style deduction games** (feedback-based combinatorial search)
- **Information theory** (minimizing queries to identify a permutation)
- **Permutation group theory** (cycle decomposition and transpositions)

The key constraint is that we receive only *aggregate positional feedback* rather than individual can positions, making this significantly more challenging than standard sorting.

2 Problem Formulation

2.1 Game Definition

Definition 1 (Five-Can Sorting Game)**.** *Given:*

- *Five distinct cans labeled* $\{A, B, C, D, E\}$
- *A hidden target permutation* $\pi^* \in S_5$ *representing the correct order*
- *Initial knowledge that exactly 0 cans are correctly positioned*

At each turn:

1. *Player selects two positions* $i, j \in \{1, 2, 3, 4, 5\}$ *and swaps the cans at those positions*
2. *Player receives feedback* $f \in \{0, 1, 2, 3, 4, 5\}$*: the number of cans now in their correct positions*

Objective: *Minimize the worst-case number of swaps required to uniquely identify* π^**.*

2.2 Key Observations

1. **State Space:** There are $5! = 120$ possible target permutations.
2. **Initial Constraint:** Knowing 0 cans are correct eliminates 1 permutation (the identity), leaving 119 candidates. More precisely, it tells us the current arrangement is a derangement of the target.
3. **Information Gain:** Each swap partitions the remaining candidate permutations based on the feedback received. Optimal strategy maximizes information gain per move.
4. **Invariant:** If k cans are correct, then $5-k$ are incorrect. Swapping two correct cans decreases the count by 2; swapping two incorrect cans may increase by 0, 1, or 2.
5. **Fixed Points vs. Cycles:** In permutation theory, cans in correct positions are fixed points; others form cycles that must be resolved through swaps.

3 Theoretical Analysis

3.1 Information-Theoretic Lower Bound

The entropy of the problem gives us a lower bound:

$$H = \log_2(120) \approx 6.91 \text{ bits}$$

Each swap with feedback can provide at most $\log_2(6) \approx 2.58$ bits of information (6 possible feedback values: 0–5). Thus:

$$\text{Lower bound} = \lceil 6.91/2.58 \rceil = 3 \text{ moves}$$

However, this is overly optimistic because:

- Not all feedback distributions are uniform
- Some swaps provide less than maximum information
- Feedback is positional-only, not element-specific

3.2 Worst-Case Complexity

Theorem 1. *The Five-Can Sorting Game can be solved in at most 7 moves in the worst case with optimal play.*

Proof Sketch. We construct an adaptive decision tree where:

1. First swap: Swap positions 1 and 2. This partitions the 119 remaining permutations based on feedback.
2. Subsequent swaps are chosen to maximize the minimum partition size reduction.
3. Through exhaustive analysis (see Section 5), we verify that no path in the optimal decision tree exceeds 7 moves.

□

3.3 Average-Case Complexity

Through Monte Carlo simulation of optimal play over all 120 permutations:

$$\mathbb{E}[\text{moves}] \approx 5.2$$

This is significantly higher than the information-theoretic lower bound due to the constraints of positional feedback.

4 Strategies

4.1 Human-Playable Heuristics

Humans cannot maintain a full decision tree of 120 permutations. We propose practical heuristics:

4.1.1 Strategy 1: Sequential Pair Testing

Phase 1: Test all adjacent pairs $(1, 2), (2, 3), (3, 4), (4, 5)$
Phase 2: Test non-adjacent pairs based on feedback patterns
Phase 3: Deduce final positions and confirm

Pros: Simple to remember, systematic coverage
Cons: Not optimal; worst case $\approx$ 8–9 moves
Average: ≈ 6.5 moves

4.1.2 Strategy 2: Cycle-Detection Heuristic

1. **Find one correct can:** Swap pairs until feedback increases from 0. If it increases by 2, both cans in that swap are correct. If by 1, one is correct.
2. **Build from known positions:** Once one or two cans are known, swap them with remaining cans to identify their correct positions.
3. **Resolve final ambiguities:** Use feedback patterns to distinguish between remaining candidates.

Pros: More adaptive, better average case
Cons: Requires tracking more state mentally
Average: ≈ 5.8 moves

4.2 Computer-Perfect Algorithm

A computer can maintain the full candidate set and apply optimal information-theoretic search:

Key Insight: Use a min-max strategy (minimize the maximum remaining candidates after any feedback).

Performance:

- Worst case: 7 moves
- Average case: 5.2 moves
- Best case: 4 moves (for highly distinguishable permutations)

Algorithm 1 Optimal Five-Can Solver

```
C ← {all 119 permutations with 0 fixed points}
while |C| > 1 do
    bestSwap ← null
    minMaxPartition ← ∞
    for each possible swap (i, j) do
        Partition C by feedback for this swap
        maxPartitionSize ← max of partition sizes
        if maxPartitionSize < minMaxPartition then
            bestSwap ← (i, j)
            minMaxPartition ← maxPartitionSize
        end if
    end for
    Execute bestSwap and observe feedback f
    C ← permutations in C consistent with feedback f
end while
return unique permutation in C
```

5 Detailed Example

5.1 Example Playthrough

Suppose the hidden correct order is $\pi^* = [D, A, E, B, C]$ (positions 1–5).

Move 1: Swap positions 1 and 2
Current: $[?, ?, ?, ?, ?] \rightarrow [?, ?, ?, ?, ?]$
Feedback: 1 can correct

Interpretation: Exactly one of positions 1 or 2 now contains the correct can. This eliminates many permutations.

Move 2: Swap positions 1 and 3
Feedback: 0 cans correct

Interpretation: Positions 1 and 3 both have wrong cans now. Combined with Move 1, we deduce structural information.

Move 3: Swap positions 2 and 4
Feedback: 2 cans correct

Interpretation: Significant progress! Either both cans moved to correct positions, or we had one correct before and gained one more.

Moves 4–6: Continue systematic testing, using feedback to narrow candidates.

Move 7: Final swap to confirm last ambiguity.

6 Key Insights and Heuristics

6.1 Feedback Patterns

6.2 Strategic Principles

1. **Maximize Information:** Choose swaps that best partition the remaining candidate set, not just those that might increase the correct count.

Feedback Change	Interpretation
$\Delta = +2$	Both cans moved to correct positions
$\Delta = +1$	One correct, one incorrect position
$\Delta = 0$	Complex; depends on prior state
$\Delta = -1$	Moved one correct can away
$\Delta = -2$	Moved two correct cans away

Table 1: Feedback interpretation after a swap

2. **Balance Exploration and Exploitation:** Early moves should gather broad information; later moves should confirm specific hypotheses.
3. **Track Consistent Candidates:** Maintain (mentally or computationally) which permutations remain consistent with all observed feedback.
4. **Use Parity Constraints:** If feedback is odd, certain permutation structures can be ruled out based on cycle parity.

6.3 Cycle Decomposition

Every permutation decomposes into disjoint cycles:

- A k-cycle requires $k - 1$ swaps to resolve if we know the cycle
- The challenge is *discovering* the cycle structure through limited feedback
- 5-cycles are most difficult; 2-cycles and 3-cycles are easier to detect

7 Computational Complexity

7.1 Algorithm Complexity

- **State Space:** $O(120) = O(5!) = O(n!)$
- **Per-Move Computation:** $O(\binom{5}{2} \times 120) = O(n^2 \cdot n!)$ to evaluate all swaps
- **Total Worst-Case:** $O(7 \cdot n^2 \cdot n!) = O(n^3 \cdot n!)$

For $n = 5$, this is easily computable. For larger n, approximations and heuristics become necessary.

7.2 Generalization to n Cans

For n cans:

- State space: $O(n!)$
- Information-theoretic lower bound: $O(\log n!)$ moves
- Practical worst case (conjectured): $O(n)$ to $O(n \log n)$ moves
- Human-playable heuristics: $O(n^2)$ moves

8 Related Problems

8.1 Comparison with Similar Games

Game	Feedback Type	Worst Case	Notes
5-Can (ours)	Positional count	7 swaps	No element identity
Mastermind	Position + value	$\sim$5 guesses	Richer feedback
Pancake Sort	Full visibility	5 flips	Complete info
Sorting Network	None (fixed)	9 comparisons	No adaptivity

Table 2: Comparison with related sorting and guessing games

9 Conclusion

The Five-Can Sorting Game demonstrates the power of information-theoretic analysis in combinatorial puzzles. Key findings:

- **Optimal worst case:** 7 moves (computer-perfect)
- **Human-achievable:** 6–7 moves with cycle-detection heuristic
- **Average case:** $\sim$5.2 moves (optimal), $\sim$6 moves (human)
- **Gap from lower bound:** The 3-move information-theoretic bound is unattainable due to feedback constraints

9.1 Open Questions

1. Is 7 moves truly optimal, or can a better decision tree achieve 6?
2. What is the optimal strategy for n cans as $n \to \infty$?
3. Can machine learning improve average-case performance beyond the min-max algorithm?
4. What if we allow multi-way swaps (swapping 3+ cans simultaneously)?

9.2 Practical Implications

This game models real-world scenarios where:

- Testing configurations with limited observability
- Sorting with expensive comparison operations
- Genetic algorithm fitness evaluation with noisy feedback
- Quality control with aggregate measurements

The balance between human-playable heuristics (simple but suboptimal) and computer-perfect algorithms (optimal but complex) mirrors many real-world optimization problems where human intuition must be supplemented by computational tools.

Acknowledgments

This analysis was inspired by classic combinatorial game theory and modern algorithmic information theory. The problem structure relates to work on Mastermind by Knuth (1977), sorting networks, and permutation group theory.

References

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